# A Coalgebraic Approach to Stateful Objects in Game Semantics

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# Semantics of Stateful Programs

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$$[\![f]\!] : W \times [\![S]\!] \to W \times [\![T]\!]$$

where $W$ denotes the program state.

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Given games $A, B$, we can form the *compound games* $A \times B$, $A \otimes B$, $A \multimap B$, $A \oslash B$,... in which the games $A$ and $B$ are played in parallel.

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 & & ? \\
? & & \\
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But what if the argument is a function? For example, take the lambda term

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...but now we are stuck. We introduce the *exponential* $!A \approx A \oslash (A \oslash (A \oslash \cdots$ Now we represent the type of functions from $A$ to $B$ using the game

$$!A \multimap B$$

# A storage cell

Consider the following (Java) class:

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public class StorageCell implements AbstractCell {
  private int state;
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The *implementation* now corresponds to a *strategy* for that game:

$$7 \checkmark \; ? \; 7 \; ? \; 7 \; 8 \checkmark \; 9 \checkmark \; ? \; 9 \ldots$$

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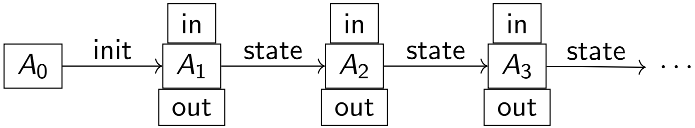

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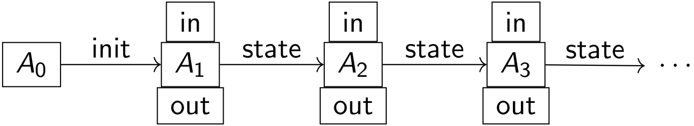


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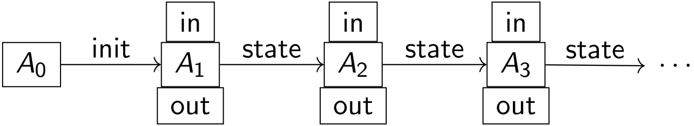


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Using some category-theoretic magic ($!A$ is the *final coalgebra* for the endofunctor $A \oslash \_$), we can automatically construct our strategy for $!X \multimap !(\mathrm{Read}[X] \times \mathrm{Write}[X])$ from this state transformer.