# PROPERTIES OF HYPERHOLOMORPHIC FUNCTIONS ON DUAL SEDENION NUMBERS

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**Abstract.** The aim of this paper is to define hyperholomorphic functions with dual sedenion variables on $\mathcal{S} \times \mathcal{S}$, where $\mathcal{S} \cong \mathbb{C}^8$. By the condition of harmonicity, we research properties of hyperholomorphic functions of dual sedenion variables in Clifford analysis.

## 1. Introduction

The dual numbers extended the real numbers by adjoining one new element $\varepsilon$ with the property $\varepsilon^2 = 0$. Every dual number has the form $z = x + \varepsilon y$ with $x$ and $y$ uniquely determined real numbers. The collection of dual numbers forms a particular two-dimensional commutative unital associative algebra over the real numbers. Hörmander [1] and Krantz [6] introduced the concepts of several complex variables and theories to complex analysis in several variables. K. Nôno [8, 9] researched the properties of hyperholomorphic functions of quaternion variable and octonion variable in the 1980s. We [2, 7, 10] found the theorem about hyperholomorphic functions of dual quaternion in $\mathbb{C}^2 \times \mathbb{C}^2$ and investigated the extensions of quaternions. Also, we [3, 4, 5] researched properties of the regularity functions on a variety of quaternionic variables and the corresponding Cauchy-Riemann system on each several types.

## 2. Preliminaries

The multiplication of these unit sedenions follows:

| $\times$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $e_0$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ |
| $e_1$ | $e_1$ | $-1$ | $e_3$ | $-e_2$ | $e_5$ | $-e_4$ | $e_7$ | $-e_6$ | $e_9$ | $-e_8$ | $e_{11}$ | $-e_{10}$ | $e_{13}$ | $-e_{12}$ | $e_{15}$ | $-e_{14}$ |
| $e_2$ | $e_2$ | $-e_3$ | $-1$ | $e_1$ | $e_6$ | $-e_7$ | $-e_4$ | $e_5$ | $e_{10}$ | $-e_{11}$ | $-e_8$ | $e_9$ | $e_{14}$ | $-e_{15}$ | $-e_{12}$ | $e_{13}$ |
| $e_3$ | $e_3$ | $e_2$ | $-e_1$ | $-1$ | $e_7$ | $e_6$ | $-e_5$ | $-e_4$ | $e_{11}$ | $e_{10}$ | $-e_9$ | $-e_8$ | $e_{15}$ | $e_{14}$ | $-e_{13}$ | $-e_{12}$ |
| $e_4$ | $e_4$ | $-e_5$ | $-e_6$ | $-e_7$ | $-1$ | $e_1$ | $e_2$ | $e_3$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ | $-e_8$ | $-e_9$ | $-e_{10}$ | $-e_{11}$ |
| $e_5$ | $e_5$ | $e_4$ | $e_7$ | $-e_6$ | $-e_1$ | $-1$ | $e_3$ | $-e_2$ | $e_{13}$ | $-e_{12}$ | $e_{15}$ | $-e_{14}$ | $e_9$ | $-e_8$ | $e_{11}$ | $-e_{10}$ |
| $e_6$ | $e_6$ | $-e_7$ | $e_4$ | $e_5$ | $-e_2$ | $-e_3$ | $-1$ | $e_1$ | $e_{14}$ | $-e_{15}$ | $-e_{12}$ | $e_{13}$ | $e_{10}$ | $-e_{11}$ | $-e_8$ | $e_9$ |
| $e_7$ | $e_7$ | $e_6$ | $-e_5$ | $e_4$ | $-e_3$ | $e_2$ | $-e_1$ | $-1$ | $e_{15}$ | $e_{14}$ | $-e_{13}$ | $-e_{12}$ | $e_{11}$ | $e_{10}$ | $-e_9$ | $-e_8$ |
| $e_8$ | $e_8$ | $-e_9$ | $-e_{10}$ | $-e_{11}$ | $-e_{12}$ | $-e_{13}$ | $-e_{14}$ | $-e_{15}$ | $-1$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
| $e_9$ | $e_9$ | $e_8$ | $e_{11}$ | $-e_{10}$ | $-e_{13}$ | $e_{12}$ | $e_{15}$ | $-e_{14}$ | $-e_1$ | $-1$ | $e_3$ | $-e_2$ | $-e_5$ | $e_4$ | $e_7$ | $-e_6$ |
| $e_{10}$ | $e_{10}$ | $-e_{11}$ | $e_8$ | $e_9$ | $-e_{14}$ | $-e_{15}$ | $e_{12}$ | $e_{13}$ | $-e_2$ | $-e_3$ | $-1$ | $e_1$ | $-e_6$ | $-e_7$ | $e_4$ | $e_5$ |
| $e_{11}$ | $e_{11}$ | $e_{10}$ | $-e_9$ | $e_8$ | $-e_{15}$ | $e_{14}$ | $-e_{13}$ | $e_{12}$ | $-e_3$ | $e_2$ | $-e_1$ | $-1$ | $-e_7$ | $e_6$ | $-e_5$ | $e_4$ |
| $e_{12}$ | $e_{12}$ | $-e_{13}$ | $-e_{14}$ | $-e_{15}$ | $e_8$ | $-e_9$ | $-e_{10}$ | $-e_{11}$ | $-e_4$ | $e_5$ | $e_6$ | $e_7$ | $-1$ | $e_1$ | $e_2$ | $e_3$ |
| $e_{13}$ | $e_{13}$ | $e_{12}$ | $e_{15}$ | $-e_{14}$ | $e_9$ | $e_8$ | $e_{11}$ | $-e_{10}$ | $-e_5$ | $-e_4$ | $e_7$ | $-e_6$ | $-e_1$ | $-1$ | $e_3$ | $-e_2$ |
| $e_{14}$ | $e_{14}$ | $-e_{15}$ | $e_{12}$ | $e_{13}$ | $e_{10}$ | $-e_{11}$ | $e_8$ | $e_9$ | $-e_6$ | $-e_7$ | $-e_4$ | $e_5$ | $-e_2$ | $-e_3$ | $-1$ | $e_1$ |
| $e_{15}$ | $e_{15}$ | $e_{14}$ | $-e_{13}$ | $e_{12}$ | $e_{11}$ | $e_{10}$ | $-e_9$ | $e_8$ | $-e_7$ | $e_6$ | $-e_5$ | $-e_4$ | $-e_3$ | $e_2$ | $-e_1$ | $-1$ |

The algebra of dual numbers is a ring that is a local ring since the principal ideal generated by $\varepsilon$ is its only maximal ideal. Dual numbers form the coefficients of dual quaternions. Using matrices, dual numbers can be represented as

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; a + b\varepsilon = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

The sum and product of dual numbers are then calculated with ordinary matrix addition and matrix multiplication; both operations are commutative and associative within the algebra of dual numbers.

For the field $\mathcal{S}$ of sedenions, we give hyperholomorphic functions and a harmonic function with dual sedenion variables on $\mathcal{S} \times \mathcal{S}$ and research properties of functions of dual sedenion variables.

The element $e_0$ is the identity of $\mathcal{S}$ and $e_1$ identifies the imaginary unit $\sqrt{-1}$ in the $\mathbb{C}$-field of complex numbers. A sedenion $z$ is given by

$$(2.1) \qquad z = x_0 + \sum_{i=0}^{15} e_i x_i, \; (x_i \in \mathbb{R}, \; i = 0, 1, \cdots, 15)$$

is an sixteen dimensional non-commutative and non-associative $\mathbb{R}$-field generated by sixteen base elements $e_l (l = 0, 1, ..., 15)$ with the following non-commutative multiplication rules:

$$e_i^2 = -1, \; e_i e_j = -e_j e_i, \; e_i e_j e_k = e_i(e_j e_k) \; (i \neq j \neq k, i \neq 0, j \neq 0, k \neq 0).$$

An sedenion $z$ given by (2.1) is regarded as $z = z_1 + z_2 e_2 + z_3 e_4 + z_4 e_6 + z_5 e_8 + z_6 e_{10} + z_7 e_{12} + z_8 e_{14} \in \mathcal{S}$, where $z_1 := x_0 + e_1 x_1$, $z_2 := x_2 + e_1 x_3$, $z_3 := x_4 + e_1 x_5$, $z_4 := x_6 + e_1 x_7$, $z_5 := x_8 + e_1 x_9$, $z_6 := x_{10} + e_1 x_{11}$, $z_7 := x_{12} + e_1 x_{13}$ and $z_8 := x_{14} + e_1 x_{15}$ are complex numbers in $\mathcal{S}$. Thus, we identify $\mathcal{S}$ with $\mathbb{C}^8$.

We write the sedenion $z = \sum_{l=0}^{15} e_l x_l$ and the sedenion conjugate $z^* = x_0 - \sum_{l=1}^{15} e_l x_l$. Also, the absolute value $|z|$ of $z$ and an inverse $z^{-1}$ of $z$ in $\mathcal{S}$ are defined by

$$|z| = \sqrt{\sum_{l=1}^{8} |z_l|^2}, \qquad z^{-1} = \frac{z^*}{|z|^2} \; (z \neq 0).$$

Thus, the sedenion $z \in \mathcal{S}$ have the following forms:

$$\begin{aligned} z \; &= \; \sum_{l=0}^{15} e_l x_l = z_1 + z_2 e_2 + z_3 e_4 + z_4 e_6 + z_5 e_8 + z_6 e_{10} + z_7 e_{12} + z_8 e_{14} \\ &= \; Z_1 + Z_2 e_4 + Z_3 e_8 + Z_4 e_{12} = P_1 + P_2 e_8 \in \mathcal{S} \end{aligned}$$

and

$$
\begin{aligned}
z^* &= x_0 - \sum_{l=1}^{15} e_l x_l \\
&= \overline{z_1} - z_2 e_2 - z_3 e_4 - z_4 e_6 - z_5 e_8 - z_6 e_{10} - z_7 e_{12} - z_8 e_{14} \\
&= Z_1^* - Z_2 e_4 - Z_3 e_8 - Z_4 e_{12} = P_1^* - P_2 e_8 \in \mathcal{S},
\end{aligned}
$$

where $Z_1 = z_1 + z_2 e_2$, $Z_2 = z_3 + z_4 e_2$, $Z_3 = z_5 + z_6 e_2$ and $Z_4 = z_7 + z_8 e_2$ are quaternion numbers in $\mathbb{C}^2$, $P_1 = Z_1 + Z_2 e_4$ and $P_2 = Z_3 + Z_4 e_4$ are octonion numbers in $\mathbb{C}^4$.

We use the following differential operators:

$$
\begin{aligned}
\frac{\partial}{\partial Z_1} &:= \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial z_2}, & \frac{\partial}{\partial Z_1^*} &= \frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial z_2}, \\
\frac{\partial}{\partial Z_2} &:= \frac{\partial}{\partial z_3} - e_2 \frac{\partial}{\partial z_4}, & \frac{\partial}{\partial Z_2^*} &= \frac{\partial}{\partial \overline{z_3}} + e_2 \frac{\partial}{\partial z_4}, \\
\frac{\partial}{\partial Z_3} &:= \frac{\partial}{\partial z_5} - e_2 \frac{\partial}{\partial z_6}, & \frac{\partial}{\partial Z_3^*} &= \frac{\partial}{\partial \overline{z_5}} + e_2 \frac{\partial}{\partial z_6}, \\
\frac{\partial}{\partial Z_4} &:= \frac{\partial}{\partial z_7} - e_2 \frac{\partial}{\partial z_8}, & \frac{\partial}{\partial Z_4^*} &= \frac{\partial}{\partial \overline{z_7}} + e_2 \frac{\partial}{\partial z_8},
\end{aligned}
$$

where $\partial/\partial z_l$, $\partial/\partial \overline{z_l}$ $(l = 1, 2, ..., 8)$ are usual differential operators used in complex analysis. And, we use the following differential operators:

$$
\begin{aligned}
\frac{\partial}{\partial P_1} &:= \frac{\partial}{\partial Z_1} - e_4 \frac{\partial}{\partial Z_2}, & \frac{\partial}{\partial P_1^*} &= \frac{\partial}{\partial Z_1^*} + e_4 \frac{\partial}{\partial Z_2}, \\
\frac{\partial}{\partial P_2} &:= \frac{\partial}{\partial Z_3} - e_4 \frac{\partial}{\partial Z_4}, & \frac{\partial}{\partial P_2^*} &= \frac{\partial}{\partial Z_3^*} + e_4 \frac{\partial}{\partial Z_4}.
\end{aligned}
$$

Also, we use the following sedenion differential operators:

$$
D := \frac{\partial}{\partial P_1} - e_8 \frac{\partial}{\partial P_2}, \qquad D^* = \frac{\partial}{\partial P_1^*} + e_8 \frac{\partial}{\partial P_2}.
$$

The operator

$$
DD^* = \sum_{l=1}^{8} \frac{\partial^2}{\partial z_l \partial \overline{z_l}} = \frac{1}{4} \sum_{l=0}^{15} \frac{\partial^2}{\partial x_l^2}
$$

is the usual complex Laplacian $\Delta$.

## 3.   Regular function with sedenion variables

Let $\Omega = \Omega_1 \times \Omega_2$ be an open set in $\mathcal{S} \times \mathcal{S}$. The function $f(z)$ is defined by the following form in $\Omega$ with valued in $\mathcal{S}$:

$$f : \Omega \ \rightarrow \ \mathcal{S}, \ (z,w) \in \Omega \ \mapsto \ f(z,w) = \phi_1(z,w) + \phi_2(z,w)e_8.$$

In detail,

$$f(z,w) \ = \ \sum_{l=0}^{15} u_l e_l = \sum_{m=1}^{8} f_m e_{2ml} = \sum_{n=0}^{3} F_n e_{4n} = \phi_1 + \phi_2 e_8 \in \mathcal{S},$$

where $u_l$ ($l = 0, 1, \cdots, 15$) are real-valued functions, $f_m = u_{2m-2} + u_{2m-1}e_1$ ($m = 1, 2, \cdots, 8$) are complex-valued functions, $F_n = f_{2n-1} + f_{2n}e_2$ ($n = 1, 2, 3, 4$) are quaternion-valued functions, $\phi_1 = F_1 + F_2 e_4$ and $\phi_2 = F_3 + F_4 e_4$ are octonion-valued functions in $\mathcal{S}$.

We use the following differential operators:

$$\frac{\partial}{\partial Z_1} := \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial z_2}, \qquad \frac{\partial}{\partial Z_1^*} = \frac{\partial}{\partial \overline{z_1}} + e_2 \frac{\partial}{\partial z_2},$$

$$\frac{\partial}{\partial Z_2} := \frac{\partial}{\partial z_3} - e_2 \frac{\partial}{\partial z_4}, \qquad \frac{\partial}{\partial Z_2^*} = \frac{\partial}{\partial \overline{z_3}} + e_2 \frac{\partial}{\partial z_4},$$

$$\frac{\partial}{\partial Z_3} := \frac{\partial}{\partial z_5} - e_2 \frac{\partial}{\partial z_6}, \qquad \frac{\partial}{\partial Z_3^*} = \frac{\partial}{\partial \overline{z_5}} + e_2 \frac{\partial}{\partial z_6},$$

$$\frac{\partial}{\partial Z_4} := \frac{\partial}{\partial z_7} - e_2 \frac{\partial}{\partial z_8}, \qquad \frac{\partial}{\partial Z_4^*} = \frac{\partial}{\partial \overline{z_7}} + e_2 \frac{\partial}{\partial z_8},$$

$$\frac{\partial}{\partial W_1} := \frac{\partial}{\partial w_1} - e_2 \frac{\partial}{\partial w_2}, \qquad \frac{\partial}{\partial W_1^*} = \frac{\partial}{\partial \overline{w_1}} + e_2 \frac{\partial}{\partial w_2},$$

$$\frac{\partial}{\partial W_2} := \frac{\partial}{\partial w_3} - e_2 \frac{\partial}{\partial w_4}, \qquad \frac{\partial}{\partial W_2^*} = \frac{\partial}{\partial \overline{w_3}} + e_2 \frac{\partial}{\partial w_4},$$

$$\frac{\partial}{\partial W_3} := \frac{\partial}{\partial w_5} - e_2 \frac{\partial}{\partial w_6}, \qquad \frac{\partial}{\partial W_3^*} = \frac{\partial}{\partial \overline{w_5}} + e_2 \frac{\partial}{\partial w_6},$$

$$\frac{\partial}{\partial W_4} := \frac{\partial}{\partial w_7} - e_2 \frac{\partial}{\partial w_8}, \qquad \frac{\partial}{\partial Z_4^*} = \frac{\partial}{\partial \overline{w_7}} + e_2 \frac{\partial}{\partial w_8},$$

where $\partial/\partial z_l$, $\partial/\partial \overline{z_l}$ ($l = 1, 2, ..., 8$) are usual differential operators used in complex analysis. And, we use the following differential operators:

$$\frac{\partial}{\partial P_1} := \frac{\partial}{\partial Z_1} - e_4 \frac{\partial}{\partial Z_2}, \qquad \frac{\partial}{\partial P_1^*} = \frac{\partial}{\partial Z_1^*} + e_4 \frac{\partial}{\partial Z_2},$$

$$\frac{\partial}{\partial P_2} := \frac{\partial}{\partial Z_3} - e_4 \frac{\partial}{\partial Z_4}, \qquad \frac{\partial}{\partial P_2^*} = \frac{\partial}{\partial Z_3^*} + e_4 \frac{\partial}{\partial Z_4},$$

$$\frac{\partial}{\partial Q_1} := \frac{\partial}{\partial W_1} - e_4 \frac{\partial}{\partial W_2}, \qquad \frac{\partial}{\partial Q_1^*} = \frac{\partial}{\partial W_1^*} + e_4 \frac{\partial}{\partial W_2},$$

$$\frac{\partial}{\partial Q_2} := \frac{\partial}{\partial W_3} - e_4 \frac{\partial}{\partial W_4}, \qquad \frac{\partial}{\partial Q_2^*} = \frac{\partial}{\partial W_3^*} + e_4 \frac{\partial}{\partial W_4}.$$

Also, we use the following sedenion differential operators:

$$D := \frac{\partial}{\partial P_1} - e_8 \frac{\partial}{\partial P_2}, \qquad D^* = \frac{\partial}{\partial P_1^*} + e_8 \frac{\partial}{\partial P_2}.$$

$$E := \frac{\partial}{\partial Q_1} - e_8 \frac{\partial}{\partial Q_2}, \qquad E^* = \frac{\partial}{\partial Q_1^*} + e_8 \frac{\partial}{\partial Q_2}.$$

$$H := D + \varepsilon E, \qquad\qquad H^* := D^* + \varepsilon E^*.$$

**Definition 3.1.** *Let $\Omega$ be an open set in $\mathcal{S} \times \mathcal{S}$. A function $f(z, w) = \phi_1(z, w) + \phi_2(z, w)e_8$ is said to be bihyperholomorphic in $\Omega$ if the following two conditions are satisfied:*

*(1) $\phi_r(z, w)$ $(r = 1, 2)$ are continuously differential functions in $\Omega$.*

*(2) The function $f$ satisfies*

$$(3.1) \qquad\qquad\qquad D^* f = 0, \ \ f E^* = 0$$

*in $\Omega$.*

**Theorem 3.1.** *Let $G$ and $T$ be functions with values in sedenions of class $\mathcal{C}^\infty$ on $\Omega$ in $\mathcal{S} \times \mathcal{S}$. Then the Cousin 1 problem according to sedenion-valued functions on $\Omega$ has a solution.*

*Proof.* Consider the inhomogeneous Cauchy-Riemann system of partial differential equations

$$(3.2) \qquad\qquad\qquad D^* f = G, \qquad f E^* = T.$$

If the system (3.1) has a solution $f$ with values in sedenions, then

$$GE^* = D^* f E^* = D^* T.$$

For the system (3.2), the solvability condition

$$(3.3) \qquad\qquad\qquad GE^* = D^* T$$

is necessary for the existence of a solution $f$ to the system (3.1). Let $\Omega$ be a product domain $\Omega_1 \times \Omega_2$ of simply connected domains in $\mathcal{S}$ and $\mathcal{S}$, respectively, and let

$$Rf := (D^*f, fE^*).$$

Then $Rf = (G, T) = 0$ satisfies (3.3). For $Rf : \Omega \rightarrow \mathcal{S}$ of class $\mathcal{C}^2$, if

$$A(G, T) = GE^* - D^*T,$$

then the Cousin 1 problem according to sedenion-valued functions on $\Omega$ has a solution.    $\square$

Let $\Omega = \Omega_1 \times \Omega_2$ be an open set in $\mathcal{S} \times \mathcal{S}$. The function $F(\zeta)$ is defined by the following form in $\Omega$ with valued in $\mathcal{S} \times \mathcal{S}$:

$$F : \Omega \rightarrow \mathcal{S} \times \mathcal{S},$$
$$\zeta = z + \varepsilon w \in \Omega \mapsto F(\zeta) = F(z + \varepsilon w) = \psi_1(z, w) + \varepsilon\psi_2(z, w)$$
$$= (\phi_1(z, w) + \phi_2(z, w)e_8) + \varepsilon(\varphi_1(z, w) + \varphi_2(z, w)e_8),$$

where

$$\psi_2(z, w) = \sum_{l=0}^{15} v_l e_l = \sum_{m=1}^{8} g_m e_{2m-2} = \sum_{n=1}^{4} G_n e_{4n-4} = \varphi_1 + \varphi_2 e_8 \in \mathcal{S}$$

with $v_l$ $(l = 0, 1, \cdots, 15)$ are real-valued functions, $g_m = v_{2m-2} + v_{2m-1}e_1$ $(m = 1, 2, \cdots, 8)$ are complex-valued functions, $G_n = g_{2n-1} + g_{2n}e_2$ $(n = 1, 2, 3, 4)$ are quaternion-valued functions, $\varphi_1 = G_1 + G_2 e_4$ and $\varphi_2 = G_3 + G_4 e_4$ are octonion-valued functions in $\mathcal{S}$.

**Definition 3.2.** *Let $\Omega$ be an open set in $\mathcal{S} \times \mathcal{S}$. A function $F(\zeta)$ is said to be L(R)-hyperholomorphic in $\Omega$ if the following two conditions are satisfied:*

*(1) $f_k(z)$, $g_k(w)$ $(k = 1, \cdots, 8)$ are continuously differential functions in $\Omega$.*

*(2)*

(3.4) $$H^*F = 0 \ (FH^* = 0) \ \text{in } \Omega.$$

When we deal with a $L$-hyperholomorphic function $F(\zeta)$ in $\Omega \subset \mathcal{S} \times \mathcal{S}$, for simplicity, we often say that $F(\zeta)$ is a hyperholomorphic function in $\Omega \subset \mathcal{S} \times \mathcal{S}$.

The Equation (3.4) is applied to $F(\zeta)$ as follows:

$$
\begin{aligned}
H^*F &= (D^* + \varepsilon E^*)(\psi_1(z,w) + \varepsilon\psi_2(z,w)) \\
&= (D^*\psi_1(z,w) + \varepsilon(E^*\psi_1(z,w) + D^*\psi_2(z,w))) \\
&= (D^*(F_1(z) + F_2(z)e_4 + F_3(z)e_8 + F_4(z)e_{12}) \\
&\quad + \varepsilon(E^*(F_1(z) + F_2(z)e_4 + F_3(z)e_8 + F_4(z)e_{12}) \\
&\quad + D^*(G_1(w) + G_2(w)e_4 + G_3(w)e_8 + G_4(w)e_{12}))) \\
&= \left(\frac{\partial f_1}{\partial P_1^*} - \frac{\partial \overline{f_5}}{\partial P_2^*}\right) + \left(\frac{\partial f_2}{\partial P_1^*} + \frac{\partial \overline{f_6}}{\partial P_2^*}\right)e_2 + \left(\frac{\partial f_3}{\partial P_1^*} + \frac{\partial \overline{f_7}}{\partial P_2^*}\right)e_4 \\
&\quad + \left(\frac{\partial f_4}{\partial P_1^*} + \frac{\partial \overline{f_8}}{\partial P_2^*}\right)e_6 + \left(\frac{\partial f_5}{\partial P_1^*} + \frac{\partial \overline{f_1}}{\partial P_2^*}\right)e_8 + \left(\frac{\partial f_6}{\partial P_1^*} - \frac{\partial \overline{f_2}}{\partial P_2^*}\right)e_{10} \\
&\quad + \left(\frac{\partial f_7}{\partial P_1^*} - \frac{\partial \overline{f_3}}{\partial P_2^*}\right)e_{12} + \left(\frac{\partial f_8}{\partial P_1^*} - \frac{\partial \overline{f_4}}{\partial P_2^*}\right)e_{14} \\
&\quad + \varepsilon\left(\left(\frac{\partial f_1}{\partial Q_1^*} - \frac{\partial \overline{f_5}}{\partial Q_2^*} + \frac{\partial g_1}{\partial P_1^*} - \frac{\partial \overline{g_5}}{\partial P_2^*}\right) + \left(\frac{\partial f_2}{\partial Q_1^*} + \frac{\partial \overline{f_6}}{\partial Q_2^*} + \frac{\partial g_2}{\partial P_1^*} + \frac{\partial \overline{g_6}}{\partial P_2^*}\right)e_2\right. \\
&\quad + \left(\frac{\partial f_3}{\partial Q_1^*} + \frac{\partial \overline{f_7}}{\partial Q_2^*} + \frac{\partial g_3}{\partial P_1^*} + \frac{\partial \overline{g_7}}{\partial P_2^*}\right)e_4 + \left(\frac{\partial f_4}{\partial Q_1^*} + \frac{\partial \overline{f_8}}{\partial Q_2^*} + \frac{\partial g_4}{\partial P_1^*} + \frac{\partial \overline{g_8}}{\partial P_2^*}\right)e_6 \\
&\quad + \left(\frac{\partial f_5}{\partial Q_1^*} + \frac{\partial \overline{f_1}}{\partial Q_2^*} + \frac{\partial g_5}{\partial P_1^*} + \frac{\partial \overline{g_1}}{\partial P_2^*}\right)e_8 + \left(\frac{\partial f_6}{\partial Q_1^*} - \frac{\partial \overline{f_2}}{\partial Q_2^*} + \frac{\partial g_6}{\partial P_1^*} - \frac{\partial \overline{g_2}}{\partial P_2^*}\right)e_{10} \\
&\quad \left. + \left(\frac{\partial f_7}{\partial Q_1^*} - \frac{\partial \overline{f_3}}{\partial Q_2^*} + \frac{\partial g_7}{\partial P_1^*} - \frac{\partial \overline{g_3}}{\partial P_2^*}\right)e_{12} + \left(\frac{\partial f_8}{\partial Q_1^*} - \frac{\partial \overline{f_4}}{\partial Q_2^*} + \frac{\partial g_8}{\partial P_1^*} - \frac{\partial \overline{g_4}}{\partial P_2^*}\right)e_{14}\right).
\end{aligned}
$$

If the following equations

$$
\frac{\partial f_1}{\partial P_1^*} = \frac{\partial \overline{f_5}}{\partial P_2^*}, \quad \frac{\partial f_2}{\partial P_1^*} = -\frac{\partial \overline{f_6}}{\partial P_2^*}, \quad \frac{\partial f_3}{\partial P_1^*} = -\frac{\partial \overline{f_7}}{\partial P_2^*}, \quad \frac{\partial f_4}{\partial P_1^*} = -\frac{\partial \overline{f_8}}{\partial P_2^*},
$$

$$
\frac{\partial f_5}{\partial P_1^*} = -\frac{\partial \overline{f_1}}{\partial P_2^*}, \quad \frac{\partial f_6}{\partial P_1^*} = \frac{\partial \overline{f_2}}{\partial P_2^*}, \quad \frac{\partial f_7}{\partial P_1^*} = \frac{\partial \overline{f_3}}{\partial P_2^*}, \quad \frac{\partial f_8}{\partial P_1^*} = \frac{\partial \overline{f_4}}{\partial P_2^*},
$$

$$
\frac{\partial g_1}{\partial P_1^*} + \frac{\partial f_1}{\partial Q_1^*} = \frac{\partial \overline{g_5}}{\partial P_2^*} + \frac{\partial \overline{f_5}}{\partial Q_2^*}, \quad \frac{\partial g_2}{\partial P_1^*} + \frac{\partial f_2}{\partial Q_1^*} = -\frac{\partial \overline{g_6}}{\partial P_2^*} - \frac{\partial \overline{f_6}}{\partial Q_2^*},
$$

$$
\frac{\partial g_3}{\partial P_1^*} + \frac{\partial f_3}{\partial Q_1^*} = -\frac{\partial \overline{g_7}}{\partial P_2^*} - \frac{\partial \overline{f_7}}{\partial Q_2^*}, \quad \frac{\partial g_4}{\partial P_1^*} + \frac{\partial f_4}{\partial Q_1^*} = -\frac{\partial \overline{g_8}}{\partial P_2^*} - \frac{\partial \overline{f_8}}{\partial Q_2^*},
$$

$$
\frac{\partial g_5}{\partial P_1^*} + \frac{\partial f_5}{\partial Q_1^*} = -\frac{\partial \overline{g_1}}{\partial P_2^*} - \frac{\partial \overline{f_1}}{\partial Q_2^*}, \quad \frac{\partial g_6}{\partial P_1^*} + \frac{\partial f_6}{\partial Q_1^*} = \frac{\partial \overline{g_2}}{\partial P_2^*} + \frac{\partial \overline{f_2}}{\partial Q_2^*},
$$

$$
(3.5) \; \frac{\partial g_7}{\partial P_1^*} + \frac{\partial f_7}{\partial Q_1^*} = \frac{\partial \overline{g_3}}{\partial P_2^*} + \frac{\partial \overline{f_3}}{\partial Q_2^*}, \quad \frac{\partial g_8}{\partial P_1^*} + \frac{\partial f_8}{\partial Q_1^*} = \frac{\partial \overline{g_4}}{\partial P_2^*} + \frac{\partial \overline{f_4}}{\partial Q_2^*},
$$

are satisfied, the function $F(\zeta)$ is a hyperholomorphic function on $\Omega$. These are the corresponding s-Cauchy-Riemann equations on $\mathcal{S} \times \mathcal{S}$.

**Definition 3.3.** *Let $\Omega$ be and open set in $\mathcal{S} \times \mathcal{S}$ and let*

$$F = (f_1, f_2, \cdots, f_8, g_1, g_2, \cdots, g_8) : \Omega \;\rightarrow\; \mathcal{S} \times \mathcal{S}.$$

*This mapping is said to be harmonic if all its components $f_j$ and $g_j$ $(j = 1, 2, \cdots, 8)$ of $F$ are harmonic on $\Omega$. The system (3.5) is called a generalized Cauchy-Riemann system if every solution $F(\zeta)$ has only harmonic components $f_j$ and $g_j$ $(j = 1, 2, \cdots, 8)$.*

**Remark 3.1.** *We redefine the Equations (3.5) as follows:*

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_5}}{\partial \overline{z_5}}, \quad \frac{\partial f_1}{\partial z_2} = \frac{\partial \overline{f_5}}{\partial z_6}, \quad \frac{\partial f_1}{\partial z_3} = \frac{\partial \overline{f_5}}{\partial z_7}, \quad \frac{\partial f_1}{\partial z_4} = \frac{\partial \overline{f_5}}{\partial z_8},$$

$$\frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial \overline{f_6}}{\partial \overline{z_5}}, \quad \frac{\partial f_2}{\partial z_2} = -\frac{\partial \overline{f_6}}{\partial z_6}, \quad \frac{\partial f_2}{\partial z_3} = -\frac{\partial \overline{f_6}}{\partial z_7}, \quad \frac{\partial f_2}{\partial z_4} = -\frac{\partial \overline{f_6}}{\partial z_8},$$

$$\frac{\partial f_3}{\partial \overline{z_1}} = -\frac{\partial \overline{f_7}}{\partial \overline{z_5}}, \quad \frac{\partial f_3}{\partial z_2} = -\frac{\partial \overline{f_7}}{\partial z_6}, \quad \frac{\partial f_3}{\partial z_3} = -\frac{\partial \overline{f_7}}{\partial z_7}, \quad \frac{\partial f_3}{\partial z_4} = -\frac{\partial \overline{f_7}}{\partial z_8},$$

$$\frac{\partial f_4}{\partial \overline{z_1}} = -\frac{\partial \overline{f_8}}{\partial \overline{z_5}}, \quad \frac{\partial f_4}{\partial z_2} = -\frac{\partial \overline{f_8}}{\partial z_6}, \quad \frac{\partial f_4}{\partial z_3} = -\frac{\partial \overline{f_8}}{\partial z_7}, \quad \frac{\partial f_4}{\partial z_4} = \frac{\partial \overline{f_8}}{\partial z_8},$$

$$\frac{\partial f_5}{\partial \overline{z_1}} = -\frac{\partial \overline{f_1}}{\partial \overline{z_5}}, \quad \frac{\partial f_5}{\partial z_2} = -\frac{\partial \overline{f_1}}{\partial z_6}, \quad \frac{\partial f_5}{\partial z_3} = -\frac{\partial \overline{f_1}}{\partial z_7}, \quad \frac{\partial f_5}{\partial z_4} = -\frac{\partial \overline{f_1}}{\partial z_8},$$

$$\frac{\partial f_6}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial \overline{z_5}}, \quad \frac{\partial f_6}{\partial z_2} = \frac{\partial \overline{f_2}}{\partial z_6}, \quad \frac{\partial f_6}{\partial z_3} = \frac{\partial \overline{f_2}}{\partial z_7}, \quad \frac{\partial f_6}{\partial z_4} = \frac{\partial \overline{f_2}}{\partial z_8},$$

$$\frac{\partial f_7}{\partial \overline{z_1}} = \frac{\partial \overline{f_3}}{\partial \overline{z_5}}, \quad \frac{\partial f_7}{\partial z_2} = \frac{\partial \overline{f_3}}{\partial z_6}, \quad \frac{\partial f_7}{\partial z_3} = \frac{\partial \overline{f_3}}{\partial z_7}, \quad \frac{\partial f_7}{\partial z_4} = \frac{\partial \overline{f_3}}{\partial z_8},$$

$$(3.6) \quad \frac{\partial f_8}{\partial \overline{z_1}} = \frac{\partial \overline{f_4}}{\partial \overline{z_5}}, \quad \frac{\partial f_8}{\partial z_2} = \frac{\partial \overline{f_4}}{\partial z_6}, \quad \frac{\partial f_8}{\partial z_3} = \frac{\partial \overline{f_4}}{\partial z_7}, \quad \frac{\partial f_8}{\partial z_4} = \frac{\partial \overline{f_4}}{\partial z_8}.$$

We call that the Equations (3.6) are the condition of harmonicity of the hyperholomorphic function $F(\zeta)$ on $\Omega$ in $\mathcal{S} \times \mathcal{S}$.

**Remark 3.2.** *We redefine the Equations (3.5) as follows:*

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_3}}{\partial z_3}, \quad \frac{\partial f_1}{\partial \overline{z_2}} = -\frac{\partial \overline{f_3}}{\partial \overline{z_4}}, \quad \frac{\partial f_3}{\partial \overline{z_1}} = -\frac{\partial \overline{f_1}}{\partial z_3}, \quad \frac{\partial f_3}{\partial \overline{z_2}} = \frac{\partial \overline{f_1}}{\partial \overline{z_4}},$$

$$\frac{\partial f_2}{\partial \overline{z_1}} = \frac{\partial \overline{f_4}}{\partial z_3}, \quad \frac{\partial f_2}{\partial \overline{z_2}} = -\frac{\partial \overline{f_4}}{\partial \overline{z_4}}, \quad \frac{\partial f_4}{\partial \overline{z_1}} = -\frac{\partial \overline{f_2}}{\partial z_3}, \quad \frac{\partial f_4}{\partial \overline{z_2}} = \frac{\partial \overline{f_2}}{\partial \overline{z_4}},$$

$$\frac{\partial g_1}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{w_1}} = \frac{\partial \overline{g_3}}{\partial z_3} + \frac{\partial \overline{f_3}}{\partial w_3}, \quad \frac{\partial g_1}{\partial \overline{z_2}} + \frac{\partial f_1}{\partial \overline{w_2}} = -\frac{\partial \overline{g_3}}{\partial z_4} - \frac{\partial \overline{f_3}}{\partial \overline{w_4}},$$

$$\frac{\partial g_3}{\partial \overline{z_1}} + \frac{\partial f_3}{\partial \overline{w_1}} = -\frac{\partial \overline{g_1}}{\partial z_3} - \frac{\partial \overline{f_1}}{\partial w_3}, \quad \frac{\partial g_3}{\partial \overline{z_2}} + \frac{\partial f_3}{\partial \overline{w_2}} = \frac{\partial \overline{g_1}}{\partial z_4} + \frac{\partial \overline{f_1}}{\partial w_4},$$

$$\frac{\partial g_2}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial \overline{w_1}} = \frac{\partial \overline{g_4}}{\partial z_3} + \frac{\partial \overline{f_4}}{\partial w_3}, \quad \frac{\partial g_2}{\partial \overline{z_2}} + \frac{\partial f_2}{\partial \overline{w_2}} = -\frac{\partial \overline{g_4}}{\partial z_4} - \frac{\partial \overline{f_4}}{\partial w_4},$$

$$\frac{\partial g_4}{\partial \overline{z_1}} + \frac{\partial f_4}{\partial \overline{w_1}} = -\frac{\partial \overline{g_2}}{\partial z_3} - \frac{\partial \overline{f_2}}{\partial w_3}, \quad \frac{\partial g_4}{\partial \overline{z_2}} + \frac{\partial f_4}{\partial \overline{w_2}} = \frac{\partial \overline{g_2}}{\partial z_4} + \frac{\partial \overline{f_2}}{\partial w_4}.$$

Now, we need the following:

$$(3.7) \quad \begin{aligned} &\frac{\partial g_1}{\partial \overline{z_1}} = \frac{\partial \overline{g_3}}{\partial z_3}, \quad \frac{\partial g_1}{\partial \overline{z_2}} = -\frac{\partial \overline{g_3}}{\partial z_4}, \quad \frac{\partial g_3}{\partial \overline{z_1}} = -\frac{\partial \overline{g_1}}{\partial z_3}, \quad \frac{\partial g_3}{\partial \overline{z_2}} = \frac{\partial \overline{g_1}}{\partial z_4}, \\ &\frac{\partial g_2}{\partial \overline{z_1}} = \frac{\partial \overline{g_4}}{\partial z_3}, \quad \frac{\partial g_2}{\partial \overline{z_2}} = -\frac{\partial \overline{g_4}}{\partial z_4}, \quad \frac{\partial g_4}{\partial \overline{z_1}} = -\frac{\partial \overline{g_2}}{\partial z_3}, \quad \frac{\partial g_4}{\partial \overline{z_2}} = \frac{\partial \overline{g_2}}{\partial z_4}. \end{aligned}$$

We call that the Equations (3.7) and (3.7) are the condition of harmonicity.

**Theorem 3.2.** *If the hyperholomorphic function*

$$\begin{aligned} F(\zeta) \;=\; & f_1(z) + f_2(z)e_2 + f_3(z)e_4 + f_4(z)e_6 + f_5(z)e_8 + f_6(z)e_{10} \\ & + f_7(z)e_{12} + f_8(z)e_{14} + \varepsilon(g_1(w) + g_2(w)e_2 + g_3(w)e_4 \\ & + g_4(w)e_6 + g_5(w)e_8 + g_6(w)e_{10} + g_7(w)e_{12} + g_8(w)e_{14}) \end{aligned}$$

*satisfies the condition of harmonicity (3.6) and (3.7) in an open set $\Omega$ in $\mathcal{S} \times \mathcal{S}$, then the functions $f_k(z)$, $g_k(w)$ ($k = 1, 2, \cdots, 8$) are harmonic in $\Omega$.*

*Proof.* We have

$$\begin{aligned} HH^* f_1 \;=\; & (D + \varepsilon E)(D^* + \varepsilon E^*)f_1 = (DD^* + \varepsilon(DE^* + D^*E)f_1 \\ =\; & \frac{\partial^2 f_1}{\partial z_1 \partial \overline{z_1}} + \frac{\partial^2 f_1}{\partial z_2 \partial \overline{z_2}} + \frac{\partial^2 f_1}{\partial z_3 \partial \overline{z_3}} + \frac{\partial^2 f_1}{\partial z_4 \partial \overline{z_4}} + \frac{\partial^2 f_1}{\partial z_5 \partial \overline{z_5}} + \frac{\partial^2 f_1}{\partial z_6 \partial \overline{z_6}} \\ & + \frac{\partial^2 f_1}{\partial z_7 \partial \overline{z_7}} + \frac{\partial^2 f_1}{\partial z_8 \partial \overline{z_8}} + \varepsilon\Big(\frac{\partial^2 f_1}{\partial z_1 \partial \overline{w_1}} + \frac{\partial^2 f_1}{\partial z_2 \partial \overline{w_2}} + \frac{\partial^2 f_1}{\partial z_3 \partial \overline{w_3}} + \frac{\partial^2 f_1}{\partial z_4 \partial \overline{w_4}} \\ & + \frac{\partial^2 f_1}{\partial z_5 \partial \overline{w_5}} + \frac{\partial^2 f_1}{\partial z_6 \partial \overline{w_6}} + \frac{\partial^2 f_1}{\partial z_7 \partial \overline{w_7}} + \frac{\partial^2 f_1}{\partial z_8 \partial \overline{w_8}} + \frac{\partial^2 f_1}{\partial \overline{z_1} \partial w_1} + \frac{\partial^2 f_1}{\partial \overline{z_2} \partial w_2} \\ & + \frac{\partial^2 f_1}{\partial \overline{z_3} \partial w_3} + \frac{\partial^2 f_1}{\partial \overline{z_4} \partial w_4} + \frac{\partial^2 f_1}{\partial \overline{z_5} \partial w_5} + \frac{\partial^2 f_1}{\partial \overline{z_6} \partial w_6} + \frac{\partial^2 f_1}{\partial \overline{z_7} \partial w_7} + \frac{\partial^2 f_1}{\partial \overline{z_8} \partial w_8}\Big) \end{aligned}$$

$$
\begin{aligned}
= \ & \frac{\partial}{\partial z_1}\Big(\frac{\partial \overline{f_5}}{\partial \overline{z_5}}\Big) + \frac{\partial}{\partial \overline{z_2}}\Big(\frac{\partial \overline{f_5}}{\partial z_6}\Big) + \frac{\partial}{\partial \overline{z_3}}\Big(\frac{\partial \overline{f_5}}{\partial z_7}\Big) + \frac{\partial}{\partial \overline{z_4}}\Big(\frac{\partial \overline{f_5}}{\partial z_8}\Big) \\
& + \frac{\partial}{\partial \overline{z_5}}\Big(-\frac{\partial \overline{f_5}}{\partial z_1}\Big) + \frac{\partial}{\partial z_6}\Big(-\frac{\partial \overline{f_5}}{\partial \overline{z_2}}\Big) + \frac{\partial}{\partial z_7}\Big(-\frac{\partial \overline{f_5}}{\partial \overline{z_3}}\Big) + \frac{\partial}{\partial z_8}\Big(-\frac{\partial \overline{f_5}}{\partial \overline{z_4}}\Big) \\
& + \varepsilon\Big(\Big(\frac{\partial}{\partial z_1}\Big(\frac{\partial \overline{f_5}}{\partial \overline{w_5}}\Big) + \frac{\partial}{\partial \overline{z_2}}\Big(\frac{\partial \overline{f_5}}{\partial w_6}\Big) + \frac{\partial}{\partial \overline{z_3}}\Big(\frac{\partial \overline{f_5}}{\partial w_7}\Big) + \frac{\partial}{\partial \overline{z_4}}\Big(\frac{\partial \overline{f_5}}{\partial w_8}\Big) \\
& + \frac{\partial}{\partial \overline{z_5}}\Big(-\frac{\partial \overline{f_5}}{\partial w_1}\Big) + \frac{\partial}{\partial z_6}\Big(-\frac{\partial \overline{f_5}}{\partial \overline{w_2}}\Big) + \frac{\partial}{\partial z_7}\Big(-\frac{\partial \overline{f_5}}{\partial \overline{w_3}}\Big) + \frac{\partial}{\partial z_8}\Big(-\frac{\partial \overline{f_5}}{\partial \overline{w_4}}\Big)\Big) \\
& + \Big(\frac{\partial}{\partial w_1}\Big(\frac{\partial \overline{f_5}}{\partial \overline{z_5}}\Big) + \frac{\partial}{\partial \overline{w_2}}\Big(\frac{\partial \overline{f_5}}{\partial z_6}\Big) + \frac{\partial}{\partial \overline{w_3}}\Big(\frac{\partial \overline{f_5}}{\partial z_7}\Big) + \frac{\partial}{\partial \overline{w_4}}\Big(\frac{\partial \overline{f_5}}{\partial z_8}\Big) \\
& + \frac{\partial}{\partial \overline{w_5}}\Big(-\frac{\partial \overline{f_5}}{\partial z_1}\Big) + \frac{\partial}{\partial w_6}\Big(-\frac{\partial \overline{f_5}}{\partial \overline{z_2}}\Big) + \frac{\partial}{\partial w_7}\Big(-\frac{\partial \overline{f_5}}{\partial \overline{z_3}}\Big) + \frac{\partial}{\partial w_8}\Big(-\frac{\partial \overline{f_5}}{\partial \overline{z_4}}\Big)\Big) \\
= \ & 0,
\end{aligned}
$$

and the functions $f_k(z)$, $g_k(w)$ $(k = 1, 2, \cdots, 8)$ are proved by the similar method as in the proof of the case of $f_1$. $\qquad\square$

**Proposition 3.1.** *Let $\Omega$ be a domain in $\mathcal{S} \times \mathcal{S}$ and $F(\zeta)$ be a L-hyperholomorphic functions in $\Omega$. If $HH^*F(\zeta) = 0$, then the system (3.5) is a generalized Cauchy-Riemann system in $\Omega$.*

*Proof.* If $HH^*F = 0$, then $\sum_{j=1}^{8}\big(\frac{\partial^2 f}{\partial z_j \overline{z_j}} + \frac{\partial^2 f}{\partial w_j \overline{w_j}}\big) = 0$. Therefore, the components $f_j$, $g_j$ $(j = 1, 2, \cdots, 8)$ are harmonic in $\Omega$. That is, the system (3.5) is a generalized Cauchy-Riemann system on $\Omega$. $\qquad\square$

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