VU Autonomous Racing Cars (2020S) **(Due Date: 05.05.2020)**

Lab 5: Scan Matching

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Course Policy: Read all the instructions below carefully before you start working on the assignment, and before you make a submission. All sources of material must be cited. The University Academic Code of Conduct will be strictly enforced.

THIS IS A GROUP ASSIGNMENT. Submit one from each team.

1 Theoretical Questions

1. $M_i = \begin{pmatrix} 1 & 0 & p_{i0} & -p_{i1} \\ 0 & 1 & p_{i1} & p_{i0} \end{pmatrix}$

   (a) Show that $B_i := M_i^T M_i$ is symmetric.

   Solution:

   If we calculate $B_i = M_i^T \cdot M_i$ we get the shown result for B_i.

$$B_i = \begin{bmatrix} 1 & 0 & p_{i0} & -p_{i1} \\ 0 & 1 & p_{i1} & p_{i0} \\ p_{i0} & p_{i1} & p_{i0}^2 + p_{i1}^2 & 0 \\ -p_{i1} & p_{i0} & 0 & p_{i0}^2 + p_{i1}^2 \end{bmatrix}$$

   As it can be seen, the matrix is symmetric, because all the elements are mirrored at the principal diagonal. So the assignment $B_i = B_i^T$ is correct.

(b) Demonstrate that B_i is positive semi-definite.

Solution:

To show that the matrix B_i is positive semi-definite, all of its eigenvalues have to be greater or equal 0. So we first have to calculate the eigenvalues and then check each of them, if it satisfies the condition.

$$Eigenvalues_{B_i} = \begin{bmatrix} 0 \\ 0 \\ p_{i0}^2 + p_{i1}^2 + 1 \\ p_{i0}^2 + p_{i1}^2 + 1 \end{bmatrix}$$

As each of the quadratic terms is bigger than 0, the sum of them is also bigger than 0, so we can directly see, that all the eigenvalues are greater or equal 0 and so we showed, that B_i is positive semi-definite.

2. The following is the optimization problem:

$$x^* = \text{argmin}_{x \in \mathbb{R}^4} \sum_{i=1}^{n} \|M_i x - \pi_i\|_2^2 \quad \text{s.t.} \quad x_3^2 + x_4^2 = 1$$

(a) Find the matrices M, W and g which give you the formulation

$$x^* = \text{argmin}_{x \in \mathbb{R}^4} x^T M x + g^T x \quad \text{s.t. } x^T W x = 1$$

Solution:
To get W we have to symbolically multiply $x^T W x = 1$ and then we can do a coefficient comparison with $x_3^2 + x_4^2 = 1$, since both of these formulas can be set equal as they evaluate to 1. Therefore, we get the 4 following equations.

$$x_1(w_{11}x_1 + w_{12}x_2 + w_{13}x_3 + w_{14}x_4) = 0 \implies w_{11} = w_{12} = w_{13} = w_{14} = 0$$

$$x_2(w_{21}x_1 + w_{22}x_2 + w_{23}x_3 + w_{24}x_4) = 0 \implies w_{21} = w_{22} = w_{23} = w_{24} = 0$$

$$x_3(w_{31}x_1 + w_{32}x_2 + w_{33}x_3 + w_{34}x_4) = x_3^2 \implies w_{31} = w_{32} = w_{34} = 0,\ w_{33} = 1$$

$$x_4(w_{41}x_1 + w_{42}x_2 + w_{43}x_3 + w_{44}x_4) = x_4^2 \implies w_{41} = w_{42} = w_{43} = 0,\ w_{44} = 1$$

So the complete matrix W looks like this

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the next step we want to define the matrices M and g from the given optimization problem.

So we start with the formula

$$\sum_{i=1}^{n} \|M_i x - \pi_i\|_{C_i}^2$$

and try to change it until we can read of the definitions of M and g.
The first step that we do is to replace the norm, with its definition $\|y\|_C^2 = y^T C y$ so we get the following formula

$$\sum_{i=1}^{n} ((M_i x - \pi_i)^T C_i (M_i x - \pi_i))$$

In the next steps, we use $(A + B)^T = A^T + B^T$ and $(AB)^T = B^T A^T$ and expand the formula.

$$\sum_{i=1}^{n} (x^T M_i^T C_i M_i x - x^T M_i^T C_i \pi_i - \pi_i^T C_i M_i x + \pi_i^T C_i \pi_i)$$

Now we can eliminate the constant term $\pi_i^T C_i \pi_i$ since we have a optimization problem and it would vanish anyway and because of $x^T M_i^T C_i \pi_i$ can be rewritten to $\pi_i^T C_i M_i x$ we can sum up these two negative terms. All in all we get

$$\sum_{i=1}^{n} (x^T M_i^T C_i M_i x - 2\pi_i^T C_i M_i x)$$

Next we split up the sum and take the x matrices out of the sums as they do not depend on i.

$$x^T \sum_{i=1}^{n} (M_i^T C_i M_i) x \sum_{i=1}^{n} (-2\pi_i^T C_i M_i) x$$

$$x^T M x + g^T x$$

Now we can again compare the two formulas above to get M and g from them. It can easily be seen, that

$$M = \sum_{i=1}^{n} M_i^T C_i M_i$$

and

$$g^T = \sum_{i=1}^{n} -2\pi_i^T C_i M_i$$

where C_i is a 2x2 matrix that is calculated with $C_i = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \begin{bmatrix} n_1 & n_2 \end{bmatrix}$

(b) Show that M and W are positive semi definite.
Solution:
Again, we have to calculate the eigenvalues for M and W and check if they are greater or equal 0. To simplify this step, we assume, $n = 1$ for M.

So we get the following eigenvalues for M and W.

$$Eigenvalues_M = \begin{bmatrix} 0 \\ 0 \\ 0 \\ n_1^2p_0^2 + n_1^2p_1^2 + n_1^2 + n_2^2p_0^2 + n_2^2p_1^2 + n_2^2 \end{bmatrix}$$

$$Eigenvalues_W = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Again all the variables are quadratic, so there cannot be any negative values, so both matrices M and W are positive semi-definite.

The eigenvalue computations and matrix multiplications are done in Matlab with the following code.

```
clc
clear

syms p0 p1
syms n1 n2
syms x1 x2 x3 x4
syms w11 w12 w13 w14 w21 w22 w23 w24 w31 w32 w33 w34 w41 w42 w43
   w44

Mi = [ 1, 0, p0, -p1;
       0, 1, p1, p0]

% 1.a)  define the matrix and check if it is symmetric
%       --> the transposed and normal matrix have to be equal
Bi = Mi.'*Mi
is_symmetric = isequal(Bi, Bi.')

% 1.b)  check if it is positive semidefinite
%       --> all the Eigenvalues have to be >= 0

eigen_Bi = eig(Bi)
% since both the squares are always >= 0 the matrix is positive
% semi-definit

% 2.a)
x = [x1; x2; x3; x4]

W = [w11,w12,w13,w14; w21,w22,w23,w24; w31,w32,w33,w34; w41,w42,w43
   ,w44]

```

```
% calculate the formula for the coefficient comparison
Form = x.'*W*x == 1
Form_2 = x3^2+x4^2 == 1

% --> from the coefficient comparison follows W
W = [ 0, 0, 0, 0;
      0, 0, 0, 0;
      0, 0, 1, 0;
      0, 0, 0, 1; ]

% calculate and define M symbolically
ni = [n1; n2]
Ci = ni*ni.'
M = Mi.'*Ci*Mi

% 2.b) check W and M if they are positiv semi-definit
eigen_W = eig(W)
%  --> all eigenvalues greater or equal 0 so W is positive semi-
    definit

eigen_M = eig(M)
%  --> all eigenvalues greater or equal 0 so M is positive semi-
    definit
```

2 Approach and implementation details

See `README.md` in the submitted archive `team3_lab5.zip`.

3 Analysis of results

See `README.md` in the submitted archive `team3_lab5.zip`.