A unified approach to the semantics of effects

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Outline

Section 1

Some Examples

Nondeterminism and innocence

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Tsukada and Ong: the problem is missing *branching time information*.

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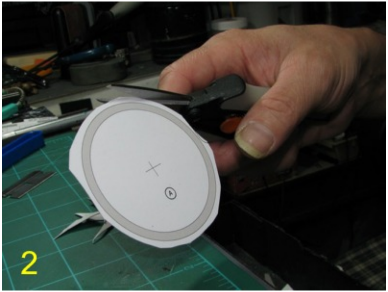

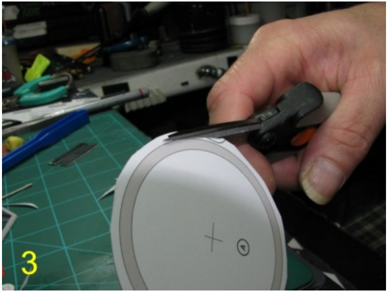

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We can use a semi-indirect approach informed by the operational semantics of the language (e.g., using linear decomposition: $\sigma, \tau\colon \mathbb{B} \to A$ are equivalent if and only if for all $\alpha\colon !\mathbb{B}$ there exists some $\beta\colon !\mathbb{B}$ such that $\alpha; \sigma = \beta; \tau$ – and vice versa).

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Or we can use more ingenious techniques. For example, we can recover the Tsukada-Ong sheaf-theoretic approach by identifying σ with the following sheaf.

$$\sigma(s) = \{t \in \sigma \; : \; t|_A = s\}$$

Another example – bell games

We want to model the language $\mathrm{PCF}^{🔔+}$, obtained by enlarging PCF with command types and a constant $\mathsf{ding} : \mathtt{com}$ with the following operational semantics.

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Game Semantics (Ghica's 'slot games'): insert an additional move $\spadesuit$ (belonging to neither player) into the play whenever we ring the bell (so $[\![\text{ding}]\!]$ is the strategy with maximal play $q\spadesuit a$).

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	com	$\rightarrow$	com	$\rightarrow$	com
					q
	q				
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			q		
🔔					
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					a

$\rightsquigarrow$

q
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Suppose instead that we attempt to model ding as a purely formal morphism $1 \to \mathbb{C}$. In our new model, a strategy for A will be a strategy for $\mathbb{C} \to A$ in the original model, where we interpret $\sigma : \mathbb{C} \to A$ as the strategy ding; $\sigma : A$.

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🔔 $= qa$ on the left.

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Can we build a model of nondeterministic innocence based on bell games?

Section 2

Cones on monoidal functors

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Note: for every object A of a CCC, the 'constant A' functor is oplax monoidal with coherence given by the diagonal and projections.

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I.e., this (a *j-category*):

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "()"] \arrow[d, "j"'] \arrow[dr, phantom, "\Downarrow\phi"] & I \arrow[d, "I"] \\
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The more objects there are in $\mathcal{C}'$, the more morphisms there are in $\mathcal{D}$. The more morphisms there are in $\mathcal{C}'$, the *fewer* morphisms there are in $\mathcal{D}$ (or: a finer equivalence relation on morphisms).

The Cone on j

We can define a morphism of j-categories $(\mathcal{D}', J', \phi') \to (\mathcal{D}, J, \phi)$ to be an oplax monoidal functor $f\colon \mathcal{D}' \to \mathcal{D}$ making the following diagram commute.

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C_j is the category obtained from $\mathcal{C}$ by 'formally adding morphisms $\psi_X \colon I \multimap jX$'.

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A morphism $A \to B$ in C_j is a pair (X, f), where X is an object of $\mathcal{C}'$ and $f : jX \otimes A \to B$ is a morphism in $\mathcal{C}'$.

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Composition is given by 'the only thing you can write down'.

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The more morphisms we can include in $\mathcal{C}'$, the closer C_j will be to our desired model.

Section 3

More examples

Unbounded nondeterminism

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Therefore, $\mathcal{D}$ is some quotient of C_{j_s}.

Unbounded nondeterminism and must-testing

C_j models may-testing: in a model of unbounded nondeterminism with must-testing, we may have $\top_A; \sigma \neq \top_B$ even if σ is surjective. The reason is that σ might have a play ending with infinitely many moves in A.

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Notice that $\mathcal{G}_{sw}$ has *fewer* morphisms than $\mathcal{G}_s$ and so the cone $C_{j_{sw}}$ has *more* morphisms than the cone C_{j_s}: we no longer identify all strategies that have the same convergent behaviours.

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Alternatively: there is a morphism of oplax monoidal functors from $c_{\mathbb{B}} \to j_{sw}$, which induces a functor $C_{c_{\mathbb{B}}} \to C_{j_{sw}}$. If we take the image of $C_{c_{\mathbb{B}}}$ inside $C_{j_{sw}}$ then we can use the finer equivalence relation with the simpler category.

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We turn the monoid $([0,1], \times)$ into a (strict) monoidal category and then add additional morphisms $\neg_p \colon p \to 1 - p$.

Take the functor $c^p_{\mathbb{B}} \colon [0,1] \to \mathcal{G}$ that sends every element of $[0,1]$ to the boolean object $\mathbb{B}$ and sends each morphism $\neg_p$ to the morphism $\mathsf{not} \colon \mathbb{B} \to \mathbb{B}$.

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The equivalence relation on morphisms in the cone automatically gives us equations such as the following.

$$\text{if } \psi_{2/3} \text{ then } (\text{if } \psi_{1/2} \text{ then } a \text{ else } b) \text{ else } c =$$
$$\text{if } \psi_{1/3} \text{ then } a \text{ else } (\text{if } \psi_{1/2} \text{ then } b \text{ else } c)$$

Game semantics of state

Let $\mathbf{Set}^{\rightleftarrows}$ be the category whose objects are sets and where the morphisms $A \to B$ are pairs of functions $f\colon A \to B, g\colon B \to A$ such that $f \circ g = \mathrm{id}_B$.

Then we get an oplax monoidal functor $\mathtt{Var}\colon \mathbf{Set}^{\rightleftarrows} \to \mathcal{G}$ sending a set X to the game $\mathtt{Var}[X] = \mathtt{com}^X \times \underline{X}$.

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So what? We can already characterize these as visible strategies.

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Then discard the explicit state to get the semantics of IA_{cbv}.

Questions?