Rounding off a corner

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Abstract

In [Lai02], Laird introduces the concept of a *sequoidal category*, a certain extension of a monoidal category and gives an example constructed using game semantics. As noted in [CLM13], cofree commutative comonoids can be constructed coalgebraically in any sequoidal category subject to certain extra hypotheses. In the first part of this note, we review the coalgebraic construction of cofree commutative comonoids in sequoidal categories. In the second part, we shall show that the extra hypotheses are necessary as well as sufficient and we shall use transfinite game semantics to construct a sequoidal category in which these hypotheses do not hold.

1 Sequoidal categories

1.1 Game semantics and the sequoidal operator

We shall present a form of game semantics in the style of [Hyl97] and [AJ13]. A game will be given by a tuple

$$A = (M_A, \lambda_A, b_A, P_A)$$

where

- M_A is a set of moves.
- $\lambda_A \colon M_A \to \{O, P\}$ is a function designating each move as either an O*-move* or a P*-move*.
- $b_A \in \{O, P\}$ is a choice of starting player.
- $P_A \subseteq M_A^*$ is a prefix-closed set of alternating plays - so if $sab \in P_A$ then $\lambda_A(a) = \neg\lambda_A(b)$.

We call $sa \in P_A$ a *P-position* if a is a P-move and an *O-position* if a is an O-move.

Note that plays in P_A do not have to start with a b_A-move, so there might be some positions in P_A that can never actually arise when playing the game.

A *strategy* for player P for a game A is identified with the set of positions that may arise when playing according to that strategy. Namely, it is a non-empty prefix-closed subset $\sigma \subseteq P_A$ satisfying the three conditions:

(sϵ) If $a \in \sigma$ is a 1-move play, then $\lambda_A(a) = b_A$.

(sO) If $s \in \sigma$ is a P-position and a is an O-move such that $sa \in P_A$, then $sa \in \sigma$.

(sP) If $sa, sb \in \sigma$ are P-positions, then $a = b$.

References

[AJ13] Samson Abramsky and Radha Jagadeesan. Games and full completeness for multiplicative linear logic. *CoRR*, abs/1311.6057, 2013.

[CLM13] Martin Churchill, Jim Laird, and Guy McCusker. Imperative programs as proofs via game semantics. *CoRR*, abs/1307.2004, 2013.

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[Lai02] J. Laird. A categorical semantics of higher-order store. In *Proceedings of CTCS '02*, number 69 in ENTCS. Elsevier, 2002.