Sequoidal Categories and Transfinite Games: Towards a Coalgebraic Approach to Linear Logic*

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Abstract

In [5], Laird introduces the concept of a *sequoidal category* as a formalization of causality in game semantics. A sequoidal category is like a monoidal category with an extra connective, $\oslash$, that allows one to construct an exponential object as a final coalgebra [3], Under a further hypothesis, it is possible to show that this final coalgebra allows one to construct cofree commutative comonoids in the category, giving us a model of the exponential connective ! from linear logic. In the first part of this note, we review the coalgebraic arguments for constructing the cofree commutative comonoid, which are known but do not yet appear in print. In the second part, we show that the extra hypotheses are necessary by outlining a definition of *transfinite game*, in which the carriers of the sequoidal coalgebra and the cofree commutative comonoid do not coincide.

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1 Defining Higher-order Stateful Objects, Coalgebraically

In this section we motivate the study of the coalgebraically derived cofree comonoid in game semantics by considering a similar but simpler and more familiar phenomenon. A *state-transformer* in a symmetric monoidal category is a morphism $f : A \otimes S \to B \otimes S$ taking an argument together with an input state to a result together with a input state. A well-studied [?] technique in semantics is to use an appropriate final coalgebra to *encapsulate* the state in such a transformer, allowing multiple successive invocations, each of which passes its output state as an input state to the next invocation.

For example, consider the category Rel of sets and relations, with symmetric monoidal structure given by the cartesian product (with unit I, the singleton set $\{*\}$). This has finite (bi)products given by the disjoint union of sets: define the functor $F(A, S) = (A \otimes S) \oplus I$

For any object (set) A, let A^* be the set A^* of finite sequences of elements of A (i.e. the carrier of the free monoid on A), and $\alpha : A^* \to F(A, A^*)$ be the morphism $\{(\varepsilon, \mathsf{inr}(*)\} \cup \{(aw, (\mathsf{inl}(a, w)) \mid a \in A, w \in A^*\}$

▶ **Lemma 1.** *(A^*, α_A) is a final coalgebra for $F(A, \_)$.*

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Proof. Rel may be cpo-enriched with the inclusion order: A^* is a *minimal invariant* for $F(A, \_)$ with respect to this order. ◀

Since we have a natural transformation (left injection) $\mathsf{inl}_{A,S} : A \otimes S \to F(A, S)$, we may encapsulate the state in the state transformer $f : S \to A \otimes S$ by taking the *anamorphism* of $f; \mathsf{inl}_{A,B} : S \to F(A, S)$, — i.e. the unique $F(A, \_)$-coalgebra morphism from $(S, f; \mathsf{inl})$ into (A^*, α_A). This is a morphism from an initial state S into A^*: by definition, composing it with $\alpha : A \to F(A, A)$ (which we can think of as *invoking* our stateful object) returns a copy of f and uses it to update the internal state.

By distributivity of $\oplus$ over $\otimes$ we have a natural transformation $t : F(A, S) \otimes S' \to F(A, S \otimes S')$. We may use this to encapsulate state transformers of general form: given $f : A \otimes S \to B \otimes S$, taking the anamorphism of the $F(B, \_)$-colagebra:

$$A^* \otimes S \xrightarrow{\alpha_A \otimes S} F(A, A^*) \otimes S \xrightarrow{t} F(A, A^* \otimes S)$$

Distributivity of $\oplus$ over $\otimes$ also implies that $F(A \oplus A', S) \cong F(A, S) \oplus F(A', S)$. This allows state transformers to be aggregated, to construct stateful objects compounded of a series of methods which share access to a common state. i.e. given morphisms $f_1 : A_1 \otimes S \to B_1 \otimes S, \ldots, f_n : A_n \otimes S \to B_n \otimes S$, encapsulating $\mathtt{dist}; (f_1 \oplus \ldots \oplus f_n); \mathtt{dist}^{-1}$, gives a morphism from $(A_1 \oplus \ldots \oplus A_n)^* \otimes S$ into $(B_1 \oplus \ldots \oplus B_n)^*$.

For example, we may represent a reference cell storing integer values as a state transformer $\mathsf{cell} : \mathbb{N} \to (\mathbb{N} \oplus \mathbb{N}) \otimes \mathbb{N}$, obtained by aggregating two "methods" which share access to a state consisting of a single integer, representing the contents of the cell — returning a "read" of the input state (and leaving it unchanged) or accepting a "write" of a new value and using it to update the state. Thus (with appropriate tagging) it is the relation $\{(i, (\mathtt{read}(i), i)) \mid i \in \mathbb{N}\} \cup \{(i, \mathtt{write}(j), j)\}$. The anamorphism of the coalgebra $\mathsf{cell}; \mathsf{inl} : \mathbb{N} \to F(\mathbb{N} \oplus \mathbb{N}, \mathbb{N})$ is the relation from $\mathbb{N}$ to $(\mathbb{N} \oplus \mathbb{N})^*$ consisting of pairs of the form $(i_1, \mathtt{read}(i_1)^* \mathtt{write}(i_2) \mathtt{read}(i_2)^* \ldots)$ Composition with this morphism is precisely the denotation of new variable declaration in the semantics of Reynolds *Syntactic Control of Interference* (SCI) in Rel given in [**?**].

Coalgebraic methods thus give us a recipe for constructing and using categorical definitions of stateful semantic objects, avoiding direct definitions which are rather combinatorial to work with. In order to fully exploit these, however, we may endow A^* with the structure of a *comonoid* by defining morphisms $\delta_A : A^* \to A^* \otimes A^* = \{(u \cdot v, (u, v)) \mid u, v \in A^*\}$ and $\epsilon : A \to I = \{(\varepsilon, *)\}$

In fact, this is the *cofree comonoid* on A — there is a morphism $\eta_A : A^* \to A = \{(a, a) \mid a \in A\}$ such that for any comonoid $(B, \delta_B, \epsilon_B)$, composition with η_A defines an equivalence (natural in B) between the morphisms from B into A, and the comonoid morphisms from $(B, \delta_B, \epsilon_B)$ into $(A^*, \delta_A, \epsilon_A)$.

▶ **Proposition 2.** *(A^*, δ, ϵ) is the cofree comonoid on* Rel.

The definitions of δ and ϵ, and the proof that this is the cofree comonoid may be derived from the fact that (A^*, α_A) is a *bifree algebra* for $F(\_, A)$ — i.e. (A^*, α^{-1}) is an initial algebra for $F(A, \_)$ (α must be an isomorphism by Lambek's lemma). We leave this as an exercise.

This structure can be used to interpret procedures which share access to a stateful resource such as a reference cell, the creation of multiple copies of such procedures ($(\_)^*$ is a monoidal comonad) etc. Its main limitation is that we have not defined a *commutative* monoid — evidently it is not the case that $\delta_A; \mathrm{sym}_{A,A} = \delta_A$ (where $\mathrm{sym}_{A,B} : A \otimes B \to B \otimes A$ is the symmetry isomorphism for the tensor) for any non-empty set A. Thus we can only model procedures with shared access to the same stateful object if the order in which they are permitted to access it is predetermined. This is precisely the situation in SCI, where the typing system allows sharing between across sequential composition, but not between functions and their arguments. In order to model sharing of state between functions without this constraint (and build a Cartesian closed category), we need to endow our final coalgebra with the structure of a cofree *commutative* comonoid. The category of sets and relations does not allow this (the cofree commutative comonoid on an object A in Rel is given by the set of finite multisets of A, which is not a final coalgebra). By properly reflecting the interleaving between function calls, game semantics does support a coalgebraic cofree commutative comonoid, leading to models of stateful higher-order languages such as Idealized Algol and core ML. As we shall show by example and counterexample, the richer structure of games also means that the relationship between initial and final coalgebras, and cofree objects is more subtle (note, for example, that any final coalgebra in Rel must be bifree, by its self-duality).

2 Sequoidal categories

2.1 Game semantics and the sequoidal operator

We shall present a form of game semantics in the style of [4] and [1]. A game will be given by a tuple

$$A = (M_A, \lambda_A, b_A, P_A)$$

where

- M_A is a set of moves.
- $\lambda_A : M_A \to \{O, P\}$ is a function designating each move as either an O-*move* or a P-*move*.
- $b_A \in \{O, P\}$ is a choice of starting player.
- $P_A \subseteq M_A^*$ is a prefix-closed set of alternating plays (so if $sab \in P_A$ then $\lambda_A(a) = \neg\lambda_A(b)$) such that if $as \in P_A$ then $\lambda_A(a) = b_A$.

We call $sa \in P_A$ a P-*position* if a is a P-move and an O-*position* if a is an O-move.

A *strategy* for player P for a game A is identified with the set of positions that may arise when playing according to that strategy. Namely, it is a non-empty prefix-closed subset $\sigma \subseteq P_A$ satisfying the two conditions:

(sO) If $s \in \sigma$ is a P-position and a is an O-move such that $sa \in P_A$, then $sa \in \sigma$.

(sP) If $sa, sb \in \sigma$ are P-positions, then $a = b$.

We shall now concentrate on games A for which $b_A = O$, called *negative games*. We shall informally describe the standard connectives on negative games:

Product If A and B are negative games then the *product* $A \times B$ is the game given by placing the game trees for A and B side by side: that is, player O may play his first move either in A or in B. Thereafter, play continues in the game that player O has chosen.

Tensor Product The tensor product $A \otimes B$ is also played by playing the games A and B in parallel, but this time player O may elect to switch games whenever it is his turn and continue play in the game he has switched to.

Linear implication The implication $A \multimap B$ is played by playing the game B in parallel with the *negation* of A - that is, the game formed by switching the roles of players P and O in A. Since play in the negation of A starts with a P-move, player O is forced to make his first move in the game B. Thereafter, player P may switch games whenever it is her turn.

If A, B, C are negative games, σ is a strategy for $A \multimap B$ and τ is a strategy for $B \multimap C$, then we may form a strategy $\tau \circ \sigma$ for $A \multimap C$ by setting

$$\sigma \| \tau = \{\mathfrak{s} \in (M_A \sqcup M_B \sqcup M_C)^* \ : \ \mathfrak{s}|_{A,B} \in \sigma, \ s|_{B,C} \in \tau\}$$

and then defining

$$\tau \circ \sigma = \{\mathfrak{s}|_{A,C} \ : \ \mathfrak{s} \in \tau \circ \sigma\}$$

It is well known (see, for example, [1]) that $\tau \circ \sigma$ is indeed a strategy for $A \multimap C$ and that this form of composition is associative and has an identity. It is also well known that the resulting category $\mathcal{G}$ of games and strategies has products given by the operator $\times$ and a symmetric monoidal closed structure given by the operations $\otimes$ and $\multimap$.

We turn now to the non-standard *sequoid* connective $\oslash$. If A and B are negative games, then the sequoid $A \oslash B$ is similar to the tensor product $A \otimes B$, but with the restriction that player O's first move must take place in the game A. We observe immediately that we have structural isomorphisms

$$\texttt{dist}\colon A \otimes B \xrightarrow{\cong} (A \oslash B) \times (B \oslash A)$$

$$\texttt{dec}\colon (A \times B) \oslash C \xrightarrow{\cong} (A \oslash C) \times (B \oslash C)$$

$$\texttt{passoc}\colon (A \oslash B) \oslash C \xrightarrow{\cong} A \oslash (B \otimes C)$$

One further question to ask is: does the sequoid operator give rise to a functor $\_ \oslash \_ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, as the tensor operator does? The answer is no: indeed, let A, B, C, D be negative games, let σ be a strategy for $A \multimap C$ and let τ be a strategy for $B \multimap D$. Our aim is to construct a natural strategy $\sigma \oslash \tau$ for $(A \oslash B) \multimap (C \oslash D)$. There is an obvious way to try and do this: player P should play according to the strategy σ whenever player O's last move was in A or C, and according to τ whenever player O's last move was in B or D.

We show that this does not in general give us a strategy for $(A \oslash B) \multimap (C \oslash D)$. Suppose that σ is such that player P's response to some opening move in C is another move in C and suppose that τ is such that player P's response to some opening move in D is a move in B (for example, τ is a copycat strategy). Then we end up with the following sequence of events in the game $(A \oslash B) \multimap (C \oslash D)$:

1. Player O starts with a move in C (as he must).
2. Player P responds according to σ with another move in C.
3. Player O decides to switch games and play a move in D.
4. Player P responds according to τ with a move in B.

But now player P's last move is not a legal move in $(A \oslash B) \multimap (C \oslash D)$, since no moves have been played in A yet.

We get round this problem by requiring that the strategy σ be *strict* – that is, whatever player O's opening move in C is, player P's reply must be a move in A.

▶ **Definition 3.** Let N, L be negative games and let σ be a strategy for $N \multimap L$. We say that σ is *strict* if player P's reply to an opening move in L is always a move in N.

Identity strategies are strict and the composition of two strict strategies is strict, so we get a full-on-objects subcategory $\mathcal{G}_s$ of $\mathcal{G}$ where the morphisms are strict strategies. Then the sequoid operator gives rise to a functor:

$$\_ \oslash \_ : \mathcal{G}_s \times \mathcal{G} \to \mathcal{G}_s$$

2.2 Sequoidal categories

We now have the motivation required to give the definition of a *sequoidal category* from [5].

▶ **Definition 4.** A *sequoidal category* consists of the following data:

- A symmetric monoidal category $\mathcal{C}$ with monoidal product $\otimes$ and tensor unit I, associators $\mathrm{assoc}_{A,B,C}\colon (A\otimes B)\otimes C \xrightarrow{\cong} A\otimes(B\otimes C)$, unitors $\mathrm{runit}_A\colon A\otimes I \xrightarrow{\cong} A$ and $\mathrm{lunit}_A\colon I\otimes A \xrightarrow{\cong} A$ and braiding $\mathrm{sym}_{A,B}\colon A\otimes B \to B\otimes A$.
- A category $\mathcal{C}_s$
- A right monoidal category action of $\mathcal{C}$ on the category $\mathcal{C}_s$. That is, a functor

  $$\_ \oslash \_ : \mathcal{C}_s \times \mathcal{C} \to \mathcal{C}_s$$

  together natural isomorphisms

  $$\mathtt{passoc}_{A,B,C}\colon A\oslash(B\otimes C) \to (A\oslash B)\oslash C$$

  and

  $$\mathtt{r}_A\colon A\oslash I \xrightarrow{\cong} A$$

  subject to the following coherence conditions:

  $$\begin{tikzcd}
A \oslash (B\otimes(C\otimes D)) \arrow[r, "\mathtt{passoc}_{A,B,C\otimes D}"] \arrow[d, "\mathrm{id}_A \oslash \mathrm{assoc}_{B,C,D}"'] & (A\oslash B)\oslash(C\otimes D) \arrow[r, "\mathtt{passoc}_{A\oslash B,C,D}"] & ((A\oslash B)\oslash C)\oslash D \\
A\oslash((B\otimes C)\otimes D) \arrow[r, "\mathtt{passoc}_{A,B\otimes C,D}"'] & (A\oslash(B\otimes C))\oslash D \arrow[ur, "\mathtt{passoc}_{A,B,C}\oslash \mathrm{id}_D"'] &
\end{tikzcd}$$

  $$\begin{tikzcd}
A\oslash(I\otimes B) \arrow[r, "\mathtt{passoc}_{A,I,B}"] \arrow[d, "\mathrm{id}_A\oslash\mathrm{lunit}_B"'] & (A\oslash I)\oslash B \arrow[dl, "\mathtt{r}_A\oslash\mathrm{id}_B"] \\
A\oslash B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\oslash(B\otimes I) \arrow[r, "\mathtt{passoc}_{A,B,I}"] \arrow[d, "\mathrm{id}_A\oslash\mathrm{runit}_B"'] & (A\oslash B)\oslash I \arrow[dl, "\mathtt{r}_{A\oslash B}"] \\
A\oslash B &
\end{tikzcd}$$

- A functor $J\colon \mathcal{C}_s \to \mathcal{C}$ (in the games example, this is the inclusion functor $\mathcal{G}_s \to \mathcal{G}$)
- A natural transformation $\mathrm{wk}_{A,B}\colon J(A)\otimes B \to J(A\oslash B)$ satisfying the coherence conditions:

  $$\begin{tikzcd}
A\otimes I \arrow[r, "\mathrm{runit}_A"] \arrow[d, "\mathrm{wk}_{A,I}"'] & A \\
A\oslash I \arrow[ur, "J(\mathtt{r}_A)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "\mathrm{wk}_{A,B}\otimes\mathrm{id}_C"] \arrow[d, "\mathrm{assoc}_{A,B,C}"'] & (A\oslash B)\otimes C \arrow[r, "\mathrm{wk}_{A\oslash B,C}"] & (A\oslash B)\oslash C \\
A\otimes(B\otimes C) \arrow[r, "\mathrm{wk}_{A,B\otimes C}"'] & A\oslash(B\otimes C) \arrow[ur, "J(\mathtt{passoc}_{A,B,C})"'] &
\end{tikzcd}$$

  Our category of games satisfies further conditions:

▶ **Definition 5.** Let $\mathcal{C} = (\mathcal{C}, \mathcal{C}_s, J, \mathrm{wk})$ be a sequoidal category. We say that $\mathcal{C}$ is an *inclusive sequoidal category* if $\mathcal{C}_s$ is a full-on-objects subcategory of $\mathcal{C}$ containing all isomorphisms and finite products of $\mathcal{C}$, and the morphisms $\mathrm{wk}_{A,B}$ and J is the inclusion functor.

We say that $\mathcal{C}$ is *decomposable* if it is affine, and for any A and B, the tensor product $A\otimes B$ is a cartesian product of $A\oslash B$ and $B\oslash A$, with projections $\mathrm{wk}_{A,B} : A\otimes B \to A\oslash B$ and $\mathrm{wk}_{A,B}\circ\mathrm{sym}_{A,B} : A\otimes B \to B\oslash A$, and if the product $A\times B$ exists, then $(A\times B)\oslash C$

is the product of $A \oslash C$ and $B \oslash C$, with projections $\pi_l \oslash C, \pi_r \oslash C$. So if $\mathcal{C}$ has all finite products, the natural transformations

$$\begin{aligned}
&\mathtt{dec}_{A,B} = \langle \mathrm{wk}_{A,B}, \mathrm{wk}_{A,B} \circ \mathrm{sym}_{A,B} \rangle \colon A \otimes B \to (A \oslash B) \times (B \oslash A) \\
&\mathtt{dist}_{A,B,C} = \langle \mathrm{pr}_1 \oslash \mathrm{id}_C, \mathrm{pr}_2 \oslash \mathrm{id}_C \rangle \colon (A \times B) \oslash C \to (A \oslash C) \times (B \oslash C) \\
&\mathtt{dist}_{A,0} \colon 1 \oslash A \to 1
\end{aligned}$$

are isomorphisms.

We have one further piece of structure available to us:

▶ **Definition 6.** Let $\mathcal{C} = (\mathcal{C}, \mathcal{C}_s, J, \mathrm{wk})$ be an inclusive sequoidal category. We say that $\mathcal{C}$ is a *sequoidal closed category* if $\mathcal{C}$ is monoidal closed (with internal hom $\multimap$ and currying $\Lambda_{A,B,C} \colon \mathcal{C}(A \otimes B, C) \xrightarrow{\cong} \mathcal{C}(A, B \multimap C))$ and if the map $f \mapsto \Lambda(f \circ \mathrm{wk})$ gives rise to a natural transformation

$$\Lambda_{A,B,C,s} \colon \mathcal{C}_s(A \oslash B, C) \to \mathcal{C}_s(A, B \multimap C)$$

It can be shown (see for example [3]) that our category $\mathcal{G}$ of games has all this structure.

▶ **Theorem 7.** *Let J be the inclusion functor $\mathcal{G}_s \to \mathcal{G}$. If A, B are games, let $\mathrm{wk}_{A,B} \colon A \otimes B \to A \oslash B$ be the natural copycat strategy. Then*

$$(\mathcal{G}, \mathcal{G}_s, J, \mathrm{wk})$$

is an inclusive, Cartesian, decomposable, distributive sequoidal closed category.

2.3 The sequoidal exponential

There are several ways to add exponentials to the basic category of games. We shall use the definition based on countably many copies of the base game (see [5], for example):

▶ **Definition 8.** Let A be a negative game. The *exponential* of A is the game $!A = (M_{!A}, \lambda_{!A}, b_{!A}, P_{!A})$, where $M_{!A}, \lambda_{!A}, b_{!A}, P_{!A}$ are defined as follows:
- $M_{!A} = M_A \times \omega$
- $\lambda_{!A} = \lambda_A \circ \mathrm{pr}_1$
- $b_{!A} = O$
- Given a sequence $s \in M_{!A}^{\omega}$, we write $s|_n$ for the largest sequence $a_1 a_2 \dots a_k \in M_A^*$ such that $(a_1, n), (a_2, n), \dots (a_k, n)$ is a subsequence of s. Then $P_{!A}$ is the set of all sequence $s \in M_{!A}^{\omega}$ that are alternating with respect to $\lambda_{!A}$, such that $s|_n \in P_A$ for all n and such that if $m < n$ and (a, n) occurs in s then (b, m) must occur earlier in s for some move b: in other words, player O can start infinitely many copies of the game A, but he must start them in order.

This last condition on the order in which games may be opened is very important, as it allows us to define morphisms that give $!A$ the semantics of the exponential from linear logic. For example, we have a natural morphism $\mu \colon !A \to !A \otimes !A$, given by the copycat strategy that starts a new copy of A on the left whenever one is started on the right. Because of the condition on the order in which copies of A may be started, there is a unique way to do this.

▶ **Proposition 9.** *μ exhibits $!A$ as a comonoid in the monoidal category $(\mathcal{G}, \otimes, I)$.*

Proof. μ shall be the comultiplication in our comonoid. The counit is given by the empty strategy $\eta\colon !A \to I$. We just need to check that μ is associative and that η is a counit for μ.

For associativity, we need to show that the following diagram commutes:

$$\begin{tikzcd}
!A \arrow[rr, "\mu"] \arrow[d, "\mu"'] & & !A \otimes !A \arrow[d, "\mathrm{id}_{!A} \otimes \mu"] \\
!A \otimes !A \arrow[r, "\mu \otimes \mathrm{id}_{!A}"] & (!A \otimes !A) \otimes !A \arrow[r, "\mathrm{assoc}_{!A,!A,!A}"'] & !A \otimes (!A \otimes !A)
\end{tikzcd}$$

This is easy to see when we notice that both branches of the square are copycat strategies on $!A \multimap !A \otimes (!A \otimes !A)$; since copies of A in $!A$ must be started in sequence, there is a unique such strategy, and so the square commutes.

For the counit, we need to show that the follosing two diagrams commute:

$$\begin{tikzcd}
!A \arrow[r, "\mu"] \arrow[dr, "\mathrm{runit}_A^{-1}"'] & !A \otimes !A \arrow[d, "\mathrm{id}_{!A} \otimes \eta"] \\
& !A \otimes I
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "\mu"] \arrow[dr, "\mathrm{lunit}_A^{-1}"'] & !A \otimes !A \arrow[d, "\eta \otimes \mathrm{id}_{!A}"] \\
& I \otimes !A
\end{tikzcd}$$

Once again, these diagrams commute because both branches are copycat strategies for $!A \multimap !A \otimes I$ or $!A \multimap I \otimes !A$ and there is a unique such strategy in each case. ◀

We shall later show that $(!A, \mu, \eta)$ is in fact the *cofree commutative comonoid* on A in the monoidal category $(\mathcal{G}, \otimes, I)$.

We shall call the exponential $!A$ the *sequoidal exponential.* The following proposition explains the name:

▶ **Proposition 10.** *Let A be a negative game. Then we get an endofunctor $A \oslash \_$ on $\mathcal{G}$ given by sending B to $A \oslash B$.*

In any inclusive sequoidal category with a cofree exponential we have a $A \oslash \_$-coalgebra $\alpha : !A \to A \oslash !A = \delta; (\eta \oslash !A); \mathrm{wk}_{A \oslash !A}$. In our category of games this is the final coalgebra for the endofunctor $A \oslash \_$. In other words, if B is a negative game and $\sigma\colon B \to A \oslash B$ is a morphism then there is a unique morphism $⦇ \sigma ⦈\colon B \to !A$ such that the following diagram commutes:

$$\begin{tikzcd}
B \arrow[r, "\sigma"] \arrow[d, "⦇ \sigma ⦈"'] & A \oslash B \arrow[d, "\mathrm{id}_A \oslash ⦇ \sigma ⦈"] \\
!A \arrow[r, "\alpha"'] & A \oslash !A
\end{tikzcd}$$

Proof. See [3]. We shall shortly give a proof in the more general case. ◀

2.4 Constructing cofree commutative comonoids in sequoidal categories

We want to construct $!A \xrightarrow{\mu} !A \otimes !A$ from the final $A \oslash \_$-coalgebra, $(!A, \alpha)$, and show that it is the cofree commutative comonoid on A. It turns out that we shall need one more fact about the category of games to prove this.

▶ **Notation 11.** We shall sometimes make the monoidal structure of the Cartesian product explicit by writing $\sigma \times \tau$ for $\langle \sigma \circ \mathrm{pr}_1, \tau \circ \mathrm{pr}_2 \rangle$.

▶ **Definition 12.** Let A, B be objects of an inclusive, Cartesian, distributive sequoidal category $(\mathcal{C}, \mathcal{C}_s, J, \mathrm{wk})$ with final coalgebras $!A \xrightarrow{\alpha_A} A \oslash !A$ for all endofunctors of the form $A \oslash \_$. Let A, B be objects of C. Then we have a composite $\kappa_{A,B} \colon !A \otimes !B \to (A \times B) \oslash (!A \otimes !B)$:

$$\begin{aligned}
\kappa_{A,B} = !A \otimes !B &\xrightarrow{\langle \mathrm{id}_{!A \otimes !B},\, \mathrm{sym}_{!A,!B} \rangle} (!A \otimes !B) \times (!B \otimes !A) \\
\cdots &\xrightarrow{(\alpha_A \otimes \mathrm{id}_{!B}) \times (\alpha_B \otimes \mathrm{id}_{!A})} ((A \oslash !A) \otimes !B) \times ((B \oslash !B) \otimes !A) \\
\cdots &\xrightarrow{\mathrm{wk}_{A \oslash !A, !B} \times \mathrm{wk}_{B \oslash !B, !A}} ((A \oslash !A) \oslash !B) \times ((B \oslash !B) \oslash !A) \\
\cdots &\xrightarrow{\mathtt{passoc}^{-1}_{A,!A,!B} \times \mathtt{passoc}^{-1}_{B,!B,!A}} (A \oslash (!A \otimes !B)) \times (B \oslash (!B \otimes !A)) \\
\cdots &\xrightarrow{\mathrm{id}_{A \oslash (!A \otimes !B)} \times (\mathrm{id}_B \oslash \mathrm{sym}_{!B,!A})} (A \oslash (!A \otimes !B)) \times (B \oslash (!A \otimes !B))
\end{aligned}$$

inducing a morphism

$$!A \otimes !B \xrightarrow{\kappa_{A,B}} (A \oslash (!A \otimes !B)) \times (B \oslash (!A \otimes !B)) \xrightarrow{\mathtt{dist}^{-1}} (A \times B) \oslash (!A \otimes !B)$$

Remembering that our category has a final coalgebra $!(A \times B)$ for the functor $(A \times B) \oslash \_$, we write $\mathtt{int}_{A,B}$ for the unique morphism $!A \otimes !B \to !(A \times B)$ making the following diagram commute

$$\begin{tikzcd}
!A \otimes !B \arrow[r, "\kappa_{A,B}"] \arrow[d, "\mathtt{int}_{A,B}"'] & (A \oslash (!A \otimes !B)) \times (B \oslash (!A \otimes !B)) \arrow[r, "\mathtt{dist}^{-1}"] & (A \times B) \oslash (!A \otimes !B) \arrow[d, "\mathrm{id}_{A \times B} \oslash \mathtt{int}_{A,B}"] \\
!(A \times B) \arrow[rr, "\alpha_{A \times B}"'] & & (A \times B) \oslash !(A \times B)
\end{tikzcd} \qquad (\star)$$

▶ **Proposition 13.** *In the category of games, the morphism $\mathtt{int}_{A,B}$ is an isomorphism for all negative games A, B.*

Proof. Observe that the morphism $\mathtt{int}_{A,B}$ is the copycat strategy on $!A \otimes !B \multimap !(A \times B)$ that starts a copy of A on the left whenever a copy of A is started on the right and starts a copy of B on the left whenever a copy of B is started on the right (indeed, the morphisms in the diagram above are all copycat morphisms, so the copycat strategy we have just described must make that diagram commute. Since there are infinitely many copies of both A and B available in $!(A \times B)$, and since a new copy of A or B may be started at any time, we may define an inverse copycat strategy on $!(A \times B) \multimap !A \otimes !B$. ◀

Our first main result for this section will be the following:

▶ **Theorem 14.** *Let $(\mathcal{C}, \mathcal{C}_s, J, \mathrm{wk})$ be an inclusive, Cartesian, distributive and decomposable sequoidal category with a final coalgebra $!A \xrightarrow{\alpha_A} A \oslash !A$ for each endofunctor of the form $A \oslash \_$. Suppose further that the morphism $\mathtt{int}_{A,B}$ as defined above is an isomorphism for all objects A, B. $A \mapsto !A$ gives rise to a strong symmetric monoidal functor from the monoidal category $(\mathcal{C}, \times, 1)$ to the monoidal category $(\mathcal{C}, \otimes, I)$.*

We start off by defining a morphism $\mu \colon !A \to !A \otimes !A$. This will turn out to be the comultiplication for the cofree commutative comonoid over A. First, we note that we have the following composite:

$$!A \xrightarrow{\alpha_A} A \oslash !A \xrightarrow{\Delta} (A \oslash !A) \times (A \oslash !A) \xrightarrow{\mathtt{dist}^{-1}} (A \times A) \oslash !A$$

where Δ is the diagonal map on the product. There is therefore a unique morphism $\sigma_A = \llparenthesis\, \mathtt{dist}^{-1} \circ \Delta \circ \alpha_A \,\rrparenthesis$ making the following diagram commute:

$$\begin{tikzcd}
!A \arrow[r, "\alpha_A"] \arrow[d, "\sigma_A"'] & A \oslash !A \arrow[r, "\Delta"] & (A \oslash !A) \times (A \oslash !A) \arrow[r, "\mathtt{dist}^{-1}"] & (A \times A) \oslash !A \arrow[d, "\mathrm{id}_{A\times A} \oslash \sigma_A"] \\
!(A \times A) \arrow[rrr, "\alpha_{A\times A}"] & & & (A \times A) \oslash !(A \times A)
\end{tikzcd} \tag{†}$$

and we may set $\mu_A = \mathtt{int}^{-1}_{A,A} \circ \sigma_A$.

We also define a morphism $\mathtt{der}_A \colon !A \to A$. Note that since I is isomorphic to 1, we have a unique morphism $*_A \colon A \to I$ for each A. We define $\mathtt{der}_A$ to be the composite

$$!A \xrightarrow{\alpha_A} A \oslash !A \xrightarrow{\mathrm{id}_A \oslash *_{!A}} A \oslash I \xrightarrow{\mathtt{r}_A} A$$

We define the action of ! on morphisms as follows: suppose that $\sigma \colon A \to B$ is a morphism in $\mathcal{C}$. Then we have a composite

$$!A \xrightarrow{\mu} !A \otimes !A \xrightarrow{\mathtt{der}_A \otimes \mathrm{id}_{!A}} A \otimes !A \xrightarrow{\sigma \otimes \mathrm{id}_{!A}} B \otimes !A \xrightarrow{\mathrm{wk}_{B,!A}} B \oslash !A$$

There is therefore a unique morphism $!\sigma \colon !A \to !B$ making the following diagram commute:

$$\begin{tikzcd}
!A \arrow[r, "\mu"] \arrow[d, "!\sigma"'] & !A \otimes !A \arrow[r, "\mathtt{der}_A \otimes \mathrm{id}_{!A}"] & A \otimes !A \arrow[r, "\sigma \otimes \mathrm{id}_{!A}"] & B \otimes !A \arrow[r, "\mathrm{wk}_{B,!A}"] & B \oslash !A \arrow[d, "\mathrm{id}_B \oslash !\sigma"] \\
!B \arrow[rrrr, "\alpha_B"] & & & & B \oslash !B
\end{tikzcd}$$

▶ **Proposition 15.** *$\sigma \mapsto !\sigma$ respects composition, so ! is a functor. Moreover, ! is a strong symmetric monoidal functor from the Cartesian category $(\mathcal{C}, \times, 1)$ to the symmetric monoidal category $(\mathcal{C}, \otimes, I)$, witnessed by* $\mathtt{int}$ *and* $\mathtt{dec}^0$.

Proof. See Appendix. ◀

This completes the proof of Theorem 14.

Since ! is a strong monoidal functor, it induces a functor CCom(!) from the category $\mathrm{CCom}(\mathcal{C}, \times, 1)$ of comonoids over $(\mathcal{C}, \times, 1)$ to the category $\mathrm{CCom}(\mathcal{C}, \otimes, I)$ of comonoids over $(\mathcal{C}, \otimes, I)$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{CCom}(\mathcal{C}, \times, 1) \arrow[r, "\mathcal{F}"] \arrow[d, "\mathrm{CCom}(!)"'] & (\mathcal{C}, \times, 1) \arrow[d, "!"] \\
\mathrm{CCom}(\mathcal{C}, \otimes, I) \arrow[r, "\mathcal{F}"] & (\mathcal{C}, \otimes, I)
\end{tikzcd}$$

where $\mathcal{F}$ is the forgetful functor.

Let A be an object of $\mathcal{C}$. Since $(\mathcal{C}, \times, 1)$ is Cartesian, the diagonal map $\Delta \colon A \to A \times A$ is the cofree commutative comonoid over A in $(\mathcal{C}, \times, 1)$.

▶ **Proposition 16.** $\mathrm{CCom}(!)\left(A \xrightarrow{\Delta} A \times A\right)$ *has comultiplication given by $\mu_A \colon !A \to !A \otimes !A$ and counit given by the unique morphism $\eta_A \colon !A \to I$.*

Proof. See appendix. ◀

In particular, this proves that the comultiplication μ_A is associative and that the counit η_A is a valid counit for μ_A.

We can now state our second main result from this section.

▶ **Theorem 17.** *Let $(\mathcal{C}, \mathcal{C}_s, J, \mathrm{wk})$ be a sequoidal category satisfying all the conditions from Theorem 14. Let A be an object of $\mathcal{C}$ (equivalently, of $\mathcal{C}_s$). Then $!A$, together with the comultiplication μ_A and counit η_A, is the cofree commutative comonoid over A.*

Proof. See Appendix. ◀

2.5 The Sequoidal Exponential as a Bifree Algebra

Observe that in our category of games, $(!A, \alpha)$ is in fact a *bifree algebra* for $A \oslash \_$ — the isomorphism $\alpha^{-1} : A \oslash !A \to !A$ is an initial algebra for $A \oslash \_$. We may show that in such cases, the condition that ! is strong monoidal — and thus the cofree exponential — always holds[1] : we may define an inverse to `int` $:!A \otimes !A \to !(A \times B)$ as the *catamorphism* of the $A \oslash \_$-algebra:

$$(A \times B) \oslash !A \otimes !B \cong (A \oslash (!A \otimes !B)) \times (B \oslash (!A \otimes !B)) \cong (!A \oslash !B) \times (!B \oslash !A)$$

It is not necessary for the final $A \oslash \_$-coalgebra to be bifree for the exponential to be strong monoidal and thus the cofree commutative comonoid. An example is provided by the category of games with winning conditions and winning strategies, which is sequoidal closed and decomposable. To show that the final $A \oslash \_$-coalgebra in this category is not bifree, it suffices to observe that from such an algebra, we may derive a *fixed point* operator $\mathsf{fix}_A : \mathcal{C}(A, A) \to \mathcal{C}(I, A)$ for each A, such that $\mathsf{fix}_A(f); f = \mathsf{fix}(f)$.

▶ **Proposition 18.** *Suppose $\mathcal{C}$ is sequoidal closed and decomposable, and $(!A, \alpha)$ is a bifree $A \oslash \_$-algebra. Then we may define a fixed point operator on $\mathcal{C}$.*

Proof. For any A, let $\Phi_A :!(A \multimap A) \to A$ be the catamorphism of the counit to the adjunction $A \oslash \_ \dashv A \multimap$, $\epsilon_{A,A} : (A \multimap A) \oslash A \to A$, which is a $(A \multimap A) \oslash \_$-algebra. For any morphism $f : A \to A$ we may define $\mathsf{fix}_A(f) = \Lambda(f)^\dagger; \Phi_A$, where $\Lambda(f) : I \to (A \multimap A)$ is the "name" of f. ◀

As one would expect, it is not possible to define a fixed point operator on the category of games and winning strategies — for example, if $\bot$ is the game with a single move then the hom-set $\mathcal{C}(I, \bot)$ is empty and hence there can be no morphism $\mathsf{fix}_\bot(\mathrm{id}_\bot)$. So the final $A \oslash \_$-coalgebra is not bifree in this case.

2.6 A formula for the sequoidal exponential

We now consider an alternative construction of the cofree commutative comonoid in a sequoidally decomposable category, which leads to some illuminating comparisons with the coalgebraic approach. It is based on the formula given by Melliés, Tabareau and Tasson [**?**], which does not depend on the presence of cartesian products, but obtains the cofree commutative comonoid as a limit of *symmetric tensor powers*.

[1] Without requiring our sequoidally decomposable category to have finite products we may equip each object $!A$ with the structure of a comonoid by defining: $\delta :!A \to !A \otimes !A$ to be the catamorphism of the $A \oslash \_$ algebra:

$$A \oslash (!A \otimes !A) \cong A \oslash (!A \otimes !A) \times A \oslash (!A \otimes !A) \cong (A \oslash !A) \oslash !A \times (A \oslash !A) \oslash !A \cong (!A \oslash !A) \cong (!A \oslash !A) \cong (!A \otimes !A)$$

This satisfies the further requirements of a *linear category* in the sense of [**?**], although it does not appear to be possible to show that it is the cofree commutative comonoid.

▶ **Definition 19.** If A is an object in a symmetric monoidal category, a n-fold symmetric tensor power of A is an *equalizer* (A^n, eq) for the group G of symmetry automorphisms on $A^{\otimes n}$. A tensor power is preserved by the tensor product if $(B \otimes A^n, \mathsf{eq} \otimes B)$ is an equalizer for the automorphisms $\{B \otimes g \mid g \in G\}$.

In any affine category[2] with tensor powers of A we may define a diagram $\Delta(A) =$

$$I \overset{p_0}{\leftarrow} A \overset{p_1}{\leftarrow} A^2 \ldots A^i \overset{p_i}{\leftarrow} A^{i+1} \ldots$$

where $p_i : A^{i+1} \to A^i$ is the unique morphism given by the universal property of the symmetric tensor power, such that $p_i; \mathsf{eq}_i : A^{i+1} \to A^{\otimes i} = \mathsf{eq}_{i+1}; (A^{\otimes i} \otimes t_A)$.

Melliés, Tabareau and Tasson [?] have shown that where the limit $(A^\infty, \{p_i^\infty : A^\infty \to A^i\}$ for this diagram exists and commutes with the tensor, — i.e. for each object B, $B \otimes A^\infty$ is the limit of

$$B \otimes I \overset{B \otimes p_0}{\leftarrow} B \otimes A \overset{B \otimes p_1}{\leftarrow} B \otimes A^2 \ldots$$

then a comultiplication $\delta : A^\infty \to A^\infty \otimes A^\infty$ may be defined making $(A^\infty, \delta, t_{!A})$ the cofree commutative comonoid. Where these conditions are satisfied, we shall call this a MTT-exponential.

▶ **Proposition 20.** *Any sequoidally decomposable category has all symmetric tensor powers, and these are preserved by the tensor.*

Proof. By sequoidal decomposability, for any n, $A^{\otimes(n+1)}$ is the cartesian product $\Pi_{i \leq n}(A \oslash A^{\otimes n})$ with projections $\mathrm{sym}_i \circ \mathrm{wk}_{A,A^{\otimes n}}$, where $\mathrm{sym}_i : A^{\otimes(n+1)} \to A^{\otimes(n+1)}$ is the symmetry isomorphism corresponding to the permutation on n which swaps 1 and i.

We inductively define the sequoid powers $A^{\oslash n}$ by $A^{\oslash 1} = A$ and $A^{\oslash(n+1)} = A \oslash A^{\oslash n}$. Given n, we inductively define a morphism $\mathsf{eq}_n \colon A^{\oslash n} \to A^{\otimes n}$ as the diagonal $\langle \mathrm{id}_A \oslash \mathsf{eq}_{n-1} \rangle$ into the product – so $\mathrm{wk} \circ \mathrm{sym}_i \circ \mathsf{eq}_n = \mathrm{id} \oslash e_{n-1}$ for each i.

If $\pi \in S_n$ is a permutation, we write sym^π for the appropriate symmetry of $A^{\otimes n}$. We now claim that $\mathrm{sym}^\pi \circ \mathsf{eq}_n = \mathsf{eq}_n$ for all $\pi \in S_n$ – so eq_n equalizes the sym^π. Indeed, let $\pi \in S_n$. Then we may write $\mathrm{sym}^\pi = \mathrm{id}_A \otimes \mathrm{sym}^\rho \circ \mathrm{sym}_i$ for some i and some $\rho \in S_{n-1}$. Now we have:

$$\begin{aligned}
\mathrm{wk} \circ \mathrm{sym}^\pi \circ \mathsf{eq}_n &= \mathrm{wk} \circ (\mathrm{id} \otimes \mathrm{sym}^\rho) \circ \mathrm{sym}_i \circ \mathsf{eq}_n \\
&= (\mathrm{id} \oslash \mathrm{sym}^\rho) \circ \mathrm{wk} \circ \mathrm{sym}_i \circ \mathsf{eq}_n \qquad \text{(by naturality of wk)} \\
&= (\mathrm{id} \oslash \mathrm{sym}^\rho) \circ (\mathrm{id} \oslash \mathsf{eq}_{n-1}) \\
&= \mathrm{id} \oslash \mathsf{eq}_{n-1}
\end{aligned}$$

Since π was arbitrary, it follows that $\mathrm{wk} \circ \mathrm{sym}_j \circ \mathrm{sym}^\pi \circ \mathsf{eq}_n = \mathrm{wk} \circ \mathrm{sym}_j \circ \mathsf{eq}_n$ for all j and therefore that $\mathrm{sym}^\pi \circ \mathsf{eq}_n = \mathsf{eq}_n$ for all π, as desired.

Now define $\mathrm{wk}^n \colon A^{\oslash n} \to A^{\otimes n}$ inductively by $\mathrm{wk}^n = \mathrm{id} \oslash \mathrm{wk}^{n-1} \circ \mathrm{wk}$. We may show by induction that $\mathrm{wk}^n \circ \mathsf{eq}_n = \mathrm{id}_{A^{\oslash n}}$ for all n: indeed, we have

$$\begin{aligned}
\mathrm{wk}^n \circ \mathsf{eq}_n &= (\mathrm{id} \oslash \mathrm{wk}^{n-1}) \circ \mathrm{wk} \circ \mathsf{eq}_n \\
&= (\mathrm{id} \oslash \mathrm{wk}^{n-1}) \circ (\mathrm{id} \oslash \mathsf{eq}_{n-1}) \\
&= \mathrm{id} \oslash \mathrm{id} = \mathrm{id} \qquad \text{(by induction)}
\end{aligned}$$

[2] This is a special case of the situation considered in [?]: that A is a "free pointed object".

Now let $f\colon C \to B \otimes A^{\otimes n}$ be such that $\mathrm{sym}^\pi \circ f = f$ for all $\pi \in S_n$. It follows from what we have just shown that if $f = (\mathrm{id} \otimes \mathsf{eq}_n) \circ g$, then $g = \mathrm{id} \otimes \mathrm{wk}^n \circ \mathrm{id} \otimes \mathsf{eq}_n \circ g = \mathrm{id} \otimes \mathrm{wk}^n \circ f$. So we just need to show that

$$f = (\mathrm{id} \otimes (\mathsf{eq}_n \circ \mathrm{wk}_n)) \circ f$$

We first assume that $B = I$, so we have $f\colon C \to B \otimes A^{\otimes n}$ and we are trying to show that $f = \mathsf{eq}_n \circ \mathrm{wk}^n \circ f$. We do this by showing that $\mathrm{wk} \circ \mathrm{sym}_i \circ \mathsf{eq}_n \circ \mathrm{wk}^n \circ f = \mathrm{wk} \circ \mathrm{sym}_i \circ f$ for all i.

Indeed, we have:

$$\begin{aligned}
\mathrm{wk} \circ \mathrm{sym}_i \circ \mathsf{eq}_n \circ \mathrm{wk}^n \circ f &= (\mathrm{id} \oslash \mathsf{eq}_{n-1}) \circ (\mathrm{id} \oslash \mathrm{wk}^{n-1}) \circ \mathrm{wk} \circ f \\
&= \mathrm{wk} \circ (\mathrm{id} \otimes (\mathsf{eq}_{n-1} \circ \mathrm{wk}^{n-1})) \circ f && \text{(by naturality of wk)} \\
&= \mathrm{wk} \circ f && \text{(by induction)} \\
&= \mathrm{wk} \circ \mathrm{sym}_i \circ f
\end{aligned}$$

Now let B be an arbitary object of the category.

—

Define $\mathrm{wk}^n : A^{\otimes n} \to A^n$ by $\mathrm{wk}^{n+1} = \mathrm{wk}_{A,A^{\otimes n}}; (A \oslash \mathrm{wk}^n)$. We show (by induction on n) that for any n, $A^{\otimes n}$ is the product of $n!$ copies of A^n, such that the projections from $A^{\otimes n}$ into A^n are the morphisms $\theta; \mathrm{wk}^n$, where θ ranges over all of the permutation isomorphisms on $A^{\otimes n}$.

The equalizer $\mathsf{eq}_n : A^n \to A^{\otimes n}$ is the diagonal into this product $\langle \mathrm{id}_{A^n} \mid i \in n! \rangle$. Given any $f : C \to A^{\otimes n} \otimes B$ such that $f; (\theta \otimes B) = f$ for any permutation symmetry θ, taking $f; (\mathrm{wk}^n \otimes B) : C \to A^n \otimes B$ gives the unique morphism such that $f; (\mathrm{wk}^n \otimes B); (\mathsf{eq}_n \otimes B) = f$. ◀

Thus, in any sequoidally decomposable category, the diagram $\Delta(A)$ exists for any A. If a limit for this diagram exists and is preserved by the sequoid, — i.e. for any B, $B \oslash A^\infty$ is the limit for $B \oslash \Delta(A)$ — then it is preserved by the tensor, and is therefore the cofree commutative comonoid. Conversely, we may show that any cofree commutative comonoid which arises as a $\otimes$-preserving limit of symmetric tensor powers is a final coalgebra.

▶ **Proposition 21.** *If a sequoidally decomposable category has a MTT-exponential, then* $(A^\infty, \delta; (\eta \otimes !A); \mathrm{wk}_{A \oslash !A})$ *is a final coalgebra for the functor* $A \oslash \_$*.*

Proof. Observe that in a sequoidally decomposable category, the morphism $\mathrm{wk}_{A,A^n} : A \oslash A^n :\to A^{n+1}$ is a section (we may define the corresponding retraction inductively from the decomposition of the tensor). Hence if the limit A^∞ is preserved by the functor $A \otimes \_$, it is preserved by the sequoid $A \oslash \_$ — i.e. A^∞ is the limit of the chain $A \oslash \Delta = \Delta$.

Thus, for any $A \oslash \_$-coalgebra $f : B \to A \oslash B$, we may define a unique coalgebra morphism $(\![f]\!) : B \to !A$ as the mediating morphism of the cone $\{f_i : B \to A^i \mid i \in \omega\}$, where $f_0 = t_B$ and $f_{i+1} = f; (A \oslash f_i)$. ◀

3 Transfinite Games

Of the conditions that we used to construct the cofree commutative comonoid in sequoidal categories, the requirement that $\mathtt{int}_{A,B}$ be an isomorphism stands out as the least satisfactory. All the other conditions are 'finitary', and relate directly to the connectives we have introduced, whereas the morphism $\mathtt{int}_{A,B}$ can only be constructed using the final coalgebra property for the exponential connective !. For this reason, we might wonder whether we can do without the condition that $\mathtt{int}_{A,B}$ be an isomorphism. In this section, we shall give a negative answer to that question: we shall construct an inclusive, distributive, decomposable sequoidal (closed) category with final coalgebras $!A$ for all functors of the form $A \oslash \_$, and shall show that $!A$ does not have a natural comonoid structure. In doing this, we hope to shed some light upon alternative algebraic or coalgebraic constructions for the cofree commutative comonoid that work in a purely 'finitary' manner.

Our sequoidal category will be closely modelled upon the category of games we have just considered: the objects will be games, with the modification that sequences of moves may now have transfinite length. This is a natural construction, occurring in the study of determinacy by Mycielski [9], Blass [2] and Weiss [10], and it appears to be present in the semantic context in the work of Roscoe [11], Levy [7] and Laird [6].

The general idea is as follows: we will show that the definition of the final coalgebra for the sequoid functor in a category of transfinite games is largely unchanged from the definition in the category of games with finite-length plays: $!A$ is the game formed from a countably infinite number of copies of A, indexed by ω, with the proviso that player O must open them in order. We observe that the copycat strategy $\mathtt{int}_{A,B} \colon !A \otimes !B \to !(A \times B)$ is not an isomorphism, and that we cannot construct the comultiplication $!A \to !A \otimes !A$ in a sensible way. Moreover, we cannot construct the comonad $!A \to !!A$, so ! does not give us a model of linear logic in even the most general sense. In all three cases, the reason why the construction fails is that we might run out of copies of the game A (or B) on the left hand side before we have run out of copies on the right hand side. In the finite-plays setting, it is impossible to run out of copies of a subgame, because there are infinitely many copies, so it is impossible to play in all of them in a finite-length play. In the transfinite setting, however, we cannot guarantee this: consider, for example, a position in $!A_0 \multimap !A_1 \otimes !A_2$ (with indices given so we can refer to the different copies of A) in which player O has opened all the copies of A in $!A_1$. Since player P is playing by copycat, she must have opened all of the copies of A in $!A_0$. If, at time $\omega + 1$, player O now plays in $!A_2$, player P will have no reply to him.

The 'correct' definition of $!A$ in the transfinite game category is one in which there is an unlimited number of copies of A to open (rather than ω-many), but this is not the final coalgebra for the functor $A \oslash \_$. [TODO: discuss ways to construct this object]

3.1 Transfinite Games

We give a brief summary of the construction of the category of transfinite games. Full details may be found in Appendix **??**.

We shall fix an additively indecomposable ordinal $\alpha = \omega^\beta$ throughout, which will be a bound on the ordinal length of positions in our game. So, for example, the original category of games is the case $\alpha = \omega$. If X is a set, we write $X^{*<\alpha}$ for the set of transfinite sequences of elements of X of length less than α.

▶ **Definition 22.** A *game* or a *game over* α or an α*-game* is a tuple $A = (M_A, \lambda_A, \zeta_A, P_A)$, where:

- M_A is a set of moves
- $\lambda_A \colon M_A \to \{O, P\}$ designates each move as an O*-move* or a P*-move*
- $P_A \subseteq M_A^{*<\alpha}$ is a non-empty prefix-closed set of transfinite sequences of moves from M_A, called *positions*. We say that s is a *successor position* if the length of s is a successor ordinal and we say that s is a *limiting* position if the length of s is a limit ordinal.
- $\zeta_A \colon P_A \to \{O, P\}$ designates each position as an O*-position* or a P*-position*.

such that:

Consistency If $sa \in P_A$ is a successor position, then $\zeta_A(sa) = \lambda_A(a)$

Alternation If $s, sa \in P_A$, then $\zeta_A(s) = \neg\zeta_A(sa)$

Limit closure If $s \in M_A^{*<\alpha}$ is a limiting position such that $t \in P_A$ for all proper prefixes $t \subsetneqq s$, then $s \in P_A$.

We say that a game A is *positive* if $\zeta_A(\epsilon) = O$ and *negative* if $\zeta_A(\epsilon) = P$. We say that A is *completely negative* if $\zeta_A(s) = P$ for all limiting plays s.

Apart from the possibly transfinite length of sequences of moves, the only new thing in this definition is the function ζ_A. Thanks to the consistency condition, ζ_A gives us no new information for successor positions; it is necessary in order to tell us which player is to move at limiting positions.

▶ **Definition 23.** A *strategy* for an α-game A is a non-empty prefix-closed subset $\sigma \subseteq P_A$ satisfying the following conditions:

Closure under O-replies If $s \in \sigma$ is a P-position and $sa \in P_A$, then $sa \in \sigma$.

Determinism If $sa, sb \in \sigma$ are P-positions, then $a = b$.

Given games A and B, we may form their product $A \times B$, tensor product $A \otimes B$, linear implication $A \multimap B$ and sequoid $A \oslash B$ in roughly the same way that we construct these connectives for finite-length games. The only point we need to take care of is the behaviour of the ζ-functions at limit ordinals. We do this according to the following formulae:

$$\zeta_{A\times B}(s) = \zeta_A(s) \wedge \zeta_B(s)$$
$$\zeta_{A\otimes B}(s) = \zeta_A(s) \wedge \zeta_B(s)$$
$$\zeta_{A\multimap B}(s) = \zeta_A(s) \Rightarrow \zeta_B(s)$$
$$\zeta_{A\oslash B}(s) = \zeta_A(s) \wedge \zeta_B(s)$$

Here, $\wedge$ and $\Rightarrow$ are the usual propositional connectives on $\{T, F\}$, but with T replaced by P and F replaced by O.

Once we have defined our connectives, we may define a *morphism* from A to B to be a strategy for $A \multimap B$ and we may define composition of morphisms in the usual way: given games A, B and C, and strategies σ for $A \multimap B$ and τ for $B \multimap C$, we define

$$\sigma \| \tau = \{\mathfrak{s} \in (M_A \sqcup M_B \sqcup M_C)^{*<\alpha} \ : \ \mathfrak{s}|_{A,B} \in \sigma,\ \mathfrak{s}|_{B,C} \in \tau\}$$

and then we define

$$\tau \circ \sigma = \{\mathfrak{s}|_{A,C} \ : \ \mathfrak{s} \in \sigma \| \tau\}$$

▶ Remark. Since α is additively decomposable, the interleaving of two sequences of length less than α must itself have length less than α. This is important: if we allow α to be an additively decomposable ordinal, then it is possible to construct two strategies whose composite is not closed under O-replies because a particular reply in the interleaving of two sequences occurs at time later than α and so is not included.

We can show that this composition is associative and moreover that we obtain an inclusive, distributive, decomposable sequoidal category. We call this category $\mathcal{G}(\alpha)$ and call the corresponding strict subcategory $\mathcal{G}_s(\alpha)$. The hardest part of this is showing that the category is monoidal closed, because the linear implication of completely negative games is not necessarily completely negative.

3.2 The final sequence for the sequoidal exponential

We now want to show that $\mathcal{G}(\alpha)$ has final coalgebras for the functor $A \oslash \_$, given by the transfinite game $!A$, which is defined as follows:

- $M_{!A} = M_A \times \omega$
- $\lambda_{!A} = \lambda_A \circ \mathrm{pr}_1$

We define $!P_A$ to be the set of all sequences $s \in M_{!A}^{*<\alpha}$ such that $s|_n \in P_A$ for all n. Then we define $\zeta_{!A} \colon !P_A \to \{O, P\}$ by

$$\zeta_{!A}(s) = \bigwedge_{n \in \omega} \zeta_A(s|_n)$$

In other words, $\zeta_{!A}(s) = P$ if and only if $\zeta_A(s|_n) = P$ for all n.

There is a natural copycat strategy $\alpha_A \colon !A \to A \oslash !A$, just as in the finite plays case. We want to show that this is the final coalgebra for $A \oslash \_$. The proof for the finite case found in [3] will not work in this case, since it implicitly uses the fact that $!A$ is the limit of the sequence

$$I \leftarrow A \leftarrow A \oslash A \leftarrow A \oslash (A \oslash A) \leftarrow \cdots$$

(cf. also [8]). In the transfinite categories, this is no longer the case.

While it is possible to prove that $\alpha_A \colon !A \to A \oslash !A$ is the final coalgebra for $A \oslash \_$ directly, we shall instead give a proof by extending the sequence given above to an ordinal-indexed sequence. This is the *final sequence*, familiar in coalgebra [12]. Specifically, we construct a functor $\mathcal{F} \colon \mathbf{Ord}^{\mathrm{op}} \to \mathcal{G}(\alpha)$, where $\mathbf{Ord}$ is the order category of the ordinals, writing

$$\mathcal{F}(\gamma) = A^{\oslash \gamma}$$
$$\mathcal{F}(\gamma \leq \delta) = j_\gamma^\delta \colon A^{\oslash \delta} \to A^{\oslash \gamma}$$

according to the following inductive recipe:

- $A^{\oslash 0} = 1$
- $A^{\oslash (\gamma+1)} = A \oslash A^{\oslash \gamma}$
- If μ is a limit ordinal, then $A^{\oslash \mu}$ is the limit of the diagram formed by the $A^{\oslash \gamma}$ for $\gamma < \mu$, together with the morphisms j_γ^δ, for $\gamma \leq \delta < \mu$.
- $j_0^\gamma = *$
- j_γ^λ is the morphism in the limiting cone
- $j_\gamma^{\delta+1} = j_\delta^{\delta+1} \circ j_\gamma^\delta$
- If we write C_λ for the limiting cone for λ over the $A^{\oslash \gamma}$ for $\gamma < \mu$, then we may form a cone $A \oslash C_\lambda$ over the same diagram by applying the functor $A \oslash \_$ to C_λ and then extending the cone to 1 in the only possible way. Then $j_\lambda^{\lambda+1}$ is the unique morphism from $A^{\oslash(\lambda+1)}$ to $A^{\oslash \lambda}$ inducing a morphism of cones from $A \oslash C_\lambda$ to C_λ.

It is well known (see [12], for instance) that if $j_\delta^{\delta+1}$ is an isomorphism for some δ, then ${j_\delta^{\delta+1}}^{-1} \colon A^{\oslash \delta} \to A \oslash A^{\oslash \delta}$ is the final coalgebra for the functor $A \oslash \_$. In this case, we say that the sequence *stabilizes at* δ. Our proof strategy is therefore to show that the sequence stabilizes at some δ, and to show that $A^{\oslash \delta}$ is isomorphic to $!A$.

We do this by giving a classification of the games $A^{\oslash \gamma}$. Let $s \in \omega^{*<\alpha}$ be any transfinite sequence of natural numbers. We define the *derivative* Δs of s to be the sequence given by removing all instances of 0 from s and subtracting 1 from all other terms. In other words, if $s \colon \gamma \to \omega$, for $\gamma < \alpha$, then we have:

$$\Delta s = s^{-1}(\omega \setminus \{0\}) \xrightarrow{s} \omega \setminus \{0\} \xrightarrow{-1} \omega$$

(where $s^{-1}(\omega \setminus \{0\})$ carries the induced order). We now define predicates $\_ \leq \gamma$ on sequences $s \in \omega^{*<\alpha}$ as follows:

- $\epsilon \leq 0$
- If $\Delta s \leq \gamma$, then $s \leq \gamma + 1$
- If μ is a limit ordinal and $s \in \omega^{*<\alpha}$ is such that for all successor-length prefixes $t \sqsubseteq s$ we have $t \leq \gamma$ for some $\gamma < \mu$, then $s \leq \mu$. In other words, $\{s \in \omega^{*<\alpha} : s \leq \mu\}$ is the limit-closure of the union of the sets $\{s \in \omega^{*<\alpha} : s \leq \gamma\}$ for $\gamma < \mu$.

We can prove some basic results about these predicates:

▶ **Proposition 24.** *i) If $s \leq \gamma$ and t is any subsequence of s (not necessarily an initial prefix), then $t \leq \gamma$.*

ii) If $s \leq \gamma$, then $\Delta s \leq \gamma$

iii) If $s \leq \gamma$ and $\gamma \leq \delta$, then $s \leq \delta$

iv) If $s \in \omega^{*<\alpha}$ has length μ, where μ is a limit ordinal, then $s \leq \mu$. If s has length $\mu + n$ for some $n \in \omega$, then $s \leq \mu + \omega$. In particular, $s \leq \alpha$ for all $s \in \omega^{*<\alpha}$.

Proof. See Appendix. ◀

Our classification result for the final sequence then becomes:

▶ **Theorem 25.** *Let A be any game. Then $A^{\oslash\gamma} \cong (M_{!A}, \lambda_{!A}, \zeta_{!A}, P_{!A,\gamma})$, where*

$$P_{!A,\gamma} = \{s \in P_{!A} : \ \mathrm{pr}_2 \circ s \leq \gamma\}$$

The morphism j^δ_γ is the copycat strategy.

Proof. See Appendix. ◀

▶ **Corollary 26.** *The final sequence for $A \oslash \_$ stabilizes at α and we have $A^{\oslash\alpha} = !A$.*

Proof. By Proposition 24(iv), $\mathrm{pr}_2 \circ s \leq \alpha$ for all $s \in P_{!A}$ and so $\mathrm{pr}_2 \circ s \leq (\alpha + 1)$, by Proposition 24(iii). It follows, by Theorem 25, that $A^{\oslash\alpha} = !A$ and that the morphism $A^{\oslash\alpha} \to A^{\oslash(\alpha+1)}$ is the morphism α_A. ◀

A Proofs

A.1 Proof of Proposition 15

▶ **Proposition 15.** *$\sigma \mapsto !\sigma$ respects composition, so ! is a functor. Moreover, ! is a strong symmetric monoidal functor from the Cartesian category $(\mathcal{C}, \times, 1)$ to the symmetric monoidal category $(\mathcal{C}, \otimes, I)$, witnessed by* `int` *and* $\mathtt{dec}^0$*.*

In order to show that $\sigma \mapsto !\sigma$ respects composition, we need the following lemma:

▶ **Lemma 27.** *Let A be an object of $\mathcal{C}$. Then $\alpha_A \colon !A \to A \oslash !A$ is equal to the following composite:*

$$!A \xrightarrow{\mu_A} !A \otimes !A \xrightarrow{\mathrm{der}_A \otimes \mathrm{id}_{!A}} A \otimes !A \xrightarrow{\mathrm{wk}_{A,!A}} A \oslash !A$$

Proof. We may paste together diagrams ($\star$) and ($\dagger$) to form the following diagram (where we shall omit subscripts where there is no ambiguity):

$$\begin{tikzcd}
!A \arrow[r, "\alpha"] \arrow[d, "\sigma_A"'] & A \oslash !A \arrow[r, "\Delta"] & (A \oslash !A) \times (A \oslash !A) \arrow[r, "\mathtt{dist}^{-1}"] & (A \times A) \oslash !A \arrow[d, "\mathrm{id}_{A\times A} \oslash \sigma_A"] \\
!(A \times A) \arrow[rrr, "\alpha"] & & & (A \times A) \oslash !(A \times A) \\
!A \otimes !A \arrow[u, "\mathtt{int}_A"] \arrow[rr, "\kappa_{A,A}"] & & (A \oslash (!A \otimes !A)) \times (A \oslash (!A \otimes !A)) \arrow[r, "\mathtt{dist}^{-1}"] & (A \times A) \oslash (!A \otimes !A) \arrow[u, "\mathrm{id}_{A\times A} \oslash \mathtt{int}_A"']
\end{tikzcd}$$

where we observe that the composites down the left and right hand sides (after inverting the lower arrows) are μ_A and $\mathrm{id}_{A\times A} \oslash \mu_A$.

Now note that we have the following commutative square:

$$\begin{tikzcd}
(A \times A) \oslash !A \arrow[r, "\mathtt{dist}"] \arrow[d, "\mathrm{id}_{A\times A} \oslash \mu_A"'] & (A \oslash !A) \times (A \oslash !A) \arrow[d, "(\mathrm{id} \oslash \mu) \times (\mathrm{id} \oslash \mu)"] \\
(A \times A) \oslash (!A \otimes !A) \arrow[r, "\mathtt{dist}"'] & (A \oslash (!A \otimes !A)) \times (A \oslash (!A \otimes !A))
\end{tikzcd}$$

(using the definition of `dist`). Putting this together with the diagram above, we get the following commutative diagram:

$$\begin{tikzcd}
!A \arrow[r, "\alpha"] \arrow[d, "\mu_A"'] & A \oslash !A \arrow[r, "\Delta"] & (A \oslash !A) \times (A \oslash !A) \arrow[d, "\mathrm{id} \oslash \mu_A \times \mathrm{id} \oslash \mu_A"] \\
!A \otimes !A \arrow[rr, "\kappa_{A,A}"] & & (A \oslash (!A \otimes !A)) \times (A \oslash (!A \otimes !A))
\end{tikzcd}$$

We now expand the definition of $\kappa_{A,A}$ and take the projections on to the first and second components, yielding the following two commutative diagrams:

$$\begin{tikzcd}
!A \arrow[rrr, "\alpha"] \arrow[d, "\mu_A"'] & & & A \oslash !A \arrow[d, "\mathrm{id} \oslash \mu_A"] \\
!A \otimes !A \arrow[r, "\alpha \otimes \mathrm{id}"] & (A \oslash !A) \otimes !A \arrow[r, "\mathrm{wk}"] & (A \oslash !A) \oslash !A \arrow[r, "\mathtt{passoc}^{-1}"] & A \oslash (!A \otimes !A)
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
!A \arrow[rrrr, "\alpha"] \arrow[d, "\mu_A"'] & & & & A \oslash !A \arrow[d, "\mathrm{id} \oslash \mu_A"] \\
!A \otimes !A \arrow[r, "\mathrm{sym}"'] & !A \otimes !A \arrow[r, "\alpha \otimes \mathrm{id}"] & (A \oslash !A) \otimes !A \arrow[r, "\mathtt{passoc}^{-1} \circ \mathrm{wk}"] & A \oslash (!A \otimes !A) \arrow[r, "\mathrm{id} \oslash \mathrm{sym}"'] & A \oslash (!A \otimes !A)
\end{tikzcd} \tag{2}$$

From diagram (**1**), we construct the following commutative diagram:

$$\begin{tikzcd}
!A \arrow[rrr, "\alpha"] \arrow[d, squiggly, "\mu_A"'] & & & A \oslash !A \arrow[d, "\mathrm{id} \oslash \mu_A"] \\
!A \otimes !A \arrow[r, "\alpha \otimes \mathrm{id}"] \arrow[ddr, squiggly, "\mathtt{der}_A \otimes \mathrm{id}"'] \arrow[urrr, phantom, "\mathbf{a}"] & (A \oslash !A) \otimes !A \arrow[r, "\mathrm{wk}"] \arrow[d, "(\mathrm{id} \oslash *) \otimes \mathrm{id}"] \arrow[dl, phantom, "\mathbf{b}"] & (A \oslash !A) \oslash !A \arrow[r, "\mathtt{passoc}^{-1}"] \arrow[d, "(\mathrm{id} \oslash *) \oslash \mathrm{id}"] \arrow[dl, phantom, "\mathbf{c}"] & A \oslash (!A \oslash !A) \arrow[d, "\mathrm{id} \oslash (* \otimes \mathrm{id})"] \arrow[dl, phantom, "\mathbf{e}"] \\
& (A \oslash I) \otimes !A \arrow[r, "\mathrm{wk}"] \arrow[d, "\mathtt{r} \otimes \mathrm{id}"] \arrow[dr, phantom, "\mathbf{d}"] & (A \oslash I) \oslash !A, \arrow[r, "\mathtt{passoc}^{-1}"] \arrow[d, "\mathtt{r} \otimes \mathrm{id}"] & A \oslash (I \otimes !A) \arrow[dl, "\mathrm{id} \oslash \mathrm{lunit}"] \arrow[dl, phantom, "\mathbf{f}"'] \\
& A \otimes !A \arrow[r, squiggly, "\mathrm{wk}"] & A \oslash !A &
\end{tikzcd}$$

a is diagram (**1**).
b commutes by the definition of $\mathtt{der}_A$.
c and **d** commute because wk is a natural transformation.
e commutes because `passoc` is a natural transformation.
f commutes by one of the coherence conditions in the definition of a sequoidal category.

We now observe that the composite of the three squiggly arrows is the composite we are trying to show is equal to α; we have α along the top, so it will suffice to show that the composite

$$\xi_A \;=\; !A \xrightarrow{\mu_A} !A \otimes !A \xrightarrow{*\otimes \mathrm{id}} I \otimes !A \xrightarrow{\mathrm{lunit}} !A$$

is equal to the identity. We do this using diagram (**2**). First we construct the diagram shown in Figure 1.

Now observe that the composite ξ_A is running along the left hand side of Figure 1, while $\mathrm{id} \oslash \xi$ is running along the right. Since we have α along the bottom, it follows by the uniqueness of $\llparenthesis \cdot \rrparenthesis$ that $\xi = \llparenthesis\, \alpha \,\rrparenthesis = \mathrm{id}_{!A}$. ◀

Now we are ready to show that $\sigma \mapsto !\sigma$ respects composition. Let A, B, C be objects, let σ be a morphism from A to B and let τ be a morphism from B to C. Using Lemma 27 and the definition of $!\sigma$, $!\tau$, we may construct a commutative diagram:

$$\begin{tikzcd}
!A \arrow[r, "\mu"] \arrow[d, "!\sigma"'] & !A \otimes !A \arrow[r, "\mathtt{der}\otimes\mathrm{id}"] & A \otimes !A \arrow[r, "\sigma\otimes\mathrm{id}"] & B \otimes !A \arrow[r, "\mathrm{wk}"] \arrow[d, dotted, "\mathrm{id}\otimes !\sigma"] & B \oslash !A \arrow[d, "\mathrm{id}\oslash !\sigma"] \\
!B \arrow[r, "\mu"] \arrow[dd, "!\tau"'] & !B \otimes !B \arrow[rr, "\mathtt{der}\otimes\mathrm{id}"] & & B \otimes !B \arrow[r, "\mathrm{wk}"] \arrow[d, "\tau\otimes\mathrm{id}"] & B \oslash !B \\
 & & & C \otimes !B \arrow[r, "\mathrm{wk}"] \arrow[d, dotted, "\mathrm{id}\otimes !\tau"'] & C \oslash !B \arrow[d, "\mathrm{id}\oslash !\tau"] \\
!C \arrow[r, "\mu"] & !C \otimes !C \arrow[rr, "\mathtt{der}\otimes\mathrm{id}"] & & C \otimes !C \arrow[r, "\mathrm{wk}"] & C \oslash !C
\end{tikzcd}$$

Here, the outermost (solid) shapes commute by the definition of $!\sigma$, $!\tau$ (after we have replaced α_B, α_C with the composite from Lemma 27). The smaller squares on the right hand side commute because wk is a natural transformation. Now observe that $\mathrm{wk}_{X,Y} = \mathrm{pr}_1 \circ \mathtt{dec}_{X,Y}$ is the composition of epimorphisms, so is an epimorphism for all X, Y. It follows that the two rectangles on the left commute as well.

Throwing away the right hand squares and adding some new arrows at the right, we arrive at the following commutative diagram:

$$\begin{tikzcd}
!A \arrow[r, "(\sigma\otimes\mathrm{id})\circ(\mathtt{der}\otimes\mathrm{id})\circ\mu"] \arrow[d, "!\sigma"'] & B \otimes !A \arrow[r, "\tau\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes !\sigma"'] & C \otimes !A \arrow[d, "\mathrm{id}\otimes !\sigma"] \\
!B \arrow[d, "!\tau"'] & B \otimes !B \arrow[r, "\tau\otimes\mathrm{id}"] \arrow[d, "\tau\otimes !\tau"'] & C \otimes !B \arrow[dl, "\mathrm{id}\otimes !\tau"] \\
!C \arrow[r, "(\mathtt{der}\otimes\mathrm{id})\circ\mu"] & C \otimes !C &
\end{tikzcd}$$

We have just shown that the square on the left commutes. The shapes on the right commute by inspection. We now throw away the internal arrows and re-apply wk on the right hand

$$\begin{tikzcd}
!A \arrow[rrrrr, "\alpha"] \arrow[d, "\mu_A"'] \arrow[drrr, phantom, "\mathbf{a}"] & & & & & A \oslash !A \arrow[d, "\mathrm{id} \oslash \mu_A"] \\
!A \otimes !A \arrow[r, "\mathrm{sym}"] \arrow[d, "* \otimes \mathrm{id}"'] \arrow[dr, phantom, "\mathbf{b}"] & !A \otimes !A \arrow[r, "\alpha \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes *"] & (A \oslash !A) \otimes !A \arrow[r, "\mathrm{wk}"] \arrow[d, "\mathrm{id} \oslash *"] \arrow[dr, phantom, "\mathbf{d}"] & (A \oslash !A) \oslash !A \arrow[r, "\mathtt{passoc}^{-1}"] \arrow[d, "\mathrm{id} \oslash *"] \arrow[dr, phantom, "\mathbf{e}"] & A \oslash (!A \otimes !A) \arrow[r, "\mathrm{id} \oslash \mathrm{sym}"] \arrow[d, "\mathrm{id} \oslash (\mathrm{id} \oslash *)"] \arrow[dr, phantom, "\mathbf{c}"] & A \oslash (!A \otimes !A) \arrow[d, "\mathrm{id} \oslash (* \oslash \mathrm{id})"] \\
I \otimes !A \arrow[r, "\mathrm{sym}"] \arrow[d, "\mathrm{lunit}"'] \arrow[dr, phantom, "\mathbf{g}"] & !A \otimes I \arrow[r, "\alpha \otimes \mathrm{id}"] \arrow[d, "\mathrm{runit}"] \arrow[dr, phantom, "\mathbf{f}"] & (A \oslash !A) \otimes I \arrow[r, "\mathrm{wk}"] \arrow[d, "\mathrm{runit}"] \arrow[dr, phantom, "\mathbf{i}"] & (A \oslash !A) \oslash I \arrow[r, "\mathtt{passoc}^{-1}"] \arrow[d, "\mathrm{r}"] \arrow[dr, phantom, "\mathbf{j}"] & A \oslash (!A \otimes I) \arrow[r, "\mathrm{id} \oslash \mathrm{sym}"] \arrow[d, "\mathrm{id} \oslash \mathrm{runit}"] \arrow[dr, phantom, "\mathbf{h}"] & A \oslash (I \otimes !A) \arrow[d, "\mathrm{id} \oslash \mathrm{lunit}"] \\
!A \arrow[r, "\mathrm{id}"'] & !A \arrow[r, "\alpha"] & A \oslash !A \arrow[r, "\mathrm{id}"'] & A \oslash !A \arrow[r, "\mathrm{id}"'] & A \oslash !A \arrow[r, "\mathrm{id}"'] & A \oslash !A
\end{tikzcd}$$

Figure 1 **a** is diagram (**2**).

b and **c** commute because sym is a natural transformation, **d** commutes because wk is a natural transformation and **e** commutes because `passoc` is a natural transformation. **f** commutes because runit is a natural transformation.

g and **h** commute by one of the coherence conditions for a symmetric monoidal category. **i** commutes by one of the coherence conditions for wk in the definition of a sequoidal category and **j** commutes by one of the coherence conditions for `passoc` in the definition of a sequoidal category.

side:

$$\begin{tikzcd}
!A \arrow[rr, "((\tau\circ\sigma)\otimes\mathrm{id})\circ(\mathtt{der}\otimes\mathrm{id})\circ\mu"] \arrow[d, "!\sigma"'] & & C\otimes !A \arrow[r, "\mathrm{wk}"] \arrow[d, "\mathrm{id}\otimes!\sigma"'] & C\oslash !A \arrow[d, "\mathrm{id}\oslash!\sigma"] \\
!B \arrow[d, "!\tau"'] & & C\otimes !B \arrow[r, "\mathrm{wk}"] \arrow[d, "\mathrm{id}\otimes!\tau"'] & !C\oslash !B \arrow[d, "\mathrm{id}\oslash!\tau"] \\
!C \arrow[rr, "(\mathtt{der}\otimes\mathrm{id})\circ\mu"] & & C\otimes !C \arrow[r, "\mathrm{wk}"] & C\oslash !C
\end{tikzcd}$$

By Lemma 27, the composite along the bottom is equal to α_C. Therefore, by uniqueness of $(\!(\,\cdot\,)\!)$, we have

$$!\tau\circ!\sigma = (\!(\,\mathrm{wk}\circ((\tau\circ\sigma)\otimes\mathrm{id})\circ(\mathtt{der}\otimes\mathrm{id})\circ\mu\,)\!) = !(\tau\circ\sigma)$$

Therefore, ! is indeed a functor.

We now want to show that ! has the structure of a strong symmetric monoidal functor from $(\mathcal{C},\times,1)$ to $(\mathcal{C},\otimes,I)$. The relevant morphisms are:

$$\mathtt{int}_{A,B}\colon !A\otimes !B\to !(A\times B) \quad \mathtt{dec}^0\colon I\to 1$$

By hypothesis, these are both isomorphisms. We just need to show that the appropriate coherence diagrams commute.

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