1 Nondeterminism

1.1 Introduction

We give a treatment of nondeterminism in the ordinal games setting. Let α be a large ordinal. If A is an α-game, then a *nondeterministic strategy* for A is a subset $\sigma \subseteq P_A$ satisfying:

- For all P-positions $s \in \sigma$, if a is an O-move such that $sa \in P_A$, then $sa \in \sigma$.

That is, we drop the condition that P-replies to O-moves should be unique.

Since the proof that the (totality of O-replies) condition above is preserved by composition did not use the uniqueness of P-replies, it follows that nondeterministic strategies may be composed to yield nondeterministic strategies and we get a category $G_{\mathrm{nd}}(\alpha)$ of games and nondeterministic strategies.

Note that if A is a game, then the collection $\mathrm{Strat}(A)$ of nondeterministic strategies for A forms a complete lattice under set-inclusion; i.e., is is closed under unions and intersections.

Let A, B be α-games. Then a strategy σ for $A \multimap B$ induces a map of sets

$$f_\sigma \colon \mathrm{Strat}(A) \to \mathrm{Strat}(B)$$

by

$$f_\sigma(\tau) = \{s|_B \ : \ s \in \sigma,\ s|_A \in \tau\}$$

If we identify $\mathrm{Strat}(X)$ with the hom-set $\mathcal{G}(\alpha)(I, X)$, then we see that this is nothing more than the action of the functor $\mathcal{G}(\alpha)(I, -)$.

Proposition 1.1. *f_σ is order preserving and preserves unions and intersections.*

Not every function $\mathrm{Strat}(A) \to \mathrm{Strat}(B)$ that preserves unions and intersections arises from a strategy for $A \multimap B$, however. Consider, for example, the map

$$\overset{!}{\circ} \colon \mathrm{Strat}(A) \to \mathrm{Strat}(!A)$$

where $\overset{!}{\sigma}$ is the strategy for $!A$ in which we play according to σ in every copy of A; i.e.:

$$\overset{!}{\sigma} = \{s \in P_{!A} \ : \ \forall\beta \ . \ s|_{A_\beta} \in \sigma\}$$

Of course, there need not be a strategy for $A \multimap !A$, so $\overset{!}{\circ}$ will not arise from a strategy in general.

Proposition 1.2. $\overset{!}{\circ}$ *is order preserving and preserves unions and intersections.*

Let us now compose the two maps we have considered: let σ be a strategy for $!A \multimap B$; then we form the map

$$f^!_\sigma = \mathrm{Strat}(A) \xrightarrow{\overset{!}{\circ}} \mathrm{Strat}(!A) \xrightarrow{f_\sigma} \mathrm{Strat}(B)$$

or, written out in full:

$$f^!_\sigma(\tau) = \{s|_B \,:\, s \in \sigma,\ \forall\beta \,.\, s|_{A_\beta} \in \tau\}$$

$f^!_\sigma$ is the composition of order preserving maps that preserve unions and intersections, so it preserves unions and intersections. We now claim that every such map arises from a strategy for $!A \multimap B$. Let $\mathcal{CL}$ denote the category of complete lattices and order-preserving maps that preserve unions and intersections. Furthermore, let $\mathcal{G}^!_{\mathrm{nd}}(\alpha)$ denote the co-Kleisli category for the exponential comonad on $\mathcal{G}_{\mathrm{nd}}(\alpha)$. Then our main result is:

Theorem 1.3. *The maps $f^!_\sigma$ define a fully faithful functor*

$$\mathcal{G}^!_{\mathrm{nd}}(\alpha) \to \mathcal{CL}$$

1.2 Functoriality of $f^!_\sigma$

We first establish that $f^!_\sigma$ gives rise to a functor $\mathcal{G}^!_{\mathrm{nd}}(\alpha) \to \mathcal{CL}$. Recall that the objects of $\mathcal{G}^!_{\mathrm{nd}}$ are completely negative α-games and that the morphisms from A to B are nondeterministic strategies for $!A \multimap B$. Our functor will send a game A to its set of nondeterministic strategies $\mathrm{Strat}(A)$ and it will send a strategy $\sigma\colon !A \multimap B$ to the map $f^!_\sigma$.

Before showing that this is a functor, we shall quickly establish that f_σ gives rise to a functor $\mathcal{G}_{\mathrm{nd}}(\alpha) \to \mathcal{CL}$ by sending a game A to $\mathrm{Strat}(A)$ and a strategy σ for $A \multimap B$ to $f_\sigma\colon \mathrm{Strat}(A) \to \mathrm{Strat}(B)$. As we remarked above, after we have identified $\mathrm{Strat}(X)$ with $\mathcal{G}_{\mathrm{nd}}(\alpha)(I, X)$, we may identify this with the hom-set functor $\mathcal{G}_{\mathrm{nd}}(\alpha)(I, -)$.

Now we come to functoriality of the functor defined by $f^!_\sigma$. Recalling the usual definition of a co-Kleisli category, we see that composition in $\mathcal{G}^!_{\mathrm{nd}}(\alpha)$ of strategies $\sigma\colon !A \multimap B$ and $\tau\colon !B \multimap C$ is given by

$$\tau \circ \sigma = !A \xrightarrow{\mathtt{mult}} !!A \xrightarrow{!\sigma} !B \xrightarrow{\tau} C$$