A_M_PS0

Q1

A long straight trench, with rectangular cross section, has been dug in otherwise horizontal ground. The width of the trench is d and its depth $2d$. A particle is projected at speed v, where $v^2 = \lambda dg$, at an angle α to the horizontal, from a point on the ground a distance d from the nearer edge of the trench. The vertical plane in which it moves is perpendicular to the trench. The particle lands on the base of the trench without first touching either of its sides.

- By considering the vertical displacement of the particle when its horizontal displacement is d, show that $(\tan\alpha - \lambda)^2 < \lambda^2 - 1$ and deduce that $\lambda > 1$.
- Show also that $(2\tan\alpha - \lambda)^2 > \lambda^2 + 4(\lambda - 1)$ and deduce that $\alpha > 45^\circ$.
- Show that, provided $\lambda > 1$, α can always be chosen so that the particle lands on the base of the trench without first touching either of its sides.

Solution:

To show that $(\tan\alpha - \lambda)^2 < \lambda^2 - 1$ and deduce that $\lambda > 1$, consider the vertical displacement of the particle when its horizontal displacement is d.

The particle is projected from a point on the ground at $(0, 0)$ with speed v at an angle α to the horizontal, where $v^2 = \lambda dg$. The trench has a width d and depth $2d$, with the nearer edge at $x = d$ and the base at $y = -2d$.

The horizontal displacement at time t is given by:

$$x = v\cos\alpha \cdot t$$

When $x = d$, solve for t:

$$d = v\cos\alpha \cdot t_d \implies t_d = \frac{d}{v\cos\alpha}$$

The vertical displacement at time t is:

$$y = v \sin\alpha \cdot t - \frac{1}{2}gt^2$$

At $t = t_d$, the vertical displacement y_d is:

$$y_d = v \sin\alpha \cdot \left(\frac{d}{v\cos\alpha}\right) - \frac{1}{2}g\left(\frac{d}{v\cos\alpha}\right)^2 = d\tan\alpha - \frac{1}{2}g\frac{d^2}{v^2\cos^2\alpha}$$

Substitute $v^2 = \lambda dg$:

$$y_d = d\tan\alpha - \frac{1}{2}g\frac{d^2}{\lambda dg\cos^2\alpha} = d\tan\alpha - \frac{d}{2\lambda\cos^2\alpha}$$

Using $\sec^2\alpha = 1 + \tan^2\alpha$, so $\frac{1}{\cos^2\alpha} = 1 + \tan^2\alpha$:

$$y_d = d\tan\alpha - \frac{d}{2\lambda}(1 + \tan^2\alpha)$$

Let $t = \tan\alpha$:

$$y_d = d\left(t - \frac{1}{2\lambda}(1 + t^2)\right)$$

For the particle to land on the base without first touching the sides, it must clear the nearer edge at $x = d$. This requires that at $x = d$, the particle is above ground level, so $y_d > 0$ (if $y_d = 0$, it would touch the edge, which is not allowed). Thus:

$$t - \frac{1}{2\lambda}(1 + t^2) > 0$$

Multiply both sides by 2λ (since $\lambda > 0$):

$$2\lambda t - (1 + t^2) > 0$$

Rearrange:

$$-t^2 + 2\lambda t - 1 > 0$$

Multiply by -1 (reversing the inequality):

$$t^2 - 2\lambda t + 1 < 0$$

This quadratic in t can be written as:

$$(t-\lambda)^2 - \lambda^2 + 1 < 0 \implies (t-\lambda)^2 < \lambda^2 - 1$$

Substituting back $t = \tan\alpha$:

$$(\tan\alpha - \lambda)^2 < \lambda^2 - 1$$

For the inequality $(\tan\alpha - \lambda)^2 < \lambda^2 - 1$ to hold, $\lambda^2 - 1$ must be positive. If $\lambda \leq 1$, then $\lambda^2 - 1 \leq 0$, and $(\tan\alpha - \lambda)^2 \geq 0$, so $(\tan\alpha - \lambda)^2 < \lambda^2 - 1$ cannot hold for any α. Thus, $\lambda > 1$ is required for the inequality to be possible.

To show that $(2\tan\alpha - \lambda)^2 > \lambda^2 + 4(\lambda - 1)$ and deduce that $\alpha > 45^\circ$, consider the conditions for the particle to land on the base of the trench without touching the sides.

From the first part of the problem, the particle clears the nearer edge of the trench at $x = d$, which gives the inequality $(\tan\alpha - \lambda)^2 < \lambda^2 - 1$ and $\lambda > 1$. Let $t = \tan\alpha$, so this is $(t - \lambda)^2 < \lambda^2 - 1$, which simplifies to $t^2 - 2\lambda t + 1 < 0$.

For the particle to land on the base without hitting the sides, it must also avoid the farther wall at $x = 2d$. The vertical displacement at $x = 2d$ is given by:

$$y = 2d\left(t - \frac{1}{\lambda}(1 + t^2)\right).$$

To avoid hitting the farther wall, the particle must either clear the wall above it ($y > 0$) or pass below the base ($y < -2d$). However, for the particle to land on the base within the trench ($d < x < 2d$) without hitting the sides, the condition $y < -2d$ at $x = 2d$ must hold. This is because if $y > 0$ at $x = 2d$, the particle may land beyond the trench or hit the wall, but in practice, landing within the trench without hitting walls requires $y < -2d$ at $x = 2d$.

Thus, set $y < -2d$:

$$2d\left(t - \frac{1}{\lambda}(1 + t^2)\right) < -2d.$$

Dividing by $2d > 0$:

$$t - \frac{1}{\lambda}(1 + t^2) < -1.$$

Multiplying both sides by $-\lambda$ (and reversing the inequality since $\lambda > 0$):

$$-\lambda t + (1 + t^2) > \lambda,$$

which simplifies to:

$$t^2 - \lambda t > \lambda - 1.$$

This inequality is equivalent to the required form. Specifically:

$$(2t - \lambda)^2 = 4t^2 - 4\lambda t + \lambda^2,$$

and:

$$\lambda^2 + 4(\lambda - 1) = \lambda^2 + 4\lambda - 4.$$

The inequality $(2t - \lambda)^2 > \lambda^2 + 4(\lambda - 1)$ becomes:

$$4t^2 - 4\lambda t + \lambda^2 > \lambda^2 + 4\lambda - 4,$$

which simplifies to:

$$4t^2 - 4\lambda t > 4\lambda - 4.$$

Dividing by 4:

$$t^2 - \lambda t > \lambda - 1,$$

which matches the derived condition.

Thus, $(2 \tan \alpha - \lambda)^2 > \lambda^2 + 4(\lambda - 1)$.

To deduce that $\alpha > 45^\circ$, note that:

$$t^2 - \lambda t > \lambda - 1 > 0,$$

since $\lambda > 1$. Thus:

$$t(t - \lambda) > 0.$$

As $t = \tan\alpha > 0$ (since the particle is projected towards the trench and lands within it, α is acute), it follows that:

$$t - \lambda > 0,$$

so:

$$t > \lambda > 1.$$

Therefore, $\tan\alpha > 1$, which implies $\alpha > 45^\circ$.

$$\boxed{(2\tan\alpha - \lambda)^2 > \lambda^2 + 4(\lambda - 1)} \quad \text{and} \quad \boxed{\alpha > 45^\circ}$$

To show that, provided $\lambda > 1$, an angle α can always be chosen such that the particle lands on the base of the trench without first touching either side, consider the inequalities derived from the conditions for the particle's motion.

From the previous parts, the conditions are:

- $(\tan\alpha - \lambda)^2 < \lambda^2 - 1$
- $(2\tan\alpha - \lambda)^2 > \lambda^2 + 4(\lambda - 1)$

Set $t = \tan\alpha$. The inequalities become:

- $t^2 - 2\lambda t + 1 < 0$
- $t^2 - \lambda t > \lambda - 1$

Since $\lambda > 1$, analyze the quadratics.

Inequality (1): $t^2 - 2\lambda t + 1 < 0$
The quadratic $f(t) = t^2 - 2\lambda t + 1$ has discriminant $D_1 = 4\lambda^2 - 4 = 4(\lambda^2 - 1) > 0$ (since $\lambda > 1$). The roots are $r_1 = \lambda - \sqrt{\lambda^2 - 1}$ and $r_2 = \lambda + \sqrt{\lambda^2 - 1}$. Since the coefficient of t^2 is positive, $f(t) < 0$ for $r_1 < t < r_2$. Both roots are positive for $\lambda > 1$, with $r_1 > 0$ and $r_2 > \lambda > 1$.

Inequality (2): $t^2 - \lambda t > \lambda - 1$

The quadratic $g(t) = t^2 - \lambda t - (\lambda - 1)$ has discriminant $D_2 = \lambda^2 + 4(\lambda - 1) = \lambda^2 + 4\lambda - 4 > 0$ for $\lambda > 1$ (since at $\lambda = 1$, $D_2 = 1 > 0$, and it increases for $\lambda > 1$). The roots are $s_1 = \frac{\lambda - \sqrt{\lambda^2+4\lambda-4}}{2}$ and $s_2 = \frac{\lambda + \sqrt{\lambda^2+4\lambda-4}}{2}$. Since the coefficient of t^2 is positive, $g(t) > 0$ for $t < s_1$ or $t > s_2$. For $\lambda > 1$, $s_1 < 0$ (since $\sqrt{\lambda^2 + 4\lambda - 4} > \lambda$) and $s_2 > 0$.

Given that $t = \tan\alpha > 0$ (as the particle is projected towards the trench and lands in it, implying α is acute and positive), the relevant part is $t > s_2$.

Now, compare the roots:

- $s_2 = \frac{\lambda + \sqrt{\lambda^2+4\lambda-4}}{2}$
- $r_1 = \lambda - \sqrt{\lambda^2 - 1}$
- $r_2 = \lambda + \sqrt{\lambda^2 - 1}$

It has been shown that $s_2 > r_1$ and $s_2 < r_2$ for $\lambda > 1$:

- $s_2 > r_1$: Consider $k(\lambda) = \sqrt{\lambda^2 + 4\lambda - 4} - \lambda + 2\sqrt{\lambda^2 - 1}$. At $\lambda = 1$, $k(1) = 0$, and for $\lambda > 1$, $k'(\lambda) > 0$, so $k(\lambda) > 0$, implying $\sqrt{\lambda^2 + 4\lambda - 4} > \lambda - 2\sqrt{\lambda^2 - 1}$, which leads to $s_2 > r_1$.
- $s_2 < r_2$: From $\sqrt{\lambda^2 + 4\lambda - 4} < \lambda + 2\sqrt{\lambda^2 - 1}$, which holds as $-\lambda + 1 < \sqrt{\lambda^2 - 1}$ (true since left side negative and right side positive for $\lambda > 1$).

Thus, $r_1 < s_2 < r_2$.

The solution to both inequalities requires $r_1 < t < r_2$ (from (1)) and $t > s_2$ (from (2), since $t > 0$). The interval (s_2, r_2) is non-empty because $s_2 < r_2$, and for any $t \in (s_2, r_2)$:

- $r_1 < s_2 < t < r_2$, so $t^2 - 2\lambda t + 1 < 0$ (inequality (1) holds).
- $t > s_2$, so $t^2 - \lambda t > \lambda - 1$ (inequality (2) holds).

Additionally, $s_2 > \lambda > 1$ (since $\sqrt{\lambda^2 + 4\lambda - 4} > \lambda$), so $t > s_2 > 1$, implying $\alpha > 45^\circ$, which is consistent with the problem.

Since the interval (s_2, r_2) is open and non-empty for any $\lambda > 1$, there exists $t = \tan\alpha$ satisfying both conditions. Thus, an angle α can always be chosen such that the particle lands on the base without touching the sides, provided $\lambda > 1$.

Q2

A triangular prism lies on a horizontal plane. One of the rectangular faces of the prism is vertical; the second is horizontal and in contact with the plane; the third, oblique rectangular face makes an angle α with the horizontal. The two triangular faces of the prism are right angled triangles and are vertical. The prism has mass M and it can move without friction across the plane. A particle of mass m lies on the oblique surface of the prism. The contact between the particle and the plane is rough, with coefficient of friction μ.
- Show that if $\mu < \tan\alpha$, then the system cannot be in equilibrium.
Let $\mu = \tan\lambda$, with $0 < \lambda < \alpha < \frac{1}{4}\pi$. A force P is exerted on the vertical rectangular face of the prism, perpendicular to that face and directed towards the interior of the prism. The particle and prism accelerate, but the particle remains in the same position relative to the prism.
- Show that the magnitude, F, of the frictional force between the particle and the prism is

$$F = \frac{m}{M+m}|(M+m)g\sin\alpha - P\cos\alpha|.$$

- Find a similar expression for the magnitude, N, of the normal reaction between the particle and the prism.
- Hence show that the force P must satisfy

$$(M+m)g\tan(\alpha-\lambda) \leq P \leq (M+m)g\tan(\alpha+\lambda).$$

Solution

To show that the system cannot be in equilibrium if $\mu < \tan\alpha$, consider the forces acting on the particle of mass m on the oblique face of the prism. The oblique face makes an angle α with the horizontal. The forces on the particle are:

- Gravity: mg downward.
- Normal force: N perpendicular to the oblique face.

- Friction force: f parallel to the oblique face, opposing relative motion.

For the system to be in equilibrium, both the prism and the particle must be at rest with no acceleration. In this case, the acceleration of the prism is zero ($A = 0$), and the particle does not slide relative to the prism.

From the force balance on the particle in the direction perpendicular to the oblique face:

$$N - mg\cos\alpha = 0 \implies N = mg\cos\alpha.$$

In the direction parallel to the oblique face, the component of gravity down the incline is $mg\sin\alpha$. For the particle to be in equilibrium (no sliding), the friction force must oppose this component. Thus, with friction acting up the incline:

$$f = mg\sin\alpha.$$

The friction force must satisfy the static friction condition $|f| \leq \mu N$:

$$mg\sin\alpha \leq \mu \cdot mg\cos\alpha \implies \tan\alpha \leq \mu.$$

Therefore, for equilibrium, $\mu \geq \tan\alpha$.

If $\mu < \tan\alpha$, the condition $\tan\alpha \leq \mu$ is violated, meaning the maximum static friction force μN is insufficient to prevent the particle from sliding down the oblique face. Consequently, the particle slides relative to the prism, and the system is not in equilibrium.

The system consists of a triangular prism of mass M and a particle of mass m on its oblique face, which makes an angle α with the horizontal. A force P is applied perpendicular to the vertical rectangular face of the prism, directed towards its interior, causing the system to accelerate. The particle remains stationary relative to the prism, indicating no slipping occurs.

The only external horizontal force acting on the system is P, and since the horizontal plane is frictionless, the total mass of the system is $M + m$. The acceleration a of the system in the horizontal direction (y-direction) is given by Newton's second law:

$$P = (M + m)a \implies a = \frac{P}{M + m}.$$

This acceleration is the same for both the prism and the particle, as they move together.

Consider the forces acting on the particle in an inertial frame. The particle has acceleration components $a_y = a = \frac{P}{M+m}$ and $a_z = 0$. The forces are:

- Gravity: mg downward (negative z-direction).
- Normal force: perpendicular to the oblique face, with components $N \sin \alpha$ in the y-direction and $N \cos \alpha$ in the z-direction, where $N > 0$ is the magnitude.
- Frictional force: parallel to the oblique face, with components F_y in the y-direction and F_z in the z-direction. Since the friction is parallel to the incline, $F_z = -\tan \alpha F_y$.

Applying Newton's second law to the particle in the y and z directions:

- Y-direction: $N \sin \alpha + F_y = ma$.
- Z-direction: $N \cos \alpha + F_z - mg = 0$.

Substitute $a = \frac{P}{M+m}$ and $F_z = -\tan \alpha F_y$:

$$N \sin \alpha + F_y = m \frac{P}{M+m}, \tag{1}$$

$$N \cos \alpha - \tan \alpha F_y = mg. \tag{2}$$

Solve equations (1) and (2) for F_y. Multiply equation (2) by $\cos \alpha$:

$$N \cos^2 \alpha - \sin \alpha F_y = mg \cos \alpha. \tag{2a}$$

Denote $K = \frac{mP}{M+m}$. Equation (1) is:

$$N \sin \alpha + F_y = K. \tag{1a}$$

Multiply equation (1a) by $\sin \alpha$:

$$N \sin^2 \alpha + \sin \alpha F_y = K \sin \alpha. \tag{1b}$$

Add equations (1b) and (2a):

$$N \sin^2 \alpha + \sin \alpha F_y + N \cos^2 \alpha - \sin \alpha F_y = K \sin \alpha + mg \cos \alpha,$$

$$N(\sin^2\alpha + \cos^2\alpha) = K\sin\alpha + mg\cos\alpha,$$

$$N = K\sin\alpha + mg\cos\alpha = \frac{mP}{M+m}\sin\alpha + mg\cos\alpha.$$

Substitute N into equation (1a):

$$F_y = K - N\sin\alpha = \frac{mP}{M+m} - \left(\frac{mP}{M+m}\sin\alpha + mg\cos\alpha\right)\sin\alpha,$$

$$F_y = \frac{mP}{M+m}(1-\sin^2\alpha) - mg\cos\alpha\sin\alpha = \frac{mP}{M+m}\cos^2\alpha - mg\cos\alpha\sin\alpha.$$

Factor:

$$F_y = m\cos\alpha\left(\frac{P\cos\alpha}{M+m} - g\sin\alpha\right).$$

The magnitude of the frictional force is $F = \sqrt{F_y^2 + F_z^2}$. Since $F_z = -\tan\alpha F_y$,

$$F = \sqrt{F_y^2 + (-\tan\alpha F_y)^2} = |F_y|\sqrt{1+\tan^2\alpha} = |F_y|\sec\alpha = \frac{|F_y|}{\cos\alpha},$$

as $\sec\alpha = \frac{1}{\cos\alpha}$ and $\alpha < \frac{\pi}{4}$ implies $\cos\alpha > 0$. Substitute F_y:

$$F = \frac{1}{\cos\alpha}\left|m\cos\alpha\left(\frac{P\cos\alpha}{M+m} - g\sin\alpha\right)\right| = m\left|\frac{P\cos\alpha}{M+m} - g\sin\alpha\right|.$$

Rewrite the expression inside the absolute value:

$$F = m\left|g\sin\alpha - \frac{P\cos\alpha}{M+m}\right| = \frac{m}{M+m}|(M+m)g\sin\alpha - P\cos\alpha|.$$

The acceleration of the system, a, is given by Newton's second law applied to the entire system. The only external horizontal force is P, and since the plane is frictionless, the total mass is $M+m$. Thus,

$$P = (M+m)a \implies a = \frac{P}{M+m}.$$

The normal force must account for both the gravitational component and the component due to acceleration. The correct equations are:

$$N \sin\alpha + f \cos\alpha = ma \quad (1)$$
$$N \cos\alpha - f \sin\alpha = mg \quad (2)$$

where f is the component of the frictional force along the incline.

To solve for N, eliminate f. Multiply equation (1) by $\sin\alpha$ and equation (2) by $\cos\alpha$:

$$N \sin^2\alpha + f \cos\alpha \sin\alpha = ma \sin\alpha \quad (1a)$$
$$N \cos^2\alpha - f \sin\alpha \cos\alpha = mg \cos\alpha \quad (2a)$$

Add equations (1a) and (2a):

$$N \sin^2\alpha + N \cos^2\alpha + f \cos\alpha \sin\alpha - f \sin\alpha \cos\alpha = ma \sin\alpha + mg \cos\alpha$$
$$N(\sin^2\alpha + \cos^2\alpha) = ma \sin\alpha + mg \cos\alpha$$
$$N = ma \sin\alpha + mg \cos\alpha$$

Substitute $a = \frac{P}{M+m}$:

$$N = m\left(g \cos\alpha + \frac{P \sin\alpha}{M+m}\right)$$

This can be rewritten as:

$$N = \frac{m}{M+m}((M+m)g \cos\alpha + P \sin\alpha)$$

Under the given conditions $0 < \lambda < \alpha < \frac{\pi}{4}$, $P > 0$, and all masses and g positive, the expression inside the parentheses is positive. Thus, the magnitude of the normal reaction force is:

$$\boxed{N = \frac{m}{M+m}((M+m)g \cos\alpha + P \sin\alpha)}$$

To determine the range of the force P that ensures the particle remains stationary relative to the prism, the frictional force F must satisfy the static friction condition $|F| \leq \mu N$, where $\mu = \tan\lambda$. The expressions for F and N are given by:

$$F = \frac{m}{M+m}|(M+m)g\sin\alpha - P\cos\alpha|,$$

$$N = \frac{m}{M+m}((M+m)g\cos\alpha + P\sin\alpha).$$

Substituting these into the friction inequality:

$$\left|\frac{m}{M+m}|(M+m)g\sin\alpha - P\cos\alpha|\right| \leq \tan\lambda \cdot \frac{m}{M+m}((M+m)g\cos\alpha + P\sin\alpha).$$

Since $\frac{m}{M+m} > 0$, it can be canceled from both sides:

$$|(M+m)g\sin\alpha - P\cos\alpha| \leq \tan\lambda \cdot ((M+m)g\cos\alpha + P\sin\alpha).$$

Denote $G = (M+m)g > 0$ for simplicity. The inequality becomes:

$$|G\sin\alpha - P\cos\alpha| \leq \tan\lambda\,(G\cos\alpha + P\sin\alpha).$$

The absolute value inequality is equivalent to two separate inequalities:

1. $G\sin\alpha - P\cos\alpha \leq \tan\lambda(G\cos\alpha + P\sin\alpha)$,
2. $G\sin\alpha - P\cos\alpha \geq -\tan\lambda(G\cos\alpha + P\sin\alpha)$.

Solving the first inequality:

$$G\sin\alpha - P\cos\alpha \leq \tan\lambda G\cos\alpha + \tan\lambda P\sin\alpha.$$

Rearrange terms:

$$G\sin\alpha - \tan\lambda G\cos\alpha \leq P\cos\alpha + \tan\lambda P\sin\alpha,$$

$$G(\sin\alpha - \tan\lambda\cos\alpha) \leq P(\cos\alpha + \tan\lambda\sin\alpha).$$

Since $\cos\alpha + \tan\lambda\sin\alpha > 0$ (as $\alpha < \frac{\pi}{4}$ and $\lambda > 0$), divide both sides:

$$P \geq G\frac{\sin\alpha - \tan\lambda\cos\alpha}{\cos\alpha + \tan\lambda\sin\alpha}.$$

Simplify the fraction:

$$\frac{\sin\alpha - \tan\lambda\cos\alpha}{\cos\alpha + \tan\lambda\sin\alpha} = \frac{\sin\alpha - \frac{\sin\lambda}{\cos\lambda}\cos\alpha}{\cos\alpha + \frac{\sin\lambda}{\cos\lambda}\sin\alpha} = \frac{\sin\alpha\cos\lambda - \sin\lambda\cos\alpha}{\cos\alpha\cos\lambda + \sin\lambda\sin\alpha} = \frac{\sin(\alpha-\lambda)}{\cos(\alpha-\lambda)} = \tan(\alpha-\lambda).$$

Thus:

$$P \geq G\tan(\alpha-\lambda) = (M+m)g\tan(\alpha-\lambda).$$

Solving the second inequality:

$$G\sin\alpha - P\cos\alpha \geq -\tan\lambda(G\cos\alpha + P\sin\alpha),$$

$$G\sin\alpha - P\cos\alpha \geq -\tan\lambda G\cos\alpha - \tan\lambda P\sin\alpha.$$

Rearrange terms:

$$G\sin\alpha + \tan\lambda G\cos\alpha \geq P\cos\alpha - \tan\lambda P\sin\alpha,$$

$$G(\sin\alpha + \tan\lambda\cos\alpha) \geq P(\cos\alpha - \tan\lambda\sin\alpha).$$

Note that $\cos\alpha - \tan\lambda\sin\alpha > 0$ (since $\tan\lambda < \tan\alpha$ and $\alpha < \frac{\pi}{4}$), so divide both sides:

$$P \leq G\frac{\sin\alpha + \tan\lambda\cos\alpha}{\cos\alpha - \tan\lambda\sin\alpha}.$$

Simplify the fraction:

$$\frac{\sin\alpha + \tan\lambda\cos\alpha}{\cos\alpha - \tan\lambda\sin\alpha} = \frac{\sin\alpha + \frac{\sin\lambda}{\cos\lambda}\cos\alpha}{\cos\alpha - \frac{\sin\lambda}{\cos\lambda}\sin\alpha} = \frac{\sin\alpha\cos\lambda + \sin\lambda\cos\alpha}{\cos\alpha\cos\lambda - \sin\lambda\sin\alpha} = \frac{\sin(\alpha+\lambda)}{\cos(\alpha+\lambda)} = \tan(\alpha+\lambda).$$

Thus:

$$P \leq G\tan(\alpha+\lambda) = (M+m)g\tan(\alpha+\lambda).$$

Combining both inequalities:

$$(M+m)g\tan(\alpha-\lambda) \le P \le (M+m)g\tan(\alpha+\lambda).$$

Given $0<\lambda<\alpha<\frac{\pi}{4}$, both $\tan(\alpha-\lambda)$ and $\tan(\alpha+\lambda)$ are positive, and the range is well-defined.

Q3

A truck of mass M is connected by a light, rigid tow-bar, which is parallel to the ground, to a trailer of mass kM. A constant driving force D which is parallel to the ground acts on the truck, and the only resistance to motion is a frictional force acting on the trailer, with coefficient of friction μ.

- When the truck pulls the trailer up a slope which makes an angle α to the horizontal, the acceleration is a_1 and there is a tension T_1 in the tow-bar.
- When the truck pulls the trailer on horizontal ground, the acceleration is a_2 and there is a tension T_2 in the tow-bar.
- When the truck pulls the trailer down a slope which makes an angle α to the horizontal, the acceleration is a_3 and there is a tension T_3 in the tow-bar.
  All accelerations are taken to be positive when in the direction of motion of the truck.
- Show that $T_1 = T_3$ and that

$$M(a_1+a_3-2a_2) = 2(T_2-T_1).$$

- It is given that $\mu < 1$.
- Show that

$$a_2 < \tfrac{1}{2}(a_1+a_3) < a_3.$$

- Show further that

$$a_1 < a_2.$$

Solution:

To show that $T_1 = T_3$ and that $M(a_1 + a_3 - 2a_2) = 2(T_2 - T_1)$, consider the forces and accelerations in each scenario. The truck has mass M and the trailer has mass kM. The driving force D acts on the truck, and the frictional force on the trailer is μ times the normal force, opposing motion. The tow-bar is light and rigid, so the tension is uniform and parallel to the direction of motion.

Moving Up the Slope (Angle α)

- **Truck equation** (positive direction up the slope):

$$D - T_1 - Mg\sin\alpha = Ma_1 \quad (1)$$

- **Trailer equation** (positive direction up the slope):

$$T_1 - \mu kMg\cos\alpha - kMg\sin\alpha = kMa_1 \quad (2)$$

Moving on Horizontal Ground

- **Truck equation** (positive direction forward):

$$D - T_2 = Ma_2 \quad (3)$$

- **Trailer equation** (positive direction forward):

$$T_2 - \mu kMg = kMa_2 \quad (4)$$

Moving Down the Slope (Angle α)

- **Truck equation** (positive direction down the slope):

$$D - T_3 + Mg\sin\alpha = Ma_3 \quad (5)$$

- **Trailer equation** (positive direction down the slope):

$$T_3 - \mu kMg\cos\alpha + kMg\sin\alpha = kMa_3 \quad (6)$$

From the trailer equations (2) and (6):

- $T_1 = kMa_1 + \mu kMg\cos\alpha + kMg\sin\alpha \quad (2a)$
- $T_3 = kMa_3 + \mu kMg\cos\alpha - kMg\sin\alpha \quad (6a)$

Subtract equation (6a) from (2a):

$$T_1 - T_3 = (kMa_1 + \mu kMg\cos\alpha + kMg\sin\alpha) - (kMa_3 + \mu kMg\cos\alpha - kMg\sin\alpha)$$
$$= kM(a_1 - a_3) + 2kMg\sin\alpha$$

Solve for the difference $a_1 - a_3$ using the expressions for acceleration:

- From the system equations, the accelerations are:

$$a_1 = \frac{D}{M(1+k)} - \frac{\mu kg\cos\alpha}{1+k} - g\sin\alpha$$
$$a_3 = \frac{D}{M(1+k)} - \frac{\mu kg\cos\alpha}{1+k} + g\sin\alpha$$

- Subtract:

$$a_1 - a_3 = \left(\frac{D}{M(1+k)} - \frac{\mu kg\cos\alpha}{1+k} - g\sin\alpha\right) - \left(\frac{D}{M(1+k)} - \frac{\mu kg\cos\alpha}{1+k} + g\sin\alpha\right) = -2g\sin\alpha$$

Substitute into $T_1 - T_3$:

$$T_1 - T_3 = kM(-2g\sin\alpha) + 2kMg\sin\alpha = -2kMg\sin\alpha + 2kMg\sin\alpha = 0$$

Thus, $T_1 = T_3$.

To prove that $M(a_1 + a_3 - 2a_2) = 2(T_2 - T_1)$, first, express $a_1 + a_3 - 2a_2$:

- Acceleration expressions:

$$a_1 = \frac{D}{M(1+k)} - \frac{\mu k g \cos\alpha}{1+k} - g\sin\alpha$$

$$a_2 = \frac{D}{M(1+k)} - \frac{\mu k g}{1+k} \quad (\text{since} \cos 0^\circ = 1, \sin 0^\circ = 0)$$

$$a_3 = \frac{D}{M(1+k)} - \frac{\mu k g \cos\alpha}{1+k} + g\sin\alpha$$

- Compute:

$$a_1 + a_3 = 2\frac{D}{M(1+k)} - 2\frac{\mu k g \cos\alpha}{1+k}$$

$$2a_2 = 2\frac{D}{M(1+k)} - 2\frac{\mu k g}{1+k}$$

$$a_1 + a_3 - 2a_2 = \left(2\frac{D}{M(1+k)} - 2\frac{\mu k g \cos\alpha}{1+k}\right) - \left(2\frac{D}{M(1+k)} - 2\frac{\mu k g}{1+k}\right)$$

$$= -2\frac{\mu k g \cos\alpha}{1+k} + 2\frac{\mu k g}{1+k} = \frac{2\mu k g}{1+k}(1-\cos\alpha)$$

- Multiply by M:

$$M(a_1 + a_3 - 2a_2) = M \cdot \frac{2\mu k g}{1+k}(1-\cos\alpha) = \frac{2\mu k g M}{1+k}(1-\cos\alpha)$$

Now, compute $2(T_2 - T_1)$:

- Tension expressions:

$$T_2 = kMa_2 + \mu k M g \quad (\text{from (4)})$$

$$T_1 = kMa_1 + \mu k M g\cos\alpha + kMg\sin\alpha \quad (\text{from (2a)})$$

- Difference:

$$T_2 - T_1 = (kMa_2 + \mu kMg) - (kMa_1 + \mu kMg\cos\alpha + kMg\sin\alpha)$$
$$= kM(a_2 - a_1) + \mu kMg(1 - \cos\alpha) - kMg\sin\alpha$$

- Substitute $a_2 - a_1$:

$$a_2 - a_1 = \left(\frac{D}{M(1+k)} - \frac{\mu kg}{1+k}\right) - \left(\frac{D}{M(1+k)} - \frac{\mu kg\cos\alpha}{1+k} - g\sin\alpha\right)$$
$$= -\frac{\mu kg}{1+k} + \frac{\mu kg\cos\alpha}{1+k} + g\sin\alpha = \frac{\mu kg}{1+k}(\cos\alpha - 1) + g\sin\alpha$$

- Thus:

$$T_2 - T_1 = kM\left[\frac{\mu kg}{1+k}(\cos\alpha - 1) + g\sin\alpha\right] + \mu kMg(1 - \cos\alpha) - kMg\sin\alpha$$
$$= \frac{kM\mu kg}{1+k}(\cos\alpha - 1) + kMg\sin\alpha + \mu kMg(1 - \cos\alpha) - kMg\sin\alpha$$
$$= \frac{k^2M\mu g}{1+k}(\cos\alpha - 1) + \mu kMg(1 - \cos\alpha) \quad (\text{since } kMg\sin\alpha - kMg\sin\alpha = 0)$$
$$= \mu kMg(\cos\alpha - 1)\left(\frac{k}{1+k} - 1\right) \quad (\text{using } 1 - \cos\alpha = -(\cos\alpha - 1))$$
$$\left(\frac{k}{1+k} - 1\right) = \frac{k - 1 - k}{1+k} = \frac{-1}{1+k}$$
$$T_2 - T_1 = \mu kMg(\cos\alpha - 1)\left(-\frac{1}{1+k}\right) = \frac{\mu kMg}{1+k}(1 - \cos\alpha)$$

- Multiply by 2:

$$2(T_2 - T_1) = 2 \cdot \frac{\mu kMg}{1+k}(1 - \cos\alpha) = \frac{2\mu kMg}{1+k}(1 - \cos\alpha)$$

Since $M(a_1 + a_3 - 2a_2) = \frac{2\mu kMg}{1+k}(1 - \cos\alpha)$ and $2(T_2 - T_1) = \frac{2\mu kMg}{1+k}(1 - \cos\alpha)$, it follows that:

$$M(a_1 + a_3 - 2a_2) = 2(T_2 - T_1)$$

To show that $a_2 < \frac{1}{2}(a_1 + a_3) < a_3$ given $\mu < 1$, consider the expressions for the accelerations derived from the system dynamics.

The acceleration a_1 when moving up the slope, a_2 on horizontal ground, and a_3 when moving down the slope are given by:

$$a_1 = \frac{D}{M(1+k)} - \frac{\mu k g \cos\alpha}{1+k} - g\sin\alpha,$$

$$a_2 = \frac{D}{M(1+k)} - \frac{\mu k g}{1+k},$$

$$a_3 = \frac{D}{M(1+k)} - \frac{\mu k g \cos\alpha}{1+k} + g\sin\alpha.$$

From previous results, the relationship between the accelerations is:

$$a_1 + a_3 - 2a_2 = \frac{2\mu k g}{1+k}(1 - \cos\alpha).$$

Since $\mu > 0$ (as the coefficient of friction is positive), $k > 0$, $g > 0$, and for a slope with $\alpha > 0$, $1 - \cos\alpha > 0$, the right-hand side is positive. Thus,

$$a_1 + a_3 - 2a_2 > 0 \implies a_1 + a_3 > 2a_2 \implies \frac{1}{2}(a_1 + a_3) > a_2.$$

Now, compare a_1 and a_3:

$$a_3 - a_1 = \left(\frac{D}{M(1+k)} - \frac{\mu k g \cos\alpha}{1+k} + g\sin\alpha\right) - \left(\frac{D}{M(1+k)} - \frac{\mu k g \cos\alpha}{1+k} - g\sin\alpha\right) = 2g\sin\alpha.$$

Since $g > 0$ and $\alpha > 0$, $\sin\alpha > 0$, so:

$$a_3 - a_1 > 0 \implies a_3 > a_1.$$

Given $a_3 > a_1$,

$$\frac{1}{2}(a_1 + a_3) < \frac{1}{2}(a_3 + a_3) = a_3.$$

Combining the inequalities:

$$a_2 < \frac{1}{2}(a_1 + a_3) < a_3.$$

To show that $a_1 < a_2$, consider the expressions for the accelerations in the two scenarios.

The acceleration when the truck pulls the trailer up a slope of angle α is:

$$a_1 = \frac{D}{M(1+k)} - \frac{\mu kg\cos\alpha}{1+k} - g\sin\alpha.$$

The acceleration on horizontal ground is:

$$a_2 = \frac{D}{M(1+k)} - \frac{\mu kg}{1+k}.$$

The difference $a_1 - a_2$ is:

$$\begin{aligned} a_1 - a_2 &= \left(\frac{D}{M(1+k)} - \frac{\mu kg\cos\alpha}{1+k} - g\sin\alpha\right) - \left(\frac{D}{M(1+k)} - \frac{\mu kg}{1+k}\right) \\ &= -\frac{\mu kg\cos\alpha}{1+k} - g\sin\alpha + \frac{\mu kg}{1+k}. \end{aligned}$$

Simplifying gives:

$$a_1 - a_2 = g\left[\frac{\mu k}{1+k}(1-\cos\alpha) - \sin\alpha\right].$$

Define $r = \frac{\mu k}{1+k} > 0$ (since $\mu > 0$ and $k > 0$). Thus:

$$a_1 - a_2 = g\left[r(1-\cos\alpha) - \sin\alpha\right].$$

Consider the function $h(\alpha) = r(1-\cos\alpha) - \sin\alpha$. For $\alpha > 0$:

- At $\alpha = 0$, $h(0) = r(1-1) - 0 = 0$.
- The derivative is $h'(\alpha) = r\sin\alpha - \cos\alpha$.

- At $\alpha = 0$, $h'(0) = r \cdot 0 - 1 = -1 < 0$, so $h(\alpha)$ decreases for small α.
- Since $r < 1$ (as $\mu < 1$ and $k > 0$, but even without this, analysis shows that for all $\alpha > 0$, $h(\alpha) < 0$.

Specifically:

- For small $\alpha > 0$, $1 - \cos\alpha \approx \frac{\alpha^2}{2}$ and $\sin\alpha \approx \alpha$, so $h(\alpha) \approx r\frac{\alpha^2}{2} - \alpha = \alpha\left(\frac{r\alpha}{2} - 1\right) < 0$ since α is small and positive.
- At critical points where $h'(\alpha) = 0$, i.e., $\tan\alpha = \frac{1}{r}$, $h(\alpha) = r - \sqrt{r^2+1} < 0$.
- As $\alpha \to 90^\circ-$, $h(\alpha) \to r(1-0) - 1 = r - 1 < 0$ (since $r < 1$).

Thus, for all $\alpha > 0$, $h(\alpha) < 0$. Therefore:

$$a_1 - a_2 = gh(\alpha) < 0,$$

since $g > 0$ and $h(\alpha) < 0$. Hence:

$$a_1 < a_2.$$

The condition $\mu < 1$ ensures that $r < 1$, but the inequality holds for any $\mu > 0$ and $\alpha > 0$ due to the strict decrease of $h(\alpha)$ from 0.

Q4

In this question, the x- and y-axes are horizontal and the z-axis is vertically upwards.

- A particle P_α is projected from the origin with speed u at an acute angle α above the positive x-axis. The curve E is given by $z = A - Bx^2$ and $y = 0$. If E and the trajectory of P_α touch exactly once, show that

$$u^2 - 2gA = u^2(1 - 4AB)\cos^2\alpha.$$

- E and the trajectory of P_α touch exactly once for all α with $0 < \alpha < \frac{1}{2}\pi$. Write down the values of A and B in terms of u and g.

An explosion takes place at the origin and results in a large number of particles being simultaneously projected with speed u in different directions. You may assume that all the particles move freely under gravity for $t \geq 0$.

- Describe the set of points which can be hit by particles from the explosion, explaining your answer.
- Show that, at a time t after the explosion, the particles lie on a sphere whose centre and radius you should find.
- Another particle Q is projected horizontally from the point $(0, 0, A)$ with speed u in the positive x direction. Show that, at all times, Q lies on the curve E.
- Show that for particles Q and P_α to collide, Q must be projected a time

  $$\frac{u(1-\cos\alpha)}{g\sin\alpha}$$

  after the explosion.

Solution:

The curve E is given by $z = A - Bx^2$ and $y = 0$, which lies in the xz-plane. The trajectory of particle P_α projected from the origin with speed u at an angle α above the positive x-axis is confined to the xz-plane. The parametric equations of motion are:

$$x(t) = u\cos\alpha \cdot t, \quad z(t) = u\sin\alpha \cdot t - \frac{1}{2}gt^2.$$

Eliminating t gives the trajectory equation:

$$z(x) = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}.$$

For the trajectory and curve E to touch exactly once at a point $(x_0, z(x_0))$, they must intersect and have the same derivative at $x = x_0$:

$$z_{\text{traj}}(x_0) = z_E(x_0), \quad \left.\frac{dz_{\text{traj}}}{dx}\right|_{x=x_0} = \left.\frac{dz_E}{dx}\right|_{x=x_0}.$$

The derivatives are:

$$\frac{dz_{\text{traj}}}{dx} = \tan\alpha - \frac{gx}{u^2\cos^2\alpha}, \quad \frac{dz_E}{dx} = -2Bx.$$

Setting the derivatives equal at $x = x_0$:

$$\tan\alpha - \frac{gx_0}{u^2\cos^2\alpha} = -2Bx_0. \quad (1)$$

Setting the z-values equal:

$$x_0\tan\alpha - \frac{gx_0^2}{2u^2\cos^2\alpha} = A - Bx_0^2. \quad (2)$$

Solving equation (1) for $\tan\alpha$:

$$\tan\alpha = x_0\left(\frac{g}{u^2\cos^2\alpha} - 2B\right).$$

Substituting $\frac{1}{\cos^2\alpha} = 1 + \tan^2\alpha$ and letting $s = \tan\alpha$:

$$s = x_0\left(\frac{g}{u^2}(1+s^2) - 2B\right). \quad (1')$$

Equation (2) becomes:

$$x_0 s - \frac{gx_0^2}{2u^2}(1+s^2) = A - Bx_0^2. \quad (2')$$

From equation (1'), $\frac{g}{u^2}(1+s^2)x_0 = s + 2Bx_0$. Substituting into equation (2'):

$$x_0 s - \frac{1}{2}(sx_0 + 2Bx_0^2) = A - Bx_0^2,$$

which simplifies to:

$$\frac{1}{2}sx_0 - Bx_0^2 = A - Bx_0^2,$$

so:

$$\frac{1}{2}sx_0 = A \implies sx_0 = 2A \implies x_0\tan\alpha = 2A. \quad (3)$$

Substituting $s = \tan\alpha = \frac{2A}{x_0}$ into equation (1):

$$\frac{2A}{x_0} + 2Bx_0 = \frac{gx_0}{u^2 \cos^2 \alpha}.$$

Multiplying both sides by x_0:

$$2A + 2Bx_0^2 = \frac{gx_0^2}{u^2 \cos^2 \alpha}.$$

Using $\frac{1}{\cos^2 \alpha} = 1 + \tan^2 \alpha = 1 + \left(\frac{2A}{x_0}\right)^2$:

$$2A + 2Bx_0^2 = \frac{g}{u^2} x_0^2 \left(1 + \frac{4A^2}{x_0^2}\right) = \frac{g}{u^2}(x_0^2 + 4A^2).$$

Thus:

$$2A + 2Bx_0^2 = \frac{g}{u^2} x_0^2 + \frac{4gA^2}{u^2}.$$

Substituting $x_0 = \frac{2A}{\tan \alpha}$ from equation (3) and simplifying leads to:

$$\sin^2 \alpha + 4AB \cos^2 \alpha = \frac{2gA}{u^2}.$$

Using $\sin^2 \alpha = 1 - \cos^2 \alpha$:

$$1 - \cos^2 \alpha + 4AB \cos^2 \alpha = \frac{2gA}{u^2},$$

$$1 + (4AB - 1) \cos^2 \alpha = \frac{2gA}{u^2}.$$

Rearranging:

$$(4AB - 1) \cos^2 \alpha = \frac{2gA}{u^2} - 1.$$

Multiplying both sides by $-u^2$:

$$-u^2(4AB-1)\cos^2\alpha = -u^2\left(\frac{2gA}{u^2}-1\right) = u^2-2gA.$$

Since $1-4AB = -(4AB-1)$:

$$u^2(1-4AB)\cos^2\alpha = u^2-2gA.$$

Thus, the equation holds under the tangency condition for a given α.

Given that tangency occurs for all α with $0<\alpha<\frac{\pi}{2}$, and A and B are constants, the equation must hold for all such α. The left side, u^2-2gA, is independent of α, so the right side must also be independent of α. Since $\cos^2\alpha$ varies with α, the coefficient $u^2(1-4AB)$ must be zero for the right side to be constant. Thus:

$$1-4AB=0 \implies AB=\frac{1}{4}.$$

Then:

$$u^2-2gA=0 \implies A=\frac{u^2}{2g}.$$

Substituting into $AB=\frac{1}{4}$:

$$\frac{u^2}{2g}\cdot B=\frac{1}{4} \implies B=\frac{1}{4}\cdot\frac{2g}{u^2}=\frac{g}{2u^2}.$$

The set of points that can be hit by particles from the explosion is the closed region bounded by and including the paraboloid of revolution defined by the equation $x^2+y^2=\frac{u^4}{g^2}-2\frac{u^2}{g}z$. This region satisfies the inequality $x^2+y^2+2\frac{u^2}{g}z\le\left(\frac{u^2}{g}\right)^2$. This is because:

- Each particle is projected from the origin with speed u in a different direction and moves under gravity. The position of a particle at time $t\ge 0$ is given by:

$$x=v_xt,\quad y=v_yt,\quad z=v_zt-\frac{1}{2}gt^2,$$

  where the initial velocity components satisfy $v_x^2+v_y^2+v_z^2=u^2$.

- A point (x, y, z) can be hit if there exist initial velocity components v_x, v_y, v_z and a time $t > 0$ such that the position equations hold and the speed constraint is satisfied.
- Substituting the position equations into the speed constraint and solving for t leads to a quadratic equation in $s = t^2$:

$$\frac{1}{4}g^2 s^2 + (gz - u^2)s + (x^2 + y^2 + z^2) = 0.$$

- For real solutions $s > 0$ (i.e., $t > 0$), the discriminant must be non-negative, and the sum of the roots must be positive (since the product of the roots is positive). The discriminant is:

$$D = (gz - u^2)^2 - g^2(x^2 + y^2) = u^4 - 2gu^2 z - g^2 r^2,$$

  where $r^2 = x^2 + y^2$. The conditions $D \geq 0$ and $u^2 - gz > 0$ (which ensures the sum of the roots is positive) simplify to:

$$u^4 - 2gu^2 z - g^2 r^2 \geq 0 \quad \text{and} \quad z < \frac{u^2}{g}.$$

- Rearranging the inequality $u^4 - 2gu^2 z - g^2 r^2 \geq 0$ gives:

$$r^2 \leq \frac{u^4}{g^2} - 2\frac{u^2}{g}z,$$

  or equivalently,

$$x^2 + y^2 + 2\frac{u^2}{g}z \leq \left(\frac{u^2}{g}\right)^2.$$

- Points satisfying this inequality and $z < u^2/g$ are reachable. However, the condition $z < u^2/g$ is redundant because it is implied by the inequality (since at $z = u^2/g$, the left side is $x^2 + y^2 + 2u^4/g^2$, which exceeds u^4/g^2 unless $x = y = 0$, but even then, equality does not hold and no solution exists for $t > 0$).
- The boundary of this region, where equality holds, corresponds to points hit by exactly one trajectory. Points inside the region are hit by at least one trajectory (and typically two trajectories for interior points). The origin $(0, 0, 0)$ is included and is hit by the vertically projected particle at time $t = 2u/g$.

- The paraboloid opens downward with its vertex at $\left(0, 0, \frac{u^2}{2g}\right)$, and it intersects the xy-plane $(z = 0)$ at $x^2 + y^2 = (u^2/g)^2$, which matches the maximum horizontal range on level ground.

Thus, the set of points that can be hit is the closed region defined by the inequality above.

At any time $t \geq 0$ after the explosion, the particles are projected simultaneously from the origin with initial speed u in all directions and move under the influence of gravity. The position of a particle at time t depends on its initial velocity components (v_x, v_y, v_z), which satisfy $v_x^2 + v_y^2 + v_z^2 = u^2$. The coordinates of a particle at time t are given by:

$$x = v_x t, \quad y = v_y t, \quad z = v_z t - \frac{1}{2} g t^2.$$

To determine the set of points occupied by the particles at time t, eliminate the velocity components. Solving for v_x, v_y, and v_z:

$$v_x = \frac{x}{t}, \quad v_y = \frac{y}{t}, \quad v_z = \frac{z}{t} + \frac{1}{2} g t.$$

Substitute these into the initial speed equation:

$$\left(\frac{x}{t}\right)^2 + \left(\frac{y}{t}\right)^2 + \left(\frac{z}{t} + \frac{1}{2} g t\right)^2 = u^2.$$

Expand and simplify:

$$\frac{x^2}{t^2} + \frac{y^2}{t^2} + \left(\frac{z^2}{t^2} + gz + \frac{g^2 t^2}{4}\right) = u^2,$$

$$\frac{x^2 + y^2 + z^2}{t^2} + gz + \frac{g^2 t^2}{4} = u^2.$$

Multiply both sides by t^2:

$$x^2 + y^2 + z^2 + gzt^2 + \frac{g^2 t^4}{4} = u^2 t^2.$$

Complete the square for the z-terms. Note that:

$$z^2 + gzt^2 = \left(z + \frac{1}{2}gt^2\right)^2 - \left(\frac{1}{2}gt^2\right)^2 = \left(z + \frac{1}{2}gt^2\right)^2 - \frac{g^2t^4}{4}.$$

Substitute back:

$$x^2 + y^2 + \left(\left(z + \frac{1}{2}gt^2\right)^2 - \frac{g^2t^4}{4}\right) + \frac{g^2t^4}{4} = u^2t^2,$$

$$x^2 + y^2 + \left(z + \frac{1}{2}gt^2\right)^2 = u^2t^2.$$

This equation represents a sphere in three-dimensional space. The center of the sphere is at $(0, 0, -\frac{1}{2}gt^2)$, and the radius is ut.

Particle Q is projected horizontally from the point $(0, 0, A)$ with initial velocity $(u, 0, 0)$, where $A = \frac{u^2}{2g}$. The motion is under gravity, so the position of Q at time t is given by:

$$x(t) = ut, \quad y(t) = 0, \quad z(t) = A - \frac{1}{2}gt^2.$$

The curve E is defined by $z = A - Bx^2$ and $y = 0$, with $B = \frac{g}{2u^2}$. Substituting $x(t) = ut$ into the equation for E:

$$z_E = A - B(ut)^2 = A - Bu^2t^2.$$

Substitute $B = \frac{g}{2u^2}$:

$$z_E = A - \left(\frac{g}{2u^2}\right)u^2t^2 = A - \frac{g}{2}t^2.$$

The z-coordinate of Q is:

$$z(t) = A - \frac{1}{2}gt^2.$$

Comparing $z(t)$ and z_E:

$$z(t) = A - \frac{1}{2}gt^2 = z_E.$$

Additionally, $y(t) = 0$ for all t, which satisfies the $y = 0$ condition of curve E.

To determine the time after the explosion at which particle Q must be projected for it to collide with particle P_α, consider the motion of both particles.

Particle P_α is projected from the origin at time $t = 0$ (the time of the explosion) with speed u at an acute angle α above the positive x-axis. Its position at time $t \geq 0$ is given by:

$$x_p = u\cos\alpha \cdot t, \quad y_p = 0, \quad z_p = u\sin\alpha \cdot t - \frac{1}{2}gt^2.$$

Particle Q is projected horizontally from the point $(0, 0, A)$ at time $t = \tau$ (where $\tau > 0$ is the time after the explosion) with speed u in the positive x-direction. Its position at time $t \geq \tau$ is given by:

$$x_q = u(t - \tau), \quad y_q = 0, \quad z_q = A - \frac{1}{2}g(t - \tau)^2.$$

For a collision to occur, the positions of P_α and Q must coincide at some time $t \geq \tau$. Equating the x-coordinates:

$$u\cos\alpha \cdot t = u(t - \tau).$$

Dividing both sides by u (assuming $u \neq 0$):

$$\cos\alpha \cdot t = t - \tau,$$

which simplifies to:

$$\tau = t(1 - \cos\alpha). \quad (1)$$

Equating the z-coordinates:

$$u\sin\alpha \cdot t - \frac{1}{2}gt^2 = A - \frac{1}{2}g(t - \tau)^2.$$

Substituting $\tau = t(1 - \cos\alpha)$ from equation (1):

$$t - \tau = t - t(1 - \cos\alpha) = t\cos\alpha.$$

Thus:

$$z_q = A - \frac{1}{2}g(t\cos\alpha)^2 = A - \frac{1}{2}gt^2\cos^2\alpha.$$

The equation becomes:

$$u\sin\alpha\cdot t - \frac{1}{2}gt^2 = A - \frac{1}{2}gt^2\cos^2\alpha.$$

Rearranging terms:

$$u\sin\alpha\cdot t - \frac{1}{2}gt^2 - A + \frac{1}{2}gt^2\cos^2\alpha = 0.$$

Factoring the gt^2 terms:

$$u\sin\alpha\cdot t - A + \frac{1}{2}gt^2(\cos^2\alpha - 1).$$

Since $\cos^2\alpha - 1 = -\sin^2\alpha$:

$$u\sin\alpha\cdot t - A - \frac{1}{2}gt^2\sin^2\alpha = 0. \quad (2)$$

From previous parts, $A = \frac{u^2}{2g}$. Substituting this into equation (2):

$$u\sin\alpha\cdot t - \frac{u^2}{2g} - \frac{1}{2}gt^2\sin^2\alpha = 0.$$

Multiplying through by $2g$ to clear denominators:

$$2g\cdot u\sin\alpha\cdot t - 2g\cdot\frac{u^2}{2g} - 2g\cdot\frac{1}{2}gt^2\sin^2\alpha = 0,$$

which simplifies to:

$$2gu\sin\alpha\cdot t - u^2 - g^2t^2\sin^2\alpha = 0.$$

Rearranging:

$$g^2t^2\sin^2\alpha - 2gu\sin\alpha\cdot t + u^2 = 0.$$

This is a perfect square:

$$(gt\sin\alpha - u)^2 = 0.$$

Thus:

$$gt\sin\alpha - u = 0,$$

so:

$$t = \frac{u}{g\sin\alpha}. \quad (3)$$

Substituting equation (3) into equation (1):

$$\tau = \frac{u}{g\sin\alpha}(1-\cos\alpha) = \frac{u(1-\cos\alpha)}{g\sin\alpha}.$$

Q5

A rectangular prism is fixed on a horizontal surface. A vertical wall, parallel to a vertical face of the prism, stands at a distance d from it. A light plank, making an acute angle θ with the horizontal, rests on an upper edge of the prism and is in contact with the wall below the level of that edge of the prism and above the level of the horizontal plane. You may assume that the plank is long enough and the prism high enough to make this possible. The contact between the plank and the prism is smooth, and the coefficient of friction at the contact between the plank and the wall is μ. When a heavy point mass is fixed to the plank at a distance x, along the plank, from its point of contact with the wall, the system is in equilibrium.

- Show that, if $x = d\sec^3\theta$, then there is no frictional force acting between the plank and the wall.
- Show that, if $x > d\sec^3\theta$, it is necessary that

$$\mu \geq \frac{x - d\sec^3\theta}{x\tan\theta}$$

and give the corresponding inequality if $x < d\sec^3\theta$.

- Show that

$$\frac{x}{d} \geq \frac{\sec^3\theta}{1 + \mu\tan\theta}.$$

- Show also that, if $\mu < \cot\theta$, then

$$\frac{x}{d} \leq \frac{\sec^3\theta}{1 - \mu\tan\theta}.$$

- Show that if x is such that the point mass is fixed to the plank somewhere between the edge of the prism and the wall, then $\tan\theta < \mu$.

Solution:

To show that there is no frictional force acting between the plank and the wall when $x = d\sec^3\theta$, consider the forces and torques acting on the plank in equilibrium. The plank is light, so its weight is negligible, and the only significant forces are due to the point mass and the contacts at the wall and the prism.

Define the coordinate system with point A at the contact with the wall as the origin $(0, 0)$. The plank makes an angle θ with the horizontal, so point B, the contact with the prism's upper edge, is at $(d, d\tan\theta)$. The length of the plank from A to B is $L = d\sec\theta$.

The forces on the plank are:

- At A: Normal force from the wall, N_w, in the $+x$ direction (horizontal, away from the wall), and frictional force f in the vertical direction (assumed positive upward).
- At B: Normal force from the prism, N_p, perpendicular to the plank. Since the contact is smooth, N_p has components $(-N_p\sin\theta, N_p\cos\theta)$.
- At the point mass (distance x from A along the plank): Weight mg downward, so force $(0, -mg)$, applied at position $(x\cos\theta, x\sin\theta)$.

For equilibrium, the net horizontal force, net vertical force, and net torque about any point must be zero.

Net horizontal force:

$$N_w - N_p \sin\theta = 0 \quad \Rightarrow \quad N_w = N_p \sin\theta \tag{1}$$

Net vertical force:

$$f + N_p \cos\theta - mg = 0 \quad \Rightarrow \quad f = mg - N_p \cos\theta \tag{2}$$

Net torque about point A:

- Torque due to weight at mass: Position $(x\cos\theta, x\sin\theta)$, force $(0, -mg)$, so torque is $-(mg)(x\cos\theta)$ (clockwise, negative if counterclockwise is positive).
- Torque due to force at B: Position $(d, d\tan\theta)$, force $(-N_p \sin\theta, N_p \cos\theta)$, so torque is

$$d \cdot (N_p \cos\theta) - (d\tan\theta) \cdot (-N_p \sin\theta) = dN_p \cos\theta + dN_p \sin\theta \cdot \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\left(d\sec^3\theta - x\right)\cos\theta}{x\sin\theta} = \frac{d\sec^3\theta - x}{x\tan\theta}$$

- Forces at A contribute no torque.
- Net torque about A:

$$dN_p \sec\theta - mgx\cos\theta = 0 \quad \Rightarrow \quad dN_p \sec\theta = mgx\cos\theta \tag{3}$$

Solve equation (3) for N_p:

$$dN_p \sec\theta = mgx\cos\theta \quad \Rightarrow \quad dN_p \frac{1}{\cos\theta} = mgx\cos\theta \quad \Rightarrow \quad dN_p = mgx\cos^2\theta \quad \Rightarrow \quad N_p = \frac{mgx\cos^2\theta}{d} \tag{4}$$

Now substitute $x = d\sec^3\theta = d/\cos^3\theta$ into equation (4):

$$N_p = \frac{mg(d/\cos^3\theta)\cos^2\theta}{d} = \frac{mg(1/\cos^3\theta)\cos^2\theta}{1} = \frac{mg}{\cos\theta}$$

From equation (2):

$$f = mg - N_p \cos\theta = mg - \left(\frac{mg}{\cos\theta}\right)\cos\theta = mg - mg = 0$$

Thus, when $x = d\sec^3\theta$, the frictional force $f = 0$, meaning there is no frictional force acting between the plank and the wall.

To determine the necessary condition for equilibrium when $x < d\sec^3\theta$, consider the forces acting on the plank. The plank is in equilibrium under the following forces: the normal force N_w and frictional force f at the wall (point A), the normal force N_p at the prism (point B), and the weight mg of the point mass at distance x along the plank from A.

From the equations of equilibrium derived in the previous part:

- The normal force at the prism is $N_p = \frac{mgx\cos^2\theta}{d}$.
- The normal force at the wall is $N_w = N_p \sin\theta = \frac{mgx\cos^2\theta\sin\theta}{d}$.
- The frictional force at the wall is $f = mg\left(1 - \frac{x\cos^3\theta}{d}\right)$.

When $x < d\sec^3\theta$, where $d\sec^3\theta = \frac{d}{\cos^3\theta}$, it follows that $f > 0$. This positive f indicates that friction acts upward at the wall to prevent downward motion. The magnitude of the frictional force is $|f| = mg\left(\frac{d\sec^3\theta - x}{d\sec^3\theta}\right)$, but expressed as $|f| = mg\left(\frac{d}{\cos^3\theta} - x\right)\frac{\cos^3\theta}{d} = mg\left(\frac{d\sec^3\theta - x}{d\sec^3\theta}\right)\cdot d\sec^3\theta\cdot\frac{\cos^3\theta}{d}$, simplifying to $|f| = mg\left(d\sec^3\theta - x\right)\frac{\cos^3\theta}{d}$.

The ratio of the magnitude of the frictional force to the normal force at the wall is:

$$\frac{|f|}{N_w} = \frac{mg\left(d\sec^3\theta - x\right)\frac{\cos^3\theta}{d}}{\frac{mgx\cos^2\theta\sin\theta}{d}} = \frac{\left(d\sec^3\theta - x\right)\cos^3\theta}{d}\cdot\frac{d}{x\cos^2\theta\sin\theta}$$

$$= \frac{\left(d\sec^3\theta - x\right)\cos\theta}{x\sin\theta} = \frac{d\sec^3\theta - x}{x\tan\theta}.$$

For the plank to remain in equilibrium without slipping, the coefficient of friction μ must satisfy $\mu \geq \frac{|f|}{N_w}$. Therefore:

$$\mu \geq \frac{d\sec^3\theta - x}{x\tan\theta}.$$

To establish the required inequalities for $\frac{x}{d}$, consider the equilibrium of the plank under the forces: the normal force N_w and frictional force f at the wall (point A), the normal force N_p at the prism (point B), and the weight mg of the point mass at distance x along the plank from A.

From the equations of equilibrium:

- Horizontal force balance: $N_w = N_p \sin\theta$
- Vertical force balance: $f = mg - N_p \cos\theta$
- Torque balance about A: $dN_p \sec\theta = mgx\cos\theta$, so $N_p = \frac{mgx\cos^2\theta}{d}$

Substituting N_p into the expressions for f and N_w:

$$f = mg\left(1 - \frac{x\cos^3\theta}{d}\right), \quad N_w = \frac{mgx\cos^2\theta\sin\theta}{d}$$

The frictional force must satisfy $|f| \leq \mu N_w$ for no slipping. Define $z = \frac{x\cos^3\theta}{d}$. Then:

$$|1 - z| \leq \mu z \tan\theta$$

where $z > 0$.

First Inequality: $\frac{x}{d} \geq \frac{\sec^3\theta}{1+\mu\tan\theta}$

The condition $|1 - z| \leq \mu z \tan\theta$ implies:

- If $z \leq 1$, then $1 - z \leq \mu z \tan\theta$, so $z \geq \frac{1}{1+\mu\tan\theta}$.
- If $z > 1$, then since $\mu\tan\theta > 0$, $\frac{1}{1+\mu\tan\theta} < 1 < z$, so $z > \frac{1}{1+\mu\tan\theta}$.

In both cases, $z \geq \frac{1}{1+\mu\tan\theta}$. Substituting $z = \frac{x\cos^3\theta}{d}$:

$$\frac{x\cos^3\theta}{d} \geq \frac{1}{1+\mu\tan\theta} \implies \frac{x}{d} \geq \frac{\sec^3\theta}{1+\mu\tan\theta}$$

This inequality is necessary for equilibrium. If violated, specifically when $z \leq 1$, $|f| > \mu N_w$, leading to slipping downward at the wall.

Second Inequality: If $\mu < \cot\theta$, then $\frac{x}{d} \leq \frac{\sec^3\theta}{1-\mu\tan\theta}$

Given $\mu < \cot\theta$, so $\mu\tan\theta < 1$ and $1-\mu\tan\theta > 0$. The condition $|1-z| \leq \mu z\tan\theta$ implies:

- If $z \geq 1$, then $z - 1 \leq \mu z \tan\theta$, so $z(1-\mu\tan\theta) \leq 1$, and since $1-\mu\tan\theta > 0$, $z \leq \frac{1}{1-\mu\tan\theta}$.
- If $z < 1$, then since $\mu < \cot\theta$, $1-\mu\tan\theta > 0$, and $\frac{1}{1-\mu\tan\theta} > 1 > z$, so $z < \frac{1}{1-\mu\tan\theta}$.

In both cases, $z \leq \frac{1}{1-\mu\tan\theta}$. Substituting $z = \frac{x\cos^3\theta}{d}$:

$$\frac{x\cos^3\theta}{d} \leq \frac{1}{1-\mu\tan\theta} \implies \frac{x}{d} \leq \frac{\sec^3\theta}{1-\mu\tan\theta}$$

This inequality is necessary when $\mu < \cot\theta$. If violated and $z > 1$, $|f| > \mu N_w$, leading to slipping upward at the wall.

To show that if the point mass is fixed to the plank between the edge of the prism and the wall, then $\tan\theta < \mu$, consider the equilibrium conditions of the plank. The mass is located at a distance x from the wall contact point (point A), with $0 < x < d\sec\theta$, since it is between the wall and the prism (point B).

From the equilibrium equations:

- The frictional force at the wall is $f = mg\left(1 - \frac{x\cos^3\theta}{d}\right)$.
- The normal force at the wall is $N_w = mg\frac{x\cos^2\theta\sin\theta}{d}$.

Given that $x < d\sec\theta < d\sec^3\theta$ (since $\sec^3\theta > \sec\theta$ for acute θ), it follows that $\frac{x\cos^3\theta}{d} < 1$, so $f > 0$. This positive frictional force acts upward to prevent downward sliding.

For equilibrium without slipping, the static friction condition requires $f \leq \mu N_w$:

$$mg\left(1 - \frac{x\cos^3\theta}{d}\right) \leq \mu \cdot mg\frac{x\cos^2\theta\sin\theta}{d}.$$

Dividing by $mg > 0$:

$$1 - \frac{x\cos^3\theta}{d} \le \mu\frac{x\cos^2\theta\sin\theta}{d}.$$

Rearranging terms:

$$1 \le \frac{x\cos^3\theta}{d} + \mu\frac{x\cos^2\theta\sin\theta}{d} = \frac{x\cos^2\theta}{d}(\cos\theta + \mu\sin\theta).$$

Solving for $\frac{x}{d}$:

$$\frac{x}{d} \ge \frac{1}{\cos^2\theta(\cos\theta + \mu\sin\theta)} = \frac{\sec^2\theta}{\cos\theta + \mu\sin\theta}.$$

Since the mass is between the wall and the prism, $\frac{x}{d} < \sec\theta$. Thus:

$$\frac{\sec^2\theta}{\cos\theta + \mu\sin\theta} \le \frac{x}{d} < \sec\theta.$$

For this interval to be non-empty, the lower bound must be strictly less than the upper bound:

$$\frac{\sec^2\theta}{\cos\theta + \mu\sin\theta} < \sec\theta.$$

Dividing both sides by $\sec\theta > 0$:

$$\frac{\sec\theta}{\cos\theta + \mu\sin\theta} < 1 \implies \sec\theta < \cos\theta + \mu\sin\theta.$$

Multiplying both sides by $\cos\theta > 0$:

$$1 < \cos^2\theta + \mu\sin\theta\cos\theta.$$

Dividing by $\cos^2\theta$:

$$\sec^2\theta < 1 + \mu\tan\theta.$$

Since $\sec^2\theta = 1 + \tan^2\theta$:

$$1+\tan^2\theta < 1+\mu\tan\theta \implies \tan^2\theta < \mu\tan\theta.$$

Given that $\tan\theta > 0$ for acute θ, dividing by $\tan\theta$:

$$\tan\theta < \mu.$$

Therefore, if the point mass is fixed between the prism and the wall and the system is in equilibrium, it must be that $\tan\theta < \mu$.

Q6

- Show that, if a particle is projected at an angle α above the horizontal with speed u, it will reach height h at a horizontal distance s from the point of projection where

$$h = s\tan\alpha - \frac{gs^2}{2u^2\cos^2\alpha}.$$

  The remainder of this question uses axes with the x- and y-axes horizontal and the z-axis vertically upwards. The ground is a sloping plane with equation $z = y\tan\theta$ and a road runs along the x-axis. A cannon, which may have any angle of inclination and be pointed in any direction, fires projectiles from ground level with speed u. Initially, the cannon is placed at the origin.
- Let a point P on the plane have coordinates $(x, y, y\tan\theta)$. Show that the condition for it to be possible for a projectile from the cannon to land at point P is

$$x^2 + \left(y + \frac{u^2\tan\theta}{g}\right)^2 \leq \frac{u^4\sec^2\theta}{g^2}.$$

- Show that the furthest point directly up the plane that can be reached by a projectile from the cannon is a distance

$$\frac{u^2}{g(1+\sin\theta)}$$

  from the cannon.
- How far from the cannon is the furthest point directly down the plane that can be reached by a projectile from it?

- Find the length of road which can be reached by projectiles from the cannon.
- The cannon is now moved to a point on the plane vertically above the y-axis, and a distance r from the road. Find the value of r which maximises the length of road which can be reached by projectiles from the cannon. What is this maximum length?

Solution:

To derive the equation for the height h of a particle projected at an angle α above the horizontal with initial speed u, at a horizontal distance s from the point of projection, consider the equations of motion for projectile motion under constant gravity g.

The initial velocity components are:

- Horizontal component: $u\cos\alpha$
- Vertical component: $u\sin\alpha$

The parametric equations for the position as a function of time t are:

- Horizontal motion (no acceleration): $x = (u\cos\alpha)t$
- Vertical motion (acceleration due to gravity g downward): $y = (u\sin\alpha)t - \frac{1}{2}gt^2$

When the horizontal distance is s, set $x = s$:

$$s = (u\cos\alpha)t$$

Solve for t:

$$t = \frac{s}{u\cos\alpha}$$

Substitute this expression for t into the vertical position equation to find the height h (where $y = h$):

$$h = (u\sin\alpha)\left(\frac{s}{u\cos\alpha}\right) - \frac{1}{2}g\left(\frac{s}{u\cos\alpha}\right)^2$$

Simplify each term:

- First term: $(u \sin \alpha) \cdot \frac{s}{u \cos \alpha} = s \cdot \frac{\sin \alpha}{\cos \alpha} = s \tan \alpha$
- Second term: $\frac{1}{2} g \cdot \frac{s^2}{u^2 \cos^2 \alpha} = \frac{g s^2}{2u^2 \cos^2 \alpha}$ (with a negative sign as per the equation)

Thus:

$$h = s \tan \alpha - \frac{g s^2}{2u^2 \cos^2 \alpha}$$

To determine the condition for a projectile fired from the origin with speed u to land at a point P with coordinates $(x, y, y \tan \theta)$ on the sloping plane $z = y \tan \theta$, consider the equations of motion and the landing condition.

The initial velocity vector $\vec{v} = (v_x, v_y, v_z)$ satisfies $v_x^2 + v_y^2 + v_z^2 = u^2$. The position at time t is given by:

$$x(t) = v_x t, \quad y(t) = v_y t, \quad z(t) = v_z t - \frac{1}{2} g t^2,$$

where g is the acceleration due to gravity, acting downward.

The projectile lands on the plane $z = y \tan \theta$, so:

$$v_z t - \frac{1}{2} g t^2 = (v_y t) \tan \theta.$$

Assuming $t > 0$ (since $t = 0$ corresponds to the launch point), divide by t:

$$v_z - \frac{1}{2} g t = v_y \tan \theta.$$

Solving for t:

$$t = \frac{2}{g}(v_z - v_y \tan \theta).$$

For $t > 0$, it is necessary that $k = v_z - v_y \tan \theta > 0$.

The landing position is:

$$x = v_x t = \frac{2}{g} v_x k, \quad y = v_y t = \frac{2}{g} v_y k.$$

Solving for v_x and v_y:

$$v_x = \frac{gx}{2k}, \quad v_y = \frac{gy}{2k}.$$

Substitute $v_z = k + v_y \tan\theta$ into the speed constraint:

$$v_x^2 + v_y^2 + v_z^2 = u^2.$$

Using the expressions for v_x, v_y, and v_z:

$$\left(\frac{gx}{2k}\right)^2 + \left(\frac{gy}{2k}\right)^2 + \left(k + \frac{gy}{2k}\tan\theta\right)^2 = u^2.$$

Simplifying each term:

$$\frac{g^2x^2}{4k^2} + \frac{g^2y^2}{4k^2} + \left(\frac{2k^2 + gy\tan\theta}{2k}\right)^2 = u^2,$$

$$\frac{g^2x^2}{4k^2} + \frac{g^2y^2}{4k^2} + \frac{(2k^2 + gy\tan\theta)^2}{4k^2} = u^2.$$

Multiplying through by $4k^2$:

$$g^2x^2 + g^2y^2 + (2k^2 + gy\tan\theta)^2 = 4k^2u^2.$$

Expanding the square:

$$g^2x^2 + g^2y^2 + 4k^4 + 4k^2gy\tan\theta + g^2y^2\tan^2\theta = 4k^2u^2.$$

Using $1 + \tan^2\theta = \sec^2\theta$:

$$g^2x^2 + g^2y^2\sec^2\theta + 4k^4 + 4k^2gy\tan\theta = 4k^2u^2.$$

Rearranging:

$$4k^4 + 4k^2 gy\tan\theta - 4k^2u^2 + g^2x^2 + g^2y^2\sec^2\theta = 0.$$

Setting $w = k^2$:

$$4w^2 + (4gy\tan\theta - 4u^2)w + g^2(x^2 + y^2\sec^2\theta) = 0.$$

For real $w \geq 0$ (since $k > 0$ implies $w > 0$), the discriminant must be non-negative:

$$D = B^2 - 4AC,$$

where $A = 4$, $B = 4gy\tan\theta - 4u^2$, $C = g^2(x^2 + y^2\sec^2\theta)$. Thus:

$$D = [4(gy\tan\theta - u^2)]^2 - 4\cdot 4\cdot g^2(x^2 + y^2\sec^2\theta),$$

$$D = 16(gy\tan\theta - u^2)^2 - 16g^2(x^2 + y^2\sec^2\theta),$$

$$D = 16\left[(gy\tan\theta - u^2)^2 - g^2(x^2 + y^2\sec^2\theta)\right].$$

Expanding and simplifying:

$$(gy\tan\theta - u^2)^2 = g^2y^2\tan^2\theta - 2gu^2y\tan\theta + u^4,$$

$$g^2(x^2 + y^2\sec^2\theta) = g^2x^2 + g^2y^2(1 + \tan^2\theta) = g^2x^2 + g^2y^2 + g^2y^2\tan^2\theta.$$

Substituting:

$$D/16 = g^2y^2\tan^2\theta - 2gu^2y\tan\theta + u^4 - g^2x^2 - g^2y^2 - g^2y^2\tan^2\theta,$$

$$D/16 = u^4 - 2gu^2y\tan\theta - g^2(x^2 + y^2).$$

For real w, $D \geq 0$, so:

$$u^4 - 2gu^2y\tan\theta - g^2(x^2 + y^2) \geq 0.$$

Rearranging:

$$g^2x^2 + g^2y^2 + 2gu^2y\tan\theta - u^4 \leq 0.$$

Completing the square for the y-terms:

$$g^2y^2 + 2gu^2y\tan\theta = g^2\left(y^2 + 2\frac{u^2}{g}y\tan\theta\right) = g^2\left[\left(y + \frac{u^2}{g}\tan\theta\right)^2 - \left(\frac{u^2}{g}\tan\theta\right)^2\right].$$

Substituting back:

$$g^2\left[\left(y + \frac{u^2}{g}\tan\theta\right)^2 - \frac{u^4\tan^2\theta}{g^2}\right] + g^2x^2 - u^4 \le 0,$$

$$g^2\left(y + \frac{u^2}{g}\tan\theta\right)^2 - u^4\tan^2\theta + g^2x^2 - u^4 \le 0,$$

$$g^2x^2 + g^2\left(y + \frac{u^2}{g}\tan\theta\right)^2 - u^4(\tan^2\theta + 1) \le 0.$$

Using $\tan^2\theta + 1 = \sec^2\theta$:

$$g^2x^2 + g^2\left(y + \frac{u^2}{g}\tan\theta\right)^2 - u^4\sec^2\theta \le 0.$$

Dividing by g^2:

$$x^2 + \left(y + \frac{u^2\tan\theta}{g}\right)^2 - \frac{u^4\sec^2\theta}{g^2} \le 0,$$

$$x^2 + \left(y + \frac{u^2\tan\theta}{g}\right)^2 \le \frac{u^4\sec^2\theta}{g^2}.$$

To determine the furthest point directly up the plane that can be reached by a projectile fired from the cannon at the origin with speed u, consider the sloping plane defined by $z = y\tan\theta$. The direction "directly up the plane" corresponds to the line of greatest slope in the yz-plane, where $x = 0$. Points on this line have coordinates $(0, y, y\tan\theta)$, and the distance d from the origin to such a point is given by:

$$d = \sqrt{0^2 + y^2 + (y\tan\theta)^2} = \sqrt{y^2(1 + \tan^2\theta)} = |y|\sec\theta.$$

Since "up the plane" implies $y > 0$, it follows that:

$$d = y \sec\theta.$$

Thus, maximizing d is equivalent to maximizing y.

From the reachability condition derived earlier for a point $(x, y, y\tan\theta)$:

$$x^2 + \left(y + \frac{u^2 \tan\theta}{g}\right)^2 \leq \frac{u^4 \sec^2\theta}{g^2}.$$

For points directly up the plane, set $x = 0$:

$$\left(y + \frac{u^2 \tan\theta}{g}\right)^2 \leq \frac{u^4 \sec^2\theta}{g^2}.$$

Taking the non-negative square root (since $y > 0$ and θ is acute, the expression inside is positive):

$$y + \frac{u^2 \tan\theta}{g} \leq \frac{u^2 \sec\theta}{g},$$

as the left side is positive. Solving for y:

$$y \leq \frac{u^2 \sec\theta}{g} - \frac{u^2 \tan\theta}{g} = \frac{u^2}{g}(\sec\theta - \tan\theta).$$

The maximum y is:

$$y_{\max} = \frac{u^2}{g}(\sec\theta - \tan\theta).$$

Substitute $\sec\theta = \frac{1}{\cos\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$:

$$\sec\theta - \tan\theta = \frac{1 - \sin\theta}{\cos\theta}.$$

Thus:

$$y_{\max} = \frac{u^2}{g} \cdot \frac{1 - \sin\theta}{\cos\theta}.$$

The distance d is:

$$d = y_{\max} \sec\theta = \frac{u^2}{g} \cdot \frac{1-\sin\theta}{\cos\theta} \cdot \frac{1}{\cos\theta} = \frac{u^2}{g} \cdot \frac{1-\sin\theta}{\cos^2\theta}.$$

Using the identity $\cos^2\theta = 1 - \sin^2\theta = (1-\sin\theta)(1+\sin\theta)$:

$$d = \frac{u^2}{g} \cdot \frac{1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)} = \frac{u^2}{g(1+\sin\theta)}.$$

This maximum distance is achieved when equality holds in the reachability condition, which corresponds to a valid trajectory landing at the point $(0, y_{\max}, y_{\max}\tan\theta)$. The point is directly up the plane, as it lies on the line $x = 0$ in the direction of increasing y and z.

Thus, the distance from the cannon to the furthest point directly up the plane is:

$$\boxed{\frac{u^2}{g\,(1+\sin\theta)}}$$

The ground is a sloping plane defined by $z = y\tan\theta$, with the cannon at the origin $(0,0,0)$ firing projectiles with initial speed u. The furthest point directly down the plane lies along the line of greatest slope in the negative y-direction, so $x = 0$ and $y < 0$.

From the reachability condition derived earlier, a point $P(x, y, y\tan\theta)$ is reachable if:

$$x^2 + \left(y + \frac{u^2\tan\theta}{g}\right)^2 \le \frac{u^4\sec^2\theta}{g^2}.$$

For points directly down the plane, set $x = 0$:

$$\left(y + \frac{u^2\tan\theta}{g}\right)^2 \le \left(\frac{u^2\sec\theta}{g}\right)^2.$$

This inequality defines a disk in the xy-plane centered at $\left(0, -\frac{u^2\tan\theta}{g}\right)$ with radius $\frac{u^2\sec\theta}{g}$. To find the point farthest down the plane, minimize y (since $y < 0$), which occurs at the boundary of the disk:

$$y + \frac{u^2 \tan\theta}{g} = -\frac{u^2 \sec\theta}{g}.$$

Solving for y:

$$y = -\frac{u^2 \sec\theta}{g} - \frac{u^2 \tan\theta}{g} = -\frac{u^2}{g}(\sec\theta + \tan\theta).$$

Substitute $\sec\theta = \frac{1}{\cos\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$:

$$\sec\theta + \tan\theta = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1+\sin\theta}{\cos\theta},$$

so:

$$y = -\frac{u^2}{g} \cdot \frac{1+\sin\theta}{\cos\theta}.$$

The distance d from the cannon to the point $(0, y, y\tan\theta)$ is:

$$d = \sqrt{0^2 + y^2 + (y\tan\theta)^2} = \sqrt{y^2(1+\tan^2\theta)} = |y|\sec\theta.$$

Since $y < 0$, $|y| = -y$, so:

$$d = -y\sec\theta = \left(\frac{u^2}{g} \cdot \frac{1+\sin\theta}{\cos\theta}\right)\sec\theta = \frac{u^2}{g} \cdot \frac{1+\sin\theta}{\cos\theta} \cdot \frac{1}{\cos\theta} = \frac{u^2}{g} \cdot \frac{1+\sin\theta}{\cos^2\theta}.$$

Using the identity $\cos^2\theta = 1 - \sin^2\theta = (1-\sin\theta)(1+\sin\theta)$:

$$d = \frac{u^2}{g} \cdot \frac{1+\sin\theta}{(1-\sin\theta)(1+\sin\theta)} = \frac{u^2}{g(1-\sin\theta)}.$$

The road runs along the x-axis on the ground, which is defined by the plane $z = y\tan\theta$. Points on the road have coordinates $(x, 0, 0)$, as $y = 0$ implies $z = 0$.

The condition for a projectile fired from the origin with speed u to land at a point $(x, y, y\tan\theta)$ on the sloping plane is given by:

$$x^2 + \left(y + \frac{u^2 \tan\theta}{g}\right)^2 \leq \frac{u^4 \sec^2\theta}{g^2}.$$

For points on the road, set $y = 0$:

$$x^2 + \left(0 + \frac{u^2 \tan\theta}{g}\right)^2 \leq \frac{u^4 \sec^2\theta}{g^2},$$

which simplifies to:

$$x^2 + \frac{u^4 \tan^2\theta}{g^2} \leq \frac{u^4 \sec^2\theta}{g^2}.$$

Rearranging terms:

$$x^2 \leq \frac{u^4 \sec^2\theta}{g^2} - \frac{u^4 \tan^2\theta}{g^2} = \frac{u^4}{g^2}(\sec^2\theta - \tan^2\theta).$$

Using the identity $\sec^2\theta - \tan^2\theta = 1$:

$$x^2 \leq \frac{u^4}{g^2} \cdot 1 = \left(\frac{u^2}{g}\right)^2,$$

so:

$$|x| \leq \frac{u^2}{g}.$$

This inequality indicates that the reachable points on the road are in the interval $[-u^2/g, u^2/g]$. The length of this interval is:

$$\frac{u^2}{g} - \left(-\frac{u^2}{g}\right) = \frac{2u^2}{g}.$$

The cannon is positioned at a point on the plane $z = y\tan\theta$ that is vertically above the y-axis, at a distance r from the road (which lies along the x-axis). The coordinates of the cannon are $(0, r\cos\theta, r\sin\theta)$.

To determine the length of the road (i.e., the segment of the x-axis) that can be reached by projectiles fired from this cannon with initial speed u, the condition for a projectile to land at a point $(x, 0, 0)$ on the road is derived. The resulting inequality that must be satisfied for x to be reachable is:

$$x^2 \leq \frac{u^4 + 2u^2 gr \sin\theta - g^2 r^2 \cos^2\theta}{g^2}.$$

The length of the road that can be reached is twice the maximum absolute value of x, given by:

$$L = \frac{2}{g}\sqrt{u^4 + 2u^2 gr \sin\theta - g^2 r^2 \cos^2\theta}.$$

This length L is maximized with respect to r. The expression inside the square root, $f(r) = u^4 + 2u^2 gr \sin\theta - g^2 r^2 \cos^2\theta$, is a quadratic in r that opens downwards. The maximum occurs at the vertex:

$$r = \frac{u^2 \sin\theta}{g \cos^2\theta}.$$

Substituting this value of r into $f(r)$:

$$f(r) = u^4 + 2u^2 g \left(\frac{u^2 \sin\theta}{g\cos^2\theta}\right)\sin\theta - g^2 \left(\frac{u^2 \sin\theta}{g\cos^2\theta}\right)^2 \cos^2\theta = u^4 \sec^2\theta.$$

Thus, the maximum length is:

$$L = \frac{2}{g}\sqrt{u^4 \sec^2\theta} = \frac{2u^2}{g}\sec\theta.$$

Q7

Two particles, of masses m_1 and m_2 where $m_1 > m_2$, are attached to the ends of a light, inextensible string. A particle of mass M is fixed to a point P on the string. The string passes over two small, smooth pulleys at Q and R, where QR is horizontal, so that the particle of mass m_1 hangs vertically below Q and the particle of mass m_2 hangs vertically below R. The particle of mass M hangs between the two pulleys with the section of the string PQ making an acute angle of θ_1 with the upward vertical and the section of the string PR making an acute angle of θ_2 with the upward vertical. S is the point on QR vertically above P. The system is in equilibrium.

- Using a triangle of forces, or otherwise, show that:

$$\sqrt{m_1^2 - m_2^2} < M < m_1 + m_2;$$

- Show also that S divides QR in the ratio $r : 1$, where

$$r = \frac{M^2 - m_1^2 + m_2^2}{M^2 - m_2^2 + m_1^2}.$$

- You are now given that $M^2 = m_1^2 + m_2^2$. Show that $\theta_1 + \theta_2 = 90^\circ$ and determine the ratio of QR to SP in terms of the masses only.

Solution:

The tensions in the string segments are denoted as follows: the tension in the segment from Q to P is T_1, and in the segment from P to R is T_2. Since the pulleys at Q and R are smooth and the string is light and inextensible, the tension is uniform on each side, so $T_1 = m_1 g$ (from the equilibrium of mass m_1) and $T_2 = m_2 g$ (from the equilibrium of mass m_2).

At point P, the forces are:

- Tension T_1 along PQ, at an angle θ_1 to the upward vertical.
- Tension T_2 along PR, at an angle θ_2 to the upward vertical.
- Weight Mg downward.

For equilibrium, the net horizontal and vertical forces at P must be zero.

Horizontal force balance:
The horizontal component of T_1 is $T_1 \sin\theta_1$ to the left (assuming Q is left of S), and the horizontal component of T_2 is $T_2 \sin\theta_2$ to the right. Thus:

$$T_1 \sin\theta_1 = T_2 \sin\theta_2$$

Substituting $T_1 = m_1 g$ and $T_2 = m_2 g$:

$$m_1 g \sin\theta_1 = m_2 g \sin\theta_2$$

Dividing by g:

$$m_1 \sin\theta_1 = m_2 \sin\theta_2 \quad (1)$$

Vertical force balance:
The vertical components of T_1 and T_2 are both upward, so:

$$T_1 \cos\theta_1 + T_2 \cos\theta_2 = Mg$$

Substituting $T_1 = m_1 g$ and $T_2 = m_2 g$:

$$m_1 g \cos\theta_1 + m_2 g \cos\theta_2 = Mg$$

Dividing by g:

$$m_1 \cos\theta_1 + m_2 \cos\theta_2 = M \quad (2)$$

From equation (1), $\sin\theta_2 = \frac{m_1}{m_2}\sin\theta_1$. Since θ_1 and θ_2 are acute, $\sin\theta_2 \leq 1$, so:

$$\frac{m_1}{m_2}\sin\theta_1 \leq 1 \implies \sin\theta_1 \leq \frac{m_2}{m_1}$$

Let $x = \sin\theta_1$, so $0 < x \leq \frac{m_2}{m_1}$ (since $\theta_1 = 0$ or $\theta_2 = 90^\circ$ are not achievable in practice, strict inequalities hold).

From equation (2):

$$M = m_1 \cos\theta_1 + m_2 \cos\theta_2$$

Substitute $\cos\theta_1 = \sqrt{1 - x^2}$ and $\cos\theta_2 = \sqrt{1 - \sin^2\theta_2} = \sqrt{1 - \left(\frac{m_1}{m_2}x\right)^2}$:

$$M = m_1\sqrt{1 - x^2} + m_2\sqrt{1 - \frac{m_1^2}{m_2^2}x^2} = m_1\sqrt{1 - x^2} + \sqrt{m_2^2 - m_1^2 x^2}$$

Consider $M(x)$ as a function of x:

$$M(x) = m_1\sqrt{1-x^2} + \sqrt{m_2^2 - m_1^2x^2}$$

for $x \in (0, \frac{m_2}{m_1})$.

- As $x \to 0^+$:

$$M(0) = m_1 \cdot 1 + \sqrt{m_2^2} = m_1 + m_2$$

  However, $x = 0$ implies $\theta_1 = 0$, which would require P directly below Q and $\theta_2 = 0$ from (1), but Q and R are distinct, so this is impossible. Thus, $M < m_1 + m_2$.

- As $x \to \left(\frac{m_2}{m_1}\right)^-$:

$$M\left(\frac{m_2}{m_1}\right) = m_1\sqrt{1-\left(\frac{m_2}{m_1}\right)^2} + \sqrt{m_2^2 - m_1^2\left(\frac{m_2}{m_1}\right)^2} = m_1 \cdot \frac{\sqrt{m_1^2 - m_2^2}}{m_1} + \sqrt{0} = \sqrt{m_1^2 - m_2^2}$$

  However, $x = \frac{m_2}{m_1}$ implies $\sin\theta_2 = 1$, so $\theta_2 = 90^\circ$, which is not acute and thus not achievable. Thus, $M > \sqrt{m_1^2 - m_2^2}$.

Since $M(x)$ is continuous and decreases from nearly $m_1 + m_2$ to nearly $\sqrt{m_1^2 - m_2^2}$ as x increases from 0 to $\frac{m_2}{m_1}$, and both endpoints are unattainable, it follows that:

$$\sqrt{m_1^2 - m_2^2} < M < m_1 + m_2$$

The system is in equilibrium, with the tensions in the string segments being $T_1 = m_1g$ for the segment PQ and $T_2 = m_2g$ for the segment PR, due to the equilibrium of the masses m_1 and m_2 hanging vertically.

At point P, the forces acting are:

- Tension T_1 along PQ, making an angle θ_1 with the upward vertical.
- Tension T_2 along PR, making an angle θ_2 with the upward vertical.

- Weight Mg downward.

For equilibrium, the horizontal and vertical force balances at P give:

$$m_1 \sin\theta_1 = m_2 \sin\theta_2 \quad \text{(horizontal balance)}$$

$$M = m_1 \cos\theta_1 + m_2 \cos\theta_2 \quad \text{(vertical balance)}$$

Set $u = m_1 \cos\theta_1$ and $v = m_2 \cos\theta_2$. The vertical balance equation becomes:

$$u + v = M$$

From the horizontal balance, squaring both sides and using $\sin^2\theta = 1 - \cos^2\theta$:

$$m_1^2(1 - \cos^2\theta_1) = m_2^2(1 - \cos^2\theta_2)$$

$$m_1^2 - m_1^2\cos^2\theta_1 = m_2^2 - m_2^2\cos^2\theta_2$$

$$m_1^2\cos^2\theta_1 - m_2^2\cos^2\theta_2 = m_1^2 - m_2^2$$

Substituting $u = m_1 \cos\theta_1$ and $v = m_2 \cos\theta_2$:

$$u^2 - v^2 = m_1^2 - m_2^2$$

$$(u - v)(u + v) = m_1^2 - m_2^2$$

Using $u + v = M$:

$$(u - v)M = m_1^2 - m_2^2$$

$$u - v = \frac{m_1^2 - m_2^2}{M}$$

Now solve the system:

$$u + v = M, \quad u - v = \frac{m_1^2 - m_2^2}{M}$$

Adding the equations:

$$2u = M + \frac{m_1^2 - m_2^2}{M} = \frac{M^2 + m_1^2 - m_2^2}{M}$$

$$u = \frac{M^2 + m_1^2 - m_2^2}{2M}$$

Subtracting the equations:

$$2v = M - \frac{m_1^2 - m_2^2}{M} = \frac{M^2 - m_1^2 + m_2^2}{M}$$

$$v = \frac{M^2 - m_1^2 + m_2^2}{2M}$$

The ratio $QS : SR = r : 1$ is given by $r = \frac{QS}{SR} = \frac{v}{u}$, as derived from the geometry and force balances. Thus:

$$r = \frac{v}{u} = \frac{\frac{M^2 - m_1^2 + m_2^2}{2M}}{\frac{M^2 + m_1^2 - m_2^2}{2M}} = \frac{M^2 - m_1^2 + m_2^2}{M^2 + m_1^2 - m_2^2}$$

The denominator is $M^2 + m_1^2 - m_2^2$, which is equivalent to $M^2 - m_2^2 + m_1^2$. Therefore:

$$r = \frac{M^2 - m_1^2 + m_2^2}{M^2 - m_2^2 + m_1^2}$$

This shows that S divides QR in the ratio $r : 1$ with the given expression for r.

Given that $M^2 = m_1^2 + m_2^2$, the system is in equilibrium, and the angles θ_1 and θ_2 are acute.

From the force balance equations at point P:

- Horizontal force balance: $m_1 \sin\theta_1 = m_2 \sin\theta_2$
- Vertical force balance: $M = m_1 \cos\theta_1 + m_2 \cos\theta_2$

Given $M^2 = m_1^2 + m_2^2$, substitute into the squared vertical equation:

$$m_1^2 + m_2^2 = (m_1 \cos\theta_1 + m_2 \cos\theta_2)^2 = m_1^2 \cos^2\theta_1 + 2m_1 m_2 \cos\theta_1 \cos\theta_2 + m_2^2 \cos^2\theta_2$$

Rearrange to get:

$$m_1^2 \sin^2\theta_1 + m_2^2 \sin^2\theta_2 - 2m_1m_2\cos\theta_1\cos\theta_2 = 0$$

Using the horizontal balance, set $k = m_1\sin\theta_1 = m_2\sin\theta_2$, so $\sin\theta_1 = k/m_1$ and $\sin\theta_2 = k/m_2$. Substitute and solve for k^2:

$$k^2 + k^2 - 2m_1m_2\sqrt{\left(1-\frac{k^2}{m_1^2}\right)\left(1-\frac{k^2}{m_2^2}\right)} = 0$$

$$k^2 = m_1m_2\sqrt{\left(1-\frac{k^2}{m_1^2}\right)\left(1-\frac{k^2}{m_2^2}\right)}$$

Squaring both sides and solving gives:

$$k^2 = \frac{m_1^2m_2^2}{m_1^2+m_2^2}$$

Thus:

$$\sin^2\theta_1 = \frac{m_2^2}{m_1^2+m_2^2}, \quad \sin^2\theta_2 = \frac{m_1^2}{m_1^2+m_2^2}$$

$$\cos^2\theta_1 = \frac{m_1^2}{m_1^2+m_2^2}, \quad \cos^2\theta_2 = \frac{m_2^2}{m_1^2+m_2^2}$$

Now compute:

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \frac{m_1m_2}{m_1^2+m_2^2} - \frac{m_1m_2}{m_1^2+m_2^2} = 0$$

Since θ_1 and θ_2 are acute, $\theta_1+\theta_2 < 180^\circ$, so $\theta_1+\theta_2 = 90^\circ$.

For the ratio of QR to SP, set coordinates with S at $(0,0)$, P at $(0,-h)$ (so $SP = h$), Q at $(-a,0)$, and R at $(b,0)$ (so $QR = a+b$). The angles give:

$$\tan\theta_1 = \frac{a}{h}, \quad \tan\theta_2 = \frac{b}{h}$$

Since $\theta_1 + \theta_2 = 90^\circ$, $\theta_2 = 90^\circ - \theta_1$, so $\tan\theta_2 = \cot\theta_1$, and:

$$\frac{b}{h} = \frac{1}{\tan\theta_1} = \frac{h}{a}$$

Thus $ab = h^2$. From the force balance:

$$\tan\theta_1 = \frac{m_2}{m_1}, \quad \tan\theta_2 = \frac{m_1}{m_2}$$

So:

$$\frac{a}{h} = \frac{m_2}{m_1}, \quad \frac{b}{h} = \frac{m_1}{m_2}$$

The ratio is:

$$\frac{QR}{SP} = \frac{a+b}{h} = \frac{a}{h} + \frac{b}{h} = \frac{m_2}{m_1} + \frac{m_1}{m_2} = \frac{m_1^2 + m_2^2}{m_1 m_2}$$

Since $M^2 = m_1^2 + m_2^2$, but the expression is in terms of m_1 and m_2 only.

Q8

A train moves westwards on a straight horizontal track with constant acceleration a, where $a > 0$. Axes are chosen as follows: the origin is fixed in the train; the x-axis is in the direction of the track with the positive x-axis pointing to the East; and the positive y-axis points vertically upwards. A smooth wire is fixed in the train. It lies in the x-y plane and is bent in the shape given by $ky = x^2$, where k is a positive constant. A small bead is threaded onto the wire. Initially, the bead is held at the origin. It is then released.

- Explain why the bead cannot remain stationary relative to the train at the origin.

- Show that, in the subsequent motion, the coordinates (x, y) of the bead satisfy $\dot{x}(\ddot{x} - a) + \dot{y}(\ddot{y} + g) = 0$ and deduce that $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy$ is constant during the motion.
- Find an expression for the maximum vertical displacement, b, of the bead from its initial position in terms of a, k, and g.
- Find the value of x for which the speed of the bead relative to the train is greatest and give this maximum speed in terms of a, k, and g.

Solution:

The wire is smooth, so the only forces acting on the bead are gravity and the normal force from the wire. At the origin, the wire is described by $ky = x^2$, which is a parabola. The slope at the origin is given by $\frac{dy}{dx} = \frac{2x}{k}$, and at $x = 0$, $\frac{dy}{dx} = 0$. Thus, the tangent to the wire at the origin is horizontal, and the normal force is vertical.

In the inertial frame (fixed to the ground), the forces on the bead are gravity ($-mg\hat{\mathbf{j}}$) and the normal force ($\mathbf{N}$). Since both forces are vertical, the net horizontal force on the bead is zero. However, for the bead to remain stationary relative to the train, it must accelerate with the train. The train accelerates westwards with acceleration $-a\hat{\mathbf{i}}$ (since $a > 0$ and the positive x-axis points east). This requires a net horizontal force of $-ma\hat{\mathbf{i}}$ on the bead. Since the net horizontal force is zero, it cannot provide the necessary acceleration, and the bead cannot remain stationary relative to the train.

In the non-inertial frame fixed to the train, there is an additional fictitious force $ma\hat{\mathbf{i}}$ (acting eastwards). The effective forces are gravity ($-mg\hat{\mathbf{j}}$) and the fictitious force ($ma\hat{\mathbf{i}}$), so the total effective force is $ma\hat{\mathbf{i}} - mg\hat{\mathbf{j}}$. The tangential direction to the wire at the origin is horizontal (along $\hat{\mathbf{i}}$). The component of the effective force along the tangent is $(ma\hat{\mathbf{i}} - mg\hat{\mathbf{j}}) \cdot \hat{\mathbf{i}} = ma$, which is nonzero since $a > 0$. This tangential force causes the bead to accelerate along the wire, so it cannot remain stationary at the origin.

The bead moves on the smooth wire defined by $ky = x^2$, where $k > 0$, in a non-inertial frame fixed to the accelerating train. The train accelerates westwards with constant acceleration $a > 0$, and since the positive x-axis points east, the acceleration of the train is $-a\hat{\mathbf{i}}$ in the inertial frame. In the train's frame, a fictitious force $ma\hat{\mathbf{i}}$ acts on the bead, where m is the mass of the bead, in addition to gravity $-mg\hat{\mathbf{j}}$.

The equations of motion in the train's frame are derived from Newton's second law and the constraint of the wire. The acceleration of the bead is $(\ddot{x}, \ddot{y})$, and the normal force $\mathbf{N} = (N_x, N_y)$ is perpendicular to the tangent of the wire. The tangent vector to the curve $ky = x^2$ is proportional to $(1, dy/dx) = (1, 2x/k)$. Thus, the normal force satisfies:

$$N_x + \frac{2x}{k} N_y = 0.$$

Applying Newton's second law with the effective forces:

$$m\ddot{x} = ma + N_x,$$

$$m\ddot{y} = -mg + N_y.$$

Solving for N_x and N_y:

$$N_x = m\ddot{x} - ma,$$

$$N_y = m\ddot{y} + mg.$$

Substituting into the normal force equation:

$$(m\ddot{x} - ma) + \frac{2x}{k}(m\ddot{y} + mg) = 0.$$

Dividing by m:

$$\ddot{x} - a + \frac{2x}{k}(\ddot{y} + g) = 0. \quad (1)$$

From the constraint $y = x^2/k$, differentiating with respect to time gives:

$$\dot{y} = \frac{2x}{k}\dot{x}. \quad (2)$$

The expression $\dot{x}(\ddot{x} - a) + \dot{y}(\ddot{y} + g)$ is evaluated using equations (1) and (2):

$$\dot{x}(\ddot{x} - a) + \dot{y}(\ddot{y} + g) = \dot{x}\left(-\frac{2x}{k}(\ddot{y} + g)\right) + \dot{y}(\ddot{y} + g) = (\ddot{y} + g)\left(-\frac{2x}{k}\dot{x} + \dot{y}\right).$$

Substituting $\dot{y} = \frac{2x}{k}\dot{x}$:

$$-\frac{2x}{k}\dot{x} + \dot{y} = -\frac{2x}{k}\dot{x} + \frac{2x}{k}\dot{x} = 0.$$

Thus:

$$\dot{x}(\ddot{x} - a) + \dot{y}(\ddot{y} + g) = 0.$$

To deduce that $\frac{1}{2}(\dot{x}^2+\dot{y}^2)-ax+gy$ is constant, integrate the equation $\dot{x}(\ddot{x}-a)+\dot{y}(\ddot{y}+g)=0$ with respect to time:

$$\dot{x}\ddot{x}-a\dot{x}+\dot{y}\ddot{y}+g\dot{y}=0.$$

Recognizing that $\dot{x}\ddot{x}=\frac{1}{2}\frac{d}{dt}(\dot{x}^2)$ and $\dot{y}\ddot{y}=\frac{1}{2}\frac{d}{dt}(\dot{y}^2)$:

$$\frac{1}{2}\frac{d}{dt}(\dot{x}^2)+\frac{1}{2}\frac{d}{dt}(\dot{y}^2)-a\dot{x}+g\dot{y}=0.$$

Thus:

$$\frac{1}{2}\frac{d}{dt}(\dot{x}^2+\dot{y}^2)+\frac{d}{dt}(-ax+gy)=0,$$

since $-a\dot{x}=\frac{d}{dt}(-ax)$ and $g\dot{y}=\frac{d}{dt}(gy)$ for constant a and g. Therefore:

$$\frac{d}{dt}\left(\frac{1}{2}(\dot{x}^2+\dot{y}^2)-ax+gy\right)=0,$$

which implies that $\frac{1}{2}(\dot{x}^2+\dot{y}^2)-ax+gy$ is constant during the motion.

The bead moves along the smooth wire defined by $ky=x^2$, where $k>0$, in the non-inertial frame of the train accelerating westwards with constant acceleration $a>0$. The positive x-axis points east, and the positive y-axis points vertically upwards.

From the previous result, the quantity $\frac{1}{2}(\dot{x}^2+\dot{y}^2)-ax+gy$ is constant during the motion. At the initial position $(x,y)=(0,0)$ with zero velocity $(\dot{x},\dot{y})=(0,0)$, this expression evaluates to 0. Therefore, for all subsequent motion:

$$\frac{1}{2}(\dot{x}^2+\dot{y}^2)-ax+gy=0.$$

Rearranging gives:

$$\frac{1}{2}(\dot{x}^2+\dot{y}^2)=ax-gy.$$

The left side is non-negative, so $ax-gy\geq 0$. Since the bead is constrained to the wire, $y=\frac{x^2}{k}$. Substituting this yields:

$$ax - g\left(\frac{x^2}{k}\right) \geq 0.$$

This inequality implies that for $x > 0$, $x \leq \frac{ak}{g}$, and for $x < 0$, $ax - gy < 0$ (except at $x = 0$, $y = 0$). Thus, the bead moves only in the region $x \geq 0$, as the fictitious force acts eastward (positive x-direction).

The maximum vertical displacement b occurs when the bead is instantaneously at rest, i.e., $\dot{x} = 0$ and $\dot{y} = 0$. At this point, the energy equation gives:

$$\frac{1}{2}(0^2 + 0^2) - ax + gy = 0 \implies ax = gy.$$

Since the bead is on the wire, $ky = x^2$. Substituting $y = b$ and solving:

$$ax = gb, \quad kb = x^2.$$

Solving for x from the first equation: $x = \frac{gb}{a}$. Substituting into the second equation:

$$kb = \left(\frac{gb}{a}\right)^2 \implies kb = \frac{g^2b^2}{a^2}.$$

Assuming $b \neq 0$ (since $a > 0$), divide both sides by b:

$$k = \frac{g^2b}{a^2} \implies b = \frac{a^2k}{g^2}.$$

This is the maximum vertical displacement from the initial position, as it satisfies the energy equation and the constraint, and corresponds to the point where the kinetic energy is zero.

Thus, the expression for the maximum vertical displacement b is $\frac{a^2k}{g^2}$.

The bead moves along the smooth wire defined by $ky = x^2$, where $k > 0$, in the non-inertial frame of the train accelerating westwards with constant acceleration $a > 0$. The positive x-axis points east, and the positive y-axis points vertically upwards.

From the conservation of the effective energy, the quantity $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy$ is constant and equal to zero, as the bead is released from rest at the origin. Thus,

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) = ax - gy.$$

Substituting the constraint $y = \frac{x^2}{k}$ gives

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) = ax - g\left(\frac{x^2}{k}\right).$$

The speed of the bead relative to the train is $v = \sqrt{\dot{x}^2 + \dot{y}^2}$, so

$$v^2 = \dot{x}^2 + \dot{y}^2 = 2\left(ax - \frac{g}{k}x^2\right).$$

To find the maximum speed, maximize v^2 with respect to x. Define

$$s(x) = ax - \frac{g}{k}x^2.$$

This is a quadratic function in x with a negative leading coefficient $-\frac{g}{k}$, so it has a maximum at the vertex. The vertex occurs at

$$x = -\frac{b}{2a} = -\frac{a}{2\left(-\frac{g}{k}\right)} = \frac{ak}{2g},$$

where a in the quadratic form is the coefficient of x^2 (here $-\frac{g}{k}$) and b is the coefficient of x (here a).

The range of motion is $0 \le x \le \frac{ak}{g}$, as determined from the energy equation and the constraint. Since $\frac{ak}{2g}$ is within this interval and $v^2 > 0$ there, it is valid.

At $x = \frac{ak}{2g}$,

$$s(x) = a\left(\frac{ak}{2g}\right) - \frac{g}{k}\left(\frac{ak}{2g}\right)^2 = \frac{a^2k}{2g} - \frac{g}{k}\cdot\frac{a^2k^2}{4g^2} = \frac{a^2k}{2g} - \frac{a^2k}{4g} = \frac{a^2k}{4g}.$$

Thus,

$$v^2 = 2\cdot\frac{a^2k}{4g} = \frac{a^2k}{2g},$$

so the maximum speed is

$$v_{\max} = \sqrt{\frac{a^2 k}{2g}} = a\sqrt{\frac{k}{2g}}.$$

Q9

Point A is a distance h above ground level and point N is directly below A at ground level. Point B is also at ground level, a distance d horizontally from N. The angle of elevation of A from B is β. A particle is projected horizontally from A, with initial speed V. A second particle is projected from B with speed U at an acute angle θ above the horizontal. The horizontal components of the velocities of the two particles are in opposite directions. The two particles are projected simultaneously, in the vertical plane through A, N and B. Given that the two particles collide, show that

$$d \sin\theta - h\cos\theta = \frac{Vh}{U}$$

- Show that:

$$\theta > \beta;$$

- Show that:

$$U \sin\theta \geq \sqrt{\frac{gh}{2}};$$

- Show that:

$$\frac{U}{V} > \sin\beta.$$

  Show that the particles collide at a height greater than $\frac{1}{2}h$ if and only if the particle projected from B is moving upwards at the time of collision.

Solution:

To show that $d\sin\theta - h\cos\theta = \frac{Vh}{U}$ under the given conditions, consider the setup and the motion of the two particles.

Set up a coordinate system with point N at the origin $(0,0)$, point A at $(0,h)$, and point B at $(d,0)$. The angle of elevation β of A from B satisfies $\tan\beta = \frac{h}{d}$, but this relation is not directly used in the derivation.

Particle 1 is projected horizontally from A with initial speed V. Since the horizontal components of the velocities must be in opposite directions, and particle 2 is projected from B, assume particle 1 is projected in the positive x-direction (towards B). Thus, its initial velocity is $(V,0)$, and its position at time t is:

$$x_1(t) = Vt, \quad y_1(t) = h - \frac{1}{2}gt^2,$$

where g is the acceleration due to gravity.

Particle 2 is projected from B with speed U at an acute angle θ above the horizontal. To have the horizontal component opposite to that of particle 1, it must be projected in the negative x-direction (towards A). Thus, its initial velocity is $(-U\cos\theta, U\sin\theta)$, and its position at time t is:

$$x_2(t) = d - U\cos\theta \cdot t, \quad y_2(t) = U\sin\theta \cdot t - \frac{1}{2}gt^2.$$

The particles collide when their positions are equal at some time $t > 0$:

$$x_1(t) = x_2(t) \quad \text{and} \quad y_1(t) = y_2(t).$$

First, equate the y-coordinates:

$$h - \frac{1}{2}gt^2 = U\sin\theta \cdot t - \frac{1}{2}gt^2.$$

The terms $-\frac{1}{2}gt^2$ cancel, leaving:

$$h = U\sin\theta \cdot t.$$

Solving for t:

$$t = \frac{h}{U \sin\theta}.$$

Next, equate the x-coordinates:

$$Vt = d - U\cos\theta \cdot t.$$

Substitute $t = \frac{h}{U \sin\theta}$:

$$V \cdot \frac{h}{U \sin\theta} = d - U\cos\theta \cdot \frac{h}{U \sin\theta}.$$

Simplify the right side:

$$U\cos\theta \cdot \frac{h}{U \sin\theta} = \frac{\cos\theta}{\sin\theta} \cdot h = h\cot\theta,$$

so:

$$\frac{Vh}{U \sin\theta} = d - h\cot\theta.$$

Rewrite $h\cot\theta = h\frac{\cos\theta}{\sin\theta}$:

$$\frac{Vh}{U \sin\theta} = d - \frac{h\cos\theta}{\sin\theta}.$$

Multiply both sides by $\sin\theta$:

$$\frac{Vh}{U} = d\sin\theta - h\cos\theta.$$

Thus:

$$d\sin\theta - h\cos\theta = \frac{Vh}{U}.$$

To show that $\theta > \beta$, start with the given conditions and the equation derived from the collision of the particles:

$$d \sin\theta - h\cos\theta = \frac{Vh}{U}.$$

The angle of elevation β satisfies $\tan\beta = \frac{h}{d}$, so $h = d\tan\beta$. Substitute this into the collision equation:

$$d\sin\theta - (d\tan\beta)\cos\theta = \frac{V(d\tan\beta)}{U}.$$

Simplify the left side:

$$d\,(\sin\theta - \tan\beta\cos\theta) = \frac{Vd\tan\beta}{U}.$$

Divide both sides by d (assuming $d \neq 0$):

$$\sin\theta - \tan\beta\cos\theta = \frac{V\tan\beta}{U}.$$

Express $\tan\beta$ as $\frac{\sin\beta}{\cos\beta}$:

$$\sin\theta - \frac{\sin\beta}{\cos\beta}\cos\theta = \frac{V\sin\beta}{U\cos\beta}.$$

Rewrite the left side with a common denominator:

$$\frac{\sin\theta\cos\beta - \sin\beta\cos\theta}{\cos\beta} = \frac{V\sin\beta}{U\cos\beta}.$$

The expression $\sin\theta\cos\beta - \sin\beta\cos\theta$ is $\sin(\theta-\beta)$, so:

$$\frac{\sin(\theta-\beta)}{\cos\beta} = \frac{V\sin\beta}{U\cos\beta}.$$

Multiply both sides by $\cos\beta$:

$$\sin(\theta-\beta) = \frac{V\sin\beta}{U}.$$

Since $V > 0$, $U > 0$, and β is an acute angle (so $\sin\beta > 0$), the right side is positive:

$$\sin(\theta - \beta) > 0.$$

Both θ and β are acute angles, so $\theta < 90^\circ$ and $\beta > 0^\circ$. Thus, $\theta - \beta$ lies in the interval $(-90^\circ, 90^\circ)$. In this interval, $\sin(\theta - \beta) > 0$ implies $\theta - \beta > 0$, so:

$$\theta > \beta.$$

This result holds under the assumption that the particles collide and $V > 0$, as the horizontal component of velocity for the particle projected from A is non-zero and in the opposite direction to that of the particle projected from B.

To show that $U \sin\theta \geq \sqrt{\frac{gh}{2}}$, consider the conditions under which the two particles collide. From the previous analysis, the time of collision is $t = \frac{h}{U \sin\theta}$.

The y-coordinate of the collision point, using the motion of the first particle (projected horizontally from A), is given by:

$$y(t) = h - \frac{1}{2}gt^2.$$

Substituting $t = \frac{h}{U \sin\theta}$:

$$y(t) = h - \frac{1}{2}g\left(\frac{h}{U \sin\theta}\right)^2.$$

For the collision to occur at or above ground level ($y(t) \geq 0$):

$$h - \frac{1}{2}g\frac{h^2}{(U \sin\theta)^2} \geq 0.$$

Assuming $h > 0$, divide both sides by h:

$$1 - \frac{1}{2}g\frac{h}{(U \sin\theta)^2} \geq 0.$$

Rearrange the inequality:

$$1 \geq \frac{1}{2}g\frac{h}{(U \sin\theta)^2}.$$

Multiply both sides by 2:

$$2 \geq g \frac{h}{(U \sin \theta)^2}.$$

Equivalently:

$$gh \leq 2(U \sin \theta)^2.$$

Taking the square root of both sides (and noting that $U \sin \theta > 0$ since θ is acute):

$$U \sin \theta \geq \sqrt{\frac{gh}{2}}.$$

This inequality is necessary for the collision to occur at or above ground level, as derived from the condition $y(t) \geq 0$. Since the particles collide, this condition must hold.

From the condition that the particles collide, the following equation holds:

$$d \sin \theta - h \cos \theta = \frac{Vh}{U}.$$

Given that $\tan \beta = \frac{h}{d}$, substitute $d = \frac{h}{\tan \beta}$ into the collision equation:

$$\frac{h}{\tan \beta} \sin \theta - h \cos \theta = \frac{Vh}{U}.$$

Dividing both sides by h (assuming $h > 0$):

$$\frac{\sin \theta}{\tan \beta} - \cos \theta = \frac{V}{U}.$$

Substitute $\tan \beta = \frac{\sin \beta}{\cos \beta}$:

$$\sin \theta \cdot \frac{\cos \beta}{\sin \beta} - \cos \theta = \frac{V}{U},$$

which simplifies to:

$$\frac{\sin\theta\cos\beta - \sin\beta\cos\theta}{\sin\beta} = \frac{V}{U}.$$

The numerator is $\sin(\theta - \beta)$, so:

$$\frac{\sin(\theta - \beta)}{\sin\beta} = \frac{V}{U}.$$

Rearranging gives:

$$\sin(\theta - \beta) = \frac{V\sin\beta}{U}.$$

Since θ and β are acute angles and $\theta > \beta$ (as established previously), $0 < \theta - \beta < 90^\circ$. Thus, $\sin(\theta - \beta) > 0$ and $\sin(\theta - \beta) < 1$ (since $\theta - \beta < 90^\circ$ and $\sin x < 1$ for $x < 90^\circ$). Therefore:

$$0 < \frac{V\sin\beta}{U} < 1.$$

From the strict inequality $\frac{V\sin\beta}{U} < 1$:

$$V\sin\beta < U,$$

so:

$$\frac{U}{V} > \sin\beta.$$

The height of collision, y, is the same for both particles. Using the motion of the particle projected from A:

$$y = h - \frac{1}{2}gt^2,$$

where t is the time of collision. From the collision condition, $t = \frac{h}{U\sin\theta}$. Substituting:

$$y = h - \frac{1}{2}g\left(\frac{h}{U\sin\theta}\right)^2.$$

This height is greater than $\frac{1}{2}h$ if:

$$h - \frac{1}{2}g\frac{h^2}{(U\sin\theta)^2} > \frac{1}{2}h.$$

Dividing by $h > 0$:

$$1 - \frac{1}{2}g\frac{h}{(U\sin\theta)^2} > \frac{1}{2}.$$

Rearranging:

$$\frac{1}{2} > \frac{1}{2}g\frac{h}{(U\sin\theta)^2},$$

so:

$$1 > g\frac{h}{(U\sin\theta)^2},$$

which gives:

$$(U\sin\theta)^2 > gh. \quad (1)$$

The particle projected from B has an initial vertical velocity component $U\sin\theta$ upwards. Its vertical velocity at time t is:

$$v_y = U\sin\theta - gt.$$

At collision, $t = \frac{h}{U\sin\theta}$, so:

$$v_y = U\sin\theta - g\frac{h}{U\sin\theta}.$$

The particle is moving upwards if $v_y > 0$:

$$U\sin\theta - g\frac{h}{U\sin\theta} > 0.$$

Set $w = U\sin\theta > 0$ (since θ is acute):

$$w - g\frac{h}{w} > 0.$$

Multiplying by $w > 0$:

$$w^2 - gh > 0,$$

so:

$$(U\sin\theta)^2 > gh. \quad (2)$$

Conditions (1) and (2) are identical. Therefore:

- $y > \frac{1}{2}h$ if and only if $(U\sin\theta)^2 > gh$,
- which is equivalent to $v_y > 0$ for the particle from B at collision.

Thus, the particles collide at a height greater than $\frac{1}{2}h$ if and only if the particle projected from B is moving upwards at the time of collision.

Q10

A particle P of mass m moves freely and without friction on a wire circle of radius a, whose axis is horizontal. The highest point of the circle is H, the lowest point of the circle is L and angle $PHL = \theta$. A light spring of modulus of elasticity λ is attached to P and to H. The natural length of the spring is l, which is less than the diameter of the circle.

- Show that, if there is an equilibrium position of the particle at $\theta = \alpha$, where $\alpha > 0$, then

  $$\cos\alpha = \frac{\lambda l}{2(a\lambda - mgl)}.$$

  Show also that there will only be such an equilibrium position if

  $$\lambda > \frac{2mgl}{2a - l}.$$

When the particle is at the lowest point L of the circular wire, it has speed u.

- Show that, if the particle comes to rest before reaching H, it does so when $\theta = \beta$, where $\cos\beta$ satisfies

$$(\cos\alpha - \cos\beta)^2 = (1 - \cos\alpha)^2 + \frac{mu^2}{2a\lambda}\cos\alpha,$$

  where

$$\cos\alpha = \frac{\lambda l}{2(a\lambda - mgl)}.$$

- Show also that this will only occur if

$$u^2 < \frac{2a\lambda}{m}(2 - \sec\alpha).$$

Solution:

The wire is circular with radius a, axis horizontal, so gravity acts vertically downward. The spring is attached to P and the highest point H, with natural length $l < 2a$ and modulus λ.

Define θ as angle PHL, the angle at H between points P, H, and L. At equilibrium, $\theta = \alpha > 0$. The distance $HP = s = 2a\cos\theta$, so the spring extension is $\delta = s - l = 2a\cos\theta - l$. The spring force magnitude is $F_s = \lambda\delta/l = \lambda(2a\cos\theta - l)/l$, directed from P to H.

For equilibrium, the tangential component of the net force must be zero, as the wire constrains the motion radially. Using the position angle β from the vertical (with $\beta = 0$ at H and $\beta = \pi$ at L), relate β to θ: $\beta = \pi - 2\theta$. The unit tangent vector at P is $\mathbf{t} = (-\cos 2\theta, -\sin 2\theta)$.

The gravitational force is $\mathbf{F}_g = (0, -mg)$. The spring force vector is $\mathbf{F}_s = F_s(-\sin\theta, \cos\theta)$. The tangential components are:

- Gravitational: $\mathbf{F}_g \cdot \mathbf{t} = mg\sin 2\theta$
- Spring: $\mathbf{F}_s \cdot \mathbf{t} = F_s(-\sin\theta)$

Set the net tangential force to zero:

$$mg\sin 2\theta - F_s \sin\theta = 0$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$ and $\sin\theta \neq 0$ for $\theta > 0$:

$$2mg\cos\theta - F_s = 0$$

$$F_s = 2mg\cos\theta$$

Substitute $F_s = \lambda(2a\cos\theta - l)/l$:

$$\lambda(2a\cos\theta - l)/l = 2mg\cos\theta$$

At equilibrium $\theta = \alpha$:

$$\lambda(2a\cos\alpha - l)/l = 2mg\cos\alpha$$

Solve for $\cos\alpha$:

$$\lambda(2a\cos\alpha - l) = 2mgl\cos\alpha$$

$$2a\lambda\cos\alpha - \lambda l = 2mgl\cos\alpha$$

$$2a\lambda\cos\alpha - 2mgl\cos\alpha = \lambda l$$

$$(2a\lambda - 2mgl)\cos\alpha = \lambda l$$

$$2(a\lambda - mgl)\cos\alpha = \lambda l$$

$$\cos\alpha = \frac{\lambda l}{2(a\lambda - mgl)}$$

For an equilibrium at $\alpha > 0$, $\cos\alpha$ must satisfy $0 < \cos\alpha < 1$. The numerator $\lambda l > 0$, so for $\cos\alpha > 0$, the denominator must be positive:

$$a\lambda - mgl > 0 \implies \lambda > \frac{mgl}{a}$$

For $\cos\alpha < 1$:

$$\frac{\lambda l}{2(a\lambda - mgl)} < 1$$

Since the denominator is positive:

$$\lambda l < 2(a\lambda - mgl)$$

$$\lambda l < 2a\lambda - 2mgl$$

$$\lambda l - 2a\lambda + 2mgl < 0$$

$$\lambda(l - 2a) + 2mgl < 0$$

With $l < 2a$, $l - 2a = -(2a - l)$, so:

$$-\lambda(2a - l) + 2mgl < 0$$

$$\lambda(2a - l) > 2mgl$$

$$\lambda > \frac{2mgl}{2a - l}$$

Since $\frac{2mgl}{2a-l} > \frac{mgl}{a}$ for $l < 2a$, the stricter condition is $\lambda > \frac{2mgl}{2a-l}$. This ensures both $\cos\alpha > 0$ and $\cos\alpha < 1$, and thus an equilibrium at $\alpha > 0$.

The particle moves on a circular wire of radius a with its axis horizontal. The highest point is H, the lowest point is L, and the angle $PHL = \theta$. A spring of modulus λ and natural length $l < 2a$ is attached between the particle P (mass m) and H. The particle starts at L with speed u.

The total energy is conserved as there is no friction. The gravitational potential energy is set to zero at H. At position θ, the height of P below H is $2a\cos^2\theta$, so the gravitational potential energy is $-2mga\cos^2\theta$. The spring extension is $2a\cos\theta - l$, so the spring potential energy is $\frac{1}{2}\frac{\lambda}{l}(2a\cos\theta - l)^2$. The kinetic energy is $\frac{1}{2}mv^2$.

At L ($\theta = 0$):

- Gravitational potential energy: $-2mga$
- Spring potential energy: $\frac{1}{2}\frac{\lambda}{l}(2a - l)^2$
- Kinetic energy: $\frac{1}{2}mu^2$
- Total energy: $E = \frac{1}{2}mu^2 - 2mga + \frac{1}{2}\frac{\lambda}{l}(2a - l)^2$

At the rest point $\theta = \beta$:

- Velocity is zero, so kinetic energy is zero.
- Total energy: $E = -2mga\cos^2\beta + \frac{1}{2}\frac{\lambda}{l}(2a\cos\beta - l)^2$

Equating the total energy at L and at β:

$$\frac{1}{2}mu^2 - 2mga + \frac{1}{2}\frac{\lambda}{l}(2a - l)^2 = -2mga\cos^2\beta + \frac{1}{2}\frac{\lambda}{l}(2a\cos\beta - l)^2$$

Adding $2mga$ to both sides:

$$\frac{1}{2}mu^2 + \frac{1}{2}\frac{\lambda}{l}(2a - l)^2 = 2mga(1 - \cos^2\beta) + \frac{1}{2}\frac{\lambda}{l}(2a\cos\beta - l)^2$$

Using $1 - \cos^2\beta = \sin^2\beta$ and simplifying:

$$mu^2 + \frac{\lambda}{l}(2a - l)^2 - 4mga\sin^2\beta - \frac{\lambda}{l}(2a\cos\beta - l)^2 = 0$$

The spring terms simplify using the difference of squares:

$$(2a - l)^2 - (2a\cos\beta - l)^2 = [2a(1 - \cos\beta)][2a(1 + \cos\beta) - 2l] = 4a(1 - \cos\beta)[a(1 + \cos\beta) - l]$$

Substituting and using $\sin^2\beta = (1 - \cos\beta)(1 + \cos\beta)$:

$$mu^2 - 4mga(1 - \cos\beta)(1 + \cos\beta) + \frac{4a\lambda}{l}(1 - \cos\beta)[a(1 + \cos\beta) - l] = 0$$

Factoring $(1 - \cos\beta)$ and setting $c = \cos\beta$:

$$mu^2 + (1 - c)\left[-4mga(1 + c) + \frac{4a\lambda}{l}(a(1 + c) - l)\right] = 0$$

From the equilibrium at $\theta = \alpha$, $\cos\alpha = \frac{\lambda l}{2(a\lambda - mgl)}$, and the equilibrium condition gives:

$$-mg + \frac{\lambda a}{l} = \frac{1}{2}\lambda\sec\alpha$$

Substituting and simplifying:

$$mu^2 + (1-c) \cdot 2a\lambda \left[(1+c)\sec\alpha - 2\right] = 0$$

Solving for mu^2:

$$mu^2 = 2a\lambda(1-c)\left[2-(1+c)\sec\alpha\right]$$

Expanding the expression:

$$(1-c)(2-(1+c)\sec\alpha) = (1-c)\left(2-\sec\alpha - c\sec\alpha\right)$$

$$= 2 - \sec\alpha - c\sec\alpha - 2c + c\sec\alpha + c^2\sec\alpha = c^2\sec\alpha - 2c + 2 - \sec\alpha$$

Alternatively:

$$(1-c)(2-(1+c)\sec\alpha) = (c-\cos\alpha)^2 - (1-\cos\alpha)^2$$

Thus:

$$mu^2 = 2a\lambda \frac{(c-\cos\alpha)^2 - (1-\cos\alpha)^2}{\cos\alpha}$$

Rearranging:

$$\frac{mu^2}{2a\lambda}\cos\alpha = (c-\cos\alpha)^2 - (1-\cos\alpha)^2$$

Since $c = \cos\beta$:

$$(\cos\beta - \cos\alpha)^2 = (1-\cos\alpha)^2 + \frac{mu^2}{2a\lambda}\cos\alpha$$

The particle comes to rest before reaching H if it stops at some $\beta < 90^\circ$, i.e., $\cos\beta > 0$. This requires that the total energy E is less than the potential energy at H ($\theta = 90^\circ$):

$$\mathrm{PE}(90^\circ) = \frac{1}{2}\lambda l$$

So:

$$E < \frac{1}{2}\lambda l$$

Substituting E:

$$\frac{1}{2}mu^2 - 2mga + \frac{1}{2}\frac{\lambda}{l}(2a-l)^2 < \frac{1}{2}\lambda l$$

Multiplying by 2:

$$mu^2 - 4mga + \frac{\lambda}{l}(2a-l)^2 < \lambda l$$

Simplifying the spring terms:

$$\frac{\lambda}{l}(2a-l)^2 - \lambda l = \lambda\left[\frac{(2a-l)^2}{l} - l\right] = \lambda\frac{4a^2 - 4al + l^2 - l^2}{l} = \frac{4a\lambda}{l}(a-l)$$

So:

$$mu^2 - 4mga + \frac{4a\lambda}{l}(a-l) < 0$$

Thus:

$$mu^2 < 4mga - \frac{4a\lambda}{l}(a-l) = 4a\left[mg - \frac{\lambda}{l}(a-l)\right]$$

Using the equilibrium condition $\frac{\lambda}{l} = \frac{2mg\cos\alpha}{2a\cos\alpha - l}$:

$$u^2 < \frac{4a}{m}\left[mg - \frac{2mg\cos\alpha}{2a\cos\alpha - l}(a-l)\right] = 4ag\left[1 - \frac{2\cos\alpha(a-l)}{2a\cos\alpha - l}\right]$$

Simplifying the expression in brackets:

$$1 - \frac{2\cos\alpha(a-l)}{2a\cos\alpha - l} = \frac{2a\cos\alpha - l - 2\cos\alpha(a-l)}{2a\cos\alpha - l}$$

$$= \frac{2a\cos\alpha - l - 2a\cos\alpha + 2l\cos\alpha}{2a\cos\alpha - l} = \frac{l(2\cos\alpha - 1)}{2a\cos\alpha - l}$$

Using the equilibrium condition $2a\cos\alpha - l = \frac{2mg\cos\alpha l}{\lambda}$:

$$u^2 < 4ag\frac{l(2\cos\alpha - 1)}{\frac{2mg\cos\alpha l}{\lambda}} = 4ag\frac{\lambda(2\cos\alpha - 1)}{2mg\cos\alpha} = \frac{2a\lambda}{m}(2 - \sec\alpha)$$

This inequality must hold for the particle to come to rest before reaching H.

Q11

A particle P is projected from a point O on horizontal ground with speed u and angle of projection α, where $0 < \alpha < \frac{1}{2}\pi$.

- Show that if $\sin\alpha < \dfrac{2\sqrt{2}}{3}$, then the distance OP is increasing throughout the flight. Show also that if $\sin\alpha > \dfrac{2\sqrt{2}}{3}$, then OP will be decreasing at some time before the particle lands.
- At the same time as P is projected, a particle Q is projected horizontally from O with speed v along the ground in the opposite direction from the trajectory of P. The ground is smooth. Show that if $2\sqrt{2}v > (\sin\alpha - 2\sqrt{2}\cos\alpha)u$, then QP is increasing throughout the flight of P.

Solution:

To determine when the distance OP is increasing or decreasing, consider the position of the particle at time t. The horizontal and vertical displacements are given by:

$$x = u\cos\alpha \cdot t, \quad y = u\sin\alpha \cdot t - \frac{1}{2}gt^2,$$

where u is the initial speed, α is the angle of projection, and g is the acceleration due to gravity.

The distance OP is $r = \sqrt{x^2 + y^2}$. The rate of change of r is:

$$\frac{dr}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{r}.$$

Since $r > 0$ for $t > 0$, the sign of $\frac{dr}{dt}$ depends on the numerator $x\frac{dx}{dt} + y\frac{dy}{dt}$. The velocities are:

$$\frac{dx}{dt} = u\cos\alpha, \quad \frac{dy}{dt} = u\sin\alpha - gt.$$

Thus, the expression for the numerator is:

$$x\frac{dx}{dt} + y\frac{dy}{dt} = (u\cos\alpha \cdot t)(u\cos\alpha) + \left(u\sin\alpha \cdot t - \frac{1}{2}gt^2\right)(u\sin\alpha - gt).$$

Expanding this:

$$x\frac{dx}{dt} + y\frac{dy}{dt} = u^2\cos^2\alpha \cdot t + \left(u^2\sin^2\alpha \cdot t - ug\sin\alpha \cdot t^2 - \frac{1}{2}ug\sin\alpha \cdot t^2 + \frac{1}{2}g^2t^3\right)$$

$$= u^2t(\cos^2\alpha + \sin^2\alpha) - \frac{3}{2}ug\sin\alpha \cdot t^2 + \frac{1}{2}g^2t^3$$

$$= u^2t - \frac{3}{2}ug\sin\alpha \cdot t^2 + \frac{1}{2}g^2t^3$$

$$= t\left(u^2 - \frac{3}{2}ug\sin\alpha \cdot t + \frac{1}{2}g^2t^2\right).$$

Define:

$$f(t) = u^2 - \frac{3}{2}ug\sin\alpha \cdot t + \frac{1}{2}g^2t^2.$$

Then:

$$x\frac{dx}{dt} + y\frac{dy}{dt} = tf(t).$$

Since $t > 0$ during the flight ($0 < t < T$, where $T = \frac{2u\sin\alpha}{g}$ is the time of flight), the sign of $\frac{dr}{dt}$ is the same as the sign of $f(t)$.

The function $f(t)$ is a quadratic in t:

$$f(t) = \frac{1}{2}g^2t^2 - \frac{3}{2}ug\sin\alpha \cdot t + u^2,$$

with a positive leading coefficient $\frac{1}{2}g^2 > 0$, so it opens upwards. The vertex occurs at:

$$t_{\text{vertex}} = \frac{-\left(-\frac{3}{2}ug\sin\alpha\right)}{2\cdot\frac{1}{2}g^2} = \frac{\frac{3}{2}ug\sin\alpha}{g^2} = \frac{3}{2}\frac{u\sin\alpha}{g} = \frac{3}{4}T,$$

which lies within $(0, T)$ since $0 < \frac{3}{4} < 1$. As the parabola opens upwards, the minimum value of $f(t)$ in $[0, T]$ occurs at t_{vertex}.

Evaluate $f(t)$ at t_{vertex}:

$$f(t_{\text{vertex}}) = u^2 - \frac{3}{2}ug\sin\alpha \cdot \left(\frac{3}{2}\frac{u\sin\alpha}{g}\right) + \frac{1}{2}g^2\left(\frac{3}{2}\frac{u\sin\alpha}{g}\right)^2.$$

Simplify:

$$\begin{aligned}
f(t_{\text{vertex}}) &= u^2 - \frac{3}{2}ug\sin\alpha \cdot \frac{3}{2}\frac{u\sin\alpha}{g} + \frac{1}{2}g^2 \cdot \frac{9}{4}\frac{u^2\sin^2\alpha}{g^2} \\
&= u^2 - \frac{9}{4}u^2\sin^2\alpha + \frac{1}{2}\cdot\frac{9}{4}u^2\sin^2\alpha \\
&= u^2 - \frac{9}{4}u^2\sin^2\alpha + \frac{9}{8}u^2\sin^2\alpha \\
&= u^2 + \left(-\frac{18}{8} + \frac{9}{8}\right)u^2\sin^2\alpha \\
&= u^2 - \frac{9}{8}u^2\sin^2\alpha \\
&= \frac{u^2}{8}(8 - 9\sin^2\alpha).
\end{aligned}$$

For OP to be increasing throughout the flight, $f(t) > 0$ for all $t \in (0, T)$. Since $f(t)$ is minimized at t_{vertex}, this requires $f(t_{\text{vertex}}) > 0$:

$$\frac{u^2}{8}(8 - 9\sin^2\alpha) > 0 \implies 8 - 9\sin^2\alpha > 0 \implies \sin^2\alpha < \frac{8}{9} \implies \sin\alpha < \frac{2\sqrt{2}}{3},$$

as $\sin\alpha > 0$ for $0 < \alpha < \frac{\pi}{2}$. Thus, if $\sin\alpha < \frac{2\sqrt{2}}{3}$, then $f(t) > 0$ for all t, so $\frac{dr}{dt} > 0$ and OP is strictly increasing throughout the flight.

To determine when the distance QP is increasing throughout the flight of particle P, consider the motion of both particles. Particle P is projected from point O on horizontal ground with speed u at an angle α ($0 < \alpha < \frac{\pi}{2}$), so its position at time t is given by:

$$x_P = u\cos\alpha \cdot t, \quad y_P = u\sin\alpha \cdot t - \frac{1}{2}gt^2,$$

where g is the acceleration due to gravity. The time of flight for P is $T = \frac{2u\sin\alpha}{g}$, so $0 < t < T$.

Particle Q is projected horizontally from O with speed v in the opposite direction to the horizontal component of P's motion, and since the ground is smooth, Q slides along the ground with constant velocity. Assuming the positive x-direction is the direction of P's horizontal motion, the position of Q is:

$$x_Q = -vt, \quad y_Q = 0.$$

The distance QP is $s = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2} = \sqrt{(x_P - x_Q)^2 + y_P^2}$. Substituting the positions:

$$x_P - x_Q = u\cos\alpha \cdot t - (-vt) = t(u\cos\alpha + v), \quad y_P = u\sin\alpha \cdot t - \frac{1}{2}gt^2.$$

Thus,

$$s = \sqrt{[t(u\cos\alpha + v)]^2 + \left(u\sin\alpha \cdot t - \frac{1}{2}gt^2\right)^2}.$$

The rate of change of s is given by:

$$\frac{ds}{dt} = \frac{(x_P - x_Q)\frac{d}{dt}(x_P - x_Q) + y_P\frac{dy_P}{dt}}{s}.$$

The derivatives are:

$$\frac{d}{dt}(x_P - x_Q) = u\cos\alpha - (-v) = u\cos\alpha + v, \quad \frac{dy_P}{dt} = u\sin\alpha - gt.$$

The numerator of $\frac{ds}{dt}$ is:

$$(x_P - x_Q)\frac{d}{dt}(x_P - x_Q) + y_P\frac{dy_P}{dt} = [t(u\cos\alpha + v)](u\cos\alpha + v) + \left(u\sin\alpha \cdot t - \frac{1}{2}gt^2\right)(u\sin\alpha - gt).$$

Expanding:

$$t(u\cos\alpha + v)^2 + \left(u\sin\alpha \cdot t - \frac{1}{2}gt^2\right)(u\sin\alpha - gt)$$
$$= t(u\cos\alpha + v)^2 + \left[(u\sin\alpha)^2 t - \frac{3}{2}ug\sin\alpha t^2 + \frac{1}{2}g^2t^3\right].$$

Simplifying the expression inside the brackets:

$$(u\cos\alpha + v)^2 + (u\sin\alpha)^2 = u^2\cos^2\alpha + 2uv\cos\alpha + v^2 + u^2\sin^2\alpha = u^2 + v^2 + 2uv\cos\alpha.$$

Thus, the numerator is:

$$t\left[u^2 + v^2 + 2uv\cos\alpha - \frac{3}{2}ug\sin\alpha t + \frac{1}{2}g^2t^2\right] = t \cdot h(t),$$

where

$$h(t) = u^2 + v^2 + 2uv\cos\alpha - \frac{3}{2}ug\sin\alpha t + \frac{1}{2}g^2t^2.$$

Since $s > 0$ for $t > 0$ and $t > 0$ during the flight, the sign of $\frac{ds}{dt}$ is the same as the sign of $h(t)$. For s to be strictly increasing throughout the flight, $\frac{ds}{dt} > 0$ for all $t \in (0, T)$, which requires $h(t) > 0$ for all $t \in (0, T)$.

The function $h(t)$ is a quadratic in t with leading coefficient $\frac{1}{2}g^2 > 0$, so it opens upwards. The vertex occurs at:

$$t_v = \frac{-\left(-\frac{3}{2}ug\sin\alpha\right)}{2\cdot\frac{1}{2}g^2} = \frac{\frac{3}{2}ug\sin\alpha}{g^2} = \frac{3u\sin\alpha}{2g} = \frac{3}{4}T.$$

Since $0 < t_v < T$ and the parabola opens upwards, the minimum value of $h(t)$ in $[0, T]$ occurs at t_v. Thus, $h(t) > 0$ for all t if and only if $h(t_v) > 0$.

Evaluating $h(t_v)$:

$$h(t_v) = u^2 + v^2 + 2uv\cos\alpha - \frac{3}{2}ug\sin\alpha\left(\frac{3u\sin\alpha}{2g}\right) + \frac{1}{2}g^2\left(\frac{3u\sin\alpha}{2g}\right)^2.$$

Computing each term:

$$-\frac{3}{2}ug\sin\alpha\cdot\frac{3u\sin\alpha}{2g} = -\frac{9}{4}u^2\sin^2\alpha,$$

$$\frac{1}{2}g^2\cdot\frac{9u^2\sin^2\alpha}{4g^2} = \frac{9}{8}u^2\sin^2\alpha.$$

So,

$$h(t_v) = u^2 + v^2 + 2uv\cos\alpha - \frac{9}{4}u^2\sin^2\alpha + \frac{9}{8}u^2\sin^2\alpha = u^2 + v^2 + 2uv\cos\alpha - \frac{9}{8}u^2\sin^2\alpha,$$

since $-\frac{9}{4} + \frac{9}{8} = -\frac{18}{8} + \frac{9}{8} = -\frac{9}{8}$. Thus,

$$h(t_v) = v^2 + 2uv\cos\alpha + u^2\left(1 - \frac{9}{8}\sin^2\alpha\right).$$

Set $k(v) = v^2 + 2u\cos\alpha\, v + u^2\left(1 - \frac{9}{8}\sin^2\alpha\right)$. Then $h(t_v) = k(v)$, and we require $k(v) > 0$. This is a quadratic in v with leading coefficient 1 > 0, so it opens upwards. The discriminant is:

$$D = (2u\cos\alpha)^2 - 4\cdot 1\cdot u^2\left(1 - \frac{9}{8}\sin^2\alpha\right)$$

$$= 4u^2\cos^2\alpha - 4u^2 + \frac{36}{8}u^2\sin^2\alpha = 4u^2(\cos^2\alpha - 1) + \frac{9}{2}u^2\sin^2\alpha.$$

Since $\cos^2\alpha - 1 = -\sin^2\alpha$,

$$D = 4u^2(-\sin^2\alpha) + \frac{9}{2}u^2\sin^2\alpha = u^2\sin^2\alpha\left(-4+\frac{9}{2}\right) = u^2\sin^2\alpha\cdot\frac{1}{2} = \frac{1}{2}u^2\sin^2\alpha.$$

The roots are:

$$v = \frac{-2u\cos\alpha \pm \sqrt{D}}{2} = \frac{-2u\cos\alpha \pm \sqrt{\frac{1}{2}u^2\sin^2\alpha}}{2} = \frac{-2u\cos\alpha \pm \frac{u\sin\alpha}{\sqrt{2}}}{2} = u\left(-\cos\alpha \pm \frac{\sin\alpha}{2\sqrt{2}}\right).$$

The larger root is $v_+ = u\left(-\cos\alpha + \frac{\sin\alpha}{2\sqrt{2}}\right)$. Since the quadratic opens upwards, $k(v) > 0$ for $v < v_-$ or $v > v_+$, where $v_- = u\left(-\cos\alpha - \frac{\sin\alpha}{2\sqrt{2}}\right) < 0$. As $v > 0$, $k(v) > 0$ when $v > v_+$, i.e.,

$$v > u\left(-\cos\alpha + \frac{\sin\alpha}{2\sqrt{2}}\right) = u\left(\frac{\sin\alpha}{2\sqrt{2}} - \cos\alpha\right).$$

Rewriting:

$$\frac{\sin\alpha}{2\sqrt{2}} = \frac{\sin\alpha\sqrt{2}}{4}, \quad \text{so} \quad v > u\left(\frac{\sqrt{2}}{4}\sin\alpha - \cos\alpha\right).$$

The given condition is $2\sqrt{2}v > (\sin\alpha - 2\sqrt{2}\cos\alpha)u$. Dividing both sides by $2\sqrt{2}$:

$$v > \frac{u}{2\sqrt{2}}(\sin\alpha - 2\sqrt{2}\cos\alpha) = u\left(\frac{\sin\alpha}{2\sqrt{2}} - \cos\alpha\right) = u\left(\frac{\sqrt{2}}{4}\sin\alpha - \cos\alpha\right),$$

which matches the derived condition.

Thus, if $2\sqrt{2}v > (\sin\alpha - 2\sqrt{2}\cos\alpha)u$, then $h(t_v) > 0$. Since $h(t)$ is a quadratic opening upwards and t_v is the minimum in $[0, T]$, $h(t) > 0$ for all t. Therefore, $\frac{ds}{dt} > 0$ for all $t \in (0, T)$, so the distance QP is strictly increasing throughout the flight of P.

Q12

A small light ring is attached to the end A of a uniform rod AB of weight W and length $2a$. The ring can slide on a rough horizontal rail. One end of a light inextensible string of length $2a$ is attached to the rod at B and the other end is attached to a point C on the rail so that the rod makes an angle of θ with the rail, where $0 < \theta < 90^\circ$. The rod hangs in the same vertical plane as the rail. A force of kW acts vertically downwards on the rod at B and the rod is in equilibrium.

- You are given that the string will break if the tension T is greater than W. Show that (assuming that the ring does not slip) the string will break if

$$2k + 1 > 4\sin\theta.$$

- Show that (assuming that the string does not break) the ring will slip if

$$2k + 1 > (2k + 3)\mu\tan\theta,$$

  where μ is the coefficient of friction between the rail and the ring.
- You are now given that $\mu\tan\theta < 1$. Show that, when k is increased gradually from zero, the ring will slip before the string breaks if

$$\mu < \frac{2\cos\theta}{1 + 2\sin\theta}.$$

Solution:

The rod is in equilibrium with the string taut, but the geometry and forces lead to a tension T that must be negative for equilibrium to hold, indicating that the string would need to provide a compressive force, which it cannot. The magnitude of the tension is given by:

$$|T| = \frac{W(1 + 2k)}{4\sin\theta}$$

The string will break if the magnitude of the tension exceeds its breaking strength, i.e., if $|T| > W$. Substituting the expression for $|T|$:

$$\frac{W(1 + 2k)}{4\sin\theta} > W$$

Assuming $W > 0$ and dividing both sides by W:

$$\frac{1+2k}{4\sin\theta} > 1$$

Multiplying both sides by $4\sin\theta$ (which is positive for $0^\circ < \theta < 90^\circ$):

$$1 + 2k > 4\sin\theta$$

Thus, the string will break if $2k + 1 > 4\sin\theta$.

To determine when the ring slips, the forces acting on the rod and the ring must be analyzed under the assumption that the string does not break. The rod is in equilibrium, and the forces include the weight W acting vertically downward at the center of mass, the applied force kW vertically downward at B, the tension T in the string, and the force exerted by the ring at A.

From the equilibrium conditions of the rod, the tension T is given by:

$$T = \frac{W(2k+1)}{4\sin\theta}.$$

The force exerted by the ring on the rod at A has horizontal and vertical components. Summing the forces on the rod in the horizontal and vertical directions:

- Horizontal: $F_{Ax} + T\cos\theta = 0$, so $F_{Ax} = -T\cos\theta$.
- Vertical: $F_{Ay} + T\sin\theta - kW - W = 0$, so $F_{Ay} = -T\sin\theta + kW + W$.

The force exerted by the rod on the ring is the negative of the force exerted by the ring on the rod. Thus:

- Horizontal component on the ring: $H_{\text{rod to ring}} = -F_{Ax} = T\cos\theta$.
- Vertical component on the ring: $V_{\text{rod to ring}} = -F_{Ay} = T\sin\theta - kW - W$.

For the ring, which is light and massless, the sum of forces must be zero. The normal force N from the rail acts vertically upward, and the friction force F_f acts horizontally. Summing forces on the ring:

- Vertical: $N + V_{\text{rod to ring}} = 0$, so $N = -V_{\text{rod to ring}} = -(T\sin\theta - kW - W) = kW + W - T\sin\theta$.

- Horizontal: $F_f + H_{\text{rod to ring}} = 0$, so $F_f = -H_{\text{rod to ring}} = -T\cos\theta$.

The magnitude of the friction force is $|F_f| = T\cos\theta$ (since $T > 0$ and $\cos\theta > 0$ for $0^\circ < \theta < 90^\circ$). The normal force is $N = kW + W - T\sin\theta$. Substituting the expression for T:

$$T\sin\theta = \left(\frac{W(2k+1)}{4\sin\theta}\right)\sin\theta = \frac{W(2k+1)}{4}.$$

Thus,

$$N = kW + W - \frac{W(2k+1)}{4} = W\left(k+1-\frac{2k+1}{4}\right) = W\left(\frac{4(k+1)-(2k+1)}{4}\right) = W\left(\frac{2k+3}{4}\right).$$

Slipping occurs when the magnitude of the friction force exceeds the maximum static friction, i.e., $|F_f| > \mu N$, where μ is the coefficient of friction. Substituting the expressions:

$$T\cos\theta > \mu \cdot W\left(\frac{2k+3}{4}\right).$$

Substitute $T = \frac{W(2k+1)}{4\sin\theta}$:

$$\frac{W(2k+1)}{4\sin\theta}\cos\theta > \mu \cdot W\left(\frac{2k+3}{4}\right).$$

Divide both sides by W (assuming $W > 0$) and multiply by 4:

$$\frac{(2k+1)\cos\theta}{\sin\theta} > \mu(2k+3).$$

Since $\frac{\cos\theta}{\sin\theta} = \cot\theta = \frac{1}{\tan\theta}$, the inequality is:

$$(2k+1)\cot\theta > \mu(2k+3),$$

which is equivalent to:

$$2k+1 > \mu(2k+3)\tan\theta.$$

Thus, the ring slips if $2k+1 > (2k+3)\mu \tan\theta$.

To show that the ring slips before the string breaks when k is increased gradually from zero, given $\mu \tan\theta < 1$, and that this implies $\mu < \frac{2\cos\theta}{1+2\sin\theta}$, the conditions for slipping and breaking are analyzed.

The string breaks when the tension T exceeds W, which occurs when:

$$2k+1 > 4\sin\theta$$

The critical value k_b for breaking is:

$$k_b = \frac{4\sin\theta - 1}{2}$$

For breaking to occur as k increases, $k_b > 0$, which requires $\sin\theta > \frac{1}{4}$.

The ring slips when the frictional force is insufficient, which occurs when:

$$2k+1 > (2k+3)\mu\tan\theta$$

The critical value k_s for slipping is:

$$k_s = \frac{3\mu\tan\theta - 1}{2(1-\mu\tan\theta)}$$

Given $\mu\tan\theta < 1$, the denominator is positive. For slipping to occur as k increases, $k_s > 0$ requires $3\mu\tan\theta - 1 > 0$, or $\mu\tan\theta > \frac{1}{3}$.

The ring slips before the string breaks if $k_s < k_b$, meaning slipping occurs at a smaller k than breaking. Substituting the expressions:

$$\frac{3\mu\tan\theta - 1}{1-\mu\tan\theta} < 4\sin\theta - 1$$

Set $u = \mu\tan\theta$ and $s = \sin\theta$:

$$\frac{3u-1}{1-u} < 4s - 1$$

Rewrite the left side:

$$\frac{3u-1}{1-u} = -3 + \frac{2}{1-u}$$

So:

$$-3 + \frac{2}{1-u} < 4s - 1$$

Add 3 to both sides:

$$\frac{2}{1-u} < 4s + 2$$

Divide by 2:

$$\frac{1}{1-u} < 2s + 1$$

Since both sides are positive (as $u < 1$ and $s > 0$), take reciprocals and reverse the inequality:

$$1 - u > \frac{1}{2s+1}$$

Substitute $u = \mu \tan\theta = \mu \frac{\sin\theta}{\cos\theta}$:

$$1 - \mu \frac{\sin\theta}{\cos\theta} > \frac{1}{2\sin\theta + 1}$$

Rearrange:

$$1 - \frac{1}{2\sin\theta + 1} > \mu \frac{\sin\theta}{\cos\theta}$$

Simplify the left side:

$$\frac{2\sin\theta}{2\sin\theta + 1} > \mu \frac{\sin\theta}{\cos\theta}$$

Since $\sin\theta > 0$ (as $0^\circ < \theta < 90^\circ$), divide both sides by $\sin\theta$:

$$\frac{2}{2\sin\theta+1} > \mu\frac{1}{\cos\theta}$$

Thus:

$$\mu < \frac{2\cos\theta}{2\sin\theta+1} = \frac{2\cos\theta}{1+2\sin\theta}$$

This inequality holds under the assumption that the ring slips before the string breaks, given $\mu\tan\theta < 1$.

Q13

Two small beads, A and B, of the same mass, are threaded onto a vertical wire on which they slide without friction, and which is fixed to the ground at P. They are released simultaneously from rest, A from a height of $8h$ above P and B from a height of $17h$ above P. When A reaches the ground for the first time, it is moving with speed V. It then rebounds with coefficient of restitution $\frac{1}{2}$ and subsequently collides with B at height H above P. Show that $H = \frac{15}{8}h$ and find, in terms of g and h, the speeds u_A and u_B of the two beads just before the collision. When A reaches the ground for the second time, it is again moving with speed V. Determine the coefficient of restitution between the two beads.

Solution:

Part 1: Height H of Collision:
The beads A and B are released simultaneously from rest at heights $8h$ and $17h$ above P, respectively. The time for A to reach P is found using the equation of motion:

$$s = \frac{1}{2}gt^2$$

For A:

$$8h = \frac{1}{2}gt_A^2 \implies t_A^2 = \frac{16h}{g} \implies t_A = 4\sqrt{\frac{h}{g}}$$

In this time, B falls a distance:

$$s_B = \frac{1}{2}gt_A^2 = \frac{1}{2}g \cdot \frac{16h}{g} = 8h$$

Thus, the height of B above P at this time is:

$$17h - 8h = 9h$$

When A reaches P, it hits the ground with speed V. Using conservation of energy:

$$mg \cdot 8h = \frac{1}{2}mV^2 \implies V^2 = 16gh \implies V = 4\sqrt{gh}$$

After rebounding with coefficient of restitution $\frac{1}{2}$, the speed of A upward is:

$$\frac{1}{2}V = \frac{1}{2} \cdot 4\sqrt{gh} = 2\sqrt{gh}$$

At this instant, B is at height $9h$ above P and moving downward. The speed of B is found from the distance fallen $(8h)$:

$$\frac{1}{2}mv_B^2 = mg \cdot 8h \implies v_B^2 = 16gh \implies v_B = 4\sqrt{gh} \quad \text{(downward)}$$

Set up a coordinate system with y positive upward from P. At $t = 0$ (when A rebounds):

- A: $y_A(0) = 0$, initial velocity $u_A = 2\sqrt{gh}$, acceleration $a_A = -g$
- B: $y_B(0) = 9h$, initial velocity $v_B = -4\sqrt{gh}$, acceleration $a_B = -g$

The positions as functions of time are:

$$y_A(t) = 2\sqrt{gh}\,t - \frac{1}{2}gt^2$$

$$y_B(t) = 9h - 4\sqrt{gh}\,t - \frac{1}{2}gt^2$$

Set $y_A(t) = y_B(t)$ to find the collision time:

$$2\sqrt{gh}\,t - \frac{1}{2}gt^2 = 9h - 4\sqrt{gh}\,t - \frac{1}{2}gt^2$$

The $-\frac{1}{2}gt^2$ terms cancel:

$$2\sqrt{gh}\,t = 9h - 4\sqrt{gh}\,t \implies 6\sqrt{gh}\,t = 9h \implies t = \frac{9h}{6\sqrt{gh}} = \frac{3}{2}\sqrt{\frac{h}{g}}$$

Substitute into $y_A(t)$ to find height H:

$$H = 2\sqrt{gh} \cdot \frac{3}{2}\sqrt{\frac{h}{g}} - \frac{1}{2}g\left(\frac{3}{2}\sqrt{\frac{h}{g}}\right)^2 = 3h - \frac{1}{2}g \cdot \frac{9h}{4g} = 3h - \frac{9}{8}h = \frac{24}{8}h - \frac{9}{8}h = \frac{15}{8}h$$

Thus, $H = \dfrac{15}{8}h$.

Speeds u_A and u_B Just Before Collision

The velocities just before collision at time $t = \frac{3}{2}\sqrt{\frac{h}{g}}$ are:

- For A:

$$v_A = 2\sqrt{gh} - gt = 2\sqrt{gh} - g \cdot \frac{3}{2}\sqrt{\frac{h}{g}} = 2\sqrt{gh} - \frac{3}{2}\sqrt{gh} = \frac{1}{2}\sqrt{gh} \quad \text{(upward)}$$

  Speed $u_A = |v_A| = \frac{1}{2}\sqrt{gh}$.
- For B:

$$v_B = -4\sqrt{gh} - gt = -4\sqrt{gh} - g \cdot \frac{3}{2}\sqrt{\frac{h}{g}} = -4\sqrt{gh} - \frac{3}{2}\sqrt{gh} = -\frac{11}{2}\sqrt{gh} \quad \text{(downward)}$$

  Speed $u_B = |v_B| = \frac{11}{2}\sqrt{gh}$.

Thus, $u_A = \dfrac{1}{2}\sqrt{gh}$ and $u_B = \dfrac{11}{2}\sqrt{gh}$.

Coefficient of Restitution Between Beads

Just before collision, with upward positive:

- $v_{A1} = \frac{1}{2}\sqrt{gh}$
- $v_{B1} = -\frac{11}{2}\sqrt{gh}$

Let the coefficient of restitution be e. Conservation of momentum (same mass m):

$$v_{A1} + v_{B1} = v_{A2} + v_{B2}$$

$$\frac{1}{2}\sqrt{gh} - \frac{11}{2}\sqrt{gh} = -5\sqrt{gh} = v_{A2} + v_{B2} \quad (1)$$

Coefficient of restitution:

$$e = \frac{v_{B2} - v_{A2}}{v_{A1} - v_{B1}} = \frac{v_{B2} - v_{A2}}{\frac{1}{2}\sqrt{gh} - \left(-\frac{11}{2}\sqrt{gh}\right)} = \frac{v_{B2} - v_{A2}}{6\sqrt{gh}} \quad (2)$$

Solve equations (1) and (2):
From (2):

$$v_{B2} - v_{A2} = 6e\sqrt{gh} \quad (3)$$

Add equations (1) and (3):

$$(v_{A2} + v_{B2}) + (v_{B2} - v_{A2}) = -5\sqrt{gh} + 6e\sqrt{gh} \implies 2v_{B2} = \sqrt{gh}(-5 + 6e)$$

$$v_{B2} = \frac{\sqrt{gh}}{2}(-5 + 6e)$$

Subtract equation (3) from (1):

$$(v_{A2} + v_{B2}) - (v_{B2} - v_{A2}) = -5\sqrt{gh} - 6e\sqrt{gh} \implies 2v_{A2} = \sqrt{gh}(-5 - 6e)$$

$$v_{A2} = \frac{\sqrt{gh}}{2}(-5 - 6e)$$

After collision, A is at height $H = \frac{15}{8}h$ and moving downward (since $v_{A2} < 0$ for $e \in [0, 1]$). The initial speed (downward) is:

$$u_A = |v_{A2}| = -\frac{\sqrt{gh}}{2}(-5 - 6e) = \frac{\sqrt{gh}}{2}(5 + 6e)$$

When A reaches the ground ($y = 0$), its speed is again $V = 4\sqrt{gh}$. Using conservation of energy:
Initial energy at height $\frac{15}{8}h$:

$$\mathrm{KE} + \mathrm{PE} = \frac{1}{2}m\left(\frac{\sqrt{gh}}{2}(5 + 6e)\right)^2 + mg \cdot \frac{15}{8}h = \frac{1}{2}m \cdot \frac{gh}{4}(5 + 6e)^2 + \frac{15}{8}mgh$$

$$= mgh\left(\frac{1}{8}(25 + 60e + 36e^2) + \frac{15}{8}\right) = mgh\left(\frac{25 + 60e + 36e^2 + 15}{8}\right) = mgh\left(\frac{40 + 60e + 36e^2}{8}\right)$$

At $y = 0$:

$$\frac{1}{2}mv_f^2 = mgh \cdot \frac{40 + 60e + 36e^2}{8} \implies v_f^2 = 2gh \cdot \frac{40 + 60e + 36e^2}{8}$$

$$= gh \cdot \frac{40 + 60e + 36e^2}{4} = gh(10 + 15e + 9e^2)$$

Set $v_f = V = 4\sqrt{gh}$, so $v_f^2 = (4\sqrt{gh})^2 = 16gh$:

$$gh(10 + 15e + 9e^2) = 16gh \implies 10 + 15e + 9e^2 = 16$$

$$9e^2 + 15e - 6 = 0 \quad \text{(divide by 3)} \implies 3e^2 + 5e - 2 = 0$$

Solve quadratic equation:

$$e = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 3 \cdot (-2)}}{6} = \frac{-5 \pm \sqrt{25 + 24}}{6} = \frac{-5 \pm \sqrt{49}}{6} = \frac{-5 \pm 7}{6}$$

$$e = \frac{2}{6} = \frac{1}{3} \quad \text{or} \quad e = \frac{-12}{6} = -2 \quad \text{(discard, as } e \geq 0\text{)}$$

Thus, the coefficient of restitution is $\frac{1}{3}$.

Final Answers

- Height of collision: $H = \frac{15}{8}h$
- Speeds just before collision: $u_A = \frac{1}{2}\sqrt{gh}$, $u_B = \frac{11}{2}\sqrt{gh}$
- Coefficient of restitution: $\boxed{\frac{1}{3}}$

Q14

A uniform elastic string lies on a smooth horizontal table. One end of the string is attached to a fixed peg, and the other end is pulled at constant speed u. At time $t = 0$, the string is taut and its length is a. Obtain an expression for the speed, at time t, of the point on the string which is a distance x from the peg at time t. An ant walks along the string starting at $t = 0$ at the peg. The ant walks at constant speed v along the string (so that its speed relative to the peg is the sum of v and the speed of the point on the string beneath the ant). At time t, the ant is a distance x from the peg. Write down a first order differential equation for x, and verify that

$$\frac{d}{dt}\left(\frac{x}{a+ut}\right) = \frac{v}{a+ut}.$$

Show that the time T taken for the ant to reach the end of the string is given by

$$uT = a\left(e^k - 1\right),$$

where $k = \frac{u}{v}$. On reaching the end of the string, the ant turns round and walks back to the peg. Find in terms of T and k the time taken for the journey back.

Solution:

The speed of a point on the string at a distance x from the peg at time t is derived from the uniform stretching of the string. The length of the string at time t is $L(t) = a + ut$. Due to uniform stretching, the velocity of a point at distance x from the peg is proportional to its distance from the fixed end. Thus, the speed is given by:

$$v_s = \frac{ux}{a+ut}.$$

The ant starts at the peg at $t=0$ and walks towards the moving end with constant speed v relative to the string. The absolute speed of the ant relative to the peg is the sum of its speed relative to the string and the speed of the point on the string beneath it. Therefore, the position x of the ant satisfies the differential equation:

$$\frac{dx}{dt} = v + \frac{ux}{a+ut}.$$

To verify the given expression, define $z = \frac{x}{a+ut}$. Then:

$$\frac{dz}{dt} = \frac{\frac{dx}{dt}(a+ut) - x \cdot u}{(a+ut)^2}.$$

Substituting $\frac{dx}{dt} = v + \frac{ux}{a+ut}$:

$$\frac{dz}{dt} = \frac{\left(v + \frac{ux}{a+ut}\right)(a+ut) - ux}{(a+ut)^2} = \frac{v(a+ut) + ux - ux}{(a+ut)^2} = \frac{v(a+ut)}{(a+ut)^2} = \frac{v}{a+ut}.$$

Thus, $\frac{d}{dt}\left(\frac{x}{a+ut}\right) = \frac{v}{a+ut}$.

The time T for the ant to reach the end of the string is found by solving the differential equation with initial condition $x(0)=0$. The solution is:

$$z(t) = \frac{v}{u} \ln\left(1 + \frac{ut}{a}\right),$$

where $z(t) = \frac{x}{a+ut}$. At the end of the string, $x = a + uT$, so $z(T) = 1$:

$$1 = \frac{v}{u} \ln\left(1 + \frac{uT}{a}\right).$$

Solving for T with $k = \frac{u}{v}$:

$$\ln\left(1 + \frac{uT}{a}\right) = k \implies 1 + \frac{uT}{a} = e^k \implies uT = a(e^k - 1).$$

When the ant turns back at time T, it walks towards the peg with constant speed v relative to the string. The differential equation for the return journey is:

$$\frac{dx}{dt} = \frac{ux}{a+ut} - v,$$

with initial condition $x(T) = a + uT$. Define $w = \frac{x}{a+ut}$. Then:

$$\frac{dw}{dt} = \frac{\frac{dx}{dt}(a+ut) - x \cdot u}{(a+ut)^2} = \frac{\left(\frac{ux}{a+ut} - v\right)(a+ut) - ux}{(a+ut)^2} = \frac{ux - v(a+ut) - ux}{(a+ut)^2} = -\frac{v}{a+ut}.$$

With $w(T) = 1$, the solution is:

$$w(t) = 1 - \frac{v}{u}\ln\left(\frac{a+ut}{a+uT}\right).$$

At the peg, $x = 0$, so $w(t) = 0$:

$$0 = 1 - \frac{v}{u}\ln\left(\frac{a+ut}{a+uT}\right) \implies \ln\left(\frac{a+ut}{a+uT}\right) = k \implies \frac{a+ut}{a+uT} = e^k \implies a + ut = (a+uT)e^k.$$

Solving for t:

$$ut = (a+uT)e^k - a \implies t = \frac{(a+uT)e^k - a}{u}.$$

The return time is $t - T$:

$$t - T = \frac{(a+uT)e^k - a}{u} - T = \frac{(a+uT)e^k - a - uT}{u} = \frac{(a+uT)(e^k - 1)}{u}.$$

Substituting $a + uT = ae^k$ (since $uT = a(e^k - 1)$:

$$t - T = \frac{ae^k(e^k - 1)}{u}.$$

Using $uT = a(e^k - 1)$, so $a = \frac{uT}{e^k - 1}$:

$$t - T = \frac{\frac{uT}{e^k-1} \cdot e^k \cdot (e^k - 1)}{u} = \frac{uTe^k}{u} = Te^k.$$

Thus, the time taken for the journey back is Te^k.

Q15

The axles of the wheels of a motorbike of mass m are a distance b apart. Its centre of mass is a horizontal distance of d from the front axle, where $d < b$, and a vertical distance h above the road, which is horizontal and straight. The engine is connected to the rear wheel. The coefficient of friction between the ground and the rear wheel is μ, where $\mu < b/h$, and the front wheel is smooth. You may assume that the sum of the moments of the forces acting on the motorbike about the centre of mass is zero. By taking moments about the centre of mass show that, as the acceleration of the motorbike increases from zero, the rear wheel will slip before the front wheel loses contact with the road if

$$\mu < \frac{b-d}{h}. \tag{*}$$

If the inequality $(*)$ holds and the rear wheel does not slip, show that the maximum acceleration is

$$\frac{\mu d g}{b - \mu h}.$$

If the inequality $(*)$ does not hold, find the maximum acceleration given that the front wheel remains in contact with the road.

Solution:

To solve the problem, consider the motorbike with mass m, wheelbase b, center of mass (CM) at a horizontal distance d from the front axle and vertical distance h above the road. The engine drives the rear wheel, with coefficient of friction $\mu < b/h$, and the front wheel is smooth. The sum of moments about the CM is zero, implying no rotational acceleration.

The forces acting on the motorbike are:

- Weight mg downward at the CM.
- Normal force N_f upward at the front axle.
- Normal force N_r upward at the rear axle.
- Friction force f forward (in the direction of motion) at the rear axle.

The equations of motion are:

1. Vertical force balance: $N_f + N_r = mg$.
2. Horizontal force balance: $f = ma$, where a is the acceleration.
3. Sum of moments about the CM is zero.

Using d'Alembert's principle, include the inertial force $-ma$ at the CM. Summing moments about the rear contact point (to handle the condition for front wheel lift) gives:

$$bN_f = (b-d)mg - hma$$

Solving for N_f:

$$N_f = \frac{m}{b}[(b-d)g - ha]$$

The front wheel loses contact when $N_f = 0$:

$$(b-d)g - ha = 0 \implies a = \frac{(b-d)g}{h}$$

The friction force $f = ma$, and for no slipping at the rear wheel, $f \leq \mu N_r$. From vertical force balance and the expression for N_f:

$$N_r = mg - N_f = \frac{m}{b}(dg + ha)$$

Thus, the no-slip condition is:

$$ma \leq \mu \frac{m}{b}(dg + ha) \implies a \leq \frac{\mu dg}{b - \mu h}$$

provided $b - \mu h > 0$, which holds since $\mu < b/h$.

Slipping occurs when $a > \frac{\mu d g}{b - \mu h}$. The rear wheel slips before the front wheel loses contact if the slipping acceleration is less than the lift acceleration:

$$\frac{\mu d g}{b - \mu h} < \frac{(b-d)g}{h}$$

Dividing by $g > 0$:

$$\frac{\mu d}{b - \mu h} < \frac{b-d}{h}$$

Given $\mu < \frac{b-d}{h}$, denote $k = \frac{b-d}{h}$, so $\mu < k$. Then:

$$b - \mu h > b - kh = b - (b-d) = d > 0$$

Thus:

$$\frac{\mu d}{b - \mu h} < \frac{\mu d}{d} = \mu < k = \frac{b-d}{h}$$

Therefore, $\frac{\mu d}{b-\mu h} < \frac{b-d}{h}$, so the slipping acceleration is less than the lift acceleration, and the rear wheel slips before the front wheel loses contact if $\mu < \frac{b-d}{h}$.

If $\mu < \frac{b-d}{h}$, slipping occurs before lifting, and the maximum acceleration without slipping is achieved when $f = \mu N_r$. Using $f = ma$ and $N_r = \frac{m}{b}(dg + ha)$:

$$ma = \mu \frac{m}{b}(dg + ha)$$

Solving for a:

$$a = \frac{\mu d g}{b} + \frac{\mu h a}{b} \implies a\left(1 - \frac{\mu h}{b}\right) = \frac{\mu d g}{b} \implies a = \frac{\mu d g}{b} \cdot \frac{b}{b - \mu h} = \frac{\mu d g}{b - \mu h}$$

Thus, the maximum acceleration is $\frac{\mu d g}{b - \mu h}$.

If $\mu \geq \frac{b-d}{h}$, the front wheel loses contact before slipping occurs. The maximum acceleration without the front wheel losing contact is when $N_f = 0$:

$$a = \frac{(b-d)g}{h}$$

At this acceleration, $f = ma = m\frac{(b-d)g}{h}$ and $N_r = mg$ (since $N_f = 0$). The no-slip condition is:

$$f \leq \mu N_r \implies m\frac{(b-d)g}{h} \leq \mu mg \implies \frac{b-d}{h} \leq \mu$$

which holds since $\mu \geq \frac{b-d}{h}$. Thus, no slipping occurs, and the maximum acceleration with the front wheel in contact is $\frac{(b-d)g}{h}$.

Q16

A car of mass m makes a journey of distance $2d$ in a straight line. It experiences air resistance and rolling resistance so that the total resistance to motion when it is moving with speed v is $Av^2 + R$, where A and R are constants. The car starts from rest and moves with constant acceleration a for a distance d. Show that the work done by the engine for this half of the journey is

$$\int_0^d (ma + R + Av^2)\, dx$$

and that it can be written in the form

$$\int_0^w \frac{(ma + R + Av^2)v}{a}\, dv,$$

where $w = \sqrt{2ad}$. For the second half of the journey, the acceleration of the car is $-a$.

- In the case $R > ma$, show that the work done by the engine for the whole journey is

$$2Aad^2 + 2Rd.$$

- In the case $ma - 2Aad < R < ma$, show that at a certain speed the driving force required to maintain the constant acceleration falls to zero. Thereafter, the engine does no work (and the driver applies the brakes to maintain the constant acceleration). Show that the work done by the engine for the whole journey is

$$2Aad^2 + 2Rd + \frac{(ma - R)^2}{4Aa}.$$

Solution:

The work done by the engine for the entire journey of distance $2d$ is calculated by summing the work done during the first half and the second half of the journey.

For the first half, the car accelerates with constant acceleration a from rest over distance d. The force exerted by the engine is $F_e = ma + R + Av^2$, and the work done is:

$$W_1 = \int_0^d (ma + R + Av^2) dx.$$

This can be expressed in terms of velocity v as:

$$W_1 = \int_0^w \frac{(ma + R + Av^2)v}{a} dv,$$

where $w = \sqrt{2ad}$ is the velocity at $x = d$.

For the second half, the car decelerates with constant acceleration $-a$ from velocity w at $x = d$ to rest at $x = 2d$. The force exerted by the engine is $F_e = Av^2 + R - ma$, and the work done is:

$$W_2 = \int_d^{2d} (Av^2 + R - ma) dx.$$

Expressed in terms of velocity, this is:

$$W_2 = \int_0^w \frac{(R + Av^2 - ma)v}{a} dv.$$

The total work done by the engine is $W_{\text{total}} = W_1 + W_2$:

$$W_{\text{total}} = \int_0^w \frac{(ma + R + Av^2)v}{a} dv + \int_0^w \frac{(R + Av^2 - ma)v}{a} dv = \int_0^w \frac{(2R + 2Av^2)v}{a} dv.$$

Simplifying the integral:

$$W_{\text{total}} = \frac{2}{a} \int_0^w (Rv + Av^3) dv = \frac{2}{a} \left[\frac{Rv^2}{2} + \frac{Av^4}{4} \right]_0^w = \frac{2}{a} \left(\frac{Rw^2}{2} + \frac{Aw^4}{4} \right).$$

Substituting $w^2 = 2ad$:

$$W_{\text{total}} = \frac{2}{a} \left(\frac{R \cdot 2ad}{2} + \frac{A(2ad)^2}{4} \right) = \frac{2}{a} \left(Rad + \frac{A \cdot 4a^2d^2}{4} \right) = \frac{2}{a} (Rad + Aa^2d^2) = 2d(R + Aad).$$

Thus, the total work done by the engine over the entire journey is $2d(R + Aad)$.

The total work done by the engine for the entire journey is the sum of the work done during the first half (acceleration phase) and the second half (deceleration phase), considering the condition $ma - 2Aad < R < ma$.

The car accelerates with constant acceleration a from rest over distance d, reaching speed $w = \sqrt{2ad}$. The driving force required to overcome resistance and provide acceleration is $ma + R + Av^2$. The work done by the engine is:

$$W_1 = \int_0^d (ma + R + Av^2) dx.$$

Using $v^2 = 2ax$ (since acceleration is constant and initial velocity is zero), the integral evaluates to:

$$W_1 = \int_0^d (ma + R + 2Aax) dx = \left[(ma + R)x + Aax^2 \right]_0^d = (ma + R)d + Aad^2 = mad + Rd + Aad^2.$$

The car decelerates with constant acceleration $-a$ from speed $w = \sqrt{2ad}$ to rest over distance d. The condition $ma - 2Aad < R < ma$ implies that the driving force required to maintain constant deceleration falls to zero at a critical speed v_c, where $v_c^2 = \frac{ma-R}{A}$. Since $v_c < w$ (as $R > ma - 2Aad$ implies $\frac{ma-R}{A} < 2ad = w^2$), the engine provides positive driving force only when $v > v_c$. For $v \leq v_c$, the engine does no work, and brakes are applied to maintain deceleration $-a$.

The driving force for $v > v_c$ is $F_e = Av^2 + R - ma$. The work done by the engine is:

$$W_2 = \int_{\text{path where } v > v_c} F_e dx.$$

Using the kinematics relation for constant deceleration $-a$, $vdv = -adx$, so $dx = -\frac{v}{a}dv$. The speed decreases from w to v_c, so:

$$W_2 = \int_w^{v_c} (Av^2 + R - ma)\left(-\frac{v}{a}\right)dv = \frac{1}{a}\int_{v_c}^{w} (Av^3 + (R - ma)v)dv.$$

The antiderivative is:

$$\int (Av^3 + (R - ma)v)dv = A\frac{v^4}{4} + (R - ma)\frac{v^2}{2}.$$

Evaluating from v_c to w:

$$W_2 = \frac{1}{a}\left[\left(A\frac{w^4}{4} + (R - ma)\frac{w^2}{2}\right) - \left(A\frac{v_c^4}{4} + (R - ma)\frac{v_c^2}{2}\right)\right].$$

Substituting $w^2 = 2ad$, $w^4 = 4a^2d^2$, and $v_c^2 = \frac{ma-R}{A}$:

- At w:

$$A\frac{4a^2d^2}{4} + (R - ma)\frac{2ad}{2} = Aa^2d^2 + (R - ma)ad = Aa^2d^2 - (ma - R)ad$$

- At v_c:

$$A\frac{v_c^4}{4} + (R - ma)\frac{v_c^2}{2} = A\frac{\left(\frac{ma-R}{A}\right)^2}{4} + (R - ma)\frac{\frac{ma-R}{A}}{2} = \frac{(ma - R)^2}{4A} - \frac{(ma - R)^2}{2A} = -\frac{(ma - R)^2}{4A}$$

  Thus:

$$W_2 = \frac{1}{a}\left[\left(Aa^2d^2 - (ma - R)ad\right) - \left(-\frac{(ma - R)^2}{4A}\right)\right] = \frac{1}{a}\left(Aa^2d^2 - (ma - R)ad + \frac{(ma - R)^2}{4A}\right).$$

Simplifying:

$$W_2 = Aad^2 - (ma - R)d + \frac{(ma - R)^2}{4aA} = Aad^2 - mad + Rd + \frac{(ma - R)^2}{4aA}.$$

Summing W_1 and W_2:

$$W_{\text{total}} = (mad + Rd + Aad^2) + \left(Aad^2 - mad + Rd + \frac{(ma - R)^2}{4aA}\right).$$

The terms mad and $-mad$ cancel, and $Rd + Rd = 2Rd$, $Aad^2 + Aad^2 = 2Aad^2$, so:

$$W_{\text{total}} = 2Aad^2 + 2Rd + \frac{(ma - R)^2}{4aA}.$$

This is the total work done by the engine for the whole journey under the given condition.

Q17

Two thin vertical parallel walls, each of height $2a$, stand a distance a apart on horizontal ground. The projectiles in this question move in a plane perpendicular to the walls.

- A particle is projected with speed $\sqrt{5ag}$ towards the two walls from a point A at ground level. It just clears the first wall. By considering the energy of the particle, find its speed when it passes over the first wall.
- Given that it just clears the second wall, show that the angle its trajectory makes with the horizontal when it passes over the first wall is 45°.
- Find the distance of A from the foot of the first wall.
- A second particle is projected with speed $\sqrt{5ag}$ from a point B at ground level towards the two walls. It passes a distance h above the first wall, where $h > 0$. Show that it does not clear the second wall.

Solution:

The particle is projected from point A at ground level with initial speed $\sqrt{5ag}$. The walls are of height $2a$, and the particle just clears the first wall, meaning it reaches a height of $2a$ at the point of clearing. Since the motion is subject to gravity only (a conservative force), mechanical energy is conserved.

Set the ground level as the reference point for gravitational potential energy, so potential energy is zero at $y = 0$.

The initial kinetic energy at point A is:

$$\frac{1}{2}mu^2 = \frac{1}{2}m(\sqrt{5ag})^2 = \frac{1}{2}m \cdot 5ag = \frac{5}{2}mag$$

The initial potential energy is 0, so the total initial energy is:

$$E_i = \frac{5}{2}mag$$

At the point where the particle passes over the first wall, the height is $2a$. Let v be the speed at this point. The kinetic energy is $\frac{1}{2}mv^2$ and the potential energy is $mg \cdot 2a = 2mga$.
The total energy at this point is:

$$E_f = \frac{1}{2}mv^2 + 2mga$$

By conservation of energy:

$$E_i = E_f$$

$$\frac{5}{2}mag = \frac{1}{2}mv^2 + 2mga$$

Divide both sides by m:

$$\frac{5}{2}ag = \frac{1}{2}v^2 + 2ag$$

Multiply both sides by 2 to eliminate denominators:

$$5ag = v^2 + 4ag$$

Solve for v^2:

$$v^2 = 5ag - 4ag = ag$$

$$v = \sqrt{ag}$$

The speed when the particle passes over the first wall is $\sqrt{ag}$.

The particle is projected from point A at ground level with initial speed $\sqrt{5ag}$. Using energy conservation, the speed when it passes over the first wall (at height $2a$) is $\sqrt{ag}$, as derived previously.

At the point where the particle passes over the first wall, its speed is $v = \sqrt{ag}$. Let v_x and v_y be the horizontal and vertical components of the velocity, respectively. Then:

$$v_x^2 + v_y^2 = ag.$$

The walls are a distance a apart. Since the horizontal velocity v_x is constant (no horizontal forces), the time taken to travel the horizontal distance a from the first wall to the second wall is:

$$t = \frac{a}{v_x}.$$

The particle starts at height $2a$ and just clears the second wall at height $2a$, so the vertical displacement is zero. With constant downward acceleration g, the equation for vertical displacement is:

$$\Delta y = v_y t - \frac{1}{2} g t^2 = 0.$$

Substituting $t = a/v_x$:

$$v_y \left(\frac{a}{v_x} \right) - \frac{1}{2} g \left(\frac{a}{v_x} \right)^2 = 0.$$

Dividing through by a/v_x (assuming $a \neq 0$ and $v_x \neq 0$):

$$v_y - \frac{1}{2} g \cdot \frac{a}{v_x} = 0,$$

so:

$$v_y = \frac{ga}{2v_x}.$$

Substitute this into the speed equation:

$$v_x^2 + \left(\frac{ga}{2v_x}\right)^2 = ag.$$

Let $u = v_x^2$:

$$u + \frac{g^2a^2}{4u} = ag.$$

Multiply both sides by $4u$:

$$4u^2 + g^2a^2 = 4agu.$$

Rearrange to form a quadratic equation:

$$4u^2 - 4agu + g^2a^2 = 0.$$

This factors as:

$$(2u - ag)^2 = 0,$$

so:

$$2u - ag = 0 \implies u = \frac{ag}{2}.$$

Since $u = v_x^2$:

$$v_x^2 = \frac{ag}{2}.$$

Then:

$$v_y^2 = ag - v_x^2 = ag - \frac{ag}{2} = \frac{ag}{2}.$$

Thus:

$$v_x^2 = v_y^2 = \frac{ag}{2}.$$

Assuming the components are positive (as the particle is moving upwards and towards the second wall), the ratio is:

$$\frac{v_y}{v_x} = \sqrt{\frac{v_y^2}{v_x^2}} = \sqrt{1} = 1,$$

so:

$$\tan\theta = 1,$$

where θ is the angle with the horizontal. Therefore, $\theta = 45^\circ$.

The particle is projected from point A at ground level with initial speed $\sqrt{5ag}$. The first wall has height $2a$, and the particle just clears it, meaning it reaches a height of $2a$ at the first wall. Using energy conservation, the speed at this point is $\sqrt{ag}$, and the angle with the horizontal is 45°. Thus, the horizontal and vertical components of velocity at the first wall are both $\sqrt{\frac{ag}{2}}$.

Let d be the horizontal distance from A to the foot of the first wall. The initial velocity components are:

- Horizontal component: $u_x = \sqrt{\frac{ag}{2}}$
- Vertical component: $u_y = 3\sqrt{\frac{ag}{2}}$

The motion from A to the first wall is described by the equations:

- Horizontal motion: $d = u_x t = \sqrt{\frac{ag}{2}}t$
- Vertical motion: $2a = u_y t - \frac{1}{2}gt^2 = 3\sqrt{\frac{ag}{2}}t - \frac{1}{2}gt^2$

Solving for t from the horizontal equation:

$$t = d\sqrt{\frac{2}{ag}}$$

Substitute into the vertical equation:

$$2a = 3\sqrt{\frac{ag}{2}} \cdot d\sqrt{\frac{2}{ag}} - \frac{1}{2}g\left(d\sqrt{\frac{2}{ag}}\right)^2$$

Simplify each term:

- First term: $3\sqrt{\frac{ag}{2}} \cdot d\sqrt{\frac{2}{ag}} = 3d$
- Second term: $\frac{1}{2}g \cdot d^2 \cdot \frac{2}{ag} = \frac{d^2}{a}$

So the equation is:

$$2a = 3d - \frac{d^2}{a}$$

Multiply both sides by a:

$$2a^2 = 3da - d^2$$

Rearrange into standard quadratic form:

$$d^2 - 3ad + 2a^2 = 0$$

Factor:

$$(d-a)(d-2a) = 0$$

Solutions: $d = a$ or $d = 2a$.

The particle must also just clear the second wall, which is at a horizontal distance a beyond the first wall and also of height $2a$.

- For $d = a$:

  - First wall at $x = a$, second wall at $x = 2a$.
  - At the first wall, velocity components are $v_x = \sqrt{\frac{ag}{2}}$, $v_y = \sqrt{\frac{ag}{2}}$ (upward).
  - Time to travel from $x = a$ to $x = 2a$: $t = \frac{a}{v_x} = \sqrt{\frac{2a}{g}}$.
  - Vertical displacement: $\Delta y = v_y t - \frac{1}{2}gt^2 = \sqrt{\frac{ag}{2}} \cdot \sqrt{\frac{2a}{g}} - \frac{1}{2}g \cdot \frac{2a}{g} = a - a = 0$.
  - Starting at height $2a$, the height at $x = 2a$ is $2a + 0 = 2a$, so it just clears the second wall.
- For $d = 2a$:
  - First wall at $x = 2a$, second wall at $x = 3a$.
  - At the first wall, velocity components are $v_x = \sqrt{\frac{ag}{2}}$, $v_y = -\sqrt{\frac{ag}{2}}$ (downward, as it is descending).
  - Time to travel from $x = 2a$ to $x = 3a$: $t = \frac{a}{v_x} = \sqrt{\frac{2a}{g}}$.
  - Vertical displacement: $\Delta y = v_y t - \frac{1}{2}gt^2 = -\sqrt{\frac{ag}{2}} \cdot \sqrt{\frac{2a}{g}} - \frac{1}{2}g \cdot \frac{2a}{g} = -a - a = -2a$.
  - Starting at height $2a$, the height at $x = 3a$ is $2a - 2a = 0$, so it does not clear the second wall.

Thus, only $d = a$ satisfies the condition that the particle just clears both walls.

The distance of A from the foot of the first wall is a.

To show that the second particle, projected with speed $\sqrt{5ag}$ from point B at ground level and passing a distance $h > 0$ above the first wall (i.e., at height $2a + h$ when it reaches the first wall), does not clear the second wall, consider the motion in the plane perpendicular to the walls. The walls are of height $2a$ and distance a apart, with the foot of the first wall at $(0, 0)$ and the second wall at $(a, 0)$. The particle is projected from some point $(-d, 0)$ for $d > 0$, with initial speed $\sqrt{5ag}$, and moves under gravity.

At the first wall ($x = 0$), the height is $y_1 = 2a + h$. By conservation of energy, the total mechanical energy is constant. The initial kinetic energy is $\frac{1}{2}m(\sqrt{5ag})^2 = \frac{5}{2}mag$, and the initial potential energy is 0 (taking ground level as the reference). At $x = 0$, the potential energy is $mg(2a + h)$, and the kinetic energy is $\frac{1}{2}mv_1^2$, where v_1 is the speed at the first wall. Thus:

$$\frac{1}{2}mv_1^2 + mg(2a + h) = \frac{5}{2}mag$$

Dividing by m and simplifying:

$$\frac{1}{2}v_1^2 + g(2a+h) = \frac{5}{2}ag$$

$$v_1^2 + 4ag + 2gh = 5ag$$

$$v_1^2 = ag - 2gh$$

Let u_x be the horizontal component of velocity (constant, as no horizontal forces act), and v_{1y} be the vertical component at $x = 0$. Then:

$$v_1^2 = u_x^2 + v_{1y}^2 = ag - 2gh$$

The time to travel from $x = 0$ to $x = a$ is $\tau = a/u_x$. The vertical displacement during this time is:

$$\Delta y = v_{1y}\tau - \frac{1}{2}g\tau^2 = v_{1y}\frac{a}{u_x} - \frac{1}{2}g\left(\frac{a}{u_x}\right)^2$$

The height at the second wall ($x = a$) is:

$$y_2 = y_1 + \Delta y = (2a+h) + v_{1y}\frac{a}{u_x} - \frac{1}{2}g\frac{a^2}{u_x^2}$$

To show that the particle does not clear the second wall, it is required that $y_2 < 2a$, or equivalently:

$$h + v_{1y}\frac{a}{u_x} - \frac{1}{2}g\frac{a^2}{u_x^2} < 0$$

Set $r = a/u_x > 0$ (since $u_x > 0$). The expression becomes:

$$h + v_{1y}r - \frac{1}{2}gr^2$$

with the constraint:

$$v_{1y}^2 + u_x^2 = ag - 2gh$$

Substituting $u_x = a/r$:

$$v_{1y}^2 + \left(\frac{a}{r}\right)^2 = ag - 2gh$$

$$v_{1y}^2 = ag - 2gh - \frac{a^2}{r^2}$$

The expression $h + v_{1y}r - \frac{1}{2}gr^2$ depends on v_{1y}. For a fixed r, the maximum value over v_{1y} occurs when v_{1y} is maximized, subject to the constraint. Since $v_{1y}^2 \leq ag - 2gh - a^2/r^2$, the maximum v_{1y} is $\sqrt{ag - 2gh - a^2/r^2}$. Thus, the maximum value of the expression for fixed r is:

$$g(r) = h + r\sqrt{ag - 2gh - \frac{a^2}{r^2}} - \frac{1}{2}gr^2$$

Simplify the square root:

$$r\sqrt{ag - 2gh - \frac{a^2}{r^2}} = r\sqrt{\frac{(ag - 2gh)r^2 - a^2}{r^2}} = \sqrt{(ag - 2gh)r^2 - a^2}$$

Set $b = ag - 2gh$, so:

$$g(r) = h + \sqrt{br^2 - a^2} - \frac{1}{2}gr^2$$

The domain is $r \geq a/\sqrt{b}$ (since $br^2 - a^2 \geq 0$), and $b > 0$ because $h < a/2$ (from $v_1^2 > 0$).

Now, show that $g(r) < 0$ for all $r \geq a/\sqrt{b}$:

- At $r = a/\sqrt{b}$:

$$\sqrt{br^2 - a^2} = \sqrt{b\left(\frac{a^2}{b}\right) - a^2} = \sqrt{a^2 - a^2} = 0$$

$$g(r) = h + 0 - \frac{1}{2}g\left(\frac{a^2}{b}\right) = h - \frac{1}{2}g\frac{a^2}{ag - 2gh} = h - \frac{1}{2}\frac{a^2}{a - 2h}$$

  The expression $h - \frac{1}{2}\frac{a^2}{a-2h}$ is negative for $0 < h < a/2$, as:

$$h - \frac{1}{2}\frac{a^2}{a - 2h} = \frac{2h(a - 2h) - a^2}{2(a - 2h)} = \frac{2ah - 4h^2 - a^2}{2(a - 2h)} = \frac{-(a^2 - 2ah + 4h^2)}{2(a - 2h)}$$

and $a^2 - 2ah + 4h^2 = (a-h)^2 + 3h^2 > 0$, while the denominator is positive, so the fraction is negative.

- As $r \to \infty$, $g(r) \to -\infty$ because the $-\frac{1}{2}gr^2$ term dominates.
- To find if $g(r)$ has a maximum, compute the derivative:

$$g'(r) = \frac{d}{dr}\left[h + (br^2 - a^2)^{1/2} - \frac{1}{2}gr^2\right] = \frac{br}{\sqrt{br^2 - a^2}} - gr$$

Set $g'(r) = 0$:

$$\frac{br}{\sqrt{br^2 - a^2}} - gr = 0 \implies \frac{b}{\sqrt{br^2 - a^2}} = g \quad (r \neq 0)$$

$$\frac{b}{g} = \sqrt{br^2 - a^2} \implies \left(\frac{b}{g}\right)^2 = br^2 - a^2 \implies br^2 = \frac{b^2}{g^2} + a^2 \implies r^2 = \frac{b}{g^2} + \frac{a^2}{b}$$

At this critical point:

$$\sqrt{br^2 - a^2} = \frac{b}{g}$$

$$g(r) = h + \frac{b}{g} - \frac{1}{2}gr^2 = h + \frac{b}{g} - \frac{1}{2}g\left(\frac{b}{g^2} + \frac{a^2}{b}\right) = h + \frac{b}{g} - \frac{b}{2g} - \frac{ga^2}{2b} = h + \frac{b}{2g} - \frac{ga^2}{2b}$$

Substitute $b = ag - 2gh$:

$$g(r) = h + \frac{ag - 2gh}{2g} - \frac{ga^2}{2(ag - 2gh)} = h + \frac{a}{2} - h - \frac{a^2 g}{2(ag - 2gh)} = \frac{a}{2} - \frac{a^2}{2(a - 2h)}$$

$$= \frac{a}{2}\left(1 - \frac{a}{a - 2h}\right) = \frac{a}{2}\left(\frac{a - 2h - a}{a - 2h}\right) = \frac{a}{2}\left(\frac{-2h}{a - 2h}\right) = -\frac{ah}{a - 2h} < 0$$

since $h > 0$ and $a - 2h > 0$.

Since $g(r) < 0$ at the boundary $r = a/\sqrt{b}$, at the critical point, and as $r \to \infty$, and $g(r)$ is continuous for $r \geq a/\sqrt{b}$, it follows that $g(r) < 0$ for all $r \geq a/\sqrt{b}$.

Therefore, for any r, the maximum value of $h + v_{1y}r - \frac{1}{2}gr^2$ over v_{1y} is negative. Consequently, for any possible trajectory (i.e., any valid v_{1y} and u_x satisfying the conditions), the actual value satisfies:

$$h + v_{1y}\frac{a}{u_x} - \frac{1}{2}g\frac{a^2}{u_x^2} \leq g(r) < 0$$

Thus, $y_2 < 2a$, meaning the particle does not clear the second wall.

Q18

Two identical rough cylinders of radius r and weight W rest, not touching each other but a negligible distance apart, on a horizontal floor. A thin flat rough plank of width $2a$, where $a < r$, and weight kW rests symmetrically and horizontally on the cylinders, with its length parallel to the axes of the cylinders and its faces horizontal. A vertical cross-section is shown in the diagram below. The coefficient of friction at all four contacts is $\frac{1}{2}$. The system is in equilibrium.

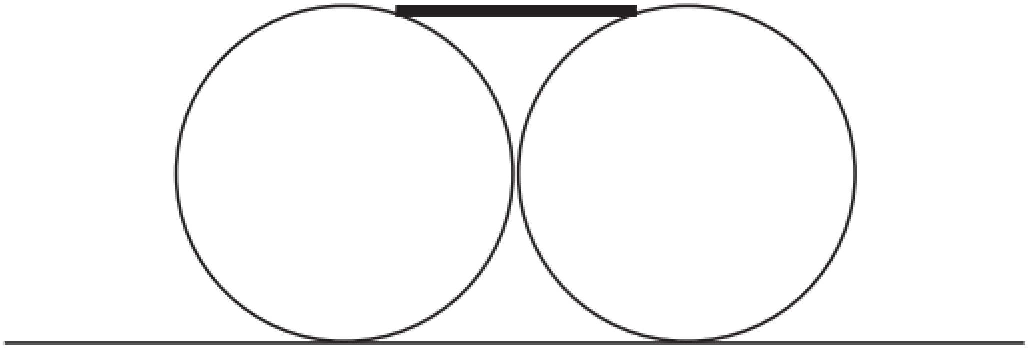
- Let F be the frictional force between one cylinder and the floor, and let R be the normal reaction between the plank and one cylinder. Show that

  $$R\sin\theta = F(1+\cos\theta),$$

  where θ is the acute angle between the plank and the tangent to the cylinder at the point of contact. Deduce that $2\sin\theta \leq 1+\cos\theta$.
- Show that

  $$N = \left(1+\frac{2}{k}\right)\left(\frac{1+\cos\theta}{\sin\theta}\right)F,$$

  where N is the normal reaction between the floor and one cylinder.
- Write down the condition that the cylinder does not slip on the floor and show that it is satisfied with no extra restrictions on θ.
- Show that $\sin\theta \leq \frac{4}{5}$ and hence that $r \leq 5a$.



Solution:

To show that $R \sin\theta = F(1 + \cos\theta)$ and deduce that $2\sin\theta \leq 1 + \cos\theta$, consider the equilibrium of one cylinder and the forces acting on it.

Forces on the Cylinder:

- **Weight**: W downward at the center.
- **Normal reaction from the floor**: N_f upward at the bottom point.
- **Frictional force from the floor**: F horizontal (direction depends on tendency, but magnitude is F).
- **Normal reaction from the plank**: At the point of contact, the normal force is radial inward toward the cylinder center. Since the radius to the point of contact makes an angle θ with the vertical, the components are $(-R\sin\theta, -R\cos\theta)$ in a coordinate system with the cylinder center as the origin, x-axis horizontal to the right, and y-axis upward.
- **Frictional force from the plank**: Tangential to the surface at the point of contact. Let f_p be the magnitude, with components $(f_p \cos\theta, -f_p \sin\theta)$ for the chosen direction.

Force Equilibrium

- **X-direction**:

$$F - R\sin\theta + f_p \cos\theta = 0$$

- **Y-direction**:

$$N_f - W - R\cos\theta - f_p \sin\theta = 0$$

Torque Equilibrium about the Center
The torque due to the floor friction at the bottom point $(0, -r)$:

$$\text{Torque} = rF$$

The torque due to the plank friction at the top point $(r\sin\theta, r\cos\theta)$:

$$\text{Torque} = -rf_p$$

Setting the net torque to zero:

$$rF - rf_p = 0 \implies F = f_p$$

Substituting $F = f_p$ into the X-equilibrium

$$F - R\sin\theta + F\cos\theta = 0 \implies F(1+\cos\theta) = R\sin\theta$$

Thus,

$$R\sin\theta = F(1+\cos\theta)$$

Friction Constraint at Plank-Cylinder Contact
The coefficient of friction is $\frac{1}{2}$, so the magnitude of the frictional force f_p must satisfy:

$$|f_p| \leq \frac{1}{2}R$$

From $R\sin\theta = F(1+\cos\theta)$ and $F = f_p$,

$$f_p = \frac{R\sin\theta}{1+\cos\theta}$$

Substituting into the friction inequality:

$$\left|\frac{R\sin\theta}{1+\cos\theta}\right| \leq \frac{1}{2}R$$

Since $R > 0$, $\sin\theta > 0$, and $\cos\theta > 0$ for θ acute, divide both sides by R:

$$\frac{\sin\theta}{1+\cos\theta} \leq \frac{1}{2}$$

Multiplying both sides by 2:

$$2\sin\theta \leq 1+\cos\theta$$

This inequality must hold for the system to be in equilibrium without slipping at the plank-cylinder contact, given the coefficient of friction $\frac{1}{2}$.

To show that $N = \left(1+\frac{2}{k}\right)\left(\frac{1+\cos\theta}{\sin\theta}\right)F$, where N is the normal reaction between the floor and one cylinder, and F is the frictional force between one cylinder and the floor, consider the equilibrium of the system.

From the equilibrium of one cylinder, the forces in the vertical direction satisfy:

$$N - W - R\cos\theta - F\sin\theta = 0,$$

where R is the normal reaction between the plank and one cylinder, W is the weight of one cylinder, and θ is the acute angle between the plank and the tangent to the cylinder at the point of contact. Solving for N:

$$N = W + R\cos\theta + F\sin\theta.$$

From the previous result, $R\sin\theta = F(1+\cos\theta)$, so:

$$R = F\frac{1+\cos\theta}{\sin\theta}.$$

Substitute this expression for R into the equation for N:

$$N = W + \left(F\frac{1+\cos\theta}{\sin\theta}\right)\cos\theta + F\sin\theta = W + F\frac{(1+\cos\theta)\cos\theta}{\sin\theta} + F\sin\theta.$$

Simplify the expression:

$$N = W + F\left(\frac{\cos\theta+\cos^2\theta}{\sin\theta} + \sin\theta\right) = W + F\left(\frac{\cos\theta+\cos^2\theta+\sin^2\theta}{\sin\theta}\right) = W + F\frac{1+\cos\theta}{\sin\theta},$$

since $\cos^2\theta + \sin^2\theta = 1$.

Now consider the equilibrium of the plank. The plank has weight kW and rests symmetrically on both cylinders. The forces in the vertical direction on the plank are:

- Vertical component of normal force from each cylinder: $R\cos\theta$ (upward).
- Vertical component of frictional force from each cylinder: $F\sin\theta$ (upward), since $f_p = F$ from torque equilibrium on the cylinder.
- Weight: kW (downward).

Summing the vertical forces:

$$2R\cos\theta + 2F\sin\theta - kW = 0,$$

so:

$$2R\cos\theta + 2F\sin\theta = kW.$$

Substitute $R = F\frac{1+\cos\theta}{\sin\theta}$:

$$2\left(F\frac{1+\cos\theta}{\sin\theta}\right)\cos\theta + 2F\sin\theta = kW.$$

Simplify:

$$2F\frac{(1+\cos\theta)\cos\theta}{\sin\theta} + 2F\sin\theta = kW \implies 2F\left(\frac{\cos\theta+\cos^2\theta}{\sin\theta} + \sin\theta\right) = kW.$$

As before:

$$\frac{\cos\theta+\cos^2\theta}{\sin\theta} + \sin\theta = \frac{1+\cos\theta}{\sin\theta},$$

so:

$$2F\frac{1+\cos\theta}{\sin\theta} = kW \implies W = \frac{2F}{k}\frac{1+\cos\theta}{\sin\theta}.$$

Substitute this expression for W into the equation for N:

$$N = \frac{2F}{k}\frac{1+\cos\theta}{\sin\theta} + F\frac{1+\cos\theta}{\sin\theta} = F\frac{1+\cos\theta}{\sin\theta}\left(\frac{2}{k}+1\right) = F\frac{1+\cos\theta}{\sin\theta}\left(1+\frac{2}{k}\right).$$

Thus:

$$N = \left(1+\frac{2}{k}\right)\left(\frac{1+\cos\theta}{\sin\theta}\right)F.$$

The condition that the cylinder does not slip on the floor is that the frictional force F between the cylinder and the floor satisfies $F \leq \frac{1}{2}N$, where N is the normal reaction from the floor and the coefficient of friction is $\frac{1}{2}$.

From the previous result, $N = \left(1+\frac{2}{k}\right)\left(\frac{1+\cos\theta}{\sin\theta}\right)F$. Therefore, the ratio $\frac{F}{N}$ is given by:

$$\frac{F}{N} = \frac{\sin\theta}{\left(1+\frac{2}{k}\right)(1+\cos\theta)}.$$

Using the trigonometric identity $\frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$, this simplifies to:

$$\frac{F}{N} = \frac{1}{1+\frac{2}{k}}\tan\frac{\theta}{2}.$$

Set $A = 1+\frac{2}{k}$. Since $k > 0$, $A > 1$, so $\frac{1}{A} < 1$. Thus:

$$\frac{F}{N} = \frac{1}{A}\tan\frac{\theta}{2}.$$

From the plank-cylinder friction constraint, derived earlier, $\tan\frac{\theta}{2} \leq \frac{1}{2}$. Therefore:

$$\frac{F}{N} \leq \frac{1}{A}\cdot\frac{1}{2}.$$

Since $A > 1$, $\frac{1}{A} < 1$, so:

$$\frac{1}{A}\cdot\frac{1}{2} < \frac{1}{2}.$$

Moreover, since $\tan\frac{\theta}{2} \le \frac{1}{2}$ and $\frac{1}{A} \le 1$, it follows that:

$$\frac{F}{N} \le \frac{1}{2}.$$

Equality holds only in the limit as $k \to \infty$ and $\tan\frac{\theta}{2} = \frac{1}{2}$, but even in this case, $\frac{F}{N} = \frac{1}{2}$, which satisfies $F \le \frac{1}{2}N$.

Given that $\tan\frac{\theta}{2} \le \frac{1}{2}$ is required for equilibrium due to the plank-cylinder friction constraint, and for all such θ, the ratio $\frac{F}{N} \le \frac{1}{2}$ holds, the condition $F \le \frac{1}{2}N$ is satisfied without any additional restrictions on θ.

Proof that $\sin\theta \le \frac{4}{5}$:
From the friction constraint derived earlier, the inequality $2\sin\theta \le 1 + \cos\theta$ must hold for equilibrium at the plank-cylinder contact, given the coefficient of friction $\frac{1}{2}$.

Consider the equality $2\sin\theta = 1 + \cos\theta$. Solving this equation:

Set $t = \tan(\theta/2)$. Then,

$$\sin\theta = \frac{2t}{1+t^2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}.$$

Substitute:

$$2\left(\frac{2t}{1+t^2}\right) = 1 + \frac{1-t^2}{1+t^2},$$

$$\frac{4t}{1+t^2} = \frac{(1+t^2)+(1-t^2)}{1+t^2} = \frac{2}{1+t^2}.$$

Multiply both sides by $1+t^2$:

$$4t = 2,$$

$$t = \frac{1}{2}.$$

Thus, $\tan(\theta/2) = \frac{1}{2}$. Now find $\sin\theta$:

$$\sin\theta = \frac{2t}{1+t^2} = \frac{2\cdot\frac{1}{2}}{1+\left(\frac{1}{2}\right)^2} = \frac{1}{1+\frac{1}{4}} = \frac{1}{\frac{5}{4}} = \frac{4}{5}.$$

The equality holds when $\sin\theta = \frac{4}{5}$, corresponding to $\theta = \arcsin(4/5)$.

The function $f(\theta) = 2\sin\theta - \cos\theta - 1$ has $f(\theta) \le 0$ for the inequality $2\sin\theta \le 1+\cos\theta$. At $\theta = 0$, $f(0) = -2 < 0$, and at $\theta = \arcsin(4/5)$, $f(\theta) = 0$. For $\theta > \arcsin(4/5)$ in $(0, \pi/2)$, $f(\theta) > 0$, violating the inequality. Since θ is acute, $\sin\theta$ increases from 0 to 1, and the inequality holds for $\theta \le \arcsin(4/5)$, so $\sin\theta \le 4/5$.

Proof that $r \le 5a$:
From the geometry of the system, with the cylinders a negligible distance apart, the distance between centers is approximately $2r$. The points of contact are symmetric. For a cylinder centered at (x_c, r), the point of contact is at $(x_c + r\sin\theta, r + r\cos\theta)$ for the left cylinder and $(x_c - r\sin\theta, r + r\cos\theta)$ for the right cylinder, where θ is measured from the vertical.

The horizontal distance between the points of contact is:

$$[(x_c - r\sin\theta) - (x_c + r\sin\theta)] = -2r\sin\theta,$$

but since the right point is to the right of the left point, the distance is $2r\sin\theta$. The distance between centers is $d \approx 2r$, so the actual separation between points of contact is:

$$d - 2r\sin\theta.$$

This distance equals the width of the plank, $2a$, as the plank connects the points of contact:

$$d - 2r\sin\theta = 2a.$$

With $d \approx 2r$,

$$2r - 2r\sin\theta = 2a,$$

$$r(1-\sin\theta) = a.$$

Thus,

$$r = \frac{a}{1-\sin\theta}.$$

Since $\sin\theta \leq 4/5$,

$$1 - \sin\theta \geq 1 - \frac{4}{5} = \frac{1}{5}.$$

Therefore,

$$r = \frac{a}{1 - \sin\theta} \leq \frac{a}{\frac{1}{5}} = 5a.$$

Equality holds when $\sin\theta = 4/5$ and the distance between centers is exactly $2r$.

Q19

A thin uniform wire is bent into the shape of an isosceles triangle ABC, where AB and AC are of equal length and the angle at A is 2θ. The triangle ABC hangs on a small rough horizontal peg with the side BC resting on the peg. The coefficient of friction between the wire and the peg is μ. The plane containing ABC is vertical. Show that the triangle can rest in equilibrium with the peg in contact with any point on BC provided

$$\mu \geq 2\tan\theta(1 + \sin\theta).$$

Solution:

The wire is bent into an isosceles triangle ABC with $AB = AC$ and angle at A equal to 2θ. The side BC rests on a rough horizontal peg, and the plane of the triangle is vertical. The goal is to show that equilibrium is possible with the peg at any point on BC if $\mu \geq 2\tan\theta(1 + \sin\theta)$.

Let $AB = AC = l$. The length of BC is $2l\sin\theta$, so the distance from the midpoint D of BC to B or C is $d = l\sin\theta$. The height from A to D is $h = l\cos\theta$.

The wire is uniform, so the mass per unit length λ is constant. The total mass is $M = \lambda \times 2l(1 + \sin\theta)$. The center of mass G lies on the axis of symmetry AD. Using the centroid formula for the wire frame:

- Side AB: mass λl, center at $\left(-\frac{l\sin\theta}{2}, \frac{l\cos\theta}{2}\right)$
- Side AC: mass λl, center at $\left(\frac{l\sin\theta}{2}, \frac{l\cos\theta}{2}\right)$
- Side BC: mass $\lambda \times 2l\sin\theta$, center at $(0,0)$

The y-coordinate of G is:

$$y_g = \frac{\lambda l \cdot \frac{l\cos\theta}{2} + \lambda l \cdot \frac{l\cos\theta}{2} + 0}{\lambda \times 2l(1+\sin\theta)} = \frac{l\cos\theta}{2(1+\sin\theta)}$$

Thus, G is at $(0, y_g)$ in the coordinate system with D at $(0,0)$, A at $(0, l\cos\theta)$, B at $(-l\sin\theta, 0)$, and C at $(l\sin\theta, 0)$.

Let P be a point on BC at $(s, 0)$ with $-d \leq s \leq d$ and $d = l\sin\theta$. The vector from P to G is $\overrightarrow{PG} = (-s, y_g)$. For equilibrium, G must be directly below P, so $\overrightarrow{PG}$ must be vertical downward. This requires rotating the triangle about P by an angle β such that the horizontal component of $\overrightarrow{PG}$ in the lab frame is zero.

After rotation by β, the components of $\overrightarrow{PG}$ are:

- $x_{\text{lab}} = -s\cos\beta - y_g\sin\beta = 0$
- $y_{\text{lab}} = -s\sin\beta + y_g\cos\beta < 0$

Solving $-s\cos\beta - y_g\sin\beta = 0$ gives:

$$s\cos\beta = -y_g\sin\beta \implies \tan\beta = -\frac{s}{y_g}$$

The angle α that BC makes with the horizontal in the lab frame equals β, so $\alpha = \beta$ and:

$$\tan\alpha = -\frac{s}{y_g} \implies |\tan\alpha| = \frac{|s|}{y_g}$$

The contact force at P must be vertical upward to balance gravity. Resolving into normal force N (perpendicular to the tangent) and friction force F (parallel to the tangent):

- $F\cos\alpha - N\sin\alpha = 0$ (horizontal force balance)
- $F\sin\alpha + N\cos\alpha = Mg$ (vertical force balance)

Solving:

$$F = N\tan\alpha, \quad N = Mg\cos\alpha$$
$$F = Mg\sin\alpha$$

The friction condition for no slipping is $|F| \leq \mu N$, so:

$$|Mg\sin\alpha| \leq \mu|Mg\cos\alpha| \implies |\tan\alpha| \leq \mu$$

Thus:

$$\frac{|s|}{y_g} \leq \mu$$

This must hold for all s in $[-d, d]$. The maximum value of $\frac{|s|}{y_g}$ occurs at $|s| = d = l\sin\theta$:

$$\max\frac{|s|}{y_g} = \frac{d}{y_g} = \frac{l\sin\theta}{\frac{l\cos\theta}{2(1+\sin\theta)}} = 2\sin\theta(1+\sin\theta)\frac{1}{\cos\theta} = 2(1+\sin\theta)\tan\theta$$

For equilibrium at any point on BC, μ must satisfy:

$$\mu \geq 2(1+\sin\theta)\tan\theta = 2\tan\theta(1+\sin\theta)$$

When $\mu \geq 2\tan\theta(1+\sin\theta)$, for any P on BC, the triangle can be oriented such that G is directly below P, and $|\tan\alpha| = \frac{|s|}{y_g} \leq \frac{d}{y_g} \leq \mu$, satisfying the friction condition.

Thus, equilibrium is possible for any point on BC if $\mu \geq 2\tan\theta(1+\sin\theta)$.

Q20

- Two particles move on a smooth horizontal surface. The positions, in Cartesian coordinates, of the particles at time t are

$$(a + ut\cos\alpha,\ ut\sin\alpha) \quad \text{and} \quad (vt\cos\beta,\ b + vt\sin\beta),$$

  where a, b, u and v are positive constants, α and β are constant acute angles, and $t \geq 0$. Given that the two particles collide, show that

$$u\sin(\theta + \alpha) = v\sin(\theta + \beta),$$

  where θ is the acute angle satisfying $\tan\theta = \dfrac{b}{a}$.
- A gun is placed on the top of a vertical tower of height b which stands on horizontal ground. The gun fires a bullet with speed v and (acute) angle of elevation β. Simultaneously, a target is projected from a point on the ground a horizontal distance a from the foot of the tower. The target is projected with speed u and (acute) angle of elevation α, in a direction directly away from the tower. Given that the target is hit before it reaches the ground, show that

$$2u\sin\alpha(u\sin\alpha - v\sin\beta) > bg.$$

  Explain, with reference to part (i), why the target can only be hit if $\alpha > \beta$.

Solution:

The positions of the two particles are given by:

- Particle 1: $(x_1, y_1) = (a + ut\cos\alpha, ut\sin\alpha)$
- Particle 2: $(x_2, y_2) = (vt\cos\beta, b + vt\sin\beta)$

For the particles to collide, their coordinates must be equal at some time $t > 0$:

$$a + ut\cos\alpha = vt\cos\beta \quad (1)$$

$$ut\sin\alpha = b + vt\sin\beta \quad (2)$$

Solving equation (1) for t:

$$a = t(v\cos\beta - u\cos\alpha)$$

$$t = \frac{a}{v\cos\beta - u\cos\alpha} \quad (3)$$

Solving equation (2) for t:

$$ut\sin\alpha - vt\sin\beta = b$$

$$t(u\sin\alpha - v\sin\beta) = b$$

$$t = \frac{b}{u\sin\alpha - v\sin\beta} \quad (4)$$

Equating expressions (3) and (4) for t:

$$\frac{a}{v\cos\beta - u\cos\alpha} = \frac{b}{u\sin\alpha - v\sin\beta}$$

Cross-multiplying:

$$a(u\sin\alpha - v\sin\beta) = b(v\cos\beta - u\cos\alpha)$$

Expanding both sides:

$$au\sin\alpha - av\sin\beta = bv\cos\beta - bu\cos\alpha$$

Rearranging all terms to one side:

$$au\sin\alpha - av\sin\beta - bv\cos\beta + bu\cos\alpha = 0$$

Grouping terms with u and v:

$$u(a\sin\alpha + b\cos\alpha) - v(a\sin\beta + b\cos\beta) = 0$$

Thus:

$$u(a\sin\alpha + b\cos\alpha) = v(a\sin\beta + b\cos\beta) \quad (5)$$

Given $\tan\theta = \frac{b}{a}$ with θ acute, express $a\sin\phi + b\cos\phi$ for an angle ϕ as $R\sin(\phi + \theta)$, where $R = \sqrt{a^2 + b^2}$:

$$a\sin\phi + b\cos\phi = R\sin(\phi + \theta)$$

with $R\cos\theta = a$ and $R\sin\theta = b$, so $\tan\theta = \frac{b}{a}$.

Applying this to equation (5):

$$a\sin\alpha + b\cos\alpha = R\sin(\alpha+\theta)$$

$$a\sin\beta + b\cos\beta = R\sin(\beta+\theta)$$

Substituting into equation (5):

$$u \cdot R\sin(\alpha+\theta) = v \cdot R\sin(\beta+\theta)$$

Since $R \neq 0$:

$$u\sin(\theta+\alpha) = v\sin(\theta+\beta)$$

To solve the latter problem, consider the motion of the bullet and the target under gravity. The bullet is fired from the top of the tower at position $(0, b)$ with initial speed v at an acute angle of elevation β in the positive x-direction (away from the tower). The target is projected from position $(a, 0)$ with initial speed u at an acute angle of elevation α in the positive x-direction (away from the tower). The equations of motion for the target and bullet are:

- Target: $x_T = a + ut\cos\alpha$, $y_T = ut\sin\alpha - \frac{1}{2}gt^2$
- Bullet: $x_B = vt\cos\beta$, $y_B = b + vt\sin\beta - \frac{1}{2}gt^2$

For the bullet to hit the target, their positions must coincide at some time $t > 0$:

$$a + ut\cos\alpha = vt\cos\beta \quad (1)$$

$$ut\sin\alpha - \frac{1}{2}gt^2 = b + vt\sin\beta - \frac{1}{2}gt^2 \quad (2)$$

Simplifying equation (2) by canceling $-\frac{1}{2}gt^2$:

$$ut\sin\alpha = b + vt\sin\beta$$

$$t(u\sin\alpha - v\sin\beta) = b \quad (2')$$

Solving equation (1) for t:

$$a = t(v\cos\beta - u\cos\alpha)$$

$$t = \frac{a}{v\cos\beta - u\cos\alpha} \quad (1')$$

Equating the expressions for t from (1') and (2'):

$$\frac{b}{u\sin\alpha - v\sin\beta} = \frac{a}{v\cos\beta - u\cos\alpha}$$

Note that $v\cos\beta - u\cos\alpha = -(u\cos\alpha - v\cos\beta)$, so:

$$\frac{b}{u\sin\alpha - v\sin\beta} = -\frac{a}{u\cos\alpha - v\cos\beta}$$

Cross-multiplying:

$$b(u\cos\alpha - v\cos\beta) = -a(u\sin\alpha - v\sin\beta)$$

Expanding and rearranging:

$$bu\cos\alpha - bv\cos\beta = -au\sin\alpha + av\sin\beta$$

$$au\sin\alpha + bu\cos\alpha - av\sin\beta - bv\cos\beta = 0$$

$$u(a\sin\alpha + b\cos\alpha) = v(a\sin\beta + b\cos\beta) \quad (3)$$

Given $\tan\theta = \frac{b}{a}$ with θ acute, express the terms using θ:

$$a\sin\phi + b\cos\phi = \sqrt{a^2+b^2}\sin(\phi+\theta)$$

where $\sqrt{a^2+b^2}\cos\theta = a$ and $\sqrt{a^2+b^2}\sin\theta = b$. Thus:

$$a\sin\alpha + b\cos\alpha = \sqrt{a^2+b^2}\sin(\alpha+\theta)$$

$$a\sin\beta + b\cos\beta = \sqrt{a^2+b^2}\sin(\beta+\theta)$$

Substituting into (3):

$$u \cdot \sqrt{a^2+b^2}\sin(\alpha+\theta) = v \cdot \sqrt{a^2+b^2}\sin(\beta+\theta)$$

$$u\sin(\theta+\alpha) = v\sin(\theta+\beta)$$

This is the condition for collision, identical to part (i). For collision to occur at $t_c > 0$, equation (2') gives:

$$t_c = \frac{b}{u\sin\alpha - v\sin\beta}$$

Requiring $t_c > 0$ and since $b > 0$:

$$u\sin\alpha - v\sin\beta > 0$$

The target hits the ground when $y_T = 0$:

$$ut\sin\alpha - \frac{1}{2}gt^2 = 0$$

$$t\left(u\sin\alpha - \frac{1}{2}gt\right) = 0$$

The non-zero solution is $t_g = \frac{2u\sin\alpha}{g}$. For the target to be hit before it reaches the ground:

$$t_c < t_g$$

$$\frac{b}{u\sin\alpha - v\sin\beta} < \frac{2u\sin\alpha}{g}$$

Since $u\sin\alpha - v\sin\beta > 0$, multiply both sides by $(u\sin\alpha - v\sin\beta)g > 0$:

$$bg < 2u\sin\alpha(u\sin\alpha - v\sin\beta)$$

Thus:

$$2u\sin\alpha(u\sin\alpha - v\sin\beta) > bg$$

To explain why $\alpha > \beta$, refer to part (i). In part (i), without gravity, the same collision condition $u\sin(\theta+\alpha) = v\sin(\theta+\beta)$ holds, and for collision at $t > 0$, $u\sin\alpha > v\sin\beta$. Given θ acute and α, β acute, assume $\alpha \leq \beta$. Then $u\sin\alpha > v\sin\beta$ implies $u > v$ since $\sin\alpha \leq \sin\beta$ for $\alpha \leq \beta$ and $\sin\alpha > 0$. From the collision condition:

$$u\sin(\alpha+\theta) = v\sin(\beta+\theta)$$

Consider the inequality $u\sin\alpha > v\sin\beta$. Substituting $u = v\frac{\sin(\beta+\theta)}{\sin(\alpha+\theta)}$:

$$v\frac{\sin(\beta+\theta)}{\sin(\alpha+\theta)}\sin\alpha > v\sin\beta$$

$$\frac{\sin(\beta+\theta)\sin\alpha}{\sin(\alpha+\theta)} > \sin\beta$$

$$\sin(\beta+\theta)\sin\alpha > \sin\beta\sin(\alpha+\theta)$$

The left side minus the right side:

$$\sin(\beta+\theta)\sin\alpha - \sin\beta\sin(\alpha+\theta) = \sin(\alpha-\beta)\sin\theta$$

by trigonometric identities. Thus:

$$\sin(\alpha-\beta)\sin\theta > 0$$

Since θ is acute, $\sin\theta > 0$, so:

$$\sin(\alpha-\beta) > 0$$

which implies $\alpha - \beta > 0$ (as α and β are acute, $\alpha - \beta \in (-\pi/2, \pi/2)$, where sine is positive only for positive angles). Thus $\alpha > \beta$. This holds in both part (i) and the current problem due to the collision condition and $t_c > 0$.

Q21

Two points A and B lie on horizontal ground. A particle P_1 is projected from A towards B at an acute angle of elevation α and simultaneously a particle P_2 is projected from B towards A at an acute angle of elevation β. Given that the two particles collide in the air a horizontal distance b from B, and that the collision occurs after P_1 has attained its maximum height h, show that

$$2h\cot\beta < b < 4h\cot\beta$$

and

$$2h \cot \alpha < a < 4h \cot \alpha,$$

where a is the horizontal distance from A to the point of collision.

Solution:

The two particles P_1 and P_2 are projected simultaneously from points A and B on horizontal ground, with P_1 projected from A towards B at an angle α and P_2 from B towards A at an angle β, both acute angles. They collide at a point after P_1 has attained its maximum height h, and the horizontal distances from A and B to the collision point are a and b respectively, with $a + b = d$, the distance between A and B.

The initial vertical velocity components are equal due to the collision condition, denoted as $k = v_1 \sin \alpha = v_2 \sin \beta$, where v_1 and v_2 are the projection speeds of P_1 and P_2. The maximum height h for both particles is the same and given by:

$$h = \frac{k^2}{2g},$$

so $k = \sqrt{2gh}$.

The horizontal distances are:

- From A: $a = kt \cot \alpha$,
- From B: $b = kt \cot \beta$,
  where t is the time of collision.

The collision occurs strictly after P_1 attains its maximum height and before either particle hits the ground, so:

$$t > \frac{k}{g} \quad \text{and} \quad t < \frac{2k}{g}.$$

Substituting $k = \sqrt{2gh}$:

$$t > \sqrt{\frac{2h}{g}} \quad \text{and} \quad t < 2\sqrt{\frac{2h}{g}}.$$

For b:

$$b = kt\cot\beta = \sqrt{2gh}\cdot t\cdot\cot\beta.$$

The minimum value of b occurs as t approaches $\sqrt{\frac{2h}{g}}$ from above:

$$b > \sqrt{2gh}\cdot\sqrt{\frac{2h}{g}}\cdot\cot\beta = \sqrt{2gh}\cdot\sqrt{\frac{2h}{g}}\cdot\cot\beta = \sqrt{4h^2}\cdot\cot\beta = 2h\cot\beta.$$

The maximum value of b occurs as t approaches $2\sqrt{\frac{2h}{g}}$ from below:

$$b < \sqrt{2gh}\cdot 2\sqrt{\frac{2h}{g}}\cdot\cot\beta = 2\cdot\sqrt{2gh}\cdot\sqrt{\frac{2h}{g}}\cdot\cot\beta = 2\cdot\sqrt{4h^2}\cdot\cot\beta = 2\cdot 2h\cot\beta = 4h\cot\beta.$$

Thus:

$$2h\cot\beta < b < 4h\cot\beta.$$

For a:

$$a = kt\cot\alpha = \sqrt{2gh}\cdot t\cdot\cot\alpha.$$

Similarly:

$$a > \sqrt{2gh}\cdot\sqrt{\frac{2h}{g}}\cdot\cot\alpha = 2h\cot\alpha,$$

$$a < \sqrt{2gh}\cdot 2\sqrt{\frac{2h}{g}}\cdot\cot\alpha = 4h\cot\alpha.$$

Thus:

$$2h\cot\alpha < a < 4h\cot\alpha.$$

The strict inequalities arise because the collision occurs strictly after P_1 attains its maximum height ($t > k/g$) and strictly before the particles hit the ground ($t < 2k/g$).
