1 Introduction

These are games with ordinal sequences of moves.

TODO: Talk about what they are and why we are interested, being careful to point out that our $\omega + 1$ games correspond to games with winning conditions.

2 Our starting category of games

Before studying games with transfinite sequences of moves, we shall illustrate some of the choices we have made by defining a category of games with finite sequences of moves. We have chosen these definitions because they extend particularly well to the transfinite case.

2.1 Games and strategies

We shall use the notation introduced in [**?**] to describe games. All our games A will have, at their heart, the following three pieces of information:

- A set M_A of possible moves
- A function $\lambda_A \colon M_A \to \{O, P\}$ assigning to each move the player who is allowed to make that move
- A prefix-closed set $P_A \subset M_A^*$ of finite sequences of moves.

We shall normally insist on an *alternating condition* on P_A:

Alternating condition If $a, b \in M_A$ are moves and $s \in M_A^*$ is a sequence of moves such that $sa, sab \in P_A$, then $\lambda_A(a) = \neg\lambda_A(b)$.

As in [**?**], we identify a *strategy* for a game A with the set of sequences of moves that can occur when player P is playing according to that strategy so that a typical definition of a (partial) strategy might be a set $\sigma \subset P_A$ such that (for all $s \in M_A^*, a, b \in M_A$):

- $\epsilon \in \sigma$ (ensures that σ is non-empty)
- If $sa \in \sigma$, $\lambda_A(a) = P$ and $sab \in P_A$ then $sab \in \sigma$ (σ contains all legal replies by player O)

- If $s, sa, sb \in \sigma$ and $\lambda_A(a) = P$ then $a = b$ (σ contains at most one legal reply by player P)

We can impose additional constraints on σ that will ensure that σ is total, strict, history free and so on. The definition given immediately above is not the only definition of a strategy found in the literature, however. For example, the games described in [**?**] have the curious property that the set P_A may contain plays that cannot actually occur when A is being played; in particular, all plays must start with a move by player O, but the set P_A may contain positions that start with a P-move. These plays do not affect the strategies for A, but they might come into play if we perform operations on A such as forming the negation $\neg A$ or the implication $A \multimap B$.

This behaviour is made implicit in Abramsky and Jagadeesan's definitions, which do not impose any conditions upon the set P_A beyond the basic alternation condition given above, but which mandate that any play occurring *in a strategy* must begin with an O-move. For the sake of clarity, we adopt a different, but completely equivalent, approach. For a game A, we define a set L_A, regarded as the set of *legal plays* occurring in P_A. In some games models, such as that found in [**?**], L_A may be defined to be the whole of P_A, while in [**?**] it is defined to be that subset of P_A consisting of plays that begin with an O-move.

The point of specifying L_A separately is that it allows us to unify the definitionn of a *strategy*, while making clearer the behaviour observed above, whereby certain plays in P_A may not occur 'in normal play'; this behaviour was previously only implicit in the definition of a strategy. Our unified definition then becomes:

Definition 2.1. If $A = (M_A, \lambda_A, P_A)$ is a game and L_A is its associated set of legal plays (in a particular games model) then a (partial) *strategy* for A is a subset $\sigma \subset L_A$ such that for all $s \in L_A$ and all $a, b \in M_A$:

- $\epsilon \in \sigma$
- If $s \in \sigma$ and a is an O-move, and if $sa \in L_A$, then $sa \in \sigma$
- If $s \in \sigma$ and a, b are P-moves, and if $sa, sb \in \sigma$, then $a = b$

2.2 Positive and negative games, ownership of plays and connectives

Abramsky-Jagadeesan games, as described in [**?**], may admit both plays that start with a P-move and plays that start with an O-move. Other games models, such as those found in [**?**] and [**?**], are more restrictive. The games in [**?**] only contain plays starting with an O-move. The plays in [**?**] may start with either a P-move or an O-move, but a play starting with a P-move and a play starting with an O-move may not occur in the same game.

Definition 2.2. We say that a game $A = (M_A, \lambda_A, P_A)$ is *positive* if every play in P_A begins with a P-move. We say that A is *negative* if every play in P_A begins with an O-move.

So the Curien model found in [**?**] admits only negative games, the Blass model in [**?**] admits positive and negative games, while the Abramsky-Jagadeesan model found in [**?**] admits not only positive and negative games, but also games that are neither negative nor positive. We shall now examine the reasons for and drawbacks of each of these choices.

The earliest games model, found in [**?**], did not include a definition of which player is to move at a given position; rather, games are defined recursively as pairs of games $\{L|R\}$, where L represents the positions that the left player may move into, while R represents the positions that the rigth player may move into. Blass's definition departs completely from this tradition; now, at every position s only one of the two players is allowed to move; extending this logic on to the empty position ϵ, it follows that all games are either positive or negative. This property means that we may freely define $L_A = P_A$, since there is never any question about whose turn it is to play. By contrast, if we were to define $L_A = P_A$ for Abramsky-Jagadeesan games, then a strategy might end up containing two branches, one of plays beginning with an O-move and one of plays beginning with a P-move, which is undesirable. The alternative definition of L_A avoids this problem.

In the case of a Blass game A, we may define a function $\zeta_A \colon P_A \to \{O, P\}$ that says which player owns each play; the idea is that if we are in position s, then the next player to move is given by $\neg\zeta_A(s)$; i.e., the opposing player to the player who has just made the move. One might want to define ζ_A by setting $\zeta_A(sa) = \lambda_A(a)$, so that ownership of a play is decided by who has made the last move in the play, but this definition does not extend in an obvious way to the empty position ϵ (and, as we shall see in the next chapter, it does not extend to plays over limit ordinals). In this case, $\zeta_A(\epsilon)$ is part of the game's data, and it determines whether the game is positive or negative: if $\zeta_A(\epsilon) = P$ then all plays must start with an O-move, and the game is negative – and vice versa.

An important question then arises: how should we extend the function ζ_A to games formed from connectives? The solution adopted by Blass is to use binary conjunctions to deduce the ownership of a play from the ownership of the restrictions of that play to the two component games. In the case of the tensor product $A \otimes B$ of two games A and B, we define $\zeta_{A\otimes B} \colon P_{A\otimes B} \to \{O, P\}$ by setting

$$\zeta_{A\otimes B}(s) = (\zeta_A(s|_A) \wedge \zeta_B(s|_B))$$

where $\wedge \colon \{O, P\} \times \{O, P\} \to \{O, P\}$ is as in Figure 1.

a	b	$a \wedge b$
O	O	O
O	P	O
P	O	O
P	P	P

a	b	$a \vee b$
O	O	O
O	P	P
P	O	P
P	P	P

a	b	$a \Rightarrow b$
O	O	P
O	P	P
P	O	O
P	P	P

Figure 1: Truth tables for binary conjunctions on $\{O, P\}$

Similarly, we may extend ζ to the implication $A \multimap B$ and the par $A \parr B$ by setting

$$\zeta_{A\multimap B}(s) = (s|_A \Rightarrow s|_B)$$
$$\zeta_{A\parr B}(s) = (s|_A \vee s|_B)$$

Note that if we use these definitions then the owner $\zeta_C(sa)$ of a play sa might not correspond to the player $\lambda_C(a)$ who played the last move a. For example, let A, B be two positive games and form their tensor product $A\otimes B$. Then we have

$$\zeta_{A\otimes B}(\epsilon) = (\zeta_A(\epsilon) \wedge \zeta_B(\epsilon)) = O \wedge O = O$$

and so $A \otimes B$ is a positive game. Player P plays an opening move in one of the two games - let us say she plays the move a in the game A. But then we have

$$\zeta_{A\otimes B}(a) = (\zeta_A(a) \wedge \zeta_B(\epsilon)) = P \wedge O = O$$

In other words, it is still player P's turn to play! Blass embrace this possibility and allows player P to make these two moves. In his paper, he introduces the notions of *strict* and *relaxed* games, where the strict games are the objects of study but the relaxed games are often used since they allow more manipulations. In this case, the game $A \otimes B$ is defined as a relaxed game that might not satisfy the alternating condition; in the process of converting it into a strict game, these two opening moves by player P are combined into a single move.

This 'double move' can only occur at the start of the game, and Blass treats it as a special case in his proofs. Perhaps unsurprisingly, this inconsistency causes major problems if we try to compose strategies. We do not get an associative composition of strategies for $A \multimap B$ with strategies for $B \multimap C$ and so we do not get a categorical semantics. An example of the failure of associativity in Blass's games model is given towards the end of [**?**].

By contrast, Abramsky-Jagadeesan games may admit moves by both players at the same position (specifically, at the beginning of the game, before any moves have been played), but this does not cause problems since we insist that our legal plays start with an O-move and be strictly alternating. The authors of [**?**] note that their model can be considered as an intermediate between Conway's games, where the position tells you nothing about which player is to move, and Blass games, where the position completely determines which player is to move. In Abramsky-Jagadeesan games, one can deduce which player is to move (by looking at which player made the last move) in every position except the empty starting position.

In the Abramsky-Jagadeesan model, a positive game is an immediate win to player P, since player O has no legal move to start the game off. As we noted before, this does not mean that the content of a positive game is meaningless, since we can use connectives to 'unlock' these illegal plays. For example, if Q is a positive game and N is a negative game then $Q \parr N$ is a negative game, and the possible positions in Q are now all achievable.

Curien's game model ([**?**]) is similar to Abramsky's and Jagadeesan's, but involves only negative games. The only slight problem is that negative Abramsky-Jagadeesan games are not closed under implication: if N, L are negative games then $N \multimap L$ may be neither negative nor positive. We may fix this by modifying the definition of $N \multimap L$ so that we delete from $P_{N \multimap L}$ all plays that start with a P-move - or, equivalently, by requiring that all plays start in L. This is the approach taken in [**?**], where it fits well with the paper's treatment of the *sequoid* operator $\oslash$, which is a version of the tensor product that has been modified so that play is required to start in the left-hand game.

We shall adopt elements of both the Blass and the Abramsky-Jagadeesan games models; specifically, we shall use Blass's games and Abramsky-Jagadeesan's strategies. This means that our games model will be more restrictive than either the Blass or the Abramsky-Jagadeesan models, but this lack of flexibility will be just what we need in order to extend these games over the transfinite ordinals. We will later consider ways we can relax our model to recover Abramsky and Jagadeesan's games model.

2.3 Our definition of games and strategies

Definition 2.3. A *game* is a triple $(M_A, \lambda_A, \zeta_A, P_A)$ where

- M_A is a set of moves,
- $\lambda_A \colon M_A \to \{O, P\}$ is a function that assigns a player to each move,
- $P_A \subset M_A^*$ is a non-empty prefix-closed set of plays that can occur in the game and
- $\zeta_A \colon P_A \to \{O, P\}$ is a function that assigns a player to each position

such that

- If $a \in M_A$ and $sa \in P_A$ then $\zeta_A(sa) = \lambda_A(a)$.
- If $a \in M_A$ and $sa \in P_A$ then $\zeta_A(s) = \neg\zeta_A(sa)$.

Remark 2.4 (Notes on the definition)**.** Given a game $A = (M_A, \lambda_A, \zeta_A, P_A)$, define $b_A = \neg\zeta_A(\epsilon)$. Then every play in P_A must start with a b_A-move, so A is either positive or negative.

Note that ζ_A is now completely specified by λ_A and b_A, so we could have specified our games mor efficiently by replacing ζ_A with b_A in our definition, as done in [**?**]. The slightly more unwieldy ζ_A will be useful when we come to extend our games over the ordinals, though, so we retain it.

If $a \in M_A$ then we may recover $\lambda_A(a)$ from ζ_A so long as a occurs in some play in P_A. Since moves that can never be played do not affect the game at all, we do not really need λ_A in our definition, but we keep it to make the connection to earlier work clearer.

If $a \in M_A$ and $\lambda_A(a) = O$, we call a an *O-move*. If $\lambda_A(a) = P$, we call a a *P-move*. If $s \in P_A$ and $\zeta_A(s) = O$, we call s an *O-play* or *O-position*. If $\zeta_A(s) = P$, we call s a *P-play* or *P-position*.

Given the game A, we define L_A to be the set of all plays $s \in P_A$ such that either s has even length and $\zeta_A(s) = P$ or s has odd length and $\zeta_A(s) = O$. Since P_A is alternating, it is evident that $L_A = P_A$ if $\zeta_A(\epsilon) = P$ and is empty otherwise. This slightly strange definition will become more meaningful once we define games over ordinals.

Definition 2.5. Let $A = (M_A, \lambda_A, \zeta_A, P_A)$ be a game and let L_A be the associated set of legal plays. A *strategy* for A is a non-empty prefix-closed subset $\sigma \subset L_A$ such that:

- If $a \in P_A$ is an O-move and $s \in \sigma$ is a P-position such that $sa \in P_A$, then $sa \in \sigma$.
- If $s \in \sigma$ is an O-position and $a, b \in M_A$ are P-moves such that $sa, sb \in \sigma$, then $a = b$.

2.4 Connectives

Our definitions of connectives on games are as in [**?**].

Definition 2.6. Let $A = (M_A, \lambda_A, \zeta_A, P_A)$ be a game. The negation of A, ${}^{\perp}A$, is the game formed by interchanging the roles of the two players.

- $M_{({}^{\perp}A)} = M_A$
- $\lambda_{({}^{\perp}A)} = \neg \circ \lambda_A$
- $\zeta_{({}^{\perp}A)} = \neg \circ \zeta_A$
- $P_{({}^{\perp}A)} = P_A$

It follows immediately from the definitions that ${}^{\perp}A$ is a well formed game.

We now define the tensor product $A \otimes B$, the par $A \parr B$ and the linear implication $A \multimap B$ of two games. All these games are obtained by playing A and B in parallel, so they all have the same set of moves:

$$M_{A \otimes B} = M_{A \parr B} = M_{A \multimap B} = M_A \sqcup M_B$$

Ownership of moves is decided via the obvious copairing functions:

$$\lambda_{A\otimes B} = \lambda_{A ⅋ B} = \lambda_A \sqcup \lambda_B$$
$$\lambda_{A\multimap B} = (\neg \circ \lambda_A) \sqcup \lambda_B$$

We define the set $P_A \| P_B$ to be the set of all plays in $(M_A \sqcup M_B)^*$ whose M_A-component is a play from P_A and whose M_B-component is a play from P_B:

$$P_A \| P_B = \{s \in (M_A \sqcup M_B)^* \;:\; s|_A \in P_A,\ s|_B \in P_B\}$$

We are now in a position to define $\zeta_{A\otimes B}, \zeta_{A ⅋ B}, \zeta_{A\multimap B}$ as functions $P_A \| P_B \to \{O, P\}$:

$$\zeta_{A\otimes B}(s) = \zeta_A(s|_A) \wedge \zeta_B(s|_B)$$
$$\zeta_{A ⅋ B}(s) = \zeta_A(s|_A) \vee \zeta_B(s|_B)$$
$$\zeta_{A\multimap B}(s) = \zeta_A(s|_A) \Rightarrow \zeta_B(s|_B)$$

(Here, $\wedge$, $\vee$ and $\Rightarrow$ are as in Figure 1.)

As things stand, $P_A \| P_B$ is not a valid set of plays for our λ and ζ functions. The first problem is that $P_A \| P_B$ contains plays which are not alternating with respect to $\zeta_{A\otimes B}, \zeta_{A ⅋ B}, \zeta_{A\multimap B}$. The second problem is that the ζ and λ functions do not always agree with one another. For example, suppose that Q, R are two positive games. So we have $\zeta_Q(\epsilon) = \zeta_R(\epsilon) = O$. Then $\zeta_{Q\otimes R}(\epsilon) = O \wedge O = O$. But now suppose that player P can make an opening move q in Q. Then we have

$$\zeta_{Q\otimes R}(q) = \zeta_Q(q) \wedge \zeta_R(\epsilon) = P \wedge O = O$$

but $\lambda_{Q\otimes R}(q) = P$.

Our solution is to throw away some plays from $P_A \| P_B$ so that what remains satisfies the alternating condition and the condition on the λ and ζ functions.

Definition 2.7. Let M be a set, let $P \subset M^*$ be prefix closed and let $\lambda \colon M \to \{O, P\}, \zeta \colon P \to \{O, P\}$ be functions. If $s \in P$, we say that s is *alternating with respect to* ζ if the set of prefixes of s satisfies the alternating condition: if $t, ta \sqsubseteq s$, then $\zeta(t) = \neg\zeta(ta)$.

We say that s is *well formed with respect to* λ, ζ if $s = \epsilon$ or if $s = s'a$ and $\zeta(s) = \lambda(a)$.

Now let A, B be games. We define:

$$P_{A\otimes B} = \left\{ s \in P_A \| P_B \;\middle|\; \begin{array}{l} s \text{ is alternating with respect to } \zeta_{A\otimes B} \\ s \text{ is well formed with respect to } \zeta_{A\otimes B}, \lambda_{A\otimes B} \end{array} \right\}$$
$$P_{A⅋B} = \left\{ s \in P_A \| P_B \;\middle|\; \begin{array}{l} s \text{ is alternating with respect to } \zeta_{A⅋B} \\ s \text{ is well formed with respect to } \zeta_{A⅋B}, \lambda_{A⅋B} \end{array} \right\}$$
$$P_{A\multimap B} = \left\{ s \in P_A \| P_B \;\middle|\; \begin{array}{l} s \text{ is alternating with respect to } \zeta_{A\multimap B} \\ s \text{ is well formed with respect to } \zeta_{A\multimap B}, \lambda_{A\multimap B} \end{array} \right\}$$

Definition 2.8. We define:

$$\begin{aligned} A \otimes B &= (M_{A\otimes B}, \lambda_{A\otimes B}, \zeta_{A\otimes B}, P_{A\otimes B}) \\ A ⅋ B &= (M_{A⅋B}, \lambda_{A⅋B}, \zeta_{A⅋B}, P_{A⅋B}) \\ A \multimap B &= (M_{A\multimap B}, \lambda_{A\multimap B}, \zeta_{A\multimap B}, P_{A\multimap B}) \end{aligned}$$

Proposition 2.9. *$A \otimes B$,$A ⅋ B$,$A \multimap B$ are well formed games. Moreover, $P_{A\otimes B}$ is the largest subset of $P_A \| P_B$ such that $(M_{A\otimes B}, \lambda_{A\otimes B}, \zeta_{A\otimes B}, P_{A\otimes B})$ is a well formed game and similarly for the connectives ⅋ and $\multimap$.*

Proof. We will prove the proposition for $A \otimes B$; the other two cases are entirely similar. Alternatively, observe that $A ⅋ B = {}^{\perp}({}^{\perp}A \otimes {}^{\perp}B)$ and $A \multimap B = {}^{\perp}A ⅋ B$.

For $A \otimes B$, it suffices to show that $P_{A\otimes B}$ is alternating with respect to $\zeta_{A\otimes B}$, since every $s \in P_{A\otimes B}$ is well-formed by definition. Suppose $s, sa \in P_{A\otimes B}$; then $s, sa \sqsubseteq sa$; since sa is alternating with respect to $\zeta_{A\otimes B}$, it follows that $\zeta_{A\otimes B}(s) = \neg\zeta_{A\otimes B}(sa)$.

For the second part of the proposition, suppose that $V \subset P_A \| P_B$ is prefix-closed and satisfies the alternating condition with respect to $\zeta_{A\otimes B}$ and that every $s \in V$ is well-formed with respect to $\lambda_{A\otimes B}, \zeta_{A\otimes B}$. We need to show that $V \subset P_{A\otimes B}$, for which it will suffice to show that every $s \in V$ is alternating. This is easy to see: since V is prefix closed, the set of all prefixes of s is a subset of V, and so it satisfies the alternating condition with respect to $\zeta_{A\otimes B}$. □

A design feature of the connectives $\otimes$, ⅋, $\multimap$ is that only player O may switch games in $A \otimes B$, while only player P may switch games in $A ⅋ B$ and $A \multimap B$. The $\multimap$ case follows immediately from the ⅋ case by noting that $A \multimap B = {}^{\perp}A ⅋ B$, and the ⅋ case then follows from the $\otimes$ case by observing that $A ⅋ B = {}^{\perp}({}^{\perp}A \otimes {}^{\perp}B)$. Thus, it will suffice to prove the following proposition for the tensor product:

Proposition 2.10. *Let A, B be games. Suppose $s \in P_{A \multimap B}$, $a \in M_A$ and $b \in M_B$. Then:*

i) If $sab \in P_{A \otimes B}$ then $\lambda_{A \otimes B}(b) = O$

ii) If $sba \in P_{A \otimes B}$ then $\lambda_{\otimes B}(a) = O$.

Proof. (i):
$$\begin{aligned} \lambda_{A\otimes B}(b) &= \zeta_{A\otimes B}(sab) \\ &= \zeta_{A\otimes B}(s|_A a) \wedge \zeta_{A\otimes B}(s|_B b) \\ &= \lambda_{A\otimes B}(a) \wedge \lambda_{A\otimes B}(b) \end{aligned}$$

By alternation, either $\lambda_{A\otimes B}(a) = O$ or $\lambda_{A\otimes B}(b) = O$, so this last expression must be equal to O.

(ii):
$$\begin{aligned} \lambda_{A\otimes B}(a) &= \zeta_{A\otimes B}(sba) \\ &= \zeta_{A\otimes B}(s|_A a) \wedge \zeta_{A\otimes B}(s|_B b) \\ &= \lambda_{A\otimes B}(a) \wedge \lambda_{A\otimes B}(b) = O \text{ (by the same argument)} \end{aligned}$$ □

2.5 A category of games and partial strategies

Following [**?**] and [**?**], we define a category $\mathcal{G}$ whose objects are games where the morphisms from a game A to a game B are strategies for $A \multimap B$. For the sake of simplicity, and to avoid varioius technical issues, we shall require that the games in our category be *negative* - namely, they should start with an opponent move. In our language, we call a game A *negative* if $\zeta_A(\epsilon) = P$ (and we call it positive if $\zeta_A(\epsilon) = O$). The equations

$$P \wedge P = P$$
$$P \Rightarrow P = P$$

tell us that the class of negative games is closed under $\otimes$ and $\multimap$ (since $P \vee P = P$, it is also closed under $\invamp$, but the par of two negative games has no legal moves in our presentation, so it does not give a good model of the $\invamp$ connective in linear logic and we will not consider it).

In order to get a category, we need a way to compose a strategy for $A \multimap B$ with a strategy for $B \multimap C$. We do this in the standard way (see [**?**] or [**?**], for example).

Given negative games A, B, C and strategies σ for $A \multimap B$ and τ for $B \multimap C$, define

$$\sigma \| \tau = \{\mathfrak{s} \in (M_A \sqcup M_B \sqcup M_C)^* \,:\, \mathfrak{s}|_{A,B} \in \sigma,\ \mathfrak{s}|_{B,C} \in \tau\}$$

We then define

$$\tau \circ \sigma = \{\mathfrak{s}|_{A,C} \,:\, \mathfrak{s} \in \sigma \| \tau\}$$

Before proving that this is indeed a strategy for $A \multimap C$, we first record a nice property of negative games that will make things particularly easy for us:

Proposition 2.11. *Let A, B be negative games and suppose that $s \in P_A \| P_B$ is alternating with respect to $\zeta_{A\otimes B}$. Then:*

i) Either $\zeta_A(s|_A) = P$ or $\zeta_B(s|_B) = P$.

ii) s is well formed with respect to $\zeta_{A\otimes B}, \lambda_{A\otimes B}$.

Furthermore, (i) and (ii) hold if we replace $\otimes$ with $\multimap$ throughout and if we replace (i) by the statement that either $\zeta_A(s|_A) = P$ or $\zeta_B(s|_B) = O$.

We gave an example earlier in which a play sa in $P_A \| P_B$ was not well formed with respect to $\zeta_{A\otimes B}, \lambda_{A\otimes B}$. The crucial thing that made that example work was that we had $\zeta_A(s|_A) = \zeta_B(s|_B) = O$. This then meant that $\zeta_{A\otimes B}(sa) = O$ even when $\lambda_{A\otimes B} = P$. Part (i) of the proposition above tells us that this situation never occurs, and part (ii) tells us that this is enough to ensure that we never get any alternating sequences that are not well formed.

Proof of Proposition 2.11. We proceed by induction on the length n of s. If $n = 0$ then $s = \epsilon$ and so $\zeta_A(s|_A) = \zeta_A(\epsilon) = P$ and similarly for B, since A, B are negative games. Now suppose that $n > 0$ - so $s = tm$ for some $t \in P_{A\otimes B}, m \in M_{A\otimes B}$. By induction, either $t|_A = P$ or $t|_B = P$, so there are three cases:

Case 1: $\zeta_A(t|_A) = P$ and $\zeta_B(t|_B) = P$. Then either $tm|_A = t|_A$ (if m is a B-move) or $tm|_B = t|_B$ (if m is an A-move). This proves part (i). For part (ii), note that in this case we have $\zeta_{A\otimes B}(t) = P$, so $\zeta_{A\otimes B}(tm) = O$, since tm is alternating. If m is an A-move, it means that $\zeta_B(tm|_B) = \zeta_B(t|_B)$, so we must have $\zeta_A(tm|_A) = O$. Then

$$\lambda_{A\otimes B}(m) = \lambda_A(m) = \zeta_A(t|_A m) = O = \zeta_{A\otimes B}(tm)$$

If m is a B-move, the argument is similar.

Case 2: $\zeta_A(t|_A) = P$ and $\zeta_B(t|_B) = O$. For part (i), it will suffice to show that m is a B-move - so $tm|_A = t|_A$. Indeed, we have $\zeta_{A\otimes B}(t) = O$, so $\zeta_{A\otimes B}(tm) = P$, since tm is alternating with respect to $\zeta_{A\otimes B}$. In particular, $\zeta_B(tm|_B) = P$, so we must have $tm|_B \neq t|_B$ and therefore m is a B-move. For part (ii), we have:

$$\lambda_{A\otimes B}(m) = \lambda_B(m) = \zeta_B(t|_B m) = \neg\zeta_B(t|_B) = \neg O = P$$
$$\zeta_{A\otimes B}(tm) = \neg\zeta_{A\otimes B}(t) = t = \neg O = P = \lambda_{A\otimes B}$$

Case 3: $\zeta_A(t|_A) = O$ and $\zeta_B(t|_B) = P$. Similar to Case 2.

Finally, note that the $\otimes$ result implies the $\multimap$ result after writing $A \multimap B = {}^{\perp}A \invamp B = {}^{\perp}(A \otimes {}^{\perp}B)$. $\square$

A corollary of our result is that we have:

$$P_{A\otimes B} = \{s \in P_A \| P_B : \ s \text{ is alternating with respect to } \zeta_{A\otimes B}\}$$
$$P_{A\multimap B} = \{s \in P_A \| P_B : \ s \text{ is alternating with respect to } \zeta_{A\multimap B}\}$$

if A, B are negative games. In other words, we can ignore the well-formedness condition on plays if we are dealing with only negative games.

This proposition allows us to prove the following useful fact:

Lemma 2.12. *Let A, B, C be negative games, and let $\mathfrak{s} \in (M_A \sqcup M_B \sqcup M_C)^*$. If any two of the following statements are true, then so is the third:*

$$\mathfrak{s}|_{A,B} \in P_{A\multimap B}$$
$$\mathfrak{s}|_{A,C} \in P_{A\multimap C}$$
$$\mathfrak{s}|_{B,C} \in P_{B\multimap C}$$

Proof. By symmetry, it will suffice to prove that if $\mathfrak{s}|_{A,B} \in P_{A\multimap B}$ and $\mathfrak{s}|_{B,C} \in P_{B\multimap C}$ then $\mathfrak{s}|_{A,C} \in P_{A\multimap C}$. If X, Y are negative games, then sequences $t \in P_{X\multimap Y}$ are specified by the following properties - $t|_X \in P_X$, $t|_Y \in P_Y$, if t has even length, then $\zeta_{X\multimap Y}(t) = P$ and if t has odd length then $\zeta_{X\multimap Y}(t) = O$. We need to show that these properties holds if $X = A$, $Y = C$ and $t = \mathfrak{s}|_{A,C}$.

Certainly, $\mathfrak{s}|_{A,C}|_A = \mathfrak{s}|_A = \mathfrak{s}|_{A,B}|_A \in P_A$ and similarly $\mathfrak{s}|_{A,C}|_C \in P_C$.

If $\mathfrak{s}|_{A,C}$ has even length, it means that the lengths of $\mathfrak{s}|_A$ and $\mathfrak{s}|_C$ have the same parity, and therefore that the lengths of $\mathfrak{s}|_{A,B}$ and $\mathfrak{s}|_{B,C}$ have the same parity. Since $\mathfrak{s}|_{A,B} \in \sigma$ and $\mathfrak{s}|_{B,C} \in \tau$, it follows that $\zeta_{A\multimap B}(\mathfrak{s}|_{A,B}) = \zeta_{B\multimap C}(\mathfrak{s}|_{B,C})$. If $\zeta_B(\mathfrak{s}|_B) = P$ then $\zeta_{A\multimap B}(\mathfrak{s}|_{A,B}) = P$ and if $\zeta_B(\mathfrak{s}|_B) = O$ then $\zeta_{B\multimap C}(\mathfrak{s}|_{B,C}) = P$, so we must have $\zeta_{A\multimap B}(\mathfrak{s}|_{A,B}) = \zeta_{B\multimap C}(\mathfrak{s}|_{B,C}) = P$. By transitivity of $\Rightarrow$, it follows that $\zeta_{A\multimap C}(\mathfrak{s}|_{A,C}) = P$.

If $\mathfrak{s}|_{A,C}$ has odd length, it means that the lengths of $\mathfrak{s}|_A$ and $\mathfrak{s}|_C$ has opposite parities and therefore that the lengths of $\mathfrak{s}|_{A,B}$ and $\mathfrak{s}|_{B,C}$ have opposite parities. So it follows that $\zeta_{A\multimap B}(\mathfrak{s}|_{A,B}) = \neg\zeta_{B\multimap C}(\mathfrak{s}|_{B,C})$. Suppose that $\zeta_{A\multimap B}(\mathfrak{s}|_{A,B}) = O$. Then $\zeta_A(\mathfrak{s}|_A) = P$ and $\zeta_B(\mathfrak{s}|_B) = O$. By Proposition 2.11 above applied to $\mathfrak{s}|_{B,C}$, we must have $\zeta_C(\mathfrak{s}|_C) = O$ and therefore $\zeta_{A\multimap C}(\mathfrak{s}|_{A,C}) = (P \Rightarrow O) = O$.

If instead we have $\zeta_{B\multimap C}(\mathfrak{s}|_{B,C}) = O$, then $\zeta_B(\mathfrak{s}|_B) = P$ and $\zeta_C(\mathfrak{s}|_C) = O$. By Proposition 2.11 applied to $\mathfrak{s}|_{A,B}$, we must have $\zeta_A(\mathfrak{s}|_A) = P$ and so $\zeta_{A\multimap C}(\mathfrak{s}|_{A,C}) = (P \Rightarrow O) = O$. □

We can use this fact to prove that the composition of two strategies is a well-defined strategy.

Proposition 2.13. *Let A, B, C be negative games, let σ be a strategy for $A \multimap B$ and let τ be a strategy for $B \multimap C$. Then $\tau \circ \sigma$ as defined above is a strategy for $A \multimap C$.*

Proof. We have $\epsilon \in \tau \circ \sigma$, so $\tau \circ \sigma$ is non-empty. Let $\mathfrak{s} \in \sigma\|\tau$, and suppose that $t \sqsubseteq \mathfrak{s}|_{A,C}$. Let m be the last move of t, and let $\mathfrak{t}$ be the prefix of $\mathfrak{s}$ consisting of all moves up to and including m. Then $\mathfrak{t}|_{A,C} = t$. Since $\sigma\|\tau$ is clearly prefix closed, it follows that $t \in \tau \circ \sigma$. This shows that $\tau \circ \sigma$ is prefix closed.

There is no distinction between the set P_X of plays and the set L_X of legal plays for any negative game X. Since $\sigma \subset P_{A\multimap B}$ and $\tau \subset P_{B\multimap C}$, Lemma 2.12 tells us that $\mathfrak{s}|_{A,C} \in P_{A\multimap C}$.

We now show that the two strategy properties hold for $\tau \circ \sigma$. Suppose $\mathfrak{s} \in \sigma\|\tau$ and that $\zeta_{A\multimap C}(\mathfrak{s}|_{A,C}) = P$. Suppose also that $a \in M_A$ and that $\mathfrak{s}|_{A,C}a \in P_{A\multimap C}$. We claim that $\mathfrak{s}a \in \sigma\|\tau$. Since $\mathfrak{s}a|_{B,C} = \mathfrak{s}|_{B,C} \in \tau$, it will suffice to show that $\mathfrak{s}a|_{A,B} \in \sigma$.

We have $\mathfrak{s}a|_{A,C} = \mathfrak{s}|_{A,C}a \in P_{A\multimap C}$ and $\mathfrak{s}a|_{B,C} = \mathfrak{s}|_{B,C} \in P_{B\multimap C}$ so, by Lemma 2.12, $\mathfrak{s}a|_{A,B} = \mathfrak{s}|_{A,B}a \in P_{A\multimap B}$. Now $\zeta_{A\multimap C}(\mathfrak{s}|_{A,C}) = P$ and so $\lambda_A(a) = P$, since $\mathfrak{s}|_{A,C}a \in P_{A\multimap C}$. Since $\mathfrak{s}|_{A,B}a \in P_{A\multimap B}$, it follows that $\zeta_{A\multimap B}(\mathfrak{s}|_{A,B}a) = P$. Since $\mathfrak{s}|_{A,B} \in \sigma$ and σ is a strategy, it follows that $\mathfrak{s}|_{A,B}a \in \sigma$. Therefore, $\mathfrak{s}a \in \sigma\|\tau$, so $\mathfrak{s}|_{A,C}a = \mathfrak{s}a|_{A,C} \in \tau \circ \sigma$.

An identical argument tells us that if $c \in M_C$ and $\mathfrak{s}|_{A,C}c \in P_{A\multimap C}$ then $\mathfrak{s}|_{A,C}c \in \tau \circ \sigma$.

Now suppose that $s \in \tau \circ \sigma$ is such that $\zeta_{A \multimap C}(s) = O$ and that $m_1, m_2 \in M_{A \multimap C}$ are such that $sm_1, sm_2 \in \tau \circ \sigma$. We wish to show that $m_1 = m_2$.

We shall use the argument given in [**?**]. Given a sequence $s \in \tau \circ \sigma$, we say that $\mathfrak{s} \in \sigma \| \tau$ *covers* s if $s \sqsubseteq \mathfrak{s}|_{A,C}$. By definition, every sequence in $\tau \circ \sigma$ is covered by some sequence in $\tau \circ \sigma$. We make two claims:

i) If $s \in \tau \circ \sigma$, then there is a sequence $\mathfrak{s}$ that is minimal among sequences covering s (with respect to the prefix ordering) - in such a case, we must have $s = \mathfrak{s}|_{A,C}$.

ii) If $s \in \tau \circ \sigma$, $\zeta_{A \multimap C}(s) = O$ and $sm_1, sm_2 \in \tau \circ \sigma$ for some $m_1, m_2 \in M_{A \multimap C}$, then $m_1 = m_2$.

Claim (ii) is what we are trying to prove. We prove (i) and (ii) by simultaneous induction on the length n of s.

If s is the empty sequence, then s is covered by $\epsilon \in \sigma \| \tau$, which is clearly minimal. Now suppose that $n \geq 1$, so $s = s'm$ for some $s' \in \tau \circ \sigma, m \in M_{A \multimap C}$. By the inductive hypothesis (i), there is some $\mathfrak{s}'$ minimal among sequences in $\sigma \| \tau$ that cover s'.

Suppose first that $\zeta_{A \multimap C}(s') = P$. Suppose that $m = a \in M_A$. We claim that $\mathfrak{s}'|_{A,B} a \in P_{A \multimap B}$. $\mathfrak{s}'a|_{A,C} = s'a \in P_{A \multimap C}$, while $\mathfrak{s}'a|_{B,C} = \mathfrak{s}'|_{B,C} \in \tau \subset P_{B \multimap C}$. Therefore, by Lemma 2.12, $\mathfrak{s}'|_{A,B} a = \mathfrak{s}'a|_{A,B} \in P_{A \multimap B}$. Since $\mathfrak{s}'|_{A,B} \in \sigma$ and $\lambda_{A \multimap B}(a) = \neg\lambda_A(a) = \lambda_{A \multimap B}(a)$, we must have $\mathfrak{s}'|_{A,B} a \in \sigma$, by the definition of a strategy. Therefore, $\mathfrak{s}'a \in \sigma \| \tau$.

Moreover, $\mathfrak{s}'a$ is minimal among sequences that cover $s'a$: any such sequence must cover s' and so must have $\mathfrak{s}'$ as a prefix by minimality of $\mathfrak{s}'$; moreover, the next move after $\mathfrak{s}'$ must be a move in A, since the opponent is not permitted to switch games. $\mathfrak{s}'a$ is clearly minimal among sequences that have these properties and also include the move a.

An identical argument applies in the case that $m = c \in M_C$.

Now suppose that $\zeta_{A \multimap C}(s') = O$. Let $\mathfrak{s}$ be any sequence covering s; after removing B-moves from the end of $\mathfrak{s}$, we may assume that m is the last move in $\mathfrak{s}$. We claim that $\mathfrak{s}$ is minimal among sequences covering s.

Since $\mathfrak{s}$ covers s', we must have $\mathfrak{s}' \sqsubseteq \mathfrak{s}$ by minimality of $\mathfrak{s}'$. Since m is the last move in $\mathfrak{s}$, we have

$$\mathfrak{s} = \mathfrak{s}' b_1 \dots b_k m$$

for some $k \geq 0, b_1, \dots, b_k \in M_B$.

Now let $m' \in M_{A\multimap C}$, and suppose that $s'm \in \tau \circ \sigma$. Let $\mathfrak{t}$ be some sequence covering $s'm$. Again, after removing B-moves from the end of $\mathfrak{t}$, we may assume that m is the last move of $\mathfrak{t}$, and as before we may write $\mathfrak{t}$ as

$$\mathfrak{t} = \mathfrak{s}'\beta_1 \dots \beta_j m'$$

for some $j \geq 0, \beta_1, \dots, \beta_j \in M_B$. We shall show that $j = k$, that $\beta_i = b_i$ for all $i = 1, \dots, j$ and that $m = m'$. This will prove claims (i) (by setting $m' = m$) and (ii) for us.

We have $\zeta_{A\multimap C}(\mathfrak{s}'|_{A,C}) = \zeta_{A\multimap C}(s') = O$, so $\zeta_A(\mathfrak{s}'|_A) = P$ and $\zeta_C(\mathfrak{s}'|_C) = O$. Let us suppose for now that $\zeta_B(\mathfrak{s}'|_B) = P$; the other case is entirely similar. In that case, $\zeta_{A\multimap B}(\mathfrak{s}'|_{A,B}) = P$ and $\zeta_{B\multimap C}(\mathfrak{s}'|_{B,C}) = O$. Since $\mathfrak{s}'|_{B,C}b_1, \mathfrak{s}'|_{B,C}\beta_1 \in \tau$, we must have $b_1 = \beta_1$. Now, by the alternating condition, we must have $\zeta_{A\multimap B}(\mathfrak{s}'b_1|_{A,B}) = O$ and $\zeta_{A\multimap B}(\mathfrak{s}'b_1|_{B,C}) = P$. Since $\mathfrak{s}'b_1|_{A,B}b_2, \mathfrak{s}'b_1|_{A,B}\beta_2 \in \sigma$, we must have $b_2 = \beta_2$. This process continues until we reach the first move outside M_B, which must be m in $\mathfrak{s}$ and m' in $\mathfrak{t}$. Since m, m' are both P-moves, they must both occur as P-replies to the same O-play in one of σ or τ, so $m = m'$. $\square$

Now that we have a working composition of strategies, we are ready to make the collection of games into a category. The identity morphism $A \to A$ is given by the copycat strategy on $A \multimap A$:

$$\mathrm{id}_A = \left\{ s \in P_{A\multimap A} : \text{ for all even-length } t \sqsubseteq s,\, t|_{\perp_A} = t|_A \right\}$$

Proposition 2.14. *Let $\mathcal{G}$ denote the class of games as defined above. If we define a morphism from a game A to a game B to be a strategy for $A \multimap B$, then the identity morphism and notion of composition given above make $\mathcal{G}$ into a category.*

Proof. We need to show that composition is associative and that id_A is the identity morphism on A. For associativity of composition, we claim that if $A \xrightarrow{\sigma} B \xrightarrow{\tau} C \xrightarrow{\upsilon} D$ are morphisms then $(\upsilon \circ \tau) \circ \sigma = S$, where

$$S = \{\mathbf{s}|_{A,D} : \mathbf{s} \in (M_A \sqcup M_B \sqcup M_C \sqcup M_D)^*,\ \mathbf{s}|_{A,B} \in \sigma,\ \mathbf{s}|_{B,C} \in \tau,\ \mathbf{s}|_{C,D} \in \upsilon\}$$

Certainly, if $\mathbf{s}|_{A,D} \in S$, then $\mathbf{s}|_{A,D} \in (\upsilon \circ \tau) \circ \sigma$: we have $\mathbf{s}|_{B,C,D} \in \tau \| \upsilon$, so $\mathbf{s}|_{B,D} \in \upsilon \circ \tau$. Then $\mathbf{s}|_{A,B,D} \in \sigma \| (\upsilon \circ \tau)$, so $\mathbf{s}|_{A,D} \in (\upsilon \circ \tau) \circ \sigma$.

Conversely, suppose $s \in (\upsilon \circ \tau) \circ \sigma$. Then there exists $\mathfrak{s} \in \sigma \| (\upsilon \circ \tau)$ such that $\mathfrak{s}|_{A,D} = s$. Then the definition of $\upsilon \circ \tau$ tells us that there exists $\mathfrak{t} \in \tau \| \upsilon$

such that $\mathfrak{s}|_{A,D} = \mathfrak{t}|_{A,D}$. We may assume that $\mathfrak{s}, \mathfrak{t}$ are the minimal covering sequences constructed in Proposition 2.13.

Note that we have $\mathfrak{s} \in (M_A \sqcup M_B \sqcup M_D)^*$ and $\mathfrak{t} \in (M_B \sqcup M_C \sqcup M_D)^*$. We inductively construct a sequence $\mathbf{s} \in (M_A \sqcup M_B \sqcup M_C \sqcup M_D)^*$ such that for each inductively-constructed subsequence $\mathbf{s}'$, we have $\mathbf{s}'|_{A,B,D} \sqsubseteq \mathfrak{s}$ and $\mathbf{s}'|_{B,C,D} \sqsubseteq \mathfrak{t}$.

If $a \in M_A, b \in M_B, c \in M_C$, then we have $\lambda_{A \multimap B}(a) = \lambda_{B \multimap C}(b) = \lambda_{C \multimap D}(c) = P$. Therefore, $\mathfrak{s}$ and $\mathfrak{t}$ must start with a D-move, and these D-moves must be equal, since $\mathfrak{s}|_{B,D} = \mathfrak{t}|_{B,D}$. This D-move - call it d - is the first move of $\mathbf{s}$, and it is clear that $d|_{A,B,D} \sqsubseteq \mathfrak{s}$ and $d|_{B,C,D} \sqsubseteq \mathfrak{t}$.

Now suppose we have constructed a non-empty sequence $\mathbf{s}'m$ such that $\mathbf{s}'|_{A,B,D} \sqsubseteq \mathfrak{s}$ and $\mathbf{s}'|_{B,C,D} \sqsubseteq \mathfrak{t}$.

If $m = a \in M_A$, then the prefix $\mathbf{s}'|_{A,B,D} \sqsubseteq \mathfrak{s}$ also ends in a. If $\mathbf{s}'|_{A,B,D} = \mathfrak{s}$, then set $\mathbf{s} = \mathbf{s}'$ and finish. Otherwise, extend $\mathbf{s}'$ by appending the next move that occurs in $\mathfrak{s}$ after $\mathbf{s}'|_{A,B,D}$. If $\lambda_{A \multimap B}(a) = P$, then the next move in $\mathfrak{s}$ must also take place in A, by the proof of Proposition 2.13. Therefore, restricting our new sequence to B, C, D gives us $\mathbf{s}'|_{B,C,D}$, which we already know is a prefix of $\mathfrak{t}$. $\square$