Games with ordinal sequence of moves

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Our starting point - the Abramsky-Jagadeesan Games Model

We start off with a few simple categories of games:

- Games with a winning condition:

  $$A = (M_A, \lambda_A, b_A, P_A, W_A)$$

  The winning condition prevents infinite internal chattering.

- Games without a winning condition:

  $$A = (M_A, \lambda_A, b_A, P_A)$$

  We have no winning condition, so we have to use partial strategies or require that our games be bounded.

- We have the tensor product ($\otimes$) and implication ($\multimap$) given by interleaving games and a categorical semantics $\mathcal{G}$ where a morphism from A to B is a strategy for $A \multimap B$.

Construction of an exponential for our game model

N is a negative game ($b_N = O$).

$$!N = (M_N \times \mathbb{N}, \lambda_N \circ \mathrm{pr}_1, O, P_{!N})$$

$!N$ is made up of countably many copies of the game N, indexed by $\mathbb{N}$ and played in parallel, with the opponent switching games.

$!N$ has the structure of the cofree commutative comonoid generated by N:

$$!N \multimap !N \otimes !N$$

The sequoid operator on games

- $N \oslash L$ is a weakening of $N \otimes L$ in which the opponent must make the first move in the game N.
- It induces a monoidal category action of $\mathcal{G}$ on (a modified version of) itself:

$$L \oslash (M \otimes N) \cong (L \oslash M) \oslash N$$

  This is the basis of the definition of a *sequoidal category*.
- Then the exponential $!N$ arises as the final coalgebra for the 'sequoid on the left by N' functor $N \oslash \_$

$$!N \multimap N \oslash !N$$

The final sequence

Let $\mathcal{C}$ be a category and $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor.

$$
\begin{tikzcd}
1 & \mathcal{F}(1) \arrow[l] & \mathcal{F}^2(1) \arrow[l] & \mathcal{F}^3(1) \arrow[l] & \cdots \arrow[l] \\
 & & \cdots & & \\
\mathcal{F}^{\omega}(1) \arrow[uu] \arrow[uur] \arrow[uurr] \arrow[uurrr] & \mathcal{F}^{\omega+1}(1) \arrow[l] & \mathcal{F}^{\omega+2}(1) \arrow[l] & \mathcal{F}^{\omega+3}(1) \arrow[l] & \cdots \arrow[l] \\
 & & \cdots & & \\
\mathcal{F}^{\omega 2}(1) \arrow[uu] \arrow[uur] \arrow[uurr] \arrow[uurrr] & \cdots \arrow[l] & & & \\
\vdots & & & &
\end{tikzcd}
$$

If a final coalgebra for $\mathcal{F}$ exists, then it occurs as $\mathcal{F}^{\alpha}(1)$ for some α and the sequence stabilizes thereafter.

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- If N is an object in the category $\mathcal{W}$ of games with a winning condition, then the final sequence for $N \oslash \_$ stabilizes by $\omega 2$.
- If $\mathcal{C}$ is a sequoidal category, do the final sequences corresponding to 'sequoid on the left' functors always stabilize by $\omega 2$? (No.)
- Can we replace $\omega 2$ by some other ordinal to give us a recipe for constructing the exponential in any sequoidal category?

Idea of transfinite games

- For games without a winning condition, we may have a move at time n for every $n \in \omega$.
- For games with a winning condition, we have things happening at time ω as well.
- We now take things further, and construct a category $\mathcal{G}(\alpha)$ for every ordinal α.
- We will get $\mathcal{G} \cong \mathcal{G}(\omega)$ and $\mathcal{W} \cong \mathcal{G}(\omega + 1)$.
- There is a connection between the ordinal α and the stabilization ordinals for functors of the form

$$N \oslash \_ : \mathcal{G}(\alpha) \to \mathcal{G}(\alpha)$$

A brief outline of the construction of the category of games played over the ordinal α

Fix an ordinal α. We construct a sequoidal category of games of length α by mimicking the Abramsky-Jagadeesan construction:

$$N = (M_N, \lambda_N, \zeta_N, P_N)$$

- M_N is a set of moves.
- $\lambda_N \colon M_N \to \{O, P\}$ is a function saying which player each move belongs to.
- P_N is a prefix closed set of ordinal length sequences (plays) that take values in M_N and are indexed by ordinals $< \alpha$
- $\zeta_N \colon P_N \to \{O, P\}$ assigns a player to each play.

The stabilization ordinal of the category $\mathcal{G}(\alpha)$

- The behaviour of the stabilization ordinal for functors of the form $N \oslash \_$ in the category $\mathcal{G}(\alpha)$ is complicated in general.
- However, the final sequence for such functors never stabilizes before α.
- Therefore, we have constructed sequoidal categories of games with arbitrarily high stabilization ordinals: there is no general recipe for constructing this exponential using only the application of the functor and passing to small limits.