

Games with ordinal sequence of moves

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Preliminaries

Our starting point - the Abramsky-Jagadeesan Games Model

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Games with ordinal sequences of moves

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Stabilization ordinals

Our starting point - the Abramsky-Jagadeesan Games Model

We start off with a few simple categories of games:

- ▶ Games with a winning condition:

$$A = (M_A, \lambda_A, b_A, P_A, W_A)$$

The winning condition prevents infinite internal chattering.

- ▶ Games without a winning condition:

$$A = (M_A, \lambda_A, b_A, P_A)$$

We have no winning condition, so we have to use partial strategies or require that our games be bounded.

- ▶ We have the tensor product (\otimes) and implication (\multimap) given by interleaving games and a categorical semantics \mathcal{G} where a morphism from A to B is a strategy for $A \multimap B$.

Construction of an exponential for our game model

N is a negative game ($b_N = O$).

$$!N = (M_N \times \mathbb{N}, \lambda_N \circ \text{pr}_1, O, P_{!N})$$

$!N$ is made up of countably many copies of the game N , indexed by \mathbb{N} and played in parallel, with the opponent switching games.

$!N$ has the structure of the cofree commutative comonoid generated by N :

$$!N \multimap !N \otimes !N$$

The sequoid operator on games

- ▶ $N \oslash L$ is a weakening of $N \otimes L$ in which the opponent must make the first move in the game N .
- ▶ It induces a monoidal category action of \mathcal{G} on (a modified version of) itself:

$$L \oslash (M \otimes N) \cong (L \oslash M) \oslash N$$

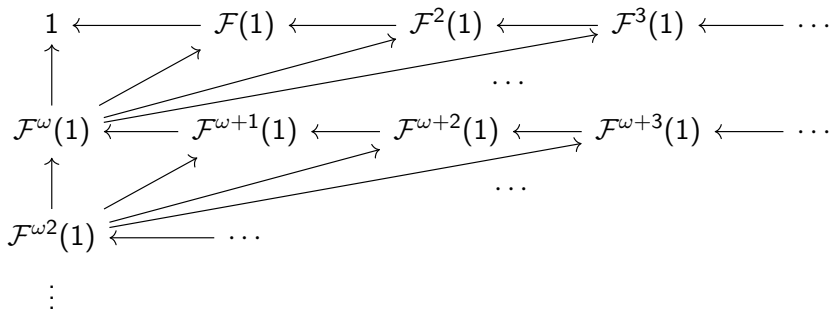
This is the basis of the definition of a *sequoidal category*.

- ▶ Then the exponential $!N$ arises as the final coalgebra for the ‘sequoid on the left by N ’ functor $N \oslash _$

$$!N \multimap N \oslash !N$$

The final sequence

Let \mathcal{C} be a category and $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{C}$ be an endofunctor.



If a final coalgebra for \mathcal{F} exists, then it occurs as $\mathcal{F}^\alpha(1)$ for some α and the sequence stabilizes thereafter.

Stabilization ordinals for sequoidal categories of games

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- ▶ If N is an object in the category \mathcal{W} of games with a winning condition, then the final sequence for $N \otimes _$ stabilizes by ω^2 .
- ▶ If \mathcal{C} is a sequoidal category, do the final sequences corresponding to 'sequoid on the left' functors always stabilize by ω^2 ? (No.)
- ▶ Can we replace ω^2 by some other ordinal to give us a recipe for constructing the exponential in any sequoidal category?

Idea of transfinite games

- ▶ For games without a winning condition, we may have a move at time n for every $n \in \omega$.
- ▶ For games with a winning condition, we have things happening at time ω as well.
- ▶ We now take things further, and construct a category $\mathcal{G}(\alpha)$ for every ordinal α .
- ▶ We will get $\mathcal{G} \cong \mathcal{G}(\omega)$ and $\mathcal{W} \cong \mathcal{G}(\omega + 1)$.
- ▶ There is a connection between the ordinal α and the stabilization ordinals for functors of the form

$$N \otimes _ : \mathcal{G}(\alpha) \rightarrow \mathcal{G}(\alpha)$$

A brief outline of the construction of the category of games played over the ordinal α

Fix an ordinal α . We construct a sequoidal category of games of length α by mimicking the Abramsky-Jagadeesan construction:

$$N = (M_N, \lambda_N, \zeta_N, P_N)$$

- ▶ M_N is a set of moves.
- ▶ $\lambda_N: M_N \rightarrow \{O, P\}$ is a function saying which player each move belongs to.
- ▶ P_N is a prefix closed set of ordinal length sequences (plays) that take values in M_N and are indexed by ordinals $< \alpha$
- ▶ $\zeta_N: P_N \rightarrow \{O, P\}$ assigns a player to each play.

The stabilization ordinal of the category $\mathcal{G}(\alpha)$

- ▶ The behaviour of the stabilization ordinal for functors of the form $N \otimes _$ in the category $\mathcal{G}(\alpha)$ is complicated in general.
- ▶ However, the final sequence for such functors never stabilizes before α .
- ▶ Therefore, we have constructed sequoidal categories of games with arbitrarily high stabilization ordinals: there is no general recipe for constructing this exponential using only the application of the functor and passing to small limits.