1 Game Semantics

Explain the properties of the game semantics that we will be using, *including* the sequoidal-exponential structure.

2 Harmer-McCusker Game Semantics of Nondeterminism

Explain about it, and explain the new contribution of considering infinite positions.

3 Explicit-oracle Game Semantics

(or better name)

3.1 Strategies and composition

We use the usual game semantics of may-equivalence, but with the following change: a *strategy* a game A is now a pair

$$(X, \sigma)$$

where σ is a strategy (in the usual sense) for $X \multimap A$. Where there is no ambiguity, we shall write σ for (X, σ).

We compose strategies in the following way: if (X, σ) is a morphism from A to B and (Y, τ) is a morphism from B to C, then the composition of these morphisms is given by the pair $(X \otimes Y, \sigma; \tau)$, where $\sigma; \tau$ is the following composite in the usual category of games:

$$Y \otimes X \xrightarrow{\tau \otimes \sigma} (B \multimap C) \otimes (A \multimap B) \longrightarrow A \multimap C$$

This composition is not immediately associative for the simple reason that while $Z \otimes (Y \otimes X)$ and $(Z \otimes Y) \otimes X$ are not isomorphic, they are still equal.

We get round this problem by imposing an equivalence relation on these strategies. One possibility would be to say that $X \xrightarrow{\sigma} A$ and $Y \xrightarrow{\tau} A$ are equivalent if there is an isomorphism $X \xrightarrow{\phi} Y$ making the following triangle commute:

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\phi"'] & A \\ Y \arrow[ur, "\tau"'] & \end{tikzcd}$$

However, this is too strong a definition for our purposes. We want to capture the idea that the game X captures the nondeterminism in the strategy and that two strategies are equivalent if they have the same behaviours in A when we play nondeterministically in X.

Definition 3.1. Let (X, σ), (Y, τ) be strategies for a game A (so σ is a strategy for $X \multimap A$ and τ is a strategy for $Y \multimap A$). Then we say that $(X, \sigma) \leq (Y, \tau)$ if there is a morphism $\phi \colon Y \multimap X$ such that

- ϕ is *total with respect to moves in* Y: i.e., if $s \in \phi$ is an O-position ending with a move in Y, then there exists some $sa \in \phi$.
- The composite (in the usual category of games) $\phi; \sigma \colon Y \multimap A$ is a subset of the strategy τ.

We say that $\sigma \equiv \tau$ if $\sigma \leq \tau$ and $\tau \leq \sigma$.

$\leq$ is easily seen to be a preorder on strategies, and it follows that $\equiv$ is an equivalence relation.

An immediate consequence of this definition is that if we have an equivalence in the earlier sense; i.e., an isomorphism between X and Y that commutes with the strategies σ and τ, then $\sigma \equiv \tau$.

Proposition 3.2. *The composition we have defined is associative, up to the equivalence relation $\equiv$.*

Proof. The composition map $(\sigma, \tau, \upsilon) \mapsto (\sigma; \tau); \upsilon$ takes strategies $\sigma \colon X \multimap (A \multimap B)$, $\tau \colon Y \multimap (B \multimap C)$ and $\upsilon \multimap (C \multimap D)$ and gives us a strategy

$$(\sigma; \tau); \upsilon \colon (Z \otimes (Y \otimes X)) \multimap (A \multimap D)$$

This mode of composition thus gives rise to a *uniform, history-free winning strategy* (see [AJ94, 3.6]) for the linear logic term

$$\begin{aligned}&(((X \multimap (A \multimap B)) \otimes (Y \multimap (B \multimap C))) \otimes (Z \multimap (C \multimap D))) \multimap \\ &\quad ((Z \otimes (Y \otimes X)) \multimap (A \multimap D))\end{aligned} \tag{1}$$

Meanwhile, the composition $(\sigma, \tau, \upsilon) \mapsto \sigma; (\tau; \upsilon)$ returns a strategy

$$\sigma; (\tau; \upsilon) \colon ((Z \otimes Y) \otimes X) \multimap (A \multimap D)$$

Composing this on the left with the isomorphism $\mathrm{assoc}_{Z,Y,X}$ then gives us a strategy for the game

$$(Z \otimes (Y \otimes X)) \multimap (A \multimap D),$$

so inducing another uniform history-free winning strategy on the term (1). By [AJ94, Theorem 1], any uniform, history-free winning strategy for a term is the denotation of some proof net for that term. Since the term (1) has exactly two occurrences of each atom, it admits at most one proof net and therefore we deduce that the two uniform, history-free winning strategies on (1) are equal. It follows that we have a commutative triangle in the original category of games:

$$\begin{tikzcd} Z \otimes (Y \otimes X) \arrow[r, "(\sigma;\tau);\upsilon"] \arrow[d, "\mathrm{assoc}_{Z,Y,X}"'] & A \multimap D \\ (Z \otimes Y) \otimes X \arrow[ur, "\sigma;(\tau;\upsilon)"'] & \end{tikzcd}$$

and therefore that $(\sigma;\tau);\upsilon \equiv \sigma;(\tau;\upsilon)$. $\square$

As we remarked, two strategies (X, σ) and (Y, τ) for a game A may be equivalent even if the games X and Y are not isomorphic.

Proposition 3.3. *Let X, Y, A be games and let σ be a strategy for $X \multimap A$. Then the strategies (X, σ) and $(X \times Y, \mathrm{pr}_1; \sigma)$ are equivalent.*

Proof. Clearly the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\sigma"] & A \\ X \times Y \arrow[u, "\mathrm{pr}_1"] \arrow[ur, "\mathrm{pr}_1;\sigma"'] & \end{tikzcd}$$

and so $\sigma \leq \mathrm{pr}_1; \sigma$.

In the other direction, we have a retraction $\rho_1 \colon X \multimap X \times Y$ in which the Proponent immediately gives up if the Opponent starts with a move in Y. The strategy ρ_1 is total with respect to moves in X on the left. Moreover, it satisfies $\rho_1; \mathrm{pr}_1 = \mathrm{id}_X$ and so the following triangle commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{pr}_1;\sigma"] & A \\ X \arrow[u, "\rho_1"] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

And therefore $\mathrm{pr}_1; \sigma \leq \sigma$. $\square$

3.2 Symmetric monoidal closed structure

As usual, our category has a monoidal structure induced by the tensor product $\_ \otimes \_$. Given strategies (X, σ) for A and (Y, τ) for B, we may form the strategy

$$(X \otimes Y, \sigma \otimes \tau)$$

Functoriality of this construction, up to the equivalence relation $\equiv$, is proved using a similar argument to that used in Proposition 3.2.

The associators, unitors and braiding are all given by the usual strategies, with the tensor unit I on the left. For example, the associator for games A, B, C is given by the strategy

$$\mathrm{assoc}_{A,B,C} \colon I \multimap (((A \otimes B) \otimes C) \multimap (A \otimes (B \otimes C)))$$

induced by the usual associator and the monoidal closed structure of the original category. The commutativity of the pentagon diagram, hexagon diagram and so on can all be deduced from the result [AJ94, Theorem 1] of Abramsky and Jagadeesan, as above.

The natural transformation $(C \otimes B) \multimap A \cong C \multimap (B \multimap A)$ gives rise, by composition with the functor $X \multimap \_$, to a natural transformation $X \multimap ((C \otimes B) \multimap A) \cong X \multimap (C \multimap (B \multimap A))$ and thence to a natural transformation between the functors $\mathcal{G}^{eo}(C \otimes B, A) \cong \mathcal{G}^{eo}(C, B \multimap A)$, proving that the category $\mathcal{G}^{eo}$ is monoidal closed.

3.3 Exponential structure

Given a strategy

$$\sigma \colon X \multimap (A \multimap B)$$

we may uncurry σ to give a strategy for

$$X \otimes A \multimap B$$

which we shall also refer to as σ. We can now construct a strategy $!\sigma$, given in the usual category of games by the composite

$$!X \otimes !A \xrightarrow{m_{X,A}} !(X \otimes A) \xrightarrow{!\sigma} !B$$

where $m_{X,A}$ is the natural transformation exhibiting ! as an endo-monoidal functor on $\mathcal{G}$.

Upon currying $!\sigma$, we arrive at a strategy for $!X \multimap (!A \multimap !B)$, which will give us the exponential functor in our new category.

Proposition 3.4. *! is an endofunctor on $\mathcal{G}^{eo}$.*

Proof. We prove this in uncurried form. The curried version will then follow, since currying is a natural isomorphism.

Let $\sigma \colon X \otimes A \to B$ and $\tau \colon Y \otimes B \to C$ be strategies. Then the composition of σ with τ is given, in uncurried form, by the following composite in $\mathcal{G}$:

$$(Y \otimes X) \otimes A \xrightarrow{\mathrm{assoc}_{Y,X,A}} Y \otimes (X \otimes A) \xrightarrow{Y \otimes \sigma} Y \otimes B \xrightarrow{\tau} C$$

(We may check this either by inspection or by using [AJ94, Theorem 1] again.)

i)

$$!(Y \otimes X) \otimes !A \xrightarrow{m_{Y \otimes X, A}} !((Y \otimes X) \otimes A) \xrightarrow{! \operatorname{assoc}_{Y,X,A}} !(Y \otimes (X \otimes A)) \xrightarrow{!(Y \otimes \sigma)} !(Y \otimes B) \xrightarrow{!\tau} !C$$

ii)

$$(!Y \otimes !X) \otimes !A \xrightarrow{\operatorname{assoc}_{!Y,!X,!A}} !Y \otimes (!X \otimes !A) \xrightarrow{!Y \otimes m_{!X,!A}} !Y \otimes !(X \otimes A) \xrightarrow{!Y \otimes !\sigma} !Y \otimes !B \xrightarrow{m_{Y,B}} !(Y \otimes B) \xrightarrow{!\tau} !C$$

iii)

$$\begin{tikzcd}
!(Y \otimes X) \otimes !A \arrow[r, "m_{Y \otimes X, A}"] & !((Y \otimes X) \otimes A) \arrow[r, "! \operatorname{assoc}_{Y,X,A}"] & !(Y \otimes (X \otimes A)) \arrow[r, "!(Y \otimes \sigma)"] & !(Y \otimes B) \arrow[r, "!\tau"] & !C \\
(!Y \otimes !X) \otimes !A \arrow[u, "m_{Y,X} \otimes !A"] \arrow[r, "\operatorname{assoc}_{!Y,!X,!A}"] & !Y \otimes (!X \otimes !A) \arrow[r, "!Y \otimes m_{!X,!A}"] & !Y \otimes !(X \otimes A) \arrow[u, "m_{Y, X \otimes A}"] \arrow[r, "!Y \otimes !\sigma"] & !Y \otimes !B \arrow[u, "m_{Y,B}"] &
\end{tikzcd}$$

Figure 1:

(i): The composite used to calculate $!(\sigma;\tau)$, where $X \xrightarrow{\sigma} A \multimap B$, $Y \xrightarrow{\tau} B \multimap C$ are strategies in $\mathcal{G}^{eo}$.

(ii): The composite used to calculate $!\sigma;!\tau$.

(iii): Commutative diagram showing that $!(\sigma;\tau) \leq !\sigma;!\tau$.

Then the exponential of this composition is given by Figure 1(i).

Alternatively, we may first compute $!\sigma$ and $!\tau$ and then compose them. Then we end up with the composition given in Figure 1(ii).

We can show that $!(\sigma;\tau) \leq !\sigma;!\tau$ using the diagram 1(iii), where the left-hand hexagon is the coherence diagram for m and the square at the right commutes because m is a natural transformation. $\square$

Proposition 3.5. *Let M be a language-term of ground type T and write $[\![M]\!]$ as $\sigma\colon X \to A$. Then for each OP-play S over M, there is a unique position $s \in \sigma$ such that S is the restriction of s to those answer moves in X and such that the final move in s is the result of the play S.*

References

[AJ94] Samson Abramsky and Radha Jagadeesan. Games and full completeness for multiplicative linear logic. *The Journal of Symbolic Logic*, 59(2):543–574, 1994.