We have a natural $\mathtt{Var} \oslash \_$-coalgebra

$$\mathsf{cell}_0 \colon !\mathbb{N} \multimap \mathtt{Var} \oslash !\mathbb{N}\,,$$

inducing a morphism

$$\mathsf{cell} \colon !\mathbb{N} \to !\mathtt{Var}\,.$$

As in Guy and Samson's work, we define the denotation of a term

$$\Gamma, s \vdash M \colon T$$

to be given by

$$I \cong I \otimes \cdots \otimes I \xrightarrow{[\![s]\!]} !\mathbb{N} \otimes \cdots \otimes !\mathbb{N} \xrightarrow{\mathsf{cell} \otimes \cdots \otimes \mathsf{cell}} !\mathtt{Var} \otimes \cdots \otimes !\mathtt{Var} \xrightarrow{[\![\Gamma \vdash M]\!]} [\![T]\!]\ .$$

The soundness result we want to prove is that if

$$\Gamma, s \vdash M \Downarrow c, s'$$

is derivable, then the denotation of $\Gamma, s \vdash M$ is the same as the denotation of $\Gamma, s' \vdash c$.

To prove this....

Let Γ be a $\mathtt{Var}$-context, and write S_Γ for the object

$$!\mathbb{N} \otimes \cdots \otimes !\mathbb{N}\,,$$

with a copy of $!\mathbb{N}$ for each variable in Γ. Then, if $\Gamma \vdash M \colon T$ is an IA term, we define a $[\![T]\!] \oslash \_$-coalgebra

$$\{\{M\}\}_\Gamma \colon S_\Gamma \to [\![T]\!] \oslash S_\Gamma$$

by induction on the structure of M.

For example:

Canonical forms $\{\{\mathsf{skip}\}\}_\Gamma$ is given by

$$S_\Gamma \xrightarrow{\text{lunit}} I \otimes S_\Gamma \xrightarrow{[\![\mathsf{skip}]\!] \otimes \text{id}} \mathbb{C} \otimes S_\Gamma \xrightarrow{\text{wk}} \mathbb{C} \oslash S_\Gamma\,.$$

Similarly for the other atomic forms (not for mkvar).

If x is a variable in Γ, then write $\Gamma = x, \Gamma'$. Then $\{\{x\}\}_\Gamma$ is given by

$$!\mathbb{N} \otimes S_{\Gamma'} \xrightarrow{\mathsf{cell}_0 \otimes S_{\Gamma'}} (\mathtt{Var} \oslash !\mathbb{N}) \otimes S_{\Gamma'} \xrightarrow{\text{wk;passoc}} \mathtt{Var} \oslash (!\mathbb{N} \otimes S_{\Gamma'})\,.$$

Sequencing Suppose we have terms $\Gamma \vdash M \colon \mathtt{com}$; $\Gamma \vdash N \colon X$ and coalgebras

$$\{\{M\}\}_\Gamma \colon S_\Gamma \to \mathbb{C} \oslash S_\Gamma \qquad\qquad \{\{N\}\}_\Gamma \colon S_\Gamma \to X \oslash S_\Gamma\,.$$

Then we define

$$\{\{M;N\}\}_\Gamma = S_\Gamma \xrightarrow{\{\{M\}\}_\Gamma} \mathbb{C} \oslash S_\Gamma \xrightarrow{\mathsf{seq}_{S_\Gamma}} S_\Gamma \xrightarrow{\{\{N\}\}_\Gamma} X \oslash S_\Gamma\,.$$

Variable dereference Suppose that we have a term $\Gamma \vdash M : \mathtt{Var}$, giving us a coalgebra

$$\{\{M\}\}_\Gamma : S_\Gamma \to \mathtt{Var} \oslash S_\Gamma\,.$$

Then we define

$$\{\{!M\}\}_\Gamma = S_\Gamma \xrightarrow{\{\{M\}\}_\Gamma} \mathtt{Var} \oslash S_\Gamma \xrightarrow{\mathrm{pr} \oslash S_\Gamma} \mathbb{N} \oslash S_\Gamma\,.$$

new Suppose that we have a term $x, \Gamma \vdash M : T$, giving us a coalgebra

$$\{\{M\}\}_{\Gamma,x} : !\mathbb{N} \otimes S_\Gamma \to [\![T]\!] \oslash (S_\Gamma \otimes !\mathbb{N})\,.$$

Then we form the coalgebra

$$!\mathbb{N} \xrightarrow{\Lambda(\{\{M\}\}_{\Gamma,x})} S_\Gamma \multimap ([\![T]\!] \oslash (!\mathbb{N} \otimes S_\Gamma)) \xrightarrow{S_\Gamma \multimap \mathsf{passoc}} S_\Gamma \multimap$$

$$(([\![T]\!] \oslash S_\Gamma) \oslash !\mathbb{N}) \xrightarrow{???} (S_\Gamma \multimap ([\![T]\!] \oslash S_\Gamma)) \oslash !\mathbb{N}\,.$$

Here, the last morphism is part of the definition of a *commutative sequoidal category* (Laird '02) and is not valid in the usual model. I'm not sure quite how to resolve this.

We take the anamorphism of this coalgebra to give us a morphism

$$!\mathbb{N} \to !(S_\Gamma \multimap (!T \oslash S_\Gamma)\,,$$

before composing on the right with $\mathtt{der}$ and on the left with 0, before uncurrying to get a coalgebra

$$S_\Gamma \to [\![T]\!] \oslash S_\Gamma\,,$$

which we will call $\{\{\mathsf{new}_T\, \lambda x.M\}\}_\Gamma$.

We now need to prove things about these coalgebras. The first thing to prove is:

Lemma 0.1. *Let $\Gamma \vdash M : T$ be a reducible term of Idealized Algol and let Δ be a $\mathtt{Var}$-context disjoint from Γ. Then*

$$\{\{M\}\}_{\Delta,\Gamma} = S_\Delta \otimes S_\Gamma \xrightarrow{S_\Delta \otimes \{\{M\}\}_\Gamma} S_\Delta \otimes ([\![T]\!] \oslash S_\Gamma) \xrightarrow{\mathrm{sym;wk}}$$

$$([\![T]\!] \oslash S_\Gamma) \oslash S_\Delta \xrightarrow{\mathsf{passoc}^{-1}} [\![T]\!] \oslash (S_\Gamma \otimes S_\Delta)\,.$$

Next, we prove.

Proposition 0.2. *Let $\Gamma \vdash M : T$ be a term of Idealized Algol. Then $(\!(\, \{\{M\}\}_\Gamma \,)\!) ; \mathtt{der}_{[\![T]\!]}$ is equal to the composite*

$$S_\Gamma \xrightarrow{\mathsf{cell} \otimes \cdots \otimes \mathsf{cell}} !\mathtt{Var} \otimes \cdots \otimes !\mathtt{Var} \xrightarrow{[\![\Gamma \vdash M]\!]} [\![T]\!]\,.$$

For example, in the case of the term $x \vdash x$, $\{\{x\}\}_\Gamma$ is the coalgebra

$$\mathsf{cell}_0 \colon !\mathbb{N} \to \mathtt{Var} \oslash !\mathbb{N} \,,$$

whose anamorphism is $\mathsf{cell} \colon !\mathbb{N} \to !\mathtt{Var}$ by definition. The general case is much harder.

We then prove:

Proposition 0.3. *Suppose that*

$$\Gamma, s \vdash M \Downarrow c, s'$$

is derivable.

Then

$$[\![s]\!] \,;\{\{M\}\}_\Gamma = [\![s']\!] \,; \{\{c\}\}_\Gamma \,.$$

Remark 0.4. This is similar to the result we are trying to prove for $[\![M]\!]$, but our result is strong enough to be proved directy by induction.

Proof. Structural induction on the derivation.

Canonical forms Obvious.

Sequencing Suppose that we have derived

$$\Gamma, s \vdash M \Downarrow \mathsf{skip}, s' \qquad\qquad \Gamma, s' \vdash N \Downarrow x, s'' \,,$$

so that we derive

$$\Gamma, s \vdash M; N \Downarrow x, s'' \,.$$

Then by induction we have

$$[\![s]\!] \,; \{\{M\}\}_\Gamma = [\![s']\!] \,; \{\{\mathsf{skip}\}\}_\Gamma \qquad\qquad [\![s']\!] \,; \{\{N\}\}_\Gamma = [\![s'']\!] \,; \{\{x\}\}_\Gamma \,.$$

Then we have

$$\begin{aligned}
& [\![s]\!] \,; \{\{M; N\}\}_\Gamma \\
&= [\![s]\!] \,; \{\{M\}\}_\Gamma; \mathsf{seq}_{S_\Gamma}; \{\{N\}\}_\Gamma \\
&= [\![s']\!] \,; \{\{\mathsf{skip}\}\}_\Gamma; \mathsf{seq}_{S_\Gamma}; \{\{N\}\}_\Gamma \\
&= [\![s']\!] \,; \{\{N\}\}_\Gamma \\
&= [\![s'']\!] \,; \{\{x\}\}_\Gamma \,.
\end{aligned}$$

Variable dereference Suppose we have derived

$$\Gamma, s \vdash M \Downarrow v, s' \,,$$

and that $s'(v) = x$, and therefore that

$$\Gamma, s \vdash !M \Downarrow x, s' \,.$$

By induction, we have

$$[\![s]\!]\,; \{\{M\}\}_\Gamma = [\![s']\!]\,; \{\{v\}\}_\Gamma\,,$$

and therefore

$$\begin{aligned}
&[\![s]\!]\,; \{\{!M\}\}_\Gamma \\
&= [\![s]\!]\,; \{\{M\}\}_\Gamma; (\mathrm{pr} \oslash !\mathbb{N}) \\
&= [\![s']\!]\,; \{\{v\}\}_\Gamma; (\mathrm{pr} \oslash !\mathbb{N}) \\
&= [\![s']\!]\,; \mathsf{cell}_0; (\mathrm{pr} \oslash !\mathbb{N}) \\
&= [\![s']\!]\,; \{\{x\}\}_\Gamma\,.
\end{aligned}$$

new Suppose that we have derived

$$\Gamma, v, (s|v \mapsto 0) \vdash M \Downarrow c, (s'|v \mapsto n)\,,$$

so that we may deduce

$$\Gamma, s \vdash \mathtt{new}\lambda x.M \Downarrow c, s'\,.$$

Here, it is important to realize that c exists in the context Γ; i.e., the variable x does not appear free in c. (Otherwise the language admits scope extrusion (?) through mkvar. In any case, it does not make sense to say $\Gamma, s \vdash \mathtt{new}\lambda x.M \Downarrow c, s'$ if c contains free variables not in Γ.) This means that c has the coalgebra

$$\{\{c\}\}_\Gamma \colon S_\Gamma \to [\![T]\!] \oslash S_\Gamma\,,$$

and that we interpret c inside the context Γ, v as described in Lemma 0.1.

We also have the coalgebra

$$\{\{M\}\}_{\Gamma,v} \colon !\mathbb{N} \otimes S_\Gamma \to ([\![t]\!] \oslash (!\mathbb{N} \otimes S_\Gamma)\,,$$

and we know by induction that

$$(0 \otimes [\![s]\!]); \{\{M\}\}_{\Gamma,v} = (n \otimes [\![s']\!]); \{\{c\}\}_{\Gamma,v}\,.$$

Then we have

$$\begin{aligned}
&[\![s]\!]\,; \{\{\mathtt{new}\lambda v.M\}\}_\Gamma \\
&= [\![s]\!]\,; \Lambda^{-1}(0; \llparenthesis\, \Lambda\{\{M\}\}_{\Gamma,v}; S_\Gamma \multimap \mathtt{passoc}; \mathtt{comm}\, \rrparenthesis; \mathtt{der}) \\
&= [\![s]\!]\,; \Lambda^{-1}(0; \llparenthesis\, \Lambda\{\{M\}\}_{\Gamma,v}; S_\Gamma \multimap \mathtt{passoc}; \mathtt{comm}\, \rrparenthesis; \alpha; \mathrm{id} \oslash (); \mathtt{r}) \\
&= [\![s]\!]\,; \Lambda^{-1}(0; \Lambda\{\{M\}\}_{\Gamma,v}; S_{\Gamma \multimap \mathtt{passoc}}; \mathtt{comm}; \mathrm{id} \oslash \llparenthesis\, \cdots\, \rrparenthesis; \mathrm{id} \oslash (); \mathtt{r}) \\
&= (0 \otimes [\![s]\!]); \{\{M\}\}_{\Gamma,v}; \mathtt{passoc}; ([\![T]\!] \oslash S_\Gamma) \oslash (); \mathtt{r} \\
&= (n \otimes [\![s']\!]); \{\{c\}\}_{\Gamma,v}; \mathtt{passoc}; ([\![T]\!] \oslash S_\Gamma) \oslash (); \mathtt{r} \\
&= (n \otimes [\![s']\!]); \{\{c\}\}_\Gamma; \mathrm{sym}; \mathrm{wk}; \mathtt{passoc}^{-1}; \mathtt{passoc}; (T \oslash S_\Gamma) \oslash (); \mathtt{r} \\
&= (n \otimes [\![s']\!]); (\mathbb{N} \otimes \{\{c\}\}_\Gamma); \mathrm{sym}; \mathrm{wk}; ([\![T]\!] \oslash S_\Gamma) \oslash (); \mathtt{r}) \\
&= (n \otimes [\![s']\!]); (() \otimes \{\{c\}\}_\Gamma); \mathrm{lunit} \\
&= [\![s']\!]\,; \{\{c\}\}_\Gamma\,.
\end{aligned}$$

And the other cases are similar. $\square$

Lastly, suppose that

$$\Gamma, s \vdash M \Downarrow c, s'$$

is derivable in Idealized Algol. Then we have

$$\begin{aligned}
& [\![\Gamma, s \vdash M]\!] \\
&= [\![s]\!]\,; \mathsf{cell} \otimes \cdots \otimes \mathsf{cell}; [\![\Gamma \vdash M]\!] \\
&= [\![s]\!]\,; (\!\!| \, \{\!\{M\}\!\}_\Gamma \, |\!\!); \mathtt{der} \\
&= [\![s]\!]\,; (\!\!| \, \{\!\{M\}\!\}_\Gamma \, |\!\!); \alpha; \mathrm{id} \oslash (); \mathtt{r} \\
&= [\![s]\!]\,; \{\!\{M\}\!\}_\Gamma; \mathrm{id} \oslash (); \mathtt{r} \\
&= [\![s']\!]\,; \{\!\{c\}\!\}_\Gamma; \mathrm{id} \oslash (); \mathtt{r} \\
&= \cdots \\
&= [\![\Gamma, s' \vdash c]\!] \ ,
\end{aligned}$$

as desired.