Rounding off a corner

John Gowers

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Abstract

In [Lai02], Laird introduces the concept of a *sequoidal category*, a certain extension of a monoidal category and gives an example constructed using game semantics. As noted in [CLM13], cofree commutative comonoids can be constructed coalgebraically in any sequoidal category subject to certain extra hypotheses. In the first part of this note, we review the coalgebraic construction of cofree commutative comonoids in sequoidal categories. In the second part, we shall show that the extra hypotheses are necessary as well as sufficient and we shall use transfinite game semantics to construct a sequoidal category in which these hypotheses do not hold.

1 Sequoidal categories

1.1 Game semantics and the sequoidal operator

We shall present a form of game semantics in the style of [Hyl97] and [AJ13]. A game will be given by a tuple

$$A = (M_A, \lambda_A, b_A, P_A)$$

where

- M_A is a set of moves.
- $\lambda_A \colon M_A \to \{O, P\}$ is a function designating each move as either an *O-move* or a *P-move*.
- $b_A \in \{O, P\}$ is a choice of starting player.
- $P_A \subseteq M_A^*$ is a prefix-closed set of alternating plays (so if $sab \in P_A$ then $\lambda_A(a) = \neg\lambda_A(b)$) such that if $as \in P_A$ then $\lambda_A(a) = b_A$.

We call $sa \in P_A$ a *P-position* if a is a P-move and an *O-position* if a is an O-move.

A *strategy* for player P for a game A is identified with the set of positions that may arise when playing according to that strategy. Namely, it is a non-empty prefix-closed subset $\sigma \subseteq P_A$ satisfying the two conditions:

(sO) If $s \in \sigma$ is a P-position and a is an O-move such that $sa \in P_A$, then $sa \in \sigma$.

(sP) If $sa, sb \in \sigma$ are P-positions, then $a = b$.

We shall now concentrate on games A for which $b_A = O$, called *negative games*. We shall informally describe the standard connectives on negative games:

Product If A and B are negative games then the *product* $A \times B$ is the game given by placing the game trees for A and B side by side: that is, player O may play his first move either in A or in B. Thereafter, play continues in the game that player O has chosen.

Tensor Product The tensor product $A \otimes B$ is also played by playing the games A and B in parallel, but this time player O may elect to switch games whenever it is his turn and continue play in the game he has switched to.

Linear implication The implication $A \multimap B$ is played by playing the game B in parallel with the *negation* of A - that is, the game formed by switching the roles of players P and O in A. Since play in the negation of A starts with a P-move, player O is forced to make his first move in the game B. Thereafter, player P may switch games whenever it is her turn.

If A, B, C are negative games, σ is a strategy for $A \multimap B$ and τ is a strategy for $B \multimap C$, then we may form a strategy $\tau \circ \sigma$ for $A \multimap C$ by setting

$$\sigma \| \tau = \{\mathfrak{s} \in (M_A \sqcup M_B \sqcup M_C)^* \,:\, \mathfrak{s}|_{A,B} \in \sigma,\ s|_{B,C} \in \tau\}$$

and then defining

$$\tau \circ \sigma = \{\mathfrak{s}|_{A,C} \,:\, \mathfrak{s} \in \tau \circ \sigma\}$$

It is well known (see, for example, [AJ13]) that $\tau \circ \sigma$ is indeed a strategy for $A \multimap C$ and that this form of composition is associative and has an identity. It is also well known that the resulting category $\mathcal{G}$ of games and strategies has products given by the operator $\times$ and a symmetric monoidal closed structure given by the operations $\otimes$ and $\multimap$.

We turn now to the non-standard *sequoid* connective $\oslash$. If A and B are negative games, then the sequoid $A \oslash B$ is similar to the tensor product $A \otimes B$, but with the restriction that player O's first move must take place in the game A. We observe immediately that we have structural isomorphisms

$$\begin{aligned}
\texttt{dist}\colon A \otimes B &\xrightarrow{\cong} (A \oslash B) \times (B \oslash A) \\
\texttt{dec}\colon (A \times B) \oslash C &\xrightarrow{\cong} (A \oslash C) \times (B \oslash C) \\
\texttt{passoc}\colon (A \oslash B) \oslash C &\xrightarrow{\cong} A \oslash (B \otimes C)
\end{aligned}$$

One further question to ask is: does the sequoid operator give rise to a functor $\_ \oslash \_ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, as the tensor operator does? The answer is no: indeed, let A, B, C, D be negative games, let σ be a strategy for $A \multimap C$ and let τ be a strategy for $B \multimap D$. Our aim is to construct a natural strategy $\sigma \oslash \tau$ for $(A \oslash B) \multimap (C \oslash D)$. There is an obvious way to try and do this: player P should play according to the strategy σ whenever player O's last move was in A or C, and according to τ whenever player O's last move was in B or D.

We show that this does not in general give us a strategy for $(A \oslash B) \multimap (C \oslash D)$. Suppose that σ is such that player P's response to some opening move in C is another move in C and suppose that τ is such that player P's response to some opening move in D is a move in B (for example, τ is a copycat strategy). Then we end up with the following sequence of events in the game $(A \oslash B) \multimap (C \oslash D)$:

1. Player O starts with a move in C (as he must).
2. Player P responds according to σ with another move in C.
3. Player O decides to switch games and play a move in D.
4. Player P responds according to τ with a move in B.

But now player P's last move is not a legal move in $(A \oslash B) \multimap (C \oslash D)$, since no moves have been played in A yet.

We get round this problem by requiring that the strategy σ be *strict* – that is, whatever player O's opening move in C is, player P's reply must be a move in A.

Definition 1.1. Let N, L be negative games and let σ be a strategy for $N \multimap L$. We say that σ is *strict* if player P's reply to an opening move in L is always a move in N.

Identity strategies are strict and the composition of two strict strategies is strict, so we get a full-on-objects subcategory $\mathcal{G}_s$ of $\mathcal{G}$ where the morphisms are strict strategies. Then the sequoid operator gives rise to a functor:

$$\_ \oslash \_ : \mathcal{G}_s \times \mathcal{G} \to \mathcal{G}_s$$

1.2 Sequoidal categories

We now have the motivation required to give the definition of a *sequoidal category* from [Lai02].

Definition 1.2. A *sequoidal category* consists of the following data:

- A symmetric monoidal category $\mathcal{C}$ with monoidal product $\otimes$ and tensor unit I, associators $\mathrm{assoc}_{A,B,C} \colon (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$, unitors $\mathrm{runit}_A \colon A \otimes I \xrightarrow{\cong} A$ and $\mathrm{lunit}_A \colon I \otimes A \xrightarrow{\cong} A$ and braiding $\mathrm{sym}_{A,B} \colon A \otimes B \to B \otimes A$.
- A category $\mathcal{C}_s$
- A right monoidal category action of $\mathcal{C}$ on the category $\mathcal{C}_s$. That is, a functor
$$\_ \oslash \_ : \mathcal{C}_s \times \mathcal{C} \to \mathcal{C}_s$$
together natural isomorphisms
$$\mathtt{passoc}_{A,B,C} \colon A \oslash (B \otimes C) \to (A \oslash B) \oslash C$$
and
$$\mathtt{r}_A \colon A \oslash I \xrightarrow{\cong} A$$
subject to the following coherence conditions:

$$\begin{tikzcd}
A \oslash (B \otimes (C \otimes D)) \arrow[r, "\mathtt{passoc}_{A,B,C\otimes D}"] \arrow[d, "\mathrm{id}_A \oslash \mathrm{assoc}_{B,C,D}"'] & (A \oslash B) \oslash (C \otimes D) \arrow[r, "\mathtt{passoc}_{A\oslash B,C,D}"] & ((A \oslash B) \oslash C) \oslash D \\
A \oslash ((B \otimes C) \otimes D) \arrow[r, "\mathtt{passoc}_{A,B\otimes C,D}"'] & (A \oslash (B \otimes C)) \oslash D \arrow[ur, "\mathtt{passoc}_{A,B,C} \oslash \mathrm{id}_D"'] &
\end{tikzcd}$$

$$\begin{tikzcd} A \oslash (I \otimes B) \arrow[r, "\mathtt{passoc}_{A,I,B}"] \arrow[d, "\mathrm{id}_A \oslash \mathrm{lunit}_B"'] & (A \oslash I) \oslash B \arrow[dl, "\mathtt{r}_A"] \\ A \oslash B & \end{tikzcd} \qquad \begin{tikzcd} A \oslash (B \otimes I) \arrow[r, "\mathtt{passoc}_{A,B,I}"] \arrow[d, "\mathrm{id}_A \oslash \mathrm{runit}_B"'] & (A \oslash B) \oslash I \arrow[dl, "\mathtt{r}_{A \oslash B}"] \\ A \oslash B & \end{tikzcd}$$

- A functor $J\colon \mathcal{C}_s \to \mathcal{C}$ (in the games example, this is the inclusion functor $\mathcal{G}_s \to \mathcal{G}$)
- A natural transformation $\mathrm{wk}_{A,B}\colon J(A) \otimes B \to J(A \oslash B)$ satisfying the coherence conditions:

$$\begin{tikzcd} A \otimes I \arrow[r, "\mathrm{runit}_A"] \arrow[d, "\mathrm{wk}_{A,I}"'] & A \\ A \oslash I \arrow[ur, "J(\mathtt{r}_A)"'] & \end{tikzcd} \qquad \begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "\mathrm{wk}_{A,B} \otimes \mathrm{id}_C"] \arrow[d, "\mathrm{assoc}_{A,B,C}"'] & (A \oslash B) \otimes C \arrow[r, "\mathrm{wk}_{A \oslash B, C}"] & (A \oslash B) \oslash C \\ A \otimes (B \otimes C) \arrow[r, "\mathrm{wk}_{A,B \otimes C}"'] & A \oslash (B \otimes C) \arrow[ur, "J(\mathtt{passoc}_{A,B,C})"'] & \end{tikzcd}$$

Our category of games satisfies further conditions:

Definition 1.3. Let $\mathcal{C} = (\mathcal{C}, \mathcal{C}_s, J, \mathrm{wk})$ be a sequoidal category. We say that $\mathcal{C}$ is an *inclusive sequoidal category* if $\mathcal{C}_s$ is a full-on-objects subcategoryof $\mathcal{C}$ containing the monoidal isomorphisms and the morphisms $\mathrm{wk}_{A,B}$, J is the inclusion functor and J reflects isomorphisms.

If $\mathcal{C}$ is an inclusive sequoidal category, we say that $\mathcal{C}$ is *Cartesian* if $\mathcal{C}_s$ has all products and these are preserved by J. In that case, we say that $\mathcal{C}$ is *decomposable* if the natural transformations

$$\mathtt{dec}_{A,B} = \langle \mathrm{wk}_{A,B}, \mathrm{wk}_{A,B} \circ \mathrm{sym}_{A,B} \rangle \colon A \otimes B \to (A \oslash B) \times (B \oslash A)$$
$$\mathtt{dec}^0 \colon I \to 1$$

are isomorphisms and we say that $\mathcal{C}$ is *distributive* if the natural transformations

$$\mathtt{dist}_{A,B,C} = \langle \mathrm{pr}_1 \oslash \mathrm{id}_C, \mathrm{pr}_2 \oslash \mathrm{id}_C \rangle \colon (A \times B) \oslash C \to (A \oslash C) \times (B \oslash C)$$
$$\mathtt{dist}_{A,0} \colon 1 \oslash A \to 1$$

are isomorphisms.

We have one further piece of structure available to us:

Definition 1.4. Let $\mathcal{C} = (\mathcal{C}, \mathcal{C}_s, J, \mathrm{wk})$ be an inclusive sequoidal category. We say that $\mathcal{C}$ is a *sequoidal closed category* if $\mathcal{C}$ is monoidal closed (with internal hom $\multimap$ and currying $\Lambda_{A,B,C} : \mathcal{C}(A \otimes B, C) \xrightarrow{\cong} \mathcal{C}(A, B \multimap C)$) and if the map $f \mapsto \Lambda(f \circ \mathrm{wk})$ gives rise to a natural transformation

$$\Lambda_{A,B,C,s} : \mathcal{C}_s(A \oslash B, C) \to \mathcal{C}_s(A, B \multimap C)$$

It can be shown (see for example [CLM13]) that our category $\mathcal{G}$ of games has all this structure.

Theorem 1.5. *If A, B are games, let J be the inclusion functor $\mathcal{G}_s \to \mathcal{G}$ and let $\mathrm{wk}_{A,B} : A \otimes B \to A \oslash B$ be the natural copycat strategy. Then*

$$(\mathcal{G}, \mathcal{G}_s, J, \mathrm{wk})$$

is an inclusive, Cartesian, decomposable, distributive sequoidal closed category.

References

[AJ13] Samson Abramsky and Radha Jagadeesan. Games and full completeness for multiplicative linear logic. *CoRR*, abs/1311.6057, 2013.

[CLM13] Martin Churchill, Jim Laird, and Guy McCusker. Imperative programs as proofs via game semantics. *CoRR*, abs/1307.2004, 2013.

[Hyl97] Martin Hyland. Game semantics. *Semantics and logics of computation*, 14:131, 1997.

[Lai02] J. Laird. A categorical semantics of higher-order store. In *Proceedings of CTCS '02*, number 69 in ENTCS. Elsevier, 2002.