Sequoidal Categories and Transfinite Games: Towards a Coalgebraic Approach to Linear Logic*

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Abstract

In [6], Laird introduces the concept of a *sequoidal category* as a formalization of causality in game semantics. A sequoidal category is like a monoidal category with an extra connective, $\oslash$, that allows one to construct an exponential object as a final coalgebra [4], Under a further hypothesis, it is possible to show that this final coalgebra allows one to construct cofree commutative comonoids in the category, giving us a model of the exponential connective ! from linear logic. In the first part of this note, we review the coalgebraic arguments for constructing the cofree commutative comonoid, which are known but do not yet appear in print. In the second part, we show that the extra hypotheses are necessary by outlining a definition of *transfinite game*, in which the carriers of the sequoidal coalgebra and the cofree commutative comonoid do not coincide.

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1 Sequoidal categories

1.1 Game semantics and the sequoidal operator

To get around the problems caused by the non-commutativity of the comonoid in the sets-and-relations model, we shall consider a game semantics model, in which we will be able to construct cofree commutative comonoids. We shall present a form of game semantics in the style of [5] and [1]. A game will be given by a tuple

$$A = (M_A, \lambda_A, b_A, P_A)$$

where

- M_A is a set of moves.
- $\lambda_A \colon M_A \to \{O, P\}$ is a function designating each move as either an *O-move* or a *P-move*.
- $b_A \in \{O, P\}$ is a choice of starting player.
- $P_A \subseteq M_A^*$ is a prefix-closed set of alternating plays (so if $sab \in P_A$ then $\lambda_A(a) = \neg\lambda_A(b)$) such that if $as \in P_A$ then $\lambda_A(a) = b_A$.

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We call $sa \in P_A$ a P*-position* if a is a P-move and an O*-position* if a is an O-move.

A *strategy* for player P for a game A is identified with the set of positions that may arise when playing according to that strategy. Namely, it is a non-empty prefix-closed subset $\sigma \subseteq P_A$ satisfying the two conditions:

(sO) If $s \in \sigma$ is a P-position and a is an O-move such that $sa \in P_A$, then $sa \in \sigma$.

(sP) If $sa, sb \in \sigma$ are P-positions, then $a = b$.

We shall now concentrate on games A for which $b_A = O$, called *negative games*. We shall informally describe the standard connectives on negative games:

Product If $(A_i : i \in I$ is a collection of negative games, then we write $\prod_{i\in I} A_i$ for the game in which player O, on his first move, may play in any of the games A_i. From then on, play continues in A_i. When we have defined the category of games, $\prod_{i\in I} A_i$ will be the category-theoretic product of the A_i. If A_1, A_2 are games, we write $A_1 \times A_2$ for $\prod_{i=1}^{2} A_i$.

Tensor Product If A, B are negative games,the tensor product $A \otimes B$ is played by playing the games A and B in parallel, where player O may elect to switch games whenever it is his turn and continue play in the game he has switched to.

Linear implication The implication $A \multimap B$ is played by playing the game B in parallel with the *negation* of A - that is, the game formed by switching the roles of players P and O in A. Since play in the negation of A starts with a P-move, player O is forced to make his first move in the game B. Thereafter, player P may switch games whenever it is her turn.

If A, B, C are negative games, σ is a strategy for $A \multimap B$ and τ is a strategy for $B \multimap C$, then we may form a strategy $\tau \circ \sigma$ for $A \multimap C$ by setting

$$\sigma \| \tau = \{\mathfrak{s} \in (M_A \sqcup M_B \sqcup M_C)^* \,:\, \mathfrak{s}|_{A,B} \in \sigma,\ s|_{B,C} \in \tau\}$$

and then defining

$$\tau \circ \sigma = \{\mathfrak{s}|_{A,C} \,:\, \mathfrak{s} \in \tau \circ \sigma\}$$

It is well known (see, for example, [1]) that $\tau \circ \sigma$ is indeed a strategy for $A \multimap C$ and that this form of composition is associative and has an identity. It is also well known that the resulting category $\mathcal{G}$ of games and strategies has products given by the operator $\times$ and a symmetric monoidal closed structure given by the operations $\otimes$ and $\multimap$.

We turn now to the non-standard *sequoid* connective $\oslash$. If A and B are negative games, then the sequoid $A \oslash B$ is similar to the tensor product $A \otimes B$, but with the restriction that player O's first move must take place in the game A. We observe immediately that we have structural isomorphisms

$$\mathtt{dist}\colon A \otimes B \xrightarrow{\cong} (A \oslash B) \times (B \oslash A)$$

$$\mathtt{dec}\colon (A \times B) \oslash C \xrightarrow{\cong} (A \oslash C) \times (B \oslash C)$$

$$\mathtt{passoc}\colon (A \oslash B) \oslash C \xrightarrow{\cong} A \oslash (B \otimes C)$$

One further question to ask is: does the sequoid operator give rise to a functor $\_ \oslash \_ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, as the tensor operator does? The answer is no: indeed, let A, B, C, D be negative games, let σ be a strategy for $A \multimap C$ and let τ be a strategy for $B \multimap D$. Our aim is to construct a natural strategy $\sigma \oslash \tau$ for $(A \oslash B) \multimap (C \oslash D)$. There is an obvious way to try and do this: player P should play according to the strategy σ whenever player O's last move was in A or C, and according to τ whenever player O's last move was in B or D.

We show that this does not in general give us a strategy for $(A \oslash B) \multimap (C \oslash D)$. Suppose that σ is such that player P's response to some opening move in C is another move in C and suppose that τ is such that player P's response to some opening move in D is a move in B (for example, τ is a copycat strategy). Then we end up with the following sequence of events in the game $(A \oslash B) \multimap (C \oslash D)$:

1. Player O starts with a move in C (as he must).
2. Player P responds according to σ with another move in C.
3. Player O decides to switch games and play a move in D.
4. Player P responds according to τ with a move in B.

But now player P's last move is not a legal move in $(A \oslash B) \multimap (C \oslash D)$, since no moves have been played in A yet.

We get round this problem by requiring that the strategy σ be *strict* – that is, whatever player O's opening move in C is, player P's reply must be a move in A.

▶ **Definition 1.** Let N, L be negative games and let σ be a strategy for $N \multimap L$. We say that σ is *strict* if player P's reply to an opening move in L is always a move in N.

Identity strategies are strict and the composition of two strict strategies is strict, so we get a full-on-objects subcategory $\mathcal{G}_s$ of $\mathcal{G}$ where the morphisms are strict strategies. Then the sequoid operator gives rise to a functor:

$$\_ \oslash \_ : \mathcal{G}_s \times \mathcal{G} \to \mathcal{G}_s$$

1.2 Sequoidal categories

We now have the motivation required to give the definition of a *sequoidal category* from [6].

▶ **Definition 2.** A *sequoidal category* consists of the following data:

- A symmetric monoidal category $\mathcal{C}$ with monoidal product $\otimes$ and tensor unit I, associators $\mathrm{assoc}_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$, unitors $\mathrm{runit}_A : A \otimes I \xrightarrow{\cong} A$ and $\mathrm{lunit}_A : I \otimes A \xrightarrow{\cong} A$ and braiding $\mathrm{sym}_{A,B} : A \otimes B \to B \otimes A$.
- A category $\mathcal{C}_s$
- A right monoidal category action of $\mathcal{C}$ on the category $\mathcal{C}_s$. That is, a functor

  $$\_ \oslash \_ : \mathcal{C}_s \times \mathcal{C} \to \mathcal{C}_s$$

  together natural isomorphisms

  $$\mathtt{passoc}_{A,B,C} : A \oslash (B \otimes C) \to (A \oslash B) \oslash C$$

  and

  $$\mathtt{r}_A : A \oslash I \xrightarrow{\cong} A$$

  subject to the following coherence conditions:

$$\begin{tikzcd}
A \oslash (B \otimes (C \otimes D)) \arrow[r, "\mathtt{passoc}_{A,B,C\otimes D}"] \arrow[d, "\mathrm{id}_A \oslash \mathrm{assoc}_{B,C,D}"'] & (A \oslash B) \oslash (C \otimes D) \arrow[r, "\mathtt{passoc}_{A\oslash B,C,D}"] & ((A \oslash B) \oslash C) \oslash D \\
A \oslash ((B \otimes C) \otimes D) \arrow[r, "\mathtt{passoc}_{A,B\otimes C,D}"'] & (A \oslash (B \otimes C)) \oslash D \arrow[ur, "\mathtt{passoc}_{A,B,C} \oslash \mathrm{id}_D"'] &
\end{tikzcd}$$

$$\begin{tikzcd} A \oslash (I \otimes B) \arrow[r, "\mathtt{passoc}_{A,I,B}"] \arrow[d, "\mathrm{id}_A \oslash \mathrm{lunit}_B"'] & (A \oslash I) \oslash B \arrow[dl, "\mathtt{r}_A \oslash \mathrm{id}_B"] \\ A \oslash B & \end{tikzcd} \qquad \begin{tikzcd} A \oslash (B \otimes I) \arrow[r, "\mathtt{passoc}_{A,B,I}"] \arrow[d, "\mathrm{id}_A \oslash \mathrm{runit}_B"'] & (A \oslash B) \oslash I \arrow[dl, "\mathtt{r}_{A \oslash B}"] \\ A \oslash B & \end{tikzcd}$$

- A functor $J \colon \mathcal{C}_s \to \mathcal{C}$ (in the games example, this is the inclusion functor $\mathcal{G}_s \to \mathcal{G}$)
- A natural transformation $\mathrm{wk}_{A,B} \colon J(A) \otimes B \to J(A \oslash B)$ satisfying the coherence conditions:

$$\begin{tikzcd} A \otimes I \arrow[r, "\mathrm{runit}_A"] \arrow[d, "\mathrm{wk}_{A,I}"'] & A \\ A \oslash I \arrow[ur, "J(\mathtt{r}_A)"'] & \end{tikzcd} \qquad \begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "\mathrm{wk}_{A,B} \otimes \mathrm{id}_C"] \arrow[d, "\mathrm{assoc}_{A,B,C}"'] & (A \oslash B) \otimes C \arrow[r, "\mathrm{wk}_{A \oslash B, C}"] & (A \oslash B) \oslash C \\ A \otimes (B \otimes C) \arrow[r, "\mathrm{wk}_{A,B \otimes C}"'] & A \oslash (B \otimes C) \arrow[ur, "J(\mathtt{passoc}_{A,B,C})"'] & \end{tikzcd}$$

Our category of games satisfies further conditions:

▶ **Definition 3.** Let $\mathcal{C} = (\mathcal{C}, \mathcal{C}_s, J, \mathrm{wk})$ be a sequoidal category. We say that $\mathcal{C}$ is an *inclusive sequoidal category* if $\mathcal{C}_s$ is a full-on-objects subcategoryof $\mathcal{C}$ containing the monoidal isomorphisms and the morphisms $\mathrm{wk}_{A,B}$, J is the inclusion functor and J reflects isomorphisms.

If $\mathcal{C}$ is an inclusive sequoidal category, we say that $\mathcal{C}$ is *Cartesian* if $\mathcal{C}_s$ has all products and these are preserved by J. In that case, we say that $\mathcal{C}$ is *decomposable* if the natural transformations

$$\mathtt{dec}_{A,B} = \langle \mathrm{wk}_{A,B}, \mathrm{wk}_{A,B} \circ \mathrm{sym}_{A,B} \rangle \colon A \otimes B \to (A \oslash B) \times (B \oslash A)$$
$$\mathtt{dec}^0 \colon I \to 1$$

are isomorphisms and we say that $\mathcal{C}$ is *distributive* if the natural transformations

$$\mathtt{dist}_{A,B,C} = \langle \mathrm{pr}_1 \oslash \mathrm{id}_C, \mathrm{pr}_2 \oslash \mathrm{id}_C \rangle \colon (A \times B) \oslash C \to (A \oslash C) \times (B \oslash C)$$
$$\mathtt{dist}_{(A_i \colon i \in I), C} = \langle (\mathrm{pr}_i \oslash \mathrm{id}_C \ : \ i \in I) \rangle \colon \left(\prod_{i \in I} A_i \right) \oslash C \to \prod_{i \in I} (A_i \oslash C)$$
$$\mathtt{dist}_{A,0} \colon 1 \oslash A \to 1$$

are isomorphisms.

We have one further piece of structure available to us:

▶ **Definition 4.** Let $\mathcal{C} = (\mathcal{C}, \mathcal{C}_s, J, \mathrm{wk})$ be an inclusive sequoidal category. We say that $\mathcal{C}$ is a *sequoidal closed category* if $\mathcal{C}$ is monoidal closed (with internal hom $\multimap$ and currying $\Lambda_{A,B,C} \colon \mathcal{C}(A \otimes B, C) \xrightarrow{\cong} \mathcal{C}(A, B \multimap C)$) and if the map $f \mapsto \Lambda(f \circ \mathrm{wk})$ gives rise to a natural transformation

$$\Lambda_{A,B,C,s} \colon \mathcal{C}_s(A \oslash B, C) \to \mathcal{C}_s(A, B \multimap C)$$

It can be shown (see for example [4]) that our category $\mathcal{G}$ of games has all this structure.

▶ **Theorem 5.** *Let J be the inclusion functor $\mathcal{G}_s \to \mathcal{G}$. If A, B are games, let $\mathrm{wk}_{A,B} \colon A \otimes B \to A \oslash B$ be the natural copycat strategy. Then*

$$(\mathcal{G}, \mathcal{G}_s, J, \mathrm{wk})$$

is an inclusive, Cartesian, decomposable, distributive sequoidal closed category.

1.3 The sequoidal exponential

There are several ways to add exponentials to the basic category of games. We shall use the definition based on countably many copies of the base game (see [6], for example):

▶ **Definition 6.** Let A be a negative game. The *exponential* of A is the game $!A = (M_{!A}, \lambda_{!A}, b_{!A}, P_{!A})$, where $M_{!A}, \lambda_{!A}, b_{!A}, P_{!A}$ are defined as follows:

- $M_{!A} = M_A \times \omega$
- $\lambda_{!A} = \lambda_A \circ \mathrm{pr}_1$
- $b_{!A} = O$
- Given a sequence $s \in M_{!A}^{\omega}$, we write $s|_n$ for the largest sequence $a_1a_2\ldots a_k \in M_A^*$ such that $(a_1, n), (a_2, n), \ldots (a_k, n)$ is a subsequence of s. Then $P_{!A}$ is the set of all sequence $s \in M_{!A}^{\omega}$ that are alternating with respect to $\lambda_{!A}$, such that $s|_n \in P_A$ for all n and such that if $m < n$ and (a, n) occurs in s then (b, m) must occur earlier in s for some move b: in other words, player O can start infinitely many copies of the game A, but he must start them in order.

This last condition on the order in which games may be opened is very important, as it allows us to define morphisms that give $!A$ the semantics of the exponential from linear logic. For example, we have a natural morphism $\mu\colon !A \to !A \otimes !A$, given by the copycat strategy that starts a new copy of A on the left whenever one is started on the right. Because of the condition on the order in which copies of A may be started, there is a unique way to do this.

▶ **Proposition 7.** *μ exhibits $!A$ as a comonoid in the monoidal category $(\mathcal{G}, \otimes, I)$.*

Proof. μ shall be the comultiplication in our comonoid. The counit is given by the empty strategy $\eta\colon !A \to I$. We just need to check that μ is associative and that η is a counit for μ.

For associativity, we need to show that the following diagram commutes:

$$\begin{tikzcd}
!A \arrow[rr, "\mu"] \arrow[d, "\mu"'] & & !A \otimes !A \arrow[d, "\mathrm{id}_{!A} \otimes \mu"] \\
!A \otimes !A \arrow[r, "\mu \otimes \mathrm{id}_{!A}"] & (!A \otimes !A) \otimes !A \arrow[r, "\mathrm{assoc}_{!A,!A,!A}"'] & !A \otimes (!A \otimes !A)
\end{tikzcd}$$

This is easy to see when we notice that both branches of the square are copycat strategies on $!A \multimap !A \otimes (!A \otimes !A)$; since copies of A in $!A$ must be started in sequence, there is a unique such strategy, and so the square commutes.

For the counit, we need to show that the follosing two diagrams commute:

$$\begin{tikzcd}
!A \arrow[r, "\mu"] \arrow[dr, "\mathrm{runit}_A^{-1}"'] & !A \otimes !A \arrow[d, "\mathrm{id}_{!A} \otimes \eta"] \\
& !A \otimes I
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mu"] \arrow[dr, "\mathrm{lunit}_A^{-1}"'] & !A \otimes !A \arrow[d, "\eta \otimes \mathrm{id}_{!A}"] \\
& I \otimes !A
\end{tikzcd}$$

Once again, these diagrams commute because both branches are copycat strategies for $!A \multimap !A \otimes I$ or $!A \multimap I \otimes !A$ and there is a unique such strategy in each case. ◀

We shall later show that $(!A, \mu, \eta)$ is in fact the *cofree commutative comonoid* on A in the monoidal category $(\mathcal{G}, \otimes, I)$.

We shall call the exponential $!A$ the *sequoidal exponential.* The following proposition explains the name:

▶ **Proposition 8.** *Let A be a negative game. Then we get an endofunctor $A \oslash \_$ on $\mathcal{G}$ given by sending B to $A \oslash B$.*

The sequoidal exponential $!A$, together with the obvious copycat strategy $\alpha\colon !A \to A \oslash !A$, is the final coalgebra for the endofunctor $A \oslash \_$. In other words, if B is a negative game and $\sigma\colon B \to A \oslash B$ is a morphism then there is a unique morphism $\llparenthesis\, \sigma \,\rrparenthesis\colon B \to !A$ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "\sigma"] \arrow[d, "\llparenthesis\, \sigma \,\rrparenthesis"'] & A \oslash B \arrow[d, "\mathrm{id}_A \oslash \llparenthesis\, \sigma \,\rrparenthesis"] \\ !A \arrow[r, "\alpha"'] & A \oslash !A \end{tikzcd}$$

We call $\llparenthesis\, \sigma \,\rrparenthesis$ the anamorphism *of σ.*

Proof. See [4]. We shall shortly give a proof in the more general case. ◀

1.4 Imperative programs as anamorphisms

Let Σ be the game

$$(\{q, a\}, \{q \mapsto O, a \mapsto P\}, O, \{\epsilon, q, qa\})$$

and let $\mathbb{N}$ be the game

$$\{q\} \cup \mathbb{N}, \{q \mapsto O,\ n \mapsto P\}, O, \{\epsilon, q\} \cup \{qn :\ n \in \mathbb{N}\})$$

That is, Σ is the game where player O makes the move q and player P must then reply with the move a, while $\mathbb{N}$ is the game where player O plays the move q and then player P must choose a natural number n to play. We model a simple stateful program as a history sensitive strategy on $!(\Sigma \times \mathbb{N})$: the program interface will consist of a button that can be pressed, and a display that can be activated to display the number of times that the button has been pressed. As a strategy on $!(\Sigma \times \mathbb{N})$, whenever player O plays q in Σ, player P responds with the move a. Whenever he plays q in $\mathbb{N}$, she responds with the number of times we have played in Σ so far.

We can construct this strategy using nothing more than the sequoidal structure on $\mathcal{G}$ that we have described and some very basic strategies on the games $\mathbb{N}$ and Σ; namely:

$\mathtt{OK}\colon I \to \Sigma$

$\mathtt{succ}\colon \mathbb{N} \to \mathbb{N}$

$\mathtt{zero}\colon I \to \mathbb{N}$

Here, `OK` is the strategy that replies a to player O's initial move q, `zero` is the strategy that replies 0 to player O's initial move q and `succ` is the strategy that queries its argument and then returns the number one greater (so maximal plays in `succ` are of the form $qqn(n+1)$ for $n \in \mathbb{N}$).

We first define a morphism `press` from $!\mathbb{N}$ to $\Sigma \oslash !\mathbb{N}$ as the composite:

$$!\mathbb{N} \xrightarrow{\text{lunit}} I \otimes !\mathbb{N} \xrightarrow{\mathtt{OK} \otimes !\mathtt{succ}} \Sigma \otimes !\mathbb{N} \xrightarrow{\text{wk}} \Sigma \oslash !\mathbb{N}$$

In this strategy, the player responds to the initial move q in Σ with a; from then on, she copies from the $!\mathbb{N}$ to $!\mathbb{N}$, increasing each argument by 1 before playing it on the right. This strategy models the effect of a single button press.

We now define a morphism from $!\mathbb{N}$ to $(\mathbb{N} \times \Sigma) \oslash !\mathbb{N}$ by the composite:

$$!\mathbb{N} \xrightarrow{\Delta} !\mathbb{N} \times !\mathbb{N} \xrightarrow{\alpha \times \mathtt{press}} \mathbb{N} \oslash !\mathbb{N} \;\times\; \Sigma \oslash !\mathbb{N} \xrightarrow{\mathtt{dist}^{-1}} (\mathbb{N} \times \Sigma) \oslash !\mathbb{N}$$

In this strategy, if player O starts with a move in $\mathbb{N}$, then player P copies plays in $!\mathbb{N}$ on the left and copies over the response. From then on, she continues to copy numbers from the left over to the right. If player O starts by playing in Σ, then player P responds with the move a. From then on, she copies numbers from the left over to the right, but now she increases the numbers by 1 each time.

This strategy represents a single user interaction (either pressing the button or checking how many times it has been pressed), together with the change of state (represented by the game $!\mathbb{N}$) that takes place after that interaction has taken place. We now take the anamorphism of this strategy, giving us a unique morphism `count` from $!\mathbb{N}$ to $!(\mathbb{N} \times \Sigma)$ making the following diagram commute:

$$\begin{tikzcd} !\mathbb{N} \arrow[r, "\Delta"] \arrow[d, "\mathtt{count}"'] & !\mathbb{N} \times !\mathbb{N} \arrow[r, "\alpha \times \mathtt{press}"] & \mathbb{N} \oslash !\mathbb{N} \;\times\; \Sigma \oslash !\mathbb{N} \arrow[r, "\mathtt{dist}^{-1}"] & (\mathbb{N} \times \Sigma) \oslash !\mathbb{N} \arrow[d, "\mathrm{id} \oslash \mathtt{count}"] \\ !(\mathbb{N} \times \Sigma) \arrow[rrr, "\alpha"] & & & (\mathbb{N} \times \Sigma) \oslash !(\mathbb{N} \times \Sigma) \end{tikzcd}$$

In the strategy `count`, if player O plays q in a copy of Σ, then player P responds with a and adds 1 to an (invisible) counter variable. If player O plays q in a copy of $\mathbb{N}$, then player P queries the argument on the left and returns that value, plus her count, on the right. If we now substitute in the initial value of the state by forming the composite $\mathtt{count} \circ !\mathtt{zero}$, then we end up with the desired behaviour.

A more consequential example is the construction of a strategy representing a storage cell. This is the strategy `cell` for $!(\mathtt{Var}[X])$ that is used in [2] to construct the denotation of the `new` term from Idealized Algol. Using this strategy, it is possible to build the model of Idealized Algol from [2] and hence construct a wealth of other stateful objects. The benefit of constructing `cell` in this way, rather than directly, is that we can now reason about it coalgebraically, rather than by direct combinatorial arguments on the strategy.

Let $(X, *)$ be a pointed set and write X for the game with maximal plays qx for $x \in X$. Recall that in Idealized Algol we represent $\mathtt{Var}[X]$, the type of variables taking values in X, by the game $\Sigma^X \times X$, where Σ denotes the command type `com` and Σ^X is the X-fold product of Σ with itself. Here, Σ^X represents the act of writing a value into the storage cell (so playing in Σ_x means writing the value x), while the copy of X represents reading a value from the storage cell.

We want to construct the strategy `cell` for $!(\Sigma^X \times X)$ that will remember what value we have written into the cell and will return the value when we request it. In the case that we request the value in the cell when nothing has been written to it, we return the default value $*$.

We shall represent the state of the storage cell by the game X, and we shall construct a state transformer on $X \multimap (\Sigma^X \times X) \oslash X$ that will allow us to recover the `cell` strategy as an anamorphism.

For each $x \in X$, we have a strategy c_x for $I \multimap X$ with maximal play qx. Now we construct a morphism $\mathtt{write}(x)$ from $!X$ to $\Sigma \oslash !X$ as the composite:

$$X \xrightarrow{*} 1 \xrightarrow{(\mathtt{dec}^0)^{-1}} I \xrightarrow{\mathrm{runit}_I} I \otimes I \xrightarrow{\mathtt{OK} \otimes c_x} \Sigma \otimes X \xrightarrow{\mathrm{wk}_{\Sigma,X}} \Sigma \oslash X$$

This strategy corresponds to filling the cell with the value x. Consequently, we ignore the previous value from the cell (the copy of X on the left) and we respond in the copy of X on the right with x.

We get a strategy:

$$\mathtt{write}\colon !X \xrightarrow{\langle \mathtt{write}(x)\ :\ x \in X\rangle} (\Sigma \oslash !X)^X \xrightarrow{\mathtt{dist}^{-1}_{(\Sigma\colon x\in X), !X}} \Sigma^X \oslash !X$$

We also want a strategy $\mathtt{read}$ for $X \multimap X \oslash X$. It doesn't appear to be possible to construct this strategy from the sequoidal axioms, but it is easy enough to say what it is: it is the strategy that returns the value of the state while leaving the state unchanged. A typical play in $\mathtt{read}$, therefore, might have the following form:

$$\begin{array}{ccccc}
X & \multimap & X & \oslash & X \\
 & & q & & \\
q & & & & \\
x & & & & \\
 & & x & & \\
 & & & & q \\
 & & & & x
\end{array}$$

Note that the content of the state (the copy of X on the left) is copied into both the output (the first copy of X on the right) and into the new state (the second copy of X on the right).

We put these strategies together to form our state transformer:

$$\mathtt{cell}'\colon !X \xrightarrow{\langle \mathtt{write},\mathtt{read}\rangle} \Sigma^X \oslash !X \ \times\ X \oslash !X \xrightarrow{\mathtt{dist}_{\Sigma^X, X, !X}} (\Sigma^X \times X) \oslash !X$$

When we take the anamorphism $⟪\, \mathtt{cell}' \,⟫$ of this strategy, we get the strategy $!X \to !(\Sigma^X \times X)$ that, when player O plays in Σ^X, stores the appropriate element of X into the (invisible) state. When player O plays in X on the right, player P responds with the current value held in the state. In the case that player O plays in X without having first played in Σ^X, we return the value of X from the left. Therefore, our desired strategy $\mathtt{cell}$ is given by:

$$\mathtt{cell} = ⟪\, \mathtt{cell}' \,⟫ \circ c_*$$

▶ **Exercise 1.** *Let $(X, *)$ be a pointed set, where $*$ is a designated error element, and write X for the game with maximal plays qx for $x \in X$. Consider a history-sensitive strategy on $!(\Sigma^X \times X)$ representing a simple stack with* push(x) *and* pop() *methods. The* push(x) *method should add an element $x \in X$ on to the stack, while the* pop() *method should return the top element of the stack, removing it from the stack in the process, or return $*$ if the stack is empty. Show that this strategy may be constructed from the strategy* OK *on Σ and the strategies c_x on X, using only the sequoidal structure of $\mathcal{G}$.*

Consider how we might add a size() *method, of type $\mathbb{N}$, to this stack, returning the number of elements currently on the stack.*

2 Constructing cofree commutative comonoids in Cartesian sequoidal categories

2.1 The Mellies-Tabareau-Tasson construction

We observed that the exponential $!A$ of a game A arises as the final coalgebra for the functor $A \oslash \_$. We also observed that $!A$ has the structure of a cofree commutative comonoid on A.

In this section, we shall consider various generalizations of this result that hold in sequoidal categories.

The first generalization we will consider is that given by Mellies, Tabareau and Tasson in [9]. Their construction is given in a general monoidal category, but we will show how to interpret it sequoidally. The first step in their construction is to pass from A to the free pointed object $A_\bullet$ on A. In our setting, our category is *affine* – we have an isomorphism $\mathtt{dec}^0$ between the tensor unit I and the terminal object 1 – and so there is no need to carry out this step. In fact, the affineness assumption is not really necessary to any of our results, though it does greatly simplify them: in the non-affine case, we would need to construct the exponential as the final coalgebra for the functor $(A \oslash \_\,) \times I$, and the construction would become rather more complicated. In this paper, we will always use the affineness assumption, so we will leave out the first step in the Mellies-Tabareau-Tasson construction.

The second step in the construction, is to form, for each object A, the *symmetrized tensor product* $A^{\leq n}$, given as the following equalizer:

$$A^{\leq n} \dashrightarrow A^{\otimes n} \underset{\text{symmetry}}{\overset{\text{symmetry}}{\rightrightarrows}} A^{\otimes n}$$

where $A^{\otimes n} = \underbrace{A \otimes \cdots \otimes A}_{n}$. Here, the '$\leq$' is because, thanks to our affine structure, we are free to ignore (weaken) any one of the copies of A.

We claim that, in a decomposable sequoidal category, the object $A^{\leq n}$ always exists and is isomorphic to the game $A^{\oslash n}$, defined inductively by:

$$A^{\oslash 0} = I$$
$$A^{\oslash (n+1)} = A \oslash A^{\oslash n}$$

Indeed, by decomposability, the morphisms $\mathrm{wk}, \mathrm{wk} \circ \mathrm{sym} \colon A \otimes A \to A \oslash A$ exhibit $A \otimes A$ as a product of $A \oslash A$ with $A \oslash A$. Using the coherence conditions, we may generalize this to higher products: write $\mathrm{wk}^n \colon A^{\otimes n} \to A^{\oslash n}$ for the natural morphism

$$\mathrm{wk}^n = \mathrm{id} \oslash (\cdots \oslash (\mathrm{id} \oslash \mathrm{wk}) \cdots) \circ \cdots \circ \mathrm{wk}$$

and for $\pi \in S_n$, write sym^π for the corresponding symmetry $A^{\otimes n} \to A^{\otimes n}$. We can show that the morphisms $\mathrm{wk}^n \circ \mathrm{sym}^\pi$ for $\pi \in S_n$ exhibit $A^{\otimes n}$ as the product of $n!$ copies of $A^{\oslash n}$.

Now the morphism $\langle \mathrm{id}, \ldots, \mathrm{id} \rangle \colon A^{\oslash n} \to (A^{\oslash n})^{n!}$ induces a morphism e from $A^{\oslash n}$ to $A^{\otimes n}$ that equalizes the morphisms $\mathrm{wk}^n \circ \mathrm{sym}^p\, i$ and therefore equalizes the morhpisms sym^π (since wk^n is an epimorphism). We claim that e is the equalizer of the sym^π.

We know that $\mathrm{wk}^n \circ \mathrm{sym}^\pi \circ e = \mathrm{id}$ for all $\pi \in S_n$; taking $\pi = \mathrm{id}$ tells us that $\mathrm{wk}^n \circ e = \mathrm{id}$. Now suppose B is an object of the category and that $f \colon B \to A^{\otimes n}$ is a morphism equalizing the sym^π.

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We want to deduce the fact that $!A \xrightarrow{\mu} !A \otimes !A$ is the cofree commutative comonoid on A from the fact that $!A$ is the final coalgebra for $A \oslash \_\,$. It turns out that we shall need one more fact about the category of games to prove this.

► **Notation 9.** We shall sometimes make the monoidal structure of the Cartesian product explicit by writing $\sigma \times \tau$ for $\langle \sigma \circ \mathrm{pr}_1, \tau \circ \mathrm{pr}_2 \rangle$.

► **Definition 10.** Let A, B be objects of an inclusive, Cartesian, distributive sequoidal category $(\mathcal{C}, \mathcal{C}_s, J, \mathrm{wk})$ with final coalgebras $!A \xrightarrow{\alpha_A} A \oslash !A$ for all endofunctors of the form $A \oslash \_\,$.

Let A, B be objects of C. Then we have a composite $\kappa_{A,B} \colon !A \otimes !B \to (A \times B) \oslash (!A \otimes !B)$:

$$\begin{aligned}
\kappa_{A,B} = !A \otimes !B &\xrightarrow{\langle \mathrm{id}_{!A\otimes !B},\, \mathrm{sym}_{!A,!B}\rangle} (!A \otimes !B) \times (!B \otimes !A) \\
\cdots &\xrightarrow{(\alpha_A \otimes \mathrm{id}_{!B}) \times (\alpha_B \otimes \mathrm{id}_{!A})} ((A \oslash !A) \otimes !B) \times ((B \oslash !B) \otimes !A) \\
\cdots &\xrightarrow{\mathrm{wk}_{A\oslash !A, !B} \times \mathrm{wk}_{B\oslash !B, !A}} ((A \oslash !A) \oslash !B) \times ((B \oslash !B) \oslash !A) \\
\cdots &\xrightarrow{\mathtt{passoc}^{-1}_{A,!A,!B} \times \mathtt{passoc}^{-1}_{B,!B,!A}} (A \oslash (!A \otimes !B)) \times (B \oslash (!B \otimes !A)) \\
\cdots &\xrightarrow{\mathrm{id}_{A\oslash(!A\otimes !B)} \times (\mathrm{id}_B \oslash \mathrm{sym}_{!B,!A})} (A \oslash (!A \otimes !B)) \times (B \oslash (!A \otimes !B))
\end{aligned}$$

inducing a morphism

$$!A \otimes !B \xrightarrow{\kappa_{A,B}} (A \oslash (!A \otimes !B)) \times (B \oslash (!A \otimes !B)) \xrightarrow{\mathtt{dist}^{-1}} (A \times B) \oslash (!A \otimes !B)$$

Remembering that our category has a final coalgebra $!(A \times B)$ for the functor $(A \times B) \oslash \_$, we write $\mathtt{coh}_{A,B}$ for the unique morphism $!A \otimes !B \to !(A \times B)$ making the following diagram commute

$$\begin{tikzcd}
!A \otimes !B \arrow[r, "\kappa_{A,B}"] \arrow[d, "\mathtt{coh}_{A,B}"'] & (A \oslash (!A \otimes !B) \times (B \oslash (!A \otimes !B)) \arrow[r, "\mathtt{dist}^{-1}"] & (A \times B) \oslash (!A \otimes !B) \arrow[d, "\mathrm{id}_{A\times B} \oslash \mathtt{coh}_{A,B}"] \\
!(A \times B) \arrow[rr, "\alpha_{A\times B}"'] & & (A \times B) \oslash !(A \times B)
\end{tikzcd} \qquad (\star)$$

▶ **Proposition 11.** *In the category of games, the morphism $\mathtt{coh}_{A,B}$ is an isomorphism for all negative games A, B.*

Proof. Observe that the morphism $\mathtt{coh}_{A,B}$ is the copycat strategy on $!A \otimes !B \multimap !(A \times B)$ that starts a copy of A on the left whenever a copy of A is started on the right and starts a copy of B on the left whenever a copy of B is started on the right (indeed, the morphisms in the diagram above are all copycat morphisms, so the copycat strategy we have just described must make that diagram commute. Since there are infinitely many copies of both A and B available in $!(A \times B)$, and since a new copy of A or B may be started at any time, we may define an inverse copycat strategy on $!(A \times B) \multimap !A \otimes !B$. ◀

Our first main result for this section will be the following:

▶ **Theorem 12.** *Let $(\mathcal{C}, \mathcal{C}_s, J, \mathrm{wk})$ be a distributive and decomposable sequoidal category with a final coalgebra $!A \xrightarrow{\alpha_A} A \oslash !A$ for each endofunctor of the form $A \oslash \_$. Suppose further that the morphism $\mathtt{coh}_{A,B}$ as defined above is an isomorphism for all objects A, B. $A \mapsto !A$ gives rise to a strong symmetric monoidal functor from the monoidal category $(\mathcal{C}, \times, 1)$ to the monoidal category $(\mathcal{C}, \otimes, I)$.*

We start off by defining a morphism $\mu \colon !A \to !A \otimes !A$. This will turn out to be the comultiplication for the cofree commutative comonoid over A. First, we note that we have the following composite:

$$!A \xrightarrow{\alpha_A} A \oslash !A \xrightarrow{\Delta} (A \oslash !A) \times (A \oslash !A) \xrightarrow{\mathtt{dist}^{-1}} (A \times A) \oslash !A$$

where Δ is the diagonal map on the product. There is therefore a unique morphism $\sigma_A = (\!| \mathtt{dist}^{-1} \circ \Delta \circ \alpha_A |\!)$ making the following diagram commute:

$$\begin{tikzcd}
!A \arrow[r, "\alpha_A"] \arrow[d, "\sigma_A"'] & A \oslash !A \arrow[r, "\Delta"] & (A \oslash !A) \times (A \oslash !A) \arrow[r, "\mathtt{dist}^{-1}"] & (A \times A) \oslash !A \arrow[d, "\mathrm{id}_{A\times A} \oslash \sigma_A"] \\
!(A \times A) \arrow[rrr, "\alpha_{A\times A}"] & & & (A \times A) \oslash !(A \times A)
\end{tikzcd} \qquad (\dagger)$$

and we may set $\mu_A = \mathtt{coh}^{-1}_{A,A} \circ \sigma_A$.

We also define a morphism $\mathtt{der}_A \colon !A \to A$. Note that since I is isomorphic to 1, we have a unique morphism $*_A \colon A \to I$ for each A. We define $\mathtt{der}_A$ to be the composite

$$!A \xrightarrow{\alpha_A} A \oslash !A \xrightarrow{\mathrm{id}_A \oslash *_{!A}} A \oslash I \xrightarrow{\mathtt{r}_A} A$$

We define the action of ! on morphisms as follows: suppose that $\sigma \colon A \to B$ is a morphism in $\mathcal{C}$. Then we have a composite

$$!A \xrightarrow{\mu} !A \otimes !A \xrightarrow{\mathtt{der}_A \otimes \mathrm{id}_{!A}} A \otimes !A \xrightarrow{\sigma \otimes \mathrm{id}_{!A}} B \otimes !A \xrightarrow{\mathrm{wk}_{B,!A}} B \oslash !A$$

There is therefore a unique morphism $!\sigma \colon !A \to !B$ making the following diagram commute:

$$\begin{tikzcd}
!A \arrow[r, "\mu"] \arrow[d, "!\sigma"'] & !A \otimes !A \arrow[r, "\mathtt{der}_A \otimes \mathrm{id}_{!A}"] & A \otimes !A \arrow[r, "\sigma \otimes \mathrm{id}_{!A}"] & B \otimes !A \arrow[r, "\mathrm{wk}_{B,!A}"] & B \oslash !A \arrow[d, "\mathrm{id}_B \oslash !\sigma"] \\
!B \arrow[rrrr, "\alpha_B"] & & & & B \oslash !B
\end{tikzcd}$$

▶ **Proposition 13.** *$\sigma \mapsto !\sigma$ respects composition, so ! is a functor. Moreover, ! is a strong symmetric monoidal functor from the Cartesian category $(\mathcal{C}, \times, 1)$ to the symmetric monoidal category $(\mathcal{C}, \otimes, I)$, witnessed by* $\mathtt{coh}$ *and* $\mathtt{dec}^0$.

Proof. See Appendix. ◀

This completes the proof of Theorem 12.

Since ! is a strong monoidal functor, it induces a functor CCom(!) from the category $\mathrm{CCom}(\mathcal{C}, \times, 1)$ of comonoids over $(\mathcal{C}, \times, 1)$ to the category $\mathrm{CCom}(\mathcal{C}, \otimes, I)$ of comonoids over $(\mathcal{C}, \otimes, I)$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{CCom}(\mathcal{C}, \times, 1) \arrow[r, "\mathcal{F}"] \arrow[d, "\mathrm{CCom}(!)"'] & (\mathcal{C}, \times, 1) \arrow[d, "!"] \\
\mathrm{CCom}(\mathcal{C}, \otimes, I) \arrow[r, "\mathcal{F}"] & (\mathcal{C}, \otimes, I)
\end{tikzcd}$$

where $\mathcal{F}$ is the forgetful functor.

Let A be an object of $\mathcal{C}$. Since $(\mathcal{C}, \times, 1)$ is Cartesian, the diagonal map $\Delta \colon A \to A \times A$ is the cofree commutative comonoid over A in $(\mathcal{C}, \times, 1)$.

▶ **Proposition 14.** $\mathrm{CCom}(!) \left(A \xrightarrow{\Delta} A \times A\right)$ *has comultiplication given by $\mu_A \colon !A \to !A \otimes !A$ and counit given by the unique morphism $\eta_A \colon !A \to I$.*

Proof. See appendix. ◀

In particular, this proves that the comultiplication μ_A is associative and that the counit η_A is a valid counit for μ_A.

We can now state our second main result from this section.

▶ **Theorem 15.** *Let $(\mathcal{C}, \mathcal{C}_s, J, \mathrm{wk})$ be a sequoidal category satisfying all the conditions from Theorem 12. Let A be an object of $\mathcal{C}$ (equivalently, of $\mathcal{C}_s$). Then $!A$, together with the comultiplication μ_A and counit η_A, is the cofree commutative comonoid over A.*

Proof. See Appendix. ◀

3 Transfinite Games

Of the conditions that we used to construct the cofree commutative comonoid in sequoidal categories, the requirement that $\mathtt{coh}_{A,B}$ be an isomorphism stands out as the least satisfactory. All the other conditions are 'finitary', and relate directly to the connectives we have introduced, whereas the morphism $\mathtt{coh}_{A,B}$ can only be constructed using the final coalgebra property for the exponential connective !. For this reason, we might wonder whether we can do without the condition that $\mathtt{coh}_{A,B}$ be an isomorphism. In this section, we shall give a negative answer to that question: we shall construct a distributive and decomposable sequoidal closed category with final coalgebras $!A$ for all functors of the form $A \oslash \_$, and shall show that $!A$ does not have a natural comonoid structure. In doing this, we hope to shed some light upon alternative algebraic or coalgebraic constructions for the cofree commutative comonoid that work in a purely 'finitary' manner.

Our sequoidal category will be closely modelled upon the category of games we have just considered: the objects will be games, with the modification that sequences of moves may now have transfinite length. This is a natural construction, occurring in the study of determinacy by Mycielski [10], Blass [3] and Weiss [11], and it appears to be present in the semantic context in the work of Roscoe [12], Levy [8] and Laird [7].

The general idea is as follows: we will show that the definition of the final coalgebra for the sequoid functor in a category of transfinite games is largely unchanged from the definition in the category of games with finite-length plays: $!A$ is the game formed from a countably infinite number of copies of A, indexed by ω, with the proviso that player O must open them in order. We observe that the copycat strategy $\mathtt{coh}_{A,B} \colon !A \otimes !B \to !(A \times B)$ is not an isomorphism, and that we cannot construct the comultiplication $!A \to !A \otimes !A$ in a sensible way. Moreover, we cannot construct the comonad $!A \to !!A$, so ! does not give us a model of linear logic in even the most general sense. In all three cases, the reason why the construction fails is that we might run out of copies of the game A (or B) on the left hand side before we have run out of copies on the right hand side. In the finite-plays setting, it is impossible to run out of copies of a subgame, because there are infinitely many copies, so it is impossible to play in all of them in a finite-length play. In the transfinite setting, however, we cannot guarantee this: consider, for example, a position in $!A_0 \multimap !A_1 \otimes !A_2$ (with indices given so we can refer to the different copies of A) in which player O has opened all the copies of A in $!A_1$. Since player P is playing by copycat, she must have opened all of the copies of A in $!A_0$. If, at time $\omega + 1$, player O now plays in $!A_2$, player P will have no reply to him.

The 'correct' definition of $!A$ in the transfinite game category is one in which there is an unlimited number of copies of A to open (rather than ω-many), but this is not the final coalgebra for the functor $A \oslash \_$. [TODO: discuss ways to construct this object]

3.1 Transfinite Games

We give a brief summary of the construction of the category of transfinite games. Full details may be found in Appendix **??**.

We shall fix an additively indecomposable ordinal $\alpha = \omega^\beta$ throughout, which will be a bound on the ordinal length of positions in our game. So, for example, the original category of games is the case $\alpha = \omega$. If X is a set, we write $X^{*<\alpha}$ for the set of transfinite sequences of elements of X of length less than α.

▶ **Definition 16.** A *game* or a *game over* α or an α-*game* is a tuple $A = (M_A, \lambda_A, \zeta_A, P_A)$, where:

- M_A is a set of moves
- $\lambda_A \colon M_A \to \{O, P\}$ designates each move as an *O-move* or a *P-move*
- $P_A \subseteq M_A^{*<\alpha}$ is a non-empty prefix-closed set of transfinite sequences of moves from M_A, called *positions*. We say that s is a *successor position* if the length of s is a successor ordinal and we say that s is a *limiting* position if the length of s is a limit ordinal.
- $\zeta_A \colon P_A \to \{O, P\}$ designates each position as an *O-position* or a *P-position*.

such that:

Consistency If $sa \in P_A$ is a successor position, then $\zeta_A(sa) = \lambda_A(a)$

Alternation If $s, sa \in P_A$, then $\zeta_A(s) = \neg\zeta_A(sa)$

Limit closure If $s \in M_A^{*<\alpha}$ is a limiting position such that $t \in P_A$ for all proper prefixes $t \subsetneqq s$, then $s \in P_A$.

We say that a game A is *positive* if $\zeta_A(\epsilon) = O$ and *negative* if $\zeta_A(\epsilon) = P$. We say that A is *completely negative* if $\zeta_A(s) = P$ for all limiting plays s.

Apart from the possibly transfinite length of sequences of moves, the only new thing in this definition is the function ζ_A. Thanks to the consistency condition, ζ_A gives us no new information for successor positions; it is necessary in order to tell us which player is to move at limiting positions.

▶ **Definition 17.** A *strategy* for an α-game A is a non-empty prefix-closed subset $\sigma \subseteq P_A$ satisfying the following conditions:

Closure under O-replies If $s \in \sigma$ is a P-position and $sa \in P_A$, then $sa \in \sigma$.

Determinism If $sa, sb \in \sigma$ are P-positions, then $a = b$.

Given games A and B, we may form their product $A \times B$, tensor product $A \otimes B$, linear implication $A \multimap B$ and sequoid $A \oslash B$ in roughly the same way that we construct these connectives for finite-length games. The only point we need to take care of is the behaviour of the ζ-functions at limit ordinals. We do this according to the following formulae:

$$\zeta_{A\times B}(s) = \zeta_A(s) \wedge \zeta_B(s)$$
$$\zeta_{A\otimes B}(s) = \zeta_A(s) \wedge \zeta_B(s)$$
$$\zeta_{A\multimap B}(s) = \zeta_A(s) \Rightarrow \zeta_B(s)$$
$$\zeta_{A\oslash B}(s) = \zeta_A(s) \wedge \zeta_B(s)$$

Here, $\wedge$ and $\Rightarrow$ are the usual propositional connectives on $\{T, F\}$, but with T replaced by P and F replaced by O.

Once we have defined our connectives, we may define a *morphism* from A to B to be a strategy for $A \multimap B$ and we may define composition of morphisms in the usual way: given games A, B and C, and strategies σ for $A \multimap B$ and τ for $B \multimap C$, we define

$$\sigma \| \tau = \{\mathfrak{s} \in (M_A \sqcup M_B \sqcup M_C)^{*<\alpha} \,:\, \mathfrak{s}|_{A,B} \in \sigma,\ \mathfrak{s}|_{B,C} \in \tau\}$$

and then we define

$$\tau \circ \sigma = \{\mathfrak{s}|_{A,C} \,:\, \mathfrak{s} \in \sigma \| \tau\}$$

▶ Remark. Since α is additively decomposable, the interleaving of two sequences of length less than α must itself have length less than α. This is important: if we allow α to be an additively decomposable ordinal, then it is possible to construct two strategies whose composite is not closed under O-replies because a particular reply in the interleaving of two sequences occurs at time later than α and so is not included.

We can show that this composition is associative and moreover that we obtain a distributive and decomposable sequoidal category. We call this category $\mathcal{G}(\alpha)$ and call the corresponding strict subcategory $\mathcal{G}_s(\alpha)$. The hardest part of this is showing that the category is monoidal closed, because the linear implication of completely negative games is not necessarily completely negative.

3.2 The final sequence for the sequoidal exponential

We now want to show that $\mathcal{G}(\alpha)$ has final coalgebras for the functor $A \oslash \_$, given by the transfinite game $!A$, which is defined as follows:

- $M_{!A} = M_A \times \omega$
- $\lambda_{!A} = \lambda_A \circ \mathrm{pr}_1$

We define $!P_A$ to be the set of all sequences $s \in M_{!A}^{*<\alpha}$ such that $s|_n \in P_A$ for all n. Then we define $\zeta_{!A} \colon !P_A \to \{O, P\}$ by

$$\zeta_{!A}(s) = \bigwedge_{n \in \omega} \zeta_A(s|_n)$$

In other words, $\zeta_{!A}(s) = P$ if and only if $\zeta_A(s|_n) = P$ for all n.

There is a natural copycat strategy $\alpha_A \colon !A \to A \oslash !A$, just as in the finite plays case. We want to show that this is the final coalgebra for $A \oslash \_$. The proof for the finite case found in [4] will not work in this case, since it implicitly uses the fact that $!A$ is the limit of the sequence

$$I \leftarrow A \leftarrow A \oslash A \leftarrow A \oslash (A \oslash A) \leftarrow \cdots$$

(cf. also [9]). In the transfinite categories, this is no longer the case.

While it is possible to prove that $\alpha_A \colon !A \to A \oslash !A$ is the final coalgebra for $A \oslash \_$ directly, we shall instead give a proof by extending the sequence given above to an ordinal-indexed sequence. This is the *final sequence*, familiar in coalgebra [13]. Specifically, we construct a functor $\mathcal{F} \colon \mathbf{Ord}^{\mathrm{op}} \to \mathcal{G}(\alpha)$, where $\mathbf{Ord}$ is the order category of the ordinals, writing

$$\mathcal{F}(\gamma) = A^{\oslash \gamma}$$
$$\mathcal{F}(\gamma \leq \delta) = j_\gamma^\delta \colon A^{\oslash \delta} \to A^{\oslash \gamma}$$

according to the following inductive recipe:

- $A^{\oslash 0} = 1$
- $A^{\oslash(\gamma+1)} = A \oslash A^{\oslash \gamma}$
- If μ is a limit ordinal, then $A^{\oslash \mu}$ is the limit of the diagram formed by the $A^{\oslash \gamma}$ for $\gamma < \mu$, together with the morphisms j_γ^δ, for $\gamma \leq \delta < \mu$.
- $j_0^\gamma = *$
- j_γ^λ is the morphism in the limiting cone
- $j_\gamma^{\delta+1} = j_\delta^{\delta+1} \circ j_\gamma^\delta$
- If we write C_λ for the limiting cone for λ over the $A^{\oslash \gamma}$ for $\gamma < \mu$, then we may form a cone $A \oslash C_\lambda$ over the same diagram by applying the functor $A \oslash \_$ to C_λ and then extending the cone to 1 in the only possible way. Then $j_\lambda^{\lambda+1}$ is the unique morphism from $A^{\oslash(\lambda+1)}$ to $A^{\oslash \lambda}$ inducing a morphism of cones from $A \oslash C_\lambda$ to C_λ.

It is well known (see [13], for instance) that if $j_\delta^{\delta+1}$ is an isomorphism for some δ, then ${j_\delta^{\delta+1}}^{-1} \colon A^{\oslash \delta} \to A \oslash A^{\oslash \delta}$ is the final coalgebra for the functor $A \oslash \_$. In this case, we say that the sequence *stabilizes at* δ. Our proof strategy is therefore to show that the sequence stabilizes at some δ, and to show that $A^{\oslash\delta}$ is isomorphic to $!A$.

We do this by giving a classification of the games $A^{\oslash\gamma}$. Let $s \in \omega^{*<\alpha}$ be any transfinite sequence of natural numbers. We define the *derivative* Δs of s to be the sequence given by removing all instances of 0 from s and subtracting 1 from all other terms. In other words, if $s\colon \gamma \to \omega$, for $\gamma < \alpha$, then we have:

$$\Delta s = s^{-1}(\omega \setminus \{0\}) \xrightarrow{s} \omega \setminus \{0\} \xrightarrow{-1} \omega$$

(where $s^{-1}(\omega \setminus \{0\})$ carries the induced order). We now define predicates $\_ \leq \gamma$ on sequences $s \in \omega^{*<\alpha}$ as follows:

- $\epsilon \leq 0$
- If $\Delta s \leq \gamma$, then $s \leq \gamma + 1$
- If μ is a limit ordinal and $s \in \omega^{*<\alpha}$ is such that for all successor-length prefixes $t \sqsubseteq s$ we have $t \leq \gamma$ for some $\gamma < \mu$, then $s \leq \mu$. In other words, $\{s \in \omega^{*<\alpha} : s \leq \mu\}$ is the limit-closure of the union of the sets $\{s \in \omega^{*<\alpha} : s \leq \gamma\}$ for $\gamma < \mu$.

We can prove some basic results about these predicates:

▶ **Proposition 18.** *i) If $s \leq \gamma$ and t is any subsequence of s (not necessarily an initial prefix), then $t \leq \gamma$.*

ii) If $s \leq \gamma$, then $\Delta s \leq \gamma$

iii) If $s \leq \gamma$ and $\gamma \leq \delta$, then $s \leq \delta$

iv) If $s \in \omega^{*<\alpha}$ has length μ, where μ is a limit ordinal, then $s \leq \mu$. If s has length $\mu + n$ for some $n \in \omega$, then $s \leq \mu + \omega$. In particular, $s \leq \alpha$ for all $s \in \omega^{*<\alpha}$.

Proof. See Appendix. ◀

Our classification result for the final sequence then becomes:

▶ **Theorem 19.** *Let A be any game. Then $A^{\oslash\gamma} \cong (M_{!A}, \lambda_{!A}, \zeta_{!A}, P_{!A,\gamma})$, where*

$$P_{!A,\gamma} = \{s \in P_{!A} : \mathrm{pr}_2 \circ s \leq \gamma\}$$

The morphism j_γ^δ is the copycat strategy.

Proof. See Appendix. ◀

▶ **Corollary 20.** *The final sequence for $A \oslash \_$ stabilizes at α and we have $A^{\oslash\alpha} = !A$.*

Proof. By Proposition 18(iv), $\mathrm{pr}_2 \circ s \leq \alpha$ for all $s \in P_{!A}$ and so $\mathrm{pr}_2 \circ s \leq (\alpha + 1)$, by Proposition 18(iii). It follows, by Theorem 19, that $A^{\oslash\alpha} = !A$ and that the morphism $A^{\oslash\alpha} \to A^{\oslash(\alpha+1)}$ is the morphism α_A. ◀

A Proofs

A.1 Proof of Proposition 13

▶ **Proposition 13.** *$\sigma \mapsto !\sigma$ respects composition, so $!$ is a functor. Moreover, $!$ is a strong symmetric monoidal functor from the Cartesian category $(\mathcal{C}, \times, 1)$ to the symmetric monoidal category $(\mathcal{C}, \otimes, I)$, witnessed by* `coh` *and* $\mathtt{dec}^0$.

In order to show that $\sigma \mapsto !\sigma$ respects composition, we need the following lemma:

▶ **Lemma 21.** *Let A be an object of $\mathcal{C}$. Then $\alpha_A \colon {!A} \to A \oslash {!A}$ is equal to the following composite:*

$$!A \xrightarrow{\mu_A} {!A} \otimes {!A} \xrightarrow{\mathtt{der}_A \otimes \mathrm{id}_{!A}} A \otimes {!A} \xrightarrow{\mathrm{wk}_{A,!A}} A \oslash {!A}$$

Proof. We may paste together diagrams ($\star$) and ($\dagger$) to form the following diagram (where we shall omit subscripts where there is no ambiguity):

$$\begin{tikzcd}
{!A} \arrow[r, "\alpha"] \arrow[d, "\sigma_A"'] & A \oslash {!A} \arrow[r, "\Delta"] & (A \oslash {!A}) \times (A \oslash {!A}) \arrow[r, "\mathtt{dist}^{-1}"] & (A \times A) \oslash {!A} \arrow[d, "\mathrm{id}_{A\times A} \oslash \sigma_A"] \\
{!(A \times A)} \arrow[rrr, "\alpha"] & & & (A \times A) \oslash {!(A \times A)} \\
{!A} \otimes {!A} \arrow[u, "\mathtt{coh}_A"] \arrow[rr, "\kappa_{A,A}"] & & (A \oslash ({!A} \otimes {!A})) \times (A \oslash ({!A} \otimes {!A})) \arrow[r, "\mathtt{dist}^{-1}"] & (A \times A) \oslash ({!A} \otimes {!A}) \arrow[u, "\mathrm{id}_{A\times A} \oslash \mathtt{coh}_A"']
\end{tikzcd}$$

where we observe that the composites down the left and right hand sides (after inverting the lower arrows) are μ_A and $\mathrm{id}_{A\times A} \oslash \mu_A$.

Now note that we have the following commutative square:

$$\begin{tikzcd}
(A \times A) \oslash {!A} \arrow[r, "\mathtt{dist}"] \arrow[d, "\mathrm{id}_{A\times A} \oslash \mu_A"'] & (A \oslash {!A}) \times (A \oslash {!A}) \arrow[d, "(\mathrm{id} \oslash \mu) \times (\mathrm{id} \oslash \mu)"] \\
(A \times A) \oslash ({!A} \otimes {!A}) \arrow[r, "\mathtt{dist}"'] & (A \oslash ({!A} \otimes {!A})) \times (A \oslash ({!A} \otimes {!A}))
\end{tikzcd}$$

(using the definition of `dist`). Putting this together with the diagram above, we get the following commutative diagram:

$$\begin{tikzcd}
{!A} \arrow[r, "\alpha"] \arrow[d, "\mu_A"'] & A \oslash {!A} \arrow[r, "\Delta"] & (A \oslash {!A}) \times (A \oslash {!A}) \arrow[d, "\mathrm{id} \oslash \mu_A \times \mathrm{id} \oslash \mu_A"] \\
{!A} \otimes {!A} \arrow[rr, "\kappa_{A,A}"] & & (A \oslash ({!A} \otimes {!A})) \times (A \oslash ({!A} \otimes {!A}))
\end{tikzcd}$$

We now expand the definition of $\kappa_{A,A}$ and take the projections on to the first and second components, yielding the following two commutative diagrams:

$$\begin{tikzcd}
{!A} \arrow[rrr, "\alpha"] \arrow[d, "\mu_A"'] & & & A \oslash {!A} \arrow[d, "\mathrm{id} \oslash \mu_A"] \\
{!A} \otimes {!A} \arrow[r, "\alpha \otimes \mathrm{id}"] & (A \oslash {!A}) \otimes {!A} \arrow[r, "\mathrm{wk}"] & (A \oslash {!A}) \oslash {!A} \arrow[r, "\mathtt{passoc}^{-1}"] & A \oslash ({!A} \otimes {!A})
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
{!A} \arrow[rrrr, "\alpha"] \arrow[d, "\mu_A"'] & & & & A \oslash {!A} \arrow[d, "\mathrm{id} \oslash \mu_A"] \\
{!A} \otimes {!A} \arrow[r, "\mathrm{sym}"'] & {!A} \otimes {!A} \arrow[r, "\alpha \otimes \mathrm{id}"] & (A \oslash {!A}) \otimes {!A} \arrow[r, "\mathtt{passoc}^{-1} \circ \mathrm{wk}"] & A \oslash ({!A} \otimes {!A}) \arrow[r, "\mathrm{id} \oslash \mathrm{sym}"'] & A \oslash ({!A} \otimes {!A})
\end{tikzcd} \tag{2}$$

From diagram (**1**), we construct the following commutative diagram:

$$
\begin{tikzcd}
!A \arrow[rrr, "\alpha"] \arrow[d, squiggly, "\mu_A"'] \arrow[drrr, phantom, "\mathbf{a}"] & & & A \oslash !A \arrow[d, "\mathrm{id} \oslash \mu_A"] \\
!A \otimes !A \arrow[r, "\alpha \otimes \mathrm{id}"] \arrow[ddr, squiggly, "\mathtt{der}_A \otimes \mathrm{id}"'] \arrow[dr, phantom, "\mathbf{b}"] & (A \oslash !A) \otimes !A \arrow[r, "\mathrm{wk}"] \arrow[d, "(\mathrm{id} \oslash *) \otimes \mathrm{id}"] \arrow[dr, phantom, "\mathbf{c}"] & (A \oslash !A) \oslash !A \arrow[r, "\mathtt{passoc}^{-1}"] \arrow[d, "(\mathrm{id} \oslash *) \oslash \mathrm{id}"] \arrow[dr, phantom, "\mathbf{e}"] & A \oslash (!A \oslash !A) \arrow[d, "\mathrm{id} \oslash (* \otimes \mathrm{id})"] \\
& (A \oslash I) \otimes !A \arrow[r, "\mathrm{wk}"] \arrow[d, "\mathtt{r} \otimes \mathrm{id}"] \arrow[dr, phantom, "\mathbf{d}"] & (A \oslash I) \oslash !A, \arrow[r, "\mathtt{passoc}^{-1}"] \arrow[d, "\mathtt{r} \otimes \mathrm{id}"] & A \oslash (I \otimes !A) \arrow[dl, "\mathrm{id} \oslash \mathrm{lunit}"] \arrow[dl, phantom, "\mathbf{f}"'] \\
& A \otimes !A \arrow[r, squiggly, "\mathrm{wk}"] & A \oslash !A &
\end{tikzcd}
$$

a is diagram (**1**).

b commutes by the definition of $\mathtt{der}_A$.

c and **d** commute because wk is a natural transformation.

e commutes because `passoc` is a natural transformation.

f commutes by one of the coherence conditions in the definition of a sequoidal category.

We now observe that the composite of the three squiggly arrows is the composite we are trying to show is equal to α; we have α along the top, so it will suffice to show that the composite

$$\xi_A \;=\; !A \xrightarrow{\mu_A} !A \otimes !A \xrightarrow{*\otimes \mathrm{id}} I \otimes !A \xrightarrow{\mathrm{lunit}} !A$$

is equal to the identity. We do this using diagram (**2**). First we construct the diagram shown in Figure 1.

Now observe that the composite ξ_A is running along the left hand side of Figure 1, while $\mathrm{id} \oslash \xi$ is running along the right. Since we have α along the bottom, it follows by the uniqueness of $\llparenthesis \cdot \rrparenthesis$ that $\xi = \llparenthesis \alpha \rrparenthesis = \mathrm{id}_{!A}$. ◀

Now we are ready to show that $\sigma \mapsto !\sigma$ respects composition. Let A, B, C be objects, let σ be a morphism from A to B and let τ be a morphism from B to C. Using Lemma 21 and the definition of $!\sigma$, $!\tau$, we may construct a commutative diagram:

$$
\begin{tikzcd}
!A \arrow[r, "\mu"] \arrow[d, "!\sigma"'] & !A \otimes !A \arrow[r, "\mathtt{der} \otimes \mathrm{id}"] & A \otimes !A \arrow[r, "\sigma \otimes \mathrm{id}"] & B \otimes !A \arrow[r, "\mathrm{wk}"] \arrow[d, dotted, "\mathrm{id} \otimes !\sigma"] & B \oslash !A \arrow[d, "\mathrm{id} \oslash !\sigma"] \\
!B \arrow[r, "\mu"] \arrow[dd, "!\tau"'] & !B \otimes !B \arrow[rr, "\mathtt{der} \otimes \mathrm{id}"] & & B \otimes !B \arrow[r, "\mathrm{wk}"] \arrow[d, "\tau \otimes \mathrm{id}"] & B \oslash !B \\
& & & C \otimes !B \arrow[r, "\mathrm{wk}"] \arrow[d, dotted, "\mathrm{id} \otimes !\tau"'] & C \oslash !B \arrow[d, "\mathrm{id} \oslash !\tau"] \\
!C \arrow[r, "\mu"] & !C \otimes !C \arrow[rr, "\mathtt{der} \otimes \mathrm{id}"] & & C \otimes !C \arrow[r, "\mathrm{wk}"] & C \oslash !C
\end{tikzcd}
$$

Here, the outermost (solid) shapes commute by the definition of $!\sigma$, $!\tau$ (after we have replaced α_B, α_C with the composite from Lemma 21). The smaller squares on the right hand side commute because wk is a natural transformation. Now observe that $\mathrm{wk}_{X,Y} = \mathrm{pr}_1 \circ \mathtt{dec}_{X,Y}$ is the composition of epimorphisms, so is an epimorphism for all X, Y. It follows that the two rectangles on the left commute as well.

$$
\begin{tikzcd}
!A \arrow[rrrrrr, "\alpha"] \arrow[d, "\mu_A"'] & & & \arrow[d, phantom, "\mathbf{a}"] & & & A \oslash !A \arrow[d, "\mathrm{id} \oslash \mu_A"] \\
!A \otimes !A \arrow[r, "\mathrm{sym}"] \arrow[d, "* \otimes \mathrm{id}"'] \arrow[dr, phantom, "\mathbf{b}"] & !A \otimes !A \arrow[r, "\alpha \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes *"] & (A \oslash !A) \otimes !A \arrow[r, "\mathrm{wk}"] \arrow[d, "\mathrm{id} \oslash *"] \arrow[dr, phantom, "\mathbf{d}"] & (A \oslash !A) \oslash !A \arrow[r, "\mathtt{passoc}^{-1}"] \arrow[d, "\mathrm{id} \oslash *"] \arrow[dr, phantom, "\mathbf{e}"] & A \oslash (!A \otimes !A) \arrow[r, "\mathrm{id} \oslash \mathrm{sym}"] \arrow[d, "\mathrm{id} \oslash (\mathrm{id} \oslash *)"] \arrow[dr, phantom, "\mathbf{c}"] & A \oslash (!A \otimes !A) \arrow[d, "\mathrm{id} \oslash (* \oslash \mathrm{id})"] \\
I \otimes !A \arrow[r, "\mathrm{sym}"] \arrow[d, "\mathrm{lunit}"'] \arrow[dr, phantom, "\mathbf{g}"] & !A \otimes I \arrow[r, "\alpha \otimes \mathrm{id}"] \arrow[d, "\mathrm{runit}"] \arrow[dr, phantom, "\mathbf{f}"] & (A \oslash !A) \otimes I \arrow[r, "\mathrm{wk}"] \arrow[d, "\mathrm{runit}"] \arrow[dr, phantom, "\mathbf{i}"] & (A \oslash !A) \oslash I \arrow[r, "\mathtt{passoc}^{-1}"] \arrow[d, "\mathtt{r}"] \arrow[dr, phantom, "\mathbf{j}"] & A \oslash (!A \otimes I) \arrow[r, "\mathrm{id} \oslash \mathrm{sym}"] \arrow[d, "\mathrm{id} \oslash \mathrm{runit}"] \arrow[dr, phantom, "\mathbf{h}"] & A \oslash (I \otimes !A) \arrow[d, "\mathrm{id} \oslash \mathrm{lunit}"] \\
!A \arrow[r, "\mathrm{id}"'] & !A \arrow[r, "\alpha"] & A \oslash !A \arrow[r, "\mathrm{id}"'] & A \oslash !A \arrow[r, "\mathrm{id}"'] & A \oslash !A \arrow[r, "\mathrm{id}"'] & A \oslash !A
\end{tikzcd}
$$

Figure 1 **a** is diagram (**2**).

b and **c** commute because sym is a natural transformation, **d** commutes because wk is a natural transformation and **e** commutes because `passoc` is a natural transformation. **f** commutes because runit is a natural transformation.

g and **h** commute by one of the coherence conditions for a symmetric monoidal category. **i** commutes by one of the coherence conditions for wk in the definition of a sequoidal category and **j** commutes by one of the coherence conditions for `passoc` in the definition of a sequoidal category.

Throwing away the right hand squares and adding some new arrows at the right, we arrive at the following commutative diagram:

$$\begin{tikzcd}
!A \arrow[rr, "(\sigma\otimes\mathrm{id})\circ(\mathtt{der}\otimes\mathrm{id})\circ\mu"] \arrow[d, "!\sigma"'] & & B\otimes !A \arrow[r, "\tau\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes!\sigma"'] & C\otimes !A \arrow[d, "\mathrm{id}\otimes!\sigma"] \\
!B \arrow[d, "!\tau"'] & & B\otimes !B \arrow[r, "\tau\otimes\mathrm{id}"] \arrow[d, "\tau\otimes!\tau"'] & C\otimes !B \arrow[dl, "\mathrm{id}\otimes!\tau"] \\
!C \arrow[rr, "(\mathtt{der}\otimes\mathrm{id})\circ\mu"] & & C\otimes !C &
\end{tikzcd}$$

We have just shown that the square on the left commutes. The shapes on the right commute by inspection. We now throw away the internal arrows and re-apply wk on the right hand side:

$$\begin{tikzcd}
!A \arrow[rr, "((\tau\circ\sigma)\otimes\mathrm{id})\circ(\mathtt{der}\otimes\mathrm{id})\circ\mu"] \arrow[d, "!\sigma"'] & & C\otimes !A \arrow[r, "\mathrm{wk}"] \arrow[d, "\mathrm{id}\otimes!\sigma"'] & C\oslash !A \arrow[d, "\mathrm{id}\oslash!\sigma"] \\
!B \arrow[d, "!\tau"'] & & C\otimes !B \arrow[r, "\mathrm{wk}"] \arrow[d, "\mathrm{id}\otimes!\tau"'] & !C\oslash !B \arrow[d, "\mathrm{id}\oslash!\tau"] \\
!C \arrow[rr, "(\mathtt{der}\otimes\mathrm{id})\circ\mu"] & & C\otimes !C \arrow[r, "\mathrm{wk}"] & C\oslash !C
\end{tikzcd}$$

By Lemma 21, the composite along the bottom is equal to α_C. Therefore, by uniqueness of $\llparenthesis\ \cdot\ \rrparenthesis$, we have

$$!\tau\circ!\sigma = \llparenthesis\ \mathrm{wk}\circ((\tau\circ\sigma)\otimes\mathrm{id})\circ(\mathtt{der}\otimes\mathrm{id})\circ\mu\ \rrparenthesis = !(\tau\circ\sigma)$$

Therefore, ! is indeed a functor.

We now want to show that ! has the structure of a strong symmetric monoidal functor from $(\mathcal{C},\times,1)$ to $(\mathcal{C},\otimes,I)$. The relevant morphisms are:

$$\mathtt{coh}_{A,B}\colon !A\otimes !B\to !(A\times B) \quad \mathtt{dec}^0\colon I\to 1$$

By hypothesis, these are both isomorphisms. We just need to show that the appropriate coherence diagrams commute. That is, for any games A, B, C, we need to show that the following diagrams commute:

$$\begin{tikzcd}
(!A\otimes !B)\otimes !C \arrow[r, "\mathtt{assoc}_{A,B,C}"] \arrow[d, "\mathtt{coh}_{A,B}\otimes\mathrm{id}_{!C}"] & !A\otimes(!B\otimes !C) \arrow[d, "\mathrm{id}_{!A}\otimes\mathtt{coh}_{B,C}"] \\
!(A\times B)\otimes !C \arrow[d, "\mathtt{coh}_{A\times B,C}"] & !A\otimes !(B\times C) \arrow[d, "\mathtt{coh}_{A,B\times C}"] \\
!((A\times B)\times C) \arrow[r, "!\mathtt{assoc}_{\times,A,B,C}"] & !(A\times(B\times C))
\end{tikzcd}$$

We first prove a small lemma, which gives us a simpler way to compute $!\sigma$ in the case that σ is a morphism in $\mathcal{C}_s$.

▶ **Lemma 22.** *Let A, B be objects of $\mathcal{C}_s$ and let σ be a morphism from A to B in $\mathcal{C}_s$. Then the following diagram commutes:*

$$\begin{tikzcd}
!A \arrow[r, "\alpha_A"] \arrow[d, "!\sigma"'] & A\oslash !A \arrow[r, "\sigma\oslash\mathrm{id}"] & B\oslash !A \arrow[d, "\mathrm{id}\oslash!\sigma"] \\
!B \arrow[rr, "\alpha_B"] & & B\oslash !B
\end{tikzcd}$$

Proof. By the definition of $!\sigma$, we have the following commutative diagram:

$$\begin{tikzcd}
!A \arrow[r, "\mu_A"] \arrow[d, "!\sigma"'] & !A \otimes !A \arrow[r, "\mathtt{der}\otimes\mathrm{id}"] & A \otimes !A \arrow[r, "\sigma\otimes\mathrm{id}"] & B \otimes !A \arrow[r, "\mathrm{wk}"] & B \oslash !A \arrow[d, "\mathrm{id}_B \oslash !\sigma"] \\
!B \arrow[rrrr, "\alpha_B"] & & & & B \oslash !B
\end{tikzcd}$$

Therefore, it will suffice to show that the following diagram (solid lines) commutes:

$$\begin{tikzcd}
!A \arrow[r, "\alpha_A"] \arrow[d, "\mu_A"'] & A \oslash !A \arrow[r, "\sigma\oslash\mathrm{id}"] & B \oslash !A \\
!A \otimes !A \arrow[r, "\mathtt{der}\otimes\mathrm{id}"] & A \otimes !A \arrow[r, "\sigma\otimes\mathrm{id}"] \arrow[u, dotted, "\mathrm{wk}"] & B \otimes !A \arrow[u, "\mathrm{wk}"]
\end{tikzcd}$$

The left hand square commutes by Lemma 21. The right hand square commutes because wk is a natural transformation. ◀

To show that the first coherence diagram commutes, we define a composite $\eta_{A,B,C}$:

$$(!A \otimes !B) \otimes !C \xrightarrow{\langle \mathrm{id},\, \mathrm{sym}\rangle} ((!A \otimes !B) \otimes !C) \times (!C \otimes (!A \otimes !B))$$
$$\cdots \xrightarrow{((\mathtt{dist}^{-1}\circ\kappa_{A,B})\otimes\mathrm{id})\times(\alpha_C\otimes\mathrm{id})} (((A \times B) \oslash (!A \otimes !B)) \otimes !C) \times ((C \oslash !C) \otimes (!A \otimes !B))$$
$$\cdots \xrightarrow{\mathrm{wk}\times\mathrm{wk}} (((A \times B) \oslash (!A \otimes !B)) \oslash !C) \times ((C \oslash !C) \oslash (!A \otimes !B))$$
$$\cdots \xrightarrow{\mathtt{passoc}^{-1}\times\mathtt{passoc}^{-1}} ((A \times B) \oslash ((!A \otimes !B) \otimes !C)) \times (C \oslash (!C \otimes (!A \otimes !B)))$$
$$\cdots \xrightarrow{\mathrm{id}\times(\mathrm{id}\oslash\mathrm{sym})} ((A \times B) \oslash ((!A \otimes !B) \otimes !C)) \times (C \oslash ((!A \otimes !B) \otimes !C))$$

Observe the similarity between the definition of $\eta_{A,B,C}$ and that of $\kappa_{A\times B,C}$. Indeed, it may be easily verified that the following diagram commutes, using the definition of $\mathtt{coh}_{A,B}$ as the anamorphism for $\mathtt{dist}^{-1} \circ \kappa_{A,B}$ and the fact that wk, $\mathtt{passoc}$ and sym are natural transformations:

$$\begin{tikzcd}
((!A \otimes !B) \otimes !C) \arrow[r, "\eta_{A,B,C}"] \arrow[d, "\mathtt{coh}_{A,B}\otimes\mathrm{id}"'] & ((A \times B) \oslash ((!A \otimes !B) \otimes !C)) \times (C \oslash ((!A \otimes !B) \otimes !C)) \arrow[d, "(\mathrm{id}\oslash(\mathtt{coh}\otimes\mathrm{id}))\times(\mathrm{id}\oslash(\mathtt{coh}\otimes\mathrm{id}))"] \\
!(A \times B) \otimes !C \arrow[r, "\kappa_{A\times B,C}"] & ((A \times B) \oslash (!(A \times B))) \times (C \oslash (!(A \times B) \otimes !C))
\end{tikzcd}$$

▶ **Lemma 23.** *The diagrams in Figure 2 commute.*

Proof. First, observe the diagram in Figure 3. Setting, $X = A$, $Y = B$ and $Z = C$, this gives us diagram (**A**) immediately. ◀

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$$\begin{tikzcd}
(!A \otimes !B) \otimes !C \arrow[r, "(\alpha_A \otimes \mathrm{id}) \otimes \mathrm{id}"] \arrow[d, "\mathrm{assoc}"'] & ((A \oslash !A) \otimes !B) \otimes !C \arrow[r, "(\mathtt{passoc}^{-1} \circ \mathrm{wk}) \otimes \mathrm{id}"] & (A \oslash (!A \otimes !B)) \otimes !C \arrow[r, "\mathtt{passoc}^{-1} \circ \mathrm{wk}"] & A \oslash ((!A \otimes !B) \otimes !C) \arrow[d, "\mathrm{id} \oslash \mathrm{assoc}"] \\
!A \otimes (!B \otimes !C) \arrow[r, "\alpha_A \otimes \mathrm{id}"] & (A \oslash !A) \otimes (!B \otimes !C) \arrow[r, "\mathrm{wk}"] & (A \oslash !A) \oslash (!B \otimes !C) \arrow[r, "\mathtt{passoc}^{-1}"] & A \oslash (!A \otimes (!B \otimes !C))
\end{tikzcd} \quad (\mathbf{A})$$

$$\begin{tikzcd}
(!A \otimes !B) \otimes !C \arrow[r, "\mathrm{sym}"] \arrow[d, "\mathrm{assoc}"'] & (!B \otimes !A) \otimes !C \arrow[r, "(\alpha_B \otimes \mathrm{id}) \otimes \mathrm{id}"] & ((B \oslash !B) \otimes !A) \otimes !C \arrow[r, "(\mathtt{passoc}^{-1} \circ \mathrm{wk}) \otimes \mathrm{id}"] & (B \oslash (!B \otimes !A)) \otimes !C \arrow[r, "\mathtt{passoc}^{-1} \circ \mathrm{wk}"] & B \oslash ((!B \otimes !A) \otimes !C) \arrow[r, "\mathrm{id} \oslash (\mathrm{sym} \otimes \mathrm{id})"] & B \oslash ((!A \otimes !B) \otimes !C) \arrow[d, "\mathrm{id} \oslash \mathrm{assoc}"] \\
!A \otimes (!B \otimes !C) \arrow[r, "\mathrm{sym}"] & (!B \otimes !C) \otimes !A \arrow[r, "(\alpha_B \otimes \mathrm{id}) \otimes \mathrm{id}"] & ((B \oslash !B) \otimes !C) \otimes !A \arrow[r, "(\mathtt{passoc}^{-1} \circ \mathrm{wk}) \otimes \mathrm{id}"] & (B \oslash (!B \otimes !C)) \otimes !A \arrow[r, "\mathtt{passoc}^{-1} \circ \mathrm{wk}"] & B \oslash ((!B \otimes !C) \otimes !A) \arrow[r, "\mathrm{id} \oslash \mathrm{sym}"] & B \oslash (!A \otimes (!B \otimes !C))
\end{tikzcd} \quad (\mathbf{B})$$

$$\begin{tikzcd}
(!A \otimes !B) \otimes !C \arrow[r, "\mathrm{sym}"] \arrow[d, "\mathrm{assoc}"'] & !C \otimes (!A \otimes !B) \arrow[r, "\alpha_C \otimes \mathrm{id}"] & (C \oslash !C) \otimes (!A \otimes !B) \arrow[r, "\mathrm{wk}"] & (C \oslash !C) \oslash (!A \otimes !B) \arrow[r, "\mathtt{passoc}^{-1}"] & C \oslash (!C \otimes (!A \otimes !B)) \arrow[r, "\mathrm{id} \oslash \mathrm{sym}"] & C \oslash ((!A \otimes !B) \otimes !C) \arrow[d, "\mathrm{id} \oslash \mathrm{assoc}"] \\
!A \otimes (!B \otimes !C) \arrow[r, "(\mathrm{sym} \otimes \mathrm{id}) \circ \mathrm{sym}"] & (!C \otimes !B) \otimes !A \arrow[r, "(\alpha_C \otimes \mathrm{id}) \otimes \mathrm{id}"] & ((C \oslash !C) \otimes !B) \otimes !A \arrow[r, "(\mathtt{passoc}^{-1} \circ \mathrm{wk}) \otimes \mathrm{id}"] & (C \oslash (!C \otimes !B)) \otimes !A \arrow[r, "\mathtt{passoc}^{-1} \circ \mathrm{wk}"] & C \oslash ((!C \otimes !B) \otimes !A) \arrow[r, "\mathrm{id} \oslash ((\mathrm{sym} \otimes \mathrm{id}) \circ \mathrm{sym})"] & C \oslash (!A \otimes (!B \otimes !C))
\end{tikzcd} \quad (\mathbf{C})$$

Figure 2 Lemma 23 asserts that these diagrams commute.

$$\begin{tikzcd}
(!X \otimes !Y) \otimes !Z \arrow[r, "(\alpha \otimes \mathrm{id}) \otimes \mathrm{id}"] \arrow[dd, "\mathrm{assoc}"'] & ((X \oslash !X) \otimes !Y) \otimes !Z \arrow[r, "\mathrm{wk} \otimes \mathrm{id}"] \arrow[dd, "\mathrm{assoc}"'] & ((X \oslash !X) \oslash !Y) \otimes !Z \arrow[r, "\mathtt{passoc}^{-1} \otimes \mathrm{id}"] \arrow[d, "\mathrm{wk}"'] & (X \oslash (!X \otimes !Y)) \otimes !Z \arrow[d, "\mathrm{wk}"] & \\
 & & ((X \oslash !X) \oslash !Y) \oslash !Z \arrow[r, "\mathtt{passoc}^{-1} \oslash \mathrm{id}"] \arrow[d, "\mathtt{passoc}^{-1}"] & (X \oslash (!X \otimes !Y)) \oslash !Z \arrow[r, "\mathtt{passoc}^{-1}"] & X \oslash ((!X \otimes !Y) \otimes !Z) \arrow[d, "\mathrm{id} \oslash \mathrm{assoc}"] \\
!X \otimes (!Y \otimes !Z) \arrow[r, "\alpha_X \otimes \mathrm{id}"] & (X \oslash !X) \otimes (!Y \otimes !Z) \arrow[r, "\mathrm{wk}"] & (X \oslash !X) \oslash (!Y \otimes !Z) \arrow[rr, "\mathtt{passoc}^{-1}"] & & X \oslash (!X \otimes (!Y \otimes !Z))
\arrow[from=1-1, to=3-2, phantom, "\mathbf{a}"]
\arrow[from=1-2, to=3-3, phantom, "\mathbf{c}"]
\arrow[from=1-3, to=2-4, phantom, "\mathbf{b}"]
\arrow[from=2-3, to=3-5, phantom, "\mathbf{d}"]
\end{tikzcd}$$

Figure 3 a commutes because assoc is a natural transformation. **b** commutes because wk is a natural transformation. **c** commutes by the coherence condition for wk. **d** commutes by the coherence condition for `passoc`.

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