A Kleisli-like construction for Parametric Monads, with Examples

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Abstract. **Keywords:** denotational semantics, category theory, game semantics

1 Introduction

This will be a fairly long section, but for now let's dive straight in.

2 Lax quotients of lax actions

Let $\mathcal{C}'$ be a monoidal category and let $\mathcal{C}$ be a category. Then a *lax iright action* of $\mathcal{C}'$ on $\mathcal{C}$ is a functor $\_.\_ : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}$ that gives rise (through currying) to a lax monoidal functor $\mathcal{C}' \to \mathrm{End}[\mathcal{C}, \mathcal{C}]$. In other words, we have natural transformations $\mathtt{passoc}_{A,X,Y} : A.X.Y \to A.(X \otimes Y)$ and $\mathtt{r}_A : A \to A.I$ making the following diagrams commute for all objects A of $\mathcal{C}$ and X, Y, Z of $\mathcal{C}'$.

$$\begin{tikzcd}
A.X.Y.Z \arrow[r, "\mathtt{passoc}_{A,X\otimes Y,Z}"] \arrow[dr, "\mathtt{passoc}_{A.X,Y,Z}"'] & A.(X \otimes Y).Z \arrow[r, "\mathtt{passoc}_{A,X\otimes Y,Z}"] & A.((X \otimes Y) \otimes Z) \arrow[d, "A.\mathtt{assoc}_{X,Y,Z}"] \\
 & A.X.(Y \otimes Z) \arrow[r, "\mathtt{passoc}_{A,X,Y\otimes Z}"] & A.(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
A.X \arrow[r, "\mathtt{r}_A.X"] \arrow[dr, "A.\mathtt{lunit}_X"'] & A.I.X \arrow[d, "\mathtt{passoc}_{A,I,X}"] \\
 & A.(I \otimes X)
\end{tikzcd}
\qquad
\begin{tikzcd}
A.X \arrow[r, "\mathtt{r}_{A.X}"] \arrow[dr, "A.\mathtt{runit}_X"'] & A.X.I \arrow[d, "\mathtt{passoc}A,X,I"] \\
 & A.(X \otimes I)
\end{tikzcd}$$

Example 1. – Any monad M is an action of the trivial category on its underlying category $\mathcal{C}$, regarding MA as the action $A.I$. The natural transformation $\mathtt{passoc}_{A,I,I}$ is precisely the monad action on A:

$$A.I.I = MMA \to MA = A.I = A.(I \otimes I)\,.$$

Dually, any lax action gives rise to a monad on the category being acted upon, by setting $MA = A.I$.

- If $\mathcal{C}$ is also a monoidal category and $J : \mathcal{C}' \to \mathcal{C}$ is a lax monoidal functor with monoidal coherence μ, then there is an action of $\mathcal{C}'$ on $\mathcal{C}$ given by

$$A.X = A \otimes JX \,,$$

and

$$\mathtt{passoc}_{A,X,Y} = (A\otimes JX)\otimes JY \xrightarrow{\mathtt{assoc}_{A,JX,JY}} A\otimes(JX\otimes JY) \xrightarrow{A\otimes\mu_{X,Y}} A\otimes J(X\otimes Y)\,.$$

- If $\mathcal{C}$ is a monoidal closed category, and j an oplax monoidal functor with coherence ν, then there is an action of $\mathcal{C}'^{co}$ (i.e., $\mathcal{C}'$ with the opposite monoidal product) on $\mathcal{C}$ given by

$$A.X = jX \multimap A$$

and

$$\mathtt{passoc}_{A,X,Y} = jY \multimap (jX \multimap A) \to (jY\otimes jX) \multimap A \xrightarrow{\nu_{Y,X} \multimap A} j(Y\otimes X) \multimap A\,.$$

We sometimes refer to a right action of the opposite category $\mathcal{C}'^{co}$ on $\mathcal{C}$ as a *left action* of $\mathcal{C}'$ on $\mathcal{C}$. In that case, we write $X.A$ instead of $A.X$, so that the coherence becomes

$$\mathtt{passoc}_{Y,X,A} : \; Y.X.A \to (Y \otimes X).A\,.$$

- The intersection of the first two examples is the *writer monad* given by $M_W X = X \otimes W$ for any monoid W in $\mathcal{C}$. The intersection of the first and third examples is the *reader monad* given by $M^R X = R \multimap X$ for any comonoid R in $\mathcal{C}$ (and, in particular, for any object R if $\mathcal{C}$ if the monoidal product in $\mathcal{C}$ is Cartesian).
We shall therefore call an action of the form $A \otimes JX$ a *writer-type action* and one of the form $JX \multimap A$ a *reader-type action*.
- We can define an *oplax action* of $\mathcal{C}'$ on $\mathcal{C}$ to be a lax action of $\mathcal{C}'$ upon the opposite category $\mathcal{C}^{op}$. In this case, the coherence `passoc` goes from $A.(X \otimes Y)$ to $A.X.Y$. As monads are lax actions of the trivial category, so comonads are oplax actions of the trivial category.