The Field Ontology in Quantum Field Theory

Xabier Oianguren-Asua

Extended Version of a Talk given in Tübingen Universität within
"Seminar Topics in Quantum Mechanics" — directed by Prof. Roderich Tumulka

Contents

The Field Ontology in Quantum Field Theory

Xabier Oianguren-Asua

Extended Version of a Talk given in Tübingen Universität within
"Seminar Topics in Quantum Mechanics" — directed by Prof. Roderich Tumulka

A. Particles, Fields or both?

After finding how Bohmian Mechanics explains all from non-relativistic quantum mechanics (QM) till quantum field theories (when equipped with a configuration POVM) just by postulating that there are actual point-like particles that exist irrespective of our aims to detect them, and that still they yield all, from the visible image of the world to the results of quantum experiments, we are now more than used to point-particles as a primitive ontology. Nomologically piloted by a configuration-space wavefunction or not, primitively identical or not, move them fully continuously or interrupted by random "mitosis" (creations) or annihilations when brought to close proximity, finite or perhaps even if infinite (recall the Dirac sea construction), particles are now our friends. And in all of the above constructions Bohmian Mechanics proudly shines its astral medal: it explains mechanistically the collapse postulate of textbook QM, providing a very cheap but valuable solution to the measurement problem just as a simple corollary.

But, could we consider that instead of infinitely small particles what there is actually are just the opposite: uninterrupted extensions that fill the whole of spacetime? In a classical setting you were used to them: the *physical-space fields* (as opposed to configuration-space fields), like the electric and magnetic field or a density field in fluid mechanics. In today's excursion, instead of particle configuration space, say, some $\mathbb{R}^{3N}$ (where each point represents a particular position for N particles in 3-space), we will consider *field configuration-space*, say $L^2(\mathbb{R}^3, \mathbb{R}^N)$, a space where each point gives N determinate fields that occupy the whole space. Instead of a "superposition" of particle configurations (represented by a wavefunction giving a complex number for each configuration), we will have a "superposition" of fields (given by wavefunction-*al* assigning a complex number to each field). And as you have possibly guessed so-far, such objects will be actually very natural to Quantum ***Field*** Theory (QFT).

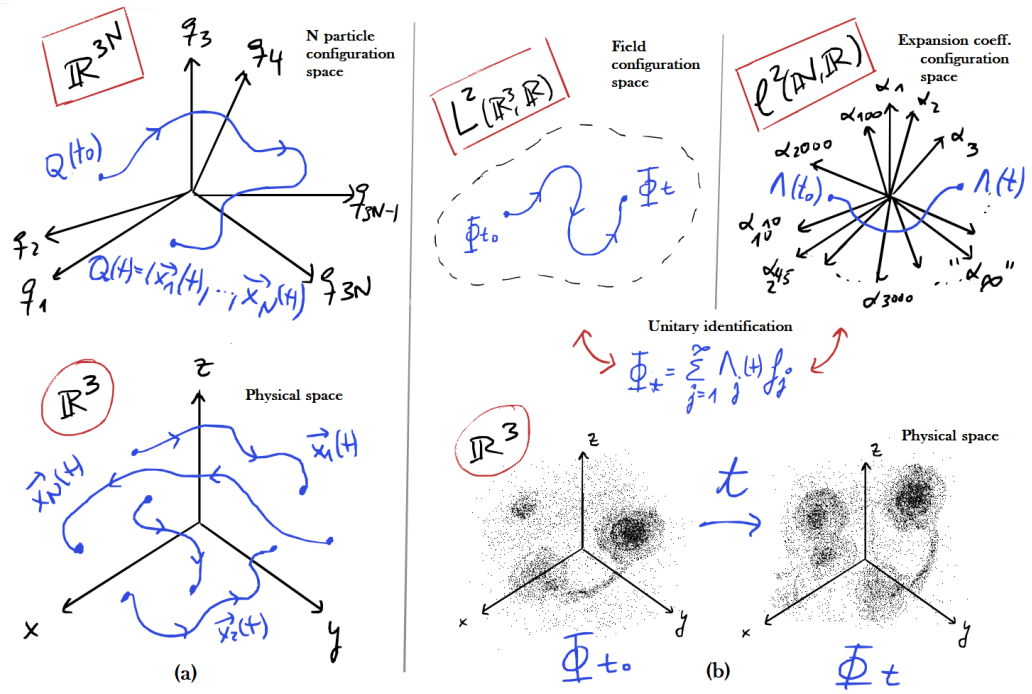

Figure 1: (a) shows a representation of configuration-space of N point-like particles, given by $\mathbb{R}^{3N}$. A point there represents a particular arrangement of the N positions of the particles in physical-space. (b) shows a representation of a field configuration-space and its associated expansion-coefficient configuration-space for some ONB $\{f_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}^3, \mathbb{R})$. Each point in either space, corresponds to a whole physical-space field.

A.1. A Provocative Example: Bohm's 1952 paper. Yeah, Bohm again

In an appendix of his seminal paper of two parts, Bohm (1952) already gave us a taste of what a "field ontology" would mean in QM, in the very same article presenting what a "particle ontology" means. Let us describe it as an introductory ignition because it is essentially an example for everything we are going to be talking about today (and clarity often emerges by going back to examples).

Consider the classical description of N point-like particles (say, electrons) of position-velocity coordinates $\{(\vec{r}_j, \dot{\vec{r}}_j)\} \subset \mathbb{R}^{6N}$ and electric charges $\{e_j\}_j$ and masses $\{m_j\}_j$. Consider also an electromagnetic 4-potential field $(\phi_{\vec{r}}, \vec{A}_{\vec{r}}) : \mathbb{R}^3 \to \mathbb{R}^4$, composed by fields on some Sobolev space. You might know that the Lagrangian of such a classical system is given by

$$L(\vec{r}_j, \dot{\vec{r}}_j, \vec{A}_{\vec{r}}, \dot{\vec{A}}_{\vec{r}}, \phi_{\vec{r}}) = \sum_{j=1}^{N} \left(\frac{1}{2} m_j |\dot{\vec{r}}_j|^{\,2} + \frac{e_j}{c}\, \dot{\vec{r}}_j \cdot \vec{A}_{\vec{r}_j} - e_j \phi_{\vec{r}_j} \right) + \frac{\varepsilon_0}{2} \int \left(|\vec{E}_{\vec{r}}|^2 - c^2 |\vec{B}_{\vec{r}}|^2 \right) d^3 r \qquad (1)$$

where the electric and magnetic field are defined respectively as $\vec{E}_{\vec{r}} := -\dot{\vec{A}}_{\vec{r}} - \vec{\nabla}\phi_{\vec{r}}$ and $\vec{B}_{\vec{r}} := \vec{\nabla} \times \vec{A}_{\vec{r}}$.[1] The Euler-Lagrange equations (we'll see in a moment how to take derivatives with respect to field variables rigorously) are exactly the Maxwell equations (the two that do not follow from the definitions of $\vec{E}$ and $\vec{B}$) and the Newton's second Law with the Lorentz and Coulomb forces in it. As might you know, the resulting equations are poorly defined because each particle is centred in a singularity of the total force field (among other things). Let us ignore that for a moment. Fix the Coulomb gauge ($\vec{\nabla} \cdot \vec{A} = 0$) and explicitly write $\phi_{\vec{r}} = \sum_k \frac{e_k}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_k|}$, which is the solution to the Gauss equation (i.e., the Poisson equation in Coulomb gauge) for a point-charge distribution $\{\vec{r}_j\}_j$. This way we get rid of ϕ as a degree of freedom. Then we can get rid of some of the problems of electrodynamics by assuming there is no "self-interaction" by "re-scaling" the energy (the first historic "renormalization"...) so that defining $V_{j,k}(|\vec{r}_j - \vec{r}_k|) := e_j e_k / (4\pi\varepsilon_0 |\vec{r}_j - \vec{r}_k|)$, the evaluation of ϕ in the Lagrangian turns

$$\text{from} \quad \frac{1}{2} \sum_j \sum_k V_{j,k} \quad \text{(infinite whenever } j = k\text{) to} \quad \frac{1}{2} \sum_j \sum_{k \neq j} V_{j,k}. \qquad (2)$$

Now, as explained in mathematical rigour for instance in Abraham et al. (2012), one can consider a symplectic structure for the infinite dimensional phase space of $\vec{A}_{\vec{r}}$ and its canonically conjugate variable $\vec{\pi}_{\vec{r}} := \dot{A}_{\vec{r}}$, as well as for the usual phase space for the particles, so that a Legendre transformation leaves us a Hamiltonian system with a Hamiltonian that after some massaging (integration by parts) looks like:

$$H = \sum_{j=1}^{N} \frac{(\vec{p}_j - e_j \vec{A}_{\vec{r}_j})^2}{2m_j} + \frac{1}{2} \sum_j \sum_{k \neq j} V_{j,k}(|\vec{r}_j - \vec{r}_k|) + \frac{\varepsilon_0}{2} \int (\dot{\vec{A}}_{\vec{r}} + c^2 |\vec{B}_{\vec{r}}|^2) \qquad (3)$$

Assume we want to describe electrodynamics in a cavity (or in a finite Universe...), so that the physical space can be considered instead of $\mathbb{R}^3$ to be $\Omega := (-\frac{L}{2}, \frac{L}{2})^3$ for some fixed $L > 0$. Then, we can artificially consider periodic boundary conditions so that using only Fourier modes of commensurate wavelengths with the box, $\vec{k} \in (\frac{2\pi}{L})\mathbb{Z}^3$, we can expand in Fourier series any L^2 function inside the box. In this line, from PDE theory one gets that there is a countable number of solutions indexed by $\vec{k} \in \mathbb{Z}^3(2\pi/L)$ to the constrained eigenvalue problem

$$\begin{cases} \Delta \vec{u}_{\vec{k}}(\vec{r}) = -|\vec{k}|^2 \vec{u}_{\vec{k}}(\vec{r}) \\ \vec{\nabla} \cdot \vec{u}_{\vec{k}}(\vec{r}) = 0 \end{cases} \qquad (4)$$

yielding an orthonormal basis (ONB) of $\{\vec{f} \in L^2(\Omega, \mathbb{R}^3) \mid \vec{\nabla} \cdot \vec{f} = 0\}$. In fact, although they are vector valued, they will be made of complex exponentials (or rather sines and cosines) —so-called plane-waves or Fourier modes. Because they form an ONB, whatever $\vec{A}_{\vec{r}} \in L^2(\Omega, \mathbb{R}^3)$ is, there exists a sequence of coefficients $\{\alpha_{\vec{k}}\}_{\vec{k} \in (\frac{2\pi}{L})\mathbb{Z}^3} \subset \mathbb{C}$ so that

$$\vec{A}_{\vec{r}} \equiv \sum_{\vec{k} \in (\frac{2\pi}{L})\mathbb{Z}^3} \alpha_{\vec{k}}\, \vec{u}_{\vec{k}}(\vec{r}) \quad \Longrightarrow \quad \vec{A}_{\vec{r}}(t) \equiv \sum_{\vec{k} \in (\frac{2\pi}{L})\mathbb{Z}^3} \alpha_{\vec{k}}(t)\, \vec{u}_{\vec{k}}(\vec{r}) \quad \begin{cases} \text{If the field depends on time, the depen-} \\ \text{dence will be charged to the coefficients!} \end{cases} \qquad (5)$$

[1] Two of the Maxwell equations are engraved in these definitions. By vector calculus, $\vec{\nabla} \times \vec{E}_{\vec{r}}(t) \equiv -\frac{\mathrm{d}}{\mathrm{d}t} \vec{B}_{\vec{r}}(t)$ and $\vec{\nabla} \cdot \vec{B}_{\vec{r}}(t) \equiv 0$.

Moreover, there is a one-to-one correspondence between $(\alpha_{\vec{k}})_{\vec{k}} \in \ell^2\left(\frac{2\pi}{L}\mathbb{Z}^3, \mathbb{C}\right)$ and $\vec{A}_{\vec{r}} \in L^2(\Omega, \mathbb{R}^3)$, so we can equally use the coefficients as the degrees of freedom to specify a field-configuration instead of the vector field $\vec{A}$. Plugging this in the Hamiltonian we get a Hamiltonian now as a function of the expansion coefficients:

$$H(\vec{r}_j, \alpha_k, \vec{p}_j, \dot{\alpha_k}) = \sum_{j=1}^{N} \frac{\left(p_j - e_k \overbrace{\sum_k \alpha_{\vec{k}}\, \vec{u}_{\vec{k}}(\vec{r}_j)}^{\vec{A}_{\vec{r}_j}}\right)^2}{2m_j} + \frac{1}{2}\sum_j \sum_{k\neq j} V_{j,k}(|\vec{r}_j - \vec{r}_k|) + \frac{1}{2}\sum_{\vec{k}\in\frac{2\pi}{L}\mathbb{Z}^3}\left(\pi_{\vec{k}}^2 + |\vec{k}|^2\, \alpha_{\vec{k}}^2\right) \tag{6}$$

(One can find that what we did is essentially a canonical transformation in infinite dimensional vector space.) If you realize, the Hamiltonian is just describing a harmonic oscillator for each of the mode coefficients! And we know a working quantization prescription in such a case! But first, by reasons we will see later (spoiler: there is no Lebesgue measure in infinite dimensions), we must assume what is called an "ultraviolet (UV) cut-off": essentially that all coefficients above certain finite $K > 0$ (in terms of the norm of $\vec{k}$) are roughly zero, so we are left with only M degrees of freedom that fully describe the vector field $\vec{A}$, say $\{\alpha_k\}_{k=1}^M$. Equivalently, since this puts a limit on the frequency of the Fourier modes we are using, it is a limit on the spatial variation of the vector $\vec{A}$. If this makes you feel discomfort (as it should), assume that K is veery big, like $K = 10^{10^{100}}$, so that the limitation is posed only for length-scales that are absurdly small.

The cut-off leaves us with finitely many degrees of freedom, so we can perform a canonical quantization *à la* Dirac, i.e., "hocus pocus, now positions and momenta are operators satisfying certain commutation relations", we get (not-fully-relativistic) quantum electrodynamics (QED). That is, the quantum counterpart of classical electrodynamics will have each system described by a wavefunction $\psi_t \in L^2(\mathbb{R}^{3N+M}, \mathbb{C}, d^{3N+M}x)$ ($d^n x$ is the Lebesgue measure for $\mathbb{R}^n$), with arguments $\psi_t(\vec{r}_1, ...\vec{r}_j, \alpha_1, ..., \alpha_M)$ that for all $t \in \mathbb{R}$ obeys the following Schrödinger Equation (SE):

$$i\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\psi_t(\{\vec{r}_j, \alpha_k\}) = \left[\sum_{j=1}^{N} \frac{\left(-i\hbar\vec{\nabla}_{\vec{r}_j} - e_j\vec{A}(\vec{r}_j; \{\alpha_k\}_k)\right)^2}{2m_j} + \frac{1}{2}\sum_j\sum_{k\neq j} V_{j,k}(|\vec{r}_j - \vec{r}_k|) + \sum_{k=1}^{M}\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\alpha_k^2} + \frac{|\vec{k}|^2}{2}c_k\right)\right]\psi_t(\{\vec{r}_j, \alpha_k\}) \tag{7}$$

with $\vec{A}(\vec{r}_j; \{\alpha_k\}_k) := \sum_{k=1}^M \alpha_{\vec{k}}\, \vec{u}_{\vec{k}}(\vec{r}_j)$.

Now, one could wonder how robust this is if we had changed the expansion basis before quantization, or we had chosen another gauge, or how different would it be if the cavity was of another size than L. And those are very good questions that do matter, but then we realize that the SE we found is exactly what physicists use in solid state and quantum optics still today! So perhaps our *Ansätze* were not that bad after all.

Back to our point: since we have a finite number of degrees of freedom, this is the type of quantum mechanics we are all used to, so that we can copy the idea of Bohmian mechanics here! That is, we find there is a continuity equation for $|\psi_t(\{\vec{r}_j, \alpha_k\})|^2$ (obtained by taking $\frac{\mathrm{d}\psi_t\psi_t^*}{\mathrm{d}t} = \psi_t\frac{\mathrm{d}\psi_t^*}{\mathrm{d}t} + \psi_t^*\frac{\partial\psi_t}{\partial t}$ and plugging the SE there)

$$\frac{\mathrm{d}|\psi_t|^2}{\mathrm{d}t} = -\vec{\nabla}\cdot\left(|\psi_t|^2\,\vec{v}\right) \quad \text{where } \vec{v} := (\vec{v}^{\,r_1}, ...\vec{v}^{\,r_N}, v^{\alpha_1}, ..., v^{\alpha_M}) \tag{8}$$

$$\vec{v}^{\,r_l}(\{\vec{r}_j, \alpha_k\}) := \mathrm{Re}\left\{\frac{-i\hbar\vec{\nabla}_{\vec{r}_l}\psi_t(\{\vec{r}_j, \alpha_k\})}{\psi_t(\{\vec{r}_j, \alpha_k\})}\right\} \quad \text{and} \quad v^{\alpha_l}(\{\vec{r}_j, \alpha_k\}) := \mathrm{Re}\left\{\frac{-i\hbar\frac{\partial}{\partial\alpha_l}\psi_t(\{\vec{r}_j, \alpha_k\})}{\psi_t(\{\vec{r}_j, \alpha_k\})}\right\}. \tag{9}$$

Next, we postulate that there is an actual, determined set of N point-particles and an actual electromagnetic field, of joint trajectory $t \mapsto \left(\vec{R}_j^{\,\xi}(t),\ \Lambda_k^{\xi}(t)\right)_{j,k}$ following the velocity field that guides the density in the continuity equation, i.e.,

$$\begin{cases} \dfrac{\mathrm{d}\vec{R}_j^{\,\xi}(t)}{\mathrm{d}t} = \vec{v}^{\,r_j}\left(\left\{\vec{r}_j = \vec{R}_j^{\,\xi}(t),\ \alpha_j = \Lambda_k^{\xi}(t)\right\}\right) \\ \dfrac{\mathrm{d}\Lambda_j^{\xi}(t)}{\mathrm{d}t} = v^{\alpha_j}\left(\left\{\vec{r}_j = \vec{R}_j^{\,\xi}(t),\ \alpha_j = \Lambda_k^{\xi}(t)\right\}\right) \\ (\vec{R}_j^{\,\xi}(t_0), \Lambda_k^{\xi}(t_0)) = \vec{\xi} \end{cases} \qquad \text{for } j \in \{1, ..., N\}, k \in \{1, ..., M\} \tag{10}$$

for some "initial condition" $\vec{\xi} \in \mathbb{R}^{3M+N}$. Then, $\vec{A}^{\xi}_{\vec{r}}(t) := \sum_k \Lambda^{\xi}_k(t)\vec{u}_k(\vec{r})$ would be the actual vector-potential field selected by the trajectory of the expansion coefficients $\Lambda^{\xi}_k(t)$ (and hence there would be a particular electric and magnetic field $\vec{E}, \vec{B}$ selected by this $\vec{A}_{\vec{r}}(t)$). Because the trajectory is a flow-line of the velocity field guiding the density, if $\vec{\xi}$ was sampled following the probability measure $|\psi_{t_0}|^2 d^{3N+M}x$, the trajectory $\left(\vec{R}^{\,\xi}_t(t), \Lambda^{\xi}_k(t)\right)_{j,k}$ will be distributed according to $|\psi_t|^2 d^{3N+M}x$ at all times t. Finally, assuming that measurement device pointers are made of some of the N particles, the measurement problem and the collapse postulate would be a corollary just as in Bohmian mechanics.

A.1.1. Digression: Bosons as Virtual Particles — The Beauty of a Field Ontology?

All in all, we have found that in the presented theory, the charged particles have a particle ontology (i.e., the particles exist as points at all times), while the electromagnetic field has a field ontology (i.e., the electric and magnetic fields exist as physical-space fields at all times). Akin to the classical theory. This is opposed to what one thinks about when we talk about a *photon*, a light particle, often said to be what emerges when we quantize the electromagnetic field. But where is it here? In this ontology a photon is nothing but an apparent particle, a *virtual particle*.

Recall that $L^2(\mathbb{R}^{3N+M}) = L^2(\mathbb{R}^{3N}) \otimes L^2(\mathbb{R}^M)$, with $L^2(\mathbb{R}^M)$ the Hilbert space over the field configurations —or more precisely their expansion coefficients. Now, from the Hamiltonian in (7), we see that each coefficient α_k behaves as a harmonic oscillator coupled to the particle velocities. If there were no charged particles, $N = 0$, we would just have the electromagnetic (EM) field and then the α_k would be isomorphic to independent harmonic oscillators with a full Hamiltonian given by $H(\{\alpha_k, \pi_k\}) = \sum_k H_{free\,k}$ and $H_{free\,k} := \frac{1}{2}\pi_k^2 + c|k|^2\alpha_k^2$. The latter has an ONB of eigenfunctions on $L^2(\mathbb{R})$ (which is the k-th mode's Hilbert space), that we denote by $\{\phi^k_n\}_{j\in\mathbb{N}_0} \subset L^2(\mathbb{R})$. They are well-known to be Gaussians times Hermite polynomials $H_n(x)$:

$$\phi^k_n(\alpha_k) := \frac{1}{\sqrt{2^n n!}\pi^{1/4}}\, e^{-\frac{1}{2}\alpha_k^2} H_n(\alpha_k) = \frac{1}{\sqrt{n!2^n}\pi^{1/4}}\left(\alpha_k - \frac{\partial}{\partial x}\right)^n e^{-\alpha_k^2/2}. \tag{11}$$

With that, we get an ONB now in the full Hilbert space of the field coefficients $L^2(\mathbb{R}^M)$ by taking tensor products of the possible combinations of "single-mode ONBs": $\{\phi^{k_1}_{j_1} \otimes \cdots \otimes \phi^{k_M}_{j_M}\}_{j_1,\ldots,j_M\in\mathbb{N}} \subset L^2(\mathbb{R}^M)$, so that any wavefunction over the field-configurations can be expanded as a superposition of them. Now, what people call "electromagnetic vacuum" is nothing but the ground state of a free electromagnetic field (i.e., where there are no charges), given by the wavefunction

$$\psi(\alpha_{k_1}, ..., \alpha_{k_N}) = \phi^{k_1}_0(\alpha_{k_1}) \cdots \phi^{k_M}_0(\alpha_{k_M}) = C\, e^{-\sum_k \alpha_k^2}, \tag{12}$$

As its associated probability density $|\psi|^2$ is a Gaussian around 0, most Bohmian trajectories of the modes $\{\alpha_k\}_k$ will be close to zero $\Lambda_k \simeq 0$, implying that typical Bohmian worlds will have an electromagnetic potential $\vec{A}_{\vec{r}} \simeq 0$ (and hence electric and magnetic fields $\vec{E}, \vec{B} \simeq 0$). But there are a significant part of trajectories Λ_k that are a bit further from the origin, which causes the so-called "vacuum fluctuations".

On the other hand, we say there is only one photon and has frequency $\vec{k}_j$ if the wavefunction is

$$\psi(\alpha_{k_1}, ..., \alpha_{k_N}) = \phi^{k_1}_0(\alpha_{k_1}) \cdots \phi^{k_j}_1(\alpha_{k_j}) \cdots \phi^{k_M}_0(\alpha_{k_M}) \propto \alpha_k\, e^{-\sum_k \alpha_k^2/2}. \tag{13}$$

two photons of frequency $\vec{k}_j$ if

$$\psi(\alpha_{k_1}, ..., \alpha_{k_N}) = \phi^{k_1}_0(\alpha_{k_1}) \cdots \phi^{k_j}_2(\alpha_{k_j}) \cdots \phi^{k_M}_0(\alpha_{k_M}) \propto (\alpha_k^2 - 1)\, e^{-\sum_k \alpha_k^2/2}. \tag{14}$$

and so on. That is, a higher excitation of the harmonic oscillator represented by the mode α_k corresponds to an additional photon of that k-th frequency. This is closely related to the Fock space representation as we will see later.

But where is in all this description the photon that is absorbed and emitted in discrete energy packets? That this happens in packets is an experimental fact! Well, let us put back one of the particles in the description. Assume it is an electron in a hydrogen atom, so that it has well-defined energy eigenstates: we denote them by $\varphi_0, \varphi_1, ...$ starting from the ground state (the s-orbital). Also, assume we have sent against the atom a pulse of a laser

and then enclosed it between mirrors. The laser has a very sharp frequency $\vec{k}$, so that we have enough leaving only one of the expansion modes for simplicity, say, the α_k (so that we only consider fields like $\vec{A}_{\vec{r}} \simeq \alpha_k \vec{u}_{\vec{k}}$). Then one can find that if we start an experiment with a joint wavefunction $\psi(\vec{r}, \alpha_k) = \varphi_0(\vec{r})\phi_1^k(\alpha_k)$, that is, a tensor product of a "single photon" state ϕ_1^k and the ground state of the electron φ_0, the Schrödinger equation will make them evolve very close to $\varphi_1(\vec{r})\phi_0^k(\alpha_k)$ —hence, a state describing no photon but the electron now in the excited state— and then back. This are the so-called Rabi oscillations, and it is just the atom absorbing and emitting a photon over and over again (it is in between mirrors). This oscillation really only happens in a meaningful way when the eigen-energy difference between ground state and first excited state of the electron and that of the mode match. But because the latter is exactly given by the frequency $\vec{k}$ of the mode (constants up or down), the Rabi oscillation essentially only happens if the laser with which we have lighted the atom contains a frequency matching the energy interval for the electron's orbitals (light-matter resonance). If we left the atom in an open environment and not between mirrors, the photon would be absorbed, then emitted and that explains the spectral lines being so specific. Likewise one can explain the photoelectric effect, among others. If you take a quantum optics course you will learn more about this.

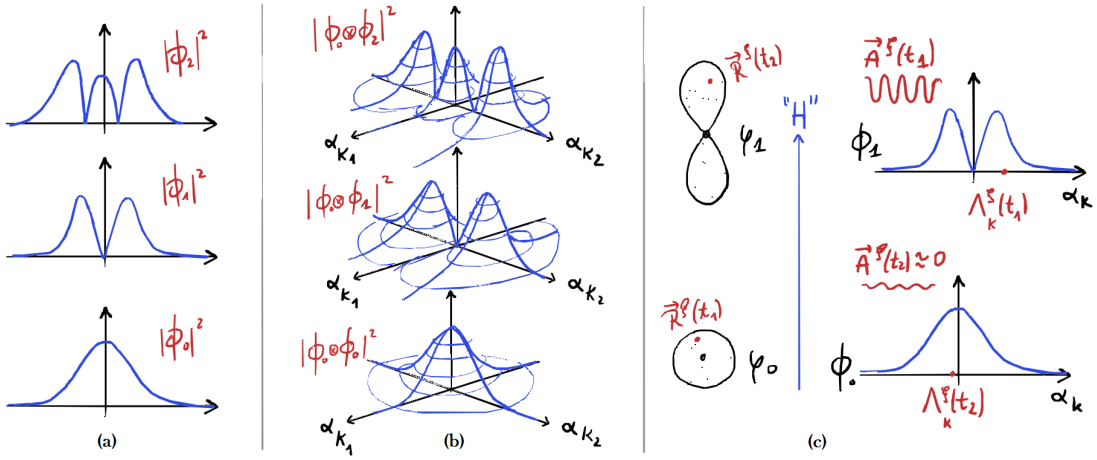

Figure 2: In (a) the magnitude squared of the first three eigenstates of the harmonic oscillator are shown. In (b), for a cut-off at $M = 2$ (two EM field modes), from bottom to top we see: the EM "vacuum" $\phi_0 \otimes \phi_0$; a photon of frequency $\vec{k}_2$ and none of $\vec{k}_1$, i.e., $\phi_0 \otimes \phi_1$; two photons of frequency $\vec{k}_2$ and none of $\vec{k}_1$, $\phi_0 \otimes \phi_2$. In (c) we see the ground state and first excited state of an electron in an atom, next to those of a single mode (the k-th one) of the EM field. The spacing between levels suggests how the eigen-energy difference in one and the other system matches for this particular $\vec{k}$. We represent example Bohmian positions of the electron and the EM mode coefficient, together with its respective vector potential (if there is a single mode, the vector potential will be a plane wave of the given frequency).

Now, in all the explanation, note there was never an abrupt absorption of anything, all the dynamics had a "soft" time evolution. Moreover, in general whenever we have an entangled wavefunction between the electron and the modes of the EM field, there is no way to attribute a wavefunction to the electron and photon separately (unless we use conditional wavefunctions, which is actually what one should do to talk about absorption and emission in the above picture more meaningfully —alternatively, one may use reduced density matrices).

The point is that in Bohmian terms, with an actual ontological EM field and an actual electron, there is never a discrete particle that is absorbed or emitted! Instead, whenever the wavefunction's part attributed to the field evolves to a no-photon state (i.e., the joint wavefunction factorizes as $\phi_0^k\varphi_1$), then the ontological field $\vec{A}_{\vec{r}}$ will likely be driven towards zero field strength (the Gaussian is centred in zero) and when back to one photon (i.e., the joint wavefunction factorizes as $\phi_1^k\varphi_0$), because the first excited state of the mode has two bumps away from zero, the ontological field $\vec{A}_{\vec{r}}$ is driven to some non-zero field strength. As we see, photons would then be only a "way of talking". What there would be, is an interaction between the ontological electrons $\{\vec{R}_j(t)\}_j$ and the EM field given by $\{\Lambda_j(t)\}_j$ mediated by the joint wavefunction $\psi(\{\vec{r}_j\}, \{\alpha_k\}_k)$, but in terms of ontology, it would look quite similar to what you would expect classically. In fact more than you think! More specifically, given a solution to the SE above, one can decompose the wavefunction in polar form $\psi(\{\vec{r}_j\}, \{\alpha_k\}_k) = R(\{\vec{r}_j\}, \{\alpha_k\}_k)e^{iS(\{\vec{r}_j\},\{\alpha_k\}_k)/\hbar}$, which separates the SE into two real PDEs obeyed by $S(\{\vec{r}_j\}, \{\alpha_k\}_k)$ and $R(\{\vec{r}_j\}, \{\alpha_k\}_k)$: one is the continuity equation we mentioned with velocity field $\vec{v}$ and density $R^2 = |\psi|^2$, while the other one is a Hamilton-Jacobi equation for the classical Hamiltonian (6) with

an extra "potential energy" term, so-called Quantum potential Q^ψ (Holland, 1995). Evaluating this Hamilton-Jacobi equation along the flow-lines of the Bohmian velocity field $\vec{v}$, i.e., as if we took the Lagrangian frame of fluid mechanics (for a fluid of Bohmian trajectories), we find an ODE obeyed by each Bohmian trajectory, which is what inhabitants of the trajectories, i.e., possible particular Bohmian worlds, could think to be the fundamental equation ruling the particles and the field.[2] It turns out, the particles still follow exactly the Newton-Lorentz equation you all know and the EM field follows the Maxwell equations, just that there is the additional force term for the particles due to the "quantum potential" and an additional inhomogeneity for one[3] of the Maxwell equations.[4] Namely,

$$m_j \frac{\mathrm{d}^2 \vec{R}_j^{\,\xi}(t)}{\mathrm{d}t^2} = e_j \left(\vec{E}^{\,\xi}(\vec{R}_j^{\,\xi}(t), t) + \frac{\mathrm{d}\vec{R}_j^{\,\xi}(t)}{\mathrm{d}t} \times \vec{B}^{\,\xi}(\vec{R}_j^{\,\xi}(t), t) \right) - \vec{\nabla}_{r_j} Q^\psi|_{\vec{R}_j^{\,\xi}(t), \Lambda_k^\xi(t)} \tag{15}$$

$$\vec{\nabla} \times \vec{B}_{\vec{r}}^{\,\xi}(t) = \frac{1}{c^2} \frac{\partial \vec{E}_{\vec{r}}^{\,\xi}(t)}{\partial t} + \mu_0 \vec{J}_{\vec{r}}^{\,\xi}(t) - F(Q^\psi)|_{\vec{R}_j^{\,\xi}(t), \Lambda_k^\xi(t)} \tag{16}$$

where $\vec{J}_{\vec{r}}^{\,\xi} := \sum_j e_j \delta(\vec{r} - \vec{R}_j^{\,\xi}(t))$ is the usual charge current of classical electrodynamics (with δ the Dirac delta "function"). Q^ψ is the quantum potential and F a map related to the inverse Fourier transform, neither of which we need to detail here.

All in all, if you are more comfortable with it, photons can indeed be considered to be a convenient fiction to describe the interaction of quantized light and matter with electronic orbital shells, but nothing more than that in ontological terms.

But somebody could insist: what about photon detection screens? I can see photons to be detected as points there! Well, technically what we really see there is the light emitted by an atom in the screen. The input EM field's wavefunction initially factorized from that of the electron in that atom, suffered a dynamics that coupled them together, generating an entangled wavefunction that performs a Rabi oscillation and makes the atom re-emit the light now in all directions. Since the emission is sourced from essentially a point (the atom), it looks like a point recording to us. But again, light was "absorbed and emitted by the atom" in a non-abrupt manner, and the ontological picture would be close to our classical image of electrodynamics.

Just to be fair, note that an alternative view can also be developed using a Fock space representation of this QFT. Employing the Bell-type QFT technique I explained in the last seminar, if we find a position POVM for the photons, we will be able to give them a particle ontology. In such a theory, there would be actual (random) photon creation and annihilation events that look more like literal absorptions and emissions. As of today, there is a problem with finding a position operator for photons (Tumulka, 2022), but still this might be sooner or later overcome. So at the end of the day it might be up to your taste: fields or particles... But please, fix one! You could try to consider that the same thing (e.g., a photon) can be two things at the same time: particle and wave, embracing the cheap orthodox metaphor of "wave-particle duality", but, if there exist remarkably simpler explanations in which one thing is just one thing, say a particle, or say a field...well, apply Occam's razor!

[2] Note the reverse does not necessarily hold, meaning that given a solution to the continuity equation and the Hamilton Jacobi equation they do not necessarily make up a wavefunction that solves the SE! Hence it is a nice way to find the classical limit of a quantum theory, because it shows the law of motion of the Bohmian trajectory as seen from within the trajectory —and it turns out to only have an extra term with respect to the classical counterpart— but this is not an equivalent PDE system to the SE nor they give an equivalent formulation of QM —in which there is no explicit wavefunction.

[3] Two of the Maxwell equations are readily given as in footnote 1. The Gauss equation was "hardcoded" in our explicit solution to the scalar potential $\phi_{\vec{r}}^{\xi}(t) = \sum_k \frac{e_k}{4\pi\varepsilon_0 |\vec{r} - \vec{R}_k^{\,\xi}(t)|}$. Hence only the Ampère-Maxwell law has a quantum "perturbation".

[4] Both extra terms depend on the wavefunction ψ governing them all across configuration-space, which makes them highly non-local. This breaks for instance the locality properties of the Maxwell equations.

B. THE LAGRANGIAN QFT APPROACH AND ITS NATURAL PILOT-WAVE THEORIES

We will now see in a more general setting that if we start from a field theoretic Lagrangian, i.e., depending on some field-configuration space, say, $L^2(\mathbb{R}^3,\mathbb{R})$, quantization naturally seems to yield wavefunctions $\Psi_t(\phi)$ over such possible field-configurations $\phi \in L^2(\mathbb{R}^3,\mathbb{R})$. In the example above we used a trick to make the degrees of freedom of the field finite, but a priori there is nothing wrong with a complex valued function with arguments in infinite dimensional vector spaces, say $\Psi_t(\alpha_1,\alpha_2,...)$ or $\Psi_t(\vec{A}_r)$, which are so-called *wavefunction**als***. In order to demystify calculus in infinite dimensions you can now proceed to Appendix B.1. There, we rigorously find what physicists call "functional derivatives" and we find a rigorous (and rather simple) derivation of the field theoretic Euler-Lagrange equations.

B.1. Canonical Quantization of Fields (Non-Rigorous)

In this section we will derive the heuristic generalization of canonical quantization to Hamiltonian systems in infinite dimensional symplectic manifolds that we exemplified in the beginning. This approach to build QFTs is also called Lagrangian QFT (Wallace, 2006) —although I would rather call it Hamiltonian QFT. Any lecture on QFT by physicists essentially approaches the topic this way, perhaps because it motivates the "field" part of quantum *field* theory, and perhaps because the whole Standard model is built essentially in an analogous way. Nevertheless, it is not a rigorous approach to QFT (although rigorizations in different directions do exist, such as the ones we will mention later). We chose to explain this because it is the approach to QFT where the field ontology emerges most naturally by generalizing Bohmian mechanics in the line of what we did in the introduction.

Assume we have a finite or infinite dimensional separable Hilbert space[5] $(Q,\langle\cdot,\cdot\rangle)$, where the points $\eta \in Q$ represent the configuration of some physical quantity at a friezed time. We will call it **configuration-space**. As driving examples we will take $(\mathbb{R}^N,\langle\cdot,\cdot\rangle_2)$ as the configuration space of N particles and $(H^2(\mathbb{R}^3,\mathbb{R}),\langle\cdot,\cdot\rangle_{H^2})$ as the possible charge density fields. The time evolution of the physical quantity η is assumed then to be given by a path $(t_0,t_f)\subset\mathbb{R}\to Q;\ t\mapsto\eta(t)$ in the space[6] Q and will usually be given as an ODE system, constituting what we call a "law of physics for η". For the example configuration-spaces, we could be interested on Newton's laws for the particles or the Klein-Gordon (KG) equation for the charge density (which is the wave equation if $m=0$),

$$m_j\ddot{q}_j(t) = -\frac{\partial V(q)}{\partial q_j}|_{q(t)} \qquad \text{and} \qquad \ddot{\phi}(t) = \Delta\phi(t) - m^2\phi(t). \tag{17}$$

Now assume we wish to find candidates to quantum theories whose "classical" counterparts are described by these ODE systems: so-said we want to "quantize" what q or ϕ represent. The trick of canonical quantization goes as follows:

1. Find the Lagrangian/action principle of whom, the ODEs in question are the Euler-Lagrange equations. In the guiding examples, respectively for $q\in\mathcal{C}^2((t_0,t_f),\mathbb{R}^n)$ and $\phi\in\mathcal{C}^2((t_0,t_f),H^2(\mathbb{R}^3,\mathbb{R}))$

$$L(q,\dot{q}) = \sum_j \frac{1}{2}m_j\dot{q}_j^2 - V(q) \qquad \text{and} \qquad L(\phi,\dot{\phi}) = \int_{\mathbb{R}^N}\mathscr{L}\Big(\phi_{\vec{x}}(t),\dot{\phi}_{\vec{x}}(t),\frac{\partial}{\partial x_j}\phi_{\vec{x}}(t)\Big)d^Nx \tag{18}$$

   would do it, where

$$\mathscr{L}(\phi,\dot{\phi},\partial j\phi) = \frac{1}{2}\left(\dot{\phi}^2 - \sum_j(\partial_j\phi)^2 - m^2\phi^2\right); \tag{19}$$

   such that the equations we want to "quantize" are, the conditions defining critical points of the functionals

$$S(q) := \int_{t_0}^{t_f} L(q,\dot{q})dt \qquad \text{and} \qquad S(\phi) := \int_{t_0}^{t_f} L(\phi,\dot{\phi})dt. \tag{20}$$

[5] We could weaken this assumption if required in several directions. For example, a generalization to Riemannian or Hilbert manifolds of these notions could be obtained by considering that Q is the image of a chart, i.e., that we are working in some local coordinates.

[6] Note how all this includes the description of a field that evolves along Cauchy surfaces Σ_t of a certain foliation of some spacetime $M=\cup_t\Sigma_t$. In particular, since every globally hyperbolic spacetime is globally isometric to an $M:=(t_0,t_f)\times\Sigma_0$, we would treat instead of $H^2(\mathbb{R}^3,\mathbb{R})$, a field configuration-space $H^2(\Sigma_0,\mathbb{R})$ so that field paths over spacetime are given by paths on $H^2(\Sigma_0,\mathbb{R})$. In local charts adapted to the foliation, in the image of the charts, we essentially have paths on $H^2(\Omega,\mathbb{R})$ for some $\Omega\subset\mathbb{R}^3$, so the treatment in the text could be seen as a local coordinate picture.

2. Assuming that the Lagrangian $L(\eta, \dot{\eta})$ has non-degenerate velocity Hessian for given points η,[7] define the canonical conjugate "momentum" map $\chi(\eta, \dot{\eta}) := grad_{\langle,\rangle}(L(\eta, \cdot))|_{\dot{\eta}}$ (using Riesz representation for non-degenarcy). In our examples

$$p_j(q, \dot{q}) = \frac{\partial L(q, \dot{q})}{\partial \dot{q}_j} = m_j \dot{q}_j \qquad \text{and} \qquad \pi(\phi, \dot{\phi}) = \frac{\delta L}{\delta \dot{\phi}}|_{(\phi, \dot{\phi})} = \frac{\partial \mathscr{L}(\phi, \dot{\phi}, \partial_j \phi)}{\partial \dot{\phi}}|_{(\phi, \dot{\phi}, \frac{\partial \phi}{\partial x_j})} = \dot{\phi}. \quad (21)$$

which maps $TQ \simeq Q \times Q$ (since Q is just a vector space) to a symplectic vector space $T^*Q \simeq Q \times Q$ (identified via $\langle , \rangle$) with symplectic form $\omega((\eta, \chi), (\eta', \chi')) := \langle \chi, \eta' \rangle - \langle \chi', \eta \rangle$. There, by a Legendre transform we get the Hamiltonian function $H(\eta, \chi) := (\langle \chi, \dot{\eta} \rangle - L(\eta, \dot{\eta}))|_{\dot{\eta} = (grad(L(q, \cdot))|_\cdot)^{-1}(\chi)}$, e.g.,

$$H(q, p) = \sum_j \frac{1}{2m_j} p_j^2 + V(q) \qquad \text{and} \qquad H(\phi, \pi) = \int_{\mathbb{R}^N} \frac{1}{2} \left[\pi^2(x) + \sum_j \left(\frac{\partial \phi(x)}{\partial x_j} \right)^2 + m^2 \phi^2(x) \right] d^N x. \quad (22)$$

The solutions to the ODEs we wanted to quantize are then given by the flow map of the symplectic gradient of these Hamiltonians, i.e., their Hamilton's equations yield the Euler-Lagrange equation.

3. Until here everything can be done rigorously! Now, the (bosonic)[8] quantization recipe tells the following heuristic steps (completely **non-rigorous** for (b) and (c)).

(a) if $\eta \in Q$ are functions over a finite set, $\eta = (\eta_1, ..., \eta_N)$, considering for simplicity that $Q = \mathbb{R}^N$ so that $\eta = q$ and $\chi = p$, we are told to consider the space $L^2(Q, \mathbb{C}, d^N q)$ with $d^N q$ (later also denoted by $d^N x$) being the Lebesgue measure and to define some (unbounded) operators $\hat{q}_j, \hat{p}_j$ given for suitable $\psi \in L^2(\mathbb{R}^N, \mathbb{C}, d^N q)$ a.e. by

$$(\hat{q}_j \psi)(q) = q_j \psi(q) \qquad \text{and} \qquad (\hat{p}_j \psi)(q) = -i\hbar \frac{\partial}{\partial q_j} \psi(q). \quad (23)$$

The key is that they must satisfy the canonical commutation relations in their joint domain, namely, $[\hat{q}_j, \hat{p}_k] = i\delta_{jk}$, $[\hat{q}_j, \hat{q}_k] = 0$ and $[\hat{p}_j, \hat{p}_k] = 0$. Then, the quantum version of the q system is described by a time evolving wavefunction $t \mapsto \psi_t \in L^2(\mathbb{R}^N, \mathbb{C}, d^N q)$ $\forall t$ satisfying the Schrödinger Equation[9]

$$(i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \psi_t)(q) = (H(q, \hat{p}))\psi_t(q). \quad (24)$$

For our driving example we get the familiar SE,

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \psi_t(q) = -\sum_j \frac{\hbar^2}{2m_j} \frac{\partial^2}{\partial x_j^2} \psi(q) + V(q)\psi(q). \quad (25)$$

Moreover, if $||\psi_t||_{L^2}^2 = 1$, the prescription tells us that the probability to find the system in configuration q at time t in a (measurable) set $B \subseteq \mathbb{R}^N$ is given by the probability measure $|\psi(q)|^2 d^n q$, i.e.,

$$Prob(q \in B \mid \psi) = \int_B |\psi(q)|^2 d^n q. \quad (26)$$

(b) If $\eta \in Q$ are functions over an uncountably infinite set X, $\eta = (\eta_x)_{x \in X}$, say, if $Q = H^2(\mathbb{R}^3, \mathbb{R})$, denoting now $\eta = \phi$ and $\chi = \pi$, we could naively generalize the same prescription and claim that now the relevant space is $L^2(H^2(\mathbb{R}^3, \mathbb{R}), \mathbb{C}, d\mu)$ with $d\mu$ some generalization of the Lebesgue measure $d^N x$

[7] If it is degenerate, as turns out to be the case in most cases of interest, then one needs to consider constrained Hamiltonian system techniques. Essentially, one must find a set of constraints in the formally corresponding phase space, that define a submanifold restricted to which the Lagrangian's gradient does give a bijection. In that submanifold of phase space, after a canonical transformation, we can proceed with the quantization almost as explained here. For several examples and references see (Struyve, 2010). For some constraint types what we get is a so-called "gauge theory", where the coordinates tangential to the constraint submanifold after the canonical transformation will be directly related to the gauge quantities, elements left very underdetermined by the time evolution.

[8] We already addressed fermionic QFTs in my last talk, but we will later mention something about them too.

[9] This is an ill-defined statement since operator ordering matters and hence we could define different quantum theories as a result of the quantization, by just exchanging variables that before "operatorization" used to commute. Lets ignore this for now.

when $N \to \infty$ (spoiler, there is no obvious choice). Then, we could introduce the operators $\hat{\phi}_x$ and $\hat{\pi}_x$ that for suitable $\Psi \in L^2(H^2(\mathbb{R}^3, \mathbb{R}), \mathbb{C}, d\mu)$, i.e., on (classes of) functions taking value on fields, act a.e. as

$$(\hat{\phi}_x \Psi)(\phi) = \phi(x)\Psi(\phi) \qquad \text{and} \qquad (\hat{\pi}_x \Psi)(\phi) = -i\hbar \frac{\delta \Psi}{\phi(x)}|_\phi. \tag{27}$$

Such operators would formally satisfy the canonical commutation relations "$[\hat{\phi}_x, \hat{\pi}_y] = i\delta(x-y)$", $[\hat{\phi}_x, \hat{\phi}_y] = 0$ and $[\hat{\pi}_x, \hat{\pi}_y] = 0$ for $x, y \in \mathbb{R}^3$. Then, the dynamical equation ruling the evolution of the wavefunctional $\Psi_t(\phi)$, a path in $L^2(H^2(\mathbb{R}^3, \mathbb{R}), d\mu)$ would be claimed to be[10]

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(\phi) = H\left(\phi, -i\hbar \frac{\delta}{\delta \phi(x)}\right)\Psi_t(\phi), \tag{28}$$

which leaves for our study case

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(\phi) = \frac{1}{2}\int_{\mathbb{R}^N} \left[-\hbar^2 \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi}\Psi|_\phi + \Big(\sum_{j=1}^{3} (\partial_j \phi)^2(x) + m^2 \phi^2(x) \Big)\Psi(\phi) \right] d^N x. \tag{29}$$

Moreover, if $||\Psi_t||^2_{L^2} = 1$, we could claim that the system will be found to be a field ϕ in some measurable substet $B \subseteq H^2(\mathbb{R}^3, \mathbb{R})$ following a probability measure $|\psi(\phi)|^2 d\mu$, i.e.,

$$Prob(\phi \in B \mid \Psi) = \int_B |\Psi(\phi)|^2 d\mu. \tag{30}$$

The generalization to the case in which there are several fields in the Lagrangian, say $\{\phi_1, ..., \phi_N\}$ would then be straightforward, yielding N field operators $\{\hat{\phi}_1(x), ..., \hat{\phi}_N(x)\}_{x \in \mathbb{R}^N}$ and $\{\hat{\pi}_1(x), ..., \hat{\pi}_N(x)\}_{x \in \mathbb{R}^N}$ in an analogous manner to the N particle case.

From the mathematical perspective, there are two main problems with this recipe for generating QFTs. On the one hand, it is not obvious how to define a field of operators like $\hat{\phi}_x, \hat{\pi}_y$ that satisfy the ill-defined condition "$[\hat{\phi}_x, \hat{\pi}_y] = i\delta(x-y)$", and take values everywhere in $x \in \mathbb{R}^d$. Most importantly, there exists no obvious generalization of the Lebesgue measure to infinite dimensional spaces, since its characteristic feature was to be the unique (complete) measure that is translation invariant in the (completion of) the Borel sigma algebra of $\mathbb{R}^n$, but no such measure exists if the space is an infinite dimensional Banach space, as we will find later.

We used (a) as a template to come up with (b), a generalization for fields, but there is a third interesting option we did not consider explicitly in the recipe yet, perhaps because at first glance it was not going to allow us the quantization of fields (assumed to live over a continuum):

(c) If $\eta \in Q$ are functions over a countably infinite set, $\eta = (\eta_j)_{j \in \mathbb{N}}$, say, if $Q = \ell^2(\mathbb{N}, \mathbb{R})$, denoting now $\eta = \alpha = (\alpha_1, \alpha_2, ...)$ and $\chi = \pi = (\pi_1, \pi_2, ...)$, we can consider a Hamiltonian system of canonical variables (α, π) with symplectic form as above for infinite dimensions, and given some Hamiltonian like

$$H(\alpha, \pi) = \frac{1}{2}\sum_{j=1}^{\infty} \Big(\pi_j^2 + (|\vec{k}_j|^2 + m^2)\alpha_j^2 \Big), \tag{31}$$

the generalization of the canonical quantization prescription would tell us that the quantum counterpart of this system lives in the space $L^2(\ell^2(\mathbb{N}), \mathbb{R}, d\mu)$, for some generalization of the Lebesgue measure $d\mu$ (still no obvious choice). Furthermore, there should be some unbounded operators $\hat{\alpha}_j$ and $\hat{\pi}_j$ acting for a suitable $\psi \in L^2(\ell^2(\mathbb{N}), \mathbb{R}, d\mu)$ as multiplication and differentiation operators

$$(\hat{\alpha}_j \psi)(\alpha) = \alpha_j \psi(\alpha) \qquad \text{and} \qquad (\hat{\pi}_j \psi)(\alpha) = -i\hbar \frac{\partial}{\partial \alpha_j}\psi(\alpha) \tag{32}$$

and satisfying (in a dense domain) the canonical commutation relations $[\hat{\alpha}_k, \hat{\pi}_j] = i\delta_{kj}$, $[\hat{\alpha}_k, \hat{\alpha}_j] = 0$ and $[\hat{\pi}_k, \hat{\pi}_j] = 0$ (note how there is no longer the same trouble as in the continuum). Then, the time evolution of the quantum system is given by the wavefunctional path $t \mapsto \Psi_t \in L^2(\ell^2(\mathbb{N}), \mathbb{R}, d\mu)$, obeying

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(\alpha) = H(\alpha, \hat{\pi})\Psi_t(\alpha), \tag{33}$$

[10] Up to the same problems related to operator ordering as in (a).

which would read for the example as

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(\alpha) \equiv \frac{1}{2}\sum_k \left(-\hbar^2\frac{\partial^2}{\partial\alpha_k^2} + (m^2+|\vec{k}_j|^2)\alpha_j^2\right)\Psi_t(\alpha). \tag{34}$$

Finally, if $||\Psi||_{L^2}^2 = 1$, we could suggest a probabilistic interpretation by which the system is found in some configuration in a (measurable) set $B \subseteq \ell^2(\mathbb{N})$ with probability measure $|\Psi|^2 d\mu$, namely,

$$Prob(\alpha \in B \mid \Psi) = \int_B |\Psi|^2(\alpha) d\mu. \tag{35}$$

The problem of the measure still remains, but in principle the condition on the operators seems a bit more straightforwardly rigorifiable without needing to talk about distributions.

The fun part about (c) is that because we demanded the configuration-spaces for fields over the continuum to be separable Hilbert spaces $\mathcal{H}$, before doing the quantization we know that there is a unitary 1 to 1 correspondence between any such field configuration-space $\mathcal{H}$ and ℓ^2, given by each fixed ONB $\{\psi_j\}_{j\in\mathbb{N}} \subset \mathcal{H}$ as

$$\begin{array}{rccc} U: & \mathcal{H} & \longrightarrow & \ell^2(\mathbb{N}) \\ & \psi = \sum_{j\in\mathbb{N}} \alpha_k \psi_j & \longmapsto & (\alpha_1, \alpha_2, ...). \end{array} \tag{36}$$

At the pre-quantization step, this allows to map any field configuration-space, i.e., of functions over a continuum, into a Hilbert space of functions over countably many elements. This map is moreover arguably given by a canonical transformation, so we get an equivalent Hamiltonian system (the example (31) actually corresponds to the KG Hamiltonian using the unitary for a cosinusoid ONB —we will find it out later). One could view this move as a change of coordinates in infinite dimensions prior to quantization. However, it is not clear at all that the quantization step would then produce equivalent quantum theories using (b) or (c) (if we could make both of them rigorous somehow, as a prerequisite to compare them!).

As a little spoiler, both (b) and (c) will be rigorously doable under suitable cut-offs (finite dimensional approximations), under which one could study this equivalence. However, as the crown of today talk, we will find that a substantial part of (c) can be built rigorously also in infinite dimensions, so it might be fair to say that we are also partially solving (b) with it, due to the pre-quantization equivalence.

Before all that, let us comment how naturally the heuristic recipes of this section suggest (at least in a heuristic level) a Bohmian-like theory (which we will denote by "pilot-wave theory").

B.2. How Canonical Quantization Shouts for a Pilot-wave Theory

For many finite dimensional cases (e.g., when the Hamiltonian is a quadratic function of the momentum), one can find that the Schrödinger equation $\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = H\psi_t$ implies the following continuity equation for $|\psi_t|^2(q)$ (obtained by taking $\frac{\mathrm{d}\psi_t\psi_t^*}{\mathrm{d}t} = \psi_t\frac{\mathrm{d}\psi_t^*}{\mathrm{d}t} + \psi_t^*\frac{\partial\psi_t}{\partial t}$ and plugging the SE there),

$$\frac{\mathrm{d}|\psi_t|^2}{\mathrm{d}t} = -\vec{\nabla}\cdot\left(|\psi_t|^2\,\vec{v}^{\,\psi_t}\right) \tag{37}$$

with v^ψ given by

$$v_j^{\psi_t}(q,t) := \mathrm{Re}\left\{\frac{\frac{i}{\hbar}[H(q,\hat{p}),\hat{p}_j]\psi_t(q)}{\psi_t(q)}\right\} \underbrace{=}_{\text{if } H \text{ as (25)}} \frac{1}{m_j}\mathrm{Re}\left\{\frac{-i\hbar\frac{\partial}{\partial q_j}\psi_t(q)}{\psi_t(q)}\right\} \underbrace{=}_{\substack{\text{wherever polar form} \\ \psi(q)=R_t(q)e^{iS_t(q)/\hbar} \\ \text{is allowed}}} \frac{1}{m_j}\frac{\partial S_t(q)}{\partial q_j}. \tag{38}$$

The interesting part of the continuity equation is that the flow lines of its vector field $v^\psi(q,t)$ are $|\psi|^2 d^n q$ equivariant.[11] It is then natural to postulate the existence of an actual trajectory $Q^\xi(t)$ with "initial" configuration

[11] Essentially meaning that if $(\xi,t) \mapsto Q(\xi,t)$ is the flow map of the velocity field v^{ψ_t}, i.e., $\frac{\partial}{\partial t}Q(\xi,t) = v^\psi(Q(\xi,t),t)$ (the trajectories are tangent to the velocity field) with initial condition $Q(\xi,t_0)=\xi$, then for all time t and (measurable) $B \subseteq \mathbb{R}^n$,

$$\int_B |\psi_{t_0}|^2(q)d^n q = \int_{Q(B,t)} |\psi_t|^2(q)d^n q.$$

$Q^\xi(t_0) = \xi$ that obeys the flow-line ODE

$$\frac{\mathrm{d}Q_k^\xi(t)}{\mathrm{d}t} = v^{\psi_t}\Big(Q^\xi(t), t\Big), \tag{39}$$

such that by definition of equivariance, if ξ was "sampled" according to $|\psi_{t_0}(q)|^2 d^N q$, then $Q^\xi(t)$ will be $|\psi_t(q)|^2 d^N q$ distributed at all times t. Such a trajectory (that is always there even if there is no measurement) would explain simultaneously that we always find a fixed configuration (say, point-like particles) in a measurement (never a superposition) but why nevertheless we find a stochastic outcome following the Born rule, such that the outcomes follow a density that could perfectly be that of a fluid (by postulating the quantum equilibrium hypothesis, as brilliantly explained by Dürr et al. (1992)).

By the flow equation above, one then says that Q^ξ, the actual configuration of the system, is piloted by the configuration-space wave $\psi(q)$ and we talk about a *pilot-wave theory*. Note that in modern treatments, the "piloting wave" is not considered to be something that materially "pushes" the ontological trajectory, but rather as part of a dynamical law of motion, i.e., it is seen as a nomological element (nobody asks "where" the piloting wave is or whether it is observable, in the same way as nobody used to ask "where" Newton's second laws were or whether they were observable).[12]

With the very same spirit, **formally** one can find a continuity equation underlying many relevant infinite dimensional configuration-space Schrödinger equations as exemplifies Struyve (2010). In particular, one can find a heuristic continuity equation for our KG study case. In direct analogy with the finite dimensional case, one could then obtain a guidance equation defining the flow-lines over field configuration-space (where each trajectory is now a time evolving field) of the obvious equivariant vector field $v^{\Psi_t}(\phi, t)$ given by the continuity equation. The pilot-wave picture would then be completed by declaring that there exists an actual field of trajectory $\Phi^\Xi(t)$ and initial configuration $\Phi^\Xi(t_0) = \Xi$ that obeys such a flow-line ODE, namely,

$$\frac{\mathrm{d}\Phi^\Xi(t)}{\mathrm{d}t} = v\Big(\Phi^\Xi(t), t\Big). \tag{40}$$

Then, if Ξ was sampled according to $|\Psi_{t_0}(\phi)|^2 d\mu$, $\Phi^\Xi(t)$ will be $|\Psi_t(\phi)|^2 d\mu$ distributed at all times t and one could perhaps propose a similar analysis as Dürr et al. (1992) to give a consistent body to the theory. We would say then that the actual field is piloted by the wavefunctional and call it a pilot-wave theory. The case of N fields in the Lagrangian would likewise yield N ontological physical space fields, all piloted by the same wavefunctional.

With all, we could give a (rather informal) definition of what we mean by a pilot wave theory:

Definition 6. A **pilot-wave theory** (PWT) is the tuple formed by:

- a measure space (Q, μ) that is (ideally) a NVS (and more ideally an IPVS), called configuration-space,
- a Hilbert space $\mathcal{H} := L^2(Q, \mathbb{C}, d\mu)$
- a (densely defined) self-adjoint operator on $\mathcal{H}$ called Hamiltonian $\hat{H}$ generating the dynamics for the vectors $\Psi_0 \in \mathcal{H}$, called wavefunction(al)s, via a Schrödinger equation: $i\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t = \hat{H}\Psi_t$. (or, equivalently by Stone's theorem, a SCOPUG $\{U_t = e^{-i\frac{t}{\hbar}\hat{H}}\}_t$)
- for each path in $\mathcal{H}$ solving the SE $t \mapsto \Psi_t \in \mathcal{H}$, a vector field v^{Ψ_t} on $Q \times \mathbb{R}$, that has a $|\Psi_t| d\mu$ equivariant flow-map
- for each "physical realization of Ψ_t", the primitive ontology is given by a single trajectory on Q, epistemologically unknown but ontologically determined and being one of the flow-lines of v^Ψ with an epistemological measure of typicality such that it crosses a subset $B \subseteq Q$ with a probability $\int_B |\Psi_t|^2 d\mu$. ♦

(Note how a priori such a pilot wave theory does **not** bring with it a connection between the ontological trajectory on Q and our phenomenological world —nor what a "physical realization of Ψ_t" means. Thus, it is not yet a complete statement of a physical theory!).

[12] Still, there are ways using the concept of conditional wavefunctions and infinite chains of potential energy fields, as explained by Norsen et al. (2015), to build a physical-space field ontology for the wavefunction within the standard SE's (25) scope.

B.2.2. Digression: Why is the quantization of fields called "second quantization"?

Aren't we doing only one quantization? Well, as we are doing with the KG equation and its associated charge density field (or whatever we are describing with that field), we could do the same procedure for the Schrödinger equation and a wavefunction (by perhaps taking the real and imaginary parts as independent fields, since after all we are considering Hamiltonian systems over real vector spaces). Then, the result would be a wavefunctional over wavefunctions! That is, each wavefunction, which is already the quantization of a set of particles, would be second quantized to get a wavefunctional over them.

Certainly, if we take seriously the term "second quantization", one could (misguidedly) end up thinking that we are considering that $|\Psi(\psi)|^2 d\mu$ is the probability measure to find a wavefunction ψ, which at the same time, via $|\psi(q)|^2 d^N q$ yields the probability measure to find the particles in configuration q. Even worse, the obvious field ontology would tell us that there is an actual wavefunction ψ at all times, which could then have a subordinate particle ontology, giving a Bohmian trajectory at all times. Moreover, the ontological wavefunction would happen to follow the Schrödinger equation except for an additional inhomogeneity due to the "quantum potential", so it would no longer posses exactly a unitary time evolution. And then, a myriad of speculative (and possibly misguided) questions would emerge: What if the extra term is exactly the inhomogeneity we get in the dynamical equation of a conditional wavefunction (a slice of a wavefunction)? Would it mean there is an additional spatial degree of freedom and our Universal wavefunction is a slice of it?

To make it even better, once such a "second quantization" is considered, we could speculate about a "**third** quantization". We could now take a wavefunction-**alal** $\mathfrak{P}(\Psi)$ over the space of possible wavefunctionals Ψ (over the space of wavefunctions ψ, over some particle configuration space Q), by choosing some measure over the wavefunctional space. Then we would postulate some "field-al" operators that act as multiplication and functional derivative with respect to wavefunctionals, give a SCOPUG/Schrödinger equation and again we could give it a wavefunctional ontology. But then, why stop here? We could consider a fourth quantization and a fifth quantization and a sixth one, until we get sufficiently mad! Would there be any limit in reasonable terms for such a recursion? What kind of new experimental predictions would it give? Mayhem!

Please, do not take this parenthetic section seriously, its objective was merely to make some comedy while explaining the misleading origin of the name.

B.3. What about Fermions?

The quintessential relativistic field describing a fermion is the Dirac 4-spinor wavefunction $\psi \in L^2(\mathbb{R}^3, \mathbb{C}^4)$. Although it has its origin in quantum mechanics, it can also be understood as a continuos charge distribution within classical mechanics, with several polarization coordinates or fields required to specify the charge field's dynamics (hence the several components in each spatial position). Then, we could proceed with the second quantization also for this field and find a wavefunctional of Dirac 4-spinors $\Psi(\psi)$ making a second quantized QFT out of it. However, there is a key difficulty that makes the field configuration-space approach very unappealing for fermions.

As there are four spinor components, namely, four fields that make up what we call the Dirac field (if we would like to make them complex then eight), we would need to consider (at least) four field operators $\{\hat{\psi}_1(\vec{x}), \hat{\psi}_2(\vec{x}), \hat{\psi}_3(\vec{x}), \hat{\psi}_4(\vec{x})\}_{\vec{x}\in\mathbb{R}^3}$. As before, the obvious generalization of canonical quantization would demand the field operators to act as multiplication operators on the wavefunctional:

$$(\hat{\psi}_j(\vec{x})\Psi)[\psi, t] = \psi_j(\vec{x})\Psi(\psi) \qquad \forall j \in \{1,2,3,4\};\ \forall \vec{x} \in \mathbb{R}^3, \tag{41}$$

together with an anticommutation relation[13]

$$\left\{\hat{\psi}_i(\vec{x}), \hat{\psi}_j(\vec{x})\right\} = 0 \qquad \forall i,j \in \{1,2,3,4\},\ \forall \vec{x}, \vec{y} \in \mathbb{R}^3. \tag{42}$$

The two above conditions together mean that for relevant $\Psi \in L^2(L^2(\mathbb{R}^3, \mathbb{C}^4), \mathbb{C})$,

$$\left\{\hat{\psi}_i(\vec{x}), \hat{\psi}_j(\vec{x})\right\}\Psi(\psi) = (\psi_i(\vec{x})\psi_j(\vec{y}) + \psi_j(\vec{y})\psi_i(\vec{x}))\Psi(\psi), \tag{43}$$

[13] Instead of commutation, due to the different statistics for identical fermions —essentially Pauli's exclusion principle.

which will not be zero in general as required by the anticommutation relation (42), unless the fields of the configuration space themselves anticommute, i.e., unless we only consider the subspace of $\psi \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ such that

$$\{\psi_i(\vec{x}), \psi_j(\vec{y})\} = 0. \tag{44}$$

Hence, the field configuration space would need to be reduced to such fields taking values on anticommuting element spinors, mathematically realizable more abstractly as so-called Grassman numbers. There are several issues with Grassman valued Dirac fields, starting from difficulties to define for each field in such a configuration space concepts like a charge or energy density, which will not be real-valued and hence are problematic for pre-quantization electromagnetic interactions. But most relevantly after quantization, there are (even more) difficulties to interpret the wavefunctional's amplitude-squared as a probability density and to give a measure to the space of Grassman valued fields. Moreover, it is a common practice to use Grassman valued wavefunctionals instead of complex valued ones, which as you have guessed, makes the problem even worse: porbability amplitudes would not even be real valued... (See Struyve (2010); Sebens (2022) for more details of the topic and how to still approach fermionic QFTs within a field ontology picture in a heuristic way.)

As opposed to all this mayhem, we found in my last talk that if we treat fermions instead with a particle ontology in mind, very sexy QFTs are obtained almost for free and in more than one way in case you do not like one of them: ranging from Fock spaces, till infinite wedge product spaces or time varying Dirac seas. Moreover, the cut-off picture there (in particular, assuming space is finite and a UV cut-off exists) allowed us to employ a theory with a fixed number of fermions to describe all QFTs (no creation and annihilation of fermions would need to happen ontologically, and they would only be an effective way to talk about fermions relative to the "sea level"). So yeah, we will not even mention fermions any more in the context of field ontologies.

As we mentioned in my last talk, it is remarkable that diametrically opposed to the complication of describing a field ontology for fermions, there are problems to give a particle ontology to bosons. In particular to those bosons that lack a clear "single particle" quantum theory, such as photons —a lack that is very nicely explained by Tumulka (2022).

All in all, the joint moral of the two talks is that a mixed ontology appears to be the nicest one in mathematical terms: describe fermions as particles ontologically while bosons are in reality a physical-space field. They are "virtual particles", in the sense that their particle aspects are rather due to "bumps" of some ontological field spreading in physical space. This turns out to match with the way we trained ourselves to picture the world in classical mechanics, so great! Or...should we be worried instead? The preference of the maths for such a mixed theory could be interpreted instead as our own preference (or its historical drift), propagated to the developed mathematical formulations. That is, as a symptom of the biased way in which we built the foundations of our physical theories, which perhaps unwillingly shaped a mathematical trap with no easy escape.

C. RIGOROUS FIELD-CONFIGURATION QM I: MODE EXPANSION CUT-OFF

In the coming two sections we present the two main paths towards solving the mathematical ill-definitions we spotted in the canonical quantization of fields. The first way is to impose enough conditions to the configuration space that only elements with finitely many degrees of freedom within it become relevant. Then we recover the finite dimensional scenario we all know and love and both raised problems: the one about the operators and the one about the measure instantly vanish.

Since we assumed the field configuration space to be a separable Hilbert space, there exists a countable orthonormal basis (ONB) so that elements of such a configuration-space are in unitary one-to-one relation with elements of $\ell^2(\mathbb{N})$. Thus, before the quantization we could equivalently express the Hamiltonian system over the space of possible expansion coefficients of the ONB $\ell^2(\mathbb{N})$ (and their canonically conjugate momenta). As we already commented before, a quantization in this apparently equivalent space would likely erase the problem of field operators $\hat{\phi}_x$ not being easily definable for all $x \in \mathbb{R}^3$, because now the indexing set for field operators would be countable. Nevertheless, we would still have the problem of an infinite dimensional vector space having no distinguished choice of "uniform" measure to build the wavefunctional space of.

To solve it, before quantization we can make an assumption. We assume that the only fields in the configuration space that matter are the ones having an expansion where the coefficients that come after the M-th ONB element are zero. This then leaves a finite dimensional configuration space (namely, the expansion coefficient space) where the Lebesgue measure is back in town!

Hence the only two ingredients we need to apply this trick to rigorify the second quantization of some field theory is (a) an ONB for the field configuration-space and (b) a point of truncation. Certainly, such a truncation of the ONB expansion would need to be justified by a very reasonable physical assumption. The physical assumption could then be used to guide the choice of an ONB.

One of the most commonly employed physical assumptions in this sense is the following one. It is arguable that spatial variations of a physical-space field below some very insignificant characteristic length-scale $l > 0$ are meaningless. This applies directly to fields over position-space like our driving example, $H^2(\mathbb{R}^3, \mathbb{R})$, and can be interpreted in two different and closely related ways (yielding different ONB to be truncated).

C.1. Ultraviolet (UV) Cut-off

One way to understand the limitation on the spatial variation is as a limitation of the "frequency" that sinusoids composing the field could have in some Fourier basis. (This is precisely the trick we used to make the introductory example sound.)

Since plane-waves are not an ONB over the unbounded set $\mathbb{R}^3$, to use this trick on $H^2(\mathbb{R}^3, \mathbb{R})$ we first need another assumption: that space is finite,[14] say $\Omega := (0, L)^3$, so that there is an ONB of cosines $\left\{\sqrt{\frac{2}{L^3}} cos(\langle \vec{k}_j, \cdot \rangle_2)\right\}_{j\in\mathbb{N}}$. Then, for any $\phi \in H^2(\Omega, \mathbb{R})$ there exist coefficients $\alpha_j \in \mathbb{R}$ such that

$$\phi(\vec{x}) \equiv \sum_{j\in\mathbb{N}} \alpha_j \sqrt{\frac{2}{L^3}} cos(\vec{k}_j \cdot \vec{x}). \tag{45}$$

(Also re-writeable as a complex exponential series with real coefficients[15]). Now yes, we can implement that only spatial variations above the length-scale l are significant by taking only $\vec{k}_j$ for which the wavelength of the sinusoid is bigger than l, i.e., $2\pi/|\vec{k}_j| > l$. That is, we impose $|\vec{k}_j| < 2\pi/l$, cutting-off the sinusoids/exponentials at a finite, M-th frequency.

Defining $u_j := \sqrt{\frac{2}{L^3}} cos(\langle \vec{k}_j, \cdot \rangle)$, the truncated expansion of (45) reads, for a trajectory in the M dimensional

[14] Justifiable because we might be considering a cavity as before, or because the space-slice we are considering is (perhaps big but) finite in extent, as is the case in several relevant cosmological Big-Bang spacetimes.

[15] Defining $\alpha_{-j} := \alpha_j$ and $c_j := \sqrt{\frac{1}{2L^3}}\alpha_j$, then:

$$\phi(\vec{x}) \equiv \sum_{j\in\mathbb{N}} \alpha_j \sqrt{\frac{2}{L^3}} cos(\vec{k}_j \cdot \vec{x}) = \sum_{j\in\mathbb{N}} \alpha_j \sqrt{\frac{1}{2L^3}} (e^{i\vec{k}_j\cdot\vec{x}} + e^{-i\vec{k}_j\cdot\vec{x}}) = \sum_{j\in\mathbb{Z}} c_j e^{i\vec{k}_j\cdot\vec{x}}$$

field configuration space $t \mapsto \phi_t \in H^2(\mathbb{R}^3, \mathbb{R})$, as $\phi_t = \sum_{j=1}^M \alpha_j(t) u_j$ with $\dot{\phi}_t = \sum_{j=1}^M \dot{\alpha}_j(t) u_j$. If we plug them in our driving case's pre-quantization Hamiltonian (22), and we apply the orthonormality of the basis (and orthonormality of the corresponding sine basis) we get

$$H\Big(\phi(\alpha_j), \pi = \dot{\phi}(\dot{\alpha}_j)\Big) = \frac{1}{2}\sum_{j=1}^{M}\Big(\dot{\alpha}_j^2 + \Big(|\vec{k}_j|^2 + m^2\Big)\alpha_j^2\Big). \tag{46}$$

One can find that this was essentially a canonical transformation if $\dot{\alpha}_j$ is taken to be the canonical momentum of α_j, which we denote by π_j. Hence, at the classical level the Hamiltonian system is still the same (albeit in different coordinates). Presumably the quantization will also need to be the same, but we will not care about this here.

Now, note that the Hamiltonian we obtained,

$$H(\alpha_j, \pi_j) = \frac{1}{2}\sum_{j=1}^{M}\Big(\pi_j^2 + \Big(|\vec{k}_j|^2 + m^2\Big)\alpha_j^2\Big) \tag{47}$$

matches exactly with the Hamiltonian of M uncoupled harmonic oscillators with positive spring coefficients depending on the frequency of the represented frequency-mode. Using the unproblematic quantization of finitely many degrees of freedom, we get that the quantized system is described by a wavefunction $\Psi \in L^2(\mathbb{R}^M, \mathbb{R}, d^M x)$, where the Lebesgue measure can be used with no problem, solving the main problem we had. On the other hand, we get the well-defined "field operators" with dense domain (say, the Schwartz functions), defined as $\hat{\alpha}_j \Psi_t(\alpha_1, ..., \alpha_M) := \alpha_j \Psi(\alpha_1, ..., \alpha_M)$ and $\hat{\pi}_j \Psi_t(\alpha_1, ..., \alpha_M) := i\hbar \frac{\partial}{\partial \alpha_j} \Psi(\alpha_1, ..., \alpha_M)$. Denoting $\alpha := (\alpha_1, ..., \alpha_M)$, the wavefunction obeys the Schrödinger equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(\alpha) \equiv \sum_k \left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial \alpha_j^2} + (m^2 + |\vec{k}_j|^2)\right)\Psi_t(\alpha). \tag{48}$$

By the Born rule, $|\Psi(\alpha)|^2 d^N \alpha_j$ gives the probability measure for finding the field in a configuration $\sum_{j=1}^M \alpha_j u_j$ and we find the continuity equation for its density $|\Psi(\alpha)|^2$ as usual, to be given by

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_t|^2(\alpha) = -\vec{\nabla} \cdot \Big(|\Psi(\alpha)|^2 \vec{v}^{\Psi}(\alpha)\Big) \tag{49}$$

where, for the polar form $\Psi_t(\alpha) =: R_t(\alpha) e^{i\frac{S(\alpha)}{\hbar}}$ we have

$$\vec{v}^{\Psi}(\alpha) = \frac{1}{m}\,\mathrm{Re}\left\{\frac{-i\hbar\vec{\nabla}\Psi(\alpha)}{\Psi(\alpha)}\right\} = \frac{1}{m}\vec{\nabla}S(\alpha). \tag{50}$$

Then as usual we introduce an ontological trajectory for a field, parametrized by some time evolving tuple of expansion coefficients $\alpha^{\xi}(t)$ with initial condition $\alpha^{\xi}(t_0) = \xi$, following the probability-equivariant velocity field, explicitly given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha^{\xi}(t) = \frac{1}{m}\vec{\nabla}S(\alpha^{\xi}(t)). \tag{51}$$

Finally, it might be interesting to mention for our coming discussions, that the ground state for the given Hamiltonian (47) is a Gaussian (since each degree of freedom is just a harmonic oscillator)

$$\Psi(\alpha_1, ..., \alpha_M) = \mathcal{N} e^{-\sum_{j=1}^M \frac{\alpha_j^2}{2}\frac{\sqrt{m^2+|\vec{k}|^2}}{\hbar}} \tag{52}$$

for some normalization constant $\mathcal{N}$. And again, using the Hermite polynomials and the Fock representation one could talk about "Klein-Gordon-on" or "charge-on"[16] particles of frequency $\vec{k}_j$ as bosonic (virtual) particles whose number is given by the excitation level of the j-th harmonic oscillator. From there one could then build a particle ontology for the KG equation's quantization using the Fock space representation.

[16] If we are describing a charged continuum.

C.1.1. Digression: The Quantum Potential as the Origin of Mass

Again, using the polar form of Ψ_t, one can find that the Schrödinger equation implies two real PDEs. If we write one of them from the perspective of a trajectory solving (51) (i.e., in the Lagrangian frame), we get that the coefficients obey the harmonic oscillator's Newton's laws, but with an additional potential term Q^Ψ (the quantum potential):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\alpha_j^\xi(t) = \sqrt{m^2+|\vec{k}_j|^2}\,\alpha_j^\xi(t) - \frac{\partial}{\partial\alpha_j}Q^\Psi(\alpha,t)|_{\alpha^\xi(t)}. \tag{53}$$

But then putting back $\Phi(t)=\alpha_j(t)u_j$ (adding the previous equation for $j\in\{1,...,M\}$), one can find that this equation implies the ontological field starting at $\Phi^\xi(t_0)=\sum_j \xi_j u_j$ obeys

$$\ddot{\Phi}_t^\xi = \Delta\Phi_t^\xi - m^2\Phi_t^\xi - F(Q^\Psi)|_{\Phi_t^\xi} \tag{54}$$

for some fixed function F of the quantum potential. That is, it obeys the Klein-Gordon equation but with an extra unexpected term giving the deviation of our ontological field from the "non-quantized" KG equation.

As a funny observation by Holland (1995), if we took a massless particle's KG equation (i.e., set $m=0$), the result would have been that the quantized version of it, in the pilot wave theory would follow an equation

$$\ddot{\Phi}_t^\xi = \Delta\Phi_t^\xi - F(Q^\Psi)|_{\Phi_t^\xi} \tag{55}$$

which if by accident (or not) would have that $F(Q^\Psi)|_{\Phi^\xi}\simeq C\Phi^\xi$ for some constant $C>0$, then the quantum version of the massless KG equation would be a massive KG equation. The speculative part: could it be that the mass of massless particles is originated in the quantum potential? Again, too speculative but fun haha.

C.2. Discrete Space

The other way to understand the limitation on the spatial variation of fields below length-scale l is to assume there is actually no field in between spatial points separated by that distance l. Perhaps because in between those points there is no space at all in the ontological sense, or perhaps because an average/coarse graining per each such tiny neighbourhood gives enough information about the field. That is, we assume that fields take value only over a discretized space of points $\mathfrak{L}:=\{\vec{x}_j\}_{j\in\mathbb{N}}\subset\mathbb{R}^3$ with grid-spacing l. The discretization of space promotes the formal Dirac deltas into a rigorous ONB (analogous to how making the space finite made the plane waves a rigorous ONB in C.1). This solves the problem of field operators in a very natural way, because unlike in the UV-cut-off case, we can now define them in position (and not frequency) space as $\{\hat{\phi}_x,\hat{\pi}_x\}_{x\in\mathfrak{L}}$, closer to the heuristic $\{\hat{\phi}_x,\hat{\pi}_x\}_{x\in\mathbb{R}^3}$.

So we now have the ingredient (a), the choice of ONB and now we need to apply a cut-off, because what corresponds to "$H^2(\{\vec{x}_j\}_{j\in\mathbb{N}},\mathbb{R},d\nu)$", is still an infinite dimensional vector space (each field still has countably infinitely many degrees of freedom).

Hence, we cut-off the Dirac delta basis at some M-th grid-point, which equivalent to assuming that space is finite! Since this leaves a finite dimensional configuration space, we can then rigorously have wavefunctionals over fields $\Psi(\phi):=\Psi(\phi_{\vec{x}_1},...,\phi_{\vec{x}_M})$ with $\Psi\in L^2(\mathbb{R}^M,\mathbb{C},d^Mx)$ equipped with the Lebesgue measure and then define field operators as we wanted,

$$(\hat{\phi}_{\vec{x}_j}\Psi)(\phi_{\vec{x}_1},...,\phi_{\vec{x}_M}) = \phi_{\vec{x}_j}\Psi(\phi_{\vec{x}_1},...,\phi_{\vec{x}_M}) \qquad \text{and} \qquad (\hat{\pi}_{\vec{x}_j}\Psi)(\phi_{\vec{x}_1},...,\phi_{\vec{x}_M}) = -i\hbar\frac{\partial}{\partial\phi_{\vec{x}_j}}\Psi(\phi_{\vec{x}_1},...,\phi_{\vec{x}_M}). \tag{56}$$

The Schrödinger equation ruling the motion of $\Psi_t\in L^2(\mathbb{R}^M,\mathbb{C},d^Mx)$ in our study case would then be the discretization of the heuristic formula (29),

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(\phi) = \frac{1}{2}\sum_{j=1}^{M}\left[-\hbar^2\frac{\partial^2}{\partial\phi_{\vec{x}_j}^2}\Psi_t(\phi) + \left(\sum_{k=1}^{3}\left(\frac{\phi_{\vec{x}_j+\vec{e}_k}-\phi_{\vec{x}_j}}{l}\right)^2 + m^2\phi_{\vec{x}_k}^2\right)\Psi_t(\phi)\right] \tag{57}$$

with $\vec{e}_k\in\mathbb{R}^3$ the canonical basis vector in direction k.

Finally, the probability to find the system in a field $\phi:=(\phi_{\vec{x}_1},...,\phi_{\vec{x}_M})$ at time t would be given by the probability measure $|\Psi_t(\phi)|^2d^Mx$. Again, one can then find a continuity equation for $|\Psi_t(\phi)|^2$ in the usual way,

and introduce the trajectory of an actual ontological field $\Phi^{\Xi}(t)$ starting in $\Phi^{\Xi}(t_0) = \Xi$, and through the obvious equivariant velocity field behind the continuity equation obtain an equation of motion for each component $\Phi^{\Xi}_{\vec{x}_j}$ with $j \in \{1, ..., M\}$.

This is the rigorization that is closest to what physicists mean by their formal manipulations. In fact, since any PDE in a computer will need to be discretized to a finite lattice, one could naively believe that in the end, use a discretized version of the theory or a rigorous infinite dimensional rigorification (as we are going to approach in D), the result will not be distinguishable in as much we use computers to obtain checkable predictions. This would apparently make attempts to rigorify infinite dimensional formulas rather unmotivated. However, although it might be the case that the infinite dimensional case should coincide with this cut-off approach, if we really had a rigorous theory in infinite dimensions, employing analytic manipulations predictions of very general kinds could be deduced, that may even be unreachable through computational manipulations of the discretization (as often happens in mathematical physics).

Finally, by the discrete Fourier transform (DFT) any discretized function on finitely many points, say, the fields over the lattice above, can be written down as finite series over cosines. Then at first glance, the two explicit prescriptions we gave to understand the length-scale cut-off seem to be essentially the same. Both end up requiring space to be finite and end up truncating a sinusoid expansion. It is an open question if this holds true for common choices of l.

C.3. QFT and the Endless Tower of Approximate Theories

Physicists routinely employ heuristic and ill-defined formulas like

$$\int_{H^2(\mathbb{R}^3,\mathbb{R})} |\Psi(\phi)|^2 \mathcal{D}\phi = \int_{H^2(\mathbb{R}^3,\mathbb{R})} |\Psi(\phi)|^2 \prod_{x\in\mathbb{R}^3} d\phi_x \tag{58}$$

where $\mathcal{D}\phi$ or $\prod_{x\in\mathbb{R}^3} d\phi_x$ pretend to represent a generalization of the Lebesgue measure to infinite dimensional spaces, something like $\lim_{N\to+\infty} \prod_{j=1}^{N} dx_j$. In the rigorous discretization we presented, such an expression would essentially be

$$\int_{\mathbb{R}^M} |\Psi(\phi)|^2 \prod_{x_j\in\{x_k\}_{k=1}^{M}} d\phi_{x_j}, \tag{59}$$

which, unlike (58), is a well-defined expression. It is likely that so many formal manipulations in the physics literature give somewhat meaningful results even if they employ ill-defined equations like (58), precisely because one can understand them as symbols that are meant to represent a discrete expression like (59).

Some branches of QFT have directly abandoned even the formal manipulations in infinite dimensions and have started to assume the discretization of space seriously, as lattice gauge theory exemplifies.

In fact, one should notice that as of today, the only way to make the Standard Model a serious/mathematically meaningful theory is by the cut-off technique, and remarkable experimental matches can be achieved indeed!

The problem with cut-off theories is that, sure, we can make the cut-off as insignificant as we wish...but each choice yields a different theory than the previous one! Wallace (2006) and references therein argue that this *per se*, should not be regarded as a problem. Similar techniques are employed in solid state physics to make sense of functional space integrals and there, where experimental results to contrast calculations are remarkably more accessible than in particle physics, it is found that as long as we choose the cut-off small enough relative to the scale of distances we are interested about, the coarse grained predictions will be independent of the particular cut-off (say, the particular grid employed). Since in solid state physics we know that assumptions about the continuum are really an approximation of a deeper theory in which there are atoms and molecules, spatial cut-offs are moreover positively motivated.

As Wallace (2006) poses very nicely, the analogue in QFT is somewhat more convoluted. It would tell us that if it really is impossible to make rigorous the infinite dimensional approach, when we are interested in shorter length-scales, we should do the discretization finer and finer. However, since there exists no "underlying" theory that works (yet) at some deeper scale, this makes QFT an endless tower of approximate theories, each less approximate than the previous one, but nevertheless only valid in a certain coarse grained scale. But then, QFT's current status is that of an intrinsically approximate theory! What a thing! This either tells us to redefine

the concept of physical theory, or to wait for a deeper theory to be found, such as a rigorous infinite dimensional version of QFT (as hoped for instance by the programme of algebraic QFT), or, if that happens to be impossible (as all efforts have shown so-far for interacting theories in 4 dimensional spacetime), perhaps a radically new theory (say, on the line of string theory). In this situation, one is left to think until which point is not QFT merely a fitting technique, as those employed in machine learning. Put enough weights to be fixed by experimental observations and you will be able to predict plenty of other experimental observations. We will come back to this in the closure.

D. Rigorous Field-configuration QM II: Insist on Infinite Degrees of Freedom

Let us break the pessimistic picture in which a field ontology can only be rigorously sustained under cut-off conditions and look towards a more serious rigorification of the heuristic QFT we found in step 3 of section B.1., "serious" meaning we insist on the **infinite** dimensional configuration space. In facing this we find two main issues to address.

D.1. Problem I: The Operator Fields

If we want to take seriously the commutation relation we wanted to impose for the field operators, "$[\hat{\phi}_x, \hat{\pi}_y] = i\delta(x-y)$", we already see that we require the field operators to be something more related to distributions than proper operators indexed by a set. Now, although there is strictly speaking no absolute impossibility proof of having a well-defined QFT with field operators like $\{\hat{\phi}_x\}_{x\in\mathbb{R}^3}$, there are several no-go theorems under quite general assumptions that discard this option. Halvorson and Müger (2006) in section 6 nicely review some relevant ones. As an example, here we give one:

Proposition 3. Let $\hat{\phi}(x)$ and $\hat{\pi}(y)$ be self-adjoint field operators, either bounded or unbounded, such that $[\hat{\phi}(x), \hat{\pi}(y)] = 0$ when $x \neq y$.[17] If $y \mapsto U(y)$ is a continuous representation of the translation group such that $U(y)\hat{\pi}(x)U(y)^* = \hat{\pi}(x+y)$ for all $x, y \in \mathbb{R}^3$, then $[\hat{\phi}(x).\hat{\pi}(x)] = 0$ for all $x \in \mathbb{R}$. ♦

But this is in flagrant contradiction with our aimed commutation relation (as we wanted $[\hat{\phi}(x), \hat{\phi}(y)]$ to "blow up" as $x \to y$). And certainly we would like to have a unitary that tells us how to change a wave-vector when we change our position-space coordinate system by a translation (hence, a representation of the translation group), and we would like the field operators to change accordingly!

Of course, as Halvorson and Müger (2006) mention, we could always weaken some of the hypothesis of a no-go theorem, say, the continuity of the representation in the above case, so that by finding example theories, we could get "go theorems". After all, history has shown in remarkable occasions, that "a no-go theorem is rather lack of imagination",[18] as Bell points out somewhere. Yet, in today's talk we will follow the general trend in mathematical approaches to QFT, that tell instead to consider **operator valued distributions**. Instead of $\{\hat{\phi}_x\}_{x\in\mathbb{R}^3}$,[19] we will define, for (a complexification of) the field configuration-space $\mathcal{H}$, the set $\{\hat{\phi}(f)\}_{f\in\mathcal{H}}$, which heuristically corresponds to

$$\hat{\phi}(f) = \text{“}\int_{\mathbb{R}^3} f(x)\ \hat{\phi}_x\ dx\text{”}, \tag{60}$$

a "smearing" of the field operator by an envelope field f. Intuitively, if we take f to be compactly supported in a small enough ball around some $\vec{x} \in \mathbb{R}^3$, we will approximate the meaning of the formal symbol $\hat{\phi}_x$. For operator valued distributions, the commutation relation would look rather like

$$[\hat{\phi}(f), \hat{\pi}(g)] = i\langle f, g\rangle\ Id, \tag{61}$$

with Id the identity on the space where the field operators act (e.g., wavefunctional space or Fock space). This is a more tractable object in the construction of a mathematically well-posed theory.

[17] In the unbounded case, it is meant that $\hat{\phi}(x)$ and $\hat{\pi}(y)$ are defined on a common dense subset D of the Hilbert space where the operators act, and that the field operators commute there.

[18] No-go theorems have often revealed more about the limitations of current frameworks than about absolute impossibilities...

[19] Apparently, the only known way to make precise these operator fields over $x \in \mathbb{R}^3$ is by considering them instead to be sequislinear form valued maps, with the problem that then there are problems to know what precisely is meant by powers of the field operators: and yeah, in heuristic presentations of interacting theories, powers of field operators are the daily grind.

In the particular case of field operators acting on our desired wavefunctional space $L^2(\mathcal{H}, \mathbb{C}, d\mu)$ ($d\mu$ yet to be specified), for Ψ in that space, instead of the problematic

$$(\hat{\phi}_x\Psi)(\phi) = \phi(x)\Psi(\phi) \qquad \text{and} \qquad (\hat{\pi}_x\Psi)(\phi) = -i\hbar\frac{\delta\Psi}{\delta\phi(x)}|_\phi, \tag{62}$$

the obvious concretization would be

$$(\hat{\phi}(f)\Psi)(\phi) = \langle f, \phi\rangle\Psi(\phi) \qquad \text{and} \qquad (\hat{\pi}(f)\Psi)(\phi) = -i\hbar\langle f, \frac{\delta\Psi}{\delta\phi}|_\phi\rangle, \tag{63}$$

which in our driving example $\mathcal{H} = H^2(\mathbb{R}^3, \mathbb{R})$ is just

$$(\hat{\phi}(f)\Psi)(\phi) = \left(\int_{\mathbb{R}^3} f(x)\phi(x)d^3x\right)\cdot\Psi(\phi) \qquad \text{and} \qquad (\hat{\pi}(f)\Psi)(\phi) = -i\hbar\int f(x)\frac{\delta\Psi}{\delta\phi(x)}|_\phi d^3x. \tag{64}$$

As a reminder, note that if the infinite dimensional configuration-space were a function space (some L^2 perhaps) over a countably infinite set (instead of an uncountable one), e.g., $\ell^2(\mathbb{N}, \mathbb{R}, \nu)$, the heuristic expression for field operators, $[\hat{\phi}_k, \hat{\pi}_j] = i\delta_{kj}$ would now be more tractable.

D.2. Problem II: The Measure

As we have mentioned several times, the main problem with an L^2 space over an infinite dimensional configuration space $Q = \mathcal{H}$ is that no obvious notion of "volume" exists in infinite dimensional spaces, while the Lebesgue measure gave us a consistent notion for all $\mathbb{R}^n$. A notion of volume should be first of all well-defined in all open-sets, because we would like to compute integrals in domains where we discuss differentiation, i.e., open sets. Hence, we demand a volume to be a Borel measure (a measure over the Borel σ-algebra of the topological space $\mathcal{H}$). Now, the key of what makes "volume" distinguished from the rest of ways to give a measure to sets, is that it is translation-invariant, meaning that if we take a (measurable) subset $B \subset \mathcal{H}$ (which will be a set of field configurations) $Volume(B) = Volume(B + \eta)\ \forall\eta \in \mathcal{H}$.

Proposition 4. (Eldredge, 2016) Let W be an infinite dimensional separable Banach space. There exists no translation invariant measure μ on the Borel σ-algebra of W that assigns positive and finite measure to open balls. In particular, any translation invariant Borel measure μ is either the zero measure or assigns $+\infty$ to every open set.

♦

Proof. **The Key problem:** By the Riesz lemma, in any ball $B(x, r)$ with $x \in W$, $r > 0$, $\exists$ a radius $s \in (0, r)$ such that inside $B(x, r)$ we can find countably infinite many disjoint balls $B(x_j, s)$ with $x_j \in B(x, r)\ \forall j \in \mathbb{N}$.

Let μ be a translation invariant Borel measure on W. By translation invariance, all $B(x_j, s)$ must have the same measure.

- If $\mu\Big(B(x_j, s)\Big) = \alpha > 0$, by σ-additivity

$$\mu\Big(\bigsqcup_{j\in\mathbb{N}} B(x_j, s)\Big) = \lim_{M\to+\infty}\sum_{j=1}^{M}\mu\Big(B(x_j, s)\Big) = \lim_{M\to+\infty}\sum_{j=1}^{M}\alpha = +\infty.$$

  Since $\sqcup_{j\in\mathbb{N}}B(x_j, s) \subseteq B(x, r)$ this implies that $\mu\Big(B(x, r)\Big) = +\infty$. Since the ball $B(x, r)$ was arbitrary and any open set has at least a ball contained in it, μ assigns measure $+\infty$ to all open sets.
- If $\mu\Big(B(x_j, s)\Big) = 0$, since W is separable, it can be covered by countably many balls of radius s and therefore $\mu\Big(W\Big) = 0$ and μ is the zero measure.

o.ε.δ.

Okay, there is no volume. But why did we want a notion of translation-invariant measure to begin with? Was it really crucial?

Translation-invariant was so special in finite dimensional $\mathbb{R}^n$ because it distinguished a measure uniquely among the zoo of ways to assign measures to sets: there exists only one translation-invariant Borel measure that assigns measure (volume) 1 to the unit cube $(0, 1) \times \cdots \times (0, 1)$ and that is the Lebesgue measure. That's

great, but in reality we only wanted to talk about measures because we wanted each state describing a quantum system on Q to have an associated **probability measure**,[20] giving probabilities to subsets of Q. Then, the straightforward way to relate probability measures to each other is by having a "background measure" $d\mu$ on Q, such that by only providing a **density** function ρ over $\mathcal{Q}$, we can get wildly different measures via $\rho d\mu$. The $L^2(Q,\mathbb{C},d\mu)$ space then entered the game because we needed such densities to be normalized for them to yield probability measures and because we found that $|\psi|^2$, for complex valued functions ψ over Q (obeying some particular PDE), gave the right measure of probability. However,

Proposition 5. Let Q be a topological space and $d\mu$ a Borel measure on Q. Then, $L^2(Q,\mathbb{C},d\mu)$ is a Hilbert space. Moreover, if $d\mu$ makes a σ-finite measure space on Q (i.e., Q has a countable cover by finite measure sets), then $L^2(Q,\mathbb{C},d\mu)$ is separable.

♦

So, given two σ-finite Borel measures $d\mu_1, d\mu_2$ over Q, both $L^2(Q,\mathbb{C},d\mu_2)$ and $L^2(Q,\mathbb{C},d\mu_1)$ are separable Hilbert spaces and hence they are unitarily equivalent. Now, if in addition, they are **mutually absolutely continuous,** meaning that they agree on who are the null-sets, there exists a function $\rho^2 : Q \to \mathbb{R}$ (that is a.e. positive wrt both μ_1 and μ_2) satisfying $d\mu_2 = \rho^2 d\mu_1$ and $d\mu_1 = \frac{1}{\rho^2} d\mu_2$. If so, the map $U : L^2(Q,\mathbb{C},d\mu_2) \to L^2(Q,\mathbb{C},d\mu_1)$ given by $U(\psi) := \rho\psi$ can be found to be a unitary map. And here is the catch: all pairs of wavefunctions related by the unitary yield the same probability measure on Q! To see why:

$$Prob\Big(q \in B \;\Big|\; \psi \in L^2(Q,\mathbb{C},d\mu_2)\Big) := \int_B |\psi|^2(q) d\mu_2 = \int_B |\psi|^2(q)\rho^2(q) d\mu_1 = \tag{65}$$

$$= \int_B (U\psi)(q) d\mu_1 =: Prob\Big(q \in B \;\Big|\; U\psi \in L^2(Q,\mathbb{C},d\mu_1)\Big),$$

i.e., as measures, $|U\psi|^2 d\mu_1 = |\psi|^2 d\mu_2$.

This means that in $\mathbb{R}^n$ if we had chosen as background measure instead of the Lebesgue measure $d^n x$, any other measure $d\mu$ mutually absolutely continuos wrt to $d^n x$, we would have obtained exactly the same quantum theory employing $L^2(\mathbb{R}^n,\mathbb{C},d\mu)$ than the usual $L^2(\mathbb{R}^n,\mathbb{C},d^n x)$. By "the same quantum theory" we mean that, given an arbitrary $\psi \in L^2(\mathbb{R}^n,\mathbb{C},d\mu)$ and its identified vector $U\psi = \rho\psi \in L^2(\mathbb{R}^n,\mathbb{C},d^n x)$:

1. The same Born rule would be given by both theories: (65) implies that ψ and $U\psi$ would attribute the same probabilities to subsets of $\mathbb{R}^n$ (i.e., although expressed in terms of different background measures, they give rise to the same measure $|\psi|^2 d\mu = |U\psi|^2 d^n x$)
2. The time evolution operator $\tilde{U}^t = e^{-itH}$ in $L^2(\mathbb{R}^n,\mathbb{C},d^n x)$ would correspond to the map $U^t := U^{-1}\tilde{U}^t U$ in $L^2(\mathbb{R}^n,\mathbb{C},d\mu)$, which is still unitary, implying (by Stone's theorem) there exists a unique self-adjoint generator K such that $U^t = e^{-itK}$. Therefore, despite the Hamiltonians H, K would look different, the dynamics in both theories would be given by a Schrödinger equation, and U would identify solutions of one SE with solutions of the other SE.
3. If a velocity field has equivariant flow-lines for the time dependent measure generated by $t \mapsto U^t\psi$, the same velocity field will have equivariant flow-lines for the respective solution in the other theory $t \mapsto \tilde{U}^t(U\psi)$, because both paths yield the same measure at all times:

$$|U_t\psi|^2 d\mu = |\psi_t|^2 \rho^2 d^n x = |\rho\psi_t|^2 d^n x = |U\psi_t|^2 d^n x = |UU^t U^{-1} U\psi|^2 d^n x = |\tilde{U}^t(U\psi)|^2 d^n x. \tag{66}$$

So moreover, they yield the same pilot wave theory! Both the experimental outcomes and the ontology in one and the other theory coincide and can genuinely be seen as a change of representation for the wavefunction.

This then proves that, other than due to its unique characterization, there is no compelling reason to use the Lebesgue measure in the canonical quantization. Instead, any other measure in the equivalence relation defined by mutual absolute continuity with the Lebesgue measure would have worked equivalently fine.

For later reference, let us check that in a sense, the unitary U identifying the theories also identifies the "algebras", by checking that it is a change of representation for the canonical commutation relation:

4. If $\hat{x}_j, \hat{p}_j$ are the usual position and momentum operators, with domains $D(\hat{x}_j), D(\hat{p}_j)$ in $L^2(\mathbb{R}^n,\mathbb{C},d^n x)$, then the corresponding operators in $L^2(\mathbb{R}^n,\mathbb{C},d\mu)$ will be the operator $U^{-1}\hat{x}_j U$ and $U^{-1}\hat{p}_j U$ on $U^{-1}D(\hat{x}_j)$

[20] Because experimentally we know that a quantum system is forcefully stochastic to epistemology.

and $U^{-1}D(\hat{p}_j)$. Explicitly, $\forall\psi \in U^{-1}D(\hat{x}_j)$, it turns out that $U^{-1}\hat{x}_jU\psi = \hat{x}_j\psi$, so formally the position operator is mapped to "itself". On the other hand, $\forall\psi \in U^{-1}D(\hat{x}_j)$, it turns out that $U^{-1}\hat{p}_jU\psi = (-i\hbar\frac{\partial\, log(\rho)}{\partial x_j} + \hat{p}_j)\psi$, so the momentum operator gets an additional multiplication operator. Indeed, they still satisfy the commutation relations.

D.2.1 The Key Strategy

Take the Lebesgue measure dx_j on $\mathbb{R}$, so that $\prod_{j=1}^{N} dx_j = d^nx$ is the n dimensional Lebesgue measure. The limit $\lim_{N\to+\infty}\prod_{j=1}^{N} dx_j$ is not a measure. But:

Proposition 6. Let μ_j be a Borel probability measure on some topological spaces W_j with $j \in \mathbb{N}$. Then, by Kolmogorov's extension theorem, there is a unique Borel measure μ on $\prod_{j=1}^{\infty} W_j$ with the product topology such that μ is the product measure $\mu := \prod_{j=1}^{\infty}\mu_j$, i.e.,

- for any given finite $I \subset \mathbb{N}$ and any Borel measurable set $B_j \subset W_j$ with $j \in I$, defining the cylinder set $B := \prod_{j\notin I} W_j \times \prod_{j\in I} B_j$, we have $\mu(B) = \prod_{j\in I}\mu_j(B_j)$.

Moreover, μ is still a probability measure on $\prod_{j=1}^{\infty} W_j$.

♦

So, if we had taken any probability measure in the equivalence class of measures of the Lebesgue measure dx, say, $d\mu_j$, then $\prod_{j=1}^{n} d\mu_j =: d^n\mu$ would still be in the same class as the n dimensional Lebesgue measure d^nx (meaning they would yield equivalent pilot wave theories), but unlike with the Lebesgue measure, the infinite product measure: $\lim_{N\to+\infty}\prod_{j=1}^{N} d\mu$ would now have a limit product measure on $\mathbb{R}^\infty := \mathbb{R}^{|\mathbb{N}|} = \prod_{j\in\mathbb{N}}\mathbb{R}$!

Then, given the ONB coefficient space $\ell^2(\mathbb{N})$, which is a subset of $\mathbb{R}^\infty$, the following hints how to find a limit theory for the tower of cut-off QFTs of D.1, i.e., a theory such that every cut-off theory is indeed an approximation of it. Fix a probability measure $d\mu$ on $\mathbb{R}$ that is equivalent to the Lebesgue measure dx. For any truncation $n \in N$, an equivalent cut-off theory results if we use as "background measure" d^nx or $d^n\mu$, i.e., using $L^2(\mathbb{R}^N, \mathbb{C}, d^nx)$ or $L^2(\mathbb{R}^N, \mathbb{C}, d^n\mu)$. The unitary $U\psi = \rho^2\psi$ would identify the field operators as in point 4 in the last subsection. Now, while the limit space (taking $N \to +\infty$) does not exist for dx, it does for $d\mu$. Then, we would just need to find in $L^2(\mathbb{R}^\infty, \mathbb{C}, d^\infty\mu)$, which are the "field operators" such that when restricted to the cut-off spaces yield the operators of the cut-off theories. Ideally, they will still satisfy the canonical commutation relation in a dense subset.

There are two loose ends in this strategy:

(a) Among the probability measures that are equivalent to the Lebesgue measure, which one gives the limit space $L^2(\mathbb{R}^\infty, \mathbb{C}, d^\infty\mu)$ that holds a reasonable representation of the field operators? This loose end would not emerge in the case that each choice of $d\mu$ yielded an equivalent infinite product measure and hence an equivalent wavefunctional space $L^2(\mathbb{R}^\infty, \mathbb{C}, d^\infty\mu)$. But whether this is the case is still an open question (for me).

(b) How to define the field operators in the limit space $L^2(\mathbb{R}^\infty, \mathbb{C}, d^\infty\mu)$? A possible idea would be to check whether the set formed by wavefunctionals only taking non-zero values on truncated sequences $(\alpha_1, ...\alpha_N, 0, 0, ...)$ (i.e., the cut-off wavefunctional set for which we know the field operators by point 4 above) is dense in $L^2(\mathbb{R}^\infty, \mathbb{C}, d^\infty\mu)$, and then define $(\hat{\alpha}_j\Psi)(\alpha) := \alpha_j\Psi(\alpha)$ in a suitable dense subset of it. On the other hand, define $(\hat{\pi}_j\psi)(\alpha) := (-i\hbar\frac{\partial}{\partial\alpha_j}log(\rho) - i\hbar\frac{\partial}{\partial\alpha_j})\psi(\alpha)$ also in some suitable dense subset (by point 4 above). A similar idea will be used in D.4.

A heuristic that could help in the search for (a) —which seems to be the most underdetermined loose end— is that if in any "finite dimensional approximation" Q_n of configuration-space, we find an a.e. positive function $\rho_n^2 := \otimes_{j=1}^{n}\rho^2 : Q_n \to \mathbb{R}$ that always appears next to d^nx when taking integrals in $L^2(Q_n, \mathbb{C}, d^nx)$, then, we could put together $\rho_n^2 d^nx = \prod_{j=1}^{n}(\rho^2 dx)$ and defining $d\mu := \rho^2 dx$, build an equivalent pilot wave theory on $L^2(Q_n, \mathbb{C}, \prod_{j=1}^{n} d\mu)$, where now the limit space $L^2(Q, \mathbb{C}, \prod_{j=1}^{\infty} d\mu)$ exists. To clarify the use of this heuristic, you can now proceed to Appendix D where we find how it can be employed for instance to derive the Feynman-Kac formula. Then, our path will be to find an already recognized way to make rigorous the QFT we are looking for (using algebraic QFT and Fock spaces) and then find a representation equivalence to some $L^2(\mathbb{R}^\infty, \mathbb{C}, d\mu)$, following the strategy employed here. This would definitively legitimize the resulting wavefunctional QFT. But before that, we must acknowledge an important detail.

D.2.2. The Price to Be Payed

To achieve a limit measure in D.2.1, a potentially tragic sacrifice was made. The product vector space in Proposition 6, $\prod_{j\in\mathbb{N}} W_j$, which in our case of interest is $\mathbb{R}^\infty$, needs to be taken in the product topology. But this topology is not normable (no norm exists s.th. its ball topology matches the product topology).[21] Hence, although there will exist a well-defined separable wavefunctional Hilbert space $L^2(\mathbb{R}^\infty, \mathbb{C}, \prod_{j=1}^\infty d\mu_j)$, the configuration space $\mathbb{R}^\infty$ will have no norm. And a norm was key for defining differentiation of wavefunctionals with respect to fields of the configuration-space.

Remarkably, it is not clear that we needed it either, since time derivatives for paths in $L^2(\mathbb{R}^\infty, \mathbb{C}, \prod_{j=1}^\infty d\mu_j)$ are very well-defined. As such, a SCOPUG on $L^2(\mathbb{R}^\infty, \mathbb{C}, \prod_{j=1}^\infty d\mu_j)$, say $U_t = e^{itH}$, still makes sense and satisfies a SE $\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = H\psi_t$ for $\psi_t = U\psi_0$ and $\psi_0 \in D(H)$. Likewise, a continuity equation can still be extracted from such a SE. Also, since $L^2(\mathbb{R}^\infty, \mathbb{C}, \prod_{j=1}^\infty d\mu_j)$ itself is a Hilbert space, then we can still do calculus for functions over it.

Moreover, we really only used to take derivatives of wavefunctions wrt configuration-space in dense subspaces (like some $H^1(\mathbb{R}^{3N})$ in the case of $L^2(\mathbb{R}^{3N})$. Hence, we would have enough with dense subspaces that do have a norm, where taking functional derivatives does make sense. Namely, we can likely find an infinite dimensional subspace of $\mathbb{R}^\infty$, ideally $\ell^2(\mathbb{N})$, that does have a norm and the norm topology coincides with the subspace topology induced by $\mathbb{R}^\infty$. If so, the Borel measure on $\mathbb{R}^\infty$ would restrict to a Borel measure for the NVS $\ell^2(\mathbb{N})$ and then we could even take $L^2(\ell^2(\mathbb{N}), \mathbb{C}, \mu|_{\ell^2})$ altogether as our wavefunctional space.

Other alternative tricks exist too. For example, although the product topology is not normable, there exists a distance function, namely,

$$d((\alpha_j),(\beta_j)) := \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{2^k(2+|x_k-y_k|)} \qquad (\alpha_j)_j, (\beta_j)_j \in \prod_{j\in\mathbb{N}} W_j, \tag{67}$$

so at least it is a metric space. Then, generalized notions to those desired from calculus might be found.

Alternatively, we could accept a non-Borel measure for the probabilistic interpretation (which in the finite dimensional restrictions is nevertheless Borel). By this I mean that one could norm $\mathbb{R}^\infty$, allowing calculus over the norm topology and only use the measure and the product topology's $\sigma-$algebra for the probabilistic interpretation.

For today's talk we will not need to care about this detail any-more. Just note that we will work on $\mathbb{R}^\infty$ as a legitimate field configuration-space, assuming it contains $\ell^2(\mathbb{N})$ (the ONB coefficient space) in a suitable way.

[21] *Proof.* Assume such a norm $||\cdot|| : W_j \to [0,+\infty)$ existed. Then, because the unit ball B_1 around the zero vector $(0,0,0,...) \in \prod_{j\in\mathbb{N}} W_j$ would be open and the product topology is generated by cylinder sets, there would exist a cylinder subset of the unit ball, say $O \subseteq B_1$ such that $O = \prod_{j\notin I} W_j \times \prod_{j\in I} B_j$ with $I \subset \mathbb{N}$ finite and $B_j \subseteq W_j$ open. But if so, for any $k \in \mathbb{N}\backslash I$ and any $x_k \in W_k \backslash \{0\}$, the point $x := (0,...,0,x_k,0,...) \in O$. Since W_k is a vector space, $tx \in O \subset B_1$ for all $t > 0$, but using the homogeneity of the norm, $||tx|| = t||x||$. Hence, the point $y := 2x/||x||$ is in the unit ball and simultaneously $||y|| = 2$. Absurd! *o.ε.δ.*

D.3. Algebraic QFT and How to Use it to Build a Distinguished QFT over Field Configurations

Consider finite dimensional good old QM that we all know. Following the narrative of Wallace (2006), one way to specify the theory would be to give a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and a set of self-adjoint operators on it $\{\hat{O}_j\}_j$ for which we give names manifesting their "experimental" correlatives: "position of j-th degree of freedom" or "canonical momentum of k-th degree of freedom". We declare that points $\psi \in \mathcal{H}$ describe laboratory preparations such that $\langle\psi, \hat{O}_j\psi\rangle$ yields the expected value for the respective "experimental correlative". Then, we would specify a function of those operators $\hat{H}$, call it "Hamiltonian", say it generates the time evolution of points ψ in the Hilbert-space as $U_t = e^{-it\hat{H}}$ and make each operator convey the effect of time evolution: $\hat{O}_j(t) := U_t\hat{O}_jU_t^{-1}$. According to the orthodox view, the set of time evolved operators contains all the predictions of the theory, so we could now forget about the U_t. Providing only the action of the operators $\hat{O}_j$ on $\mathcal{H}$ together with their experimental correlatives and their algebraic relations, we would have all an orthodox needs. Now, what if we also forgot about the action on $\mathcal{H}$ and the fact they are operators at all, merely keeping the abstract algebraic relations between the operators (or more particularly, between the bounded functions built from those position and momentum operators), keeping as labels their experimental correlatives? Would we be able to recover the theory back? Remarkably, the answer is that for finitely many position-momentum pairs yes! For a follower of Heisenberg's theory, this is not merely a causality, but it is the true way to approach QM. Let us introduce their approach, so that we can see how they already found a way to rigorify our driving example QFT in a rather different space than the wavefunctionals leading to field ontology. The idea is to use it as a solid scaffold or mirror, relative to which we can build our wavefunctional rigorization.

Definition 7. We say that a complex Banach space $(A, ||\cdot||)$ (complete NVS over $\mathbb{C}$) equipped with an associative bilinear product $\cdot$ such that $||x\cdot y|| \leq ||x||\cdot||y|| \; \forall x, y \in A$ is a **Banach algebra**.

♦

Definition 8. A **C^*-algebra** is a Banach algebra $(A, ||\cdot||, \cdot)$ equipped with an involution (a bijection that is its own inverse), $\cdot^* : A \to A$, that has the properties of the adjoint (i-iii) and is compatible with the norm (iv):

$$(i)\;\; (x+y)^* = x^* + y^* \qquad (ii)\;\; (xy)^* = y^*x^* \qquad (iii)\;\; (\lambda x)^* = \bar{\lambda}x^* \qquad (iv)\;\; ||xx^*|| = ||x||\cdot||x|| = ||x||^2$$

for all $x, y \in A$, $\lambda \in \mathbb{C}$.

♦

Definition 9. We define a **representation** of the C^*-algebra $(A, ||\cdot||, \cdot, \cdot^*)$ as a linear map $\wr : A \to \mathscr{L}(\mathcal{H})$ (only bounded operators!) with $\mathcal{H}$ a Hilbert space, such that

$$\wr(a_1 \cdot a_2) = \wr(a_1)\cdot\wr(a_2) \qquad and \qquad \wr(a^*) = \wr(a)^* \qquad \forall a_1, a_2, a \in A.$$

We say it is an **irreducible representation** when $\mathcal{H}$ does not split in orthogonal subspaces preserved by the action of the operators in $\wr(A)$. (This holds iff there exists a cyclic vector for the represented operators, i.e., a vector $\psi \in \mathcal{H}$ whose image under the operators $\{\wr(a)\psi \mid a \in A\}$ is dense).

♦

Definition 10. We say that two representations $(\wr, \mathcal{H})$, $(\wr', \mathcal{H}')$ of a C^*-algebra are **equivalent** if there exists a unitary isomorphism $U : \mathcal{H} \to \mathcal{H}'$ such that $\wr(a) = U^{-1}\wr'(a)U$.

♦

To put it into context, assume we take a quantum theory and do the procedure we mentioned before of forgetting everything except the bounded operators and their relations. We build an abstract C^*-algebra out of them. If we are able to prove that there exists a unique representation of the C^*-algebra up to unitary equivalence, then, whichever representation we find of it, we will have essentially recovered the original quantum theory (as long as we remember the experimental correlatives of the operators!).

For finitely many quantized degrees of freedom ("finitely many position and momentum operators") this is precisely the case by the Stone-von Neumann theorem (Reed and Simon, 1981). So, the algebraic and "Schrödinger way" of specifying the theory match in finite dimensions. However, for infinitely many quantized degrees of freedom, and hence in particular for QFT (the field operators being the "infinitely many position-momentum operators"), the theorem fails: for a C^*-algebra abstracted in such a context, there even exist uncountably many inequivalent representations. The algebraic structure alone is not enough to specify a QFT.

The reason why an algebraist still sees this abstraction technique as the foundational truth is that for them, the core of the canonical quantization process is the declaration of some "position" and "momentum"-like operators (say, the field operators) that satisfy a fixed commutation relation: hence, they claim, the key of canonical

quantization is that it imposes algebraic constraints on the generated theory's operators, independently of the Hilbert space in which we implement them. Believing this to its last consequences, they have built a very strong programme with the goal to rigorify QFT known as Algebraic QFT (AQFT) —see (Baez et al., 2014) for a (very) technical introduction.

Following their modus operandi, our strategy will be to abstract the algebraic relations between the heuristic field operators we developed in B, find their corresponding C^*-algebra; there we will find a particular representation that AQFT has used to rigorify our KG QFT, even if it happens very far away from wavefunctionals: it uses the so-called, Fock spaces. This representation is legitimized because it is actually the closest one to what experimentalists use to check their cross-section experiments (they use heuristic Fock spaces to describe the asymptotically free states that are detected in the walls of particle colliders). It is completely alien to the wavefunctional approach we are suggesting, but we will find that each discretization of this Fock space QFT has an obvious equivalent wavefunctional cut-off representation. Then, using our trick in D.2.1 to take the "no cut-off" limit, we will find an equivalent representation of the infinite dimensional C^*-algebra, but now in wavefunctional space. That is how we are going to tie the loose ends (a) and (b) to an already recognized rigorization!

D.3.1. The C^*-algebras that Abstract the Commutation Relations

This section follows from Bachmann (2017). If we take a manual on C^*-algebras, the first thing we will find in our interest line is the Canonical **Anti**-commutation Relation (CAR), which is

Definition 11. Given a Hilbert space $\mathcal{H}$, we define the CAR algebra $\mathcal{A}_-(\mathcal{H})$ as the C^*-algebra generated by the identity Id and the elements $a(f)$ with $f \in \mathcal{H}$ such that for all $f, g \in \mathcal{H}$

- $f \mapsto a(f)$ is anti-linear.
- $\{a(f), a(g)\} = 0 = \{a(f)^*, a(g)^*\}$.
- $\{a(f), a(f)^*\} = \langle f, g\rangle\, Id$.

♦

This is the algebra of creation and annihilation operators abstracted from a fermionic theory. It is particularly nice because it can be given by bounded self-adjoint operators. Yet, it is not the commutation relation we are looking for.

We would then leaf through the manual, looking for a similar thing but using commutators, something like the C^*-algebra for bosonic creation and annihilation operators, perhaps called Canonical Commutation Relations (CCR). Instead of a C^*-algebra we would find something in the line of

Definition 12. Given a Hilbert space $\mathcal{H}$ and a set of operators $a(f)$ with $f \in \mathcal{H}$ s.th. for $f, g \in \mathcal{H}$ satisfy

- $f \mapsto a(f)$ is anti-linear.
- $[a(f), a(g)] = 0 = [a(f), a(g)]$
- $[a(f), a(g)^*] = \langle f, g\rangle Id$

understood as equalities over **dense domains**, we say they satisfy the **CCR.**

♦

We will find that the problem is that such operators cannot form a proper C^*-algebra because they will need to be unbounded operators! Instead, the clever AQFT community has find an "almost equivalent" C^*-algebra, which are essentially the exponentials (by the functional calculus) of such creation and annihilation operators. By Stone's theorem the exponentials are bounded (unitary) operators and hence can indeed form a C^*-algebra. The exponential version of the CCR is called Weyl algebra.

Definition 13. Given a Hilbert space $\mathcal{H}$, we define the exponential or **Weyl CCR** algebra $\mathcal{A}_+(\mathcal{H})$ as the C^*-algebra generated by[22] the elements $W(f)$ with $f \in \mathcal{H}$ such that for all $f, g \in \mathcal{H}$

- $W(-f) = W(f)^*$.
- $W(f)W(g) = exp\big(\frac{-i}{2} Im\langle f, g\rangle\big) W(f+g)$.

Note that this implies the commutation relation $W(f)W(g) = exp(-iIm\langle g, f\rangle)W(g)W(f)$.

♦

[22] As a comment on well-definition, one can prove that if A_1 and A_2 satisfy $\mathcal{A}_+(\mathcal{H})$ there exists a unique $*$-isomorphism $\gamma : A_1 \to A_2$ such that $a_2(f) = \gamma(a_1(f))\ \forall f \in \mathcal{H}$. The same holds for $\mathcal{A}_-(\mathcal{H})$.

Under usual conditions (the representation being regular[23]) a representation of the Weyl CCR algebra will have[24] densely defined self-adjoint generators $\hat{\varphi}(f)$ that will give both, bosonic creation and annihilation operators satisfying the CCR of Definition 12, and some "conjugate field operators" $\hat{\phi}(f), \hat{\pi}(f)$ satisfying the CCR we wanted to impose in the canonical quantization, $[\hat{\phi}(f), \hat{\pi}(g)] = i\langle f, g\rangle$. We will see more in a moment.

Note that the **number of quantized degrees of freedom** of a quantum theory representing the CAR or CCR (informally defined before as "number of position and momentum operators") is the dimension of the Hilbert space $\mathcal{H}$ that represents the C^*-algebra.

Immediately after that, the manual of C^*-algebras will show us how to build the so-called **Fock spaces** that satisfy the CAR and Weyl CCR. This then proves that at least one representation of each, the CCR and CAR, exists for any Hilbert space $\mathcal{H}$ that we want to quantize (even for infinitely many degrees of freedom to be quantized).

Definition 14. Let there be an arbitrary complex Hilbert space $\mathcal{H}$.

- Define for a permutation $\sigma \in S_n$ the vector permutation map Π_σ on $\mathcal{H}^{\otimes n}$ defined as the extension by linearity of $\Pi_\sigma : \psi_1 \otimes \cdots \otimes \psi_n \to \psi_{\sigma^{-1}(1)} \otimes \cdots \otimes \psi_{\sigma^{-1}(n)}$.
- For $n \in \mathbb{N}$, define $\mathcal{H}_\pm^{(n)} := \{\psi^{(n)} \in \mathcal{H}^{\otimes n} \,\big|\, \Pi_\sigma \psi^{(n)} = (\pm 1)^{sign(\sigma)} \psi^{(n)} \; \forall \sigma \in S_n\}$. That is, the symmetric (+) and antisymmetric (-) subspace of $\mathcal{H}^{\otimes n}$. Define also $\mathcal{H}_\pm^{(0)} := \mathbb{C}$.
- We define the **bosonic** (+) and **fermionic** (-) **Fock space** over $\mathcal{H}$ by $\mathcal{F}_\pm(\mathcal{H}) := \bigoplus_{n=0}^\infty \mathcal{H}_\pm^{(n)}$. An element $\psi \in \mathcal{F}_\pm(\mathcal{H})$ is a tuple $\psi = (\psi^{(0)}, \psi^{(1)}, ...)$ with $\psi^{(j)} \in \mathcal{H}_\pm^{(j)}$ and $\sum_{n\in\mathbb{N}} ||\psi||_{\mathcal{H}} < +\infty$.
- If we equip $\mathcal{F}_\pm(\mathcal{H})$ with the inner product $\langle \psi, \phi\rangle := \sum_{j=0}^\infty \langle \psi^{(n)}, \phi^{(n)}\rangle$, it is a Hilbert space.
- If $\mathcal{H}$ represents the Hilbert space of some quantum theory for a single "particle", $\mathcal{H}_\pm^{(n)}$ is the quantum theory of n identical fermionic or bosonic "particles". In such a case, if $||\psi|| = 1$, then $||\psi^n||^2$ can be interpreted as the probability that there are exactly n "particles" (for us they will be virtual particles). Hence, for such $\mathcal{H}$, Fock space describes the Hilbert space of a variable but finite number of particles:
$$\text{Prob}\Big(\#particles \geq N \,\Big|\, \psi\Big) = \sum_{n\geq N} ||\psi^{(n)}||^2 \underbrace{\to}_{N\to+\infty} 0.$$
- In analogy, whatever $\mathcal{H}$ is, we refer to the n-th factor of the direct sum of $\mathcal{F}_\pm(\mathcal{H})$, namely $\mathcal{H}_\pm^{(n)}$, as the n **"particle" sector**.
- We call $\Omega := (1, 0, 0, ...)$ the **vacuum state** (it would represent a state with no "particle").
- Define the set of states that are non-zero only in a finite number of sectors (describing finitely many "particles") by $\mathcal{F}_\pm^{fin}(\mathcal{H}) := \{\psi \in \mathcal{F}_\pm(\mathcal{H}) \mid \exists N \in \mathbb{N}\; s.th.\; \psi^{(n)} = 0 \; \forall n \geq N\}$. (It is dense in $\mathcal{F}_\pm(\mathcal{H})$!)
- We define the extension of the operator $N : \mathcal{F}_\pm^{fin}(\mathcal{H}) \to \mathcal{F}_\pm^{fin}(\mathcal{H})$ such that $N\psi = n\psi \; \forall \psi \in \mathcal{H}_\pm^{(n)}$, to be the **number operator**. The Fock space can be seen as the spectral diagonalization space of the number operator.[25] The number operator is the lift[26] of the Id operator.
- For $f \in \mathcal{H}$ define the operator $a_\pm(f) : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes(n-1)}$ removing a "particle" in state f as
$$a_\pm(f)(\psi_1 \otimes \cdots \otimes \psi_n) := \sqrt{n}\langle f, \psi_1\rangle \; \psi_2 \otimes \cdots \otimes \psi_n$$

[23] A representation $(\mathscr{H}, \mathscr{R})$ of $\mathcal{A}_+(\mathcal{H})$ is said to be **regular** if $t \mapsto \mathscr{R}(W(tf))$ is a SCOPUG for all $f \in \mathcal{H}$.

[24] If the representation is regular, by Stone's theorem, there exists a densely defined s.a. generator $\hat{\varphi}(f)$ s.th. $\mathscr{R}(W(tf)) = exp(it\hat{\varphi}(f))$ for all $f \in \mathcal{H}$. Moreover, $[\hat{\varphi}(f), \hat{\varphi}(g)] = iIm\langle f, g\rangle$ on $D(\hat{\varphi}(f)) \cap D(\hat{\varphi}(g))$. By defining $a^*(f) := 2^{-1/2}(\hat{\varphi}(f) - i\hat{\varphi}(if))$ and $a(f) := 2^{-1/2}(\hat{\varphi}(f) + i\hat{\varphi}(if))$ on $D(\hat{\varphi}(f)) \cap D(\hat{\varphi}(if))$, which is dense in $\mathscr{H}$, then $[a(f), a^*(g)] = \langle f, g\rangle$, satisfying Def. 12 (the CCR for creation and annihilation operators). Our desired CCR for canonical "conjugate" field operators can be obtained then as explained after equation (68).

[25] The wavefunctional space on the other hand could be viewed as the space where the field operators are diagonal.

[26]
- Given a densely defined operator (A, D) on $\mathcal{H}$, we define its **lift** or **second quantization** to be the operator $\Gamma(A)$ with domain $\Gamma(D) := \{\psi \in \mathcal{F}^{fin}(\mathcal{H}) \mid \psi^{(n)} \in \otimes_{k=1}^n D \; \forall n \in \mathbb{N}\}$. For example $N = \Gamma(Id)$.
- Given a unitary U on $\mathcal{H}$, we call the unitary $\Gamma(U)$ on $\mathcal{F}_\pm(\mathcal{H})$ s.th. it acts as $\otimes_{k=1}^n U$ on $\mathcal{H}_\pm^{(n)}$ the **lifted** or **second quantized propagator**. If e^{itA} is a SCOPUG on $\mathcal{H}$ then $\Gamma(e^{itA}) = e^{it\Gamma(A)}$ is a SCOPUG on $\mathcal{F}_\pm(\mathcal{H})$. Hence, there is a naive way to implement free particle dynamics in the Fock space.

and $a_\pm(f)\mathcal{H}_\pm^{(0)} = 0$ (e.g., $a_\pm(f)\Omega = 0$). Its obvious extension $a(f) : \mathcal{F}_\pm(\mathcal{H}) \to \mathcal{F}_\pm(\mathcal{H})$ is called the **annihilation operator** of $f \in \mathcal{H}$. (As a map $f \mapsto a_\pm(f)$ it is anti-linear and $Na_\pm(f) = a_\pm(f)(N - Id)$).

- Its adjoint $a_\pm^*(f) := a_\pm(f)^* : \mathcal{H}^{(n-1)} \to \mathcal{H}^{(n)}$ given for $\sigma_k^{-1} := (k, 1, ..., \hat{k}, ..., n)$ as

$$(a_\pm^*(f)\psi^{(n-1)}) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (\pm 1)^{k-1} \Pi_{\sigma_k} f \otimes \psi^{(n-1)}$$

is called the **creation operator** of $f \in \mathcal{H}$. Its action is exactly to take the $n-1$ particle sector vector, to add a "particle" in the state f and then (anti)symmetrize the result. (Note $f \mapsto a_\pm^*(f)$ is linear). ♦

Proposition 7. Within the notation of the last definition,

- $a_\pm(f)$ and $a_\pm^*(f)$ satisfy the CAR and CCR respectively for (-) and (+), in particular, $[a_+(f), a_+^*(f)] = \langle f, g\rangle Id$ and $\{a_-(f), a_-^*(f)\} = \langle f, g\rangle Id$.
- Ω is a cyclic vector for the creation operators, in the sense that $\{a_\pm^*(f_1)\cdots a_\pm^*(f_n)\Omega \mid f_j \in \mathcal{H},\ n \in \mathbb{N}\}$ is dense in the Fock space (any state of the Fock space can be written as a series of "creations of particles from the vacuum"). Hence, Fock space is an irreducible representation of the CAR and CCR!
- Moreover, given any "1-particle" ONB $B := \{f_j\}_{j\in\mathbb{N}} \subset \mathcal{H}$, the following is an ONB of $\mathcal{F}_+(\mathcal{H})$:

$$\left\{ \frac{1}{\sqrt{n_1!\cdots n_N!}} a_+^*(f_{j_1})^{n_1}\cdots a_+^*(f_{j_N})^{n_N}\Omega \mid N, n_k \in \mathbb{N} \text{ and } f_{j_k} \in B \text{ and } j_1 < \cdots < j_N \right\}$$

while the following is an ONB of the fermionic $\mathcal{F}_-(\mathcal{H})$

$$\left\{ a_-^*(f_{j_1})\cdots a_-^*(f_{j_N})\Omega \mid N \in \mathbb{N} \text{ and } f_{j_k} \in B \text{ and } j_1 < \cdots < j_N \right\}.$$

(Hence, we will be able to write any state in terms of the vacuum state, and this will give us the "ρ^2 that always appears next the Lebesgue measure" we talked about in D.2.1).

♦

D.3.2. A rigorous QFT over Fock-Space

Centring in the bosonic Fock space (following now the presentation by Reed and Simon (1975)), we will find that there is a rigorous way to arrive at the "canonical conjugate" field operators we are interested on, but in Fock space instead of field configuration-space. This is not exactly what we were expecting from the quantization recipe in B, but as we explained, it will be the perfect proxy to find the rigorous field-configuration space theory.

Definition 15. Taking the Fock space construction of Def. 14, we define the **Segal field operators** (that end up yielding what physicists call the *real* free bosonic field) as $\hat{\varphi}_+(f) := \frac{1}{\sqrt{2}}(a_+(f) + a_+^*(f))$ for $f \in \mathcal{H}$. ♦

They also satisfy the CCR in $\mathcal{F}_+^{fin}(\mathcal{H})$,[27] but now in a shape that is closer to field operators (note the i):

$$[\hat{\varphi}_+(f), \hat{\varphi}_+(g)] = i\, Im\langle f, g\rangle. \tag{68}$$

Now, the main difference between $\hat{\varphi}_+$ and our desired field operators $\hat{\phi}, \hat{\pi}$ is that in general, the Segal field operator takes any field in the "1-particle" Hilbert space $\mathcal{H}$, and $\mathcal{H}$ is taken to be a **complex** Hilbert space in the present construction,[28] such as $H^2(\mathbb{R}^3, \mathbb{C}, d^3x)$ for our driving example. Instead, we wanted to build a QFT over the configuration space of **real** valued fields $H^2(\mathbb{R}^3, \mathbb{R}, d^3x)$. Hence, we need to "uncomplexify" $\mathcal{H}$, taking its underlying real Hilbert space (chosen by some conjugation, which in our case is) given by complex conjugation.

[27] In particular, they are symmetric on $\mathcal{F}_+^{fin}(\mathcal{H})$ and extend to self-adjoint operators. Hence, they generate unitaries $W_+(f) := exp(i\hat{\varphi}(f))$ on the whole $\mathcal{F}_+(\mathcal{H})$ which are a representation of the Weyl algebra (the exponential CCR).

[28] The Segal field operators on the bosonic Fock space can be used to build a rigorous quantization of the Klein-Gordon equation taking as $\mathcal{H}$ the complex L^2 space over the hyperbola of Minkowski space representing the momentum-space mass-shell. Then, one uses field operators that take fields on Minkowski spacetime instead of a single slice as us, but the input to the field operators can be restricted adequately so that a rigorous version of the quantization in single-time slices like the one we are aiming at, can be obtained as a corollary (see (Reed and Simon, 1975)).

Definition 16. Let $\mathcal{H}$ be a complex Hilbert space and C a conjugation (a conjugate-linear, isometric involution, e.g., the point-wise complex conjugation in $L^2(\mathbb{R}^n)$). Define $\mathcal{H}_R := \{f \in \mathcal{H} \mid Cf = f\}$ (which for the complex conjugation yields the subspace of real valued functions). In particular, we get that $\mathcal{H} = \mathcal{H}_R \oplus i\mathcal{H}_R$. Then, for $f \in \mathcal{H}_R$, we define $\hat{\phi}(f) := \hat{\varphi}_+(f)$ and $\hat{\pi}(f) := \hat{\varphi}_+(if)$ as the **canonical conjugate field operators**. ♦

Proposition 8. Given the notation of the last definition,

- $\forall f \in \mathcal{H}_R$ the operators $\hat{\phi}(f), \hat{\pi}(f)$ are essentially self-adjoint on $\mathcal{F}_+^{fin}$.
- $\{\hat{\phi}(f) \mid f \in \mathcal{H}_R\}$ is a commuting family of self-adjoint operators (likewise for $\hat{\pi}(\cdot)$).
- Ω is cyclic for $\{\hat{\phi}(f)\}$. In particular, given an ONB of $\mathcal{H}$ s.th. $\mathcal{B} := \{f_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_R$, the span of

$$\left\{ P\Big(\hat{\phi}(f_{k_1}), ..., \hat{\phi}(f_{k_N})\Big)\Omega \;\Big|\; N \in \mathbb{N};\ f_k \in \mathcal{B};\ \text{P a complex valued polynomial in } N \text{ real variables} \right\}$$

  is dense in $\mathcal{H}$.
- For all $f, g \in \mathcal{H}_R$, the CCR we were looking for, $[\hat{\phi}(f), \hat{\pi}(g)] = i\,\langle f, g\rangle$, holds on the dense set $\mathcal{F}^{fin}$.
- For all $f, g \in \mathcal{H}_R$, also the Weyl CCR holds, $e^{i\hat{\phi}(f)}e^{i\hat{\pi}(g)} = e^{-i\langle f,g\rangle}e^{i\hat{\pi}(g)}e^{i\hat{\phi}(f)}$.

♦

If we did not care about field configuration-space wavefunctionals, then, for $\mathcal{H} = H^2(\mathbb{R}^3, \mathbb{C}, d^3x)$, the Fock space $\mathcal{F}_+(\mathcal{H})$ with the field operators $\hat{\phi}(f)$ and $\hat{\pi}(f)$ and $f \in H^2(\mathbb{R}^3, \mathbb{R}, d\mu)$ would already satisfy most of the other conditions of canonical quantization heuristically given in B. In fact, as we mentioned already, it is the standard way to make it mathematically rigorous and physicists employ a heuristic version of this construction to define asymptotic states of their experiments. However, we lost two main things in skipping wavefunctional spaces: (i) how simple it was to obtain the Hamiltonian for second quantized dynamics, and (ii) the opportunity to have a pilot wave theory with field ontology behind it, which is our leitmotif today. Let us recover them.

D.3.3. A Baby Version of the Grand Finale

What we are going to do here for one degree of freedom (find an equivalence between the Schrödinger and Fock representations) is exactly what gives the wavefunctional picture if done for infinitely many degrees of freedom. The Schrödinger picture will be exactly the wavefunctional picture.

Quantum mechanics for a single degree of freedom/single particle is usually given in the so-called Schrödinger representation, which is a representation of the $\mathcal{A}_+(\mathbb{C})$ Weyl CCR algebra on $L^2(\mathbb{R}, \mathbb{C}, dx)$ given, as explained by Bachmann (2017), via

$$\mathscr{r}\Big(W(s+it)\Big) := e^{\frac{i}{2}st}U(s)V(t) \quad \text{where } \forall \psi \in L^2(\mathbb{R}) \begin{cases} (U(s)\psi)(x) = e^{is}\psi(x) & \text{(global phase)} \\ (V(t)\psi)(x) = \psi(x+t) & \text{(translation)} \end{cases}. \tag{69}$$

As SCOPUGs, by Stone's theorem there exists a unique generator for $U(s), V(t)$, being the position operator $\hat{x} = x$ and the momentum operator $\hat{p} = -i\hbar\partial_x$, respectively. In a dense set (say Schwartz functions) they satisfy the ordinary CCRs, $[\hat{x}, \hat{p}] = i\,Id$ etc., as you know.

But now you also know that there must exist a Fock space representation of $\mathcal{A}_+(\mathbb{C})$ and it is certainly not the one we described. It is given in $\mathcal{F}_+(\mathbb{C}) := \bigoplus_{n=0}^{\infty} \mathbb{C}^{\otimes n}$. Note that $\mathbb{C}^{\otimes n}$ is 1 dimensional for all $n \in \mathbb{N}$ and has ONB $\{1 \otimes \cdots \otimes 1\}$. Hence, an ONB for $\mathcal{F}_+(\mathbb{C})$ is given by $\{1,\ 1 \otimes 1,\ 1 \otimes 1 \otimes 1, ...\}$ and a state $\psi \in \mathcal{F}_+(\mathbb{C})$ is $\psi = (c_0,\ c_1 1 \otimes 1,\ c_2 1 \otimes 1 \otimes 1, ...)$, so a square summable sequence of complex numbers $(c_j)_j \subset \ell^2(\mathbb{N}_0, \mathbb{C})$. Almost as if they were the expansion coefficients of a vector in $L^2(\mathbb{R})$. (Spoiler: they secretly are!)

So we obtained two different representations of the Weyl CCR and both can be checked to be irreducible.[29] Hence, because the number of quantized degrees of freedom is finite, i.e., $dim(\mathcal{H}) = dim(\mathbb{C}) = 1$, by the Stone-von Neumann theorem they must be equivalent representations of the CCR. We can explicitly find their unitary equivalence as follows.

For this, we are going to find a Fock space structure hiding in $L^2(\mathbb{R})$. Define the operators

$$\hat{a}_S(1) := \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) \qquad \text{and} \qquad \hat{a}_S^*(1) := \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}). \tag{70}$$

[29] Actually, looking for the obvious cyclic vectors will reveal the unitary equivalence.

They also satisfy the CCR (in the same common dense domain), but now in the creation and annihilation operator version we saw in Def. 12, namely $[\hat{a}_S(1), \hat{a}_S^*(1)] = Id$ etc. Moreover, there is a vector that acts as the "vacuum" too: the solution to $a_S(1)\Omega_S = 0$,

$$\left(x + \frac{\partial}{\partial x}\right)\Omega_S(x) \equiv 0 \Longleftrightarrow \Omega_S(x) \equiv \frac{1}{\pi^{1/4}} e^{-x^2/2}. \tag{71}$$

This is nothing but the ground-state of the harmonic oscillator (note there is no Hamiltonian in game!). What is more, this analogue of the vacuum is also cyclic wrt repeated application of $a^*(1)$ (the span of $\{a^*(1)^n\Omega_S \mid n \in \mathbb{N}\}$ is dense in $L^2(\mathbb{R})$) because up to a norm factor $1/\sqrt{n!}$ they form an ONB:

$$\psi_n := a^*(1)^n\Omega_S = \frac{1}{2^{n/2}\pi^{1/4}}\left(x - \frac{\partial}{\partial x}\right)^n e^{-x^2/2} = \frac{1}{2^{n/2}\pi^{1/4}} e^{-x^2/2} H_n(x) \tag{72}$$

are the eigen-energy states of the harmonic oscillator, where $H_n(x) := e^{x^2/2}(x - \partial_x)^n e^{-x^2/2}$ is the n-th Hermite polynomial.[30] Because they form an ONB of $L^2(\mathbb{R})$, defining $\mathcal{H}^{(n)} := span\{\psi_n\} = span\{a_S^*(1)^n\Omega_s\}$, we get $L^2(\mathbb{R}) = \oplus_{n=0}^{\infty}\mathcal{H}^{(n)}$. But this is exactly a Fock space structure! Hence, we found the unitary relating the representations:

$$U : L^2(\mathbb{R}) \to \mathcal{F}_+(\mathbb{C}) \quad \text{mapping} \quad \psi = \sum_{n=0}^{\infty} c_n\psi_n \mapsto (c_0,\ c_1 1\otimes 1,\ c_2 1\otimes 1\otimes 1, ...). \tag{73}$$

As promised, Fock space elements turned out to represent expansion coefficients! Importantly, the creation and annihilation operators (70) are mapped to those of Fock space. Interestingly, the canonical conjugate fields of Fock space are identified with the Schrödinger representation's position and momentum operators:

$$\hat{\phi}_S(1) := \hat{\varphi}_S(1) := \frac{1}{\sqrt{2}}(\hat{a}_S(1) + \hat{a}_S^*(1)) = \hat{x} \quad \text{and} \quad \hat{\pi}_S(1) := \hat{\varphi}_S(i) := \frac{1}{\sqrt{2}}(-i\hat{a}_S(1) + i\hat{a}_S^*(1)) = \hat{p}. \tag{74}$$

Then, it should not be a surprise that the canonical conjugate field operators of the Fock space will be exactly the ones mapped to the generalized position and momentum operators, i.e., the field operators defined in B, for the wavefunctional picture of the grand finale.

With all, $U : L^2(\mathbb{R}) \to \mathcal{F}_+(\mathbb{C})$ maps:

$$\begin{array}{rcl}
\Omega_S = \pi^{-1/4}e^{-x^2/2} = \psi_0 & \longmapsto & \Omega = (1,0,0,0,...) \\
(a_S^*)(1)\Omega_S = \psi_1 & \longmapsto & (a_+^*)(1)\Omega = (0,1,0,0,...) \\
(a_S^*)(1)^2\Omega_S = \psi_2 & \longmapsto & (a_+^*)(1)^2\Omega = (0,0,1,0,...) \\
(a_S^*)(1)^n\Omega_S = \psi_n & \longmapsto & (a_+^*)(1)^n\Omega = (\underbrace{0,0,...}_{n-1},1,0,...) \\
\hat{N}_S := a_S^*(1)a_S(1) = \frac{1}{2}(\hat{x} - i\hat{p})(\hat{x} + i\hat{p}) & \longmapsto & \hat{N} = a_+^*(1)a_+(1)
\end{array} \tag{75}$$

$$\begin{array}{rcl}
\hat{\varphi}_S(1)\Omega_S = \pi^{-1/4}\, x\, e^{-x^2/2} & \longmapsto & \hat{\varphi}(1)\Omega = \frac{1}{\sqrt{2}}(a_+(1) + a_+^*(1)) = (0,1,0,...) \\
\hat{\varphi}_S(1)^2\Omega_S = \pi^{-1/4}\, x^2\, e^{-x^2/2} & \longmapsto & \hat{\varphi}(1)^2\Omega = (1,0,1,0,...) \\
\hat{\pi}_S(1)\Omega_S = \pi^{-1/4}\,(-i\hbar\partial_x)\, e^{-x^2/2} & \longmapsto & \hat{\pi}(1)\Omega = \frac{1}{\sqrt{2}}(a_+(i) + a_+^*(i)) = (0,i,0,...)
\end{array} \tag{76}$$

So from the Schrödinger picture's perspective what the creation and annihilation operators "create" or "annihilate" is an excitation or relaxation on the ONB. Creating a particle in the Fock picture can be seen as taking the wavefunction of the Schrödinger picture to a higher excited eigenstate (jargon that only makes strict sense to use if we really had that the governing Hamiltonian was the harmonic oscillator's). Hence, the number of bosons accounts for the excitation number of the harmonic oscillator: having a Fock state with three bosons is nothing but having the particle of the Schrödinger picture in the third excited state. Having the vacuum state, hence no bosons, is nothing but having the Schrödinger particle in the ground state. Hence, from this perspective, the particles we were apparently counting in the Fock space are not actual particles: we call them **virtual particles**.

[30] Note there is no Hamiltonian in game, so we are just using the ONB with no additional meaning, but if there was, $H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$ would be also equal to $H = a^*a + 1/2$.

The same will happen in the final demonstration: there, we will find an equivalence between the Fock space of an infinite dimensional Hilbert space (instead of the 1D Hilbert space $\mathbb{C}$) and a Schrödinger picture (the wave-functional picture) having infinitely many position and momentum operators. There, the Fock space particles will have the chance to occupy different states from an ONB, $\{g_k\}_k$, and each occupation of a state g_k will correspond to an excitation of a different "harmonic oscillator" (one oscillator per each of the g_k). Creating a Fock boson in the k-th mode g_k will mean an excitation of the wavefunctional to a "higher energy" along the k-th axis.

A few remarks:

- I would like to appreciate explicitly that there was no reference to any Hamiltonian in this development. The harmonic oscillator only appeared because we chose the position and momentum as the Schrödinger "Segal operators" (to be identified with those in Fock space). They happen to have as associated creation and annihilation operators, and associated vacuum those defining the ONB of the harmonic oscillator. However, unitary maps between Hilbert spaces are not unique and neither the representation equivalences. One could take any ONB $\{f_n\}_{n\in\mathbb{N}_0} \subset L^2(\mathbb{R})$ (other than the harmonic oscillator) and define its creation and annihilation operators by: $\hat{A}f_0 := 0$ and $\hat{A}f_n := \sqrt{n}f_{n-1}$ for $n > 0$, which extend then by linearity. That is, define $\hat{A} := \sqrt{j}f_{j-1}\langle f_j, \cdot\rangle$, or in matrix form:

$$\Big(\langle f_i, Af_j\rangle\Big)_{ij} := \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{77}$$

  It is certainly an unbounded operator, but has a dense domain in the subspaces formed by the span of finitely many ONB elements (i.e., finitely many virtual particles or excitations), just like in Fock space. And again, A and A^* would satisfy the CCR in that domain, $[A, A^*] = Id$. Then, we would find that the unitary

$$U : L^2(\mathbb{R}) \to \mathcal{F}_+(\mathbb{C}) \quad \text{that maps} \quad \psi = \sum_{n=0}^{\infty} c_n f_n \mapsto (c_0,\ c_1 1\otimes 1,\ c_2 1\otimes 1\otimes 1, ...) \tag{78}$$

  would allow us to use Fock space as a bookkeeper of the excitation levels and linear combinations of this other ONB. It would then be interesting to analyse what the "field operators" $\hat{\varphi}(1)$ and $\hat{\pi}(1)$ correspond to back in $L^2(\mathbb{R})$ in this correspondence, namely, what

$$\hat{\phi}'_S(1) := \hat{\varphi}'_S(1) := \frac{1}{\sqrt{2}}(A + A^*) \quad \text{and} \quad \hat{\pi}'_S(1) := \hat{\varphi}'_S(i) := \frac{1}{\sqrt{2}}(-iA + iA^*) \tag{79}$$

  would correspond to in terms of $\hat{x}$ and $\hat{p}$. I leave that for you.

- To complete the picture, consider now $\mathcal{A}_+(\mathbb{C}^n)$ (the CCR for n quantized degrees of freedom). It has a Fock space representation over $\mathcal{F}_+(\mathbb{C}^n)$, which by the Stone-von Neumann theorem must be equivalent to the Schrödinger representation of $L^2(\mathbb{R}^n, \mathbb{C}, d^n x)$ describing n actual position and momentum operators. The explicit representation equivalence is essentially given by tensor products of the equivalence for $n = 1$.[31] The final trick today is then to take the limit $n \to +\infty$ using the strategy of D.2.1.

- Last but not least, notice that in Fock space there is a distinguished and **fixed** state called "vacuum", while in the analogous Schrödinger picture, the "vacuum" is just a particular state like any other. What does characterize the "vacuum" in both spaces is that it coincides with the ground state of some Hamiltonian. Hence, in a setting where there are interactions, and therefore potentially time varying ground states of the Hamiltonian, perhaps working in the Schrödinger representation is more convenient, because the "vacuum" would change in time, which is conflictive if we have a fixed state called vacuum like in Fock space. This might be critical in the case of QFTs with interactions!

[31] And the same thing will happen: creating/annihilating bosons in the k-th among the now n available modes, will account in the Schrödinger picture to promote or relax a tensor product ONB along the axis of the k-th mode. In our context, this Schrödinger picture will not represent particles, but instead they will be a cut-off of a mode expansion to n mode coefficients.

D.4. A rigorous QFT with configuration-space $\mathbb{R}^\infty$: a Ceiling for the Infinite Tower?!

Aand the Grand Finale: an attempt to construct rigorously a meaningful QFT over the configuration-space of expansion mode coefficients (and beyond): $L^2(\mathbb{R}^\infty, \mathbb{C}, d\mu)$ —a ceiling for the infinite tower of cut-off theories?

On the one hand, we found a way to build measures $d\mu$ on $\mathbb{R}^\infty$ for a pilot wave-theory that can be understood as the limit of discrete/finite dimensional cut-off approximations/restrictions of an ideal limit theory, by choosing equivalent measures to the Lebesgue measure that unlike the latter, allow an infinite product measure. Still, we found a big number of possible measures to build our arrow towards the ceiling.

To remedy this, for the complexification of the field configuration-space $\mathcal{H}$, we have found a rigorous "second quantization" by using Fock space $\mathcal{F}_+(\mathcal{H})$. This gathered many of the features we looked for in the heuristic prescription of B, but lacked the leitmotif: it is not built on a field configuration space like $L^2(\mathcal{H}, \mathbb{C}, d\mu)$, neither in its analogous (extended[32]) "expansion coefficient" space $L^2(\mathbb{R}^\infty, \mathbb{C}, d\mu)$. But, if we find a unitary giving a representation equivalence between the field operators in Fock space and some field operators that act as multiplication by functions in $L^2(\mathbb{R}^\infty, \mathbb{C}, d\mu)$ for some $d\mu$, our mission for today will arrive to its end!

Proposition 9. Given:

- a complex infinite dimensional Hilbert space $\mathcal{H}$,
- a conjugation C on $\mathcal{H}$ with "real subspace" $\mathcal{H}_R := \{\psi \in \mathcal{H} \mid C\psi = \psi\}$ (which is the configuration-space we want to "second quantize"),
- the bosonic Fock space $\mathcal{F}_+(\mathcal{H})$ equipped with canonical conjugate field operators $\{\hat{\phi}(f)\}_{f\in\mathcal{H}_R}$ and $\{\hat{\pi}(f)\}_{f\in\mathcal{H}_R}$,
- any fixed ONB $\{f_j\}_{j\in\mathbb{N}_0}$ of $\mathcal{H}$ that is made of elements of $\mathcal{H}_R$,

there exist:

- a Borel (probability) measure $d\mu$ on $\mathbb{R}^\infty := \mathbb{R}^{|\mathbb{N}|} := \prod_{j\in\mathbb{N}} \mathbb{R}$ with product topology, giving the separable Hilbert space $L^2(\mathbb{R}^\infty, \mathbb{C}, d\mu)$, (roughly, the wavefunctionals over expansion coefficients),
- a unitary map $S : \mathcal{F}_+(\mathcal{H}) \to L^2(\mathbb{R}^\infty, \mathbb{C}, d\mu)$,

such that,

- $\forall f \in \mathcal{H}_R$ the field operators $\hat{\phi}(f)$ are identified with multiplication operators. For the ONB $\{f_j\}_j$ in particular, $S\hat{\phi}(f_j)S^{-1} = \alpha_j$ (which are the "position operators" along the corresponding "axis" of $L^2(\mathbb{R}^\infty, \mathbb{C}, d\mu)$),
- the vacuum of Fock space Ω is identified with the function $\Psi_0(\alpha) \equiv 1$ in $L^2(\mathbb{R}^\infty, \mathbb{C}, d\mu)$, i.e., $S\Omega = \Psi_0$.

♦

Proof. Courtesy of Reed and Simon (1975). We first construct the unitary for finite dimensional subspaces:

- **Step 1:** Fix $N \in \mathbb{N}$ and let $\{g_k\}_{k=1}^N$ be a finite subset of the ONB $\{f_j\}_{j\in\mathbb{N}}$. Define

$$\mathcal{F}_N := closure\Big\{P\Big(\hat{\phi}(g_1), ..., \hat{\phi}(g_N)\Big)\Omega \mid P \text{ a complex polynomial of N arguments}\Big\} \subset \mathcal{F}_+(\mathcal{H}). \tag{80}$$

  (The span of repeated application on the vacuum by the field operators of the first N modes of the ONB.)

- **Step 2:** By Proposition 8, $\hat{\phi}(g_k), \hat{\pi}(g_l)$ are essentially self-adjoint on $\mathcal{F}_+^{fin}(\mathcal{H}) \cap \mathcal{F}_N$ and they satisfy the Weyl relation on $\mathcal{F}_N$:

$$e^{i\hat{\phi}(g_k)}e^{i\hat{\pi}(g_l)} = e^{-ist\delta_{kl}}e^{is\hat{\pi}(g_l)}e^{it\hat{\phi}(g_k)}, \tag{81}$$

  so they are a representation of the Weyl CCR with finitely many (N) "position and momentum operator pairs" $\hat{\phi}(g_k), \hat{\pi}(g_l)$. Moreover, Ω is cyclic by definition of $\mathcal{F}_N$, so it is an irreducible representation of the Weyl CCR. By the Stone-Von Neumann theorem there exists a unitary equivalence with the N "particle" Hilbert space $L^2(\mathbb{R}^N, \mathbb{C}, d^N x)$ that maps the field and momentum operators to the position and momentum operators

[32] Technically, the expansion coefficient space should be $L^2(\ell^2(\mathbb{N}))$, which is a subspace of the vector space $\mathbb{R}^\infty$ (precisely, the subspace for which the ℓ^2-norm is finite). We can guarantee an infinite product measure for $\mathbb{R}^\infty$, so one could restrict it to $\ell^2(\mathbb{N})$, but the measure lives on the Borel σ-algebra of the product topology of $\mathbb{R}^\infty$, which is not normable, so the topic deserves its own treaty. Also, it might be that the measure of $\ell^2(\mathbb{N})$ is zero altogether! Haha.

there, i.e., a unitary isomorphism $\tilde{S}^{(N)} : \mathcal{F}_N \to L^2(\mathbb{R}^N, \mathbb{C}, d^N x)$:

$$\begin{cases} \tilde{S}^{(N)} \hat{\phi}(g_k) (\tilde{S}^{(N)})^{-1} = \alpha_k =: \hat{\alpha}_k \\ \tilde{S}^{(N)} \hat{\pi}(g_k) (\tilde{S}^{(N)})^{-1} = -i\hbar \dfrac{\partial}{\partial \alpha_k} =: \hat{p}_k \end{cases} \quad \text{for all } k \in \{1, ..., N\}. \tag{82}$$

Note that we will denote the points of $\mathbb{R}^\infty$ (and similarly of R^N) by $\alpha := (\alpha_1, \alpha_2, \alpha_3, ...)$ with $\alpha_j \in \mathbb{R}$, to make manifest our intention to use this as the space of coefficients of some expansion modes.

- **Step 3:** Since by definition 16, $\hat{\phi}(g_k) = (a_+(g_k) + a_+^*(g_k))/\sqrt{2}$ and $\hat{\pi}(g_k) = i(-a_+(g_k) + a_+^*(g_k))/\sqrt{2}$, using that $a_+(g_k)\Omega = 0$, we can verify the identity

$$\Big(\hat{\phi}(g_k)^2 + \hat{\pi}(g_k)^2 - 1\Big)\Omega = 0. \tag{83}$$

Now, define Ψ_0 to be the vector identified with the vacuum of Fock space Ω, i.e., $\Psi_0 := \tilde{S}^{(N)}\Omega$. Then, plugging $\hat{\alpha}_k = (\tilde{S}^{(N)})^{-1}\hat{\psi}(g_k)\tilde{S}^{(N)}$ and $\hat{p}_k = (\tilde{S}^{(N)})^{-1}\hat{\pi}(g_k)\tilde{S}^{(N)}$ in equation (83), we find it to imply

$$\Big(\frac{1}{2}\hat{p}_k^2 + \frac{1}{2}\hat{\alpha}_k^2\Big)\Psi_0 = \frac{1}{2}\Psi_0. \tag{84}$$

Adding together the equations for $k \in \{1, ..., N\}$ we get the eigenfunction equation

$$\sum_{k=1}^{N} \Big(\frac{1}{2}\hat{p}_k^2 + \frac{1}{2}\hat{\alpha}_k^2\Big)\Psi_0 = \frac{N}{2}\Psi_0, \tag{85}$$

which is the equation defining the ground state of N independent harmonic oscillators. It has as solution the Gaussian $\Psi_0 = \pi^{-N/4} exp(-\sum_{k=1}^N \frac{\alpha_k^2}{2})$. Hence:

$$\tilde{S}^{(N)}\Omega = \frac{1}{\pi^{N/4}} e^{-\sum_{k=1}^N \frac{1}{2}\alpha_k^2}. \tag{86}$$

(The Fock vacuum is identified with a tensor product of N Gaussians in the cut-off wavefunctional space.)

- **Step 4:** By definition, any state in $\mathcal{F}_N$ is a limit of polynomials of the field operators acting on Ω. By the equivalence $\tilde{S}$, any state in $L^2(\mathbb{R}^N, \mathbb{C}, d^N x)$ is spanned by acting with polynomial multiplication operators on the Gaussian Ψ_0. Recalling the idea of D.2.1 about finding a $\prod_{j=1}^N \rho^2$ that would "always" be present in the states besides the Lebesgue measure, we found it: ρ^2 is the Gaussian $\pi^{-1/2} exp(-\alpha_k^2)$!

  We define the measure $d\mu_k := \pi^{-1/2} e^{-x_k^2} dx_k$. It is mutually absolutely continuos wrt the 1D Lebesgue measure dx_k and it is a probability measure, so-called Gaussian measure. Define the unitary map

$$\begin{array}{rccc} U: & L^2\big(\mathbb{R}^N, \mathbb{C},\, d^N x\big) & \longrightarrow & L^2\Big(\mathbb{R}^N, \mathbb{C},\, \prod_{k=1}^N d\mu_k\Big) = L^2\Big(\mathbb{R}^N, \mathbb{C},\, \pi^{-N/2}\, e^{-\sum_{k=1}^N x_k^2} d^N x\Big) \\ & \eta(\alpha) & \longmapsto & \pi^{N/4} e^{+\sum_{k=1}^N \frac{1}{2}\alpha_k^2} \eta(\alpha) \end{array} \tag{87}$$

giving an equivalent "cut-off" quantum theory (even an equivalent underlying pilot-wave theory), but now with a nicer background measure to take the limit. As we explained in D.2, $L^2(\mathbb{R}^N, \mathbb{C}, d^N x)$ does not have any limit as $N \to \infty$, while $L^2(\mathbb{R}^N, \mathbb{C}, \prod_{k=1}^N d\mu_k)$ does have. Hence, the unitary we are really interested on to go from Fock space to a limit wavefunctional space is $S^{(N)} := U\tilde{S}^{(N)}$. According to this unitary,

$$\begin{array}{rccc} S^{(N)}: & \mathcal{F}_N & \longrightarrow & L^2(\mathbb{R}^N, \mathbb{C}, \prod_{k=1}^N d\mu_k) \\ & \eta & \longmapsto & U\tilde{S}^{(N)}\eta \end{array} \tag{88}$$

$$\begin{cases} (S^{(N)}\Omega)(\alpha) & \equiv 1 \\ S^{(N)}\hat{\phi}(g_k)(S^{(N)})^{-1} & = \alpha_k =: \hat{\alpha}_k \\ S^{(N)}\hat{\pi}(g_k)(S^{(N)})^{-1} & = i\hbar\alpha_k - i\hbar \dfrac{\partial}{\partial \alpha_k} =: \tilde{\hat{p}}_k \end{cases} \tag{89}$$

(The price we pay is the extra term in the momentum operator.)

- **Step 5:** We compute an identity. For any set of polynomials in one variable $P_1, ..., P_N$:

$$\Big\langle \Omega,\ P_1(\hat{\phi}(g_1))\cdots P_N(\hat{\phi}(g_N))\Omega\Big\rangle_{\mathcal{F}} \underbrace{=}_{S^{(N)} \text{ is unitary}} \Big\langle 1,\ P_1(\alpha_1)\cdots P_N(\alpha_N)1\Big\rangle_{L^2(\mathbb{R}^N,\prod_{j=1}^N d\mu_j)} = \tag{90}$$

$$= \int_{\mathbb{R}^N} P_1(\alpha_1)\cdots P_N(\alpha_N)\prod_{k=1}^N d\mu_k = \star^{33}$$

- **Step 6:** Now consider $\mathbb{R}^\infty = \prod_{k\in\mathbb{N}}\mathbb{R}$ with the σ-algebra generated by countable products of Borel measurable sets of $\mathbb{R}$, call it $\mathcal{A}$. It is the same σ-algebra as the one generated by the product topology, so by Kolmogorov's extension theorem the infinite product measure $d\mu := \prod_{k=1}^\infty d\mu_k$ exists uniquely and is a probability measure on $(\mathbb{R}^\infty, \mathcal{A})$.

- **Step 7:**
  (a) The functionals making use of only N coordinates of its input points $\alpha := (\alpha_1, \alpha_2, ...) \in \mathbb{R}^\infty$, defined by $G^{P;k_1,...,k_N} : \mathbb{R}^\infty \to \mathbb{C}$ with $G^{P;k_1,...,k_N}(\alpha) := P(\alpha_{k_1}, ..., \alpha_{k_N})$ for fixed polynomials P over N variables, form a vector space of functionals

$$B_{poly} := \Big\{ G^{P;k_1,...,k_N} \ \Big|\ N\in\mathbb{N};\ k_1,...,k_N\in\mathbb{N} \text{ and } P \text{ a complex polynomial of } N \text{ arguments} \Big\} \tag{91}$$

  that is dense in $L^2(\mathbb{R}^\infty, \mathbb{C}, d\mu)$.

  (b) Similarly, the space of elements $\eta \in \mathcal{F}_+(\mathcal{H})$ that can be written as $P\Big(\hat{\phi}(f_{k_1}), ..., \hat{\phi}(f_{k_N})\Big)\Omega$ with $N \in \mathbb{N}$ for some complex polynomial P of N variables and some ONB elements f_{k_j}, i.e.,

$$\tilde{B}_{poly} := \Big\{ P\Big(\hat{\phi}(f_{k_1}), ..., \hat{\phi}(f_{k_N})\Big)\Omega \ \Big|\ N\in\mathbb{N};\ k_1,...,k_N\in\mathbb{N} \text{ and } P \text{ a complex poly. of } N \text{ arguments} \Big\} \tag{92}$$

  is dense in $\mathcal{F}_+(\mathcal{H})$.

  (c) Define the following linear isomorphism between both dense subsets:

$$\begin{array}{rccc} S: & \tilde{B}_{poly}\subset\mathcal{F}_+(\mathcal{H}) & \longrightarrow & B_{poly}\subset L^2(\mathbb{R}^\infty,\mathbb{C},d\mu) \\ & P\Big(\hat{\phi}(f_{k_1}), ..., \hat{\phi}(f_{k_N})\Big)\Omega & \longmapsto & G^{P;k_1,...,k_N}. \end{array} \tag{93}$$

- **Step 8:** Given an arbitrary polynomial of N variables and order M, there are coefficients $c_{l_1,...,l_N} \in \mathbb{C}$ such that,$P(x_{k_1}, ..., x_{k_N}) := \sum_{l_1,...,l_N=1}^M c_{l_1,...,l_N} x_{k_1}^{l_1}\cdots x_{k_N}^{l_N}$. Then,

$$\Big|\Big|P\Big(\hat{\phi}(f_{k_1}), ..., \hat{\phi}(f_{k_N})\Big)\Omega\Big|\Big|^2_{\mathcal{F}_+(\mathcal{H})} = \sum_{l_1,...,l_N=1}^M\sum_{m_1,...,m_N=1}^M c_l\bar{c_m}\Big\langle\Omega,\ \hat{\phi}(f_{k_1})^{l_1+m_1}\cdots\hat{\phi}(f_{k_N})^{l_N+m_N}\Omega\Big\rangle_{\mathcal{F}} = \tag{94}$$

$$\underbrace{=}_{\text{Step 5}} \sum_{l_1,...,l_N=1}^M\sum_{m_1,...,m_N=1}^M c_l\bar{c}_m\int_{\mathbb{R}^N}\alpha_{k_1}^{l_1+m_1}\cdots\alpha_{k_N}^{l_N+m_N}\prod_{j=1}^N d\mu_{k_j} \underbrace{=}_{\substack{\text{Leb. integr.}\\ \text{linear}}} \int_{\mathbb{R}^N}|P(\alpha_{k_1},...,\alpha_{k_N})|^2\prod_{j=1}^N d\mu_{k_j}$$

$$\underbrace{=}_{\substack{\int_{\mathbb{R}} d\mu_j=1 \text{ and}\\ d\mu \text{ is product measure:}\\ d\mu=\prod_{j=1}^\infty d\mu_j}} \int_{\mathbb{R}^\infty}|P(\alpha_{k_1},...,\alpha_{k_N})|^2 d\mu = \int_{\mathbb{R}^\infty}|G^{P;k_1,...,k_N}|^2 d\mu = \Big|\Big|G^{P;k_1,...,k_N}\Big|\Big|^2_{L^2(\mathbb{R}^\infty,d\mu)} =$$

$$\underbrace{=}_{by\ def} \Big|\Big|\ S\Big(P\Big(\hat{\phi}(f_{k_1}), ..., \hat{\phi}(f_{k_N})\Big)\Omega\Big)\ \Big|\Big|^2_{L^2(\mathbb{R}^\infty,d\mu)}.$$

  Hence, the operator norm of S is 1.

[33] One could continue until the equality $\star = \prod_{k=1}^N\int_{\mathbb{R}} P_k(\alpha_k)d\mu_k = \prod_{k=1}^N\Big\langle 1, P_k(\alpha_k)1\Big\rangle_{L^2(\mathbb{R}^N,d\mu_j)} = \prod_{k=1}^N\Big\langle\Omega, P_k(\hat{\phi}(g_k))\Omega\Big\rangle_{\mathcal{F}}$.

- **Step 9:** Then, by the theorem we saw in MQT (some Hahn-Banach extension theorem), since S is a linear isomorphism between two dense subspaces with operator norm 1, there exists a unique extension operator with the same operator norm, which we call again S, between the closure spaces. That is, there exists a unique unitary $S : \mathcal{F}_+(\mathcal{H}) \to L^2(\mathbb{R}^\infty, \mathbb{C}.d\mu)$ extending the map in (93). By construction we have that $S\hat{\phi}(f_k)S^{-1} = \alpha_k$ and $(S\Omega)(\alpha) \equiv 1$.

o.ε.δ.

Therefore, what we called single boson states in Fock space, i.e., those spanned by linear combinations of $\hat{\phi}(f_j)\Omega$ are mapped to linear combinations of polynomials $\psi(\alpha) = \alpha_j \Psi_0$ (yes, $\Psi_0 = 1$, but we keep a ghost of it to remember that *heuristically* it corresponds to the Gaussian $e^{-\sum_{j=1}^{\infty} |\alpha_j|^2}$ that we pushed to the measure). As such, a single boson state in the functional representation is a first order polynomial of arbitrary variables. Likewise, two boson states $\sum_{j,k} c_{j,k}\hat{\phi}(f_k)\hat{\phi}(f_j)\Omega$ are identified with second order polynomials of possibly infinitely many variables $\sum_{j,k} c_{j,k}\alpha_j\alpha_k\Psi_0$. And so on.

To shed a light on what we found and link it with all the previous, denote by H_n^k the n-th Hermite polynomial along the axis α_k, i.e., $H_n^k(\alpha) := H_n(\alpha_k)$. Then, the "creation of a boson" from the vacuum Ω to a state f_k of the ONB, which is given by the Fock creation operator as $a_+^*(f_k)\Omega$, is mapped into the promotion of the "ground state" wavefunctional Ψ_0 to the "excited" state $\Psi_{k\text{-th dof 1 excitation, the rest 0}} := H_1^k\Psi_0$. Likewise, the creation of say, 3 bosons in state f_k and 5 in state f_j, namely $a_+(f_k)^3 a_+^*(f_j)^5\Omega$ is mapped to $\Psi_{\text{k-th dof 3 excitats, j-th 5 excit, rest 0}} := H_3^k H_5^j \Psi_0$. Hence, as promised, we are playing the same game as in D.3.3, but now with infinitely many modes that can take excitations on them.

Finally, since by Proposition 7, the possible combinations of bosons in states of the ONB $\mathcal{B} := \{f_j\}_{j\in\mathbb{N}}$ that we employed to build the unitary S, namely,

$$\left\{ \frac{1}{\sqrt{n_1! \cdots n_N!}} a_+^*(f_{j_1})^{n_1} \cdots a_+^*(f_{j_N})^{n_N}\Omega \;\middle|\; N, n_k \in \mathbb{N} \text{ and } f_{j_k} \in B \text{ and } j_1 < \cdots < j_N \right\}$$

is an ONB of the Fock space $\mathcal{F}_+(\mathcal{H})$ and they are mapped unitarily to products of the Hermite polynomials

$$\{H_{n_1}^{j_1} \cdots H_{n_N}^{j_N} \mid N, n_k \in \mathbb{N} \text{ and } j_1 < \cdots < j_N\},$$

the latter are an ONB of $L^2(\mathbb{R}^\infty, \mathbb{C}, d\mu)$.

D.4.1. A Rigorous Pilot-Wave Theory in $\mathbb{R}^\infty$?

To be continued in the MathPhys Colloquium next semester...

...

E. A Meditation in the Summit: there were Higher Peaks Above the Ceiling...

Now that we finally arrived to the ceiling of the cut-off expansion sequence (for the free theory), taking advantage of the sights from here, let us close the talk with some observations.

- Certainly, there are still many open questions (at least for me) regarding to what extent the wavefunctional QFT we found can be understood as a QFT over the space of mode expansions. Everything points to that, but as we found earlier, $\mathbb{R}^\infty$ has in it more than expansion coefficients (more points than those in ℓ^2) and as the Wiener measures showed in the Feynman-Kac construction, it might very well be the case that our preferred subspace, ℓ^2, has measure 0 altogether. If so, cleverer ideas will need to be thought, perhaps in the direction of distributions instead of fields. Perhaps, just like in the case of the Feynman-Kac formula, we might need to face that only kinky fields, say, nowhere differentiable ones, are the typical ones. However, as Dürr and Lazarovici (2020) posed it, this feels like a "deja vu" in fact: recall the time when we decided to leave rational numbers to go to real numbers, just because maths got more beautiful doing so. Not that we really needed them for making experimental predictions, but that way we would avoid talking about infinite cut-off towers of fractions (familiar...). But then we found ourselves in a hurry to define a new measure. The counting measure so perfectly meaningful to intuition was meaningless in the continuum and needed to be substituted by a new measure. We sadly said goodbye to the nice properties of the counting measure and embraced the awkward new measure: the Lebesgue measure. But the most dramatic part of it was still about to come (just like it happened to us...): our beautiful rational numbers suddenly were lost in measure and typicality. All of them together were made to weigh zero, and almost every number according to the new strange measure were numbers with crazy random sequences of decimal numbers, today we know in fact that almost all of them are not even computable! But hey, the measure of some open sets was made finite in turn (so familiar indeed!).

- Regarding whether we really achieved the ceiling of the cut-off tower, for free theories it seems so. However, as we saw explicitly for the case of fermion QFTs in my last talk, and more in general by the so-called Haag's theorem, although QFTs with no cut-offs can nicely be made rigorous for free species of particles using Fock space techniques, there is no way to make a rigorous mathematical description of interactions using the free particle QFTs (unless we only want to talk about asymptotically free states and scattering matrices). And therefore, by unitary equivalence, neither can be the case if we employ the wavefunctional space we just built. In fact, as of today, every interacting QFT in 4D (or higher) spacetime (so the whole Standard Model) is pre-mathematical unless we accept the cut-off towers as satisfactory theories.

  Still, we found in the previous talk that part of the problem with interactions and Fock spaces was that interactions forced us to consider different (even inequivalent) Fock spaces at each time, tightly related to the fact that the vacuum (say the Dirac sea) needed to be re-defined (infinitely many virtual particles emerge from the vacuum). Now, we found that in the discretized versions of the wavefunctional theory we have built (where there was still a Lebesgue measure), there was nowhere a preferred state to be called "vacuum". Only the identification with Fock space gave so. It is also true that if we had a Hamiltonian with a ground-state then there would be a distinguished state, but there would be no problem for such a ground state to change in time, unlike in Fock space, where the vacuum is given by a fixed vector. Now, in the "no cut-off" limit, we designed the Gaussian measure so that the state equal to 1 everywhere was made to represent the Fock vacuum, so we forced the resulting limit wavefunctional space to have a distinguished state, which was moreover encoded in the measure. But that was arguably only with the purpose of giving it a Fock space look in the search of a unitary equivalence. There are several physicists, like Sebens (2022) or Jackiw (1995) that hope the property of the cut-offs, where different "vacuums" can be described in the very same Hilbert space, will soon be translated into an infinite dimensional space. That is, they hope that there exists a way to talk about different vacuums and perhaps inequivalent Fock spaces as parts of the same wavefunctional space. Indeed, in their heuristic formulas, they already assume so. This would be an amazing step to solve the problem of interactions in QFT and hence worth to be further studied.

- Another advantage of the wavefunctional approach is that actually the Standard Model is built on the Lagrangian QFT style of heuristic formulas (the interaction Hamiltonians etc.), which are simpler to translate to wavefunctional language than to Fock space. However, it must be conceded that most verifications of experimental results, which are done using scattering matrices, cross sections and asymptotically free QFTs, are commonly done in (more or less heuristic) Fock space pictures. And the problem is that it is hard to

see where the probabilistic interpretation for the field configuration-space, that we fought so hard to make rigorous, fits with the probabilistic interpretation of Fock space. The point is the following: maybe bosons are merely virtual particles, right, excitations of the underlying ontological fields, but if so, we need a mechanism to go from probabilities to find n particles in a configuration $(x_1, ..., x_n)$ as given by a Fock wavefunction $|\psi_t^{(n)}(x_1, x_2, ..., x_n)|^2 d^n x$ to the probabilities to find the ontological fields in a configuration with coefficients $(\alpha_1, \alpha_2, ...)$, as given by the wavefunctional $|\Psi_t(\alpha_1, \alpha_2, ...)|^2 d\mu$. A heuristic discussion in this direction can be found in (Struyve, 2010).

- So, as you see there are only problems in QFT haha. For all tastes and disciplines: from experiments to philosophy. Of course, we could be happy with cut-off theories and renormalization and believe the tower of approximate QFTs is all there is. Certainly, the trick will continue to work as we keep making the predictive algorithms and the number of parameters and ad-hoc tweaks increase. But at some point we will need to consider seriously (and beware, because the time is about to come) which is the difference between fitting a model to the experimental data (hence, being able to predict phenomenology), just like any neural network is capable of doing with enough weights and tweaks; and doing physics. I claim that physics is not **fitting and predicting** "at all costs". Instead, it is the development of a concise mathematical and mechanistic narrative to **understand** the edge between phenomenology and whatever lays beyond it, which as a necessary by-product then allows the prediction of phenomenology.

  Let me close with an adapted version of a fine bar by Dürr and Lazarovici (2020) in this direction: "The infinities which appear abundantly in" QFT cannot be "merely mathematical problems that we can try to solve by new techniques" (as both particle physicists and mathematicians in AQFT would like to believe). "They are clear signs of fundamental physical problems which will lead to a distortion of the notion of physical theory if we keep trying to push them to one side: non-convergent perturbative expansions and unspeakable limits of renormalization procedures become the substitute for perhaps two or three fundamental equations" and a clear picture of the world. To which I add: let us not *renormalize* what we mean by *physical theory!*

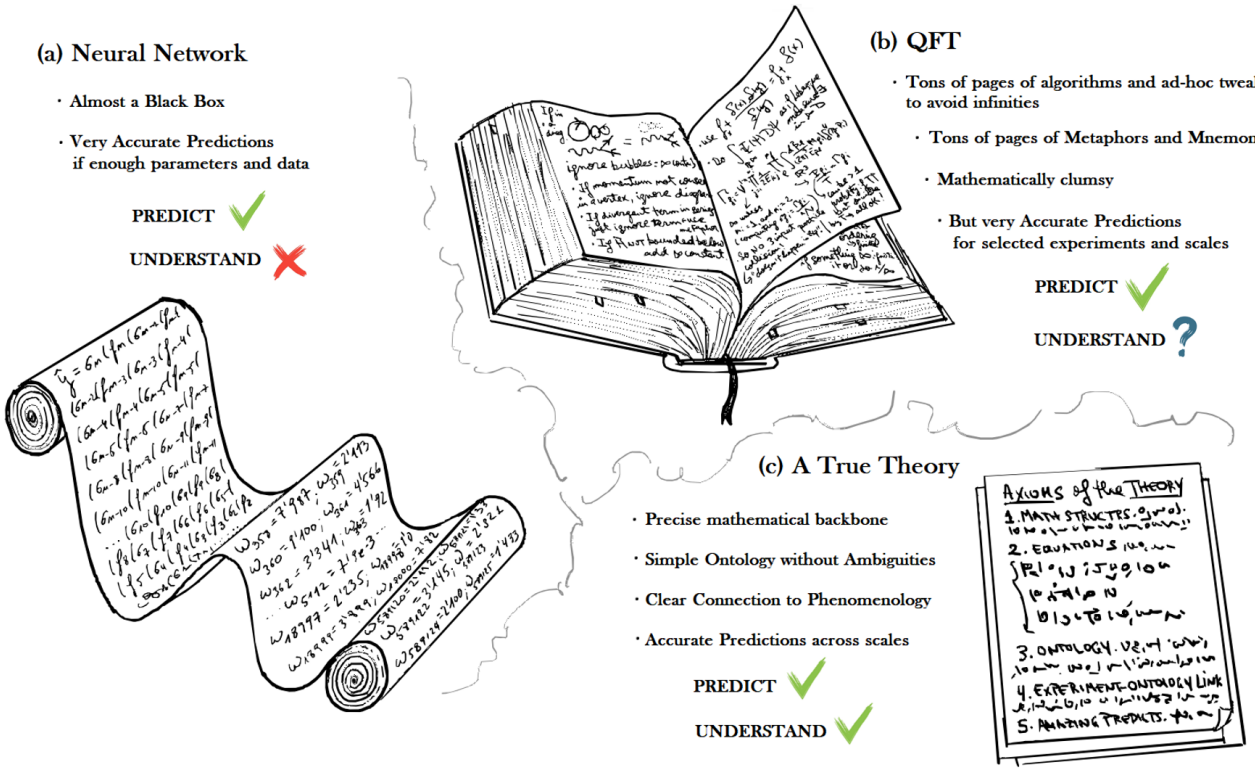

Figure 3: A satirical sketch I made to describe the deformation (or perhaps "renormalization") of the notion of "physical theory". As of 2024, we are still in (b) so perhaps we are still in time to go back to (c).

F. BIBLIOGRAPHY

Abraham, R., Marsden, J. E., and Ratiu, T. (2012). *Manifolds, tensor analysis, and applications*, volume 75. Springer Science & Business Media.

Bachmann, S. (2017). Lecture Notes on C^*-algebras. Mathematical Statistical Physics, LMU. https://www.math.lmu.de/ cuenin/MSP2017/Algebras.pdf.

Baez, J. C., Segal, I. E., and Zhou, Z. (2014). *Introduction to algebraic and constructive quantum field theory*, volume 56. Princeton University Press.

Bohm, D. (1952). A suggested interpretation of the quantum theory in terms of "hidden" variables. ii. *Phys. Rev.*, 85:180–193.

Dürr, D., Goldstein, S., and Zanghì, N. (1992). Quantum equilibrium and the origin of absolute uncertainty. *Journal of Statistical Physics*, 67:843–907.

Dürr, D. and Lazarovici, D. (2020). Understanding quantum mechanics. *Cham: Springer*.

Eldredge, N. (2016). Analysis and probability on infinite-dimensional spaces. *arXiv preprint arXiv:1607.03591*.

Halvorson, H. and Müger, M. (2006). Algebraic quantum field theory. *arXiv preprint math-ph/0602036*.

Holland, P. R. (1995). *The quantum theory of motion: an account of the de Broglie-Bohm causal interpretation of quantum mechanics*. Cambridge university press.

Jackiw, R. W. (1995). *Diverse topics in theoretical and mathematical physics*. World Scientific.

Norsen, T., Marian, D., and Oriols, X. (2015). Can the wave function in configuration space be replaced by single-particle wave functions in physical space? *Synthese*, 192:3125–3151.

Reed, M. and Simon, B. (1975). *II: Fourier analysis, self-adjointness*, volume 2. Elsevier.

Reed, M. and Simon, B. (1981). *I: Functional analysis*, volume 1. Academic press.

Sebens, C. T. (2022). The fundamentality of fields. *Synthese*, 200(5):380.

Struyve, W. (2010). Pilot-wave theory and quantum fields. *Reports on Progress in Physics*, 73(10):106001.

Tumulka, R. (2022). *Foundations of quantum mechanics*, volume 1003. Springer.

Wallace, D. (2006). In defence of naiveté: The conceptual status of lagrangian quantum field theory. *Synthese*, 151:33–80.

G. APPENDICES

Appendix to B.1: Calculus in Infinite Dimensional Vector Spaces

This section drinks mainly from chapter 2 in the highly recommended book Abraham et al. (2012). Recall a normed vector space (NVS) is a vector space equipped with a norm, which induces a distance function and hence a ball topology, i.e., it is also a metric space. We call it Banach if the topology is Cauchy complete. In particular any inner product vector space (IPVS), which is a vector space with an inner product, has a natural norm and hence is a NVS. We call it Hilbert if Cauchy complete. We say a NVS is finite dimensional (e.g., $(\mathbb{R}^n, ||\cdot||_2)$ or $(\mathbb{C}, |\cdot|)$) if there exists a set of finitely many linearly independent elements spanning the whole space (any element is writeable as a finite linear combination of them), else it is infinite dimensional (e.g., $(L^2(\mathbb{R}^n), ||\cdot||_{L^2})$ or the space of continuos linear maps between two NVS E, F in the operator norm, denoted by $(\mathscr{L}(E,F), ||\cdot||_{op})$). So, a (wave)functional $f : L^2(\mathbb{R}^n) \to \mathbb{C}$ is nothing but a map between two NVS (for our purposes they will even be Hilbert spaces —although it would be interesting to investigate wavefunctionals over general NVS). Think of such examples of infinite dimensional NVS in the following definitions.

Definition 1. Let E, F be NVS, $U \subseteq E$ open and $f : U \to F$ a mapping. We say that f is **differentiable** at $u_0 \in U$ when there exists a continuous linear map $Df|_{u_0} \in \mathscr{L}(E,F)$ such that

$$\lim_{u \to u_0} \frac{f(u) - f(u_0) - Df|_{u_0}(u - u_0)}{||u - u_0||} = 0. \tag{95}$$

♦

If it exists, there is a unique such linear map (hence the notation $Df|_{u_0}$) and we call it the derivative or (total) **differential** of f at u_0. It is the definition you all know from calculus, but now also working for infinite dimensional vector spaces (for whom we call it Fréchet derivative).

Definition 2. In the notation, of Def. 1.,

- If f is differentiable at each $u_0 \in U$ then $Df : U \to \mathscr{L}(E,F)$ is called the (total Fréchet) derivative or differential of f. If it is a continuous map, we say f is **continuously differentiable** and denote the set of continuously differentiable maps by $\mathcal{C}^1(E,F)$.
- We can then define (if it exists) the r-th differential $D^r f := D(D^{r-1}f) : U \subset E \to \mathscr{L}(E, \mathscr{L}(E^{r-1}, F)) \simeq \mathscr{L}(E^r, F)$, such that $D^r f|_{u_0}$ is an r-linear continuous map for each $u_0 \in U$. If $D^r f$ is continuous, we say f is r **times continuously differentiable** or of **class** $\mathcal{C}^r(E,F)$.

♦

Definition 3. In the notation of Def. 1, we say that f has a **directional derivative in direction** $v \in E$ **at the point** $u \in U$ when

$$\frac{\mathrm{d}}{\mathrm{d}t} f(u + tv)|_{t=0} := \lim_{t \to 0} \frac{f(u+tv) - f(u)}{t} \quad \text{exists.} \tag{96}$$

♦

Proposition 1. In the notation of Def. 1,

(i) If f is differentiable at $u \implies$ all directional derivatives exist and

$$\frac{\mathrm{d}}{\mathrm{d}t} f(u + tv)|_{t=0} = Df|_u \, v. \tag{97}$$

(ii) $f \in \mathcal{C}^1(U,F) \iff$ for all $v \in E$, the directional derivative exists at all points $u \in U$ and are continuos functions of u for fixed $v \in E$.

(ii) If $f \in \mathcal{C}^r(U,F)$, for all $v_1, ..., v_n \in E$ and $u \in U$

$$D^r f|_u(v_1, ..., v_n) = \frac{\mathrm{d}}{\mathrm{d}t_1} \cdots \frac{\mathrm{d}}{\mathrm{d}t_n} f\Big(u + \sum_j t_j v_j\Big)|_{t_1 = \cdots = t_n = 0}. \tag{98}$$

♦

Note that all the multivariate calculus results still hold almost verbatim in arbitrary NV (Abraham et al., 2012). Among others,

- The Schwarz cross derivative rule: $f \in \mathcal{C}^r \Rightarrow D^r f|_u$ is a symmetric r-linear map in $\mathscr{L}(E^r, F)$).
- Taylor's expansion theorem: If $U \subseteq E$ is convex and $f \in \mathcal{C}^r(U, F)$, for all $(u, v) \in U \times E$ such that $u + v \in U$, there exists a symmetric $R|_{(u,v)} \in \mathscr{L}(E^r, F)$ such that

$$f(u+v) = f(u) + \frac{Df|_u}{1!}h + \frac{Df^2|_u}{2!}(h,h) + \cdots + \frac{Df^r|_u}{r!}\underbrace{(h,...,h)}_{r\ times} + R|_{(u,v)}\underbrace{(h,...,h)}_{r\ times} \tag{99}$$

  for $h := v - u$ and such that $R|_{(u,v)}(h,...,h) = o(||h||_E^r)$.
- $f : U \subseteq E \to \mathbb{R}$ is differentiable in $u \in U$ and has a local extremum in $u \in U \Longrightarrow Df|_u = 0$.
- Things like the sufficient conditions for local maxima or minima, the inverse function theorem, the implicit function theorem, differentiation under the integral sign etc.

Then, the functional derivative is nothing but the gradient related to the differential of a map by some inner product, or more generally by some pairing (a slightly weaker notion).

Definition 4. Let E, F be NVS.

- A **pairing (between elements of E and F)** is a continuous, bilinear functional $\langle \cdot, \cdot \rangle : E \times F \to \mathbb{R}$. (For example if $E = F = L^2(\mathbb{R}^d, \mathbb{R})$, the inner product of L^2 is a pairing).
- A pairing is called **E-non-degenerate** when $\langle x, y \rangle = 0\ \forall y \in F \Longrightarrow x = 0$. That is, when $x \mapsto \langle x, \cdot \rangle$ is an injective map $E \to F^*$.

♦

For example if $E = F = L^2(\mathbb{R}^d, \mathbb{R})$, the inner product of L^2 is non-degenerate in both slots by the Riesz representation theorem. But if we restrict the second arguments to only $\mathcal{C}_c^\infty(\mathbb{R}^d)$ (smooth compactly supported functions), still it is L^2 non-degenerate by the du Bois-Reymond lemma we saw in PDEs: i.e,

$$\text{if } \psi \in L^2(\mathbb{R}^d, \mathbb{R}) \text{ is s.th. } \int_{\mathbb{R}^d} \psi(x)\eta(x)d^n x = 0 \quad \forall \eta \in \mathbb{C}_c^\infty(\mathbb{R}^n) \Longrightarrow \psi = 0. \tag{100}$$

We defined the general notion of pairing so we can find functional derivatives using the inner product of some bigger space in smaller subspaces. For instance, we could also take the L^2 inner product restricted to $\mathcal{C}_c^\infty(\mathbb{R}^d)$ in the second slot but to $H^2(\mathbb{R}^3, \mathbb{R}) \subset L^2(\mathbb{R}^3, \mathbb{R})$ in the first slot, the called the second **Sobolev space**[34]. A priori H^2 has another associated inner product, but by (100) the one of L^2 already gives a non-degenerate pairing with elements of $\mathcal{C}_c^\infty(\mathbb{R})$.

Definition 5. Let E, F be NVS with an E-degenerate pairing $\langle \cdot, \cdot \rangle : E \times F \to \mathbb{R}$ and $f : E \to \mathbb{R}$ be differentiable at $\varphi \in E$. We define the **functional derivative** of f in φ, denoted by $\frac{\delta f}{\delta \phi}|_\varphi$ to be, if it exists, the unique element of E such that

$$Df|_\varphi\, \eta = \left\langle \frac{\delta f}{\delta \phi}|_\varphi, \eta \right\rangle \quad \forall \eta \in F \quad i.e., \quad Df|_\varphi = \left\langle \frac{\delta f}{\delta \phi}|_\varphi, \cdot \right\rangle. \tag{101}$$

It is also called the **gradient** of f in φ denoted $grad_{\langle , \rangle} f|_\varphi \equiv \frac{\delta f}{\delta \phi}|_\varphi$. If E is a function space, we denote $\frac{\delta f}{\delta \phi(x)}|_\varphi := \frac{\delta f}{\delta \phi}|_\varphi(x)$.

♦

Hence, the functional derivative is the generalization of the gradient and is a mapping from points in $U \subseteq E$ to vectors in E (unlike the differential which gives operators, or the directional derivative, which gives vectors of F).

Then, if $f \in \mathcal{C}^1$ we can equivalently characterize functional derivatives by directional derivatives (which yields a simpler formula to compute them). Namely,

$$Df|_\varphi\, \eta = \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi + t\eta)|_{t=0} = \left\langle \frac{\delta f}{\delta \phi}|_\varphi, \eta \right\rangle \quad \forall \eta \in F. \tag{102}$$

For instance, if $S : L^2(\mathbb{R}^3, \mathbb{R}) \to \mathbb{R}$ is $\mathcal{C}^1$ (say, represents some action function over fields), then

$$\frac{\mathrm{d}}{\mathrm{d}t} S(\varphi + t\eta)|_{t=0} = \int_{\mathbb{R}^n} \frac{\delta S}{\delta \phi(x)}|_\varphi\, \eta(x) d^n x, \tag{103}$$

[34] The set of square integrable functions with two (weak) derivatives that are also square integrable.

so to find $\frac{\delta S}{\delta \phi}|_\varphi$ we would develop the left hand side's explicit expression until we get an expresion like the r.h.s. By uniqueness of directional derivative we would have found it.

Now, we say that S has a **critical point** at $\varphi \in L^2(\mathbb{R}^3, \mathbb{R})$, when $Df|_\varphi = 0$ (which is a necessary condition for local extremum). If the gradient exists for some non-degenerate $\langle , \rangle$, a critical point will be equivalent to $\frac{\delta f}{\delta \phi}|_\varphi = 0$. (By first year calculus, we could see it via $\frac{\mathrm{d}}{\mathrm{d}t} S(\varphi + t\eta)|_{t=0} = 0 \; \forall \eta \in \mathcal{C}_c^\infty$). Before proving the field theoretic Euler-Lagrange equations as an example use-case of all this macihnery, we need a last ingredient: is there a notion of "partial derivative" with respect to only some arguments?

Definition 6. Let $E_1, ..., E_n, F_1, ..., F_n$ be NVS, $f : E_1 \times \cdots \times E_n \to \mathbb{R}$ be $\mathcal{C}^1$ and $\langle \cdot, \cdot \rangle_i : E_i \times F_i \to \mathbb{R}$ E_j-non-degenerate pairings. We define the **(partial) functional derivative wrt ϕ_j at the point** $\vec{\varphi} := (\varphi_1, ..., \varphi_n) \in \prod_{j=1}^n E_n$ as the unique element (if it exists), denoted by $\frac{\delta f}{\delta \phi_j}|_{(\vec{\phi})}$ such that

$$D_j f|_{\vec{\varphi}} \cdot \eta_j := Df|_{\vec{\varphi}}(0, ..., 0, \eta_j, 0, ..., 0) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} f(\varphi_1, ..., \varphi_j + \varepsilon\eta_j, ..., \varphi_n) = \Big\langle \frac{\delta f}{\delta \phi_j}|_{\vec{\varphi}}, \; \eta_j \Big\rangle_j. \tag{104}$$

♦

Example: The Field Euler-Lagrange Equations. Let there be some open set $\Omega \subset \mathbb{R}^n$ with compact closure ($\mathbb{R}^n$ can include a time direction, so it could be a space-time) and let $M := H^2(\Omega, \mathbb{R})$ denote the NVS of fields over (the piece of space-time) $\Omega \subset \mathbb{R}^n$ that are square integrable and twice (weakly) differentiable with square integrable (weak) derivatives. Now, let there be a function $S : M^m \to \mathbb{R}$, which we call the **action**, defined by

$$S(\varphi_1, ..., \varphi_n) := \int_\Omega \mathscr{L}\Big(x, \varphi_j(x), \frac{\partial \varphi_j}{\partial x_k}\Big) d^n x \tag{105}$$

for some smooth function $\mathscr{L} : \Omega \times \mathbb{R}^m \times \mathbb{R}^{n+m} \to \mathbb{R}$, of arguments $\mathscr{L}(x, \phi_1, \frac{\partial \phi_j}{\partial x_r})$, called the **Lagrangian density**. According to the postulate known as **action principle**, the actual fields $\vec{\varphi}$ we observe in the world are critical points of the action S. But as we saw above, $\vec{\varphi}$ is a critical point of S if and only if for some pairing for which the functional derivative exists, $\frac{\delta S}{\delta \phi_k}|_{\vec{\varphi}} = 0$ for all $k \in \{1, ..., m\}$. Hence, if we find some functional derivatives, we will obtain an explicit equation governing the physically occurring fields.

A simple way to find a functional derivative wrt the k-th argument in the case of an action-like function (given by the integration of some density), is to take the directional derivative in an arbitrary direction $\eta \in \mathcal{C}_c^\infty(\Omega)$ along the k-th argument and to manipulate the expression until we get the L^2 pairing between the η and some other vector ψ. By the uniqueness of functional derivative, the ψ must be the searched functional derivative. As a showcase, let $\eta_k \in \mathcal{C}_c^\infty(\Omega, \mathbb{R})$ and fix a point in the configuration space(-time) $\vec{\varphi} := (\varphi_1, ..., \varphi_m) \in M^m$, then,

$$D_k S|_{\vec{\varphi}} \eta_k := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} \; S\Big(\varphi_k + \varepsilon\eta_k; \; \{\varphi_j\}_{j\neq k}\Big) = \tag{106}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} \int_\Omega \mathscr{L}\Big(x, \; \varphi_k(x) + \varepsilon\eta_k(x), \; \frac{\partial \varphi_k(x)}{\partial x_r} + \varepsilon \frac{\partial \eta_k(x)}{\partial x_r}; \Big\{\varphi_j(x), \frac{\partial}{\partial x_r}\varphi_j(x)\Big\}_{j\neq k}\Big) d^n x =$$

$$= \underbrace{\cdots}_{\substack{\text{different. under integral +} \\ \text{chain \& Leibniz rules +} \\ \eta_k(\partial\Omega)=0 \text{ bec. comp. supp}}} = \int_\Omega \left[\frac{\partial \mathscr{L}}{\partial \phi_k}|_{\vec{\varphi}}(x) - \sum_{r=1}^n \frac{\partial}{\partial x_r}\left(\frac{\partial \mathscr{L}}{\partial\Big(\frac{\partial \phi_k}{\partial x_r}\Big)}|_{\vec{\varphi}(x)}\right)\right] \eta_k(x) d^n x =$$

$$= \Big\langle \frac{\partial \mathscr{L}}{\partial \phi_k}|_{\vec{\varphi}} - \sum_{r=1}^n \frac{\partial}{\partial x_r}\left(\frac{\partial \mathscr{L}}{\partial\Big(\frac{\partial \phi_k}{\partial x_r}\Big)}|_{\vec{\varphi}}\right), \; \eta_k \Big\rangle_{L^2(\Omega,\mathbb{R})} =: \Big\langle \frac{\delta S}{\delta \phi_k}, \; \eta_k \Big\rangle_{L^2}.$$

Hence, $\frac{\delta S}{\delta \phi_k}|_{\vec{\varphi}} = 0$ for all $k \in \{1, ..., m\}$ if and only if

$$\frac{\partial \mathscr{L}}{\partial \phi_k}|_{\vec{\varphi}} - \sum_{r=1}^n \frac{\partial}{\partial x_r}\left(\frac{\partial \mathscr{L}}{\partial\Big(\frac{\partial \phi_k}{\partial x_r}\Big)}|_{\vec{\varphi}}\right) = 0 \qquad \text{for all } k \in \{1, ..., m\}. \tag{107}$$

This system of PDEs is called **the Euler-Lagrange equations.**

Appendix to B.2: Not all PWTs explain the collapse and "measurement" probabilities!

It is not at all clear that a non-position configuration pilot-wave theory,[35] in the context of a quantum "measurement", can predict without postulating it, the collapse of the wavefunction or the probabilities associated to its outcomes. And like it or not, both are experimentally verified facts. Let us find why this is not clear by reviewing the key aspects of why Bohmian mechanics does predict them. (Spoiler: because its configuration space is over **position-space**!).

· First Act: An Interpretation Agnostic Setting of an ideal Quantum Measurement

Let Q be the configuration-space of the laboratory (or if preferred, the whole Universe), whatever it is: position configuration or others and either particle or field configuration-space. As an obvious assumption in physics, we can split it as $Q = Q_{object} \times Q_{device}$, with Q_{object} the configuration-space of an *object-system* we want to study and Q_{device} containing the rest of degrees of freedom (in particular, the devices of the laboratory).[36] We assume the object is sufficiently isolated from the rest of the environment to have "its own unitary SE", meaning that it has some normalized effective wavefunction $\psi \in L^2(Q_{object})$ (an obvious assumption in any quantum treatment where we want to consider only the object system with no external influence).

Now, assume we are interested on a countable (or finite) set of orthonormal wavefunctions $\{\varphi_j\}_{j\in J} \subset L^2(Q_{object})$ (e.g., eigenfunctions of the object system's Hamiltonian). We can bookkeep them in the self-adjoint operator $\hat{D} := \sum_{j\in J} j\, \varphi_j\langle\varphi_j, \cdot\rangle$ —so that we can recover them up to global phase from the spectral decomposition of $\hat{D}$. We build a (macroscopic) device (i.e., such that we can distinguish its different dial positions with the "naked eye"). The configuration of the device is described within Q_{env}. We build the device with a dial that can take the values in J and is built so that **if** $\psi \in \{\varphi_j\}_{j\in J}$, after the device is brought to interact with the object, we can unambiguously determine in which of those states was the object prior to the interaction. We call this assumption ($\star$), telling that by calibration/design of the device:

($\star$) Whenever the object's ψ is prepared so that it is proportional to φ_j (i.e., it is in the eigenspace of eigenvalue j of the bookkeeping s.a. operator $\hat{D}$), after the device-object coupling, the dial of the device shows the direction $j \in J$ almost surely (probability 1), i.e., it happens so in (almost) all repetitions of the experiment.

Arguably, this qualifies the device to be called $\hat{D}$-"measurement" device, but in general it is rather called $\hat{D}$-"quantum measurement" device (because as we will mention later, the object can also initially be in a superposition of the states in $\psi \in \{\varphi_j\}_{j\in J}$ and then...we will see what happens).

If initially the object has an effective wavefunction because it is sufficiently isolated from the devices, so must its complement (the devices) have by reciprocal isolation from the object. We call it $\phi_R \in L^2(Q_{device})$, which we will assume to be compatible with the "Ready to take measurements"-state of the device.

· Second Act: Interpretation Agnostic Dynamics of an ideal Quantum Measurement

For any preparation $\psi \in L^2(Q_{object})$ (among the eigenstates of $\hat{D}$ or not), the process of "measurement" starts with a joint device-object wavefunction $\phi_R \otimes \psi \in L^2(Q_{device} \times Q_{object})$. As an obvious fact, for the extraction of information from the object there must be an influence of the object on the device, meaning that the device will need to break the isolation of the object system by interacting with it for some time $\tau > 0$.[37] By the Schrödinger equation dictating the joint dynamics, the coupling happens through a unitary propagator U^τ acting on $L^2(Q_{device} \times Q_{object})$ that will leave towards the end of the measurement a joint state $U^\tau(\phi_R \otimes \psi)$. This is generally an entangled device-object wavefunction.

In particular, ($\star$) and the linearity of the Schrödinger equation (thus, linearity of the unitary U^τ) imply

($\star\star$) If the object system is prepared in a superposition of the states for which our device was built, say, if $\psi = \sum_{j\in J} c_j\varphi_j$ for $\{c_j\}_j \subset \mathbb{C}$, then, at the end of the coupling the device-object state is described by the superposition $\sum_{j\in J} c_j\, U^\tau(\phi_R \otimes \varphi_j)$.

[35] Meaning that the primitive ontology is not a trajectory unambiguously translatable to position-space. An example would be a canonical momentum trajectory.

[36] For example, if we chose $Q = \mathbb{R}^{3N}$, with $N \in \mathbb{N}$ the number of particles in the lab (or the Universe), such that n of those particles belong to the object system, then $Q_{object} = \mathbb{R}^{3n}$. If instead, we assumed Q to be position-space fields, say $Q = H^2(\mathbb{R}^3, \mathbb{R})$, then Q_{object} would be the set of position-space fields localized in the object's domain $\Omega \subset \mathbb{R}^3$, say $H^2(\Omega, \mathbb{R})$.

[37] As always in the known physics, by mutuality of interactions, an interaction will not only cause the object system to influence the device and its pointer, but this will reciprocally influence/perturb/back-act the object system.

But then we have the experimental constraint that:

($\star\star\star$) At the end of the "measurement", we see that the device points unambiguously to one particular $k \in J$ with a probability $|c_k|^2$.

· Resolution according to Orthodox QM: an unnecessary and unimaginative "Deus ex machina"

Let us say that in a particular experiment we get $k \in J$. By calibration of the device, every state $U^\tau(\phi_R \otimes \varphi_j)$ with $j \neq k$ is "incompatible" with the observed macroscopic situation and only $U^\tau(\phi_R \otimes \varphi_k)$ is compatible with it. Then, as one of the most shameless and unprofessional "Deus ex machina"-s in the whole history of science, the textbook QM theory **postulates** that there has been a so-called "collapse of the wavefunction" abruptly taking the superposition $\sum_{j\in J} c_j\ U^\tau(\phi_R \otimes \varphi_j)$ into $U^\tau(\phi_R \otimes \varphi_k)$, and that this so-happens with probability $|c_k|^2$.

· Resolution according to Bohmian Mechanics: ($\star\star\star$) is an obvious consequence of ($\star$) and the SE

In Bohmian mechanics, where $Q = \mathbb{R}^{3N}$ and $Q_{object} = \mathbb{R}^{3n}$ are the **position**-configuration space, with $N > 10^{23}$ (because we are describing a whole macroscopic device) and $n < 10^{11}$ (for it is a microscopic object system —a human DNA molecule has around 10^{11} particles), the plot is resolved with no need of divine intervention where "God plays some die". Within Bohmian mechanics ($\star$) has the precise meaning that:

($\star$) By design of the device, the state $U^\tau(\phi_R \otimes \varphi_j)$ is such that almost all possible device-object Bohmian trajectories are compatible with a macroscopic situation in which the dial shows j.

But then, the following chain of observations and arguments follows:

- The dial must be in macroscopically different positions to show us different j.
- The dial's position is unambiguously defined by the device-object Bohmian trajectory **because the trajectory lives in position-space.**
- Hence, different dial positions happen if and only if macroscopically different Bohmian trajectories.
- Therefore, by $\star$, almost every device-object Bohmian trajectory supporting the state $U^\tau(\phi_R \otimes \varphi_j)$ must be macroscopically far away in configuration space from those that support $U^\tau(\phi_R \otimes \varphi_k)$ (with $k \neq j$).
- Finally, because the typicality measure of Bohmian trajectories is exactly the magnitude (squared) of the wavefunction, the Bohmian trajectories supporting a state make up its functional support, so that:

  ($\overset{\star\star}{\star\star}$) The supports of $U^\tau(\phi_R \otimes \varphi_j)$ and $U_\tau(\phi_R \otimes \varphi_k)$ must be macroscopically disjoint in Q, i.e., a macroscopic distance (several orders of magnitude more than the characteristic dispersion-scales of φ_j) must separate their supports. We say they are macroscopically disjoint wavepackets.

So, we proved that in Bohmian mechanics, ($\star$) $\Rightarrow$($\overset{\star\star}{\star\star}$).

On the other hand, ($\star\star$) and ($\overset{\star\star}{\star\star}$) together mean that at the end of the measurement, the joint state $\psi^\tau := \sum_{j\in J} c_j\ U^\tau(\phi_R \otimes \varphi_j)$ is the sum of macroscopically disjoint wavepackets. By the dispersive time evolution of the Schrödinger equation, macroscopically disjoint wavepackets made of a macroscopic amount of particles remain macroscopically disjoint for almost always (this loss of capability to self-interfere again, for the macro-disjoint portions of a many-particle wavefunction, is called *"decoherence"*). As such, after the "measurement", the packets $c_j\ U_\tau\phi_0 \otimes \varphi_j$ evolve independently of each other in the joint configuration-space. But now, in Bohmian mechanics there is a determinate position for all the particles at all times and they follow the typicality measure of the magnitude squared of the wavefunction, so the device-object Bohmian trajectory will end up in one of the supports of the wavepackets, say, in the k-th one. This explains why we see a determinate dial at all times that ends up pointing in some $k-$th direction (in agreement with the experimental constraint ($\star\star\star$)). Moreover, since the different $c_j\ U^\tau(\phi_R \otimes \varphi_j)$ in the superposition evolve independently of each other, effectively the device-object wavefunction is $c_k U_\tau\phi_0 \otimes \varphi_k$ (explaining why the collapse postulate of Orthodox QM is indeed acceptable to be assumed effectively).

Finally, denote by $y \in \mathbb{R}^{3m}$ the configuration of the device (including the dial) and $x \in \mathbb{R}^{3n}$ the configuration of the object. Define $B_k \subset \mathbb{R}^{3m}$ as the set of configurations of the device corresponding to a dial pointing in the k-th direction. By ($\overset{\star\star}{\star\star}$),

$$supp\ U^\tau(\phi_0 \otimes \varphi_k)\ \cap\ B_k = \begin{cases} \varnothing & if\ j \neq k \\ supp\ U^\tau(\phi_0 \otimes \varphi_k) & if\ j = k \end{cases}. \tag{108}$$

Hence, the probability that at the end of the "measurement" the dial points in the k-th direction, according to Bohmian mechanics, is:

$$\mathbb{P}rob(y \in B_k) = \int_{B_k}\int_{\mathbb{R}^n} |\psi^\tau(x,y)|^2 d^n x d^m y = \int_{B_k}\int_{\mathbb{R}^n} \sum_{j\in J} |c_j\, U_\tau\phi_0 \otimes \varphi_j|^2 d^n x d^m y = \tag{109}$$

$$\underbrace{=}_{(108)} \int_{supp\ U^\tau(\phi_0\otimes\varphi_k)}\int_{\mathbb{R}^n} |c_k|^2\, |U_\tau\phi_0 \otimes \varphi_k|^2 d^n x d^m y \underbrace{=}_{U^\tau \text{ is unitary}} |c_k|^2.$$

Exactly as found experimentally in $(\star\star\star)$.

· Necessary Conditions for a Pilot Wave Theory to use the Bohmian Resolution

Proposition 2. Given a pilot wave theory, the following is a necessary condition to be able to use the same proof as Bohmian mechanics to derive from the postulates of the pilot-wave theory, the collapse of the wavefunction and the probabilistic outcomes of ideal quantum measurements (by just substituting "Bohmian trajectory" by "primitive ontology's trajectory"):

> *The macroscopic position of a dial is unambiguously defined by the trajectory of the device-object primitive ontology.*

♦

The key reason why this holds is that in physics, the image of the macroscopic world is based on positions,[38] so that different macroscopic situations can only be unambiguously given by position-configurations of it. In very simple terms, knowing for instance the classical canonical momentum trajectory of a set of particles does not tell us how far apart their associated position trajectories could be, hence, we could not tell if the dial says this or that if we only a momentum trajectory for it. The problem is that we could attribute more than one position trajectory to the same momentum trajectory. With this in mind, the obvious quality that tells us if a dial is pointing in this or that direction with least doubt, is the position (and subordinate coarse-grainings of it).

Examples that fall in the necessary condition are the pilot wave theories where the primitive ontology has an immediate position representation. For example, if we take fields over position-space $\rho(x,y,z)$ to describe the density of matter in physical space, by using fields that show higher concentrations wherever our macroscopic image of the world goes, we will achieve the above proposition.[39] Likewise, if we have a field configuration-space of mode expansion coefficients $(\alpha_1, \alpha_2, ...)$ that represent a density field over position-space via some $\rho(x,y,z) = \sum_j \alpha_j u_j(x,y,z)$, there is a 1 to 1 related position representation that we can use for the Bohmian-like explanation of collapse. Yet another example where there are fields that do not represent densities of matter but still the Proposition 2 holds is the QED we gave in the introduction. There, the device dial would be assumed to be made of electrons and that would be enough. A non-example is a cousin pilot wave theory to Bohmian mechanics where we describe wavefunctions in Fourier space and the Bohmian trajectory is a "momentum space" trajectory. Similar examples could be built for fields over non-position configuration-spaces.

Let us generalize this a bit. The condition of Proposition 2 was necessary to prove $(\star) \Rightarrow (\overset{\star\star}{\star\star})$, which was the crucial thing allowing the rest of the derivation of the collapse and its probabilities. But if we find a pilot wave theory that explains $(\star) \Rightarrow (\overset{\star\star}{\star\star})$ in an alternative way, we will have also solved the problem.

Proposition 3. Given a pilot wave theory, $(\star) \Rightarrow (\overset{\star\star}{\star\star})$ is a necessary condition to use, substituting "Bohmian trajectory" by "primitive ontology's trajectory", the same proof as Bohmian mechanics (the part after we proved that implication) to derive from the postulates of the pilot-wave theory, the collapse of the wavefunction and the probabilistic outcomes of ideal quantum measurements.

♦

Then, a (stronger) condition to check that indeed, a pilot wave theory is able to explain the collapse and its probabilities would be that:

[38] For a very good reason: whether something occupies a spatial extension or not is the perceptual/phenomenological quality that most humans will agree on. We could take colour, sound, smell etc., but none will be as simply agreeable.

[39] There are some claims that under the Gaussian measures we will develop in the rigorification of the wavefunctional spaces, the set of all wavefunctionals compactly supported in discrete islands (hence giving clearly localized objects) has measure zero. But of course, the fields do not necessarily need to go down to zero nor be differentiable to show well-defined macro-situations. If the difference between the regions of space where they are very dense and those in which they are not is sufficiently large, we will not be able to distinguish the result.

For any pair of different macroscopic situations A and B, there exist wavefunction(al)s $\Psi_A, \Psi_B \in L^2(Q, \mathbb{C}, d\mu)$ in the global (device-object) configuration space, such that:

- *Almost every ontological trajectory piloted by Ψ_A represents the same macroscopic situation A (likewise for B).*
- *Ψ_A and Ψ_B have disjoint support in the configuration-space of primitive ontology Q.*

In our context of QFTs, this would imply to check whether "whenever a wavefunctional yields the same macro-situation for a.e. ontological field, this wavefunctional is disjoint form others yielding other macro-situations for a.e. ontological field." To check this stronger condition is more conventional in the literature —although it is harder— (see some examples for various QFTs in Struyve (2010)).

Appendix to D.2: How the Trick is Employed in the Feynman-Kac Fromula

In D.2.1 we suggested a trick that one can employ in the search for a measure that gives a rigorization of Feynman's path integral over path-space (which is infinite dimensional) for the heat equation. Let us find which is the problem we want to solve with it and how the trick could be employed. What follows drinks from Reed and Simon (1975)'s section[40] on the topic.

(α) Feynman's Heuristic

Consider a single Newtonian particle following a trajectory $\tau \mapsto \vec{X}(\tau) \in \mathbb{R}^3$ for $\tau \geq 0$. Given a potential energy field $V : \mathbb{R}^3 \to \mathbb{R}$ (with suitable regularity), we define the usual action functional

$$S\Big(\vec{X};\ t\Big) := \int_0^t \left(\frac{1}{2} m \left| \frac{\mathrm{d}\vec{X}(\tau)}{\mathrm{d}t} \right|^2 - V(\vec{X}(\tau)) \right) d\tau. \tag{110}$$

(We will consider the mass m to be $1/2$ hereafter.)

Now, if we wanted to compute such an integral numerically, we would first discretize $[0, t]$, considering a grid of $n \in \mathbb{N}$ intervals in time of equal length $\Delta t := t/n$ and equispaced nodes $t_j := j\Delta t$ for $j \in \{0, ..., n\}$. Next, we would define the discretization of the trajectory as $x_j := \vec{X}(t_j)$ (see Figure 4.(a)). Employing a trapezoid rule for the integral[41] and an Euler finite difference[42] for the velocity of the trajectory, we would discretize S as

$$S_n(x_0, ..., x_n, t) := \sum_{j=0}^{n-1} \frac{t}{n} \left[\frac{1}{4} \left(\frac{|x_{j+1} - x_j|}{t/n} \right)^2 - V(x_j) \right]. \tag{111}$$

Then, by numerical calculus (for differentiable/nice enough $\vec{X}$), $\lim_{n\to+\infty} S_n(x_0, ..., x_n, t) = S(\vec{X};\ t)$. Hereafter, we will denote by x the **end-point** of the trajectories, namely, the node $x_n = \vec{X}(t)$.

In quantum mechanics, one can find that if we define $H_0 := -\Delta$ as the single free particle Hamiltonian in $L^2(\mathbb{R}^3)$ (of domain $D(H_0)$),

Theorem 1. For $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, and any initial wavefunction $\psi_0 \in L^2(\mathbb{R}^3)$, the time evolved wavefunction $\psi_t := \left(e^{-it(H_0+V)}\psi_0\right)$ is also given by

$$\psi_t(x) = \lim_{n\to+\infty} \left(\frac{n}{4\pi i t} \right)^{\frac{3n}{2}} \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} e^{iS_n(x_0,...,x_{n-1},x,t)}\ \psi_0(x_0)\ dx_0 \cdots dx_{n-1} \qquad \text{(in } L^2 \text{ sense).} \tag{112}$$

♦

Proof. On the one hand, we found in MQT that the free time evolution operator e^{itH_0} is equal to the integral operator:

$$\left(e^{itH_0}\varphi\right)(x) = \frac{1}{(4\pi i t)^{3/2}} \int_{\mathbb{R}^3} e^{\frac{i|x-y|^2}{4t}} \varphi(y) dy. \tag{113}$$

[40] Their derivations can be quite confusing for those aiming a rigorification of the physics literature path integral, because they consider that it is the initial point of the paths (and not the finial point) what is fixed. I have modified their derivations with care to match the most common physical intuitions.

[41] For a regular enough $f : (0, t) \to \mathbb{R}$, to leading n order and large $n \in \mathbb{N}$, $\int_0^t f(\tau)d\tau$ is given by $\sum_{j=0}^{n-1} f(t_j)\Delta t$.

[42] For a regular enough $t \mapsto \vec{X}(t)$, to leading Δt order in large $n \in \mathbb{N}$, the velocity $\frac{\mathrm{d}}{\mathrm{d}t}\vec{X}(t_j)$ is given by $\frac{\vec{X}(t_{j+1})-\vec{X}(t_j)}{\Delta t}$.

On the other hand, as we proved in MQT, a potential splitable as in the hypothesis makes $H_0 + V$ self-adjoint on $D(H_0)$, so that we can employ the Trotter formula (4.36 in Teufel's MQT notes), i.e.,

$$e^{-it(H_0+V)}\psi_0 = \lim_{n\to+\infty}\left(e^{-i\frac{t}{n}H_0}\, e^{-i\frac{t}{n}V}\right)^n \psi_0. \tag{114}$$

But in this expression we can evaluate each of the n propagators $e^{-i\frac{t}{n}H_0}$ using (113) in a chained way to get (112) (namely, evaluate (113) first in the left-most free propagator, i.e., take as if $e^{-i\frac{t}{n}V}\left(e^{-i\frac{t}{n}H_0}\, e^{-i\frac{t}{n}V}\right)^{n-1}\psi_0$ was the φ of (113) and call y instead x_{n-1}; evaluate the next free propagator, taking as if the φ was now $e^{-i\frac{t}{n}V}\left(e^{-i\frac{t}{n}H_0}\, e^{-i\frac{t}{n}V}\right)^{n-2}\psi_0$ and using x_{n-2} instead of y. Iterate n times). *o.ε.δ.*

Let us analyse (112). In the dx_j integral of (112) we are sweeping over all the possible values that a path $\vec{X}$ could take at $\vec{X}(t_j)$. Hence, as we take the limit $n \to +\infty$, the spatial integrals of the trajectory over the time grid, end up sweeping the values that the trajectory could take at each $\tau \in [0, t)$. The only clear constraint is that the end-point is fixed as $\vec{X}(t) = x$, because we are not integrating over it. This is as if we were integrating the exponential of the action iS over the space of possible paths that end in x (note how this would include all paths, crazily oscillating paths as well). See Figure 4.(a). With all, as the limit exists by (112), and it seems to be interpretable as an integral over the space of paths ending at x, which we denote by Ω_x, you must already have on the tip of your tongue that a formula like

$$\text{“}\psi_t(x) = \int_{\vec{X}\in\Omega_x} e^{iS(\vec{X};\, t)}\psi_0\Big(\vec{X}(0)\Big)d\mu\text{”} \tag{115}$$

should exist as the limit of (112). It is said that R. Feynman was the first one having such an insight. But now you know that no matter how much we look for a measure $d\mu$ which is the limit of infinitely many Lebesgue measures “$d\mu = \lim_{n\to\infty} dx_0 \cdots dx_{n-1}$”, such a fantasy does not exist.

But heuristic (115) would be very nice if true. Among others, it would provide a very fancy way to explain the “classical limit” of QM. If we did not take units where $\hbar = 1$, the propagators would be things like $e^{-i\frac{t}{\hbar}H}$, so that the heuristic would be

$$\text{“}\psi_t(x) = \int_{\vec{X}\in\Omega_x} e^{i\frac{S(\vec{X};\, t)}{\hbar}}\psi_0\Big(\vec{X}(0)\Big)d\mu.\text{”} \tag{116}$$

Then, as $\hbar$ is made smaller and smaller compared to the characteristic values of S (say, because the mass m to which S is proportional, becomes larger and larger as we consider larger systems), the phases would “tend to cancel each other”, except around critical points of S, where S is constant at first order and corresponds to the classical mechanics paths (by virtue of the action principle). So, we would have that in the limit $\hbar \to 0$, the contribution to dynamics would come exclusively from paths nearer and nearer to the classical path.

Despite physicists use heuristic formula (115) to derive (ill-defined) perturbation series[43] assuming somebody will make it rigorous somehow, to my knowledge, we are still unable to make “Feynman's dream” come true. What is possible, is to rigorify such a limit for a very closely related expression: for the analogue propagator of the heat equation (with a potential) instead of the Schrödinger equation. The only formal difference between these two equations is that the i multiplying the time derivative in the SE is substituted by a -1. The point is that this change will be the crucial thing that will allow an immediate usage of the trick in D.2.1 to find a limit measure for the heat equation's case.

(β) A More Promising Path Integral

Let us see how we can use the same proof as that of Theorem 1 to derive an analogous result for the heat equation. Recall we also found in MQT (using convolution and Fourier multiplier techniques) that the free heat equation $\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = \Delta\rho_t$ has a propagator e^{-tH_0} given by a very similar integral operator to that of the Schrödinger equation, so that for $\eta \in L^2(\mathbb{R}^3)$

$$\big(e^{tH_0}\eta\big)(x) = \frac{1}{(4\pi t)^{3/2}}\int_{\mathbb{R}^3} e^{-\frac{\|x-y\|^2}{4t}}\eta(y)dy. \tag{117}$$

[43] They could perhaps be made sense of- by considering discretized versions of such heuristic formulas (e.g., considering (112) to be the discretization of (115)). If so, the heuristic formulas employed in the physics literature would be understandable as symbols representing such discretizations. However, a priori, the derived expressions would be discretization-dependent and it is not clear if they would converge to a common result. Such a situation is analogous to the cut-off techniques mentioned in C.

- Alternatively, one can find this propagator formula using the following insight based on the fact that the heat equation emerges as the time evolution of the probability density of a Brownian particle (as seen in the "Chaos" course). Take a Brownian particle of which we know the $t = 0$ position to be y, so that at $t = 0$ its position is distributed according a Dirac delta measure at y. If we now consider all the possible random walks it could take from there till time t (where each "infinitesimal step" is normally distributed), its position x will follow a normal/Gaussian distribution with density (called "heat-kernel")

$$K(x; y, t) := \frac{1}{(4\pi t)^{3/2}} e^{-\frac{\|x-y\|^2}{4t}}. \tag{118}$$

  Now, if we initially did not know its position y, but instead, it was distributed according to a probability density η, we could predict the probability measure at (x, t) to be the "sum" of the densities arriving to that point via random walks that start at each possible y, weighted by the actual probability density there was at y. This would then leave $\eta_t(x) = \int_{\mathbb{R}^3} K(x; y, t)\eta(y)dy$, which is equation (117). See Figure 4.(c) for a graphical representation of this intuition.

We could then take the Trotter formula and plug the integral operator (117) in it to get a formula akin to the one in Theorem 1. For the Trotter formula we need to assume V is a self-adjoint multiplication operator that is bounded above, with $H := H_0 - V$ essentially self adjoint in $D(H_0) \cap D(V)$. Then, the PDE governing the probability density of a Brownian particle in a potential, which is the heat equation $\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -H\rho_t = (\Delta + V)\rho_t$ (i.e., the SE replacing i by -1), has a propagator e^{-tH} (defined using the functional calculus). Moreover, the Trotter formula leaves

$$e^{-tH}\rho_0 = \lim_{n\to+\infty} \left(e^{-\frac{t}{n}H_0} e^{\frac{t}{n}V}\right)^n \rho_0. \tag{119}$$

Denoting $\rho_t := e^{-tH}\rho_0$, in the last expression we can evaluate (117) for each of the n free propagators (just as done in Theorem 1), to get that rigorously,

$$\rho_t(x) = \lim_{n\to+\infty} \left(\frac{n}{4\pi t}\right)^{3n/2} \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} e^{-\sum_{j=0}^{n-1} \frac{|x_{j+1}-x_j|^2}{4t/n}} e^{\sum_{j=0}^{n-1} \frac{t}{n} V(x_j)} \rho_0(x_0) \prod_{j=0}^{n-1} dx_j = \tag{120}$$

$$= \lim_{n\to+\infty} \left(\frac{n}{4\pi t}\right)^{3n/2} \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} e^{-S_n(x_0,\ldots,x_n,t)}\, \rho_0(x_0)\, dx_0 \cdots dx_{n-1}.$$

This is the same provocative formula as (112) but now with a -1 instead of an i. Using the same arguments, we would find on the tip of our tongues the heuristic (115) (now changing i by -1) as the limit expression of (120). Writing S explicitly, such a heuristic would read:

$$\text{“}\rho_t(x) = \int_{\vec{X}\in\Omega_x} e^{-\frac{1}{4}\int_0^t |\frac{\mathrm{d}\vec{X}(\tau)}{\mathrm{d}t}|^2 d\tau} e^{\int_0^t V(\vec{X}(\tau))d\tau} \rho_0\Big(\vec{X}(0)\Big) \prod_{j=0}^{\infty} dx_j\text{”}. \tag{121}$$

And again, there would be no limit measure $\lim_{n\to+\infty} \prod_{j=0}^{n-1} dx_j$. However, we are now allowed to apply the trick of D.2: if we rewrite the (rigorous) equation (120) by placing the kinetic energy part next to the Lebesgue measure, we get

$$\rho_t(x) = \lim_{n\to+\infty} \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} e^{\sum_{j=0}^{n-1} \frac{t}{n} V(x_j)} \rho_0(x_0) \prod_{j=0}^{n-1} \left(\frac{n}{4\pi t}\right)^{3/2} e^{-\frac{|x_{j+1}-x_j|^2}{4t/n}} dx_j. \tag{122}$$

If we considered instead of dx_j, the measure $d\mu_j := \left(\frac{n}{4\pi t}\right)^{3/2} e^{-\frac{|x_{j+1}-x_j|^2}{4t}} dx_j$, which is a probability measure for fixed x_j (a Gaussian centred in x_j with standard deviation $2t/n$), we know that the limit product measure would now exist! And hence not the formula (121), but the same formula with the kinetic energy term and the normalization constants absorbed into the measures would indeed have a chance to exist!

The reason why such a trick works here and not in the quantum case can be heuristically understood as follows. In both ideal formulas (115) and (121), differing only in the i changed for -1, the Lebesgue measure product blows up to infinity (every open set would have infinite measure, as we saw in D.2). Likewise, for arbitrary paths $\vec{X} \in \Omega_x$ (say, non-differentiable kinky paths), the "derivative" $\frac{\mathrm{d}\vec{X}(\tau)}{\mathrm{d}t}$ would blow-up to infinity

as well, which would make the integral $\int_0^t |\frac{d\vec{X}(\tau)}{dt}|^2 d\tau$ highly singular. However, the exponential with the minus sign in (121) would make the latter blow-up arbitrarily close to zero, and "lumping" together this zero of the kinetic energy with the blowing up Lebesgue measures, gives a well-defined limit measure. In the quantum case (115) on the other hand, since the integral of the kinetic energy in the exponential does not have a minus sign, but instead an i, it does not decay for singular derivatives and there is no term to compensate the blowing up Lebesgue measure.

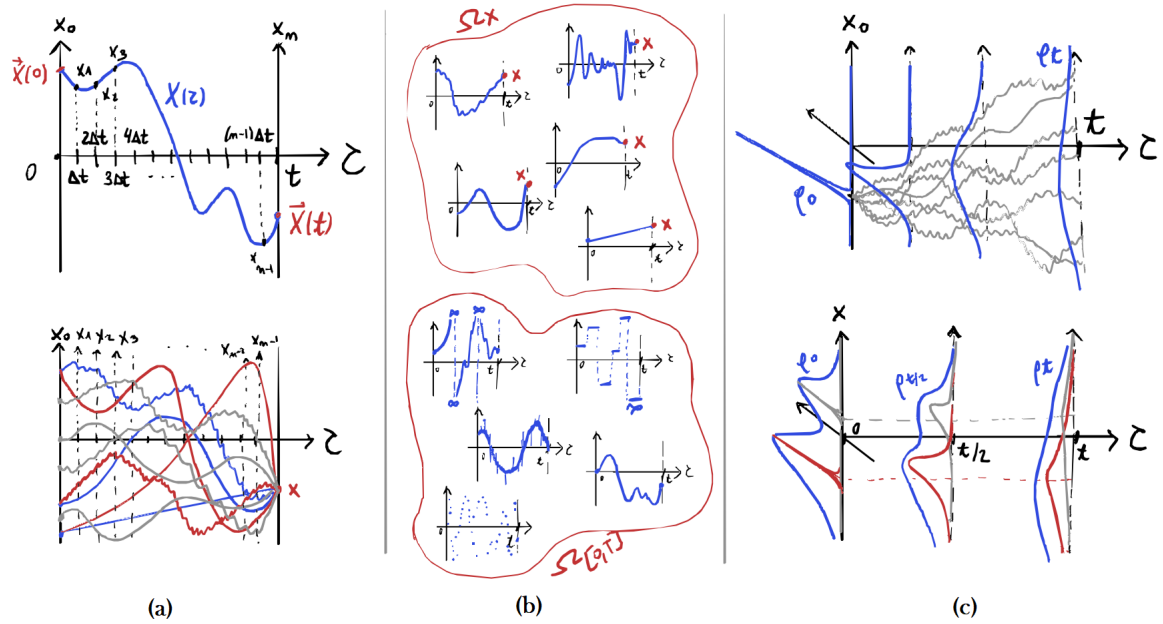

Figure 4: In (a), above, a sketch of the discretization we considered for a trajectory; below, an ensemble of paths ending in a fixed x. Note how sweeping the possible combinations of $x_0, x_1, ..., x_{n-1}$ essentially exhausts all such paths when $n \to +\infty$. In (b), above, a sketch of the set Ω_x mentioned in the text, consisting of trajectories ending in x; below, a sketch of the set of paths $\Omega_{[0,T]} := \prod_{j\in[0,t]} \overline{\mathbb{R}^m}$ considered in the construction of the Wiener measure, which includes "totally discontinuous" and diverging paths. Despite all, we also found that according to the Wiener measure, continuos paths have measure 1, so the lower sketch does not represent its typicality notion (rather the one above does for $d\mu_{x,t}$). In (c) above, a sketch of the time evolution of the density of a Brownian particle starting at a known location; below, how we can employ the time evolution of a "point source" to get the evolution of a general initial density profile.

(γ) Construction of the Wiener Measure — an Alternative to get Nice Measures in "Infinite Dimensions"

In this section we build a very general limit measure for the path space (now for an arbitrary dimension $m \in \mathbb{N}$ instead of 3) known as the *Wiener measure*. In particular, it allows to find a rigorous version of (121). Now, instead of using explicitly our "trick" above (which will be exemplified in D either-way), here we present an alternative method for limit measures employed by Reed and Simon (1975) in the present context (still, we will find that in its core, our proposed "trick" will tacitly be employed). The essence of the alternative approach relays in using the Riesz-Markov theorem to find a limit measure instead of the Kolmogorov extension theorem. To use the Riesz-Markov theorem, we need that the "field configuration-space", which in this case is the space of paths (along $\mathbb{R}^m$ and indexed by times in $[0, t]$), has a compact and Hausdorff topology. To achieve it, we will consider the one point compactification of $\mathbb{R}^m$, adding a point representing the infinity. However, in doing so we will need to accept an even more dramatic tragedy than our lose of norm in D.2: the compactified $\mathbb{R}^m$ is not a vector space any-more! Still, just as in $D.4$, the L^2 space of functionals over the space of paths will indeed be a vector space (even a complete Hilbert space!).

That said, lets build the Wiener measures:

(1). Consider the one-point compactification of $\mathbb{R}^m$, which we denote by $\overline{\mathbb{R}^m} := \mathbb{R}^m \cup \{\infty\}$ and let $\Omega_{[0,t]} := \prod_{j\in[0,t]} \overline{\mathbb{R}^m}$ be the set of all paths (non-necessarily continuous) γ indexed by a time $\tau \in [0, t]$ and taking values $\gamma(\tau) \in \overline{\mathbb{R}^m}$ (see Figure 4.(b)). We give the path space $\Omega_{[0,t]}$ the product topology, which by Tychonoff's theorem makes $\Omega_{[0,t]}$ a compact and Hausdorff space.

(2). Let $F : \prod_{j=1}^{n} \overline{\mathbb{R}^m} \to \mathbb{R}$ denote a continuous function of n variables. Using it, we can define a function on the path space which only depends on n of the points of a path, $G^{F,t_0,...,t_{n-1}} : \Omega_{[0,t]} \to \mathbb{R}$, via

$$G^{F,t_0,...t_{n-1}}(\gamma) := F(\gamma(t_0), ..., \gamma(t_{n-1})), \tag{123}$$

for some parameters $t_0 \leq \cdots \leq t_{n-1} \in [0, t]$. It is a continuous function (by the continuity of the evaluation map). Then, we can define the set $\mathcal{C}^{fin}(\Omega_{[0,t]})$ as the set of such continuous functionals that depend only on a finite number of entries of the paths and are given by some F. Note that $\mathcal{C}^{fin}(\Omega_{[0,t]})$ is a vector space (even if path-space $\Omega_{[0,t]}$ is not). We equip it with the infinity norm $\|G\|_\infty := \sup_{\gamma\in\Omega_{[0,t]}} |G(\gamma)|$ (which is always finite because G are continuous maps from a compact to $\mathbb{R}$).

(3). Fix any $x \in \mathbb{R}^m$, denote $t_n := t$, $x_n := x$ and define the map $L_{x,t} : \mathcal{C}^{fin}(\Omega_{[0,t]}) \to \mathbb{R}$,

$$L_{x,t}(G^{F,t_0,...,t_{n-1}}) := \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} F(x_0, ..., x_{n-1}) \prod_{j=0}^{n-1} \left(\frac{1}{4\pi(t_{j+1} - t_j)}\right)^{m/2} e^{-\frac{1}{4}\frac{|x_{j+1}-x_j|^2}{|t_{j+1}-t_j|}} dx_j. \tag{124}$$

This is a well-defined [44] linear functional acting on $\mathcal{C}^{fin}(\Omega_{[0,t]})$. (Note the formula we obtained in (122) would be the case for $F(x_0, ..., x_{n-1}) := e^{\sum_{j=0}^{n-1} \frac{t}{n} V(x_j)} \rho_0(x_0)$ and time nodes $\{0, \frac{t}{n}, 2\frac{t}{n}, ..., (n-1)\frac{t}{n}\}$.)

(4). Trivially, if $G^{F,t_0,...,t_{n-1}}(\gamma) \geq 0 \ \forall \gamma \in \Omega_{[0,t]}$ then $L_{x,t}(G^{F,t_0,...,t_{n-1}}) \geq 0$, so it is a positive linear functional on $\mathcal{C}^{fin}(\Omega_{[0,t]})$. One can prove that[45] this implies the operator norm is $\|L_{x,t}\|_{\mathscr{L}(\mathcal{C}^{fin},\mathbb{R})} = L_{x,t}(1)$, where 1 is the constant function equal to 1 for all paths. But a representative for such a constant function is $F = 1$, and by normalization of the Gaussians, we get $L_{x,t}(1) = 1$. Thus, $L_{x,t}$ is a positive linear functional of norm 1 on $\mathcal{C}^{fin}(\Omega_{[0,t]})$, i.e.

$$|L_{x,t}(G)| \leq \|G(\gamma)\|_\infty \qquad \forall G \in \mathcal{C}^{fin}(\Omega_{[0,t]}). \tag{125}$$

(5). By the Stone-Weierstrass theorem[46] $\mathcal{C}^{fin}(\Omega_{[0,t]})$ is dense in the set of all continuos functions from $\Omega_{[0,t]}$ to $\mathbb{R}$, denoted by $\mathcal{C}(\Omega_{[0,t]}, \mathbb{R})$. But then, by the operator extension theorem we saw in MQT (a Hahn-Banach extension theorem), $L_{x,t}$ has a unique positive linear functional extension to all $\mathcal{C}(\Omega_{[0,t]}, \mathbb{R})$ with norm 1 (which we still denote by $L_{x,t}$).

(6). Finally, by the Riesz-Markov theorem[47] there exists a unique (regular) Borel measure $d\mu_{x,t}$ on $\Omega_{[0,t]}$ with $d\mu_{x,t}(\Omega_{[0,t]}) = 1$ (probability measure) so that

$$L_{x,t}(G) = \int_{\gamma\in\Omega_{[0,t]}} G(\gamma)\, d\mu_{x,t} \qquad \forall G \in \mathcal{C}(\Omega_{[0,t]}, \mathbb{R}). \tag{126}$$

Every such measure $d\mu_{x,t}$ is called a **Wiener measure** on $\Omega_{[0,t]}$.

---

[44] Given an F of n variables, we could also think of it as a function F' of $N > n$ variables by declaring that F' does not depend on the extra variables (yeah, silly, but we need to consider this to check well-definition of (124)). The extra variables of F' could happen in between those of F, so consider taking $(t_0, ..., t_{n-1})$ and $(t'_0, ..., t'_{N-1})$ such that the t'_j are equal to the t_k whenever the j-th and k-th are corresponding variables in F and F'. Then, $F'(\gamma(t'_0), ..., \gamma(t'_{N-1})) = F(\gamma(t_0), ..., \gamma(t_{n-1}))\ \forall \gamma \in \Omega_{[0,t]}$ and hence, $G^{F,t_0,...,t_{n-1}} = G^{F',t'_0,...,t'_{N-1}}$ are the same function (obviously!). The point of this silliness is that because we defined the functional (124) in terms of the representative F, we should check that $L_{x,t}(G^{F,t_0,...,t_{n-1}}) = L_{x,t}(G^{F',t'_0,...,t'_{N-1}})$. But this is indeed the case because if F' does not depend on some x'_k, we can integrate it out using that for the m dimensional version of (118),

$$\int_{\mathbb{R}^n} K(x_{k-1}, x_k;\ t_k - t_{k-1}) K(x_k, x_{k+1};\ t_{k+1} - t_k) dx_k = K(x_{k-1}, x_{k+1};\ t_{k+1} - t_{k-1}).$$

[45] Since for any $G \in \mathcal{C}^{fin}(\Omega_{[0,t]})$ we have $-\|G\|_\infty \leq G \leq \|G\|_\infty$, by positivity and linearity, $-L_{x,t}(1)\|G\|_\infty \leq L_{x,t}(G) \leq L_{x,t}(1)\|G\|_\infty$, or equivalently, $|L_{x,t}(G)| \leq L_{x,t}(1)\|G\|_\infty$, which by definition means $\|L_{x,t}\|_{\mathscr{L}(\mathcal{C}^{fin},\mathbb{R})} = L_{x,t}(1)$.

[46] Which essentially says that if X is a compact Hausdorff space (in our case $X = \Omega_{[0,t]}$, explaining the need to compactify $\mathbb{R}^m$), and A is a subalgebra of $\mathcal{C}(X, \mathbb{R})$ (e.g., $C^{fin}(\Omega_{[0,t]})$) containing a non-zero constant function (e.g., the function 1), then, A is dense in $\mathcal{C}(A, \mathbb{R})$ if and only if it separates points (i.e., for every two points $a, b \in X$ there exists a function $f \in A$ with $f(x) \neq f(y)$ —which in our case holds straightforwardly).

[47] Essentially it states that given X is a compact Hausdorff space (like our $\Omega_{[0,t]}$, explaining the need to compactify $\mathbb{R}^m$), for any positive linear functional $L : \mathcal{C}(X, \mathbb{R}) \to \mathbb{R}$, there exists a unique (positive real regular) Borel measure $d\mu$ on X s.th. $L(G) = \int_{\gamma\in X} G(\gamma) d\mu\ \forall G \in \mathcal{C}(X, \mathbb{R})$.

But now, which ensembles of paths in $\Omega_{[0,t]}$ are Borel and what is their measure according to the Wiener measures? These are important questions because in $\Omega_{[0,t]}$ we have wildly crazy paths, starting from nowhere continuous paths till paths that go to infinity for a third of their trip and then come back. Are they significant? Might it be that all continuos paths have measure zero, so that we are working on an unreasonably big path space?

To answer some of these questions, one can first check that for all $\alpha \in (0,1]$, the set $\Omega_\alpha \subset \Omega_{[0,t]}$ of α-Hölder continuos paths[48] is a Borel set (an element in the σ-algebra generated by the open sets). Following this,

Theorem 2. Fix any $x \in \mathbb{R}^m$. Then,

- for all $\alpha \in (0, 1/2)$, $d\mu_{x,t}(\Omega_\alpha) = 1$.
- for all $\alpha \in [1/2, 1]$, $d\mu_{x,t}(\Omega_\alpha) = 0$.

♦

Hence, according to a Wiener measure, almost every path is continuos. That is a relief, because we can then consider only continuos paths in our considerations and still be rigorous. However, almost no path is differentiable! As a consequence, a typical trajectory (i.e., one sampled randomly) following the measure given by any Wiener measure, would be a non-differentiable trajectory. As an example consequence: trajectories for whom the velocity is well-defined and the kinetic energy term in the action is well-defined are almost non-existent under this notion of typicality! So, expressions like (121) are ill-defined. Fortunately, we lumped the kinetic energy part inside the measure and dodged this problem!

(δ) The Feynman-Kac formula

In particular, fixing back $m = 3$, result (126) tells us that for any continuous function $\rho_0 : \mathbb{R}^3 \to \mathbb{R}$ (as a particular case of an F with a single variable and $t_0 = 0$), now denoting back paths by $\vec{X}$ instead of γ,

$$\int_{\vec{X}\in\Omega_{[0,t]}} \rho_0\Big(\vec{X}(0)\Big)\, d\mu_{x,t} = \int_{\mathbb{R}^n} \rho_0(x_0)\left(\frac{1}{4\pi t}\right)^{3/2} e^{-\frac{1}{4}\frac{|x-x_0|^2}{t}} dx_0 = \tag{127}$$

$$= \int_{\mathbb{R}^n} \rho_0(x_0) K(x_0, x, t) dx_0.$$

One can show this formula also holds for any ρ_0 that is measurable on $\mathbb{R}^3$ and has integrable $\rho_0(\cdot)K(\cdot, x, t)$ (using that such a ρ_0 can be approximated by continuous functions and then applying the dominated convergence theorem). But then (127) holds for all $\rho_0 \in L^2(\mathbb{R}^3)$ and it turns out to be exactly the integral operator of (117). Therefore, we have that rigorously, the path integral formula giving the free time evolution is

$$e^{-tH_0}\rho_0(x) = \int_{\vec{X}\in\Omega_{[0,t]}} \rho_0\Big(\vec{X}(0)\Big) d\mu_{x,t} \qquad \forall \rho_0 \in L^2(\mathbb{R}^3). \tag{128}$$

So, we have found that the kernel of the operator e^{-tH_0}, which was $K(x_0, x, t)$ for the Lebesgue measure over $\mathbb{R}$, according to the Wiener measure is 1 over $\Omega_{[0,t]}$.[49] The measure took over the role of free propagator. In a similar fashion we can find the rigorification of the full path integral (121), as the limit expression of the also rigorous equation (120):[50]

Theorem 3. (The Feynman-Kac formula) Let V be a real valued potential in $L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, $H_0 = -\Delta$ the "free Hamiltonian" and $H = H_0 - V$ the "Hamiltonian" of the heat equation $-\frac{\mathrm{d}\rho_t}{\mathrm{d}t} = H\rho_t$. Then, for all $\rho_0 \in L^2(\mathbb{R}^3)$,

$$(e^{-tH}\rho_0)(x) = \int_{\vec{X}\in\Omega_{[0,t]}} e^{\int_0^t V(\vec{X}(\tau))d\tau} \rho_0\Big(\vec{X}(0)\Big) d\mu_{x,t}. \tag{129}$$

♦

(This is exactly the ideal formula (121) after we applied the D.2 trick on (122).)

[48] For instance, if $\alpha = 1$, these are the Lipschitz paths, and for any $\alpha \in (0, 1]$ they are uniformly continuos paths.

[49] Analogous to how the Fock "vacuum", represented by a Gaussian in wavefunctional space was mapped to the function 1 in D.4.

[50] Note that in these rigorifications we take the whole space of paths over $[0, t]$, instead of only the paths that end in x. The "knowledge" of which is the end-point of the paths that we care about is inside $d\mu_{x,t}$ (note the x dependence). This was evident from the "discrete formulas" like (124), which we employed to build the measure, or (122): the dependence on x was only explicit in the Gaussian that ended up being fused with the measure. Possibly (should be checked), the subset of $\Omega_{[0,t]}$ consisting of continuous paths ending in a fixed $x \in \mathbb{R}^m$ has measure 1 for $d\mu_x$, while zero for $d\mu_y$ if $y \neq x$. What holds true for sure is that because $\Omega_x := \prod_{j\in[0,t)} \overline{R^m} \times \{x\}$ is also compact and Hausdorff, with the exact same steps as above, we could have done the construction of a measure $d\mu'_{x,t}$ using now only the space of paths ending in x at t. Then we could undoubtedly replace $\Omega_{[0,t]}$ by Ω_x in Theorem 3.

Proof. We do the proof for compactly supported continuous potentials V. The extension of the result to the rest of potentials, is not as enlightening and is given in Reed and Simon (1975) X.68.

For compactly supported continuous potentials V, we are in the conditions where (120) is rigorous, and hence also (122). As we observed during the construction of the Wiener measure, each element of the sequence in (122), is a particular case of (124) with $F(x_0, ..., x_{n-1}) = e^{\sum_{j=0}^{n-1} \frac{t}{n} V(x_j)} \rho_0(x_0)$. Hence, by construction of (126), putting back $x_j = \vec{X}(j\frac{t}{n})$ in the potential,

$$e^{-tH}\rho_0(x) = \lim_{n\to+\infty} \int_{\vec{X}\in\Omega_{[0,t]}} e^{\sum_{j=0}^{n-1} \frac{t}{n} V(\vec{X}(j\frac{t}{n}))} \rho_0(\vec{X}(0))\, d\mu_{x,t}, \tag{130}$$

where as always in this appendix, the limit holds in the L^2 sense. This implies in particular that there is a subsequence of (130) for which the convergence to $e^{-tH}\rho_0$ happens almost everywhere in $x \in \mathbb{R}^3$.

We now want to use the dominated convergence theorem on the mentioned subsequence of the limit in (130), in order to "put the limit under the integral" and obtain the desired result (129). We check the ingredients for dominated convergence:

- If $\vec{X}$ is a continuous path, then $t \mapsto V(\vec{X}(t))$ is continuous and hence, by definition of the Riemann integral (which matches with the Lebesgue integral in this case),

$$\lim_{n\to+\infty} \frac{t}{n} \sum_{j=0}^{n-1} V\left(\vec{X}\left(j\frac{t}{n}\right)\right) = \int_0^t V\Big(\vec{X}(\tau)\Big) d\tau. \tag{131}$$

- For every fixed $x \in \mathbb{R}^3$, we found in Theorem 2 that $d\mu_{x,t}$-almost all paths $\vec{X}$ of $\Omega_{[0,t]}$ are continuous. Thus, we get that point-wise $d\mu_{x,t}$-almost everywhere in $\Omega_{[0,t]}$,

$$\lim_{n\to+\infty} \rho_0(\vec{X}(t))\, e^{\frac{t}{n}\sum_{j=0}^{n-1} V(\vec{X}(j\frac{t}{n}))} = \rho_0(\vec{X}(t))\, e^{\int_0^t V(\vec{X}(\tau))d\tau}. \tag{132}$$

- Moreover, for almost every $x \in \mathbb{R}^3$,

$$\int_{\vec{X}\in\Omega_{[0,t]}} \left|\rho_0(\vec{X}(0))\, e^{\frac{t}{n}\sum_{j=0}^{n-1} V(\vec{X}(j\frac{t}{n}))}\right| d\mu_{x,t} \leq e^{t\ max|V|} \int_{\vec{X}\in\Omega_{[0,t]}} \left|\rho_0(\vec{X}(0))\right| d\mu_{x,t} \underbrace{=}_{(128)} e^{t\ max|V|}(e^{-tH_0}|\rho_0|)(x) < +\infty. \tag{133}$$

  certifying there is a dominating L^2 function.

Hence, applying the dominated convergence theorem on (130) to we get that (129) holds for almost all $x \in \mathbb{R}^3$. *o.ε.δ.*