Games with ordinal sequences of moves

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1 Introduction

Game Semantics, or the use of games and strategies to model logical systems and programming languages, has its origins in a model of intuitionistic logic by Lorenzen [Lor67] using games. Towards the end of the twentieth century, several authors, starting with Blass [Bla92], realized that certain natural operations on games could be used to model the connectives in Girard's linear logic. Major results from this enterprise include:

- A fully complete categorical semantics for the multiplicative fragment of linear logic, together with the MIX rule, by Abramsky and Jagadeesan [AJ13]
- A fully complete categorical semantics for the multiplicative fragment of linear logic without the MIX rule, by Hyland and Ong [HO93]
- A modification of Abramsky and Jagadeesan's construction, giving a fully complete model of the multiplicative-exponential fragment of linear logic, by Baillot, Danos, Ehrhard and Regnier [BDER97]
- Fully complete models of the negative fragment of linear logic by others, starting with independent work by Curien [Cur92] and Lamarche [Lam92].

Another major early result was the use of game semantics to give a fully abstract model of the programming language PCF (This result was obtained independently, using different techniques, by Abramsky, Jagadeesan and Malacaria [AJM00], by Hyland and Ong [HO00], and by Nickau [Nic94]).

Central to much of the current work on game semantics for linear logic is the treatment of the exponential connective !. Several approaches have been made to model this connective in a game-theoretic way (see Section 3 of [CLM13] for a brief survey). We shall use the construction from [AJM00] and [Hyl97]: if A is a game, then $!A$ is the game formed by playing a countably infinite number of copies of A — $A_0, A_1, A_2, \dots$ — in parallel, where only the opponent O may switch games and where neither player may play in game A_{n+1} until a move has been made in game A_n.

This definition, and in particular the restriction on the order in which games may be started, allows us to give the game $!A$ the structure of the *cofree commutative comonoid* over A [Lai] and gives rise to a *linear exponential comonad* on our category of games. The linear exponential comonad [Sch04] and the cofree commutative comonoid functor [Laf88, MTT09] are both sufficient to model the exponential connective from linear logic.

The restriction that says that the copies of A in $!A$ must be opened in order, motivates the following definition [Lai02]: if A, B are games, then the *sequoid* $A \oslash B$ of A and B is the game in which A and B are played in parallel, only the opponent O may switch games, and play must start in A. It turns out that if A is a game, then the operation $A \oslash \_$ gives rise to an endofunctor on the category $\mathcal{G}$ of games, and that the exponential $!A$ is the final coalgebra for this endofunctor. That is to say, we have a morphism $\alpha\colon !A \to A \oslash !A$ such that whenever $\sigma\colon B \to A \oslash B$ is a morphism, there is a unique morphism $⦅ \sigma ⦆\colon B \to !A$ such that the following diagram commutes:

$$\begin{tikzcd}
B \arrow[r, "\sigma"] \arrow[d, "⦅ \sigma ⦆"'] & A \oslash B \arrow[d, "A \oslash ⦅ \sigma ⦆"] \\
!A \arrow[r, "\alpha"'] & A \oslash !A
\end{tikzcd}$$

Existing proofs of this fact in a finitary games setting (e.g., in [CLM13]) make implicit use of the fact that $!A$ is the limit of the sequence

$$I \leftarrow A \leftarrow A \oslash A \leftarrow A \oslash (A \oslash A) \leftarrow \dots$$

Indeed, any finite-length play in $!A$ must involve only finitely many copies of A, so exists as a play in $(A \oslash \_)^n I$ for some n.

On the other hand, when we start imposing winning conditions upon infinite plays, $!A$ is no longer the limit of the sequence above. Indeed, define a game $¡A$ which is exactly like $!A$ except that all infinite plays that contain moves from infinitely many copies of A are wins for the player P, regardless of who has won the individual games. This game still admits a cone over the sequence above, since any play occurring in $(A \oslash \_)^n I$ for some n must involve only finitely many copies of A. Moreover, it does not admit a morphism of cones to $!A$. In fact, in the infinitary setting, $¡A$, and not $!A$, is the limit of the above sequence.

$$\begin{tikzcd}
\text{¡}A \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & \cdots & & \\
I & A \arrow[l] & A \oslash A \arrow[l] & A \oslash (A \oslash A) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Applying the functor $A \oslash \_$ to the cone from $\text{¡}A$ gives rise to another cone over the sequence:

$$\begin{tikzcd}
 & A \oslash \text{¡}A \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \cdots & \\
I & A \arrow[l] & A \oslash A \arrow[l] & A \oslash (A \oslash A) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

and therefore a morphism $A \oslash \text{¡}A \to \text{¡}A$. We can then continue to apply the functor $A \oslash \_$ to build up a second layer of the sequence:

$$\begin{tikzcd}
\text{¡}A \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & A \oslash \text{¡}A \arrow[l] & A \oslash (A \oslash \text{¡}A) \arrow[l] & A \oslash (A \oslash (A \oslash \text{¡}A)) \arrow[l] & \cdots \arrow[l] \\
 & & \cdots & & \\
I & A \arrow[l] & A \oslash A \arrow[l] & A \oslash (A \oslash A) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

What is the limit of this entire diagram? An O-winning infinite play s in $(A \oslash \_)^n \text{¡}A$ may be identified with an O-winning play in $!A$ such that either the opponent O wins in one of the games $A_1, \ldots, A_n$ or that only finitely many copies of A are opened. It follows that the limit of the diagram *is* the final coalgebra $!A$.

The diagrams we have drawn above are special cases of the *final sequence* [Wor05]: given a category $\mathcal{C}$ with a terminal object 1 and enough limits (in a sense which will become clear) and an endofunctor $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$, we may construct an ordinal-indexed sequence

$$1, \mathcal{F}(1), \mathcal{F}^2(1), \ldots \mathcal{F}^\omega(1), \mathcal{F}^{\omega+1}(1), \ldots, \mathcal{F}^\delta(1), \ldots$$

together with commuting morphisms $\mathcal{F}^\gamma(1) \to \mathcal{F}^\delta(1)$ for $\delta \leq \gamma$, by setting $\mathcal{F}^{\delta+1}(1) = \mathcal{F}(\mathcal{F}^\delta(1))$ and letting $\mathcal{F}^\mu(1)$ be the limit of the diagram formed by the $\mathcal{F}^\gamma(1)$ for $\gamma < \mu$ if μ is a limit-ordinal. It is well known that if δ is an ordinal and the map

$$\mathcal{F}^{\delta+1}(1) \to \mathcal{F}^\delta(1)$$

is an isomorphism, then the inverse of this isomorphism exhibits $\mathcal{F}^\delta(1)$ as the final coalgebra of $\mathcal{F}$. In this case, we say that the final sequence *stabilizes at δ*.

In the case of finitary games, then, we have

$$!A \cong (A \oslash \_\,)^{\omega}(I)$$

and in the infinitary case, we have

$$!A \cong (A \oslash \_\,)^{\omega 2}(I)$$

The eventual goal of this research is to study the behaviour of the final sequence of the functor $A \oslash \_$ in categories of games. Since the sequence stablizes so early on for existing categories of games, this question is not very interesting in that setting. We want to modify our definitions of games in such a way that this final sequence will take much longer to stabilize, and it turns out that one way to do that is to allow plays of transfinite length.

Games with ordinal sequences of moves are not a new idea. They have, for example, been studied in [Ne04] from the point of view of determinacy. Even in game semantics, as we shall see, we may identify games in which we impose a winning condition on the infinite plays with games in which the players are allowed to make plays of length ω – we shall call these $(\omega+1)$-games, in contrast to the finitary games, which we shall call ω-games. Allowing plays whose lengths are longer transfinite ordinals will give rise to a richer variety of plays, which will be reflected in the final sequence.

One important aspect of the shift from finite to transfinite plays is that the idea of a *play* or *position* takes on a new importance, in contrast to the finitary case, where we can define everything in terms of *moves*. Normally, a move is designated as either a P-move or an O-move; we will extend this so that every position is either a P-position or an O-position. A position s will be an ordinal-indexed sequence of moves; if there is a last move a, then s is a P-position or an O-position according as a is a P-move or an O-move. If the length of s is a limit ordinal, however, then its sign cannot be determined from the signs of its constituent moves. This is useful if we want to model games with a winning condition as $(\omega+1)$-games: a play of length ω in a game corresponds to an infinite play in that game, and its designation as a P-position or O-position is essentially the same thing as saying whether it is a P-winning or O-winning play.

This approach is similar to that of Blass [Bla92], and indeed our treatment of P-positions and O-positions under the connectives will follow his approach. For example, a position s in a tensor product $A \otimes B$ is a P-position if and only if its restrictions to A and B are both P-positions.

To avoid certain mathematical difficulties, we shall follow Curien, Lamarche and others by requiring that our games be *negative* – that is, they must always begin with an O-move. In fact, we shall go further than that: under the usual alternating condition, saying that O always starts is the same thing as requiring that the empty position ϵ is a P-position. In order to avoid problems constructing our category, we shall need to require that *all* positions whose lengths are limit ordinals be P-positions. We shall call games satisfying this property *completely negative*.

This condition on the sign of games allows us to construct a well-behaved category of games, but it has one serious problem that occurs as soon as we consider games with transfinite-length plays: if A, B are completely negative games, then the linear implication $A \multimap B$ is not necessarily completely negative.

We want $A \multimap B$ to be the internal hom in our closed monoidal category, so we need to find a way around this. Our solution is to construct, for every game A, a 'minimal extension' A^{cn} of A that is completely negative. We prove various results about the relationship A^{cn} and A that make this precise and then use them to show that the game $A \overset{\mathrm{cn}}{\multimap} B := (A \multimap B)^{\mathrm{cn}}$ does give rise to a closed monoidal structure in our category. We also give a tentative way of constructing a category whose objects are *all* games, not just the completely negative ones, and use our results to show that this category is in fact equivalent to our existing category of completely negative games.

Having constructed a symmetric monoidal closed category of games, we introduce the sequoid operator and the exponential, and study the final sequence. This section will involve some material that is more ordinal-theoretic than game-theoretic TODO: write up this section and then write the introduction

1.1 Conventions

I shall denote the composition of morphisms $A \xrightarrow{\sigma} B \xrightarrow{\tau} C$ by $\tau \circ \sigma$, the disjoint union of two sets X, Y by $X \sqcup Y$ and the corresponding copairing of two functions $X \xrightarrow{f} Z, Y \xrightarrow{g} Z$ by $f \sqcup g \colon X \sqcup Y \to Z$.

Following the $\forall$belard/$\exists$loise convention that is sometimes used for Opponent/Player, I shall refer to the opponent O as 'he' and the player P as 'she' throughout.

2 Our starting category of games

Before studying games with transfinite sequences of moves, we shall illustrate some of the choices we have made by defining a category of games with finite sequences of moves. We have chosen these definitions because they extend particularly well to the transfinite case.

2.1 Games and strategies

We shall use the notation introduced in [AJ13] to describe games. All our games A will have, at their heart, the following three pieces of information:

- A set M_A of possible moves
- A function $\lambda_A \colon M_A \to \{O, P\}$ assigning to each move the player who is allowed to make that move
- A prefix-closed set $P_A \subseteq M_A^*$ of finite sequences of moves.

We shall normally insist on an *alternating condition* on P_A:

Alternating condition If $a, b \in M_A$ are moves and $s \in M_A^*$ is a sequence of moves such that $sa, sab \in P_A$, then $\lambda_A(a) = \neg\lambda_A(b)$.

As in [AJ13], we identify a *strategy* for a game A with the set of sequences of moves that can occur when player P is playing according to that strategy so that a typical definition of a (partial) strategy might be a set $\sigma \subseteq P_A$ such that (for all $s \in M_A^*, a, b \in M_A$):

- $\epsilon \in \sigma$ (ensures that σ is non-empty)
- If $sa \in \sigma$, $\lambda_A(a) = P$ and $sab \in P_A$ then $sab \in \sigma$ (σ contains all legal replies by player O)
- If $s, sa, sb \in \sigma$ and $\lambda_A(a) = P$ then $a = b$ (σ contains at most one legal reply by player P)

We can impose additional constraints on σ that will ensure that σ is total, strict, history free and so on. The definition given immediately above is not the only definition of a strategy found in the literature, however. For example, the games described in [AJ13] have the curious property that the set P_A may contain plays that cannot actually occur when A is being played; in particular, all plays must start with a move by player O, but the set P_A

may contain positions that start with a P-move. These plays do not affect the strategies for A, but they might come into play if we perform operations on A such as forming the negation $\neg A$ or the implication $A \multimap B$.

This behaviour is made implicit in Abramsky and Jagadeesan's definitions, which do not impose any conditions upon the set P_A beyond the basic alternation condition given above, but which mandate that any play occurring *in a strategy* must begin with an O-move. For the sake of clarity, we adopt a different, but completely equivalent, approach. For a game A, we define a set L_A, regarded as the set of *legal plays* occurring in P_A. In some games models, such as that found in [Bla92], L_A may be defined to be the whole of P_A, while in [AJ13] it is defined to be that subset of P_A consisting of plays that begin with an O-move.

The point of specifying L_A separately is that it allows us to unify the definition of a *strategy*, while making clearer the behaviour observed above, whereby certain plays in P_A may not occur 'in normal play'; this behaviour was previously only implicit in the definition of a strategy. Our unified definition then becomes:

Definition 2.1. If $A = (M_A, \lambda_A, P_A)$ is a game and L_A is its associated set of legal plays (in a particular games model) then a (partial) *strategy* for A is a subset $\sigma \subseteq L_A$ such that for all $s \in L_A$ and all $a, b \in M_A$:

- $\epsilon \in \sigma$
- If $s \in \sigma$ and a is an O-move, and if $sa \in L_A$, then $sa \in \sigma$
- If $s \in \sigma$ and a, b are P-moves, and if $sa, sb \in \sigma$, then $a = b$

2.2 Positive and negative games, ownership of plays and connectives

Abramsky-Jagadeesan games, as described in [AJ13], may admit both plays that start with a P-move and plays that start with an O-move. Other games models, such as those found in [Bla92] and [Cur92], are more restrictive. The games in [Cur92] only contain plays starting with an O-move. The plays in [Bla92] may start with either a P-move or an O-move, but a play starting with a P-move and a play starting with an O-move may not occur in the same game.

Definition 2.2. We say that a game $A = (M_A, \lambda_A, P_A)$ is *positive* if every play in P_A begins with a P-move. We say that A is *negative* if every play in P_A begins with an O-move.

So the Curien model found in [Cur92] admits only negative games, the Blass model in [Bla92] admits positive and negative games, while the Abramsky-Jagadeesan model found in [AJ13] admits not only positive and negative games, but also games that are neither negative nor positive. We shall now examine the reasons for and drawbacks of each of these choices.

The earliest games model, found in [Con76], did not include a definition of which player is to move at a given position; rather, games are defined recursively as pairs of games $\{L|R\}$, where L represents the positions that the left player may move into, while R represents the positions that the right player may move into. Blass's definition departs completely from this tradition; now, at every position s only one of the two players is allowed to move; extending this logic on to the empty position ϵ, it follows that all games are either positive or negative. This property means that we may freely define $L_A = P_A$, since there is never any question about whose turn it is to play. By contrast, if we were to define $L_A = P_A$ for Abramsky-Jagadeesan games, then a strategy might end up containing two branches, one of plays beginning with an O-move and one of plays beginning with a P-move, which is undesirable. The alternative definition of L_A avoids this problem.

In the case of a Blass game A, we may define a function $\zeta_A \colon P_A \to \{O, P\}$ that says which player owns each play; the idea is that if we are in position s, then the next player to move is given by $\neg\zeta_A(s)$; i.e., the opposing player to the player who has just made the move. One might want to define ζ_A by setting $\zeta_A(sa) = \lambda_A(a)$, so that ownership of a play is decided by who has made the last move in the play, but this definition does not extend in an obvious way to the empty position ϵ (and, as we shall see in the next chapter, it does not extend to plays over limit ordinals). In this case, $\zeta_A(\epsilon)$ is part of the game's data, and it determines whether the game is positive or negative: if $\zeta_A(\epsilon) = P$ then all plays must start with an O-move, and the game is negative – and vice versa.

An important question then arises: how should we extend the function ζ_A to games formed from connectives? The solution adopted by Blass is to use binary conjunctions to deduce the ownership of a play from the ownership of the restrictions of that play to the two component games. In the case of the tensor product $A \otimes B$ of two games A and B, we define $\zeta_{A\otimes B} \colon P_{A\otimes B} \to$

$\{O, P\}$ by setting

$$\zeta_{A \otimes B}(s) = (\zeta_A(s|_A) \wedge \zeta_B(s|_B))$$

where $\wedge \colon \{O, P\} \times \{O, P\} \to \{O, P\}$ is as in Figure 1.

a	b	$a \wedge b$
O	O	O
O	P	O
P	O	O
P	P	P

a	b	$a \vee b$
O	O	O
O	P	P
P	O	P
P	P	P

a	b	$a \Rightarrow b$
O	O	P
O	P	P
P	O	O
P	P	P

Figure 1: Truth tables for binary conjunctions on $\{O, P\}$

Similarly, we may extend ζ to the implication $A \multimap B$ and the par $A \parr B$ by setting

$$\zeta_{A \multimap B}(s) = (s|_A \Rightarrow s|_B)$$
$$\zeta_{A \parr B}(s) = (s|_A \vee s|_B)$$

Note that if we use these definitions then the owner $\zeta_C(sa)$ of a play sa might not correspond to the player $\lambda_C(a)$ who played the last move a. For example, let A, B be two positive games and form their tensor product $A \otimes B$. Then we have

$$\zeta_{A \otimes B}(\epsilon) = (\zeta_A(\epsilon) \wedge \zeta_B(\epsilon)) = O \wedge O = O$$

and so $A \otimes B$ is a positive game. Player P plays an opening move in one of the two games - let us say she plays the move a in the game A. But then we have

$$\zeta_{A \otimes B}(a) = (\zeta_A(a) \wedge \zeta_B(\epsilon)) = P \wedge O = O$$

In other words, it is still player P's turn to play! Blass embrace this possibility and allows player P to make these two moves. In his paper, he introduces the notions of *strict* and *relaxed* games, where the strict games are the objects of study but the relaxed games are often used since they allow more manipulations. In this case, the game $A \otimes B$ is defined as a relaxed game that might not satisfy the alternating condition; in the process of converting it into a strict game, these two opening moves by player P are combined into a single move.

This 'double move' can only occur at the start of the game, and Blass treats it as a special case in his proofs. Perhaps unsurprisingly, this inconsistency causes major problems if we try to compose strategies. We do not get an associative composition of strategies for $A \multimap B$ with strategies for $B \multimap C$ and so we do not get a categorical semantics. An example of the failure of associativity in Blass's games model is given towards the end of [AJ13].

By contrast, Abramsky-Jagadeesan games may admit moves by both players at the same position (specifically, at the beginning of the game, before any moves have been played), but this does not cause problems since we insist that our legal plays start with an O-move and be strictly alternating. The authors of [AJ13] note that their model can be considered as an intermediate between Conway's games, where the position tells you nothing about which player is to move, and Blass games, where the position completely determines which player is to move. In Abramsky-Jagadeesan games, one can deduce which player is to move (by looking at which player made the last move) in every position except the empty starting position.

In the Abramsky-Jagadeesan model, a positive game is an immediate win to player P, since player O has no legal move to start the game off. As we noted before, this does not mean that the content of a positive game is meaningless, since we can use connectives to 'unlock' these illegal plays. For example, if Q is a positive game and N is a negative game then $Q \parr N$ is a negative game, and the possible positions in Q are now all achievable.

Curien's game model ([Cur92]) is similar to Abramsky's and Jagadeesan's, but involves only negative games. The only slight problem is that negative Abramsky-Jagadeesan games are not closed under implication: if N, L are negative games then $N \multimap L$ may be neither negative nor positive. We may fix this by modifying the definition of $N \multimap L$ so that we delete from $P_{N \multimap L}$ all plays that start with a P-move - or, equivalently, by requiring that all plays start in L. This is the approach taken in [CLM13], where it fits well with the paper's treatment of the *sequoid* operator $\oslash$, which is a version of the tensor product that has been modified so that play is required to start in the left-hand game.

We shall adopt elements of both the Blass and the Abramsky-Jagadeesan games models; specifically, we shall use Blass's games and Abramsky-Jagadeesan's strategies. This means that our games model will be more restrictive than either the Blass or the Abramsky-Jagadeesan models, but this lack of flexibility will be just what we need in order to extend these games over the transfinite ordinals. We will later consider ways we can relax our model to recover Abramsky and Jagadeesan's games model.

2.3 Our definition of games and strategies

Definition 2.3. A *game* is a triple $(M_A, \lambda_A, \zeta_A, P_A)$ where

- M_A is a set of moves,
- $\lambda_A \colon M_A \to \{O, P\}$ is a function that assigns a player to each move,
- $P_A \subseteq M_A^*$ is a non-empty prefix-closed set of plays that can occur in the game and
- $\zeta_A \colon P_A \to \{O, P\}$ is a function that assigns a player to each position

such that

- If $a \in M_A$ and $sa \in P_A$ then $\zeta_A(sa) = \lambda_A(a)$.
- If $a \in M_A$ and $sa \in P_A$ then $\zeta_A(s) = \neg\zeta_A(sa)$.

Remark 2.4 (Notes on the definition)**.** Given a game $A = (M_A, \lambda_A, \zeta_A, P_A)$, define $b_A = \neg\zeta_A(\epsilon)$. Then every play in P_A must start with a b_A-move, so A is either positive or negative.

Note that ζ_A is now completely specified by λ_A and b_A, so we could have specified our games more efficiently by replacing ζ_A with b_A in our definition, as done in [CLM13]. The slightly more unwieldy ζ_A will be useful when we come to extend our games over the ordinals, though, so we retain it.

If $a \in M_A$ then we may recover $\lambda_A(a)$ from ζ_A so long as a occurs in some play in P_A. Since moves that can never be played do not affect the game at all, we do not really need λ_A in our definition, but we keep it to make the connection to earlier work clearer.

If $a \in M_A$ and $\lambda_A(a) = O$, we call a an *O-move.* If $\lambda_A(a) = P$, we call a a *P-move.* If $s \in P_A$ and $\zeta_A(s) = O$, we call s an *O-play* or *O-position.* If $\zeta_A(s) = P$, we call s a *P-play* or *P-position.*

We give the usual definition of a strategy as an appropriate subset of P_A, where we have identified the strategy with the set of all plays that can arise when player P plays according to that strategy:

Definition 2.5. Let $A = (M_A, \lambda_A, \zeta_A, P_A)$ be a game. A *strategy* for A is a non-empty prefix-closed subset $\sigma \subseteq P_A$ such that:

- If $a \in P_A$ is an O-move and $s \in \sigma$ is a P-position such that $sa \in P_A$, then $sa \in \sigma$.

- If $s \in \sigma$ is an O-position and $a, b \in M_A$ are P-moves such that $sa, sb \in \sigma$, then $a = b$.

The definition above will be the most technically useful, but it is also convenient to give a strategy by a partial function that tells player P which move to make in each O-position. If we write $P_A^- = \zeta_A^{-1}(\{O\})$ for the set of all O-positions and $M_A^+ = \lambda_A^{-1}(\{P\})$ for the set of all P-moves, then the strategy σ above gives rise to a partial function $\hat{\sigma} \colon P_A^- \rightharpoonup M_A^+$ given by:

$$\hat{\sigma}(s) = \begin{cases} a & \text{if } a \in M_A^+ \text{ and } sa \in P_A \\ \text{undefined} & \text{if } sb \notin P_A \text{ for all } b \in M_A^+ \end{cases}$$

The second condition on strategies tells us that this is well defined. Going in the other direction, if we are given a partial function $f \colon P_A^- \rightharpoonup M_A^+$ then we can define a strategy $\overline{f}$ by

$$\overline{f} = \{s \in M_A^* \ : \text{ for all } ta \sqsubseteq s \text{ with } \zeta_A(t) = O,\ f(t) \text{ is defined and equal to } a\}$$

2.4 Connectives

Our definitions of connectives on games are as in [Bla92].

Definition 2.6. Let $A = (M_A, \lambda_A, \zeta_A, P_A)$ be a game. The negation of A, ${}^{\perp}A$, is the game formed by interchanging the roles of the two players.

- $M_{({}^{\perp}A)} = M_A$
- $\lambda_{({}^{\perp}A)} = \neg \circ \lambda_A$
- $\zeta_{({}^{\perp}A)} = \neg \circ \zeta_A$
- $P_{({}^{\perp}A)} = P_A$

It follows immediately from the definitions that ${}^{\perp}A$ is a well formed game.

We now define the tensor product $A \otimes B$, the par $A \parr B$ and the linear implication $A \multimap B$ of two games. All these games are obtained by playing A and B in parallel, so they all have the same set of moves:

$$M_{A \otimes B} = M_{A \parr B} = M_{A \multimap B} = M_A \sqcup M_B$$

Ownership of moves is decided via the obvious co-pairing functions:

$$\lambda_{A \otimes B} = \lambda_{A \parr B} = \lambda_A \sqcup \lambda_B$$
$$\lambda_{A \multimap B} = (\neg \circ \lambda_A) \sqcup \lambda_B$$

We define the set $P_A \| P_B$ to be the set of all plays in $(M_A \sqcup M_B)^*$ whose M_A-component is a play from P_A and whose M_B-component is a play from P_B:

$$P_A \| P_B = \{s \in (M_A \sqcup M_B)^* \ : \ s|_A \in P_A,\ s|_B \in P_B\}$$

We are now in a position to define $\zeta_{A \otimes B}, \zeta_{A ⅋ B}, \zeta_{A \multimap B}$ as functions $P_A \| P_B \to \{O, P\}$:

$$\begin{aligned}
\zeta_{A \otimes B}(s) &= \zeta_A(s|_A) \wedge \zeta_B(s|_B) \\
\zeta_{A ⅋ B}(s) &= \zeta_A(s|_A) \vee \zeta_B(s|_B) \\
\zeta_{A \multimap B}(s) &= \zeta_A(s|_A) \Rightarrow \zeta_B(s|_B)
\end{aligned}$$

(Here, $\wedge$, $\vee$ and $\Rightarrow$ are as in Figure 1.)

As things stand, $P_A \| P_B$ is not a valid set of plays for our λ and ζ functions. The first problem is that $P_A \| P_B$ contains plays which are not alternating with respect to $\zeta_{A \otimes B}, \zeta_{A ⅋ B}, \zeta_{A \multimap B}$. The second problem is that the ζ and λ functions do not always agree with one another. For example, suppose that Q, R are two positive games. So we have $\zeta_Q(\epsilon) = \zeta_R(\epsilon) = O$. Then $\zeta_{Q \otimes R}(\epsilon) = O \wedge O = O$. But now suppose that player P can make an opening move q in Q. Then we have

$$\zeta_{Q \otimes R}(q) = \zeta_Q(q) \wedge \zeta_R(\epsilon) = P \wedge O = O$$

but $\lambda_{Q \otimes R}(q) = P$.

Our solution is to throw away some plays from $P_A \| P_B$ so that what remains satisfies the alternating condition and the condition on the λ and ζ functions.

Definition 2.7. Let M be a set, let $P \subseteq M^*$ be prefix closed and let $\lambda \colon M \to \{O, P\}, \zeta \colon P \to \{O, P\}$ be functions. If $s \in P$, we say that s is *alternating with respect to* ζ if the set of prefixes of s satisfies the alternating condition: if $t, ta \sqsubseteq s$, then $\zeta(t) = \neg\zeta(ta)$.

We say that s is *well formed with respect to* λ, ζ if whenever $ta \sqsubseteq s$, for some $a \in M$, we have $\zeta(ta) = \lambda(a)$.

Now let A, B be games. We define:

$$P_{A\otimes B} = \left\{ s \in P_A \| P_B \,\middle|\, \begin{array}{l} s \text{ is alternating with respect to } \zeta_{A\otimes B} \\ s \text{ is well formed with respect to } \zeta_{A\otimes B}, \lambda_{A\otimes B} \end{array} \right\}$$
$$P_{A\parr B} = \left\{ s \in P_A \| P_B \,\middle|\, \begin{array}{l} s \text{ is alternating with respect to } \zeta_{A\parr B} \\ s \text{ is well formed with respect to } \zeta_{A\parr B}, \lambda_{A\parr B} \end{array} \right\}$$
$$P_{A\multimap B} = \left\{ s \in P_A \| P_B \,\middle|\, \begin{array}{l} s \text{ is alternating with respect to } \zeta_{A\multimap B} \\ s \text{ is well formed with respect to } \zeta_{A\multimap B}, \lambda_{A\multimap B} \end{array} \right\}$$

Definition 2.8. We define:

$$\begin{aligned} A \otimes B &= (M_{A\otimes B}, \lambda_{A\otimes B}, \zeta_{A\otimes B}, P_{A\otimes B}) \\ A \parr B &= (M_{A\parr B}, \lambda_{A\parr B}, \zeta_{A\parr B}, P_{A\parr B}) \\ A \multimap B &= (M_{A\multimap B}, \lambda_{A\multimap B}, \zeta_{A\multimap B}, P_{A\multimap B}) \end{aligned}$$

Proposition 2.9. *$A \otimes B$,$A \parr B$,$A \multimap B$ are well formed games. Moreover, $P_{A\otimes B}$ is the largest subset $X \subseteq P_A \| P_B$ such that $(M_{A\otimes B}, \lambda_{A\otimes B}, \zeta_{A\otimes B}, X)$ is a well formed game and similarly for the connectives $\parr$ and $\multimap$.*

Proof. We will prove the proposition for $A \otimes B$; the other two cases are entirely similar. Alternatively, observe that $A \parr B = {}^{\perp}({}^{\perp}A \otimes {}^{\perp}B)$ and $A \multimap B = {}^{\perp}A \parr B$.

For $A \otimes B$, it suffices to show that $P_{A\otimes B}$ is alternating with respect to $\zeta_{A\otimes B}$, since every $s \in P_{A\otimes B}$ is well-formed by definition. Suppose $s, sa \in P_{A\otimes B}$; then $s, sa \sqsubseteq sa$; since sa is alternating with respect to $\zeta_{A\otimes B}$, it follows that $\zeta_{A\otimes B}(s) = \neg\zeta_{A\otimes B}(sa)$.

For the second part of the proposition, suppose that $V \subseteq P_A \| P_B$ is prefix-closed and satisfies the alternating condition with respect to $\zeta_{A\otimes B}$ and that every $s \in V$ is well-formed with respect to $\lambda_{A\otimes B}, \zeta_{A\otimes B}$. We need to show that $V \subseteq P_{A\otimes B}$, for which it will suffice to show that every $s \in V$ is alternating. This is easy to see: since V is prefix closed, the set of all prefixes of s is a subset of V, and so it satisfies the alternating condition with respect to $\zeta_{A\otimes B}$. □

A design feature of the connectives $\otimes, \parr, \multimap$ is that only player O may switch games in $A \otimes B$, while only player P may switch games in $A \parr B$ and $A \multimap B$. The $\multimap$ case follows immediately from the $\parr$ case by noting that $A \multimap B = {}^{\perp}A \parr B$, and the $\parr$ case then follows from the $\otimes$ case by observing that $A \parr B = {}^{\perp}({}^{\perp}A \otimes {}^{\perp}B)$. Thus, it will suffice to prove the following proposition for the tensor product:

Proposition 2.10. *Let A, B be games. Suppose $s \in P_{A \multimap B}$, $a \in M_A$ and $b \in M_B$. Then:*

i) If $sab \in P_{A \otimes B}$ then $\lambda_{A \otimes B}(b) = O$

ii) If $sba \in P_{A \otimes B}$ then $\lambda_{\otimes B}(a) = O$.

Proof. (i):
$$\begin{aligned}\lambda_{A\otimes B}(b) &= \zeta_{A\otimes B}(sab)\\ &= \zeta_{A\otimes B}(s|_A a) \wedge \zeta_{A\otimes B}(s|_B b)\\ &= \lambda_{A\otimes B}(a) \wedge \lambda_{A\otimes B}(b)\end{aligned}$$

By alternation, either $\lambda_{A\otimes B}(a) = O$ or $\lambda_{A\otimes B}(b) = O$, so this last expression must be equal to O.

(ii):
$$\begin{aligned}\lambda_{A\otimes B}(a) &= \zeta_{A\otimes B}(sba)\\ &= \zeta_{A\otimes B}(s|_A a) \wedge \zeta_{A\otimes B}(s|_B b)\\ &= \lambda_{A\otimes B}(a) \wedge \lambda_{A\otimes B}(b) = O \text{ (by the same argument)}\end{aligned}$$ □

2.5 A category of games and partial strategies

Following [Joy77] and [AJ13], we define a category $\mathcal{G}$ whose objects are games where the morphisms from a game A to a game B are strategies for $A \multimap B$. For the sake of simplicity, and to avoid various technical issues, we shall require that the games in our category be *negative* - namely, they should start with an opponent move. In our language, we call a game A *negative* if $\zeta_A(\epsilon) = P$ (and we call it positive if $\zeta_A(\epsilon) = O$). The equations

$$P \wedge P = P$$
$$P \Rightarrow P = P$$

tell us that the class of negative games is closed under $\otimes$ and $\multimap$ (since $P \vee P = P$, it is also closed under $⅋$, but the par of two negative games has no legal moves in our presentation, so it does not give a good model of the $⅋$ connective in linear logic and we will not consider it).

In order to get a category, we need a way to compose a strategy for $A \multimap B$ with a strategy for $B \multimap C$. Our treatment of composition is heavily influenced by work done by Hyland and Schalk (see [Hyl97], [HS02] and the excellent set of notes [Sch01]).

We shall use the usual definition of composition. If σ is a strategy for $A \multimap B$ and τ is a strategy for $B \multimap C$, we define a set

$$\sigma \| \tau = \{\mathfrak{s} \in (M_A \sqcup M_B \sqcup M_C)^* \, : \, \mathfrak{s}|_{A,B} \in \sigma, \; \mathfrak{s}|_{B,C} \in \tau\}$$

Then we define the composite strategy on $A \multimap C$ to be

$$\tau \circ \sigma = \{\mathfrak{s}|_{A,C} \, : \, \mathfrak{s} \in \sigma \| \tau\}$$

The process of showing that this is indeed a strategy for $A \multimap C$, and that composition is associative, will be quite involved. We shall start by recording some nice properties of negative games that will tell us that $\tau \circ \sigma$ is alternating:

Proposition 2.11. *Let A, B be negative games and suppose that $s \in P_A \| P_B$ is alternating with respect to $\zeta_{A \otimes B}$. Then:*

i) Either $\zeta_A(s|_A) = P$ or $\zeta_B(s|_B) = P$.

ii) s is well formed with respect to $\zeta_{A \otimes B}, \lambda_{A \otimes B}$.

Furthermore, (i) and (ii) hold if we replace $\otimes$ with $\multimap$ throughout and if we replace (i) by the statement that either $\zeta_A(s|_A) = P$ or $\zeta_B(s|_B) = O$.

We gave an example earlier in which a play sa in $P_A \| P_B$ was not well formed with respect to $\zeta_{A \otimes B}, \lambda_{A \otimes B}$. The crucial thing that made that example work was that we had $\zeta_A(s|_A) = \zeta_B(s|_B) = O$. This then meant that $\zeta_{A \otimes B}(sa) = O$ even when $\lambda_{A \otimes B} = P$. Part (i) of the proposition above tells us that this situation never occurs, and part (ii) tells us that this is enough to ensure that we never get any alternating sequences that are not well formed.

Proof of Proposition 2.11. We proceed by induction on the length n of s. If $n = 0$ then $s = \epsilon$ and so $\zeta_A(s|_A) = \zeta_A(\epsilon) = P$ and similarly for B, since A, B are negative games. Now suppose that $n > 0$ - so $s = tm$ for some $t \in P_{A \otimes B}, m \in M_{A \otimes B}$. By induction, either $t|_A = P$ or $t|_B = P$, so there are three cases:

Case 1: $\zeta_A(t|_A) = P$ and $\zeta_B(t|_B) = P$. Then either $tm|_A = t|_A$ (if m is a B-move) or $tm|_B = t|_B$ (if m is an A-move). This proves part (i). For part (ii), note that in this case we have $\zeta_{A \otimes B}(t) = P$, so $\zeta_{A \otimes B}(tm) = O$, since tm is alternating. If m is an A-move, it means that $\zeta_B(tm|_B) = \zeta_B(t|_B) = P$, so we must have $\zeta_A(tm|_A) = O$ by the definition of $\zeta_{A\otimes B}$. Then

$$\lambda_{A\otimes B}(m) = \lambda_A(m) = \zeta_A(t|_A m) = O = \zeta_{A\otimes B}(tm)$$

If m is a B-move, the argument is similar.

Case 2: $\zeta_A(t|_A) = P$ and $\zeta_B(t|_B) = O$. For part (i), it will suffice to show that m is a B-move - so $tm|_A = t|_A$. Indeed, we have $\zeta_{A\otimes B}(t) = O$, so $\zeta_{A\otimes B}(tm) = P$, since tm is alternating with respect to $\zeta_{A\otimes B}$. In particular, $\zeta_B(tm|_B) = P$, so we must have $tm|_B \neq t|_B$ and therefore m is a B-move. For part (ii), we have:

$$\lambda_{A\otimes B}(m) = \lambda_B(m) = \zeta_B(t|_B m) = \neg\zeta_B(t|_B) = \neg O = P$$
$$\zeta_{A\otimes B}(tm) = \neg\zeta_{A\otimes B}(t) = t = \neg O = P = \lambda_{A\otimes B}$$

Case 3: $\zeta_A(t|_A) = O$ and $\zeta_B(t|_B) = P$. Similar to Case 2.

Finally, note that the $\otimes$ result implies the $\multimap$ result after writing $A \multimap B = {}^{\perp}A \parr B = {}^{\perp}(A \otimes {}^{\perp}B)$. □

A corollary of our result is that we have:

$$P_{A\otimes B} = \{s \in P_A \| P_B : \ s \text{ is alternating with respect to } \zeta_{A\otimes B}\}$$
$$P_{A\multimap B} = \{s \in P_A \| P_B : \ s \text{ is alternating with respect to } \zeta_{A\multimap B}\}$$

if A, B are negative games. In other words, we can ignore the well-formedness condition on plays if we are dealing with only negative games.

This proposition allows us to prove the following useful fact:

Lemma 2.12. *Let A, B, C be negative games, and let $\mathfrak{s} \in (M_A \sqcup M_B \sqcup M_C)^*$. If any two of the following statements are true, then so is the third:*

$$\mathfrak{s}|_{A,B} \in P_{A\multimap B}$$
$$\mathfrak{s}|_{A,C} \in P_{A\multimap C}$$
$$\mathfrak{s}|_{B,C} \in P_{B\multimap C}$$

Proof. By symmetry, it will suffice to prove that if $\mathfrak{s}|_{A,B} \in P_{A\multimap B}$ and $\mathfrak{s}|_{B,C} \in P_{B\multimap C}$ then $\mathfrak{s}|_{A,C} \in P_{A\multimap C}$. If X, Y are negative games, then sequences $t \in P_{X\multimap Y}$ are specified by the following properties - $t|_X \in P_X$, $t|_Y \in P_Y$, if t has even length, then $\zeta_{X\multimap Y}(t) = P$ and if t has odd length then $\zeta_{X\multimap Y}(t) = O$. We need to show that these properties holds if $X = A$, $Y = C$ and $t = \mathfrak{s}|_{A,C}$.

Certainly, $\mathfrak{s}|_{A,C}|_A = \mathfrak{s}|_A = \mathfrak{s}|_{A,B}|_A \in P_A$ and similarly $\mathfrak{s}|_{A,C}|_C \in P_C$.

If $\mathfrak{s}|_{A,C}$ has even length, it means that the lengths of $\mathfrak{s}|_A$ and $\mathfrak{s}|_C$ have the same parity, and therefore that the lengths of $\mathfrak{s}|_{A,B}$ and $\mathfrak{s}|_{B,C}$ have the same parity. Since $\mathfrak{s}|_{A,B} \in \sigma$ and $\mathfrak{s}|_{B,C} \in \tau$, it follows that $\zeta_{A\multimap B}(\mathfrak{s}|_{A,B}) = \zeta_{B\multimap C}(\mathfrak{s}|_{B,C})$. If $\zeta_B(\mathfrak{s}|_B) = P$ then $\zeta_{A\multimap B}(\mathfrak{s}|_{A,B}) = P$ and if $\zeta_B(\mathfrak{s}|_B) = O$ then $\zeta_{B\multimap C}(\mathfrak{s}|_{B,C}) = P$, so we must have $\zeta_{A\multimap B}(\mathfrak{s}|_{A,B}) = \zeta_{B\multimap C}(\mathfrak{s}|_{B,C}) = P$. By transitivity of $\Rightarrow$, it follows that $\zeta_{A\multimap C}(\mathfrak{s}|_{A,C}) = P$.

If $\mathfrak{s}|_{A,C}$ has odd length, it means that the lengths of $\mathfrak{s}|_A$ and $\mathfrak{s}|_C$ has opposite parities and therefore that the lengths of $\mathfrak{s}|_{A,B}$ and $\mathfrak{s}|_{B,C}$ have opposite parities. So it follows that $\zeta_{A\multimap B}(\mathfrak{s}|_{A,B}) = \neg\zeta_{B\multimap C}(\mathfrak{s}|_{B,C})$. Suppose that $\zeta_{A\multimap B}(\mathfrak{s}|_{A,B}) = O$. Then $\zeta_A(\mathfrak{s}|_A) = P$ and $\zeta_B(\mathfrak{s}|_B) = O$. By Proposition 2.11 above applied to $\mathfrak{s}|_{B,C}$, we must have $\zeta_C(\mathfrak{s}|_C) = O$ and therefore $\zeta_{A\multimap C}(\mathfrak{s}|_{A,C}) = (P \Rightarrow O) = O$.

If instead we have $\zeta_{B\multimap C}(\mathfrak{s}|_{B,C}) = O$, then $\zeta_B(\mathfrak{s}|_B) = P$ and $\zeta_C(\mathfrak{s}|_C) = O$. By Proposition 2.11 applied to $\mathfrak{s}|_{A,B}$, we must have $\zeta_A(\mathfrak{s}|_A) = P$ and so $\zeta_{A\multimap C}(\mathfrak{s}|_{A,C}) = (P \Rightarrow O) = O$. $\square$

Now we can move on to the main work. We start with a surprising lemma about strategies on the implication $A \multimap B$.

Lemma 2.13. *Let A, B be games and let σ be a strategy for $A \multimap B$. Suppose that $s, t \in \sigma$ are such that $s|_A = t|_A$ and $s|_B = t|_B$. Then $s = t$.*

Proof. Suppose that $s \neq t$. Let $r \sqsubseteq s, t$ be the longest common subsequence of s and t - so we have $rx \sqsubseteq s, ry \sqsubseteq t$ for some moves $x, y \in M_{A\multimap B}$ with $x \neq y$. Since rx, ry are both substrings of the strategy σ, r must be a P-position and therefore the moves x, y must take place in the same game. Without loss of generality, suppose that $x, y \in M_A$. Then we have $r|_A x \sqsubseteq s|_A, r|_A y \sqsubseteq t|_A$ and so $s|_A \neq t|_A$. $\square$

Thus we see that the plays according to some given strategy σ for $A \multimap B$ are characterized by their A- and B-components. More is true, however: given games A and B, the strategies on $A \multimap B$ are characterized by the A- and B-components of their constituent plays:

Lemma 2.14. *Let A, B be games and let σ, σ' be strategies for $A \multimap B$ such that*

$\{(s|_A, s|_B) : s \in \sigma \textit{ is a } P\textit{-position}\} = \{(s'|_A, s'|_B) : s' \in \sigma' \textit{ is a } P\textit{-position}\}$

Then $\sigma = \sigma'$.

Proof. Our proof is essentially that given in [CM10]. Suppose for a contradiction that $\sigma \neq \sigma'$, so without loss of generality there is some P-position $s \in \sigma \setminus \sigma'$ (using the definition of a strategy). By hypothesis, there is some $s' \in \sigma'$ such that $s'|_A = s|_A$ and $s'|_B = s|_B$. Clearly, $s \neq s'$, since $s \notin \sigma'$, so let $r \sqsubseteq s$ be the largest common prefix of s and s'. We therefore have $rx \sqsubseteq s, ry \sqsubseteq s'$, where $x, y \in M_{A \multimap B}$ and $x \neq y$.

Since $s|_A = s'|_A$ and $s|_B = s'|_B$, the moves x and y must take place in different games (otherwise we would be forced to have $x = y$). They are therefore both P-moves. Suppose without loss of generality that x is an A-move and that y is a B-move. By hypothesis, there is some P-position $t \in \sigma$ such that $t|_A = ry|_A = r|_A$ and $t|_B = ry|_B = r|_B y$.

Now clearly $rx \neq t$ (since their restrictions to A and B are different). Let $q \sqsubseteq rx, t$ be their largest common prefix; then we have $qu \sqsubseteq rx, qv \sqsubseteq t$ for some $u \neq v$. But now since $qu, qv \in \sigma$, u and v must both be O-moves, so they must be contained in the same component. But this is impossible: if they are both A-moves then we have $q|_A u \sqsubseteq r|_A x$ and $q|_A v \sqsubseteq r|_A$, so $u = v$ and if they are both B-moves then we have $q|_B u \sqsubseteq r|_B$ and $q|_B v \sqsubseteq r|_B y$, so $u = v$ again. This is a contradiction. $\square$

Given games A, B and a strategy σ for $A \multimap B$, the set

$$\{(s|_A, s|_B) : s \in \sigma \text{ is a } P\text{-position}\} \subseteq P_A \times P_B$$

is a *relation* between P_A and P_B. Writing $\underline{\sigma} \subseteq P_A \times P_B$, we will show that if A, B, C are games and $A \xrightarrow{\sigma} B \xrightarrow{\tau} C$ are morphisms, then:

$$\underline{\tau \circ \sigma} = \underline{\tau} \circ \underline{\sigma}$$

under the usual composition of relations:

$$\underline{\tau} \circ \underline{\sigma} = \{(s, t) \in P_A \times P_C : \exists u \in P_B \,.\, (s, u) \in \underline{\sigma},\ (u, t) \in \underline{\tau}\}$$

Thus we will get a functor $\mathcal{F} \colon \mathcal{G} \to \mathbf{Rel}$, where $\mathcal{G}$ is our category of negative games and $\mathbf{Rel}$ is the category of sets and relations. Proposition 2.14 then tells us that this functor is faithful.

It is clear from the definition of $\tau \circ \sigma$ that $\underline{\tau \circ \sigma} \subseteq \underline{\tau} \circ \underline{\sigma}$; indeed if $\mathfrak{s} \in \sigma \| \tau$, then $(\mathfrak{s}|_A, \mathfrak{s}|_B) \in \underline{\sigma}$ and $(\mathfrak{s}|_B, \mathfrak{s}|_C) \in \underline{\tau}$, so $(\mathfrak{s}|_A, \mathfrak{s}|_C) \in \underline{\tau} \circ \underline{\sigma}$. To show the reverse inclusion, we need to show the following: if $s \in \sigma$ and $t \in \tau$ are such that $s|_B = t|_B$, then there is some $\mathfrak{s} \in (M_A \sqcup M_B \sqcup M_C)^*$ such that $\mathfrak{s}|_{A,B} = s$ and $\mathfrak{s}|_{B,C} = t$.

This can be shown in an easy way. Roughly speaking, we write out the sequence s, and then insert immediately after each B-move the C-moves from t that occur after that B-move. This technique works for arbitrary sequences and is sufficient to prove that $\underline{\tau \circ \sigma} = \underline{\tau} \circ \underline{\sigma}$. However, it turns out that if s and t are alternating plays in $A \multimap B$ and $B \multimap C$ then this interleaving is unique - and even more is true:

Lemma 2.15. *Let A, B, C be negative games, let σ be a strategy for $A \multimap B$ and let τ be a strategy for $B \multimap C$. Suppose that $s \in \sigma$ and $t \in \tau$ are such that $s|_B = t|_B$. Let $n = \mathrm{length}(s) + \mathrm{length}(t|_C)$; i.e., the total number of unique terms of s and t after we have identified their common B-component. Then for all k with $0 \leq k \leq n$, there is a unique sequence $\mathfrak{s}^k \in \sigma \| \tau$ of length k such that $\mathfrak{s}^k|_A \sqsubseteq s|_A$ and $\mathfrak{s}^k|_C \sqsubseteq t|_C$. Moreover, for this $\mathfrak{s}^k$ we have $\mathfrak{s}^k|_{A,B} \sqsubseteq s$ and $\mathfrak{s}^k|_{B,C} \sqsubseteq t$, and if $j \leq k$ then $\mathfrak{s}^j \sqsubseteq \mathfrak{s}^k$.*

Note that Lemma 2.15 is stronger than saying that there is a unique sequence $\mathfrak{s}^k$ of length k such that $\mathfrak{s}^k|_{A,B} \sqsubseteq s$ and $\mathfrak{s}^k|_{B,C} \sqsubseteq t$. Instead, we are saying that there exists a sequence $\mathfrak{s}^k$ with those properties, but that it is uniquely determined even when we make no requirement on its B-component other than that the whole sequence is contained in $\sigma \| \tau$.

As a corollary, we see that the sequences s and t may be interleaved: set $\mathfrak{s} = \mathfrak{s}^n$; then we have $\mathfrak{s}|_{A,B} \sqsubseteq s$ and $\mathfrak{s}|_{B,C} \sqsubseteq t$. In fact, we have $\mathfrak{s}|_{A,B} = s$ and $\mathfrak{s}|_{B,C} = t$ by a length argument:

$$\mathrm{length}(\mathfrak{s}|_{A,B}) \leq \mathrm{length}(s) = n - \mathrm{length}(t|_C) \leq n - \mathrm{length}(\mathfrak{s}|_C) = \mathrm{length}(\mathfrak{s}|_{A,B})$$

so we must have equality everywhere, and similarly for $\mathfrak{s}|_{B,C}$ and t.

Proof of Lemma 2.15. We prove this by induction on k. There is a unique sequence of length 0, namely the empty sequence, and its A- and C-components are both empty, so are prefixes of s and t.

Suppose now that $k = 1$. We make the observation that if a is an O-move in A and if b is an O-move in B then a is a P-move in $A \multimap B$ and b is a P-move in $B \multimap C$. Since the starting move in $A \multimap B$ or $B \multimap C$ must be an O-move in both its component game and in the game as a whole, we see that any sequence in $\sigma\|\tau$ must begin with a move in C. So we have $\mathfrak{s}^1 = c$, for some C-move c, and the condition that $\mathfrak{s}^1|_C \sqsubseteq t$ means that c must be the first move in t, which is always a move from C. We have $\mathfrak{s}^1|_A = \epsilon \sqsubseteq s|_A$ and $\mathfrak{s}^1|_B = \epsilon \sqsubseteq s|_B$, as desired.

Now suppose that we have constructed the sequence $\mathfrak{s}^k$ for some k such that $1 \leq k < n$. We seek a sequence $\mathfrak{s}^{k+1} \in \sigma\|\tau$ of length $k+1$ such that $\mathfrak{s}^{k+1}|_A \sqsubseteq s$ and $\mathfrak{s}^{k+1}|_C \sqsubseteq t$. If we write $\mathfrak{s}'$ for the sequence obtained by removing the last move in $\mathfrak{s}^{k+1}$, it is clear that $\mathfrak{s}'|_A \sqsubseteq s$ and $\mathfrak{s}'|_C \sqsubseteq t$, so $\mathfrak{s}' = \mathfrak{s}^k$ by uniqueness. Therefore, $\mathfrak{s}^{k+1}$ is of the form $\mathfrak{s}^k x$, for some move $x \in M_A \sqcup M_B \sqcup M_C$.

Write $\mathfrak{s}^k = \mathfrak{s}'' y$, where y is the last move in $\mathfrak{s}^k$. Our move x will depend on which game y is played on as follows:

- Suppose y is a move in A or an O-move in B. By induction, $\mathfrak{s}'' y|_{A,B}$ is a prefix of s ending in y. We claim that it is a proper prefix.

  Indeed, let b be the last B-move occurring in $\mathfrak{s}'' y$ (since play in $A \multimap B$ starts with a B-move, there must be such a move). If $b = y$, then b is an O-move in $A \multimap B$. Otherwise, y must be a move in A, and then b is again an O-move in $A \multimap B$, since player P switches games. Therefore, b is a P-move in $B \multimap C$. Now $\mathfrak{s}'' y|_B$ is a prefix of t ending in b. Since b is a P-move in $B \multimap C$, it may only be followed in $B \multimap C$ by another move from B, so there are two possibilities: either $\mathfrak{s}'' y|_{B,C} = t$ or there is some other B-move b' that occurs later than b in the sequence t. In the first case, $\mathfrak{s}'' y|_{A,B}$ must be a proper prefix of s by length considerations:

  $$\begin{aligned} \text{length}(\mathfrak{s}'' y|_{A,B}) &= \text{length}(\mathfrak{s}'' y) - \text{length}(\mathfrak{s}'' y|_C) \\ &= \text{length}(\mathfrak{s}'' y) - \text{length}(t|_C) \\ &= k - \text{length}(t|_C) < n - \text{length}(t|_C) = \text{length}(s) \end{aligned}$$

  while in the second case $\mathfrak{s}'' y|_B$ must be a proper prefix of $t|_B = s|_B$, and so $\mathfrak{s}'' y|_{A,B}$ must be a proper prefix of s.

  Therefore, we have $\mathfrak{s}'' y|_{A,B} x \sqsubseteq s$ for some move $x \in M_A \sqcup M_B$. If x is an A-move then $\mathfrak{s}'' yx|_{B,C} = \mathfrak{s}'' y|_{B,C} \sqsubseteq t$, as desired. If x is a move in B, then it must be a P-move in $A \multimap B$ (since y is either a move in A or an O-move in B), so it is an O-move in $B \multimap C$. Since $s|_B = t|_B$, we must have $\mathfrak{s}'' yx|_B \sqsubseteq t|_B$, and so y must be a P-move in $B \multimap C$, and should therefore be followed by another move in B, which must be x. Therefore, $\mathfrak{s}'' yx|_{B,C} \sqsubseteq t$.

For uniqueness, suppose that $\mathfrak{s}''yz \in \sigma\|\tau$ is such that $\mathfrak{s}''yz|_A \sqsubseteq s|_A$ and $\mathfrak{s}''yz|_C \sqsubseteq t|_C$. As before, let b be the last B-move occurring in $\mathfrak{s}''y$; we have already shown that b must be a P-move in $B \multimap C$, and it follows that it must be the last move occurring in $\mathfrak{s}''y|_{B,C}$ (since it can only be followed by another B-move). Suppose that $z \in M_B \sqcup M_C$. Then $\mathfrak{s}''yz|_{B,C} = \mathfrak{s}''y|_{B,C}z$; since the last move in $\mathfrak{s}''y|_{B,C}$ is b, this means that z must be an O-move in B. So z is either an O-move in B or a move in A.

If y was an O-move in A, then y is a P-move in $A \multimap B$, so it must be followed in $A \multimap B$ by another move from A. Then the condition that $\mathfrak{s}''yz|_A \sqsubseteq s|_A$ tells us that $z = x$. If instead y was a P-move in A or a move in B then y is an O-move in $A \multimap B$; now, since we have $\mathfrak{s}''y|_{A,B}z, \mathfrak{s}''y|_{A,B}x \in \sigma$, it must be the case that $x = z$ by the definition of a strategy.

- If instead y is a move in C or a P-move in B, we use a symmetrical argument, choosing x to be the next move along in t. We need to take a little extra care here when we talk about the last B-move in $\mathfrak{s}''y$; there may in fact be no B-moves in $\mathfrak{s}''y$. Since any play in $A \multimap B$ must begin with a B-move, this only happens when $\mathfrak{s}''y$ is entirely made up of C-moves. In this case, it is not difficult to show that $\mathfrak{s}''y|_{B,C}$ is a proper prefix of t by length considerations. □

We now have all the ingredients ready to prove that $\tau \circ \sigma$ is a well-defined strategy.

Proposition 2.16. *Let A, B, C be games, let σ be a strategy for $A \multimap B$ and let τ be a strategy for $B \multimap C$. Then $\tau \circ \sigma$ (as defined above) is a strategy for $A \multimap C$.*

Proof. By Lemma 2.12, $\tau \circ \sigma \in P_{A \multimap C}$. Moreover, it is non-empty, since it contains ϵ. We need to show that the two strategy conditions hold for $\tau \circ \sigma$.

Firstly, suppose that $\mathfrak{s} \in \sigma\|\tau$ and $\mathfrak{s}|_{A,C}$ is a P-play in $A \multimap C$. Suppose that $\mathfrak{s}|_{A,C}a \in P_{A \multimap C}$ for some $a \in M_A$. We claim that $\mathfrak{s}a \in \sigma\|\tau$.

Indeed, certainly $\mathfrak{s}a|_{B,C} = \mathfrak{s}|_{B,C} \in \tau$. Moreover, we have $\mathfrak{s}a|_{A,C} \in P_{A \multimap C}$, so Lemma 2.12 tells us that $\mathfrak{s}a|_{A,B} \in P_{A \multimap B}$.

Since a is an O-move in $A \multimap B$, it is an O-move in $A \multimap C$. Since $\mathfrak{s}|_{A,B} \in \sigma$, we must have $\mathfrak{s}a|_{A,B} \in \sigma$, by the definition of a strategy. So $\mathfrak{s}a \in \sigma\|\tau$.

By a symmetrical argument, if $\mathfrak{s}|_{A,C}c \in P_{A \multimap C}$ for some $c \in M_C$, then $\mathfrak{s}c \in \sigma \| \tau$.

For the second strategy property, suppose that $s \in \tau \circ \sigma$ is n O-play and that $sx, sy \in \tau \circ \sigma$. We claim that $x = y$. By the definition of $\tau \circ \sigma$, there must be sequences $\mathfrak{s}, \mathfrak{t} \in \sigma \| \tau$ with $\mathfrak{s}|_{A,C} = sx$ and $\mathfrak{t}|_{A,C} = sy$. Moreover, by removing any B-moves form the end of $\mathfrak{s}, \mathfrak{t}$, we may assume that the last move in $\mathfrak{s}$ is x and the last move in $\mathfrak{t}$ is y. We may write $\mathfrak{s} = \mathfrak{s}'\mathfrak{b}x$ and $\mathfrak{t} = \mathfrak{t}'\boldsymbol{\beta}y$, where $\mathfrak{b}$ and $\boldsymbol{\beta}$ are sequences composed entirely out of B-moves and neither $\mathfrak{s}'$ nor $\mathfrak{t}'$ ends with a B-move. In particular, $\mathfrak{s}'|_{A,C} = \mathfrak{t}'|_{A,C} = s$.

Without loss of generality, $\text{length}(\mathfrak{s}') \leq \text{length}(\mathfrak{t}')$. Let $\mathfrak{t}''$ be the prefix of $\mathfrak{t}'$ that has the same length as $\mathfrak{s}'$, writing $\mathfrak{t}' = \mathfrak{t}''\mathfrak{r}$.

Now $\mathfrak{s}'$ and $\mathfrak{t}''$ have the same length, and we have

$$\begin{array}{ll} \mathfrak{s}'|_A \sqsubseteq \mathfrak{s}|_A & \mathfrak{t}''|_A \sqsubseteq \mathfrak{s}|_A \\ \mathfrak{s}'|_C \sqsubseteq \mathfrak{s}|_C & \mathfrak{t}''|_C \sqsubseteq \mathfrak{s}|_C \end{array}$$

so we must have $\mathfrak{s}' = \mathfrak{t}''$, by uniqueness in Lemma 2.15. Now observe that we have

$$\mathfrak{s}'|_{A,C} = \mathfrak{t}'|_{A,C} = \mathfrak{t}''|_{A,C}\mathfrak{r}|_{A,C} = \mathfrak{s}'|_{A,C}\mathfrak{r}|_{A,C}$$

Since $\mathfrak{t}' = \mathfrak{t}''\mathfrak{r}$ does not end with a B-move, $\mathfrak{r}|_{A,C}$ must be non-empty if $\mathfrak{r}$ is non-empty. Therefore $\mathfrak{r} = \epsilon$, and so $\mathfrak{t}' = \mathfrak{t}'' = \mathfrak{s}'$.

We now claim that $\text{length}(\mathfrak{b}) = \text{length}(\boldsymbol{\beta})$. Indeed, suppose instead that $\text{length}(\mathfrak{b}) < \text{length}(\boldsymbol{\beta})$. Then there would be a prefix $\boldsymbol{\beta}' \sqsubseteq \boldsymbol{\beta}$ of length $\text{length}(\mathfrak{b}) + 1$. Then we would have

$$\begin{array}{ll} \mathfrak{s}'\mathfrak{b}x|_A = \mathfrak{s}|_A & \mathfrak{s}'\boldsymbol{\beta}'|_A = \mathfrak{s}'|_A \sqsubseteq \mathfrak{s}|_A \\ \mathfrak{s}'\mathfrak{b}x|_C = \mathfrak{s}|_C & \mathfrak{s}'\boldsymbol{\beta}'|_C = \mathfrak{s}'|_C \sqsubseteq \mathfrak{s}|_C \end{array}$$

and so $\mathfrak{s}'\mathfrak{b}x = \mathfrak{s}'\boldsymbol{\beta}'$ by Lemma 2.15, which is clearly impossible. Therefore, $\text{length}(\mathfrak{b}) = \text{length}(\boldsymbol{\beta})$. Now we can apply Lemma 2.15 in the same way to tell us that $\mathfrak{b} = \boldsymbol{\beta}$.

Suppose that $\zeta_B(\mathfrak{s}'\mathfrak{b}|_B) = P$. Since $\mathfrak{s}'\mathfrak{b} \in \sigma \| \tau$, this means that $\zeta_A(\mathfrak{s}'\mathfrak{b}|_A) = P$, by Proposition 2.11(i). Therefore, $\zeta_{A \multimap B}(\mathfrak{s}'\mathfrak{b}) = P$, so x and y must both be moves in C. Then we have $\mathfrak{s}'\mathfrak{b}|_{B,C}x, \mathfrak{s}'\mathfrak{b}|_{B,C}y \in \tau$, and so $x = y$ by the definition of a strategy.

By a symmetrical argument, if $\zeta_B(\mathfrak{s}'\mathfrak{b}|_B) = O$, then x and y must both be moves in A. We then have $\mathfrak{s}'\mathfrak{b}|_{A,B}x, \mathfrak{s}'\mathfrak{b}|_{A,B}y \in \sigma$, so $x = y$. $\square$

Now we are ready to define our category $\mathcal{G}$ of games. We have defined composition of strategies, so we need to define identity morphisms. Identity morphisms are given by the copycat strategy, as usual:

$$\mathrm{id}_A = \left\{ s \in P_{A \multimap A} \ : \ \text{for all even length } t \sqsubseteq s,\, t|_{\left({}^{\perp}A\right)} = t|_A \right\}$$

We can now state our first main result.

Theorem 2.17. *The collection of negative games, with morphisms from A to B given by strategies on $A \multimap B$ with the notions of composition and identity given above, forms a category $\mathcal{G}$. Moreover, there is a faithful functor $\mathcal{F} \colon \mathcal{G} \to$ **Rel**, where **Rel** is the category of sets and relations, given by sending a game A to the set of plays P_A and sending a strategy $\sigma \colon A \multimap B$ to the set*

$$\underline{\sigma} = \{(s|_A, s|_B) \ : \ s \in \sigma \textit{ is a } P\textit{-position}\} \subseteq P_A \times P_B$$

Proof. We shall make much use of the relational content of a strategy. We first need to show that the identity is indeed an identity and that composition is associative. For the identity, note that $\underline{\mathrm{id}_A} = \Delta_{P_A} \subseteq P_A \times P_A$, the identity relation in **Rel**. If B, C are negative games and $\sigma \colon B \multimap A, \tau \colon A \multimap C$ are strategies, then from our earlier discussion we have

$$\underline{\mathrm{id}_A \circ \sigma} = \underline{\mathrm{id}_A} \circ \underline{\sigma} = \Delta_{P_A} \circ \underline{\sigma} = \underline{\sigma}$$

and therefore $\mathrm{id}_A \circ \sigma = \sigma$, by Lemma 2.14. Similarly, $\tau \circ \mathrm{id}_A = \tau$.

For associativity of composition, we hijack the associativity in **Rel**: if A, B, C, D are negative games, and $A \xrightarrow{\sigma} B \xrightarrow{\tau} C \xrightarrow{\upsilon} D$ are morphisms, then we have:

$$\begin{aligned}
\underline{(\upsilon \circ \tau) \circ \sigma} &= \underline{\upsilon \circ \tau} \circ \sigma \\
&= (\underline{\upsilon} \circ \underline{\tau}) \circ \underline{\sigma} \\
&= \underline{\upsilon} \circ (\underline{\tau} \circ \underline{\sigma}) \quad \text{by associativity in } \mathbf{Rel} \\
&= \underline{\upsilon} \circ \underline{\tau \circ \sigma} \\
&= \underline{\upsilon \circ (\tau \circ \sigma)}
\end{aligned}$$

and so $(\upsilon \circ \tau) \circ \sigma = \upsilon \circ (\tau \circ \sigma)$, by Lemma 2.14.

We have already shown that the map $\mathcal{F}$ respects composition, so it is a functor. It is faithful by Lemma 2.14. $\square$

2.6 Total strategies and winning conditions

Let A be a game. We call a strategy σ for A *total* if it satisfies the additional requirement that whenever $s \in \sigma$ is an O-play, there exists some $a \in M_A$ such that $sa \in \sigma$ (before we only required that such a play be unique if it existed).

We might want to talk about a category of negative games and total strategies. However, we cannot do this immediately, since the composition of two total strategies need not be total.

Example 2.18. Let Σ^* be the negative game where each player has a unique move at every position:

$$M_{\Sigma^*} = \{q, a\}$$
$$\lambda_{\Sigma^*}(q) = O,\ \lambda_{\Sigma^*}(a) = P$$
$$P_{\Sigma^*} = \{\epsilon, q, qa, qaq, qaqa, \dots\}$$

Let I be the negative game with no moves and let $\perp$ be the negative game with a single opponent move and no further play:

$$\begin{array}{cc} M_I = \varnothing & M_\perp = \{*\} \\ \lambda_I = \varnothing & \lambda_\perp(*) = O \\ P_I = \{\epsilon\} & P_\perp = \{\epsilon, *\} \end{array}$$

It is easy to see that if A is a negative game then strategies for $I \multimap A$ are the same thing as strategies for A and that strategies for $A \multimap \perp$ are the same thing as opponent strategies for A. So we get morphisms $\sigma\colon I \to \Sigma^*, \tau\colon \Sigma^* \to \perp$ given by the unique player and opponent strategies for Σ^*:

$$\sigma = P_{\Sigma^*} \quad \tau = \{*s\ :\ s \in P_{\Sigma^*}\}$$

Moreover, both these strategies are total.

The plays in $\sigma \| \tau$ are now those sequences $\mathfrak{s} \in (M_I \sqcup M_{\Sigma^*} \sqcup M_\perp)^*$ such that:

$$\mathfrak{s}|_A = \epsilon$$
$$\mathfrak{s}|_B \in P_{\Sigma^*}$$
$$\text{If } \mathfrak{s} \neq \epsilon \text{ then } \mathfrak{s}|_C = *$$

Upon restricting these sequences to A and C, we see that $\tau \circ \sigma$ contains only two plays - ϵ and $*$. $*$ is an O-play with no P-reply in $\tau \circ \sigma$, so $\tau \circ \sigma$ is not a total strategy.

Clearly it is the infinite play in the game Σ^* that is causing the problem. We can get around this by insisting that our games contain no infinite plays: if A is a game, we say A has an infinite play if $(P_A, \sqsubseteq)$ has an infinite chain.

Proposition 2.19. *Let A, B, C be negative games and assume that B has no infinite play. Let σ be a total strategy for $A \multimap B$ and let τ be a total strategy for $B \multimap C$. Then $\tau \circ \sigma$ is a total strategy for $A \multimap C$.*

Proof. We have already shown that $\tau \circ \sigma$ is a strategy for $A \multimap C$, so we need to show that it is total. Let $s \in \tau \circ \sigma$ be an O-play. We seek a P-move x such that $sx \in \tau \circ \sigma$.

By the definition of $\tau \circ \sigma$, there is some $\mathfrak{s} \in \sigma \| \tau$ such that $\mathfrak{s}|_{A,C} = s$. Since P_B contains no infinite chain, we may take $\mathfrak{s}$ to be maximal such that $\mathfrak{s}|_{A,B} = s$.

Since s is an O-play, $\mathfrak{s}|_A$ must be a P-play in A and $\mathfrak{s}|_C$ must be an O-play in C. Consider the B-component $\mathfrak{s}|_B$ of $\mathfrak{s}$. If this is an O-play, then $\mathfrak{s}|_{A,B}$ is an O-play in σ, so, by totality of σ, there exists x such that $\mathfrak{s}|_{A,B}x \in \sigma$ and therefore $\mathfrak{s}x \in \sigma \| \tau$. Similarly, if $\mathfrak{s}|_B$ is a P-play, then $\mathfrak{s}|_{B,C}$ is an O-play in τ, so there exists x such that $\mathfrak{s}|_{B,C}x \in \tau$ and so $\mathfrak{s}x \in \sigma \| \tau$.

If x is a B-move, we have $\mathfrak{s}x|_{A,C} = s$, contradicting maximality of s. Therefore, x is an A- or C-move, and we have $sx \in \tau \circ \sigma$. $\square$

The result of this proposition means that we could create a category of games and total strategies by requiring that all our games have no infinite plays. However, this approach will not work for us, since our particular version of the exponential connective ! introduces infinite plays into all non-empty games. Moreover, it seems a bit wasteful to require that none of the games A, B, C have infinite plays when it is only infinite plays in game B that are causing problems.

Fortunately, there is a cleverer way of getting around the problem. Recall that we defined a game to be given by a tuple $(M_A, \lambda_A, \zeta_A, P_A)$. Given such a game, and given some subset $X \subseteq P_A$, define X^ω to be the set of all infinite plays in the game – i.e., the set of all infinite sequences $s \in M_A^\omega$ all of whose prefixes lie in X.

Definition 2.20. A *win-game* is a tuple $A = (M_A, \lambda_A, \zeta_A, P_A, W_A)$ where $(M_A, \lambda_A, \zeta_A, P_A)$ is a game as defined before and $W_A \colon P_A^\omega \to \{O, P\}$ is a function telling us whether each infinite play is a win for player P or for player O.

To distinguish our original games from these win-games, we shall refer to the earlier games as *finitary games.*

We can define an appropriate notion of strategies for these games. If A is a win-game, then a *winning strategy* for A is a total strategy for $(M_A, \lambda_A, \zeta_A, P_A)$ such that $W_A(s) = P$ for all $s \in \sigma^\omega$.

We can also extend our connectives to our new win games. At the level of finitary games, our definitions of the connectives are unchanged, and it remains to define the W-functions. If A is a win game, then $W_{\perp_A} = \neg \circ W_A$.

If A and B are win games, we would like to define $W_{A\otimes B}$ by

$$W_{A\otimes B}(s) = W_A(s|_A) \wedge W_B(s|_B)$$

However, this is not well defined at present because one out of $s|_A$ and $s|_B$ might be a finite sequence even if s is infinite. For this purpose, it will be convenient to extend our function W_A to include finite plays as well using ζ_A:

$$W^*_A = \zeta_A \sqcup W_A \colon P_A \sqcup P^\omega_A \to \{O, P\}$$

We can then define

$$W_{A\otimes B}(s) = W^*_A(s|_A) \wedge W^*_B(s|_B)$$

Indeed, we could do away with ζ_A and W_A altogether, replacing them with W^*_A. Since the rule for $\zeta_{A\otimes B}$ is also given by a conjunction, our rule for the tensor product becomes:

$$W^*_{A\otimes B}(s) = W^*_A(s|_A) \wedge W^*_B(s|_B)$$

This will be the starting point for our development of transfinite games in the next section. In our formulation, finitary games are games played over the ordinal ω, while win-games will be games played over the ordinal $\omega + 1$. We make no special distinction between finite and infinite plays, so the function ζ_A for our transfinite games incorporates both the function ζ_A and the function W_A from our win-games, in the same manner as the function W^*_A defined above. One advantage of this point of view is that we no longer have to treat the finite- and infinite-play cases separately, but can give a uniform presentation.

For now, let us concentrate on developing the theory of win-games in the traditional way. Having defined the tensor product, we may recover the definitions of the par $⅋$ and linear implication $\multimap$ of two games as usual. In particular, this means that we have

$$W^*_{A\multimap B}(s) = (W_A(s|_A) \Rightarrow W_B(s|_B))$$

It is worth examining the definitions of the W-functions for tensor and implication. Recall that in the tensor product, it is always player O who switches games. Therefore, if s is an infinite play in $A \otimes B$ and $s|_A$ is finite and non-empty, it must be the case that player P made the last move in s, and so $W^*_A(s|_A) = P$. If we assume our games are negative, then this extends to the case that $s|_A$ is empty. This tells us that player P wins an infinite play s in $A \otimes B$ if and only if each component is either P-winning or is finite.

Similarly, player P wins an infinite play s in $A \multimap B$ if and only if $s|_A$ is infinite and O-winning or if $\mathfrak{s}|_B$ is either finite or is infinite and P-winning. Since it is player P who switches games in $A \multimap B$, if $s|_A$ is finite then it will end with an O-move, so will count as an O-winning play.

The point now is that if A, B, C are negative win-games and σ and τ are strategies for $A \multimap B$ and $B \multimap C$, we will never end up in the situation we were in in Example 2.18 where we have an infinite play in B that stops us from replying in the composition $\tau \circ \sigma$. In that example, the unique infinite play in Σ^* would have to be either O-winning or P-winning, so either σ or τ would not be a valid winning strategy.

Proposition 2.21. *Let A, B, C be negative win-games, let σ be a winning strategy for $A \multimap B$ and let τ be a winning strategy for $B \multimap C$. Then $\tau \circ \sigma$ is a winning strategy for $A \multimap C$.*

Proof. First, we want to show that $\tau \circ \sigma$ is a total strategy. Let $s \in \tau \circ \sigma$ be an O-position; then we have some $\mathfrak{s} \in \sigma \| \tau$ with $\mathfrak{s}|_{A,C} = s$.

We claim that there is a maximal such $\mathfrak{s}$. By Lemma 2.15, $\mathfrak{s}$ is determined by its length, and so the only way that there could not be a maximal such $\mathfrak{s}$ is if there is some infinite sequence $\mathfrak{S} \in (M_A \sqcup M_B \sqcup M_C)^\omega$ with $\mathfrak{S}|_{A,B} \in \sigma \cup \sigma^\omega$, $\mathfrak{S}|_{B,C} \in \tau \cup \tau^\omega$ and $\mathfrak{S}|_{A,C} = s$. Since $\mathfrak{S}$ has some finite prefix $\mathfrak{s}$ with $\mathfrak{s}|_{A,C} = s$, $\mathfrak{S}|_A$ and $\mathfrak{S}|_C$ must both be finite, so they are O-plays in the games $A \multimap B$ and $B \multimap C$ respectively.

Now $\mathfrak{S}|_B$ is infinite. If $W_B(\mathfrak{S}|_B) = O$, then $W_{A\multimap B}(\mathfrak{S}|_{A,B}) = O$, contradicting the fact that σ is a winning strategy. Symmetrically, if $W_B(\mathfrak{S}|_B) = P$, then $W_{B\multimap C}(\mathfrak{S}|_{B,C}) = O$, contradicting the fact that τ is a winning strategy. We have a contradiction in either case, so if σ and τ are indeed winning strategies, there must be a maximal (finite) $\mathfrak{s} \in \sigma \| \tau$ such that $\mathfrak{s}|_{A,C} = s$.

The argument from Proposition 2.19 now applies, showing that $sx \in \tau \circ \sigma$ for some P-move x.

Lastly, we want to show that any infinite play in $\tau \circ \sigma$ is P-winning. Suppose s is some infinite play in $\tau \circ \sigma$. For each finite prefix $t \sqsubseteq s$ there is some $\mathfrak{t} \in \sigma \| \tau$ with $\mathfrak{t}|_{A,C} = t$, and by Lemma 2.15, we have $\mathfrak{t}' \sqsubseteq \mathfrak{t}$ if $t' \sqsubseteq t$. Therefore, these $\mathfrak{t}$ glue to give an infinite sequence $\mathfrak{s} \in (M_A \sqcup M_B \sqcup M_C)^\omega$ such that $\mathfrak{s}|_{A,B} \in \sigma^\omega$ and $\mathfrak{s}|_{B,C} \in \tau^\omega$. We claim that $W^*_{A\multimap B}(\mathfrak{s}|_{A,B}) = W^*_{B\multimap C}(\mathfrak{s}|_{B,C}) = P$ - then it follows that $W^*_{A\multimap C}(\mathfrak{s}|_{A,C}) = P$ by transitivity of $\Rightarrow$. If $\mathfrak{s}|_{A,B}$ and $\mathfrak{s}|_{B,C}$ are both infinite, then this is automatically true, since σ and τ are winning strategies. Otherwise, one of them must be infinite and the other finite.

Suppose that $\mathfrak{s}|_{A,B}$ is finite. Let $\mathfrak{t}$ be the shortest finite prefix of $\mathfrak{s}$ with $\mathfrak{t}|_{A,B} = \mathfrak{s}|_{A,B}$; then we have $\mathfrak{t}c \sqsubseteq \mathfrak{s}$ for some move $c \in M_C$. If $\mathfrak{t}|_{A,B} = \epsilon$, then $W^*_A(\mathfrak{t}|_{A,B}) = P$ since A and B are negative. Otherwise, there is at least one B-move in $\mathfrak{t}|_{A,B}$. Let b be the last B-move occurring in $\mathfrak{t}|_{A,B}$ - then it is the last B-move occurring in $\mathfrak{t}|_{B,C}$; since it is followed by the move c, it must be an O-move in $B \multimap C$ and therefore a P-move in B. Therefore, $W^*_{A\multimap B}(\mathfrak{s}|_{A,B}) = P$.

If instead $\mathfrak{s}|_{B,C}$ is finite, once again we have some $\mathfrak{t} \sqsubseteq \mathfrak{s}$ with $\mathfrak{t}|_{A,B} = \mathfrak{s}|_{A,B}$ and $\mathfrak{t}a \sqsubseteq \mathfrak{s}$ for some move $a \in M_A$. Since $\mathfrak{s}|_{A,B}$ is infinite, it must be non-empty, so it must contain some B-move. Let b be the last B-move occurring in $\mathfrak{s}|_{A,B}$; since b is followed by an A-move, it must be an O-move in B. Now b is also the last B-move occurring in $\mathfrak{s}|_{B,C}$ and so $W^*_{B\multimap C}(\mathfrak{s}|_{B,C}) = P$. $\square$

We get another result:

Theorem 2.22. *The collection of all negative win-games, with morphisms from A to B given by winning strategies on $A \multimap B$ and composition defined as above, is a category.*

Proof. We have shown already that the composition of two winning strategies is a winning strategy. Associativity of composition is inherited from finitary games and the identity strategy is winning for $A \multimap A$. $\square$

3 Transfinite games

Note 3.1*: In this section, we assume familiarity with ordinals, ordinal arithmetic and transfinite induction/recursion. We shall frequently identify an ordinal α with the set $\{\beta : \beta < \alpha\}$ of all ordinals less than α without comment.*

In the last section, we came across win-games, where in addition to the usual function $\zeta_A \colon P_A \to \{O, P\}$ we have a function $W_A \colon P_A^\omega \to \{O, P\}$, where P_A^ω is the set of infinite limits of plays in P_A. We saw that it is useful to combine these two functions into one:

$$W_A^* = \zeta_A \sqcup W_A \colon P_A \sqcup P_A^* \to \{O, P\}$$

This suggests that it might make sense to combine P_A and P_A^ω into one a single set $\overline{P_A}$ and use the function W_A^* instead of ζ_A and W_A. This new set will contain both finite and infinite plays, and the infinite plays will be precisely the limits of the finite plays. In this way, the set of possible lengths of plays in $\overline{P_A}$ is the set $\{0, 1, 2, 3, \dots, \omega\}$, or the ordinal $\omega + 1$.

Contrast this with the finitary case, in which the set of possible lengths of plays is the set $\{0, 1, 2, 3, \dots\}$, or the ordinal ω. In the coming sections, we will generalize these results so that we end up with games where the plays can have much larger ordinal lengths. Win-games will be taken to be the motivating example throughout.

3.1 Transfinite Games and Strategies

We fix some ordinal α throughout. Given a set M, we write $M^{*<\alpha}$ for the set of all ordinal-length sequences of elements of M whose length is less than α.

Definition 3.2. A *game over* α, or an α*-game*, is given by a tuple

$$A = (M_A, \lambda_A, \zeta_A, P_A)$$

where

- M_A is a set of moves.
- $\lambda_A \colon M_A \to \{O, P\}$ is a function telling us which player may make each move.

- P_A is a set of pairs (β, s), where $\beta < \alpha$ is some ordinal and $s\colon \beta \to M_A$ is a sequence of moves of length β.
- $\zeta_A\colon P_A \to \{O, P\}$ tells us which player owns each position.

We will normally abuse notation and write s instead of (β, s) and $\text{length}(s)$ instead of β. If $a \in M_A$, we will write a for the 1-move play $(1, a)$, and st for the concatenation of two sequences s and t (where $\text{length}(st) = \text{length}(s) + \text{length}(t)$). We will write $t \sqsubseteq s$ if $\text{length}(t) \leq \text{length}(s)$ and $s|_{\text{length}(t)} = t$.

To be called a game, $(M_A, \lambda_A, \zeta_A, P_A)$ has to satisfy four rules.

Prefix closure P_A must be closed under the relation $\sqsubseteq$.

Well-formedness If $a \in M_A$ and $sa \in P_A$ then $\zeta_A(sa) = \lambda_A(a)$.

Alternating condition If $a \in M_A$ and $sa \in P_A$ then $\zeta_A(s) = \neg\zeta_A(sa)$.

Limit condition If $\mu < \alpha$ is a limit ordinal and $s \in (M_A)^\mu$ is a sequence of length μ such that every proper prefix $t \sqsubsetneq s$ is in P_A, then s is in P_A.

Remark 3.3. In the $\omega + 1$ case, the limit condition tells us that the plays of length ω in P_A are precisely those infinite plays all of whose finite prefixes lie in P_A. It also guarantees that P_A is non-empty, since the empty play ϵ will always satisfy the conditions.

In the $\omega + 1$ case, ζ_A will take the role of what we previously called W_A^*, while P_A takes the role of what we called $P_A \sqcup P_A^\infty$.

We define strategies for these games in much the same way that we defined them before:

Definition 3.4. Let A be an α-game. A (partial) *strategy* for A is a subset $\sigma \subseteq P_A$ satisfying three conditions:

σ contains all O-replies If $s \in \sigma$ is a P-play and $sa \in P_A$ is an O-response, then $sa \in \sigma$.

P-replies are unique If $s \in \sigma$ is an O-play and $sa, sb \in \sigma$ are P-responses, then $a = b$.

Limit condition If $\mu < \alpha$ is a limit ordinal and $s \in P_A$ is a sequence of length μ such that every proper prefix $t \sqsubsetneq s$ lies in σ, then $s \in \sigma$.

We call σ *total* if it satisfies the extra requirement:

P always has a reply If $s \in \sigma$ is an O-play, then there is some P-move $a \in M_A$ with $sa \in \sigma$.

The only new part of this is the limit condition. Let us examine what it means in the $\omega + 1$ case. Play according to a total strategy σ gives rise to a sequence of plays $\epsilon \sqsubseteq s_1 \sqsubseteq s_2 \sqsubseteq \dots$ (were s_i has length i). The limit condition then tells us that the limit s_ω of these plays must be contained in σ. If $\zeta_A(s_\omega) = O$ then P has no response to s_ω (since there are no plays of length $\omega + 2$), contradicting totality of σ. So for σ to be a total strategy, we must have $\zeta_A(s_\omega) = P$ for every s_ω arising as the limit of finite plays according to σ. So total strategies for $\omega + 1$-games are exactly the same thing as the winning strategies for win-games that we defined earlier.

If $s \in P_A$ and length(s) is a successor ordinal, we will call s a *successor play* or *successor position.* If length(s) is a limit ordinal, we will call s a *limiting play* or *limiting position.*

3.2 Connectives

We may define connectives for α-games in much the same way that we defined them for finitary games and win-games.

Definition 3.5. Let A be an α-game. The negation ${}^\perp A$ of A is given by

- $M_{({}^\perp A)} = M_A$
- $\lambda_{({}^\perp A)} = \neg \circ \lambda_A$
- $\zeta_{({}^\perp A)} = \neg \circ \zeta_A$
- $P_{({}^\perp A)} = P_A$

For the other connectives – $\otimes$, $⅋$ and $\multimap$ – we will use interleaved plays. If A and B are α games, then our interleaved plays will be sequences taking values in $M_A \sqcup M_B$. If $\beta < \alpha$ and $s \in (M_A \sqcup M_B)^\beta$, we may write β_A for $s^{-1}(M_A)$ and β_B for $s^{-1}(M_B)$.

Lemma 3.6. *Let β be an ordinal, and let γ be any subset of β. Then γ inherits a well-ordering from β and, as ordinals, we have $\gamma \leq \beta$.*

Proof. Any subset of γ is a subset of β, so it has a least element. Therefore, the inherited order on γ is a well-order.

Suppose for a contradiction that $\beta < \gamma$. Then we have order preserving injections

$$i\colon \gamma \hookrightarrow \beta \quad j\colon \beta \hookrightarrow \gamma$$

where i is inclusion of γ as a subset of β and j is inclusion of β as a proper initial prefix of γ. Let $f\colon \gamma \hookrightarrow \gamma$ be the composition $\gamma \overset{i}{\hookrightarrow} \beta \overset{j}{\hookrightarrow} \gamma$. Then f is order preserving.

Let $S = \{\delta \in \gamma :\ f(\delta) < \delta\}$. S is non-empty, since $\beta \in S$ (considering β as an element of γ), so S has a least element ξ. Now $f(\xi) < \xi$, so $f(f(\xi)) < f(\xi)$, since f is order-preserving and injective. Therefore, $\xi \leq f(\xi)$ by minimality of ξ, which is a contradiction. $\square$

Using Lemma 3.6, we see that $\beta_A, \beta_B \leq \beta$; in particular, they are both less than α, so we get induced sequences $s|_A$ of length β_A and $s|_B$ of length β_B. We define

$$P_A \| P_B = \left\{ s \in (M_A \sqcup M_B)^{*<\alpha} :\ s|_A \in P_A,\ s|_B \in P_B \right\}$$

We may now define functions $\zeta_{A\otimes B}, \zeta_{A⅋B}, \zeta_{A\multimap B}\colon P_A\|P_B \to \{O, P\}$ as before:

$$\begin{aligned}
\zeta_{A\otimes B}(s) &= \quad \zeta_A(s|_A) \wedge \zeta_B(s|_B) \\
\zeta_{A⅋B}(s) &= \quad \zeta_A(s|_A) \vee \zeta_B(s|_B) \\
\zeta_{A\multimap B}(s) &= (\zeta_A(s|_A) \Rightarrow \zeta_B(s|_B))
\end{aligned}$$

Our definitions of the functions $\lambda_{A\otimes B}, \lambda_{A⅋B}, \lambda_{A\multimap B}\colon M_A \sqcup M_B \to \{O, P\}$ are the same as before:

$$\begin{gathered}
\lambda_{A\otimes B} = \lambda_{A⅋B} = \lambda_A \sqcup \lambda_B \\
\lambda_{A\multimap B} = (\neg \circ \lambda_A) \sqcup \lambda_B
\end{gathered}$$

Given a set M, a prefix-closed set $P \subseteq M^{*<\alpha}$ and functions $\lambda\colon M \to \{O, P\}, \zeta\colon P \to \{O, P\}$, we say that a play $s \in P$ is *well-formed with respect to* λ, ζ if whenever $ta \sqsubseteq s$ (with $a \in M$) we have $\zeta(ta) = \lambda(a)$. We say that s is *alternating with respect to* ζ if whenever t, ta are prefixes of s (where $a \in M$), we have $\zeta(t) = \neg\zeta(ta)$.

As before, we now define:

$$P_{A\otimes B} = \left\{ s \in P_A \| P_B \;\middle|\; \begin{array}{l} s \text{ is alternating with respect to } \zeta_{A\otimes B} \\ s \text{ is well formed with respect to } \zeta_{A\otimes B}, \lambda_{A\otimes B} \end{array} \right\}$$
$$P_{A⅋B} = \left\{ s \in P_A \| P_B \;\middle|\; \begin{array}{l} s \text{ is alternating with respect to } \zeta_{A⅋B} \\ s \text{ is well formed with respect to } \zeta_{A⅋B}, \lambda_{A⅋B} \end{array} \right\}$$
$$P_{A\multimap B} = \left\{ s \in P_A \| P_B \;\middle|\; \begin{array}{l} s \text{ is alternating with respect to } \zeta_{A\multimap B} \\ s \text{ is well formed with respect to } \zeta_{A\multimap B}, \lambda_{A\multimap B} \end{array} \right\}$$

Setting $M_{A\otimes B} = M_{A⅋B} = M_{A\multimap B} = M_A \sqcup M_B$, and restricting $\zeta_{A\otimes B}, \zeta_{A⅋B}, \zeta_{A\multimap B}$ to $P_{A\otimes B}, P_{A⅋B}, P_{A\multimap B}$ respectively, we arrive at our definitions of the connectives:

$$\begin{aligned} A \otimes B &= (M_{A\otimes B}, \lambda_{A\otimes B}, \zeta_{A\otimes B}, P_{A\otimes B}) \\ A ⅋ B &= (M_{A⅋B}, \lambda_{A⅋B}, \zeta_{A⅋B}, P_{A⅋B}) \\ A \multimap B &= (M_{A\multimap B}, \lambda_{A\multimap B}, \zeta_{A\multimap B}, P_{A\multimap B}) \end{aligned}$$

Proposition 3.7. *$A\otimes B$, $A ⅋ B$ and $A \multimap B$ are well formed games. Moreover, $P_{A\otimes B}$ is the largest subset $X \subseteq P_A \| P_B$ such that $(M_{A\otimes B}, \lambda_{A\otimes B}, \zeta_{A\otimes B}, X)$ is a well formed game and similarly for the connectives ⅋ and $\multimap$.*

Proof. As before, we prove this proposition for $A \otimes B$ and the other two cases are entirely similar. Alternatively, observe that $A ⅋ B = {}^{\perp}({}^{\perp}A \otimes {}^{\perp}B)$ and that $A \multimap B = {}^{\perp}A ⅋ B$.

Examining the definitions of well formed and alternating sequences, it is immediate that $P_{A\otimes B}$ is prefix-closed. Moreover, every sequence contained in $P_{A\otimes B}$ is well formed, so the well-formedness condition holds. If $s, sa \in P_{A\otimes B}$ then s, sa are both prefixes of sa and so we must have $\zeta_{A\otimes B}(s) = \neg\zeta_{A\otimes B}(sa)$, since sa is alternating.

To show that the limit condition holds, suppose that $\mu < \alpha$ is a limit ordinal and that $s \in (M_A)^{\mu}$ is a sequence of length μ such that $t \in P_{A\otimes B}$ for every proper prefix $t \subsetneqq s$. For each $t \subsetneqq s$, then, we must have $t|_A \in P_A$ and $t|_B \in P_B$.

We claim that $s|_A \in P_A$. Indeed, if $\text{length}(s|_A)$ is a successor ordinal, then $s|_A$ has a last move a. Let t be the prefix of s consisting of all moves up to and including that last move. Then $\text{length}(t)$ is a successor ordinal, so t must be a *proper* prefix of s, and therefore $s|_A = t|_A \in P_A$. If instead $\text{length}(s|_A)$ is a limit ordinal, then for every proper prefix $u \subsetneqq s|_A$, we have some $t \sqsubseteq s$ with $t|_A = u$ (take t to be the prefix consisting of all moves in s that occur in u or that occur before some move in u). Since $s|_A \neq u$, t must be a *proper* prefix of s, and so $t|_A \in P_A$. Since u is arbitrary, the limit condition for A tells us that $s|_A \in P_A$.

By an identical argument, $s|_B \in P_B$, and so $s \in P_A \| P_B$. We claim that s is alternating with respect to $\zeta_{A\otimes B}$. Indeed, if $t, ta \sqsubseteq s$, then t, ta must both be *proper* prefixes of s, since length(s) is a limit ordinal and length(ta) is a successor ordinal. So ta is alternating and therefore $\zeta_{A\otimes B}(t) = \neg\zeta_{A\otimes B}(ta)$.

Lastly, we show that s is well formed with respect to $\lambda_{A\otimes B}, \zeta_{A\otimes B}$. Suppose $ta \sqsubseteq s$. Once again, ta must be a *proper* prefix of s, since it clearly has a different length. Therefore, ta is well formed and we have $\zeta_{A\otimes B}(ta) = \lambda_{A\otimes B}(a)$. Therefore, $s \in P_{A\otimes B}$.

For the second part of the proposition, suppose that $V \subseteq P_A \| P_B$ is prefix closed and satisfies the alternating condition with respect to $\zeta_{A\otimes B}$ and that if $sa \in V$ then $\zeta_{A\otimes B}(sa) = \lambda_{A\otimes B}(a)$. We want to show that $V \subseteq P_{A\otimes B}$. Indeed, if $s \in V$ and $ta \sqsubseteq s$, then $ta \in V$ (since V is prefix closed) and therefore $\zeta_{A\otimes B}(ta) = \lambda_{A\otimes B}(a) = \neg\zeta_{A\otimes B}(s)$. $\square$

As before, we want to show that if player O switches games in the tensor product and that player P switches games in the par and linear implication. Recalling that if A, B are α-games then $A \parr B = {}^{\perp}({}^{\perp}A \otimes {}^{\perp}B)$ and that $A \multimap B = {}^{\perp}A \parr B$, we only need to prove this for the tensor product:

Proposition 3.8. *Let A, B be α-games. Suppose that $s \in P_{A\multimap B}$, $a \in M_A$ and $b \in M_B$. Then:*

i) If $sab \in P_{A\otimes B}$ then $\lambda_{A\otimes B}(b) = O$

ii) If $sba \in P_{A\otimes B}$ then $\lambda_{A\otimes B}(a) = O$.

Proof. Unchanged from the finitary case - see Proposition 2.10 $\square$

3.3 Categories of transfinite games and strategies

We want to build a category $\mathcal{G}(\alpha)$ whose objects are games and whose morphisms are strategies on the linear implication of two games. Given α-games

A, B and C and strategies σ for $A \multimap B$ and τ for $B \multimap C$, we define

$$\sigma \| \tau = \left\{ \mathfrak{s} \in (M_A \sqcup M_B \sqcup M_C)^{*<\alpha\#\alpha} \;:\; \mathfrak{s}|_{A,B} \in \sigma,\ \mathfrak{s}|_{B,C} \in \tau \right\}$$
$$\tau \circ \sigma = \{ \mathfrak{s}|_{A,C} \;:\; \mathfrak{s} \in \sigma \| \tau \}$$

Here, $\alpha\#\alpha$ is intended to be an upper bound and indeed any sufficiently large ordinal will do without changing the elements of $\sigma\|\tau$. The symbol $\#$ denotes the so-called *natural addition* on ordinals: if β, γ are ordinals then the order type of $\beta\#\gamma$ is the largest well-ordering on the disjoint union $\beta \sqcup \gamma$ such that the restrictions to β and γ give the original orderings[1]. For example, we have $\omega\#1 = \omega + 1$, but $1\#\omega = \omega + 1$ as well (and *not* $1 + \omega$, which equals ω). In any case, this is not an important point - we want to allow the sequences $\mathfrak{s}$ to be as long as we need to be, but we need to introduce some bound to make it clear that the resulting collection of plays is a small set. $\alpha\#\alpha$ turns out to be a suitable upper bound.

For the functor $\mathcal{F}\colon \mathcal{G}(\alpha) \to \mathbf{Rel}$, we shall want to show that

$$\tau \circ \sigma = \{ s \in P_{A \multimap C} \;:\; \exists t \in \sigma, u \in \tau \;.\; s|_A = t|_A,\ t|_B = u|_B,\ s|_C = u|_C \}$$

To show this, we will want to show that for any such s, t, u there is some interleaving $\mathfrak{s}$ such that $\mathfrak{s}|_{A,C} = s$, $\mathfrak{s}|_{A,B} = t$ and $\mathfrak{s}|_{B,C} = u$. Even though s, t and u all have length less than α, $\mathfrak{s}$ might have longer length; it will, however, have length less than $\alpha\#\alpha$.

Note that it is not immediately clear that the elements of $\tau \circ \sigma$ have length less than α. We will have to show this. First, though, we will need to restrict our class of games so that we do get a well-defined composition.

3.3.1 Completely negative and almost completely negative α-games

Recall that when defining a finitary game $A = (M_A, \lambda_A, \zeta_A, P_A)$, the function ζ_A was entirely determined by two pieces of information: the function λ_A (to determine $\zeta_A(sa)$ for any non-empty play sa) and the designation of the game as either *positive* or *negative* (to determine $\zeta_A(\epsilon)$). In the $\omega+1$ case, we also needed to specify a function $W_A\colon P_A^\infty \to \{O, P\}$ to get the sign of each ω-play. The general case is similar - once we have specified $\zeta_A(s)$ for any

[1] Alternatively, $\#$ represents the coproduct in the category of well-ordered sets and order-preserving maps.

play s whose length is a limit ordinal, the values of $\zeta_A(t)$ for t a play over a successor ordinal are determined by the alternating condition (alternatively, they are determined automatically by well-formedness).

When building a category of games, we restricted our attention to the negative games. This is necessary in order to avoid technical complications[2]. Of course, these technical complications carry over to the transfinite case, so we need a new negativity condition for our α-games.

Definition 3.9. Let A be an α-game. We say that A is *completely negative* if $\zeta_A(s) = P$ whenever $s \in P_A$ and length(s) is a limit ordinal.

Since the only limiting play in the finitary case is the empty play, we see that the completely negative ω-games are precisely the negative games. But now consider the $\omega + 1$ case. If A is a win-game, then A is completely negative only if it is negative and all infinite plays are P-winning. Recall that we *were* able to build a category of negative win-games without making the requirement that player P wins all infinite plays, so if we insist on complete negativity then we lose much of the richness of our category for no technical benefit. We would be better off with some weaker condition.

Definition 3.10. Let α be an ordinal. If $\alpha = \alpha' + 1$ and A is an α-game, we say that $s \in P_A$ is *length maximal* if it has the largest possible length - i.e., if length$(s) = \alpha'$.

If α is a limit ordinal then there are no length maximal plays in α-games.

Let A be an α-game. We say that A is *almost completely negative* (*acn*) if $\zeta_A(s) = P$ whenever length(s) is a limit ordinal *and s is not length-maximal.*

Note that for most ordinals α there is no difference between being completely negative and being acn. The only ordinals for which there is a difference are those of the form $\mu + 1$, where μ is a limit ordinal.

Our category will have acn α-games as objects and the morphisms from A to B will be partial strategies for $A \multimap B$. Later we will consider the total strategy case.

Proposition 3.11. *Let A, B be acn α-games. Then $A \otimes B$ is acn*

Proof. Let $s \in P_{A \otimes B}$ be a limiting play that is not length maximal. Then at

[2] The category in [AJ13] has both positive and negative games as objects, together with games where either player may start, but it does not admit a faithful functor into **Rel** in the way that we want.

least one of $s|_A$ and $s|_B$ must be a limiting play (otherwise they both have last moves and the later of the two will be the last move in s). Suppose that $s|_A$ and $s|_B$ are both limiting plays. By Lemma 3.6, $\mathrm{length}(s|_A), \mathrm{length}(s|_B) \leq \mathrm{length}(s)$, so neither $s|_A$ nor $s|_B$ is length maximal, since s is not. Therefore, $\zeta_A(s|_A) = \zeta_B(s|_B) = P$, since A and B are acn. Therefore, $\zeta_{A\otimes B}(s) = P$, since $P \wedge P = P$.

Suppose instead that $s|_A$ is a successor play. Then $s|_B$ is a limiting play. Since s is a limiting play, there must be moves from B in s that occur after the last move in $s|_A$. Therefore, $\zeta_A(s|_A) = P$ by Proposition 3.8. Now $s|_B$ is a limiting play and not length maximal, so $\zeta_B(s|_B) = P$ and therefore $\zeta_{A\otimes B}(s) = P$. The case where $s|_B$ is a successor play and $s|_A$ is a limiting play is similar. $\square$

Note, however, that $A \multimap B$ is not necessarily acn, even if A, B are completely negative. This is a direct consequence of the extension of the definitions to ordinals greater than ω. For negative ω-games A, B, the only limiting play is the empty play and its components are both necessarily empty, so they are also limiting plays. Therefore, $A \multimap B$ is a negative game, by the equation $(P \Rightarrow P) = P$.

However, when we move beyond ω, it is no longer true that the components of a limiting play are always limiting plays, so we cannot be sure that they are P-positions, and this means that $A \multimap B$ is not necessarily completely negative, even if A, B are. As an example, we take the completely negative game

$$\perp = \{\{*\}, \{* \mapsto O\}, \{\epsilon \mapsto P,\ * \mapsto O\}, \{\epsilon, *\}\}$$

which has a single O-move with no P-replies. If A is an acn α-game, then $A \multimap \perp$ is the same as the game ${}^{\perp}A$, but with the extra O-move $*$ attached to the start. For ω-games, this extra O-move means that $A \multimap \perp$ is negative if A is negative, but if A contains plays of length greater than ω then $A \multimap \perp$ is not completely negative if A is completely negative: indeed, if $s \in P_A$ is a non-empty limiting play, then we have

$$\zeta_{A\multimap\perp}(*s) = (\zeta_A(s) \Rightarrow \zeta_\perp(*)) = (P \Rightarrow O) = O$$

But $*s$ is a limiting play, so $A \multimap \perp$ is not completely negative.

This will pose a problem for our categorical semantics. We want to build a category where the objects are acn α-games and where the morphisms from a game A to a game B are strategies for $A \multimap B$. We also want the category to be monoidal closed, which means that we would really like $A \multimap B$ to be an object of the category. As we have just seen, this is not the case in general.

We take the following approach: we define a morphism in our category between objects A and B to be a strategy for $A \multimap B$, ignoring the fact that this is not an object of our category. We will get a working definition of the composition of strategies. Then, when we come to define the monoidal closed structure on the category, we will use a slightly modified form of $A \multimap B$ for our linear implication. This modified form will be, in a sense we shall make precise, the smallest modification of $A \multimap B$ that is completely negative, and this will be enough to show that it does indeed give us a monoidal closed structure.

Before moving on to the discussion of composition of strategies, we shall want a version of Proposition 2.11 to record the properties of negative games that make them so useful for our definition of composition.

Proposition 3.12. *a) Let A, B be acn α-games and suppose that $s \in P_A \| P_B$ is alternating with respect to $\zeta_{A\otimes B}$. Then:*

i) If s is not a length maximal limiting play, then either $\zeta_A(s|_A) = P$ or $\zeta_B(s|_B) = P$.

ii) s is well formed with respect to $\zeta_{A\otimes B}, \lambda_{A\otimes B}$ (so $s \in P_{A\otimes B}$).

b) Let A be an acn α-game, **let B be an arbitrary α-game** *and suppose that $s \in P_A \| P_B$ is alternating with respect to $\zeta_{A\multimap B}$. Then:*

i) If s is not a length maximal limiting play, then either $\zeta_A(s|_A) = P$ or $\zeta_B(s|_B) = O$.

ii) s is well formed with respect to $\zeta_{A\multimap B}, \lambda_{A\multimap B}$ (so $s \in P_{A\multimap B}$).

Note that the extra hypothesis on (i) is necessary: suppose $\alpha = \omega+1$ and let Σ^* be as defined above where the unique infinite play is O-winning. Then $\Sigma^* \otimes \Sigma^*$ contains an infinite play s both of whose components are infinite and we have $\zeta_A(s|_A) = \zeta_B(s|_B) = O$.

Proof of Proposition 3.12. (a): By Proposition 3.7, for part (ii) it suffices to show that the set of alternating plays in $P_A \| P_B$ is well-formed with respect to $\zeta_{A\otimes B}, \lambda_{A\otimes B}$ - i.e., that if $s, sx \in P_A \| P_B$ are alternating then $\zeta_{A\otimes B}(sx) = \lambda_{A\otimes B}(x)$.

If s is a limiting play, then one of $s|_A, s|_B$ must be a limiting play. By Lemma 3.6, $\text{length}(s|_A), \text{length}(s|_B) \leq \text{length}(s)$, so $s|_A$ and $s|_B$ are not length maximal if s is not. Therefore, either $\zeta_A(s|_A) = P$ or $\zeta_B(s|_B) = P$, since A and B are both acn.

We may now prove (i) and (ii) for all successor plays by induction on their length, exactly as we did in Proposition 2.11.

(b): Again, by Proposition 3.7, for part (ii) it suffices to show that the set of alternating plays in $P_A \| P_B$ is well-formed with respect to $\zeta_{A\multimap B}, \lambda_{A\multimap B}$ - i.e., that if $s, sx \in P_A \| P_B$ are alternating then $\zeta_{A\multimap B}(sx) = \lambda_{A\otimes B}(x)$.

We prove this by induction on the length of the play s; the successor case is exactly as in Proposition 2.11. Suppose that s is a non-length maximal limiting play. If $s|_A$ is a limiting play, then we are done, since A is acn and $s|_A$ is not length maximal by Lemma 3.6. Otherwise, if $s|_A$ is a successor play, then it has a last move a. Since s is a limiting play, there must be moves from B occurring later in s than a, so a is an O-move in $A \multimap B$ by Proposition 3.8, so it is a P-move in A. $\square$

3.3.2 Interleavings and the functor into Rel

We now set out to extend to the transfinite case some of the results that we proved in the previous section about plays and strategies for the game $A \multimap B$.

Lemma 3.13. *Let A, B be α-games and let σ be a strategy for $A \multimap B$. Suppose that $s, t \in \sigma$ are such that $s|_A = t|_A$ and $s|_B = t|_B$. Then $s = t$.*

Proof. Exactly as in Lemma 2.13: suppose $s \neq t$. By well-ordering, there is an earliest index β such that $s(\beta) \neq t(\beta)$. So we have a sequence r and moves $x \neq y$ such that $rx \sqsubseteq s$ and $ry \sqsubseteq t$. Since $x \neq y$, r must be a P-position by the definition of strategy, and so x, y must take place in the same game. Therefore, either $s|_A \neq t|_A$ or $s|_B \neq t|_B$. $\square$

Given acn α-games A and B and a strategy σ for $A \multimap B$, we may define a relation $\underline{\sigma} \subseteq P_A \times P_B$ by

$$\underline{\sigma} = \{(s|_A, s|_B) : \ s \in \sigma \text{ is a } P\text{-position}\}$$

Once we have our category $\mathcal{G}(\alpha)$ set up, this will allow us to define a functor $\mathcal{F}(\alpha)\colon \mathcal{G}(\alpha) \to \mathbf{Rel}$. Then the following lemma, which is a version of Lemma 2.14, will tell us that this functor is faithful.

Lemma 3.14. *Let A, B be α-games and let σ, σ' be strategies for $A \multimap B$. Suppose that $\underline{\sigma} = \underline{\sigma'}$ as subsets of $P_A \times P_B$. Then $\sigma = \sigma'$.*

Proof. Exactly as in Lemma 2.14. □

We will postpone discussion of the functor $\mathcal{F}(\alpha)\colon \mathcal{G}(\alpha) \to \mathbf{Rel}$ until we have proved the main technical result, which will be the transfinite version of Lemma 2.15.

Lemma 3.15. *Let A, B be acn α-games, let C be an arbitrary α-game, let σ be a strategy for $A \multimap B$ and let τ be a strategy for $B \multimap C$. Suppose that $s \in \sigma$ and $t \in \tau$ are such that $s|_B = t|_B$. Then there exists some ordinal $\beta < \alpha\#\alpha$ such that for all γ with $0 \le \gamma \le \beta$ there is a unique sequence $\mathfrak{s}^\gamma \in \sigma\|\tau$ of length γ such that $\mathfrak{s}^\gamma|_A \sqsubseteq s|_A$ and $\mathfrak{s}\gamma|_C \sqsubseteq t|_C$. Moreover, for this $\mathfrak{s}^\gamma$ we have $\mathfrak{s}^\gamma|_{A,B} \sqsubseteq s$ and $\mathfrak{s}^\gamma \sqsubseteq t$ and if $\delta \le \gamma$ then $\mathfrak{s}^\delta \sqsubseteq \mathfrak{s}^\gamma$. Lastly, we have $\mathfrak{s}^\beta|_{A,B} = s$ and $\mathfrak{s}^\beta|_{B,C} = t$.*

Note that we have no explicit formula for the ordinal β as we did for the the numeral n in Lemma 2.15, owing to the fact that addition of ordinals is less well-behaved than addition of numerals. Nevertheless, this lemma shall be sufficient for our purposes.

Proof of Lemma 3.15. We proceed by induction on γ. At some point we will be unable to proceed further with the induction, so we terminate and the final γ we reach will be our β.

There is a unique sequence of length 0, namely the empty sequence and its A- and C-components are both empty, so are prefixes of s and t.

Now suppose that μ is a limit ordinal and that we have a chain of sequences $\mathfrak{s}^\gamma$ for all $\gamma < \mu$ such that $\mathfrak{s}^\gamma|_{A,B} \sqsubseteq s$ and $\mathfrak{s}^\gamma|_{B,C} \sqsubseteq t$. Then we may take the limit $\mathfrak{s}^\mu$ of these sequences and we have $\mathfrak{s}^\mu|_{A,B} \sqsubseteq s$ and $\mathfrak{s}^\mu|_{B,C} \sqsubseteq t$. For uniqueness, suppose that $\mathfrak{t}^\mu$ has length μ and that $\mathfrak{t}^\mu|_A \sqsubseteq s|_A$ and $\mathfrak{t}^\mu|_C \sqsubseteq t|_C$. Let $\gamma < \mu$, and let $\mathfrak{t}^\gamma$ be the initial prefix of $\mathfrak{t}^\mu$ of length γ. Then $\mathfrak{t}^\gamma|_A \sqsubseteq s|_A$ and $\mathfrak{t}^\gamma|_C \sqsubseteq t|_C$, so $\mathfrak{t}^\gamma = \mathfrak{s}^\gamma$, by uniqueness. Since $\gamma < \mu$ was arbitrary, it follows that $\mathfrak{t}^\mu = \mathfrak{s}^\mu$.

Lastly, suppose that $\gamma = \delta + 1$ for some ordinal δ and suppose that we have already constructed the sequence $\mathfrak{s}^\delta$. If X is an acn α-games, y is an arbitrary

α-game and $r \in P_{X \multimap Y}$ is not length maximal, then Proposition 3.12(i) tells us that either $\zeta_X(r|_X) = P$ or $\zeta_Y(r|_Y) = O$. We have $\mathfrak{s}^\delta|_{A,B} \in P_{A \multimap B}$ and $\mathfrak{s}^\delta|_{B,C} \in P_{B \multimap C}$, so if we assume that $\mathfrak{s}^\delta|_{A,B}$ and $\mathfrak{s}^\delta|_{B,C}$ are not length maximal plays, this gives us four possible 'sign profiles' for the components of the sequence $\mathfrak{s}^\delta$:

Case	$\zeta_A(\mathfrak{s}^\delta\|_A)$	$\zeta_B(\mathfrak{s}^\delta\|_B)$	$\zeta_C(\mathfrak{s}^\delta\|_C)$
A	P	P	P
B	P	P	O
C	P	O	O
D	O	O	O

We now split the argument up according to cases:

Cases A and B: We have $\mathfrak{s}^\delta|_{B,C} \sqsubseteq t$. If in fact $\mathfrak{s}^\delta|_{B,C} = t$, then we terminate and set $\beta = \delta$. We claim that in this case we must have $\mathfrak{s}^\beta|_{A,B} = s$. Certainly, $\mathfrak{s}^\beta|_{A,B} \sqsubseteq s$; if this prefix relation is proper then we must have $\mathfrak{s}^\beta|_{A,B}x \sqsubseteq s$ for some move x. Since $\zeta_A(\mathfrak{s}^\beta|_A) = \zeta_B(\mathfrak{s}^\beta|_B) = P$, x must be an O-move and it must therefore take place in B, which is a contradiction, since $s|_B = t|_B$:

$$\mathfrak{s}^\beta|_B x \sqsubseteq s|_B = t|_B = \mathfrak{s}^\beta|_{B,C}|_B = \mathfrak{s}^\beta|_B \quad \textbf{(contradictory)}$$

Therefore, $\mathfrak{s}^\beta|_{A,B} = s$. We must have $\beta < \alpha\#\alpha$, since we may decompose the set of (distinct) moves in $\mathfrak{s}^k$ as the disjoint union of the set of moves that occur in A and the set of moves that occur in B or C. The induced ordering on each of these sets corresponds with the natural ordering of the moves as a play in P_A or in $P_{B \multimap C}$, and since A and $B \multimap C$ are both α-games, this implies that these two orders are both less than α.

Now suppose instead that $\mathfrak{s}^\delta|_{B,C}$ is a *proper* prefix of t. So $\mathfrak{s}^\delta|_{B,C}x \sqsubseteq t$ for some move x. We set $\mathfrak{s}^{\delta+1} = \mathfrak{s}^\delta x$. Now certainly $\mathfrak{s}^{\delta+1}|_{B,C} \sqsubseteq t$; we claim that $\mathfrak{s}^{\delta+1}|_{A,B} \sqsubseteq s$.

If x is a move from C then $\mathfrak{s}^{\delta+1}|_{A,B} = \mathfrak{s}^\delta|_{A,B} \sqsubseteq s$. So suppose that x is a move from B - so $\mathfrak{s}^{\delta+1}|_{A,B} = \mathfrak{s}^\delta|_{A,B}x$. Since $s|_B = t|_B$, this must be a prefix of s unless there is some move from A that occurs immediately after $\mathfrak{s}^\delta|_{A,B}$. But this is impossible in **Cases A and B**, since $\mathfrak{s}^\delta|_{A,B}$ is a P-position in $A \multimap B$ while $\mathfrak{s}^\delta|_A$ is an O-position in ${}^\perp A$, so any move following $\mathfrak{s}^\delta|_{A,B}$ must be a move in B.

Lastly, we show uniqueness. Suppose that $\mathfrak{t}^{\delta+1} \in \sigma \| \tau$ has length $\delta+1$ and that $\mathfrak{t}|_A \sqsubseteq s|_C$ and $\mathfrak{t}|_C \sqsubseteq t|_C$. Let y be the last move of $\mathfrak{t}^{\delta+1}$ and write $\mathfrak{t}^{\delta+1} = \mathfrak{t}^\delta y$. Then we have

$$\mathfrak{t}^\delta|_A \sqsubseteq \mathfrak{t}^{\delta+1}|_A \sqsubseteq s|_A \quad \mathfrak{t}^\delta|_C \sqsubseteq \mathfrak{t}^{\delta+1}|_C \sqsubseteq t|_C$$

and so $\mathfrak{t}^\delta = \mathfrak{s}^\delta$ by uniqueness for sequences of length δ. We claim that $y = x$. By our argument above, y cannot be a move from A, so it is a move from B or C. Then we have:

$$\mathfrak{s}^\delta y|_{B,C} = \mathfrak{t}^{\delta+1}|_{B,C} \in \tau$$
$$\mathfrak{s}^\delta x|_{B,C} \sqsubseteq t \in \tau$$

In **Case B**, $\mathfrak{s}^\delta|_{B,C}x, \mathfrak{s}^\delta|_{B,C}y$ are both P-plays in $B \multimap C$, so they must be equal, since τ is a strategy. In **Case A**, x and y must both be O-moves in C. So the fact that $\mathfrak{t}^{\delta+1}|_C = \mathfrak{s}^{\delta+1}|_C$ means that $x = y$.

Therefore, $\mathfrak{t}^{\delta+1} = \mathfrak{s}^{\delta+1}$.

Cases C and D: Entirely symmetrical. This time, we get the next move from s instead of t, and uniqueness follows from the fact that σ is a strategy.

We still need to deal with the case that one of $\mathfrak{s}^\delta|_{A,B}, \mathfrak{s}^\delta|_{B,C}$ is length maximal. In that case, the 'sign profile' for the components of $\mathfrak{s}^\delta$ might not fit one of the cases in the table above, since Proposition 3.12(i) only holds for non-length maximal pays. If the components of $\mathfrak{s}^\delta$ do fit **Case A, B, C or D** above, then our existing argument is valid. Otherwise, there are four new cases to consider:

Case	$\zeta_A(\mathfrak{s}^\delta\|_A)$	$\zeta_B(\mathfrak{s}^\delta\|_B)$	$\zeta_C(\mathfrak{s}^\delta\|_C)$	
E	O	P	P	where $\mathfrak{s}^\delta\|_{A,B}$ is a length maximal limiting play
F	O	P	O	
G	P	O	P	where $\mathfrak{s}^\delta\|_{B,C}$ is a length maximal limiting play
H	O	O	P	

In **Cases E and F**, we must have $\mathfrak{s}^\delta|_{A,B} = s$, since it is already length maximal. If we also have $\mathfrak{s}^\delta|_{B,C} = t$ then we set $\beta = \delta$ and terminate. Otherwise, we have $\mathfrak{s}^\delta|_{B,C}x \sqsubseteq t$ for some move x, and we set $\mathfrak{s}^{\delta+1} = \mathfrak{s}^\delta x$. Then we certainly have $\mathfrak{s}^{\delta+1}|_{A,B} \sqsubseteq s$ and $\mathfrak{s}^{\delta+1}|_{B,C} \sqsubseteq t$.

For uniqueness, suppose that $\mathfrak{t}^{\delta+1} \in \sigma \| \tau$ has length $\delta+1$ and that $\mathfrak{t}^{\delta+1}|_A \sqsubseteq s|_A$ and $\mathfrak{t}^{\delta+1}|_C \sqsubseteq t|_C$. As before, the induction hypothesis tells us that $\mathfrak{t}^{\delta+1}$

must be of the form $\mathfrak{s}^\delta y$ for some move y. We claim that x and y must both be moves in C: if x is a move in A or B then we have $\mathfrak{s}^\delta|_{A,B}x \sqsubseteq s$, contradicting length maximality of $\mathfrak{s}^\delta|_{A,B}$. If y is a move in A or B then we have $\mathfrak{t}^{\delta+1}|_{A,B} = \mathfrak{s}^\delta|_{A,B}y \in \sigma$, and this also contradicts length maximality of $\mathfrak{s}^\delta|_{A,B}$. It follows that x and y are both moves in C, so they are identical, since $\mathfrak{s}^{\delta+1}|_C = \mathfrak{t}^{\delta+1}|_C$.

We deal with **Cases G and H** using a symmetrical argument. $\square$

We shall use (the proof of) this lemma to show that composition is well defined for acn α-games. First, though, we shall need to impose some conditions on the ordinal α. To start with, we shall assume that $\alpha = \omega^\beta$ for some β. It is well known that these are precisely the *additively closed* ordinals – so if $\gamma, \delta < \omega^\beta$ then $\gamma + \delta < \omega^\beta$. We shall want a stronger property though – the ordinals of the form ω^β are closed under the natural addition # that we introduced earlier.

To see this, suppose that $\gamma, \delta < \omega^\beta$. We may write γ and δ in their *Cantor normal forms*:

$$\gamma = \omega^{\beta_1}.j_1 + \omega^{\beta_2}.j_2 + \cdots + \omega^{\beta_n}.j_n$$
$$\delta = \omega^{\beta_1}.k_1 + \omega^{\beta_2}.k_2 + \cdots + \omega^{\beta_n}.k_n$$

Here, $\beta_1 > \beta_2 > \cdots > \beta_n$, and $n, j_1, \ldots, j_n, k_1, \ldots, k_n < \omega$ (We allow j_i, k_i to be zero so that we can use the same sequence β_i in both cases.) Then we may write

$$\gamma\#\delta = \omega^{\beta_1}.(j_1 + k_1) + \omega^{\beta_2}.(j_2 + k_2) + \cdots + \omega^{\beta_n}.(j_n + k_n)$$

using the fact that $\omega^\xi + \omega^\eta = \omega^\xi\#\omega^\eta$ if $\xi \geq \eta$ and the fact that the natural addition # is commutative. Since $\gamma, \delta < \omega^\beta$, we must have $\beta_i < \beta$ for each i and therefore $\gamma\#\delta < \omega^\beta$, since ω^β is additively closed.

The reason that this is good for us is that it means that if A, B, C are completely negative ω^β-games, and if σ is a strategy for $A \multimap B$ and τ is a strategy for $B \multimap C$ then any sequence $\mathfrak{s} \in \sigma\|\tau$ must have length less than ω^β. Indeed, we may split $\mathfrak{s}$ into its A, B-component and its C-component – if these have lengths $\gamma, \delta < \omega^\beta$ then the full sequence has length at most $\gamma\#\delta < \omega^\beta$. This allows us to ensure that $\mathfrak{s}|_{A,C}$ has length at most ω^β – so that it is a valid play in $A \multimap C$.

To get an idea of why we need to make some restriction on the ordinal α, consider the situation when $\alpha = \omega^2+1$. We define a game Σ^* as before:

$$M_{\Sigma^*} = \{q, a\}$$
$$\lambda_{\Sigma^*}(q) = O,\ \lambda_{\Sigma^*}(a) = P$$

P_{Σ^*} will be the set of all plays $s \in M_{\Sigma^*}^{*<\omega^2+1}$ such that q is always followed by a, a is always followed by q and a limiting play is always followed by q. We define ζ_{Σ^*} to make Σ^* into a completely negative game.

Let $A = B = C = \Sigma^*$. Consider the sequence of plays $s_0, s_1, \ldots, s_{\omega^2}$ in $A \multimap B$ shown in Figure 2 and the sequence $t_0, t_1, \ldots, t_{\omega+1}$ in $B \multimap C$ shown in Figure 3. These sequences have matching B-components, so Lemma 3.15 tells us that we can interleave them to give a sequence $\mathfrak{s}$. But now $\mathfrak{s}|_{A,C}$ consists in a single move in C followed by a play of length ω^2 in A, followed by a single move in C - in other words, it has length $\omega^2 + 1$, which is too long.

Therefore we shall restrict for now to ordinals α of the form ω^β. This means in particular that there is no distinction between completely negative α-games and acn α-games, so we shall deal with completely negative games for simplicity. Later on, we shall mimic the construction of win-games to construct categories of games over ordinals of the form $\omega^\beta + 1$, but for the reasons outlined above this will not be as simple as changing the ordinal from ω^β to $\omega^\beta + 1$ in our definitions.

Now we have made this restriction on α, the fact that composition is well-defined follows as a corollary of Lemma 3.15.

Corollary 3.16. *Let α be an ordinal of the form ω^β. Let A, B be completely negative α-games, let C be an arbitrary α-game, let σ be a strategy for $A \multimap B$ and let τ be a strategy for $B \multimap C$. Set*

$$\tau \circ \sigma = \{\mathfrak{s}|_{A,C} : \mathfrak{s} \in \sigma \| \tau\}$$

as before. Then $\tau \circ \sigma$ is a strategy for $A \multimap C$.

Proof. We first claim that if $\mathfrak{s} \in \sigma \| \tau$ then $\text{length}(\mathfrak{s}) < \alpha$. Indeed, suppose that $\text{length}(\mathfrak{s}|_{A,B}) = \gamma$ and $\text{length}(\mathfrak{s}|_C) = \delta$. Then $\gamma, \delta < \alpha$, since $\mathfrak{s}|_{A,B} \in P_{A \multimap C}$ and $\mathfrak{s}|_C \in P_C$. We have order-preserving inclusions

$$\gamma \hookrightarrow \text{length}(\mathfrak{s})$$
$$\delta \hookrightarrow \text{length}(\mathfrak{s})$$

δ	A	B	$\zeta_A(s_\delta\|_A)$	$\zeta_B(s_\delta\|_B)$
0			P	P
1		q	P	O
2	q		O	O
3	a		P	O
4	q		O	O
5	a		P	O
$\vdots$	$\vdots$		$\vdots$	$\vdots$
ω			P	O
$\omega+1$		a	P	P
$\omega+2$		q	P	O
$\omega+3$	q		O	O
$\omega+4$	a		P	O
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\omega 2$			P	O
$\omega 2+1$		a	P	P
$\omega 2+2$		q	P	O
$\omega 2+3$	q		O	O
$\omega 2+4$	a		P	O
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\omega 3$			P	O
		$\vdots$		
		$\vdots$		
ω^2			P	P

Figure 2: Diagrammatic representation of a play s of length ω^2 in $A \multimap B = \Sigma^* \multimap \Sigma^*$. Here $\mathfrak{s}_\delta$ is the initial segment of s of length δ. Note that $s|_A$ has length ω^2, while $s|_B$ has length ω, since only finitely many moves from B are played in each ω-segment.

that make the underlying set of length($\mathfrak{s}$) into the disjoint union $\gamma \sqcup \delta$. Then the induced orderings on γ, δ are the same as their original orderings, so we must have length($\mathfrak{s}$) $\leq \gamma \# \delta$. Since α is closed under natural addition $\#$, this means that length($\mathfrak{s}$) $< \alpha$.

Now we see that $\tau \circ \sigma \subseteq P_A \| P_C$: if $\mathfrak{s} \in \sigma \| \tau$, then $\mathfrak{s}|_{A,C}|_A = \mathfrak{s}|_{A,B}|_A \in P_A$ and similarly $\mathfrak{s}|_{A,C}|_C = \mathfrak{s}|_{B,C}|_C \in P_C$. To show that $\tau \circ \sigma \subseteq P_{A \multimap C}$, it suffices by

δ	B	C	$\zeta_B(t_\delta\|_B)$	$\zeta_C(t_\delta\|_C)$
0			P	P
1		q	P	O
2	q		O	O
3	a		P	O
4	q		O	O
5	a		P	O
$\vdots$	$\vdots$		$\vdots$	$\vdots$
ω			P	O
$\omega+1$		a	P	P

Figure 3: Diagrammatic representation of a play t of length $\omega + 1$ in $B \multimap C = \Sigma^* \multimap \Sigma^*$. Compare with Figure 2 and notice that the B-component of this sequence is exactly the same as the B-component in that sequence.

Proposition 3.12(ii) to show that $\tau \circ \sigma$ is alternating with respect to $\zeta_{A \multimap C}$.

Suppose $\mathfrak{s}a \in \sigma \| \tau$ for some move $a \in M_A$. By the proof of Lemma 3.15, we must be in **Case C or D** (there are no length maximal plays, so **Cases G and H** never occur). The following table shows that we must have $\zeta_{A \multimap B}(\mathfrak{s}|_{A,B}) = \zeta_{A \multimap C}(\mathfrak{s}|_{A,C})$:

Case	$\zeta_A(\mathfrak{s}\|_A)$	$\zeta_B(\mathfrak{s}\|_B)$	$\zeta_C(\mathfrak{s}\|_C)$	$\zeta_{A \multimap B}(\mathfrak{s}\|_{A,B})$	$\zeta_{A \multimap C}(\mathfrak{s}\|_{A,C})$
C	P	O	O	O	O
D	O	O	O	P	P

Then we get:

$$\lambda_{A \multimap C}(a) = \lambda_{A \multimap B}(a) = \neg \zeta_{A \multimap B}(\mathfrak{s}) = \neg \zeta_{A \multimap C}(\mathfrak{s})$$

as desired. The case when we have $\mathfrak{s}c \in \tau \| \sigma$ for some $c \in M_C$ is entirely symmetrical, working in **Cases A and B** this time. Therefore, $\tau \circ \sigma \subseteq P_{A \multimap C}$.

We now need to show that the two strategy conditions are satisfied. Let $\mathfrak{s} \in \sigma \| \tau$ and suppose that $\mathfrak{s}|_{A,C}$ is a P-position and that $a \in M_A$ is an O-move in A such that $\mathfrak{s}|_{A,C}a \in P_{A \multimap C}$. We want to show that $\mathfrak{s}|_{A,C}a \in \tau \circ \sigma$. Since $\mathfrak{s}|_{A,C}$ is a P-position, we must be in **Case A or Case D**. In that case, $\mathfrak{s}|_{A,B}$ is a P-position and we know that $\mathfrak{s}|_A a \in P_A$, so $\mathfrak{s}|_{A,B}a \in P_{A \multimap B}$. Since σ is a strategy, that implies that $\mathfrak{s}|_{A,B}a \in \sigma$ and therefore $\mathfrak{s}a \in \sigma \| \tau$. So we have $\mathfrak{s}|_{A,C}a = \mathfrak{s}a|_{A,C} \in \tau \circ \sigma$.

A symmetrical argument shows that if $c \in M_C$ is an O-move in C and $\mathfrak{s}|_{A,C}c \in P_{A \multimap C}$ then $\mathfrak{s}|_{A,C}c \in \tau \circ \sigma$.

Lastly, suppose that we have $s \in P_{A \multimap C}$ is an O-position and that $x, y \in M_{A \multimap C}$ are P-moves such that $sx, sy \in \tau \circ \sigma$. We need to show that $x = y$. By the definition of $\tau \circ \sigma$, there are sequences $\mathfrak{s}, \mathfrak{t} \in \sigma \| \tau$ with $\mathfrak{s}|_{A,C} = sx$ and $\mathfrak{t}|_{A,C} = sy$. We may write:

$$\begin{aligned} \mathfrak{s} &= \mathfrak{s}'x \\ \mathfrak{t} &= \mathfrak{t}'y \end{aligned}$$

Without loss of generality, assume that length$(\mathfrak{s}') \leq$ length$(\mathfrak{t}')$ and write $\mathfrak{t}' = \mathfrak{t}''\mathfrak{r}$, where $\mathfrak{t}''$ has the same length as $\mathfrak{s}'$. Then $\mathfrak{t}''|_A \sqsubseteq \mathfrak{s}'|_A$ and $\mathfrak{t}''|_C \sqsubseteq \mathfrak{s}'|_C$, so $\mathfrak{t}'' = \mathfrak{s}'$ by Lemma 3.15.

We claim that $\mathfrak{r} = \epsilon$. Indeed, suppose $\mathfrak{r}$ is non-empty. We have

$$\mathfrak{t}''\mathfrak{r}|_{A,C} = \mathfrak{t}'|_{A,C} = s = \mathfrak{s}'|_{A,C} = \mathfrak{t}''|_{A,C}$$

and so $\mathfrak{r}$ must be entirely composed of moves from B. Let b be the first move in $\mathfrak{r}$. Then $\mathfrak{s}'x$ and $\mathfrak{s}'b$ have the same length and we have $\mathfrak{s}'b|_A \sqsubseteq \mathfrak{s}'x|_A$ and $\mathfrak{s}'b|_C \sqsubseteq \mathfrak{s}'x|_C$, so $b = a$, which is impossible.

Therefore, we must have $\mathfrak{s}'x, \mathfrak{s}'y \in \sigma \| \tau$. Since x, y are P-moves in $A \multimap C$, each must be either an O-move in A or a P-move in C. It follows that either $\mathfrak{s}'$ is in **Case B** and x and y are both moves in C or that $\mathfrak{s}'$ is in **Case C** and x and y areboth moves in A. Suppose $\mathfrak{s}'$ is in **Case B** and x and y are moves in C. Then $\mathfrak{s}'|_{B,C}x, \mathfrak{s}'|_{B,C} \in \tau$ and so $x = y$, since they are P-moves and τ is a strategy. Similarly, if $\mathfrak{s}'$ is in **Case C** and x and y are moves in A, we have $\mathfrak{s}'|_{A,B}x, \mathfrak{s}'|_{A,B}y \in \sigma$ and so $x = y$ since σ is a strategy. $\square$

As a second corollary of Lemma 3.15, we show that the relational presentation of games and strategies respects composition:

Corollary 3.17. *Let $\alpha = \omega^\beta$ for some β and let A, B, C be completely negative α-games. Let σ be a strategy for $A \multimap B$ and let τ be a strategy for $B \multimap C$. Then we have:*

$$\underline{\tau \circ \sigma} = \underline{\tau} \circ \underline{\sigma}$$

Proof. We have

$$\begin{aligned} \underline{\tau \circ \sigma} &= \{(s|_A, s|_C) \,:\, s \in \tau \circ \sigma \text{ is a } P\text{-play}\} \\ \underline{\tau} \circ \underline{\sigma} &= \{(s,t) \in P_A \times P_C \,:\, \exists u \in P_B \,.\, (s,u) \in \underline{\tau},\ (u,t) \in \underline{\sigma}\} \end{aligned}$$

Suppose $s \in \tau \circ \sigma$ is a P-play and write $s = \mathfrak{s}|_{A,C}$, where $\mathfrak{s} \in \sigma \| \tau$. Then we have $\mathfrak{s}|_{A,B} \in \sigma$ and $\mathfrak{s}|_{B,C} \in \tau$. Since $s = \mathfrak{s}|_{A,C}$ is a P-play, $\mathfrak{s}$ must be in **Case A or Case D**, so $\mathfrak{s}|_{A,B}$ and $\mathfrak{s}|_{B,C}$ are both P-plays. Therefore, we have

$$(s|_A, \mathfrak{s}|_B) \in \underline{\sigma}$$
$$(\mathfrak{s}|_B, s|_C) \in \underline{\tau}$$

and therefore $(s|_A, s|_C) \in \underline{\tau} \circ \underline{\sigma}$. Conversely, suppose $(s, t) \in \underline{\tau} \circ \underline{\sigma}$. Then there exist $S \in \sigma, T \in \tau$ such that $S|_A = s$, $S|_B = T|_B$ and $T|_C = t$. By Lemma 3.15, there is some $\mathfrak{s} \in \sigma \| \tau$ such that $\mathfrak{s}|_A = s$ and $\mathfrak{s}|_C = t$. So $(s, t) \in \underline{\tau \circ \sigma}$. □

Corollary 3.18. *Let $\alpha = \omega^\beta$ for some β. Then we can define a category $\mathcal{G}(\alpha)$ as follows - the objects are completely negative α-games, and if A, B are two objects then a morphism from A to B is given by a strategy for $A \multimap B$. We compose strategies as above. If A is an object, then the identity morphism on A is given by*

$$\mathrm{id}_A = \left\{ s \in P_{A \multimap A} : \textit{ for all } P\textit{-positions } t \sqsubseteq s,\ t|_A = t|_{({}^\perp A)} \right\}$$

Proof. We need to check that this is indeed an identity and that composition is associative. We do this using the relational presentation exactly as in Theorem 2.17. □

3.3.3 Symmetric monoidal structure

In this section, we shall fix an ordinal $\alpha = \omega^\beta$. The tensor product $\otimes$ on α-games makes the category $\mathcal{G}(\alpha)$ into a monoidal category. If A, B are objects of $\mathcal{G}(\alpha)$, then Proposition 3.11 tells us that $A \otimes B$ is an object of $\mathcal{G}(\alpha)$. In fact, $\otimes$ gives us a functor $\mathcal{G}(\alpha) \times \mathcal{G}(\alpha) \to \mathcal{G}(\alpha)$ as follows: let A, A', B, B' be completely negative α-games, let σ be a strategy for $A' \multimap A$ and let τ be a strategy for $B' \multimap B$. Then we define a strategy $\sigma \otimes \tau$ for $(A' \otimes B') \multimap (A \otimes B)$ as follows:

$$\sigma \otimes \tau = \{ s \in P_{(A' \otimes B') \multimap (A \otimes B)} : s|_{A',A} \in \sigma,\ s|_{B',B} \in \tau \}$$

To show that this is indeed a strategy for $(A' \otimes B') \multimap (A \otimes B)$, suppose that $s \in \sigma \otimes \tau$ is a P-position and that $sx \in P_{(A' \otimes B') \multimap (A \otimes B)}$. We need to show that $sx|_{A',A} \in \sigma$ and that $sx|_{B',B} \in \tau$.

We claim that $s|_{A',A}$ and $s|_{B',B}$ are P-positions in $A' \multimap A$ and $B' \multimap B$. Indeed, suppose without loss of generality that $s|_{A',A}$ is an O-position in $A' \multimap A$. We will show that s is an O-position in $(A' \otimes B') \multimap (A \otimes B)$, contradicting our assumptions.

If $s|_{A',A}$ is an O-position, then $s|_{A'}$ is a P-position in A' and $s|_A$ is an O-position in A. Now, $s|_{A,B}$ is a play in $A \otimes B$, so Proposition 3.12(i) tells us that $s|_B$ must be a P-position in B.

Now $s|_{B',B}$ is a play in $\tau \subseteq P_{B' \multimap B}$, so Proposition 3.12(i) now tells us that $s|_{B'}$ is a P-position in B'. It follows that $s|_{A',B'}$ is a P-position in $A' \otimes B'$ and that $s|_{A,B}$ is an O-position in $A \otimes B$, so s is an O-position in $(A' \otimes B') \multimap (A \otimes B)$, which is the desired contradiction.

Without loss of generality, suppose that the move x occurs in A' or A. Since $s|_{A',A}$ is a P-position, we must have $sx|_{A',A} \in \sigma$, by the definition of a strategy, and then $sx|_{B',B} = s|_{B',B} \in \tau$. So $sx \in \sigma \otimes \tau$.

For the second strategy condition, suppose that $sx, sy \in \sigma \otimes \tau$, where s is an O-position in $(A' \otimes B') \multimap (A \otimes B)$ and x, y are P-moves. Then $s|_{A',B'}$ is a P-position in $A' \otimes B'$ and $s|_{A,B}$ is an O-position in $A \otimes B$.

This immediately tells us that $s|_{A'}$ is a P-position in A' and that $s|_{B'}$ is a P-position in B'. Moreover, by Proposition 3.12(i), we know that either $s|_A$ is a P-position in A and $s|_B$ is an O-position in B or that $s|_A$ is an O-position in A and $s|_B$ is a P-position in B.

That is, we find ourselves in one of the following cases:

Case	$\zeta_{A'}(s\vert_{A'})$	$\zeta_{B'}(s\vert_{B'})$	$\zeta_A(s\vert_A)$	$\zeta_B(s\vert_B)$
1	P	P	P	O
2	P	P	O	P

Suppose we are in **Case 1**. Then x cannot be a move in A or A': indeed, it is a P-move in $(A' \otimes B') \multimap (A \otimes B)$, so it would be a P-move in $A' \multimap A$. But $s|_{A',A}$ is a P-position in $A' \multimap A$. Similarly, y cannot be a move in A or A'. Therefore, x and y are both P-moves in $B \multimap B'$, and then the fact that τ is a strategy implies that $x = x'$.

In **Case 2**, we can use a symmetrical argument, this time using the fact that σ is a strategy.

It is easy to show that this tensor action on strategies respects composition:

indeed, if we have morphisms

$$A'' \xrightarrow{\sigma'} A' \xrightarrow{\sigma} A$$
$$B'' \xrightarrow{\tau'} B' \xrightarrow{\tau} B$$

then we have

$$(\sigma \circ \sigma') \otimes (\tau \circ \tau') = \left\{ s \in P_{(A'' \otimes B'') \multimap (A \otimes B)} \ : \ s|_{A'',A} \in \sigma \circ \sigma',\ s|_{B'',B} \in \tau \circ \tau' \right\}$$

Meanwhile:

$$(\sigma' \otimes \tau') \| (\sigma \otimes \tau) = \left\{ \mathfrak{s} \in (M_{A'' \otimes B''} \sqcup M_{A' \otimes B'} \sqcup M_{A \otimes B})^{*<\alpha} \ \middle| \ \begin{array}{c} \mathfrak{s}|_{A'' \otimes B'', A' \otimes B'} \in \sigma' \otimes \tau' \\ \mathfrak{s}|_{A' \otimes B', A \otimes B} \ \in \ \sigma \otimes \tau \end{array} \right\}$$

$$= \left\{ \mathfrak{s} \in (M_{A'' \otimes B''} \sqcup M_{A' \otimes B'} \sqcup M_{A \otimes B})^{*<\alpha} \ \middle| \ \begin{array}{c} \mathfrak{s}|_{A'',A'} \in \sigma' \\ \mathfrak{s}|_{A',A} \ \in \ \sigma \\ \mathfrak{s}|_{B'',B'} \in \tau' \\ \mathfrak{s}|_{B',B} \ \in \ \tau \end{array} \right\}$$

$$= \left\{ \mathfrak{s} \in (M_{A'' \otimes B''} \sqcup M_{A' \otimes B'} \sqcup M_{A \otimes B})^{*<\alpha} \ \middle| \ \begin{array}{c} \mathfrak{s}|_{A'',A',A} \in \sigma' \| \sigma \\ \mathfrak{s}|_{B'',B',B} \in \tau' \| \tau \end{array} \right\}$$

It follows that

$$(\sigma \otimes \tau) \circ (\sigma' \otimes \tau') = \left\{ \mathfrak{s}|_{A'' \otimes B'', A \otimes B} \ : \ \mathfrak{s} \in (\sigma' \otimes \tau') \| (\sigma \otimes \tau) \right\} = (\sigma \circ \sigma') \otimes (\tau \circ \tau')$$

To define the associators and unitors (which are the obvious choices) for our tensor product, we introduce a bit more technology. Let A', A be α-games. We call a bijective function $f \colon P_{A'} \to P_A$ a *tree isomorphism* if f preserves length and the prefix ordering and commutes with $\zeta_A, \zeta_{A'}$. If A, A' are completely negative, then we have a strategy σ_f for $A' \multimap A$ given by

$$\sigma_f = \{ s \in P_{A' \multimap A} \ : \ \text{for all } P\text{-positions } t \sqsubseteq s, \text{ we have } t|_A = f(t|_{A'}) \}$$

We check that σ_f is indeed a strategy. It is clearly prefix-closed and satisfies the limit condition by the definition. If $s \in \sigma_f$ is a P-position, then we must have $sx \in \sigma_f$ for all $sx \in P_{A' \multimap A}$. Now suppose that $sx, sy \in \sigma_f$. We first claim that x and y take place in the same game. Indeed, suppose instead that x is a move in A' and y is a move in A. Then we have

$$s|_A = f(s|_{A'} x)$$
$$s|_A y = f(s|_{A'})$$

But this is impossible, since f preserves lengths. Therefore, x and y take place in the same game. If they are moves in A, we have

$$s|_A x = f(s|_{A'}) = s|_A y$$

and so $x = y$, and if they are moves in A', we have

$$s|_{A'} x = f^{-1}(s|_A) = s|_{A'} y$$

and so $x = y$ again.

We call a strategy σ_f arising from a tree isomorphism f a *copycat strategy*. The relational presentation $\overline{\sigma_f}$ of such a strategy σ_f is the graph of the function $f\colon P_{A'} \to P_A$, and therefore the function f is determined uniquely by the strategy f. Moreover, if we have completely negative α-games A'', A', A and tree isomorphisms $A'' \xrightarrow{f'} A' \xrightarrow{f} A$, then we must have

$$\sigma_f \circ \sigma_{f'} = \sigma_{f \circ f'}$$

We therefore have a full-on-objects subcategory $\mathcal{G}_{\mathrm{cc}}(\alpha)$ of $\mathcal{G}(\alpha)$ of games and copycat strategies that comes equipped with a natural functor $\mathcal{H}\colon \mathcal{G}_{\mathrm{cc}}(\alpha) \to$ **Set** that sends a game A to P_A and a copycat strategy σ_f to the function f. We get a commutative square in **CAT**:

$$\begin{tikzcd}
\mathcal{G}_{\mathrm{cc}}(\alpha) \arrow[r, hook] \arrow[d, "\mathcal{H}"'] & \mathcal{G}(\alpha) \arrow[d, "\mathcal{F}"] \\
\mathbf{Set} \arrow[r, hook] & \mathbf{Rel}
\end{tikzcd}$$

where all the functors in the diagram are faithful.

Given sets X, Y and a function $\phi\colon X \to Y$, we write $\overline{\phi}\colon X^{*<\alpha} \to Y^{*<\alpha}$ for the induced map that acts pointwise via ϕ. Observe that disjoint union induces the structure of a symmetric monoidal category on **Set** with associators, unitors and braiding:

$$\begin{aligned}
\mathrm{assoc}_{X,Y,Z}\colon (X \sqcup Y) \sqcup Z &\to X \sqcup (Y \sqcup Z) \\
\mathrm{lunit}_X\colon \varnothing \sqcup X &\to X \\
\mathrm{runit}_X\colon X \sqcup \varnothing &\to X \\
\mathrm{sym}_{X,Y}\colon X \sqcup Y &\to Y \sqcup X
\end{aligned}$$

Suppose A, B, C are completely negative α-games. We have a function

$$\mathrm{assoc}_{M_A,M_B,M_C}\colon M_{(A\otimes B)\otimes C} \to M_{A\otimes(B\otimes C)}$$

and it is clear from the definition that this gives rise to a function

$$\overline{\mathrm{assoc}_{M_A,M_B,M_C}} \colon P_{(A\otimes B)\otimes C} \to P_{A\otimes(B\otimes C)}$$

The tensor unit in our category will be the empty game $I = (\varnothing, \varnothing, \varnothing, \{\epsilon\})$. If A is a completely negative α-game, we get natural maps

$$\mathrm{lunit}_{M_A} \colon M_{I\otimes A} \to M_A$$
$$\mathrm{runit}_{M_A} \colon M_{A\otimes I} \to M_A$$

and these give rise to maps

$$\overline{\mathrm{lunit}_{M_A}} \colon P_{I\otimes A} \to P_A$$
$$\overline{\mathrm{runit}_{M_A}} \colon P_{A\otimes I} \to P_A$$

Lastly, we have a map

$$\mathrm{sym}_{M_A,M_B} \colon M_{A\otimes B} \to M_{B\otimes A}$$

inducing a map

$$\overline{\mathrm{sym}_{M_A,M_B}} \colon P_{A\otimes B} \to P_{B\otimes A}$$

It is clear that these maps on sets of plays are all tree isomorphisms, so they give rise to isomorphisms in $\mathcal{G}(\alpha)$:

$$\mathrm{assoc}_{A,B,C} \colon (A\otimes B)\otimes C \to A\otimes(B\otimes B)$$
$$\mathrm{lunit}_A \colon I\otimes A \to A$$
$$\mathrm{runit}_A \colon A\otimes I \to A$$
$$\mathrm{sym}_{A,B} \colon A\otimes B \to B\otimes A$$

Since the original isomorphisms in **Set** satisfy the pentagon, triangle and hexagon identities, and since all the functors we are considering are faithful, our isomorphisms in $\mathcal{G}(\alpha)$ must satisfy the pentagon, triangle and hexagon identities as well. We give the argument in the case of the pentagon identity.

Given completely negative α-games A, B, C, D, we have the following commutative diagram in **Set**:

$$
\begin{tikzcd}
& M_{(A\otimes B)\otimes(C\otimes D)} \arrow[dr, "\mathrm{assoc}_{M_A,M_B,M_{C\otimes D}}"] & \\
M_{((A\otimes B)\otimes C)\otimes D} \arrow[ur, "\mathrm{assoc}_{M_{A\otimes B},M_C,M_D}"] \arrow[d, "\mathrm{assoc}_{M_A,M_B,M_C}\otimes\mathrm{id}_D"'] & & M_{A\otimes(B\otimes(C\otimes D))} \\
M_{(A\otimes(B\otimes C))\otimes D} \arrow[rr, "\mathrm{assoc}_{M_A,M_{B\otimes C},M_D}"'] & & M_{A\otimes((B\otimes C)\otimes D)} \arrow[u, "\mathrm{id}_A\otimes\mathrm{assoc}_{M_B,M_C,M_D}"']
\end{tikzcd}
$$

which immediately gives rise to a second commutative diagram in **Set**:

$$
\begin{tikzcd}
& P_{(A\otimes B)\otimes(C\otimes D)} \arrow[dr, "\overline{\mathrm{assoc}_{M_A,M_B,M_{C\otimes D}}}"] & \\
P_{((A\otimes B)\otimes C)\otimes D} \arrow[ur, "\overline{\mathrm{assoc}_{M_{A\otimes B},M_C,M_D}}"] \arrow[d, "\overline{\mathrm{assoc}_{M_A,M_B,M_C}\otimes\mathrm{id}_D}"'] & & P_{A\otimes(B\otimes(C\otimes D))} \\
P_{(A\otimes(B\otimes C))\otimes D} \arrow[rr, "\overline{\mathrm{assoc}_{M_A,M_{B\otimes C},M_D}}"'] & & P_{A\otimes((B\otimes C)\otimes D)} \arrow[u, "\overline{\mathrm{id}_A\otimes\mathrm{assoc}_{M_B,M_C,M_D}}"']
\end{tikzcd}
$$

But this is the image under the functor $\mathcal{H}$ of the following diagram in $\mathcal{G}_{\mathrm{cc}}(\alpha)$:

$$
\begin{tikzcd}
& (A\otimes B)\otimes(C\otimes D) \arrow[dr, "\mathrm{assoc}_{A,B,C\otimes D}"] & \\
((A\otimes B)\otimes C)\otimes D \arrow[ur, "\mathrm{assoc}_{A\otimes B,C,D}"] \arrow[d, "\mathrm{assoc}_{A,B,C}\otimes\mathrm{id}_D"'] & & A\otimes(B\otimes(C\otimes D)) \\
(A\otimes(B\otimes C))\otimes D \arrow[rr, "\mathrm{assoc}_{A,B\otimes C,D}"'] & & A\otimes((B\otimes C)\otimes D) \arrow[u, "\mathrm{id}_A\otimes\mathrm{assoc}_{B,C,D}"']
\end{tikzcd}
$$

Since $\mathcal{H}$ is a faithful functor and the second diagram commutes, this last diagram must commute as well.

It remains to show that $\mathrm{assoc}_{A,B,C}, \mathrm{lunit}_A, \mathrm{runit}_A, \mathrm{sym}_{A,B}$ are natural transformations. We show this for $\mathrm{assoc}_{A,B,C}$ and the other cases are entirely similar.

Showing that $\mathrm{assoc}_{A,B,C}$ is a natural transformation from $((\_ \otimes \_) \otimes \_)$ to $(\_ \otimes (\_ \otimes \_)\colon \mathcal{G}(\alpha) \times \mathcal{G}(\alpha) \times \mathcal{G}(\alpha) \to \mathcal{G}(\alpha)$ means showing that the following diagram commutes for all completely negative α-games A', B', C', A, B, C and all morphisms $\sigma\colon A' \to A, \tau\colon B' \to B, \upsilon\colon C' \to C$:

$$\begin{tikzcd}
(A' \otimes B') \otimes C' \arrow[r, "(\sigma\otimes\tau)\otimes\upsilon"'] \arrow[d, "\mathrm{assoc}_{A',B',C'}"'] & (A \otimes B) \otimes C \arrow[d, "\mathrm{assoc}_{A,B,C}"] \\
A' \otimes (B' \otimes C') \arrow[r, "\sigma\otimes(\tau\otimes\upsilon)"'] & A \otimes (B \otimes C)
\end{tikzcd}$$

It is easy to check that this diagram commutes by applying the functor $\mathcal{F}\colon \mathcal{G}(\alpha) \to \mathbf{Rel}$. The relational content $\underline{\mathrm{assoc}_{A,B,C}}$ of the strategy $\mathrm{assoc}_{A,B,C}$ is the graph of the function

$$\overline{\mathrm{assoc}}_{M_A,M_B,M_C}\colon P_{(A\otimes B)\otimes C} \to P_{A\otimes(B\otimes C)}$$

and its action by left composition on $\underline{(\sigma \otimes \tau) \otimes \upsilon} \subseteq P_{(A'\otimes B')\otimes C'} \times P_{(A\otimes B)\otimes C}$ is pointwise application of this function. We find that

$$\begin{aligned}
\underline{\mathrm{assoc}_{A,B,C} \circ (\sigma \otimes \tau) \otimes \upsilon} &= \underline{\mathrm{assoc}_{A,B,C}} \circ \underline{(\sigma \otimes \tau) \otimes \upsilon} \\
&= \left\{ (s,t) \in P_{(A'\otimes B')\otimes C'} \times P_{A\otimes(B\otimes C)} \;\middle|\; \begin{array}{l} (s|_{A'}, t|_A) \in \underline{\sigma} \\ (s|_{B'}, t|_B) \in \underline{\tau} \\ (s|_{C'}, t|_C) \in \underline{\upsilon} \end{array} \right\}
\end{aligned}$$

A similar argument now shows that $\underline{\sigma \otimes (\tau \otimes \upsilon) \circ \mathrm{assoc}_{A',B',C'}}$ is equal to the same set, and so we have a commutative diagram in **Rel**:

$$\begin{tikzcd}
P_{(A'\otimes B')\otimes C'} \arrow[r, "\underline{(\sigma\otimes\tau)\otimes\upsilon}"'] \arrow[d, "\underline{\mathrm{assoc}_{A',B',C'}}"'] & P_{(A\otimes B)\otimes C} \arrow[d, "\underline{\mathrm{assoc}_{A,B,C}}"] \\
P_{A'\otimes(B'\otimes C')} \arrow[r, "\underline{\sigma\otimes(\tau\otimes\upsilon)}"'] & P_{A\otimes(B\otimes C)}
\end{tikzcd}$$

Therefore, our original diagram in $\mathcal{G}(\alpha)$ is commutative, since the functor $\mathcal{F}$ is faithful.

We therefore have the structure of a symmetric monoidal category on $\mathcal{G}(\alpha)$.

3.3.4 Symmetric monoidal closed structure

In existing finitary models of game semantics, the category of games is made into a symmetric monoidal closed category where the linear implication of two objects is given by the connective $\multimap$. As we remarked earlier, this approach will not work immediately for us, since $A \multimap B$ need not be completely negative even if A and B are. Our solution will be to modify $A \multimap B$ so that it becomes completely negative.

Remember that when we proved that the composition of strategies $\sigma\colon A \multimap B$ and $\tau\colon B \multimap C$ is well defined, we were careful not to require that C be acn (or completely negative). From the point of view of constructing a category whose objects are α-games, this is not useful, but it does allow us to construct certain alternative categories.

Definition 3.19. Recall that if $\mathcal{C}$ is a category and A is an object of $\mathcal{C}$, then the *slice category* $\mathcal{C}/A$ is the category where the objects are morphisms

$$B \xrightarrow{f} A$$

where B is an object of $\mathcal{C}$ and where the morphisms between an object $B \xrightarrow{f} A$ and an object $C \xrightarrow{g} A$ are commutative triangles:

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "h"'] & A \\ C \arrow[ur, "g"] & \end{tikzcd}$$

Let A be a (not necessarily completely negative) α-game. Abusing notation slightly, we write $\mathcal{G}(\alpha)/A$ for the category whose objects are strategies

$$\sigma\colon B \multimap A$$

for B a *completely negative* α-game and where the morphisms from an object $B \xrightarrow{\sigma} A$ to an object $C \xrightarrow{\tau} A$ are commutative triangles of the form

$$\begin{tikzcd} B \arrow[r, "\sigma"] \arrow[d, "\upsilon"'] & A \\ C \arrow[ur, "\tau"] & \end{tikzcd}$$

The main technical result we shall use to construct our closed monoidal structure is the following theorem:

Theorem 3.20. *Let A be an α-game. Then the category $\mathcal{G}(\alpha)/A$ has a terminal object.*

Note that if A is completely negative, then the terminal object of $\mathcal{G}(\alpha)/A$ is the identity strategy $A \xrightarrow{\mathrm{id}_A} A$, as is standard for slice categories. If A is not completely negative, then we will need to use a different construction.

Definition 3.21. Let A be an α-game. We form a completely negative α-game A^{cn} as follows:

- $M_{A^{\mathrm{cn}}} = M_A \sqcup \{*_t \,:\, t \in P_A \text{ is a limiting } O\text{-play}\}$
- $\lambda_{A^{\mathrm{cn}}}(x) = \lambda_A(x)$ for all $x \in M_A$. $\lambda_{A^{\mathrm{cn}}}(*_s) = O$
- $\zeta_{A^{\mathrm{cn}}}(sx) = \lambda_{A^{\mathrm{cn}}}(x)$ for $sx \in P_{A^{\mathrm{cn}}}$ a successor play.
- $\zeta_{A^{\mathrm{cn}}}(s) = P$ for $s \in P_{A^{\mathrm{cn}}}$ a limiting play.

$$P_{A^{\mathrm{cn}}} = \left\{ s \in (M_{A^{\mathrm{cn}}})^{*<\alpha} \;\middle|\; \begin{array}{l} t|_{M_A} \in P_A \\ \text{If } u*_t \sqsubseteq s \text{ then } u = t. \\ \text{If } vx \sqsubseteq s \text{ and } v|_{M_A} \text{ is a limiting } O\text{-play in} \\ P_A \text{ then } x = *_v \end{array} \right\}$$

In other words, we force A to be completely negative by making all limiting plays be P-plays. If u was a limiting O-play in A, then it becomes a P-play in A^{cn}. Since the moves that follow u in A are P-moves, we add in an extra O-move $*_u$ in order to preserve the alternating condition.

Example 3.22. If $\alpha = \omega$ and A is a negative game then we have $\left({}^{\perp}A\right)^{\mathrm{cn}} \cong A \multimap \perp$: the single move in $\perp$ takes the role of $*_\epsilon$. However, if α is a higher ordinal, then this does not hold, since $A \multimap \perp$ is no longer a completely negative game.

If A, B are completely negative α-games, we define $A \overset{\mathrm{cn}}{\multimap} B$ to stand for $(A \multimap B)^{\mathrm{cn}}$. Then we do have $\left({}^{\perp}A\right)^{\mathrm{cn}} \cong A \overset{\mathrm{cn}}{\multimap} \perp$. The object $A \overset{\mathrm{cn}}{\multimap} B$ will be the linear implication in our category.

Given an α-game A, there is a natural strategy cn_A for $A^{\mathrm{cn}} \multimap A$ given by

$$\mathrm{cn}_A = \{s \in P_{A^{\mathrm{cn}} \multimap A} \,:\, \text{for all } P\text{-positions } t \sqsubseteq s,\ t|_{A^{\mathrm{cn}}}|_{M_A} = t|_A\}$$

Concretely, this strategy is the copycat strategy for A, with the difference that whenever we reach a limiting O-play u in A, player P plays the extra move $*_u$ in A^{cn} before continuing.

We claim that $A^{\mathrm{cn}} \xrightarrow{\mathrm{cn}_A} A$ is the terminal object in the category $\mathcal{G}(\alpha)/A$. To do that, it will suffice to prove the following proposition:

Proposition 3.23. cn_A *is a total strategy for* $A^{\mathrm{cn}} \multimap A$*. Moreover,* cn_A *satisfies the following universal property: given a completely negative* α*-game* B *and a strategy* σ *for* $B \multimap A$*, there is a unique strategy* σ^{cn} *for* $B \multimap A^{\mathrm{cn}}$ *such that the following diagram commutes:*

$$\begin{tikzcd} B \arrow[r, "\sigma^{\mathrm{cn}}"] \arrow[dr, "\sigma"'] & A^{\mathrm{cn}} \arrow[d, "\mathrm{cn}_A"] \\ & A \end{tikzcd}$$

Proof. We first show that cn_A is a valid strategy for $A^{\mathrm{cn}} \multimap A$. It is certainly prefix closed and satisfies the limiting condition by its definition. Moreover, if $s \in \mathrm{cn}_A$ is a P-position and $sx \in P_{A^{\mathrm{cn}} \multimap A}$ then sx satisfies the conditions to be an element of cn_A.

Suppose that $sx, sy \in \mathrm{cn}_A$ are P-positions. Suppose first that neither x nor y is a move of the form $*_t$. We claim that sx, sy take place in the same game. Indeed, suppose instead that x is a move in A^{cn} and y is a move in A. Then we have:

$$\begin{aligned} s|_{A^{\mathrm{cn}}}|_{M_A} x = sx|_{A^{\mathrm{cn}}}|_{M_A} = sx|_A = s|_A \\ s|_{A^{\mathrm{cn}}}|_{M_A} = sy|_{A^{\mathrm{cn}}}|_{M_A} = sy|_A = s|_A y \end{aligned}$$

which is a contradiction since it implies that $s|_{A^{\mathrm{cn}}}|_{M_A}$ is simultaneously a shorter sequence than $s|_A$ and a longer sequence than $s|_A$.

If x, y are both moves in A^{cn} then we have

$$\begin{aligned} s|_{A^{\mathrm{cn}}}|_{M_A} x &= sx|_{A^{\mathrm{cn}}}|_{M_A} \\ &= sx|_A \\ &= s|_A \end{aligned}$$

and $s|_{A^{\mathrm{cn}}}|_{M_A} y = s|_A$ by an identical argument, and so $x = y$. Similarly, if x, y are both moves in A, then we have:

$$\begin{aligned} s|_A x &= sx|_A \\ &= sx|_{A^{\mathrm{cn}}}|_{M_A} \\ &= s|_{A^{\mathrm{cn}}}|_{M_A} \end{aligned}$$

and similarly for y, so $x = y$.

Now, without loss of generality assume that $x = *_t$ for some t. Then $s|_{A^{\mathrm{cn}}}|_{M_A}$ is a limiting play. Suppose for a contradiction that y is a move in A. Then we have:

$$s|_{A^{\mathrm{cn}}}|_{M_A} = sy|_{A^{\mathrm{cn}}}|_{M_A} = sy|_A = s|_A y$$

which is a contradiction, since $s|_A y$ is a successor play. Therefore, y is a move in A^{cn}. We have:

$$\begin{aligned} t &= s|_{A^{\mathrm{cn}}}|_{M_A} \\ &= s *_t |_{A^{\mathrm{cn}}}|_{M_A} \\ &= s *_t |_{A^{\mathrm{cn}}}|_{M_A} \\ &= s|_{A^{\mathrm{cn}}}|_{M_A} \end{aligned}$$

Therefore, $s|_{A^{\mathrm{cn}}}|_{M_A}$ is a limiting O-play in A and therefore we have $x = y = *_t$ by the definition of A^{cn}.

Now we show that cn_A is total. Suppose that $s \in \mathrm{cn}_A$ is an O-position. Then $s|_{A^{\mathrm{cn}}}$ is a P-position and $s|_A$ is an O-position. Suppose that $s = ty$ is a successor play. Note that y cannot be a $*$-move, since these are all O-moves occurring in A^{cn}. If y takes place in A^{cn} then we have $t|_{A^{\mathrm{cn}}}|_{M_A} = t|_A$ and therefore player P can play y in A and still be playing according to cn. Similarly, if y takes place in A, then player P can play y in A^{cn} and be in accordance with cn_A.

Suppose instead that s is a limiting play. It is clear from the definition that $s|_{A^{\mathrm{cn}}}|_A$ and $s|_A$ are both cofinal in s, so they are both limiting plays. Therefore, $s|_A$ is a limiting O-play in A and $s|_{A^{\mathrm{cn}}}|_{M_A}$ must be the corresponding P-play in A^{cn}. Therefore, player P may play $*_t$ (where $t = s|_A$) in A^{cn} and still be in accordance with A^{cn}. So cn_A is total.

Let $q \in P_A$. We claim that there is some $s \in \mathrm{cn}_A$ such that $s|_A = q$. Indeed, since Player P makes all $*$-moves in $A^{\mathrm{cn}} \multimap A$, player O has a strategy for $A^{\mathrm{cn}} \multimap A$ in which he plays the moves of the sequence q in order. Since the strategy cn_A is total, player P always has a reply to Player O's moves and the resulting sequence s satisfies $s|_A = q$.

Therefore, the relational presentation of cn_A has the following property: for all sequences $q \in P_A$, there exists some $r \in P_{A^{\mathrm{cn}}}$ such that $(r, q) \in \underline{\mathrm{cn}_A}$. Moreover, it is not difficult to see that this r is unique, since play passes from A to A^{cn} without fail at each turn, so a P-position in cn_A is uniquely determined by its sequence of O-moves, which must agree with its A-component.

In other words, the relational presentation $\underline{\mathrm{cn}_A}$ is the graph of an injective function $h\colon P_A \to P_{A^{\mathrm{cn}}}$.

Let B be a completely negative α-game and let σ be a strategy for $B \multimap A$. We define the strategy σ^{cn} for $B \multimap A^{\mathrm{cn}}$ as follows:

$$\sigma^{\mathrm{cn}} = \{s \in P_{B\multimap A^{\mathrm{cn}}} \;:\; s|_{B,M_A} \in \sigma\}$$

where we write $s|_{B,M_A}$ for the sequence obtained by deleting all $*$-moves from s.

σ^{cn} is prefix closed, since σ is, and satisfies the limit condition, since σ does. Suppose that $s \in \sigma^{\mathrm{cn}}$ is a P-position and that $sx \in P_{B\multimap A^{\mathrm{cn}}}$. If x is a $*$-move, then we have $sx|_{B,M_A} = s|_{B,M_A} \in \sigma$, so $sx \in \sigma^{\mathrm{cn}}$. Otherwise, we have $sx|_{B,M_A} = s|_{B,M_A}x$.

If $s|_{B,M_A}$ is a P-position in $B \multimap A$ then $s|_{B,M_A}x \in \sigma$ and therefore $sx \in \sigma^{\mathrm{cn}}$. So suppose instead that $s|_{B,M_A}$ is an O-position. If $s = ty$ is a successor position then y is a P-move in $B \multimap A^{\mathrm{cn}}$, so in particular y is not a $*$-move and therefore $s|_{B,M_A} = t|_{B,M_A}y$ is a P-position. Therefore, s must be a limiting play.

Since s is a P-position, $s|_B$ must be a P-position, $s|_{A^{\mathrm{cn}}}|_{M_A}$ must be an O-position in A and $s|_{A^{\mathrm{cn}}}$ must be a P-position in A^{cn}. It follows that $u \coloneqq s|_{A^{\mathrm{cn}}}|_{M_A}$ is a limiting O-play. It follows that x is a move in A^{cn}, and therefore that $x = *_u$, which is a case we have already dealt with.

Next, suppose that $sx, sy \in \sigma^{\mathrm{cn}}$ are P-positions. In particular, x, y are not $*$-moves and we have $s|_{B,M_A}x, s|_{B,M_A}y \in \sigma$. Therefore, $x = y$. So σ^{cn} is a strategy.

Lastly, suppose that σ is a total strategy. Let $s \in \sigma^{\mathrm{cn}}$ be an O-position in $B \multimap A^{\mathrm{cn}}$. Then $s|_B$ is a P-position in B and $s|_{A^{\mathrm{cn}}}$ is an O-position in A^{cn}. So $s|_{A^{\mathrm{cn}}}$ is a successor play in A^{cn}, since A^{cn} is completely negative. If the last move of $s|_{A^{\mathrm{cn}}}$ is not a $*$-move then $s|_{A^{\mathrm{cn}}}|_{M_A}$ has the same last move as $s|_{A^{\mathrm{cn}}}$ and is therefore an O-position as well. Then $s|_{B,M_A}$ is an O-position and so there is some $x \in P_{B\multimap A}$ such that $s|_{B,M_A}x \in \sigma$. Then $sx \in P_{B\multimap A^{\mathrm{cn}}}$ and so $sx \in \sigma^{\mathrm{cn}}$.

Suppose instead that $s|_{A^{\mathrm{cn}}} = t*_u$, where $u = t|_{M_A}$. Then u is an O-position in A, so $s|_{B,M_A}$ is an O-position in $B \multimap A$ and therefore there is some

$x \in P_{B \multimap A}$ such that $s|_{B,M_A} x \in \sigma$. Then $sx \in \sigma^{\mathrm{cn}}$. So σ^{cn} is a total strategy.

We claim that $\underline{\mathrm{cn}_A} \circ \underline{\sigma^{\mathrm{cn}}} = \underline{\sigma}$. Then we will have $\mathrm{cn}_A \circ \sigma^{\mathrm{cn}} = \sigma$ by Lemma 3.14 and Corollary 3.17. Suppose $(s,t) \in \underline{\mathrm{cn}_A} \circ \underline{\sigma^{\mathrm{cn}}}$. Then there is some $u \in P_{A^{\mathrm{cn}}}$ such that $(s,u) \in \underline{\sigma^{\mathrm{cn}}}$ and $(u,t) \in \underline{\mathrm{cn}_A}$. Therefore, $(s, u|_{M_A}) \in \underline{\sigma}$ and $t = u|_{M_A}$. It follows that $(s,t) \in \underline{\sigma}$, so $\underline{\mathrm{cn}_A} \circ \underline{\sigma^{\mathrm{cn}}} \subseteq \underline{\sigma}$.

To show the reverse inclusion, we shall use the following lemma, which we shall prove later.

Lemma 3.24. *Let B be a completely negative α-game, let A be an arbitrary α-game and let σ be a strategy for $B \multimap A$. Let $s \in \sigma$ be a P-position. Then there is a unique P-position $t \in B \multimap A^{\mathrm{cn}}$ such that $t|_{B,M_A} = s$.*

We use Lemma 3.24 to show the reverse inclusion as follows: let $s \in \sigma$ be a P-position – so $(s|_B, s|_A)$ is an arbitrary element of $\underline{\sigma}$. By Lemma 3.24 there exists $t \in B \multimap A^{\mathrm{cn}}$ such that $t|_{B,M_A} = s$. In particular, $t|_{A^{\mathrm{cn}}}|_{M_A} = s|_A$ and $t|_B = s|_B$. So we have $(s|_B, t|_{A^{\mathrm{cn}}}) \in \underline{\sigma^{\mathrm{cn}}}$ and $(t|_{A^{\mathrm{cn}}}, s|_A) \in \underline{\mathrm{cn}_A}$. It follows that $(s|_B, s|_A) \in \underline{\sigma}$. So $\underline{\sigma} \subseteq \underline{\mathrm{cn}_A} \circ \underline{\sigma^{\mathrm{cn}}}$.

Lastly, we need to show that the strategy σ^{cn} is unique. Suppose that there were some other strategy τ for $B \multimap A^{\mathrm{cn}}$ making the diagram commute:

$$\begin{tikzcd} B \arrow[r, "\tau"] \arrow[dr, "\sigma"'] & A^{\mathrm{cn}} \arrow[d, "\mathrm{cn}_A"] \\ & A \end{tikzcd}$$

We shall show that $\underline{\tau} = \underline{\sigma^{\mathrm{cn}}}$ – so $\tau = \sigma^{\mathrm{cn}}$. First suppose that $s \in \sigma^{\mathrm{cn}}$ is a P-position. Since $\sigma = \mathrm{cn}_A \circ \tau$, there is some sequence $\mathfrak{s} \in \tau \| \mathrm{cn}_A$ such that $\mathfrak{s}|_{B,A} = s|_{B,M_A}$. We have $\mathfrak{s}|_{A^{\mathrm{cn}}}|_{M_A} = \mathfrak{s}|_A$ and therefore $s|_{B,M_A} = \mathfrak{s}|_{B,A^{\mathrm{cn}}}|_{B,M_A}$. Therefore, $s = \mathfrak{s}|_{B,A^{\mathrm{cn}}} \in \tau$, by uniqueness in Lemma 3.24. So $\underline{\sigma^{\mathrm{cn}}} \subseteq \underline{\tau}$.

Conversely, suppose that $s \in \tau$ is a P-position. Then our argument above tells us that we have $(s|_{A^{\mathrm{cn}}}, s|_{A^{\mathrm{cn}}}|_A) \in \underline{\mathrm{cn}_A}$. So $(s|_B, s|_{A^{\mathrm{cn}}}|_{M_A}) \in \underline{\sigma}$. Therefore there is some P-position $u \in \sigma$ such that $u|_B = s|_B$ and $u|_A = s|_{A^{\mathrm{cn}}}|_{M_A}$. By Lemma 3.24, there is a sequence $t \in \sigma^{\mathrm{cn}}$ with $t|_{B,M_A} = s$. Then we have

$$(s|_B, s|_{A^{\mathrm{cn}}}) = (t|_B, t|_{A^{\mathrm{cn}}}) \in \underline{\sigma^{\mathrm{cn}}}$$

and therefore $\underline{\tau} \subseteq \underline{\sigma^{\mathrm{cn}}}$. $\square$

It remains to prove Lemma 3.24.

Proof of Lemma 3.24. This proof is similar to the proof of 3.15. We inductively construct a sequence $t^0 \sqsubseteq t^1 \sqsubseteq \ldots \sqsubseteq t^\gamma \sqsubseteq \ldots \sqsubseteq t^\beta$, where t^γ has length γ and t^β is our desired sequence t. We shall prove inductively that t^γ is the unique element $u \in \sigma^{\mathrm{cn}}$ of length γ such that $u|_{B,M_A} = t$.

If γ is a limit ordinal, then set t^γ to be the limit of all the sequences t^δ for $\delta < \gamma$. We have $r|_{B,M_A} \sqsubseteq s$ for all proper prefixes $r \sqsubsetneq t^\gamma$ and therefore $t^\gamma|_{B,M_A} \sqsubseteq s$. For uniqueness, suppose that u^γ has length γ and $u^\gamma|_{B,M_A} \sqsubseteq s$. If $\delta < \gamma$ and we write u^δ for the prefix of u^γ of length δ then we have $u^\delta = t^\delta$ by uniqueness and it follows that $u^\gamma = t^\gamma$.

Suppose instead that $\gamma = \delta + 1$ is a successor ordinal. We have a sequence t^δ of length δ.

Suppose first that t^δ is an O-position. Then, as usual, $t^\delta|_{B,M_A}$ is an O-position. We have $t^\delta|_{B,M_A} \sqsubseteq s$; since s is a P-position, this must be a proper prefix, so there is some move x such that $t^\delta|_{B,M_A} x \sqsubseteq s$. By the argument above (showing that σ^{cn} is total if σ is total), we must have $t^\delta x \in \sigma$. We set $t^{\delta+1} = t^\delta x$. Then $t^{\delta+1}|_{B,M_A} \sqsubseteq s$.

For uniqueness, suppose that $u^{\delta+1}$ is some other sequence of length $\delta + 1$ with $u^{\delta+1}|_{B,M_A} \sqsubseteq s$. Writing $u^{\delta+1} = u^\delta y$, we have $u^\delta = t^\delta$ by uniqueness for sequences of length δ. Then $t^\delta x, t^\delta y \in \sigma$ and so $x = y$. So $u^{\delta+1} = t^{\delta+1}$.

Now suppose that t^δ is a P-position. We have $t^\delta|_{B,M_A} \sqsubseteq s$. If $t^\delta|_{B,M_A} = s$ then we terminate, setting $\beta = \delta$. Otherwise, there is some move x such that $t^\delta|_{B,M_A} x \sqsubseteq s$.

Suppose that $t^\delta|_{B,M_A}$ is a P-position. Then x is an O-move, so $t^\delta x$ is alternating. **Since B is completely negative**, Proposition 3.12 tells us that $t^\delta x \in P_{B \multimap A^{\mathrm{cn}}}$. Therefore, $t^\delta x \in \sigma^{\mathrm{cn}}$, since σ^{cn} is a strategy. We set $t^{\delta+1} = t^\delta x$. Then we certainly have $t^{\delta+1}|_{B,M_A} \sqsubseteq s$. For uniqueness, suppose that $u^{\delta+1}$ has length $\delta+1$ and that $u^{\delta+1}|_{B,M_A} \sqsubseteq s$. Then $u^{\delta+1} = t^\delta y$ for some y by uniqueness for sequences of length δ. Since $t^\delta|_{B,M_A}$ is a P-position, it is in particular not a limiting O-play, so y cannot be a $*$-move. Therefore, we have $u^\delta|_{B,M_A} y \sqsubseteq s$ and so $x = y$.

Suppose instead that $t^\delta|_{B,M_A}$ is an O-position. Then $v \coloneqq t^\delta|_{A^{\mathrm{cn}}}|_{M_A}$ must be a limiting O-play. We set $t^{\delta+1} = t^\delta *_v$. Since $*_v$ is the only move possible in position t^δ, this must certainly be unique, and we have $t^\delta *_v |_{B,M_A} = t^\delta|_{B,M_A} \sqsubseteq s$.

This completes the proof by induction. $\square$

3.3.5 A digression

Earlier on, we defined a category $\mathcal{G}(\alpha)/A$ for A an arbitrary α-game. At this point, it is tempting to define a category as follows:

Definition 3.25. The objects of the category $\tilde{\mathcal{G}}(\alpha)$ are (not necessarily completely negative) α-games. If A, B are α-games, a morphism in $\tilde{\mathcal{G}}(\alpha)$ from A to B is a functor

$$\mathcal{F} \colon \mathcal{G}(\alpha)/A \to \mathcal{G}(\alpha)/B$$

that commutes with the natural projection functors $p_A \colon \mathcal{G}(\alpha)/A \to \mathcal{G}(\alpha)$ and $p_B \colon \mathcal{G}(\alpha)/B \to \mathcal{G}(\alpha)$.

This definition is useful because we have a natural embedding (the 2-Yoneda embedding) of $\mathcal{G}(\alpha)$ into the category $\tilde{\mathcal{G}}(\alpha)$ that sends a completely negative game N to the category $\mathcal{G}/N$ and sends a morphism $\sigma \colon M \multimap N$ to the 'post-composition with σ' functor between the slice categories $\mathcal{G}(\alpha)//M$ and $\mathcal{G}(\alpha)/N$.

It is well known that this is a fully faithful embedding. I claim that it is also essentially surjective, so it is in fact an equivalence of categories. Indeed, this follows from what we have just shown:

Proposition 3.26. *Let A be an arbitrary α game. Then the categories $\mathcal{G}(\alpha)/A$ and $\mathcal{G}(\alpha)/A^{\mathrm{cn}}$ are isomorphic as categories, via an isomorphism that commutes with the projection functors.*

Proof. We have a natural functor from $\mathcal{G}(\alpha)/A^{\mathrm{cn}}$ to $\mathcal{G}(\alpha)/A$ given by post-composition with cn_A, where functoriality follows from associativity of composition. This functor certainly commutes with the projection functors. Then Proposition 3.23 tells us that it is fully faithful and a bijection on objects, so it is a categorical isomorphism. $\square$

Therefore, the categories $\mathcal{G}(\alpha)$ and $\tilde{\mathcal{G}}(\alpha)$ are equivalent. This suggests one strategy for defining a symmetric monoidal closed structure on $\mathcal{G}(\alpha)$: since the objects of $\tilde{\mathcal{G}}(\alpha)$ are not required to be completely negative, we may freely define a closed monoidal structure on $\tilde{\mathcal{G}}(\alpha)$ by setting the linear implication between games A and B to be the game $A \multimap B$ and the tensor product to be the game $A \otimes B$. This symmetric monoidal closed structure will give rise to a symmetric monoidal closed structure on $\mathcal{G}(\alpha)$ via the equivalence of categories.

However, there is still quite a lot of work to show that this does indeed give a symmetric monoidal closed structure on $\tilde{\mathcal{G}}(\alpha)$, and it will be easier to construct the closed structure on $\mathcal{G}(\alpha)$ directly.

3.3.6 The completely negative linear implication $A \overset{\mathrm{cn}}{\multimap} B$

Let A, B be completely negative α-games. We have a completely negative α-game $A \overset{\mathrm{cn}}{\multimap} B := (A \multimap B)^{\mathrm{cn}}$. We shall construct a monoidal closed structure on $\mathcal{G}(\alpha)$ where the linear implication between A and B is $A \overset{\mathrm{cn}}{\multimap} B$.

We shall start off with the relevant result about the (usual) implication $A \multimap B$:

Proposition 3.27. *Let A, B, C be completely negative α games. Then the games $(A \otimes B) \multimap C$ and $A \multimap (B \multimap C)$ are tree isomorphic.*

Note 3.28*: Since $(A \otimes B) \multimap C$ is not a completely negative game in general, we cannot say that these games are isomorphic as objects of $\mathcal{G}(\alpha)$, but it will help us to know that they are tree-isomorphic as games.*

Proof of Proposition 3.27. The sets of plays $P_{(A\otimes B)\multimap C}$ and $P_{A\multimap(B\multimap C)}$ are both appropriate subsets of $P_A \| P_B \| P_C$. We just need to check that $\lambda_{(A\otimes B)\multimap C} = \lambda_{A\multimap(B\multimap C)}$ and that $\zeta_{(A\otimes B)\multimap C} = \zeta_{A\multimap(B\multimap C)}$.

We have

$$\begin{aligned} \lambda_{(A\otimes B)\multimap C} &= \neg \circ \lambda_{A\otimes B} \sqcup \lambda_C \\ &= \neg \circ \lambda_A \sqcup \neg \circ \lambda_B \sqcup \lambda_C \\ &= \neg \circ \lambda_A \sqcup \lambda_{B\multimap C} \\ &= \lambda_{A\multimap(B\multimap C)} \end{aligned}$$

and

$$\begin{aligned} \zeta_{(A\otimes B)\multimap C}(s) &= ((\zeta_A(s|_A) \wedge \zeta_B(s|_B)) \Rightarrow \zeta_C(s|_C)) \\ &= (\zeta_A(s|_A) \Rightarrow (\zeta_B(s|_B) \Rightarrow \zeta_C(s|_C))) \\ &= \zeta_{A\multimap(B\multimap C)}(s) \end{aligned}$$

where we can prove the second equality as follows: if $p, q, r \in \{O, P\}$, then $((p \wedge q) \Rightarrow r) = O$ if and only if $p = q = P$ and $r = O$. Meanwhile, $(p \Rightarrow (q \Rightarrow r)) = O$ if and only if $p = q = P$ and $r = O$. So $((p \wedge q) \Rightarrow r) = (p \Rightarrow (q \Rightarrow r))$. $\square$

To construct the monoidal closed structure, it suffices to give a morphism

$$\mathrm{ev}_{A,B}\colon (A \overset{\mathrm{cn}}{\multimap} B) \otimes A \to B$$

for all pairs (A, B) of completely negative α-games, such that for all completely negative α-games X and all morphisms $\sigma\colon X \otimes A \to B$ there is a unique morphism $\upsilon\colon X \to A \overset{\mathrm{cn}}{\multimap} B$ making the following diagram commute:

$$\begin{tikzcd} X \otimes A_3 \arrow[r, "\sigma"] \arrow[d, "\upsilon\otimes\mathrm{id}_A"'] & B_2 \\ (A_1 \overset{\mathrm{cn}}{\multimap} B_1) \otimes A_2 \arrow[ur, "\mathrm{ev}_{A,B}"'] & \end{tikzcd}$$

(Here we have numbered the copies of A and B so we can refer to them individually).

The strategy $\mathrm{ev}_{A,B}$ for $((A \overset{\mathrm{cn}}{\multimap} B) \otimes A) \multimap B$ is constructed as follows: we have our strategy $\mathrm{cn}_{A \multimap B}$ for $(A \overset{\mathrm{cn}}{\multimap} B) \multimap (A \multimap B)$. By Proposition 3.27, this game is tree isomorphic to the game $((A \overset{\mathrm{cn}}{\multimap} B) \otimes A) \multimap B$, so we get the strategy $\mathrm{ev}_{A,B}$ by applying this tree isomorphism to the strategy $\mathrm{cn}_{A \multimap B}$.

We need to show that the strategy $\mathrm{ev}_{A,B}$ satisfies the property given above. Let X be a completely negative α-game and let σ be a strategy for $(X \otimes A) \multimap B$. Then σ gives rise, via the tree isomorphism above, to a strategy $\tilde{\sigma}$ for $X \multimap (A \multimap B)$. Since the tree isomorphism does nothing more than identify $P_{(X\otimes A)\multimap B}$ with $P_{X \multimap (A \multimap B)}$ as subsets of $(M_X \sqcup M_B \sqcup M_C)^{*<\alpha}$, we may identify σ and $\tilde{\sigma}$ as subsets of $(M_X \sqcup M_A \sqcup M_B)$.

Similarly, we may identify $\mathrm{ev}_{A,B}$ with $\mathrm{cn}_{A \multimap B}$ as subsets of

$$(M_{A_1} \sqcup M_{B_1} \sqcup M_{A_2} \sqcup M_{B_2})^{*<\alpha}$$

where we have again labelled the different copies of A and B to avoid confusion.

The strategy $\tilde{\sigma}$ gives rise to a strategy $\tilde{\sigma}^{\mathrm{cn}}$ for $X \multimap (A \overset{\mathrm{cn}}{\multimap} B)$. We claim that this $\tilde{\sigma}^{\mathrm{cn}}$ satisfies the property above.

First, we show that $\mathrm{ev}_{A,B} \circ (\tilde{\sigma}^{\mathrm{cn}} \otimes \mathrm{id}_A) = \sigma$. By the definition of $\tilde{\sigma}^{\mathrm{cn}}$, we know that it makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\tilde{\sigma}^{\mathrm{cn}}"] \arrow[dr, "\tilde{\sigma}"'] & A_1 \overset{\mathrm{cn}}{\multimap} B_1 \arrow[d, "\mathrm{cn}_{A\multimap B}"] \\ & A_2 \multimap B_2 \end{tikzcd}$$

where we have labelled the copies of A and B so we can tell them apart. Therefore, we have

$$\tilde{\sigma} = \{\mathfrak{s}|_{X,A_2,B_2} \,:\, \mathfrak{s} \in \tilde{\sigma}^{\mathrm{cn}} \| \,\mathrm{cn}_{A\multimap B}\}$$

where

$$\tilde{\sigma}^{\mathrm{cn}} \| \,\mathrm{cn}_{A\multimap B} = \left\{ \mathfrak{s} \in (M_X \sqcup M_{A_1} \sqcup M_{B_1} \sqcup M_{A_2} \sqcup M_{B_2})^{*<\alpha} \;\middle|\; \begin{array}{l} \mathfrak{s}|_{X,A_1,B_1} \in \tilde{\sigma}^{\mathrm{cn}} \\ \mathfrak{s}|_{A_1,B_1,A_2,B_2} \in \mathrm{cn}_{A\multimap B} \end{array} \right\}$$

Now we have

$$\mathrm{ev}_{A,B} \circ (\tilde{\sigma}^{\mathrm{cn}} \otimes \mathrm{id}_A) = \{\mathfrak{t}|_{X,A_2,B_2} \,:\, \mathfrak{t} \in (\tilde{\sigma}^{\mathrm{cn}} \otimes \mathrm{id}_A) \,\|\, \mathrm{ev}_{A,B}\}$$

where

$$(\tilde{\sigma}^{\mathrm{cn}} \otimes \mathrm{id}_A) \,\|\, \mathrm{ev}_{A,B} = \left\{ \mathfrak{t} \in (M_X \sqcup M_{A_3} \sqcup M_{A_1} \sqcup M_{B_1} \sqcup M_{A_2} \sqcup M_{B_2})^{*<\alpha} \;\middle|\; \begin{array}{l} \mathfrak{t}|_{X,A_1,B_1} \in \tilde{\sigma}^{\mathrm{cn}} \\ \mathfrak{t}|_{A_3,A_2} \in \mathrm{id}_A \\ \mathfrak{t}|_{A_2,B_2} \in \mathrm{ev}_{A,B} \end{array} \right\}$$

Remembering that we may identify $\mathrm{ev}_{A,B}$ with $\mathrm{cn}_{A\multimap B}$ as sets of plays, we see that

$$\mathrm{ev}_{A,B} \circ (\tilde{\sigma}^{\mathrm{cn}} \otimes \mathrm{id}_A) \subseteq \tilde{\sigma} = \sigma$$

since if $\mathfrak{t} \in (\tilde{\sigma}^{\mathrm{cn}} \otimes \mathrm{id}_A) \,\|\, \mathrm{ev}_{A,B}$ then we have $\mathfrak{t}|_{X,A_1,B_1,A_2,B_2} \in \tilde{\sigma}^{\mathrm{cn}} \| \,\mathrm{cn}_{A\multimap B}$ and so $\mathfrak{t}|_{X,A_2,B_2} \in \tilde{\sigma}$.

Conversely, let $s \in \tilde{\sigma}$ and write $s = \mathfrak{s}|_{X,A_2,B_2}$, where $\mathfrak{s} \in \tilde{\sigma}^{\mathrm{cn}} \| \,\mathrm{cn}_{A\multimap B}$. We may construct a sequence $\mathfrak{t} \in (\tilde{\sigma}^{\mathrm{cn}} \otimes \mathrm{id}_A) \,\|\, \mathrm{ev}_{A,B}$ with $\mathfrak{t}|_{X,A_1,B_1,A_2,B_2} = \mathfrak{s}$ by inserting before each P-move in A_2 the corresponding move in A_3 and inserting *after* each O-move in A_2 the corresponding move in A_3. Therefore:

$$\sigma \subseteq \mathrm{ev}_{A,B} \circ (\tilde{\sigma}^{\mathrm{cn}} \otimes \mathrm{id}_A)$$

For uniqueness, suppose that we have a strategy υ for $X \multimap (A \overset{\mathrm{cn}}{\multimap} B)$ making the following diagram commute:

$$\begin{tikzcd} X \otimes A_3 \arrow[r, "\sigma"] \arrow[d, "\upsilon \otimes \mathrm{id}_A"'] & B_2 \\ (A_1 \overset{\mathrm{cn}}{\multimap} B_1) \otimes A_2 \arrow[ur, "\mathrm{ev}_{A,B}"'] & \end{tikzcd}$$

Exactly the same argument then tells us that

$$\mathrm{cn}_{A\multimap B} \circ \upsilon = \mathrm{ev}_{A,B} \circ \upsilon \otimes \mathrm{id}_A = \sigma = \tilde{\sigma}$$

(as sets of plays) and so $\upsilon = \tilde{\sigma}^{\mathrm{cn}}$ by the uniqueness part of Proposition 3.23.

Therefore $\mathcal{G}(\alpha)$ has the structure of a symmetric monoidal category with tensor product given by $\otimes$ and linear implication given by $\overset{\mathrm{cn}}{\multimap}$.

3.3.7 Win-games and winning strategies

TODO: Write this section.

4 Exponentials

We want to model the exponential connective ! from linear logic in our game semantics. We adapt the approach taken in [Lai02] and [CLM13] to our transfinite games.

Definition 4.1. Let $\mathcal{G}(\alpha)$ be a category of transfinite games and let A be an object of $\mathcal{G}(\alpha)$. Then the *exponential* $!A$ is given as follows:

- $M_{!A} = A \times \omega$
- $\lambda_{!A} = \lambda_A \circ \mathrm{pr}_1$

Given an ordinal-indexed sequence s taking values in $M_{!A}$ and some $n \in \omega$, we write $s|_n$ for the subsequence of s consisting of terms of the form (a, n). Then define

$$!P_A = \{s \in M_{!A}^{*<\alpha} \,:\, s|_n \in P_A \text{ for all } n \in \omega\}$$

We define $\zeta_{!A} \colon !P_A \to \{O, P\}$ by

$$\zeta_{!A}(s) = \bigwedge_{n\in\omega} \zeta_A(s|_n)$$

(Here, if $(b_n) \in \{O, P\}^\omega$ then we have $\bigwedge_{n\in\omega} b_n = P$ if $b_n = P$ for all n and $\bigwedge_{n\in\omega} = O$ otherwise.)

We define $P_{!A}$ to be the set of all sequences $s \in !P_A$ that are well formed with respect to $\zeta_{!A}, \lambda_{!A}$ and are alternating with respect to $\zeta_{!A}$ and that satisfy the following property:

- If the term (a, n) occurs in s, and if $m < n$, then there is some term (b, m) that occurs earlier in the sequence s than (a, n).

Then $!A = (M_{!A}, \lambda_{!A}, \zeta_{!A}, P_{!A})$.

So $!A$ is like an infinite tensor product of copies of A with the restriction that the first move must take place in the first copy and that moves may not take place in a subsequent copy of A until moves have been made in all earlier copies.

Proposition 4.2. *$!A$ is a well formed game.*

Proof. Exactly the same argument from Proposition 3.7. □

Just like the tensor product, the exponential $!A$ has the property that only player O may switch games:

Proposition 4.3. *Suppose that $s(a, m)(b, n) \in P_{!A}$, where $m \neq n$. Then $\lambda_{!A}(b, n) = O$.*

Proof.
$$\begin{aligned} \lambda_{!A}(b, n) &= \zeta_{!A}(s(a, m)(b, n)) \\ &= \bigwedge_{k \in \omega} \zeta_A(s(a, m)(b, n)|_k) \end{aligned}$$

Now by alternation we must have either $\zeta_A(s|_m(a, m)) = O$ or $\zeta_A(s|_n(b, n)) = O$. So this last expression must be equal to O. □

We can also prove a version of Proposition 3.12 for the exponential connective using the same argument.

The next Proposition goes some way towards showing why $!A$ is a suitable model for the exponential connective in linear logic:

Proposition 4.4. *$!A$ gives rise to a cofree commutative comonoid on A in our monoidal category.*

Sketch proof. The comultiplication $\mu \colon !A \to !A \otimes !A$ is the copycat strategy that opens a new copy of A on the left hand side whenever a new copy of A is opened on the right hand side. (Here we see why the condition on the order in which games may be opened is important – if we had no such restriction then there would be no canonical way to choose which copy of A to open on the left hand side and the comultiplication would not be associative.)

The counit $\eta \colon !A \to I$ is the unique empty strategy on $!A \multimap I$.

For associativity of comultiplication, we want to check that the following diagram commutes:

$$
\begin{tikzcd}
{!A} \arrow[r, "\mu"] \arrow[dd, "\mu"'] & {!A \otimes !A} \arrow[d, "\mathrm{id}_{!A} \otimes \mu"] \\
 & {!A \otimes (!A \otimes !A)} \arrow[d, "\mathrm{assoc}^{-1}_{A,A,A}"] \\
{!A \otimes !A} \arrow[r, "\mu \otimes \mathrm{id}_{!A}"'] & {(!A \otimes !A) \otimes !A}
\end{tikzcd}
$$

For this, note that each path round the diagram is a copycat strategy $!A \to (!A \otimes !A) \otimes !A$. Since we are forced to open copies of $!A$ in order, there is only one such copycat strategy, and so the two paths must be equal.

Exactly the same argument tells us that comultiplication is commutative and that the counit is indeed a counit for this comultiplication.

We have a natural forgetful functor from the category of comonoids over $\mathcal{G}(\alpha)$ to $\mathcal{G}(\alpha)$ itself. We first want to show that we have a functor in the opposite direction that sends a game A to this comonoid over $!A$.

Let A, B be completely negative α-games and let σ be a strategy for $A \multimap B$. Then we have a natural strategy $!\sigma$ for $!A \multimap !B$ given by

$$!\sigma = \{s \in P_{!A} \; : \; s|_n \in \sigma \text{ for all } n \in \omega\}$$

where we have written $s|_n$ for the subsequence of s consisting of all terms that occur in the n-th copy of A or the n-th copy of B.

The proof that this is a strategy may be adapted from the discussion of the strategy $\sigma \otimes \tau$ above. Similarly, we may show that if A, B, C are completely negative α-games and we have strategies $A \xrightarrow{\sigma} B \xrightarrow{\tau} C$ then

$$!\tau \circ !\sigma = !(\tau \circ \sigma)$$

using the same argument we used to show that taking the tensor product of two strategies preserves composition.

This gives us a functor $! \colon \mathcal{G}(\alpha) \to \mathcal{G}(\alpha)$. We want to show that it gives rise to a functor from $\mathcal{G}(\alpha)$ into the category of comonoids over $\mathcal{G}(\alpha)$ sending a game A to the comonoid $!A \to !A \otimes !A$. For this, we need to show that if

A, B are completely negative α-games and σ is a strategy for $A \multimap B$ then the following diagram commutes:

$$\begin{array}{ccc} !A & \xrightarrow{\mu} & !A \otimes !A \\ {\scriptstyle !\sigma}\downarrow & & \downarrow{\scriptstyle !\sigma \otimes !\sigma} \\ !B & \xrightarrow[\mu]{} & !B \otimes !B \end{array}$$

Each branch of this diagram is a strategy for $!A \multimap (!B \otimes !B)$ in which player P plays according to σ, opening a new copy of A on the left each time player O opens a copy of B on the right. Since the order in which player P may open copies of A is pre-determined, there is a unique such strategy, and so this diagram must commute.

Lastly, we need to show that this functor is adjoint to the forgetful functor from the category of commutative comonoids on $\mathcal{G}(\alpha)$ to the category $\mathcal{G}(\alpha)$ itself. We start off by providing a natural transformation from the functor $A \mapsto !A$ on $\mathcal{G}(\alpha)$ to the identity functor on $\mathcal{G}(\alpha)$. Given a completely negative α-game A, we have a strategy `der` for $!A \multimap A$ that is the copycat strategy where player P forgets about the extra copies of A. Then, if A, B are completely negative α-games and σ is a strategy for $A \multimap B$, we have the following commutative square:

$$\begin{array}{ccc} !A & \xrightarrow{\texttt{der}} & A \\ {\scriptstyle !\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ !B & \xrightarrow[\texttt{der}]{} & B \end{array}$$

(by inspecting the definition of $!\sigma$) and so `der` gives rise to a natural transformation from $A \mapsto !A$ to id_A.

To complete the proof of adjointness, we need to show that the reverse composition of these functors admits a natural transformation out of the identity functor on the category of commutative comonoids over $\mathcal{G}(\alpha)$. Let $X \xrightarrow{\mu} X \otimes X$ be a commutative comonoid over $\mathcal{G}(\alpha)$. $\square$

TODO: complete this section.

4.1 Products in $\mathcal{G}(\alpha)$

Let $(A_i : i \in I)$ be a collection of objects of $\mathcal{G}(\alpha)$. We define the product

$$\prod_i A_i$$

to be the game given by

- $M_{\prod_i A_i} = \bigsqcup_i M_{A_i}$
- $\lambda_{\prod_i A_i} = \bigsqcup_i \lambda_{A_i}$
- $P_{\prod_i A_i} = \bigsqcup_i P_{A_i}$
- $\zeta_{\prod_i A_i} = \bigsqcup_i \zeta_{A_i}$

This is the game where Player O may decide which of the A_i to play his first move in; play then continues in that game and the other games are never used.

We claim that $\prod_i A_i$ is the categorical product of the A_i in $\mathcal{G}(\alpha)$. First note that we have natural projection strategies pr_j for $\prod_i A_i \to A_j$ for each $j \in I$, in which player P plays copycat in game A_j on the left. Now if B is an object of $\mathcal{G}(\alpha)$ and σ_i are strategies for $B \multimap A_i$ for each $i \in I$, then we have a strategy σ for $B \multimap \prod_i A_i$ in which player P uses strategy j when player O's initial move takes place in A_j:

$$\sigma = \bigsqcup_i \sigma_i$$

We can see that $\mathrm{pr}_j \circ \sigma = \sigma_j$ for each j: indeed, $\underline{\mathrm{pr}_j}$ is the graph of the inclusion $P_{A_j} \hookrightarrow P_{\prod_i A_i}$, while $\underline{\sigma}$ is the union of the relations $\underline{\sigma_i}$, so this equation holds in the relational presentation:

$$\underline{\mathrm{pr}_j} \circ \underline{\sigma} = \underline{\sigma_j}$$

Moreover, it is easy to see that $\underline{\sigma}$ is the unique relation between P_B and $P_{\prod_i A_i}$ such that this equation holds for all j, and therefore *sigma* is the unique strategy for $B \multimap \prod_i A_i$ such that $\mathrm{pr}_j \circ \sigma = \sigma_j$ for each $j \in I$. Therefore, $\mathcal{G}(\alpha)$ has all small products.

Given games A, B, we shall write $A \times B$ for the product of A and B.

4.2 The exponential as a final coalgebra

Our definition of the exponential connective ! was different from the definitions of the other connectives in that it involved an explicit restriction on the order in which new games could be opened. We saw that this restriction was necessary in order to guarantee associativity of the comultiplication in our comonoid, but it means that it is hard to study the exponential $!A$ using the connectives $\otimes$ and $\multimap$.

Let A, B be objects of some category $\mathcal{G}(\alpha)$ of transfinite games. Following [Lai02], we define the *sequoid* $A \oslash B$ by:

- $M_{A \oslash B} = M_{A \otimes B}$
- $\lambda_{A \oslash B} = \lambda_{A \otimes B}$
- $P_{A \oslash B} = \{s \in P_{A \otimes B} : \ s = \epsilon \text{ or the first move of } s \text{ takes place in } A\}$
- $\zeta_{A \oslash B} = \zeta_{A \otimes B}|_{P_{A \oslash B}}$

It is clear that $A \oslash B$ is a well-formed game - namely, it is the weakening of the tensor product $A \otimes A$ such that player O is forced to start play in the game A. We have a natural copycat strategy $\mathrm{wk} \colon A \otimes B \to A \oslash B$ and the morphism

$$A \otimes B \to (A \oslash B) \times (B \oslash A)$$

induced by the morphisms

$$A \otimes B \xrightarrow{\mathrm{wk}} A \oslash B \quad A \otimes B \xrightarrow{\mathrm{sym}} B \otimes A \xrightarrow{\mathrm{wk}} B \oslash A$$

is an isomorphism.

Other natural copycat isomorphisms may be verified by inspection:

$$\mathtt{passoc}_{A,B,C} \colon (A \oslash B) \oslash C \xrightarrow{\cong} A \oslash (B \otimes C)$$

$$\mathtt{pcomm}_{A,B,C} = \mathtt{passoc}^{-1}_{A,B,C} \circ (\mathrm{id}_A \otimes \mathrm{sym}) \circ \mathtt{passoc}_{A,B,C} \colon (A \oslash B) \oslash C \xrightarrow{\cong} (A \oslash C) \oslash B$$

$$\mathtt{l}_A \colon I \oslash A \xrightarrow{\cong} I$$

$$\mathtt{r}_A \colon A \oslash I \xrightarrow{\cong} A$$

One important point is that the sequoid does not give rise to a functor $\mathcal{G}(\alpha) \times \mathcal{G}(\alpha) \to \mathcal{G}(\alpha)$ in the way that the tensor product does. To see why, let A, B, C, D be games, let σ be a strategy for $A \multimap B$ and let τ be a strategy for $C \multimap D$. We would like to construct a strategy $\sigma \oslash \tau$ for

$$(A \oslash C) \multimap (B \oslash D)$$

in which player P plays according to σ in A and B and according to τ in B and D, exactly as in the strategy $\sigma \otimes \tau$. The problem is that $\sigma \otimes \tau$ is not necessarily a valid strategy for $(A \oslash C) \multimap (B \oslash D)$, even if σ is a valid strategy for $A \multimap B$ and τ is a valid strategy for $C \multimap D$.

Indeed, suppose that player P's reply in σ to a particular opening move in B is also a move in B, and suppose further that player P's reply in τ to a particular opening move in D is a move in C (for example, if τ is a copycat strategy). Now consider the following narrative for the game $(A \oslash C) \multimap (B \oslash D)$:

1. Player O plays the opening move in B, as he must.
2. Player P replies in B, according to σ.
3. Player O decides to switch and start the game D.
4. According to τ, player P should reply with a move in C. But she cannot play in C, since no moves have been played in A yet.

In order to fix this situation, we place a restriction on the strategy σ.

Definition 4.5. Let A, B be objects of $\mathcal{G}(\alpha)$. A strategy σ for $A \multimap B$ is called *strict* if player P's reply to each opening O-move in B is a move in A - so if $bx \in \sigma$ then $x \in M_A$.

It is easy to see that the composition of strict strategies σ for $A \multimap B$ and τ for $B \multimap C$ is a strict strategy for $A \multimap C$ and that the identity strategy is strict. Therefore, we have a full-on-objects subcategory $\mathcal{G}_s(\alpha)$ of $\mathcal{G}(\alpha)$ of games and strict strategies.

Proposition 4.6. *Let A, B, C, D be objects of $\mathcal{G}(\alpha)$, let σ be a strict strategy for $A \multimap B$ and let τ be a strict strategy for $C \multimap D$. Then $\sigma \oslash \tau = \sigma \otimes \tau \cap P_{(A \oslash C) \multimap (B \oslash D)}$ is a strict strategy for $(A \oslash C) \multimap (B \oslash D)$ and the following diagram commutes (so* wk *is a natural transformation between $\_ \otimes \_$ and $\_ \oslash \_$):*

$$\begin{tikzcd}
A \otimes C \arrow[r, "\sigma \otimes \tau"] \arrow[d, "\mathrm{wk}"'] & B \otimes D \arrow[d, "\mathrm{wk}"] \\
A \oslash C \arrow[r, "\sigma \oslash \tau"'] & B \oslash D
\end{tikzcd}$$

Proof. We know that $\sigma \otimes \tau$ is a valid strategy for $(A \otimes C) \multimap (B \otimes D)$, so we just need to check that no moves in B take place before moves in D and that no moves in A take place before moves in C. Indeed, player O's opening move must take place in B; then, since σ is strict, player P's reply will take place in A. So $\sigma \oslash \tau$ is a valid strategy for $(A \oslash C) \multimap (B \oslash D)$; moreover, it is itself a strict strategy. $\square$

What this means is that $\_ \oslash \_$ gives us a functor, not from $\mathcal{G}(\alpha) \times \mathcal{G}(\alpha)$ to $\mathcal{G}(\alpha)$, but from $\mathcal{G}_s(\alpha) \times \mathcal{G}(\alpha)$ to $\mathcal{G}_s(\alpha)$. In fact, the isomorphism $\mathtt{passoc}\colon (A \oslash B) \oslash C \cong A \oslash (B \otimes C)$ tells us that $\_ \oslash \_$ is a *right action* of the monoidal category $\mathcal{G}(\alpha)$ upon the category $\mathcal{G}_s(\alpha)$.

The closed monoidal category $\mathcal{G}(\alpha)$, the subcategory $\mathcal{G}_s(\alpha)$, the morphisms `passoc` and `r`, and the natural transformation wk in fact give rise to a structure known as a *sequoidal category* (This definition was first introduced in [Lai02], but with the direction of the sequoid operator reversed; see, for example, [CLM13] for the definition where the orientation of the sequoid agrees with ours).

Moreover, the isomorphism $A \otimes B \cong (A \oslash B) \times (B \oslash A)$ means that our sequoidal category is *decomposable*.

[Lai02] gives examples of what we can prove in general sequoidal categories, particularly in the cpo-enriched setting. They are used in [CLM13] to study the logic modelled by games and history-sensitive strategies. The most useful property of the sequoid from the point of view of our study of the exponential is that it allows us to express our exponential as a final coalgebra.

Let $\mathcal{C}$ be a category and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor. A *coalgebra* for $\mathcal{F}$ is an object A of $\mathcal{C}$ together with a morphism $A \xrightarrow{f} \mathcal{F}(A)$. Given two coalgebras

$$A \xrightarrow{f} \mathcal{F}(A)$$
$$B \xrightarrow{g} \mathcal{F}(B)$$

a *coalgebra homomorphism* from $A \xrightarrow{f} \mathcal{F}(A)$ to $B \xrightarrow{g} \mathcal{F}(A)$ is a morphism $h\colon A \to B$ that makes the following square commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(h)"] \\ B \arrow[r, "g"'] & \mathcal{F}(B) \end{tikzcd}$$

A *final coalgebra* for $\mathcal{F}$ is a terminal object for the category of coalgebras and coalgebra homomorphisms. Namely, it is an object Z of $\mathcal{C}$ together with a morphism $Z \xrightarrow{\alpha} \mathcal{F}(Z)$ such that for each coalgebra $A \xrightarrow{f} \mathcal{F}(A)$ there is a unique morphism

$$\llparenthesis f \rrparenthesis \colon A \to Z$$

making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\llparenthesis f \rrparenthesis"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(\llparenthesis f \rrparenthesis)"] \\ Z \arrow[r, "\alpha"'] & \mathcal{F}(Z) \end{tikzcd}$$

Theorem 4.7. *Let A be an object of a category $\mathcal{G}(\alpha)$ of games. Then $!A$ may be given the structure of a final coalgebra for the functor*

$$A \oslash \_ : \mathcal{G}(\alpha) \to \mathcal{G}(\alpha)$$

Proof. The coalgebra morphism $\alpha \colon !A \to A \oslash !A$ is given by

$$\alpha \colon !A \xrightarrow{\mu} !A \otimes !A \xrightarrow{\mathtt{der} \otimes \mathrm{id}_{!A}} A \otimes !A \xrightarrow{\mathrm{wk}} A \oslash !A$$

In other words, it is the copycat strategy where moves from the extra copy of A on the right are copied into the first copy of A on the left, moves from the first copy of A in the exponential $!A$ on the right are copied into the second copy of A on the left and so on.

Now let B be a game and let σ be a strategy for $B \multimap A \oslash B$. We define a strategy $\llparenthesis \sigma \rrparenthesis$ for $B \multimap !A$ as follows: we first define

$$\|\sigma\| = \{\mathfrak{s} \in (M_B \times \omega \sqcup M_A \times \omega)^{*<\alpha} \ : \ \forall i \geq 0, \mathfrak{s}|_{B_i, A_i, B_{i+1}} \in \sigma\}$$

Then we set

$$\llparenthesis \sigma \rrparenthesis = P_{B \multimap !A} \cap \{\mathfrak{s}|_{B_0, !A} \ : \ \mathfrak{s} \in \|\sigma\|\}$$

where we write $\mathfrak{s}|_{B_0, !A}$ for $\mathfrak{s}|_{B_0, A_0, A_1, \dots}$.

We shall postpone the proof that $\llparenthesis \sigma \rrparenthesis$ is a strategy. We claim that the final coalgebra diagram commutes:

$$\begin{tikzcd} B \arrow[r, "\sigma"] \arrow[d, "\llparenthesis \sigma \rrparenthesis"'] & A \oslash B \arrow[d, "\mathrm{id}_A \oslash \llparenthesis \sigma \rrparenthesis"] \\ !A \arrow[r, "\alpha"'] & A \oslash !A \end{tikzcd}$$

First, we have:

$$\llparenthesis\, \sigma \,\rrparenthesis = P_{B \multimap !A} \cap \left\{ \mathfrak{s}|_{B_0, A_0, A_1, \ldots} \;\middle|\; \begin{array}{l} \mathfrak{s} \in (M_B \times \omega \sqcup M_A \times \omega)^{*<\alpha} \\ \forall i \geq 0,\ \mathfrak{s}|_{B_i, A_i, B_{i+1}} \in \sigma \end{array} \right\}$$

We claim that:

$$\mathrm{id}_A \oslash \llparenthesis\, \sigma \,\rrparenthesis = P_{(A \oslash B) \multimap (A \oslash !A)} \cap \left\{ \mathfrak{s}|_{A_{-2}, B_0, A_{-1}, A_0, A_1, \ldots} \;\middle|\; \begin{array}{l} \mathfrak{s} \in (M_{A_{-2}} \sqcup M_{A_{-1}} \sqcup M_B \times \omega \sqcup M_A \times \omega)^{*<\alpha} \\ \mathfrak{s}|_{A_{-2}, A_{-1}} \in \mathrm{id}_A \\ \forall i \geq 0,\, \mathfrak{s}|_{B_i, A_i, B_{i+1}} \in \sigma \end{array} \right\}$$

Using the definition:

$$\mathrm{id} \oslash \llparenthesis\, \sigma \,\rrparenthesis = \{ s \in P_{(A_{-2} \oslash B_0) \multimap (A_{-1} \oslash !A)} \,:\, s|_{A_{-2}, A_{-1}} \in \mathrm{id}_A,\ s|_{B_0, !A} \in \llparenthesis\, \sigma \,\rrparenthesis \}$$

it is clear that the RHS above is a subset of the LHS. To show the reverse inclusion, suppose we have some sequence $s \in \mathrm{id} \oslash \llparenthesis\, \sigma \,\rrparenthesis$. Since $s|_{B_0, !A} \in \llparenthesis\, \sigma \,\rrparenthesis$, there is some sequence $\mathfrak{s} \in \|\sigma\|$ with $\mathfrak{s}|_{B_0, A_0, A_1, \ldots} = s$. Then it is possible to interleave this sequence with the sequence $s|_{A_{-2}, A_{-1}}$ exactly as we did towards the end of Section 3.3.6: we insert *before* each P-move in A_{-1} the corresponding move in A_{-2} and insert *after* each O-move in A_{-1} the corresponding move in A_{-2}. This interleaved sequence – call it $\mathfrak{t}$ – satisfies $\mathfrak{t}|_{A_{-2}, A_{-1}} \in \mathrm{id}_A$ and $\mathfrak{t}|_{B_i, A_i, B_{i+1}} \in \sigma$ for all $i \geq 0$, so it gives rise to an element of the RHS. Moreover, we have $t|_{A_{-2}, B_0, A_{-1}, A_0, A_1, \ldots} = s$, and therefore s is contained in the RHS.

We now have $(\mathrm{id}_A \oslash \llparenthesis\, \sigma \,\rrparenthesis) \circ \sigma =$

$$P_{B \multimap A \oslash !A} \cap \left\{ \mathfrak{s}|_{B_{-1}, A_{-1}, A_0, A_1, \ldots} \;\middle|\; \begin{array}{l} \mathfrak{s} \in (M_{B_{-1}} \sqcup M_{A_{-2}} \sqcup M_{A_{-1}} \sqcup M_B \times \omega \sqcup M_A \times \omega)^{*<\alpha} \\ \mathfrak{s}|_{B_{-1}, A_{-2}, B_0} \in \sigma \\ \mathfrak{s}|_{A_{-2}, A_{-1}} \in \mathrm{id}_A \\ \forall i \geq 0,\, \mathfrak{s}|_{B_i, A_i, B_{i+1}} \in \sigma \end{array} \right\}$$

We claim that this is equal to the following strategy:

$$S = P_{B \multimap A \oslash !A} \cap \left\{ \mathfrak{s}|_{B_{-1}, A_{-1}, A_0, A_1, \ldots} \;\middle|\; \begin{array}{l} \mathfrak{s} \in (M_{B_{-1}} \sqcup M_{A_{-1}} \sqcup M_B \times \omega \sqcup M_A \times \omega)^{*<\alpha} \\ \forall i \geq -1,\, \mathfrak{s}|_{B_i, A_i, B_{i+1}} \in \sigma \end{array} \right\}$$

S is a strategy since it is nothing more than a relabelling of the indices of $\llparenthesis\, \sigma \,\rrparenthesis$. To show that these two strategies are the same, it suffices to show that they contain the same P-positions.

Suppose we have some P-position $s \in (\mathrm{id}_A \oslash \llparenthesis\, \sigma \,\rrparenthesis) \circ \sigma$ and write $s = \mathfrak{s}|_{B_{-1}, A_{-1}, A_0, A_1, \ldots}$. By the argument above, we may assume that all moves from A_{-2} are adjacent to a move from A_{-1}; since s is a P-position, it cannot end with an O-move in A_{-1}, and therefore $\mathfrak{s}|_{A_{-2},A_{-1}}$ must have even length. Therefore

$$\mathfrak{s}|_{A_{-2}} = \mathfrak{s}|_{A_{-1}}$$

by the definition of id_A.

It follows (after hiding $M_{A_{-2}}$) that $(\mathrm{id}_A \oslash \llparenthesis\, \sigma \,\rrparenthesis) \circ \sigma \subseteq S$. Conversely, given an element $s = \mathfrak{s}|_{B_{-1},A_{-1},A_0,A_1,\ldots} \in S$, we may interleave in a play in A_{-2} that is identical to the play in $\mathfrak{s}|_{A_{-1}}$ as above, showing that $s \in (\mathrm{id}_A \oslash \llparenthesis\, \sigma \,\rrparenthesis) \circ \sigma$.

It remains only to observe that S is precisely the strategy $\alpha \circ \llparenthesis\, \sigma \,\rrparenthesis$, and so the square above does indeed commute.

Now we show uniqueness. Suppose that τ is some strategy for $B \multimap\, !A$ such that the diagram commutes:

$$\begin{tikzcd} B \arrow[r, "\sigma"] \arrow[d, "\tau"'] & A \oslash B \arrow[d, "\mathrm{id}_A \oslash \tau"] \\ !A \arrow[r, "\alpha"'] & A \oslash !A \end{tikzcd}$$

We claim that $\tau = \llparenthesis\, \sigma \,\rrparenthesis$. The argument above tells us that

$$(\mathrm{id}_A \oslash \tau) \circ \sigma = \left\{ \mathfrak{s}|_{B_{-1},A_{-1},A_0,A_1,\ldots} \;\middle|\; \begin{array}{l} \mathfrak{s} \in (M_{B_{-1}} \sqcup M_{A_{-1}} \sqcup M_{B_0} \sqcup M_A \times \omega)^{*<\alpha} \\ \mathfrak{s}|_{B_{-1},A_{-1},B_0} \in \sigma \\ \mathfrak{s}|_{B_0,A_0,A_1,\ldots} \in \tau \end{array} \right\}$$

Meanwhile, we have

$$\alpha \circ \tau = \{ t^\dagger \,:\, t \in \tau \}$$

where $t^\dagger \in (M_{B_{-1}} \sqcup M_{A_{-1}} \sqcup M_A \times \omega)^{*<\alpha}$ is formed by reducing the indices on the copies of A and B in $t \in (M_{B_0} \sqcup M_A \times \omega)^{*<\alpha}$.

We have $(\mathrm{id} \oslash \tau) \circ \sigma = \alpha \circ \tau$, so, after promoting all indices by 1, we may write:

$$\tau = \left\{ \mathfrak{s}|_{B_0,A_0,A_1,\ldots} \;\middle|\; \begin{array}{l} \mathfrak{s} \in (M_{B_0} \sqcup M_{B_1} \sqcup M_A \times \omega)^{*<\alpha} \\ \mathfrak{s}|_{B_0,A_0,B_1} \in \sigma \\ (\mathfrak{s}|_{B_1,A_1,A_2\ldots})^\dagger \in \tau \end{array} \right\} \tag{$*$}$$

Let $t \in \tau$. Then we have just shown that we may interleave t with a sequence $r \in M_{B_1}$ such that the interleaved sequence $\mathfrak{t} \in (M_{B_0} \sqcup M_{B_1} \sqcup M_A \times \omega)^{*<\alpha}$ satisfies

$$\mathfrak{t}|_{B_0,!A} = t \quad \mathfrak{t}|_{B_1} = r$$
$$\mathfrak{t}|_{B_0,A_0,B_1} \in \sigma$$
$$(\mathfrak{t}|_{B_1,A_1,A_2,\ldots})^\dagger \in \tau$$

Let $t_1 = (\mathfrak{t}|_{B_1,A_1,A_2,\ldots})^\dagger$. Applying the result repeatedly allows us to build up a sequence $t = t_0, t_1, t_2, \ldots$ where $t_i \in \tau$ for each i, together with a sequence $\mathfrak{t} = \mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2, \cdots \in (M_{B_0} \sqcup M_{B_1} \sqcup M_A \times \omega)^{*<\alpha}$ such that for each i:

$$\mathfrak{t}_i|_{B_0,!A} = t_i$$
$$\mathfrak{t}_i|_{B_0,A_0,B_1} \in \sigma$$

By construction, the A-components of the $\mathfrak{t}_i$ agree (indeed, the A_i-component of $\mathfrak{t}_j$ is the same as the A_{i+j}-component of $\mathfrak{t}_0$), and the B_1-component of $\mathfrak{t}_i$ is the same as the B_0-component of t_{i+1}. Therefore, the $\mathfrak{t}_i$ interleave to give a sequence

$$\mathfrak{T} \in (M_B \times \omega \sqcup M_A \times \omega)^{*<\alpha}$$

such that $\mathfrak{T}|_{B_0,!A} = t$ and for all $i \geq 0$, $\mathfrak{T}|_{B_i,A_i,B_{i+1}} \in \sigma$. It follows that $t \in \llparenthesis\, \sigma \,\rrparenthesis$ and so $\tau \subseteq \llparenthesis\, \sigma \,\rrparenthesis$.

Now suppose that $s \in \llparenthesis\, \sigma \,\rrparenthesis$. We show by transfinite induction on the length of s that $s \in \tau$. Since τ satisfies the limiting condition, the limiting case of the induction is easy. So suppose instead that $s = s'a$ is a successor play, and suppose that the last move a takes place in the game A_n. By induction, we assume that $s' \in \tau$. We now apply the construction above to give us a sequence

$$s' = t_0, t_1, \ldots, t_{n+1}$$

of plays in τ whose A-components agree so that the A_i-component of t_j is the A_{i+j}-component of s'. It follows that the A-components of t_{n+1} also agree with the components of $A_{n+1}, A_{n+2}, \ldots$ of sa, since the move a takes place in A_n.

We have $t_{n+1} \in \tau$. We now claim that $t_n a$ satisfies the conditions on the right in $(*)$, so $t_n a \in \tau$ as well. Indeed, we have $s'a \in \llparenthesis\, \sigma \,\rrparenthesis$, so there exists some $\mathfrak{s}a \in (M_B \times \omega \sqcup M_A \times \omega)^{*<\alpha}$ with $\mathfrak{s}|_{B_0,!A} = s$ and $\mathfrak{s}a|_{B_i,A_i,B_{i+1}} \in \sigma$ for all i.

Consider the sequence

$$\mathfrak{s}a|_{B_n,B_{n+1},A_n,A_{n+1},A_{n+2},\ldots}$$

We have $\mathfrak{s}a|_{B_n,A_n,A_{n+1}} \in \sigma$ (since $\mathfrak{s}a \in \llparenthesis\, \sigma \,\rrparenthesis$) and $\mathfrak{s}a|_{B_{n+1},A_{n+1},A_{n+2},\ldots} = s|_{B_{n+1},A_{n+1},A_{n+2},\ldots} \in \tau$ by $(*)$. Therefore, $\mathfrak{s}a|_{B_n,B_{n+1},A_n,A_{n+1},A_{n+2},\ldots}$ satisfies the conditions for the right hand side of $(*)$ (after subtracting n from each of the indices) and therefore

$$t_n a = \mathfrak{s}a|_{B_n,A_n,A_{n+1},\ldots} \in \tau$$

Moreover, this sequence is contained in $\langle\!\langle\, \sigma \,\rangle\!\rangle$ (by checking the definition).

By repeatedly applying this argument, we eventually show that

$$sa = \mathfrak{s}a|_{B_0,A_0,A_1,\ldots} \in \tau$$

This completes the proof. $\square$

We still need to show that $\langle\!\langle\, \sigma \,\rangle\!\rangle$ is indeed a strategy. We shall start with a lemma that will take the role of Lemma 3.15.

Lemma 4.8. *Let $s \in \langle\!\langle\, \sigma \,\rangle\!\rangle$. Then there exists a unique shortest sequence $\mathfrak{s} \in \|\sigma\|$ such that $\mathfrak{s}|_{B,!A} = s$.*

Proof. The definition of $\langle\!\langle\, \sigma \,\rangle\!\rangle$ tells us that there exists some $\mathfrak{s} \in \|\sigma\|$ with $\mathfrak{s}|_{B,!A} = s$. We may assume that $\mathfrak{s}$ is minimal with this property by removing moves from the end.

Let β be the length of $\mathfrak{s}$ We shall prove that if $\mathfrak{t}$ has length β and $\mathfrak{t}|_{B,!A} = s$ then $\mathfrak{t} = \mathfrak{s}$.

Write $\mathfrak{s}^\delta$ for the initial prefix of $\mathfrak{s}$ of length δ. We shall prove by induction on δ that $\mathfrak{s}^\delta$ is the unique sequence $\mathfrak{t}^\delta$ of length δ such that $\mathfrak{t}^\delta|_{B,!A} \sqsubseteq s$.

Suppose we have some sequence $\mathfrak{t}^{\delta+1}$ of length $\delta + 1$ such that $\mathfrak{t}^{\delta+1}|_{B,!A} \sqsubseteq s$. Then, by the induction hypothesis, we must have $\mathfrak{t}^{\delta+1} = \mathfrak{s}^\delta x$ for some x. We now split into cases:

Case 1: $\mathfrak{s}^\delta$ is a limiting play or $\mathfrak{s}^\delta$ ends with a P-move in a copy of A. In this case, the condition on $\|\sigma\|$ tells us that x must be a move in a copy of A, and so

$\square$

5 The final sequence

Familiarity with the final sequence is assumed; see [Wor05] for a less terse presentation of the following material.

Let $\mathcal{C}$ be a category and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor. Suppose that $\mathcal{C}$ has a terminal object 1. We shall further assume that we are able to construct all the limits we want in $\mathcal{C}$. Let **Ord** denote the ordered class of ordinals, considered as a category. We shall inductively construct a functor $\mathcal{FS}_{\mathcal{F}} \colon \mathbf{Ord}^{\mathrm{op}} \to \mathcal{C}$ (known as the *final sequence of* $\mathcal{F}$) as follows:

- $\mathcal{FS}_{\mathcal{F}}(0) = 1$
- $\mathcal{FS}_{\mathcal{F}}(\beta + 1) = \mathcal{F}\left(\mathcal{FS}_{F}(\beta)\right)$
- On the level of morphisms, $\mathcal{FS}_{\mathcal{F}}(0 \le \beta)$ is the unique morphism $\mathcal{FS}_{\mathcal{F}}(\beta) \to 1$
- $\mathcal{FS}_{\mathcal{F}}(\beta + 1 \le \beta + 2) = \mathcal{F}(\mathcal{FS}_{\mathcal{F}}(\beta \le \beta + 1))$
- If μ is a limit ordinal, then $\mathcal{FS}_{\mathcal{F}}(\mu)$ is the limit of the diagram

  $$\mu \hookrightarrow \mathbf{Ord} \xrightarrow{\mathcal{FS}_{\mathcal{F}}} \mathcal{C}$$

  and $\mathcal{FS}_{\mathcal{F}}(\beta \le \mu)$ is the arrow in the limiting cone
- Applying the functor $\mathcal{F}$ to the cone $\mathcal{FS}_{\mathcal{F}}(\mu)$ gives rise to a new cone over the above diagram, inducing a unique arrow $\mathcal{FS}_{\mathcal{F}}(\mu + 1) \to \mathcal{FS}_{\mathcal{F}}(\mu)$. We set $\mathcal{FS}_{\mathcal{F}}(\mu \le \mu + 1)$ to be equal to this arrow.

Our goal is to characterize the final sequence for functors of the form $A \oslash \_$ in categories $\mathcal{G}(\alpha)$ of games, in the hope that this will allow us to study the exponential $!A$ in more depth.

5.1 The rank of a transfinite sequence of natural numbers

We fix a (large) ordinal α. The goal of this section is to give an ordinal *rank* to a sequence $\beta \to \omega$ of natural numbers (for $\beta < \alpha$). This rank will turn out to be the key to characterizing the final sequence of the $A \oslash \_$ functor. This rank will not depend on α, and we could do without mentioning α at all (although mentioning α allows us to avoid size issues).

To define our rank, we will give a chain of subsets

$$R_0 \subseteq R_1 \subseteq R_2 \subseteq \ldots \subseteq R_\delta \subseteq \ldots \subseteq \omega^{*\alpha}$$

where δ will run over all the ordinals. We think of R_δ as being the set of all sequences *with rank at most* δ; accordingly, we shall define the *rank* of a sequence s to be the smallest ordinal δ such that $s \in R_\delta$.

Definition 5.1. Let $(s \colon \beta \to \omega) \in \omega^{*<\alpha}$. We define the *derivative* of s to be the sequence Δs given by:

$$s^{-1}\left(\omega \setminus \{0\}\right) \xrightarrow{s} \omega \setminus \{0\} \xrightarrow{n \mapsto n-1} \omega$$

In other words, we remove all occurences of 0 from the sequence s and then subtract 1 from each term.

We shall now define the sets R_δ:

Definition 5.2.

- $R_0 = \{\epsilon\}$
- $R_{\delta+1} = \{s \in \omega^{*<\alpha} : \Delta s \in R_\delta\}$
- If μ is a limit ordinal then R_μ is the limit closure of the union of the R_δ for $\delta < \mu$. In other words, a sequence s is contained in R_μ if **either**
  - length(s) is a successor ordinal and $s \in R_\delta$ for some $\delta < \mu$ **or**
  - length(s) is a limit ordinal and for all proper prefixes $t \subsetneqq s$, $t \in R_\delta$ for some $\delta < \mu$

If $s \in \omega^{*<\alpha}$, we define $\text{rank}(s) = \min\{\delta : s \in R_\delta\}$.

One consequence of the above definition is that if μ is a limit ordinal then all the sequences of rank μ have limiting length.

We need to prove some results so we can be sure that the rank of a sequence is well defined.

Proposition 5.3.

i) If $s \in R_\delta$ and t is any subsequence of s (not necessarily an initial prefix), then $t \in R_\delta$

ii) If $s \in R_\delta$, then $\Delta s \in R_\delta$.

iii) If $\gamma \leq \delta$, then $R_\gamma \subseteq R_\delta$

iv) If $s \in \omega^{*<\alpha}$, then $s \in R_\delta$ for some δ.

Proof.

(i): We prove this by induction on δ. Suppose $s \in R_{\delta+1}$ and t is a subsequence of s. Then $\Delta s \in R_\delta$. Now Δt is a subsequence of Δs, so $\Delta t \in R_\delta$ by the induction hypothesis and therefore $\Delta t \in R_{\delta+1}$.

Let μ be a limit ordinal and suppose that $s \in R_\mu$. If s has successor length, then $s \in R_\delta$ for some $\delta < \mu$. Then $t \in R_\delta$ by the induction hypothesis. If t has successor length, this automatically means that $t \in R_\mu$, while if t has limiting length, then every subsequence of t is contained in R_δ by induction, so $t \in R_\mu$.

Suppose instead that s has limiting length. Then for all proper prefixes $u \subsetneqq s$, we have $u \in R_\delta$ for some $\delta < \mu$. If t has successor length, then t cannot be cofinal in s, so it is a subsequence of some proper prefix u of s. Therefore, $t \in R_\delta$ by the induction hypothesis, and so $t \in R_\mu$ as before.

If instead t has limiting length, then every proper prefix $v \subsetneqq t$ is a subsequence of some proper prefix $u \subsetneqq s$. Therefore, by induction, for every proper prefix $v \subsetneqq t$, we have $v \in R_\delta$ for some $\delta < \mu$ and therefore $t \in \mu$.

(ii): Induction on δ. If $sinR_{\delta+1}$ then $\Delta s \in R_\delta$; by the induction hypothesis, this means that $\Delta\Delta s \in R_\delta$ and therefore that $\Delta s \in R_{\delta+1}$.

Now let μ be a limit ordinal. Suppose $s \in R_\mu$. If s has successor length, it means that $s \in R_\delta$ for some $\delta < \mu$, so $\Delta s \in R_\delta$ by induction. By (i), all the prefixes of Δs are in R_δ as well, so $\Delta s \in R_\mu$.

Suppose instead that s has limiting length. Then for all proper prefixes $t \subsetneqq s$, we have $t \in R_\delta$ for some $\delta < \mu$. If Δs has successor length, then we may choose some proper prefix $t \subsetneqq s$ such that $\Delta s = \Delta t$ and therefore $\Delta s \in R_\delta$, so it is in R_μ. If instead Δs has limiting length, then for all proper prefixes $v \subsetneqq \Delta s$, we may choose some $u \subsetneqq s$ with $v = \Delta u$. Then we have $u \in R_\delta$ for some $\delta < \mu$, so $v \in R_\delta$ by the induction hypothesis. Therefore, $t \in R_\mu$.

(iii): It is clear that $R_\delta \subseteq R_\mu$ if $\delta < \mu$ and μ is a limit ordinal. Therefore, by induction it will suffice to show that $R_\delta \subseteq R_{\delta+1}$ for all δ.

Suppose $s \in R_\delta$. Then $\Delta s \in R_\delta$ by (ii) and therefore $s \in R_{\delta+1}$.

(iv): We show this by induction on the length β of s. If $\beta = 0$ then $s \in R_0$. Suppose that $s = s'n$. If we apply Δ n times to s then we end up with the sequence $\Delta^n s.0$. Applying Δ never lengthens the sequence, by Lemma 3.6. Now if we apply Δ one more time, we end up with a strictly shorter sequence, so by induction it is contained in some R_δ. Then s is contained in $R_{\delta+n+1}$.

Now suppose that s has limiting length. Then, by induction, for all proper prefixes $t \subsetneqq s$ there exists δ_t such that $t \in R_{\delta_t}$. If we let μ be some limit ordinal that is greater than all the δ_t, then $s \in R_\mu$. □

Example 5.4. Any sequence with only finitely many distinct terms has finite rank: indeed, suppose $s \colon \beta \to n$ is such a sequence. Then, after applying the derivative n times, we end up with the empty sequence.

The 'identity' sequence $\omega \to \omega$ has rank ω: indeed, every proper prefix has finite rank, but applying the derivative operator has no effect.

More generally, the division algorithm for ordinal arithmetic tells us that if γ is an ordinal, then there is a unique ordinal γ' and a unique natural number n such that $\gamma = \omega\gamma' + n$. If β is an ordinal, we may define a sequence $c_\beta \colon \beta \to \omega$ by setting:

$$c_\beta(\omega\gamma' + n) = n$$

We show by induction on β that the sequence c_β has rank β. First, we claim that

$$c_\beta = \Delta c_{\beta+1}$$

The first thing to show is that the order type of $c_{\beta+1}^{-1}(\omega \setminus \{0\})$ is β. Indeed, $c_{\beta+1}^{-1}(\omega \setminus \{0\})$ is precisely the set of successor ordinals that are less than $\beta+1$. We therefore have an order preserving bijection

$$\begin{aligned} \beta &\to c_{\beta+1}^{-1}(\omega \setminus \{0\}) \\ \gamma &\mapsto \gamma + 1 \end{aligned}$$

This map sends $\omega\gamma' + n$ to $\omega\gamma' + (n+1)$. The map $\Delta c_{\beta+1}$ is therefore given by:

$$\begin{array}{ccccccc} \beta & \xrightarrow{\sim} & c_{\beta+1}^{-1}(\omega \setminus \{0\}) & \to & \omega \setminus \{0\} & \to & \omega \\ \omega\gamma' + n & \mapsto & \omega\gamma' + (n+1) & \mapsto & n+1 & \mapsto & n \end{array}$$

In other words, it is the same as c_β. This deals with the successor step of the induction.

Now let μ be a limit ordinal. A proper prefix of c_μ is a sequence of the form c_δ for $\delta < \mu$. By the induction hypothesis, $c_\delta \in R_\delta$ for all $\delta < \mu$ and therefore $c_\mu \in R_\mu$.

Now suppose for a contradiction that $c_\mu \in R_\delta$ for some $\delta < \mu$. Then $c_{\delta+1} \sqsubseteq c_\mu$ so, by Proposition 5.3(i), $c_{\delta+1} \in R_\delta$, which contradicts our induction hypothesis.

Therefore, $\operatorname{rank}(c_\mu) = \mu$.

We call c_δ the *canonical sequence of length* δ. We want to use it to obtain a neater characterization of the sets R_δ. What we would like to say is that rank$(s) \geq \delta$ if and only if c_δ is a subsequence of s. However, this is not quite true. For example, the sequence

$$23456\ldots$$

over ω has rank ω, but does not contain the sequence $c_\omega = 01234\ldots$ as a subsequence. We will have to modify the condition a little bit; the cost will be that our characterization only works for limit ordinals, but this is not a big problem.

Definition 5.5. Let $s\colon \beta \to \omega$ be a sequence. We say that s is *weakly increasing* if $s(\delta) < s(\delta+1)$ for all $\delta < \beta$.

Example 5.6. c_β is weakly increasing for any β. If we take, for example, the sequence $c_{\omega+\omega}$:

$$01234\ldots\ 01234\ldots$$

then we see where the 'weakly' comes from: we have $c_{\omega+\omega}(2) > c_{\omega+\omega}(\omega)$, even though $2 < \omega$. *Weakly increasing* means 'increasing on every ω-segment, but unconstrained at limits'.

There are plenty of other weakly increasing sequences. For example:

$$2468\ldots$$
$$12358\ldots\ 2357\ldots$$
$$01234\ldots\ 12345\ldots\ 23456\ldots\ 34567\ldots\ 45678\ldots\ \ldots\ldots$$

Lemma 5.7.

i) Let $s \in \omega^{*<\alpha}$ be weakly increasing. Then Δs is weakly increasing.

ii) Suppose μ is a limit ordinal and that $s \in \omega^{*<\alpha}$ is such that rank$(s) = \mu$. *Then* rank$(\Delta s) = \mu$.

iii) Let μ be a limit ordinal and let $s\colon \mu \to \omega$ be a weakly increasing sequence of length μ. Then rank$(s) = \mu$.

Proof.

(i): Let m, n be adjacent terms in Δs. That means that $(m+1).p$ occurs somewhere in the sequence s, so $p > m+1$, since s is weakly increasing. In particular, $p \neq 0$, so it is not removed from the sequence. It follows that $n = p - 1$ and therefore that $m < n$.

(ii): We have $s \in R_\mu$, so $\Delta s \in R_\mu$; i.e., rank$(\Delta s) \leq \mu$. Suppose that $\Delta s \in R_\delta$ for $\delta < \mu$. Then $s \in R_{\delta+1}$, which is a contradiction.

(iii): We prove the following statement by induction on β: if $s\colon \beta \to \omega$ is a weakly increasing sequence of length β, then

- If $\beta = \omega\beta' + m$ is a successor ordinal, and $s = s'n_0 \dots n_{m-1}$, then $\operatorname{rank}(s) = \omega\beta' + n_{m-1} + 1$
- If $\beta = \omega\beta'$ is a limit ordinal then $\operatorname{rank}(s) = \beta$

Suppose that s has length $\omega\beta' + m$ and write s as $s'n_0 \dots n_{m-1}$, where s' is a limiting play. Write $n = n_{m-1}$ – so $n_i < n$ for $i < m-1$. By the induction hypothesis, s' has rank $\omega\beta'$.

If we apply the derivative n times to s then we end up with the sequence $\Delta^n s'.0$. By part (ii), $\Delta^n s'$ has rank $\omega\beta'$, and so $\Delta^n s'.0$ has rank at least $\omega\beta'$. Moreover, since it is a successor play, it cannot have limiting rank, and therefore its rank is at least $\omega\beta + 1'$. If we take the derivative one more time, then we end up with the sequence $\Delta^{n+1}s'$, which has rank $\omega\beta'$, and it follows that $\Delta^n s'.0$ has rank $\omega\beta' + 1$. Therefore, s has rank $\omega\beta' + n + 1$.

Now suppose that μ is a limit ordinal and that s has length μ. If we take some proper prefix $t \sqsubsetneq s$ of length $\delta < \mu$, then t is weakly increasing, so by the induction hypothesis it has rank at most $\delta + m$ for some m. $\delta + m < \mu$ and t was arbitrary, so it follows that $s \in R_\mu$; i.e., that $\operatorname{rank}(s) \leq \mu$.

We claim that $\operatorname{rank}(s) = \mu$. Indeed, suppose that $s \in R_\delta$ for some $\delta < \mu$. Let u be the prefix of s of length $\delta + 1$. By the induction hypothesis, $\operatorname{rank}(u) \geq \delta + 1 > \operatorname{rank}(s)$, which is a contradiction, since $u \sqsubseteq s$. ∎

We can now give our classification result:

Theorem 5.8. *Let $s \in \omega^{*<\alpha}$ and let $\mu < \alpha$ be a limit ordinal. Then* $\operatorname{rank}(s) \geq \mu$ *if and only if s has a weakly increasing subsequence of length μ.*

For $n \geq 1$, $\operatorname{rank}(s) \geq \mu + n$ *if and only if s has a weakly increasing subsequence t of length μ and a term of magnitude at least $n-1$ that occurs later in the sequence than all the terms of t.*

Proof. If: Suppose t is a weakly increasing subsequence of length μ. By Lemma 5.7, t has rank μ, so by Proposition 5.3, s has rank at least μ.

Write $s = s'u$, where t is a subsequence of s' and u contains some term $m \geq n-1$. Then s' has rank at least μ. Then $\Delta^{n-1}s'$ has rank at least μ and $\Delta^{n-1}u \neq \epsilon$, since u contains the term m. Let x be the first term of $\Delta^{n-1}u$. Then $\Delta^{n-1}s'.x$ is a subsequence of $\Delta^{n-1}s$, and it must have rank at least $\mu + 1$ (since it is a successor play and contains $\Delta^{n-1}s'$ as a subsequence. Therefore, $\Delta^{n-1}s$ has rank at least $\mu + 1$ and so s has rank at least $\mu + 1 + (n-1) = \mu + n$.

Only if: Let $\delta = \mu + n$ be an ordinal $\square$

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