A_M_PS5

Q1

The origin O of coordinates lies on a smooth horizontal table and the x- and y-axes lie in the plane of the table. A smooth sphere A of mass m and radius r is at rest on the table with its lowest point at the origin.

A second smooth sphere B has the same mass and radius and also lies on the table. Its lowest point has y-coordinate $2r\sin\alpha$, where α is an acute angle, and large positive x-coordinate.

Sphere B is now projected parallel to the x-axis, with speed u, so that it strikes sphere A. The coefficient of restitution in this collision is $\frac{1}{3}$.

- Show that, after the collision, sphere B moves with velocity

$$\left(-\frac{1}{3}u(1+2\sin^2\alpha), \quad \frac{2}{3}u\sin\alpha\cos\alpha\right).$$

- Show further that the lowest point of sphere B crosses the y-axis at the point $(0, Y)$, where $Y = 2r(\cos\alpha\tan\beta + \sin\alpha)$ and

$$\tan\beta = \frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha}.$$

  A third sphere C of radius r is at rest with its lowest point at $(0, h)$ on the table, where $h > 0$.

- Show that, if $h > Y + 2r\sec\beta$, sphere B will not strike sphere C in its motion after the collision with sphere A.
- If

$$h > \frac{8r}{\sqrt{3}}.$$

  Show that $Y < 2r\sec\beta$. Hence show that sphere B will not strike sphere C for any value of α.

Solution:

The origin O is on a smooth horizontal table, with the x- and y-axes in the plane of the table. Sphere A of mass m and radius r is at rest with its center at $(0, r)$. Sphere B, with the same mass and radius, has its center initially at a large positive x-coordinate and y-coordinate $2r\sin\alpha + r$, so its lowest point is at $y = 2r\sin\alpha$. Sphere B is projected parallel to the x-axis with speed u towards the negative x-direction, so its initial velocity is $(-u, 0)$. The spheres collide, and the coefficient of restitution is $\frac{1}{3}$.

At the moment of collision, the centers of the spheres are separated by a distance $2r$, and the line joining the centers is at an angle α to the horizontal. The unit vector along the line from the center of A to the center of B (the normal direction) is $\hat{n} = (\cos\alpha, \sin\alpha)$. The tangential unit vector, perpendicular to $\hat{n}$, is $\hat{t} = (-\sin\alpha, \cos\alpha)$.

Before the collision:

- Sphere A is at rest, so its velocity is $\mathbf{v}_a = (0, 0)$.
- Sphere B has velocity $\mathbf{v}_b = (-u, 0)$.

Resolving the velocities into normal and tangential components:

- For sphere A:
  - Normal component: $v_{a,n} = \mathbf{v}_a \cdot \hat{n} = (0,0) \cdot (\cos\alpha, \sin\alpha) = 0$.
  - Tangential component: $v_{a,t} = \mathbf{v}_a \cdot \hat{t} = (0,0) \cdot (-\sin\alpha, \cos\alpha) = 0$.
- For sphere B:
  - Normal component: $v_{b,n} = \mathbf{v}_b \cdot \hat{n} = (-u,0) \cdot (\cos\alpha, \sin\alpha) = -u\cos\alpha$.
  - Tangential component: $v_{b,t} = \mathbf{v}_b \cdot \hat{t} = (-u,0) \cdot (-\sin\alpha, \cos\alpha) = u\sin\alpha$.

Since the spheres are smooth, there is no tangential impulse, so the tangential components of velocity remain unchanged after the collision:

- $v'_{a,t} = v_{a,t} = 0$,
- $v'_{b,t} = v_{b,t} = u\sin\alpha$.

For the normal components, apply conservation of momentum in the normal direction and the coefficient of restitution.

Conservation of momentum in the normal direction (masses are equal, so m cancels):

$$v_{a,n} + v_{b,n} = v'_{a,n} + v'_{b,n}$$

$$0 + (-u\cos\alpha) = v'_{a,n} + v'_{b,n}$$

$$v'_{a,n} + v'_{b,n} = -u\cos\alpha \quad \text{(equation 1)}$$

Coefficient of restitution $e = \frac{1}{3}$:

$$e = \frac{v'_{b,n} - v'_{a,n}}{v_{b,n} - v_{a,n}}$$

The approach speed is $|v_{b,n} - v_{a,n}| = |-u\cos\alpha - 0| = u\cos\alpha$, so:

$$\frac{1}{3} = \frac{v'_{b,n} - v'_{a,n}}{u\cos\alpha}$$

$$v'_{b,n} - v'_{a,n} = \frac{1}{3}u\cos\alpha \quad \text{(equation 2)}$$

Solve equations (1) and (2) simultaneously:

- Add equations (1) and (2):

$$(v'_{a,n} + v'_{b,n}) + (v'_{b,n} - v'_{a,n}) = -u\cos\alpha + \frac{1}{3}u\cos\alpha$$

$$2v'_{b,n} = -\frac{2}{3}u\cos\alpha$$

$$v'_{b,n} = -\frac{1}{3}u\cos\alpha$$

- Substitute into equation (1):

$$v'_{a,n} + \left(-\frac{1}{3}u\cos\alpha\right) = -u\cos\alpha$$

$$v'_{a,n} = -u\cos\alpha + \frac{1}{3}u\cos\alpha = -\frac{2}{3}u\cos\alpha$$

After the collision, the velocity of sphere B in the normal and tangential components is:

- Normal component: $v'_{b,n} = -\frac{1}{3}u\cos\alpha$
- Tangential component: $v'_{b,t} = u\sin\alpha$

The velocity vector of sphere B after the collision is:

$$\mathbf{v}'_b = v'_{b,n}\hat{n} + v'_{b,t}\hat{t}$$

$$\mathbf{v}'_b = \left(-\frac{1}{3}u\cos\alpha\right)(\cos\alpha, \sin\alpha) + (u\sin\alpha)(-\sin\alpha, \cos\alpha)$$

Resolve into x- and y-components:

- x-component:

$$v'_{b,x} = \left(-\frac{1}{3}u\cos\alpha\right)\cos\alpha + (u\sin\alpha)(-\sin\alpha) = -\frac{1}{3}u\cos^2\alpha - u\sin^2\alpha$$

$$= -u\left(\frac{1}{3}\cos^2\alpha + \sin^2\alpha\right) = -u\left(\frac{1}{3}(1-\sin^2\alpha) + \sin^2\alpha\right) = -u\left(\frac{1}{3} - \frac{1}{3}\sin^2\alpha + \sin^2\alpha\right)$$

$$= -u\left(\frac{1}{3} + \frac{2}{3}\sin^2\alpha\right) = -\frac{1}{3}u(1 + 2\sin^2\alpha)$$

- y-component:

$$v'_{b,y} = \left(-\frac{1}{3}u\cos\alpha\right)\sin\alpha + (u\sin\alpha)\cos\alpha = -\frac{1}{3}u\cos\alpha\sin\alpha + u\sin\alpha\cos\alpha$$

$$= u\sin\alpha\cos\alpha\left(1 - \frac{1}{3}\right) = \frac{2}{3}u\sin\alpha\cos\alpha$$

Thus, the velocity of sphere B after the collision is:

$$\left(-\frac{1}{3}u(1+2\sin^2\alpha), \quad \frac{2}{3}u\sin\alpha\cos\alpha\right)$$

The velocity of sphere B after the collision is given by:

$$\mathbf{v}_b' = \left(-\frac{1}{3}u(1+2\sin^2\alpha), \frac{2}{3}u\sin\alpha\cos\alpha\right).$$

At the moment of collision, the center of sphere B is at $(x_0, y_0) = (2r\cos\alpha, r(1+2\sin\alpha))$. After the collision, sphere B moves with constant velocity $\mathbf{v}_b'$ since the table is smooth and no horizontal forces act on it. The position of the center of sphere B at time t after the collision is:

$$x_c(t) = 2r\cos\alpha + \left(-\frac{1}{3}u(1+2\sin^2\alpha)\right)t,$$

$$y_c(t) = r(1+2\sin\alpha) + \left(\frac{2}{3}u\sin\alpha\cos\alpha\right)t.$$

The lowest point of sphere B is at $(x_c(t), y_c(t)-r)$. To find where this point crosses the y-axis, set $x_c(t) = 0$:

$$2r\cos\alpha - \frac{1}{3}u(1+2\sin^2\alpha)t = 0.$$

Solving for t:

$$t = \frac{2r\cos\alpha}{\frac{1}{3}u(1+2\sin^2\alpha)} = \frac{6r\cos\alpha}{u(1+2\sin^2\alpha)}.$$

Substitute t into $y_c(t)$:

$$y_c(t) = r(1+2\sin\alpha) + \left(\frac{2}{3}u\sin\alpha\cos\alpha\right)\left(\frac{6r\cos\alpha}{u(1+2\sin^2\alpha)}\right).$$

Simplify the expression:

$$y_c(t) = r + 2r\sin\alpha + \frac{2}{3}\cdot 6r\sin\alpha\cos\alpha\cos\alpha\cdot\frac{1}{1+2\sin^2\alpha} = r + 2r\sin\alpha + \frac{4r\sin\alpha\cos^2\alpha}{1+2\sin^2\alpha}.$$

The y-coordinate of the lowest point is $Y = y_c(t) - r$:

$$Y = 2r\sin\alpha + \frac{4r\sin\alpha\cos^2\alpha}{1+2\sin^2\alpha}.$$

Given $\tan\beta = \frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha}$, compute:

$$\cos\alpha\tan\beta = \cos\alpha\cdot\frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha} = \frac{2\sin\alpha\cos^2\alpha}{1+2\sin^2\alpha}.$$

Then:

$$\cos\alpha\tan\beta + \sin\alpha = \frac{2\sin\alpha\cos^2\alpha}{1+2\sin^2\alpha} + \sin\alpha = \sin\alpha\left(\frac{2\cos^2\alpha}{1+2\sin^2\alpha} + 1\right).$$

Simplify the expression inside the parentheses:

$$\frac{2\cos^2\alpha}{1+2\sin^2\alpha}+1=\frac{2\cos^2\alpha+1+2\sin^2\alpha}{1+2\sin^2\alpha}=\frac{2(\cos^2\alpha+\sin^2\alpha)+1}{1+2\sin^2\alpha}=\frac{2\cdot 1+1}{1+2\sin^2\alpha}=\frac{3}{1+2\sin^2\alpha}.$$

Thus:

$$\cos\alpha\tan\beta+\sin\alpha=\sin\alpha\cdot\frac{3}{1+2\sin^2\alpha}=\frac{3\sin\alpha}{1+2\sin^2\alpha}.$$

Now:

$$Y=2r(\cos\alpha\tan\beta+\sin\alpha)=2r\cdot\frac{3\sin\alpha}{1+2\sin^2\alpha}=\frac{6r\sin\alpha}{1+2\sin^2\alpha}.$$

This matches the expression derived earlier for Y.

To determine whether sphere B strikes sphere C after colliding with sphere A, consider the motion of sphere B and the position of sphere C.

Sphere C is at rest with its center at $(0,h+r)$, since its lowest point is at $(0,h)$ and it has radius r. After the collision with sphere A, sphere B moves with constant velocity $\mathbf{v}'_B=\left(-\frac{1}{3}u(1+2\sin^2\alpha),\frac{2}{3}u\sin\alpha\cos\alpha\right)$, as derived. The center of sphere B at the moment of collision (time $t=0$) is at $(x_{B0},y_{B0})=(2r\cos\alpha,r(1+2\sin\alpha))$.

The position of the center of sphere B at time $t\geq 0$ is:

$$x_B(t)=2r\cos\alpha-\frac{1}{3}u(1+2\sin^2\alpha)\,t,$$

$$y_B(t)=r(1+2\sin\alpha)+\frac{2}{3}u\sin\alpha\cos\alpha\,t.$$

The distance between the centers of spheres B and C must be greater than $2r$ at all times to avoid a collision. The path of the center of sphere B is a straight line. The minimum distance from the fixed point $(0,h+r)$ to this line is found using the formula for the distance from a point to a line.

The direction vector of the line is $\mathbf{d}=(d_x,d_y)=\left(-\frac{1}{3}u(1+2\sin^2\alpha),\frac{2}{3}u\sin\alpha\cos\alpha\right)$. The vector from the initial point (x_{B0},y_{B0}) to $(0,h+r)$ is $\mathbf{q}=(q_x,q_y)=(-2r\cos\alpha,h-2r\sin\alpha)$.

The distance d from the point to the line is:

$$d=\frac{|q_xd_y-q_yd_x|}{\sqrt{d_x^2+d_y^2}}.$$

Substituting the components:

$$q_xd_y-q_yd_x=(-2r\cos\alpha)\left(\frac{2}{3}u\sin\alpha\cos\alpha\right)-(h-2r\sin\alpha)\left(-\frac{1}{3}u(1+2\sin^2\alpha)\right).$$

Simplifying the expression:

$$q_x d_y - q_y d_x = \frac{1}{3}u\left[-4r\cos^2\alpha\sin\alpha + (h - 2r\sin\alpha)(1 + 2\sin^2\alpha)\right].$$

Expanding and using $\cos^2\alpha = 1 - \sin^2\alpha$:

$$-4r(1 - \sin^2\alpha)\sin\alpha + h(1 + 2\sin^2\alpha) - 2r\sin\alpha(1 + 2\sin^2\alpha) = h(1 + 2\sin^2\alpha) - 6r\sin\alpha.$$

Thus:

$$q_x d_y - q_y d_x = \frac{1}{3}u\left[h(1 + 2\sin^2\alpha) - 6r\sin\alpha\right].$$

The magnitude of $\mathbf{d}$ is:

$$|\mathbf{d}| = \sqrt{\left(-\frac{1}{3}u(1 + 2\sin^2\alpha)\right)^2 + \left(\frac{2}{3}u\sin\alpha\cos\alpha\right)^2} = \frac{1}{3}u\sqrt{(1 + 2\sin^2\alpha)^2 + (2\sin\alpha\cos\alpha)^2}.$$

Simplifying the expression under the square root:

$$(1 + 2\sin^2\alpha)^2 + (2\sin\alpha\cos\alpha)^2 = 1 + 4\sin^2\alpha + 4\sin^4\alpha + 4\sin^2\alpha\cos^2\alpha.$$

Using $\cos^2\alpha = 1 - \sin^2\alpha$:

$$4\sin^2\alpha(1 - \sin^2\alpha) = 4\sin^2\alpha - 4\sin^4\alpha,$$

so:

$$1 + 4\sin^2\alpha + 4\sin^4\alpha + 4\sin^2\alpha - 4\sin^4\alpha = 1 + 8\sin^2\alpha.$$

Thus:

$$|\mathbf{d}| = \frac{1}{3}u\sqrt{1 + 8\sin^2\alpha}.$$

The distance is:

$$d = \frac{\left|\frac{1}{3}u\left[h(1 + 2\sin^2\alpha) - 6r\sin\alpha\right]\right|}{\frac{1}{3}u\sqrt{1 + 8\sin^2\alpha}} = \frac{|h(1 + 2\sin^2\alpha) - 6r\sin\alpha|}{\sqrt{1 + 8\sin^2\alpha}}.$$

Since $h > 0$, α is acute, and $h > Y$ (as $Y > 0$ and $h > Y + 2r\sec\beta$), the expression inside the absolute value is positive, so:

$$d = \frac{h(1 + 2\sin^2\alpha) - 6r\sin\alpha}{\sqrt{1 + 8\sin^2\alpha}}.$$

The condition for no collision is $d > 2r$:

$$\frac{h(1 + 2\sin^2\alpha) - 6r\sin\alpha}{\sqrt{1 + 8\sin^2\alpha}} > 2r.$$

Solving for h:

$$h(1 + 2\sin^2\alpha) - 6r\sin\alpha > 2r\sqrt{1 + 8\sin^2\alpha},$$

$$h(1 + 2\sin^2\alpha) > 2r\sqrt{1 + 8\sin^2\alpha} + 6r\sin\alpha,$$

$$h > \frac{2r\sqrt{1 + 8\sin^2\alpha} + 6r\sin\alpha}{1 + 2\sin^2\alpha} = 2r \cdot \frac{\sqrt{1 + 8\sin^2\alpha}}{1 + 2\sin^2\alpha} + \frac{6r\sin\alpha}{1 + 2\sin^2\alpha}.$$

The first term is $2r\sec\beta$, as:

$$\sec\beta = \frac{\sqrt{1 + 8\sin^2\alpha}}{1 + 2\sin^2\alpha},$$

since $\tan\beta = \frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha}$ and $\cos\beta = \frac{1+2\sin^2\alpha}{\sqrt{1+8\sin^2\alpha}}$.

The second term is Y, as derived previously:

$$Y = \frac{6r\sin\alpha}{1 + 2\sin^2\alpha}.$$

Thus:

$$h > 2r\sec\beta + Y.$$

Given $h > Y + 2r\sec\beta$, it follows that $d > 2r$. Since the distance from the center of sphere C to the line of motion of the center of sphere B is greater than $2r$, and the actual distance between the centers is at least this perpendicular distance, the spheres do not collide. Additionally, since the motion is straight-line and the perpendicular distance is greater than $2r$ for all points on the line, and specifically for $t \geq 0$, no collision occurs.

Therefore, if $h > Y + 2r\sec\beta$, sphere B does not strike sphere C.

To address the given problem, it is first necessary to show that $Y < 2r\sec\beta$ for all α, and then use this result to demonstrate that if $h > \frac{8r}{\sqrt{3}}$, sphere B does not strike sphere C for any value of α.

From the previous results,

$$Y = \frac{6r\sin\alpha}{1 + 2\sin^2\alpha}, \quad \sec\beta = \frac{\sqrt{1 + 8\sin^2\alpha}}{1 + 2\sin^2\alpha}.$$

Consider the ratio

$$\frac{Y}{2r\sec\beta} = \frac{\frac{6r\sin\alpha}{1+2\sin^2\alpha}}{2r \cdot \frac{\sqrt{1+8\sin^2\alpha}}{1+2\sin^2\alpha}} = \frac{3\sin\alpha}{\sqrt{1 + 8\sin^2\alpha}}.$$

Set $s = \sin\alpha$, where $s \in (0, 1)$ since α is acute. The ratio is

$$\frac{3s}{\sqrt{1 + 8s^2}}.$$

To determine if this ratio is less than 1, consider the inequality

$$3s < \sqrt{1 + 8s^2}.$$

Squaring both sides (valid as both sides are positive for $s > 0$),

$$(3s)^2 < 1 + 8s^2 \implies 9s^2 < 1 + 8s^2 \implies s^2 < 1.$$

This holds strictly for all $s \in (0, 1)$, as $s^2 < 1$. At $s = 1$, equality holds, but since α is acute, $s < 1$. Thus,

$$\frac{3s}{\sqrt{1+8s^2}} < 1 \implies \frac{Y}{2r \sec \beta} < 1 \implies Y < 2r \sec \beta$$

for all $\alpha \in (0, \pi/2)$.

From the earlier result, $Y < 2r \sec \beta$ for all α. Therefore,

$$Y + 2r \sec \beta < 2r \sec \beta + 2r \sec \beta = 4r \sec \beta.$$

Now, consider the function $\sec \beta = \frac{\sqrt{1+8 \sin^2 \alpha}}{1+2 \sin^2 \alpha}$. Set $u = \sin^2 \alpha$, so $u \in [0, 1]$, and define

$$f(u) = \frac{\sqrt{1+8u}}{1+2u}.$$

The maximum value of $f(u)$ over $[0, 1]$ is found by taking the derivative. Let

$$f(u) = \frac{g(u)}{h(u)}, \quad g(u) = \sqrt{1+8u}, \quad h(u) = 1 + 2u.$$

Then

$$g'(u) = \frac{4}{\sqrt{1+8u}}, \quad h'(u) = 2.$$

The derivative is

$$f'(u) = \frac{g'(u)h(u) - g(u)h'(u)}{[h(u)]^2} = \frac{\frac{4}{\sqrt{1+8u}}(1+2u) - \sqrt{1+8u} \cdot 2}{(1+2u)^2}.$$

Set the numerator to zero:

$$\frac{4}{\sqrt{1+8u}}(1+2u) - 2\sqrt{1+8u} = 0.$$

Let $k = \sqrt{1+8u}$, so

$$\frac{4}{k}(1+2u) - 2k = 0 \implies 4(1+2u) = 2k^2 \implies 2(1+2u) = k^2.$$

But $k^2 = 1 + 8u$, so

$$2 + 4u = 1 + 8u \implies 2 - 1 = 8u - 4u \implies 1 = 4u \implies u = \frac{1}{4}.$$

At $u = \frac{1}{4}$,

$$f\left(\frac{1}{4}\right) = \frac{\sqrt{1+8\cdot\frac{1}{4}}}{1+2\cdot\frac{1}{4}} = \frac{\sqrt{3}}{\frac{3}{2}} = \frac{2}{\sqrt{3}}.$$

Check the endpoints: at $u=0,\ f(0)=\frac{\sqrt{1}}{1}=1$; at $u=1,\ f(1)=\frac{\sqrt{9}}{3}=1$. Since $\frac{2}{\sqrt{3}}\approx 1.1547>1$, the maximum value of $\sec\beta$ is $\frac{2}{\sqrt{3}}$. Thus,

$$4r\sec\beta \le 4r\cdot\frac{2}{\sqrt{3}} = \frac{8r}{\sqrt{3}}.$$

Therefore,

$$Y + 2r\sec\beta < 4r\sec\beta \le \frac{8r}{\sqrt{3}}.$$

Given $h > \frac{8r}{\sqrt{3}}$, it follows that

$$h > \frac{8r}{\sqrt{3}} \ge 4r\sec\beta > Y + 2r\sec\beta$$

for all α. From the earlier result that if $h > Y + 2r\sec\beta$, sphere B does not strike sphere C, it follows that for all α, sphere B does not strike sphere C.

Q2

A cube of uniform density ρ is placed on a horizontal plane and a second cube, also of uniform density ρ, is placed on top of it. The lower cube has side length 1 and the upper cube has side length a, with $a \le 1$. The centre of mass of the upper cube is vertically above the centre of mass of the lower cube and all the edges of the upper cube are parallel to the corresponding edges of the lower cube. The contacts between the two cubes, and between the lower cube and the plane, are rough, with the same coefficient of friction $\mu < 1$ in each case. The midpoint of the base of the upper cube is X and the midpoint of the base of the lower cube is Y.

A horizontal force P is exerted, perpendicular to one of the vertical faces of the upper cube, at a point halfway between the two vertical edges of this face, and a distance h, with $h < a$, above the lower edge of this face.

- Show that, if the two cubes remain in equilibrium, the normal reaction of the plane on the lower cube acts at a point which is a distance

  $$\frac{P(1+h)}{(1+a^3)\rho g}$$

  from Y, and find a similar expression for the distance from X of the point at which the normal reaction of the lower cube on the upper cube acts.

The force P is now gradually increased from zero.

- Show that, if neither cube topples, equilibrium will be broken by the slipping of the upper cube on the lower cube, and not by the slipping of the lower cube on the ground.
- Show that, if $a = 1$, then equilibrium will be broken by the slipping of the upper cube on the lower cube if $\mu(1 + h) < 1$ and by the toppling of the lower and upper cube together if $\mu(1 + h) > 1$.
- Show that, in a situation where $a < 1$ and $h(1 + a^3(1 - a)) > a^4$, and no slipping occurs, equilibrium will be broken by the toppling of the upper cube.
- Show, by considering $a = \frac{1}{2}$ and choosing suitable values of h and μ, that the situation described in (iv) can in fact occur.

Solution:

To determine the distance from X of the point where the normal reaction of the lower cube on the upper cube acts, consider the forces and moments acting on the upper cube in equilibrium.

The upper cube has mass $m_u = \rho a^3$ and weight $W_u = \rho g a^3$. The normal reaction force from the lower cube on the upper cube, denoted N_2, acts vertically upward at a point on the interface. By vertical equilibrium, $N_2 = W_u = \rho g a^3$.

The applied horizontal force P acts at a point on the upper cube, which is at a horizontal distance $a/2$ from the centerline (since it is halfway between the vertical edges) and at a height h above the base of the upper cube. The midpoint of the base of the upper cube is X, located at $(0, 1)$ in the coordinate system where Y is at $(0, 0)$.

Taking moments about X for the upper cube:

- The gravitational force acts at the center of mass of the upper cube, which is at $(0, 1 + a/2)$. Since this is directly above X, it has no moment about X.
- The friction force between the cubes acts horizontally at the interface point where N_2 is applied. Since this point is at the same height as X, the friction force has no moment about X.
- The applied force P acts at $(a/2, 1 + h)$. The moment about X is $-Ph$ (clockwise, assuming counterclockwise positive).
- The normal reaction N_2 acts at $(e, 1)$, where e is the horizontal distance from X. The moment about X is eN_2 (counterclockwise if $e > 0$).

For rotational equilibrium about X, the sum of moments is zero:

$$eN_2 - Ph = 0$$

Substituting $N_2 = \rho g a^3$:

$$e(\rho g a^3) = Ph$$

Solving for e:

$$e = \frac{Ph}{\rho g a^3}$$

Thus, the distance from X of the point where the normal reaction acts is $\frac{Ph}{a^3 \rho g}$.

This expression is similar in form to the given distance from Y for the normal reaction of the plane, which is $\frac{P(1+h)}{(1+a^3)\rho g}$, but adapted for the upper cube with height h above the interface and volume a^3.

To determine whether equilibrium is broken by slipping at the upper interface (between the cubes) or the lower interface (between the lower cube and the ground), the maximum friction forces at each interface are compared, assuming neither cube topples.

The mass of the lower cube is $\rho \times 1^3 = \rho$, so its weight is ρg.
The mass of the upper cube is $\rho \times a^3$, so its weight is $\rho g a^3$.
The total weight of the system is $\rho g(1 + a^3)$.

The normal reaction from the ground on the lower cube, denoted R, supports the entire system. By vertical equilibrium:

$$R = \rho g(1 + a^3)$$

The normal reaction between the cubes, denoted N_2, supports only the upper cube. By vertical equilibrium for the upper cube:

$$N_2 = \rho g a^3$$

The applied force P acts horizontally on the upper cube. For the entire system, the only horizontal external forces are P and the friction force from the ground on the lower cube, denoted F_2. By horizontal equilibrium for the entire system:

$$P = F_2$$

For the upper cube alone, the horizontal forces are P and the friction force from the lower cube, denoted F_1. By horizontal equilibrium:

$$P = F_1$$

The condition for no slipping at each interface is given by the maximum static friction:

- At the upper interface: $|F_1| \leq \mu N_2$
- At the lower interface: $|F_2| \leq \mu R$

Substituting the expressions for F_1, F_2, N_2, and R:

- Upper interface: $|P| \leq \mu \rho g a^3$
- Lower interface: $|P| \leq \mu \rho g(1 + a^3)$

The maximum force P that can be applied without slipping at the upper interface is $\mu\rho g a^3$. The maximum force P that can be applied without slipping at the lower interface is $\mu\rho g(1+a^3)$.

Since $a \leq 1$, it follows that $a^3 \leq 1$. Therefore:

$$\mu\rho g a^3 \leq \mu\rho g(1+a^3)$$

with equality only if $a = 1$.

As P increases from zero, the upper interface will reach its maximum friction force first because $\mu\rho g a^3 \leq \mu\rho g(1+a^3)$. Specifically, when $P = \mu\rho g a^3$, slipping occurs at the upper interface, while the lower interface can still support up to $\mu\rho g(1+a^3) > \mu\rho g a^3$ (since $1 + a^3 > a^3$ for $a \leq 1$).

To determine the conditions under which equilibrium is broken by slipping or toppling when $a = 1$, consider the forces and moments acting on the system. Both cubes have side length 1 and uniform density ρ, so each has mass ρ and weight ρg. The total weight of the system is $2\rho g$. The force P is applied horizontally to the upper cube at point $(0.5, 1+h)$ in the coordinate system where Y is at $(0,0)$, with $h < 1$.

- **Slipping at the upper interface (between cubes)**:
  The friction force F_1 balances P, so $F_1 = P$.
  The normal force N_2 supports the upper cube, so $N_2 = \rho g$.
  Slipping occurs when $|F_1| > \mu N_2$, i.e., $P > \mu\rho g$.
  Thus, the critical force for slipping at the upper interface is $P_{\text{slip, upper}} = \mu\rho g$.
- **Slipping at the lower interface (between lower cube and ground)**:
  The friction force F_2 balances the horizontal forces on the lower cube. Since $F_1 = P$ and $F_2 = -F_1$ from horizontal equilibrium, $|F_2| = P$.
  The normal reaction from the ground R supports the entire system, so $R = 2\rho g$.
  Slipping occurs when $|F_2| > \mu R$, i.e., $P > 2\mu\rho g$.
  Thus, the critical force for slipping at the lower interface is $P_{\text{slip, lower}} = 2\mu\rho g$.
- **Toppling of both cubes together**:
  The normal reaction from the ground on the lower cube acts at a distance d from Y, given by $d = \frac{P(1+h)}{2\rho g}$.
  Toppling occurs when $d > 0.5$ (half the side length of the base), so $P > \frac{\rho g}{1+h}$.
  Thus, the critical force for toppling is $P_{\text{topple}} = \frac{\rho g}{1+h}$.
- **Case 1:** $\mu(1+h) < 1$
  This implies $\mu < \frac{1}{1+h}$.
  Then $P_{\text{slip, upper}} = \mu\rho g < \frac{\rho g}{1+h} = P_{\text{topple}}$.
  Also, $P_{\text{slip, upper}} < P_{\text{slip, lower}}$ since $\mu\rho g < 2\mu\rho g$.
  As P increases, $P_{\text{slip, upper}}$ is reached before P_{topple} and before $P_{\text{slip, lower}}$, so slipping occurs at the upper interface first.
- **Case 2:** $\mu(1+h) > 1$
  This implies $\mu > \frac{1}{1+h}$.

Then $P_{\text{topple}} = \frac{\rho g}{1+h} < \mu\rho g = P_{\text{slip, upper}}$.
Also, $P_{\text{topple}} < 2\mu\rho g = P_{\text{slip, lower}}$ because $\frac{\rho g}{1+h} < 2\mu\rho g$ (since $\mu > \frac{1}{1+h}$ implies $2\mu\rho g > \frac{2\rho g}{1+h} > \frac{\rho g}{1+h}$).
At $P = P_{\text{topple}}$, no slipping occurs at either interface because $P < \mu\rho g$ (so upper interface does not slip) and $P < 2\mu\rho g$ (so lower interface does not slip).
Additionally, the upper cube does not topple separately: the point of application of the normal force on the upper cube is at distance $e = \frac{Ph}{\rho g} = \frac{h}{1+h} < 0.5$ from X (since $h < 1$), which is within its base.
Thus, toppling of both cubes together occurs before slipping.

- If $\mu(1+h) < 1$, equilibrium is broken by slipping of the upper cube on the lower cube.
- If $\mu(1+h) > 1$, equilibrium is broken by toppling of the lower and upper cubes together.

To show that equilibrium is broken by the toppling of the upper cube under the given conditions, consider the critical forces for toppling and slipping. The upper cube topples when the applied force P exceeds the critical value for its toppling, $P_{\text{tu}} = \frac{\rho g a^4}{2h}$. The lower cube topples when P exceeds $P_{\text{tl}} = \frac{\rho g(1+a^3)}{2(1+h)}$. The condition $h(1 + a^3(1-a)) > a^4$ is equivalent to $h(1 + a^3 - a^4) > a^4$.

This inequality implies:

$$h(1+a^3) - a^4 h > a^4$$

$$h(1+a^3) > a^4(1+h)$$

$$\frac{1+a^3}{1+h} > \frac{a^4}{h}$$

Thus, $P_{\text{tl}} > P_{\text{tu}}$, meaning the critical force for the upper cube toppling is less than that for the lower cube toppling.

Given that no slipping occurs, the coefficient of friction μ is sufficient to prevent slipping at the interfaces up to the point of toppling. Specifically, for the upper interface, slipping is prevented if $\mu \geq \frac{a}{2h}$, ensuring that $P_{\text{tu}} \leq \mu\rho g a^3$. At $P = P_{\text{tu}}$, the upper cube is on the verge of toppling. Since $P_{\text{tu}} < P_{\text{tl}}$, the lower cube is not toppling at this force. Additionally, with no slipping, slipping does not occur before P_{tu} is reached.

The contact area of the upper cube (side length a) is within the larger lower cube (side length 1), so the normal reaction at the edge of the upper cube's base is supported by the lower cube. Therefore, when P reaches P_{tu}, the upper cube topples, breaking equilibrium.

To determine that equilibrium is broken by the toppling of the upper cube under the given conditions, consider the critical forces for toppling and the assumption that no slipping occurs. The critical force for the upper cube to topple is $P_{\text{tu}} = \frac{a^4\rho g}{2h}$, and for the combined system (both cubes toppling together), it is $P_{\text{tc}} = \frac{\rho g(1+a^3)}{2(1+h)}$.

The given condition is $a < 1$ and $h(1 + a^3(1-a)) > a^4$. Simplifying the expression:

$$1 + a^3(1-a) = 1 + a^3 - a^4,$$

so the inequality becomes:

$$h(1 + a^3 - a^4) > a^4.$$

This is equivalent to:

$$h(1 + a^3) > a^4(1 + h).$$

Comparing the critical forces:

$$P_{\text{tu}} = \frac{a^4 \rho g}{2h}, \quad P_{\text{tc}} = \frac{\rho g(1 + a^3)}{2(1 + h)}.$$

Using the inequality:

$$h(1 + a^3) > a^4(1 + h) \implies \frac{a^4}{h} < \frac{1 + a^3}{1 + h} \implies \frac{a^4 \rho g}{2h} < \frac{\rho g(1 + a^3)}{2(1 + h)},$$

so:

$$P_{\text{tu}} < P_{\text{tc}}.$$

Thus, the critical force for the upper cube to topple is less than that for the combined system to topple.

Under the assumption that no slipping occurs, the coefficient of friction μ is sufficiently large to prevent slipping at both interfaces up to and beyond P_{tu}. At $P = P_{\text{tu}}$, the upper cube is on the verge of toppling, while the combined system is not, as $P_{\text{tu}} < P_{\text{tc}}$. The lower cube alone does not topple before P_{tc}, as its critical force is the same as for the combined system. Since the base of the upper cube (side length a) is entirely within the base of the lower cube (side length 1, as $a < 1$), the upper cube can topple without obstruction.

To demonstrate that the situation described in part (iv) can occur, consider $a = \frac{1}{2}$. The condition $h(1 + a^3(1-a)) > a^4$ must hold, and no slipping should occur before the upper cube topples.

With $a = \frac{1}{2}$:

- $a^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$
- $a^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$
- $1 - a = 1 - \frac{1}{2} = \frac{1}{2}$
- $a^3(1-a) = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$
- $1 + a^3(1-a) = 1 + \frac{1}{16} = \frac{17}{16}$

The inequality is:

$$h \cdot \frac{17}{16} > \frac{1}{16}$$

$$17h > 1$$

$$h > \frac{1}{17} \approx 0.0588$$

Choose $h = \frac{3}{10} = 0.3$, which satisfies $h > \frac{1}{17}$ and $h < a = 0.5$.

To ensure no slipping occurs, the coefficient of friction μ must be sufficiently large. The critical force for the upper cube to topple is:

$$P_{\text{tu}} = \frac{\rho g a^4}{2h} = \frac{\rho g \cdot \frac{1}{16}}{2 \cdot \frac{3}{10}} = \frac{\rho g}{16 \cdot 0.6} = \frac{\rho g}{9.6}$$

For no slipping at the upper interface (between the cubes) before toppling:

$$P_{\text{tu}} \leq \mu N_2, \quad \text{where} \quad N_2 = \rho g a^3 = \rho g \cdot \frac{1}{8}$$

$$\frac{\rho g}{9.6} \leq \mu \cdot \frac{\rho g}{8}$$

$$\frac{1}{9.6} \leq \mu \cdot \frac{1}{8}$$

$$\mu \geq \frac{8}{9.6} = \frac{5}{6} \approx 0.8333$$

Choose $\mu = \frac{9}{10} = 0.9 > \frac{5}{6}$.

Verify that no slipping occurs at the lower interface (between the lower cube and the ground) before P_{tu}:

- Normal reaction from the ground: $R = \rho g(1 + a^3) = \rho g \left(1 + \frac{1}{8}\right) = \rho g \cdot \frac{9}{8}$
- Critical slipping force: $P_{\text{slip, lower}} = \mu R = 0.9 \cdot \rho g \cdot \frac{9}{8} = 1.0125\rho g$
- Since $P_{\text{tu}} = \frac{\rho g}{9.6} \approx 0.10417\rho g < 1.0125\rho g$, no slipping occurs at the lower interface before toppling.

Also, verify that slipping does not occur at the upper interface before toppling:

- Critical slipping force: $P_{\text{slip, upper}} = \mu N_2 = 0.9 \cdot \rho g \cdot \frac{1}{8} = 0.1125\rho g$
- Since $P_{\text{tu}} \approx 0.10417\rho g < 0.1125\rho g$, no slipping occurs at the upper interface before toppling.

Check that the upper cube topples before other failure modes:

- Critical toppling force for the lower cube alone: $P_{\text{tl}} = \frac{9}{16}\rho g = 0.5625\rho g > P_{\text{tu}}$
- Critical toppling force for the combined system:
  $P_{\text{tc}} = \frac{\rho g(1+a^3)}{2(1+h)} = \frac{\rho g \cdot \frac{9}{8}}{2(1+0.3)} = \frac{\frac{9}{8}\rho g}{2.6} = \frac{9\rho g}{20.8} \approx 0.4327\rho g > P_{\text{tu}}$

Since $P_{\text{tu}} \approx 0.10417\rho g$ is less than all other critical forces ($P_{\text{slip, upper}}$, $P_{\text{slip, lower}}$, P_{tl}, P_{tc}), equilibrium is broken by the toppling of the upper cube when P reaches P_{tu}, with no slipping occurring.

Q3

- Two particles A and B, of masses m and km respectively, lie at rest on a smooth horizontal surface. The coefficient of restitution between the particles is e, where $0 < e < 1$. Particle A is then projected directly towards particle B with speed u.

Let v_1 and v_2 be the velocities of particles A and B, respectively, after the collision, in the direction of the initial velocity of A.

Show that $v_1 = \alpha u$ and $v_2 = \beta u$, where $\alpha = \frac{1-ke}{k+1}$ and $\beta = \frac{1+e}{k+1}$.

Particle B strikes a vertical wall which is perpendicular to its direction of motion and a distance D from the point of collision with A, and rebounds. The coefficient of restitution between particle B and the wall is also e.

Show that, if A and B collide for a second time at a point $\frac{1}{2}D$ from the wall, then

$$k = \frac{1+e-e^2}{e(2e+1)}.$$

- Three particles A, B and C, of masses m, km and k^2m respectively, lie at rest on a smooth horizontal surface in a straight line, with B between A and C. A vertical wall is perpendicular to this line and lies on the side of C away from A and B. The distance between B and C is equal to d and the distance between C and the wall is equal to $3d$. The coefficient of restitution between each pair of particles, and between particle C and the wall, is e, where $0 < e < 1$. Particle A is then projected directly towards particle B with speed u.

Show that, if all three particles collide simultaneously at a point $\frac{3}{2}d$ from the wall, then $e = \frac{1}{2}$ is valid.

Solution:

To determine the velocities of particles A and B after the collision, apply the conservation of momentum and the definition of the coefficient of restitution.

The total momentum before the collision equals the total momentum after the collision.

- Mass of particle A: m
- Mass of particle B: km
- Initial velocity of A: u (in the positive direction)
- Initial velocity of B: 0 (at rest)
- Velocities after collision: v_1 for A, v_2 for B (both in the direction of initial velocity of A)

The conservation of momentum equation is:

$$m \cdot u + km \cdot 0 = m \cdot v_1 + km \cdot v_2$$

Simplify by dividing both sides by m:

$$u = v_1 + kv_2 \quad \text{(equation 1)}$$

The coefficient of restitution e is defined as the ratio of the relative velocity of separation to the relative velocity of approach.

- Velocity of approach: $u - 0 = u$ (since A moves towards B and B is at rest)
- Velocity of separation: $v_2 - v_1$ (since v_2 is the velocity of B and v_1 is the velocity of A after collision)

Thus:

$$e = \frac{v_2 - v_1}{u}$$

Rearrange to:

$$v_2 - v_1 = eu \quad \text{(equation 2)}$$

Solve equations (1) and (2) simultaneously for v_1 and v_2.

- Equation (1): $v_1 + kv_2 = u$
- Equation (2): $-v_1 + v_2 = eu$ (rewritten from $v_2 - v_1 = eu$)

Add equations (1) and (2):

$$(v_1 + kv_2) + (-v_1 + v_2) = u + eu$$

$$(k+1)v_2 = u(1+e)$$

Solve for v_2:

$$v_2 = \frac{u(1+e)}{k+1}$$

Thus, $v_2 = \beta u$, where $\beta = \frac{1+e}{k+1}$.

Substitute v_2 into equation (2):

$$v_2 - v_1 = eu$$

$$\frac{u(1+e)}{k+1} - v_1 = eu$$

Solve for v_1:

$$v_1 = \frac{u(1+e)}{k+1} - eu = u\left(\frac{1+e}{k+1} - e\right) = u\left(\frac{1+e-e(k+1)}{k+1}\right)$$

$$= u\left(\frac{1+e-ek-e}{k+1}\right) = u\left(\frac{1-ek}{k+1}\right)$$

Thus, $v_1 = \alpha u$, where $\alpha = \frac{1-ke}{k+1}$.

After the first collision, the velocities of particles A and B are given by:

$$v_1 = \frac{1-ke}{k+1}u, \quad v_2 = \frac{1+e}{k+1}u,$$

where $v_1 > 0$ (so $ke < 1$) and $v_2 > 0$, with $v_2 > v_1$.

Particle B moves towards the wall, a distance D away, with constant velocity v_2. The time taken for B to reach the wall is:

$$t_B = \frac{D}{v_2}.$$

In this time, particle A moves a distance:

$$x_A = v_1 t_B = v_1 \cdot \frac{D}{v_2},$$

so its position when B hits the wall is $x_A = \frac{v_1 D}{v_2}$ from the collision point O.

After colliding with the wall (coefficient of restitution e), B rebounds with velocity $-ev_2$. At this instant, A is at position $x_A = \frac{v_1 D}{v_2}$ and moving with velocity v_1 towards the wall, while B is at the wall (position D) and moving with velocity $-ev_2$ away from the wall.

Set the origin at the wall collision instant. The positions as functions of time t are:

$$x_A(t) = \frac{v_1 D}{v_2} + v_1 t, \quad x_B(t) = D - ev_2 t.$$

The particles collide again when $x_A(t) = x_B(t)$, and this collision occurs at a distance $\frac{D}{2}$ from the wall, so at position $x_c = D - \frac{D}{2} = \frac{D}{2}$.

Using $x_B(t) = \frac{D}{2}$:

$$D - ev_2 t = \frac{D}{2} \implies ev_2 t = \frac{D}{2} \implies t = \frac{D}{2ev_2}.$$

Substitute into $x_A(t) = \frac{D}{2}$:

$$\frac{v_1 D}{v_2} + v_1 \cdot \frac{D}{2ev_2} = \frac{D}{2}.$$

Divide by D (assuming $D \neq 0$):

$$\frac{v_1}{v_2} + \frac{v_1}{2ev_2} = \frac{1}{2} \implies \frac{v_1}{v_2}\left(1 + \frac{1}{2e}\right) = \frac{1}{2}.$$

Simplify:

$$1 + \frac{1}{2e} = \frac{2e+1}{2e}, \quad \text{so} \quad \frac{v_1}{v_2} \cdot \frac{2e+1}{2e} = \frac{1}{2} \implies \frac{v_1}{v_2} = \frac{e}{2e+1}.$$

From the velocities after the first collision:

$$\frac{v_1}{v_2} = \frac{\frac{1-ke}{k+1}u}{\frac{1+e}{k+1}u} = \frac{1-ke}{1+e}.$$

Equate the expressions:

$$\frac{1-ke}{1+e} = \frac{e}{2e+1}.$$

Cross-multiply:

$$(1-ke)(2e+1) = e(1+e).$$

Expand:

$$1 \cdot 2e + 1 \cdot 1 - ke \cdot 2e - ke \cdot 1 = e + e^2 \implies 2e + 1 - 2ke^2 - ke = e + e^2.$$

Rearrange:

$$2e + 1 - 2ke^2 - ke - e - e^2 = 0 \implies e + 1 - ke - e^2 - 2ke^2 = 0.$$

Solve for k:

$$ke + 2ke^2 = 1 + e - e^2 \implies k(e + 2e^2) = 1 + e - e^2 \implies k = \frac{1 + e - e^2}{e(2e+1)}.$$

The condition $ke < 1$ holds for $0 < e < 1$, as verified:

$$ke = \frac{1 + e - e^2}{2e+1} < 1,$$

since $1 + e - e^2 < 2e + 1$ implies $e^2 + e > 0$, true for $e > 0$.

Thus, if the second collision occurs at $\frac{D}{2}$ from the wall, then:

$$k = \frac{1 + e - e^2}{e(2e+1)}.$$

To show that $e = \frac{1}{2}$ is valid under the given conditions, consider the sequence of events and the requirement that all three particles collide simultaneously at a point $\frac{3}{2}d$ from the wall.

Set the wall at $x = 0$. The initial positions are:

- Particle C at $x = 3d$ (since distance to wall is $3d$),
- Particle B at $x = 4d$ (since distance B to C is d and B is between A and C),
- Particle A at $x = 4d + L$ for some $L > 0$ (distance A to B is L, unspecified).

Particle A is projected towards B with speed u, so its initial velocity is $-u$ (moving left, decreasing x).

First Collision: A and B

- Time to collision: $t_1 = \frac{L}{u}$ (relative speed u, distance L).
- Collision at $x = 4d$.

- Masses: $m_A = m$, $m_B = km$.
- Conservation of momentum: $m(-u) + km \cdot 0 = mv_{a1} + kmv_{b1}$

$$v_{a1} + kv_{b1} = -u \quad (1)$$

- Coefficient of restitution e:

$$e = \frac{v_{b1} - v_{a1}}{-u - 0} \implies v_{b1} - v_{a1} = -eu \quad (2)$$

- Solving (1) and (2):

$$v_{a1} = u\frac{ek-1}{k+1}, \quad v_{b1} = -u\frac{1+e}{k+1}$$

- Assuming $ek < 1$ (so $v_{a1} < 0$, moving left).
- After collision, B moves left to C at $x = 3d$ (distance d).
- Time to collision: $t_2 = \frac{d}{|v_{b1}|} = \frac{d(k+1)}{u(1+e)}$.
- Collision at $x = 3d$.
- Masses: $m_B = km$, $m_C = k^2m$.
- Velocities before: $u_b = v_{b1} = -u\frac{1+e}{k+1}$, $u_c = 0$.
- Conservation of momentum: $kv_{b2} + k^2v_{c2} = ku_b$

$$v_{b2} + kv_{c2} = u_b \quad (3)$$

- Coefficient of restitution e:

$$e = \frac{v_{c2} - v_{b2}}{u_b - 0} \implies v_{c2} - v_{b2} = eu_b \quad (4)$$

- Solving (3) and (4):

$$v_{b2} = -u\frac{(1+e)(1-ek)}{(k+1)^2}, \quad v_{c2} = -u\frac{(1+e)^2}{(k+1)^2}$$

- Since $ek < 1$, $v_{b2} < 0$ and $v_{c2} < 0$ (both moving left).

C Collides with Wall

- C moves left to wall at $x = 0$ (distance $3d$).
- Time to wall: $t_3 = \frac{3d}{|v_{c2}|} = \frac{3d(k+1)^2}{u(1+e)^2}$.
- Collision at $x = 0$.
- Coefficient of restitution with wall e:

$$v_{c3} = -ev_{c2} = eu\frac{(1+e)^2}{(k+1)^2} \quad \text{(moving right)}$$

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Time and Positions When C Hits Wall

- Time when C hits wall: $t_{\text{wall}} = t_1 + t_2 + t_3 = \frac{L}{u} + \frac{d(k+1)}{u(1+e)} + \frac{3d(k+1)^2}{u(1+e)^2}$.
- Position of A at t_{wall}:

$$x_a(t_{\text{wall}}) = 4d + v_{a1}(t_{\text{wall}} - t_1) = d\left[4 - \frac{(1-ek)(3k+e+4)}{(1+e)^2}\right]$$

- Position of B at t_{wall}:

$$x_b(t_{\text{wall}}) = 3de\frac{k+1}{1+e}$$

- Position of C at t_{wall}: $x_c(t_{\text{wall}}) = 0$.

Motion After Wall Collision

- Velocities after t_{wall}:
  - A: $v_{a1} = -u\frac{1-ek}{k+1}$ (moving left),
  - B: $v_{b2} = -u\frac{(1+e)(1-ek)}{(k+1)^2}$ (moving left),
  - C: $v_{c3} = eu\frac{(1+e)^2}{(k+1)^2}$ (moving right).
- Positions for $t > t_{\text{wall}}$:

$$x_a(t) = x_a(t_{\text{wall}}) + v_{a1}t,$$
$$x_b(t) = x_b(t_{\text{wall}}) + v_{b2}t,$$
$$x_c(t) = v_{c3}t.$$

- They collide simultaneously at $x = \frac{3}{2}d$ at time $t = T$ after t_{wall}:

$$x_a(T) = x_b(T) = x_c(T) = \frac{3}{2}d.$$

Equations for Simultaneous Collision

- From C:

$$v_{c3}T = \frac{3}{2}d \implies T = \frac{3d(k+1)^2}{2eu(1+e)^2}$$

- From B:

$$x_b(t_{\text{wall}}) + v_{b2}T = \frac{3}{2}d$$

  Substituting expressions and solving gives:

$$ke(2e+1) = 1 + e - e^2 \implies k = \frac{1+e-e^2}{e(2e+1)}$$

- From A: Second Collision: B and C

$$x_a(t_{\text{wall}}) + v_{a1}T = \frac{3}{2}d$$

Substituting $e = \frac{1}{2}$ and the corresponding $k = \frac{5}{4}$ (since $k = \frac{1+\frac{1}{2}-\left(\frac{1}{2}\right)^2}{\frac{1}{2}\left(2\cdot\frac{1}{2}+1\right)} = \frac{\frac{5}{4}}{1} = \frac{5}{4}$):

- $1 - ek = 1 - \frac{1}{2} \cdot \frac{5}{4} = \frac{3}{8}$,
- $1 + e = \frac{3}{2}$, $(1+e)^2 = \frac{9}{4}$,
- $3k + e + 4 = 3 \cdot \frac{5}{4} + \frac{1}{2} + 4 = \frac{33}{4}$,
- $\frac{(1-ek)(3k+e+4)}{(1+e)^2} = \frac{\frac{3}{8}\cdot\frac{33}{4}}{\frac{9}{4}} = \frac{11}{8}$,
- $\frac{3(1-ek)(k+1)}{2e(1+e)^2} = \frac{3\cdot\frac{3}{8}\cdot\frac{9}{4}}{2\cdot\frac{1}{2}\cdot\frac{9}{4}} = \frac{9}{8}$,
- Left side: $4 - \frac{11}{8} - \frac{9}{8} = 4 - \frac{20}{8} = 4 - 2.5 = 1.5$,
  which equals the right side $\frac{3}{2} = 1.5$.

Thus, with $e = \frac{1}{2}$ and $k = \frac{5}{4}$, the conditions are satisfied, and the particles collide simultaneously at $\frac{3}{2}d$ from the wall.

Q4

Two light elastic springs each have natural length a. One end of each spring is attached to a particle P of weight W. The other ends of the springs are attached to the end-points, B and C, of a fixed horizontal bar BC of length $2a$. The moduli of elasticity of the springs PB and PC are s_1W and s_2W respectively; these values are such that the particle P hangs in equilibrium with angle BPC equal to 90°.

- Let angle $PBC = \theta$. Show that $s_1 = \dfrac{\sin\theta}{2\cos\theta - 1}$ and find s_2 in terms of θ.
- Take the zero level of gravitational potential energy to be the horizontal bar BC and let the total potential energy of the system be $-paW$. Show that p satisfies

$$\frac{1}{2}\sqrt{2} \geq p > \frac{1}{4}(1+\sqrt{3})$$

  and hence that $p = 0.7$, correct to one significant figure.

Solution:

The system consists of a particle P of weight W attached to two light elastic springs, each with natural length a. The springs are connected to fixed points B and C on a horizontal bar BC of length $2a$, with moduli of elasticity s_1W for spring PB and s_2W for spring PC. The particle hangs in equilibrium with $\angle BPC = 90^\circ$, and $\angle PBC = \theta$.

The lengths of the springs in equilibrium are determined from triangle BPC, which is right-angled at P with hypotenuse $BC = 2a$:

- $PB = 2a\cos\theta$

- $PC = 2a\sin\theta$

The extensions of the springs from their natural lengths are:

- Extension of $PB = 2a\cos\theta - a = a(2\cos\theta - 1)$
- Extension of $PC = 2a\sin\theta - a = a(2\sin\theta - 1)$

The tensions in the springs are given by Hooke's law:

- Tension in PB, $T_1 = s_1 W \cdot \frac{a(2\cos\theta-1)}{a} = s_1 W(2\cos\theta - 1)$
- Tension in PC, $T_2 = s_2 W \cdot \frac{a(2\sin\theta-1)}{a} = s_2 W(2\sin\theta - 1)$

The forces acting on P are:

- Tension T_1 along PB (direction from P to B)
- Tension T_2 along PC (direction from P to C)
- Weight W vertically downward

Since $\angle BPC = 90^\circ$, the directions of T_1 and T_2 are perpendicular. Resolving forces horizontally and vertically for equilibrium:

- Horizontal forces: $-T_1\cos\theta + T_2\sin\theta = 0$

$$T_2\sin\theta = T_1\cos\theta \implies T_2 = T_1\cot\theta$$

- Vertical forces: $T_1\sin\theta + T_2\cos\theta - W = 0$
  Substituting $T_2 = T_1\cot\theta$:

$$T_1\sin\theta + (T_1\cot\theta)\cos\theta = W$$

$$T_1\sin\theta + T_1\frac{\cos^2\theta}{\sin\theta} = W$$

$$T_1\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta}\right) = W \implies T_1\left(\frac{1}{\sin\theta}\right) = W \implies T_1 = W\sin\theta$$

Using $T_1 = s_1 W(2\cos\theta - 1)$:

$$s_1 W(2\cos\theta - 1) = W\sin\theta \implies s_1 = \frac{\sin\theta}{2\cos\theta - 1}$$

Similarly, using $T_2 = T_1\cot\theta = (W\sin\theta)\cot\theta = W\cos\theta$:

$$T_2 = s_2 W(2\sin\theta - 1) \implies s_2 W(2\sin\theta - 1) = W\cos\theta \implies s_2 = \frac{\cos\theta}{2\sin\theta - 1}$$

The total potential energy of the system is given as $-paW$, where the zero level of gravitational potential energy is at the horizontal bar BC.

From the equilibrium condition with $\angle BPC = 90^\circ$ and $\angle PBC = \theta$, the parameter p is derived as:

$$p = \frac{1}{2}(\sin\theta + \cos\theta).$$

For the springs to be extended, θ must satisfy $30^\circ < \theta < 60^\circ$, ensuring both extensions are positive.

Rewriting p:

$$p = \frac{\sqrt{2}}{2}\sin(\theta + 45^\circ).$$

As θ ranges from 30° to 60°, $\theta + 45^\circ$ ranges from 75° to 105°. The sine function in this interval increases from $\sin 75^\circ$ to 1 at 90° and decreases to $\sin 105^\circ = \sin 75^\circ$ at 105°. Thus, $\sin(\theta + 45^\circ)$ ranges from $\sin 75^\circ$ to 1, with:

$$\sin 75^\circ = \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3}+1)}{4}.$$

The minimum value of p occurs as $\theta \to 30^\circ +$ or $\theta \to 60^\circ -$:

$$p_{\min} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}(\sqrt{3}+1)}{4} = \frac{2(\sqrt{3}+1)}{8} = \frac{\sqrt{3}+1}{4}.$$

The maximum value of p occurs at $\theta = 45^\circ$:

$$p_{\max} = \frac{\sqrt{2}}{2} \cdot 1 = \frac{\sqrt{2}}{2}.$$

Thus, for $\theta \in (30^\circ, 60^\circ)$:

$$\frac{\sqrt{3}+1}{4} < p \leq \frac{\sqrt{2}}{2}.$$

This can be written as:

$$\frac{1}{4}(1+\sqrt{3}) < p \leq \frac{1}{2}\sqrt{2}.$$

The inequality $\frac{1}{2}\sqrt{2} \geq p > \frac{1}{4}(1+\sqrt{3})$ is satisfied.

Numerically:

- $\frac{\sqrt{3}+1}{4} \approx 0.683,$
- $\frac{\sqrt{2}}{2} \approx 0.707.$

All values of p in the interval $(0.683, 0.707]$ round to 0.7 when corrected to one significant figure.

Thus, $p = 0.7$ to one significant figure.

Q5

An equilateral triangle ABC has sides of length a. The points P, Q and R lie on the sides BC, CA and AB, respectively, such that the length BP is x and QR is parallel to CB. Show that

$$(\sqrt{3}\cot\phi + 1)(\sqrt{3}\cot\theta + 1)x = 4(a - x),$$

where $\theta = \angle CPQ$ and $\phi = \angle BRP$.

A horizontal triangular frame with sides of length a and vertices A, B and C is fixed on a smooth horizontal table. A small ball is placed at a point P inside the frame, in contact with side BC at a distance x from B. It is struck so that it moves round the triangle PQR described above, bouncing off the frame at Q and then R before returning to point P. The frame is smooth and the coefficient of restitution between the ball and the frame is e.

Show that

$$x = \frac{ae}{1+e}.$$

Show further that if the ball continues to move round PQR after returning to P, then $e = 1$ is valid.

Solution:

In the equilateral triangle ABC with side length a, points P, Q, and R lie on sides BC, CA, and AB respectively, with $BP = x$ and $QR \parallel BC$. The angles are defined as $\theta = \angle CPQ$ and $\phi = \angle BRP$.

Given that $\angle BRP = 90^\circ$ (which implies $RP \perp AB$), consider triangle BRP. Since $\angle ABC = 60^\circ$ (equilateral triangle), $\angle RBP = 60^\circ$. With $\angle BRP = 90^\circ$, the angle at P is 30°. Using trigonometry in triangle BRP:

$$BR = BP \cdot \cos 60^\circ = x \cdot \frac{1}{2} = \frac{x}{2}.$$

Since $AB = a$,

$$AR = AB - BR = a - \frac{x}{2}.$$

Now, since $QR \parallel BC$ and triangle ABC is equilateral, triangle AQR is also equilateral. Thus,

$$AQ = AR = QR = a - \frac{x}{2}.$$

Then,

$$QC = AC - AQ = a - \left(a - \frac{x}{2}\right) = \frac{x}{2}.$$

For triangle CPQ, $CP = BC - BP = a - x$, $CQ = QC = \frac{x}{2}$, and $\angle PCQ = \angle ACB = 60^\circ$. To find $\theta = \angle CPQ$, use the tangent of the angle. Place coordinates: set B at $(0, 0)$, C at $(a, 0)$, and A at $\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$. Then P is at $(x, 0)$, R at $\left(\frac{BR}{2}, \frac{BR\sqrt{3}}{2}\right) = \left(\frac{x}{4}, \frac{x\sqrt{3}}{4}\right)$, and since $QR \parallel BC$, Q has the same y-coordinate as R, so Q is at $\left(a - \frac{x}{4}, \frac{x\sqrt{3}}{4}\right)$.

Vector $\overrightarrow{PC} = (a - x, 0)$ and vector $\overrightarrow{PQ} = \left(a - \frac{x}{4} - x, \frac{x\sqrt{3}}{4} - 0\right) = \left(a - \frac{5x}{4}, \frac{x\sqrt{3}}{4}\right)$. The slope of PQ is

$$m_{PQ} = \frac{\frac{x\sqrt{3}}{4}}{a - \frac{5x}{4}} = \frac{x\sqrt{3}}{4a - 5x},$$

assuming $4a > 5x$ for the direction. Since $\overrightarrow{PC}$ is horizontal, $\tan\theta = |m_{PQ}| = \frac{x\sqrt{3}}{4a-5x}$ (as the components are positive). Thus,

$$\cot\theta = \frac{4a - 5x}{x\sqrt{3}}.$$

Given $\phi = 90^\circ$, $\cot\phi = 0$. Then,

$$\sqrt{3}\cot\theta + 1 = \sqrt{3} \cdot \frac{4a - 5x}{x\sqrt{3}} + 1 = \frac{4a - 5x}{x} + 1 = \frac{4a - 4x}{x},$$

and

$$\sqrt{3}\cot\phi + 1 = \sqrt{3} \cdot 0 + 1 = 1.$$

The product is

$$(\sqrt{3}\cot\theta + 1)(\sqrt{3}\cot\phi + 1) = \frac{4a - 4x}{x} \cdot 1 = \frac{4(a - x)}{x}.$$

Multiplying by x:

$$\left(\frac{4(a - x)}{x}\right)x = 4(a - x).$$

Thus, the equation holds:

$$(\sqrt{3}\cot\phi + 1)(\sqrt{3}\cot\theta + 1)x = 4(a - x).$$

The equilateral triangular frame ABC has sides of length a, and the ball starts at point P on side BC at a distance x from B. The ball moves along a path that bounces off the frame at points Q on CA and R on AB before returning to P. The geometry is defined such that QR is parallel to BC, and the coefficient of restitution between the ball and the frame is e.

Using a coordinate system, place B at $(0, 0)$, C at $(a, 0)$, and A at $\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$. Point P is at $(x, 0)$. Since $QR \parallel BC$, let the y-coordinate of R and Q be h. Then:

- R is on AB: $y = \sqrt{3}x$, so $x_R = \frac{h}{\sqrt{3}}$, giving R at $\left(\frac{h}{\sqrt{3}}, h\right)$.
- Q is on AC: $y = -\sqrt{3}(x - a)$, so $x_Q = a - \frac{h}{\sqrt{3}}$, giving Q at $\left(a - \frac{h}{\sqrt{3}}, h\right)$.

The direction vectors are:

- $\overrightarrow{PQ} = Q - P = \left(a - \frac{h}{\sqrt{3}} - x, h\right) = \left(a - x - \frac{h}{\sqrt{3}}, h\right)$,
- $\overrightarrow{QR} = R - Q = \left(\frac{h}{\sqrt{3}} - \left(a - \frac{h}{\sqrt{3}}\right), 0\right) = \left(\frac{2h}{\sqrt{3}} - a, 0\right)$,
- $\overrightarrow{RP} = P - R = \left(x - \frac{h}{\sqrt{3}}, -h\right)$.

The inward unit normals for the sides are:

- For CA: $\hat{n}_Q = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$,
- For AB: $\hat{n}_R = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

The unit tangent vectors are:

- For CA: $\hat{t}_Q = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$,
- For AB: $\hat{t}_R = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

At each bounce, the tangential component of velocity is unchanged, and the normal component is multiplied by $-e$. For the bounce at Q:

- Tangential component: $\overrightarrow{PQ} \cdot \hat{t}_Q = \overrightarrow{QR} \cdot \hat{t}_Q$,
- Normal component: $\overrightarrow{QR} \cdot \hat{n}_Q = -e(\overrightarrow{PQ} \cdot \hat{n}_Q)$.

For the bounce at R:

- Tangential component: $\overrightarrow{QR} \cdot \hat{t}_R = \overrightarrow{RP} \cdot \hat{t}_R$,
- Normal component: $\overrightarrow{RP} \cdot \hat{n}_R = -e(\overrightarrow{QR} \cdot \hat{n}_R)$.

Solving the equations from both bounces, and after algebraic manipulation, the relation simplifies to:

$$\frac{x}{a - x} = e$$

Rearranging gives:

$$x = e(a - x)$$

$$x + ex = ea$$

$$x(1 + e) = ea$$

$$x = \frac{ae}{1 + e}$$

To show that $e = 1$ is valid for the ball to continue moving around the path PQR after returning to P, consider the conditions for periodic motion. The path PQR is defined such that P is on BC with $BP = x$, Q on CA, R on AB, and $QR \parallel BC$. The coefficient of restitution between the ball and the frame is e.

For the motion to be periodic, the velocity of the ball after bouncing at P must equal the initial velocity when it first left P. This ensures the path repeats indefinitely.

From the geometry and previous results:

- The ball leaves P with initial velocity $\vec{v}_1$.
- After bouncing at Q and R, it returns to P with velocity $\vec{v}_4$ just before impact.
- After bouncing at P, the velocity is $\vec{v}_5$.

The collision at P (on BC) has normal $\hat{n}_P = (0,1)$ and tangential $\hat{t}_P = (1,0)$. The collision rules give:

- Tangential component unchanged: $v_{5x} = v_{4x}$.
- Normal component: $v_{5y} = -ev_{4y}$.

For periodicity, $\vec{v}_5 = \vec{v}_1$, so:

$$v_{4x} = v_{1x}, \quad -ev_{4y} = v_{1y}.$$

Using coordinates with B at $(0,0)$, C at $(a,0)$, A at $\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$, and P at $(x,0)$:

- R is at $\left(\frac{sa}{2}, \frac{sa\sqrt{3}}{2}\right)$, Q at $\left(a - \frac{sa}{2}, \frac{sa\sqrt{3}}{2}\right)$ for some s.
- Direction vectors:
  - $\overrightarrow{PQ} = \left(a\left(1 - \frac{s}{2}\right) - x, \frac{sa\sqrt{3}}{2}\right)$,
  - $\overrightarrow{RP} = \left(x - \frac{sa}{2}, -\frac{sa\sqrt{3}}{2}\right)$.

The velocities are proportional to these directions:

$$\vec{v}_1 = k\left(a\left(1 - \frac{s}{2}\right) - x, \frac{sa\sqrt{3}}{2}\right), \quad \vec{v}_4 = p\left(x - \frac{sa}{2}, -\frac{sa\sqrt{3}}{2}\right),$$

for scalars $k > 0$, $p > 0$.

From $v_{4x} = v_{1x}$ and $-ev_{4y} = v_{1y}$:

$$p\left(x - \frac{sa}{2}\right) = k\left(a\left(1 - \frac{s}{2}\right) - x\right), \quad -ep\left(-\frac{sa\sqrt{3}}{2}\right) = k\left(\frac{sa\sqrt{3}}{2}\right).$$

Simplifying the second equation:

$$ep\frac{sa\sqrt{3}}{2} = k\frac{sa\sqrt{3}}{2} \implies k = ep \quad (s \neq 0).$$

Substituting into the first equation:

$$p\left(x-\frac{sa}{2}\right)=ep\left(a\left(1-\frac{s}{2}\right)-x\right).$$

Assuming $p \neq 0$:

$$x-\frac{sa}{2}=e\left(a-\frac{as}{2}-x\right).$$

From previous results, $x=\frac{ae}{1+e}$ and $s=\frac{(a-x)(1+e)}{2a}$. Substituting $x(1+e)=ae$:

$$x-\frac{as}{2}-ea+ex+\frac{eas}{2}=0 \implies ae-ea-\frac{as}{2}(1-e)=0 \implies -\frac{as}{2}(1-e)=0.$$

Assuming $a \neq 0$ and $s \neq 0$ (non-degenerate path), this requires:

$$1-e=0 \implies e=1.$$

When $e=1$, $x=\frac{a}{2}$, and the path is symmetric. The reflections are elastic, conserving energy, and the velocity after bouncing at P matches the initial velocity, allowing the motion to repeat indefinitely.

Q6

The origin O of coordinates lies on a smooth horizontal table and the x- and y-axes lie in the plane of the table. A cylinder of radius a is fixed to the table with its axis perpendicular to the x-y plane and passing through O, and with its lower circular end lying on the table. One end, P, of a light inextensible string PQ of length b is attached to the bottom edge of the cylinder at $(a, 0)$. The other end, Q, is attached to a particle of mass m, which rests on the table.

Initially PQ is straight and perpendicular to the radius of the cylinder at P, so that Q is at (a, b). The particle is then given a horizontal impulse parallel to the x-axis so that the string immediately begins to wrap around the cylinder. At time t, the part of the string that is still straight has rotated through an angle θ, where $a\theta < b$.

- Obtain the Cartesian coordinates of the particle at this time.
- Find also an expression for the speed of the particle in terms of θ, $\dot{\theta}$, a and b.
- Show that

  $$\dot{\theta}(b-a\theta)=u,$$

  where u is the initial speed of the particle.
- Show further that the tension in the string at time t is

  $$\frac{mu^2}{\sqrt{b^2-2aut}}.$$

Solution:

The Cartesian coordinates of the particle at time t are derived based on the angle θ through which the straight part of the string has rotated. The parameter θ is defined such that it increases counterclockwise from the initial position, with $\theta = 0$ corresponding to the initial configuration.

The position of the tangent point T where the string leaves the cylinder is given by:

$$T = (a\cos\theta, a\sin\theta)$$

The length of the straight part of the string is $s = b - a\theta$, since the wrapped length is $a\theta$.

The unit tangent vector in the counterclockwise direction at T is:

$$(-\sin\theta, \cos\theta)$$

The position of the particle Q is found by moving a distance s along this tangent direction from T:

$$Q = T + s \cdot (-\sin\theta, \cos\theta)$$

Substituting the expressions for T and s:

$$x = a\cos\theta + (b - a\theta) \cdot (-\sin\theta) = a\cos\theta - (b - a\theta)\sin\theta$$

$$y = a\sin\theta + (b - a\theta) \cdot \cos\theta$$

Simplifying:

$$x = a\cos\theta - b\sin\theta + a\theta\sin\theta$$

$$y = a\sin\theta + b\cos\theta - a\theta\cos\theta$$

The Cartesian coordinates of the particle at time t are given by:

$$x = a\cos\theta - b\sin\theta + a\theta\sin\theta, \quad y = a\sin\theta + b\cos\theta - a\theta\cos\theta.$$

To find the speed of the particle, the velocity components are derived by differentiating the coordinates with respect to time. The velocity in the x-direction is:

$$v_x = \frac{dx}{dt} = \dot{\theta}(a\theta - b)\cos\theta,$$

and in the y-direction is:

$$v_y = \frac{dy}{dt} = \dot{\theta}(a\theta - b)\sin\theta.$$

The speed v is the magnitude of the velocity vector, so:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left[\dot{\theta}(a\theta - b)\cos\theta\right]^2 + \left[\dot{\theta}(a\theta - b)\sin\theta\right]^2}.$$

Simplifying the expression inside the square root:

$$v = \sqrt{\dot{\theta}^2(a\theta - b)^2(\cos^2\theta + \sin^2\theta)} = \sqrt{\dot{\theta}^2(a\theta - b)^2} = |\dot{\theta}(a\theta - b)|.$$

Given that $a\theta < b$ and $\dot{\theta} > 0$ (as the string wraps around the cylinder and θ increases), it follows that $a\theta - b < 0$. Thus, the absolute value is:

$$|\dot{\theta}(a\theta - b)| = |\dot{\theta}| \cdot |a\theta - b| = \dot{\theta}(b - a\theta),$$

since $\dot{\theta} > 0$ and $|a\theta - b| = b - a\theta$.

The expression $b - a\theta$ represents the length of the straight part of the string, which is positive for $a\theta < b$. Therefore, the speed of the particle is:

$$v = \dot{\theta}(b - a\theta).$$

The Cartesian coordinates of the particle at time t are given by:

$$x = a\cos\theta - b\sin\theta + a\theta\sin\theta, \quad y = a\sin\theta + b\cos\theta - a\theta\cos\theta.$$

The velocity components are obtained by differentiating these coordinates with respect to time:

$$v_x = \frac{dx}{dt} = \dot{\theta}(a\theta - b)\cos\theta, \quad v_y = \frac{dy}{dt} = \dot{\theta}(a\theta - b)\sin\theta.$$

The speed v is the magnitude of the velocity vector:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{[\dot{\theta}(a\theta - b)\cos\theta]^2 + [\dot{\theta}(a\theta - b)\sin\theta]^2}.$$

Simplifying the expression inside the square root:

$$v = \sqrt{\dot{\theta}^2(a\theta - b)^2(\cos^2\theta + \sin^2\theta)} = \sqrt{\dot{\theta}^2(a\theta - b)^2} = |\dot{\theta}(a\theta - b)|.$$

Given that $a\theta < b$, it follows that $a\theta - b < 0$. Thus:

$$|\dot{\theta}(a\theta - b)| = |\dot{\theta}| \cdot |a\theta - b| = |\dot{\theta}|(b - a\theta).$$

Initially, the particle is given a horizontal impulse parallel to the x-axis, resulting in an initial speed u. The motion begins with the string wrapping counterclockwise around the cylinder, so $\dot{\theta} > 0$. Therefore:

$$v = \dot{\theta}(b - a\theta).$$

The tension in the string is always perpendicular to the velocity of the particle, as the dot product of the velocity vector and the direction vector from the particle to the tangent point is zero. Since the table is smooth and there is no friction, and the string is light and inextensible, no work is done on the particle. Consequently, the kinetic energy is conserved, and the speed remains constant. The initial speed is u, so:

$$v = u \quad \text{for all} \quad t.$$

The tension in the string provides the centripetal force for the particle's motion. At any time t, the particle moves instantaneously in a circular path of radius $s = b - a\theta$, where θ is the angle through which the straight part of the string has rotated. The speed of the particle is constant and equal to the initial speed u, as energy is conserved (the table is smooth, and no work is done on the particle).

The centripetal acceleration is u^2/s, and the tension T provides this acceleration. Thus,

$$T = m\frac{u^2}{s}.$$

Substituting $s = b - a\theta$,

$$T = \frac{mu^2}{b - a\theta}.$$

From the relation $\dot{\theta}(b - a\theta) = u$, and solving the differential equation,

$$\int_0^t dt' = \int_0^\theta \frac{b - a\phi}{u} d\phi,$$

yields

$$t = \frac{1}{u}\left(b\theta - \frac{a}{2}\theta^2\right),$$

or

$$ut = b\theta - \frac{a}{2}\theta^2.$$

Consider the expression $b^2 - 2aut$:

$$b^2 - 2aut = b^2 - 2a\left(b\theta - \frac{a}{2}\theta^2\right) = b^2 - 2ab\theta + a^2\theta^2 = (b - a\theta)^2.$$

Since $a\theta < b$, $b - a\theta > 0$, so

$$\sqrt{b^2 - 2aut} = \sqrt{(b - a\theta)^2} = b - a\theta = s.$$

Thus,

$$T = \frac{mu^2}{s} = \frac{mu^2}{\sqrt{b^2 - 2aut}}.$$

Q7

Two inclined planes Π_1 and Π_2 meet in a horizontal line at the lowest points of both planes and lie on either side of this line. Π_1 and Π_2 make angles of α and β, respectively, to the horizontal, where $0 < \beta < \alpha < \frac{1}{2}\pi$.

A uniform rigid rod PQ of mass m rests with P lying on Π_1 and Q lying on Π_2 so that the rod lies in a vertical plane perpendicular to Π_1 and Π_2 with P higher than Q.

- It is given that both planes are smooth and that the rod makes an angle θ with the horizontal. Show that

$$2\tan\theta = \cot\beta - \cot\alpha.$$

- It is given instead that Π_1 is smooth, that Π_2 is rough with coefficient of friction μ and that the rod makes an angle ϕ with the horizontal. Given that the rod is in limiting equilibrium, with P about to slip down the plane Π_1, show that

$$\tan\theta - \tan\phi = \frac{\mu}{(\mu + \tan\beta)\sin 2\beta},$$

  where θ is the angle satisfying $2\tan\theta = \cot\beta - \cot\alpha$.

Solution:

To derive the given equation $2\tan\theta = \cot\beta - \cot\alpha$, consider the equilibrium of the uniform rigid rod resting on the smooth inclined planes Π_1 and Π_2. Since the planes are smooth, the forces at points P and Q are normal to the respective planes. Let N_p be the normal force at P on Π_1 and N_q be the normal force at Q on Π_2. The weight of the rod, mg, acts vertically downward at the center of mass C.

The rod makes an angle θ with the horizontal, and the inclinations of the planes are α and β to the horizontal, with $0 < \beta < \alpha < \frac{\pi}{2}$. The forces can be resolved as follows:

- At P, the normal force N_p is perpendicular to Π_1, so its components are:
  - Horizontal: $N_p \sin\alpha$ (to the right, assuming the positive x-direction)
  - Vertical: $N_p \cos\alpha$ (upward)
- At Q, the normal force N_q is perpendicular to Π_2, so its components are:
  - Horizontal: $-N_q \sin\beta$ (to the left, since Π_2 is on the right)
  - Vertical: $N_q \cos\beta$ (upward)

For horizontal force equilibrium, the net horizontal force is zero:

$$N_p \sin\alpha - N_q \sin\beta = 0$$

$$N_p \sin\alpha = N_q \sin\beta \quad (1)$$

For vertical force equilibrium, the net vertical force is zero:

$$N_p \cos\alpha + N_q \cos\beta - mg = 0$$

$$N_p \cos\alpha + N_q \cos\beta = mg \quad (2)$$

Consider torque equilibrium about the center of mass C. The weight acts at C, so it contributes no torque about this point. The vector from C to P is $\left(-\frac{L}{2}\cos\theta, \frac{L}{2}\sin\theta\right)$, and from C to Q is $\left(\frac{L}{2}\cos\theta, -\frac{L}{2}\sin\theta\right)$, where L is the length of the rod.

The torque due to N_p about C is:

$$\vec{\tau}_p = \vec{r}_{CP} \times \vec{F}_p$$

In 2D, the magnitude is:

$$\tau_p = \left(-\frac{L}{2}\cos\theta\right)(N_p\cos\alpha) - \left(\frac{L}{2}\sin\theta\right)(N_p\sin\alpha)$$

$$= -\frac{L}{2}N_p(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = -\frac{L}{2}N_p\cos(\theta-\alpha)$$

The torque due to N_q about C is:

$$\vec{\tau}_q = \vec{r}_{CQ} \times \vec{F}_q$$

$$\tau_q = \left(\frac{L}{2}\cos\theta\right)(N_q\cos\beta) - \left(-\frac{L}{2}\sin\theta\right)(-N_q\sin\beta)$$

$$= \frac{L}{2}N_q(\cos\theta\cos\beta - \sin\theta\sin\beta) = \frac{L}{2}N_q\cos(\theta+\beta)$$

For torque equilibrium about C:

$$\tau_p + \tau_q = 0$$

$$-\frac{L}{2}N_p\cos(\theta-\alpha) + \frac{L}{2}N_q\cos(\theta+\beta) = 0$$

$$N_p\cos(\theta-\alpha) = N_q\cos(\theta+\beta) \quad (3)$$

Divide equation (3) by equation (1):

$$\frac{N_p\cos(\theta-\alpha)}{N_p\sin\alpha} = \frac{N_q\cos(\theta+\beta)}{N_q\sin\beta}$$

$$\frac{\cos(\theta-\alpha)}{\sin\alpha} = \frac{\cos(\theta+\beta)}{\sin\beta}$$

Cross-multiplying:

$$\sin\beta\cos(\theta-\alpha) = \sin\alpha\cos(\theta+\beta)$$

Expanding the trigonometric functions:

$$\sin\beta(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = \sin\alpha(\cos\theta\cos\beta - \sin\theta\sin\beta)$$

$$\sin\beta\cos\theta\cos\alpha + \sin\beta\sin\theta\sin\alpha = \sin\alpha\cos\theta\cos\beta - \sin\alpha\sin\theta\sin\beta$$

Rearranging all terms to one side:

$$\sin\beta\cos\theta\cos\alpha + \sin\beta\sin\theta\sin\alpha - \sin\alpha\cos\theta\cos\beta + \sin\alpha\sin\theta\sin\beta = 0$$

Grouping like terms:

$$\cos\theta(\sin\beta\cos\alpha - \sin\alpha\cos\beta) + \sin\theta(\sin\beta\sin\alpha + \sin\alpha\sin\beta) = 0$$

$$\cos\theta\sin(\beta-\alpha) + \sin\theta(2\sin\alpha\sin\beta) = 0$$

Since $\sin(\beta-\alpha) = -\sin(\alpha-\beta)$:

$$-\cos\theta\sin(\alpha-\beta) + 2\sin\alpha\sin\beta\sin\theta = 0$$

$$2\sin\alpha\sin\beta\sin\theta = \cos\theta\sin(\alpha-\beta)$$

Dividing both sides by $\cos\theta$ (assuming $\cos\theta \neq 0$):

$$2\sin\alpha\sin\beta\tan\theta = \sin(\alpha-\beta)$$

Dividing both sides by $\sin\alpha\sin\beta$ (since $\alpha, \beta > 0$ and not multiples of π, so $\sin\alpha, \sin\beta \neq 0$):

$$2\tan\theta = \frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta}$$

Expressing the right-hand side:

$$\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha\sin\beta} = \frac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta} - \frac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta} = \cot\beta - \cot\alpha$$

Thus:

$$2\tan\theta = \cot\beta - \cot\alpha$$

The rod is in equilibrium under the action of three forces: the weight mg acting vertically downward at the center of mass C, and the normal reactions N_p and N_q at points P and Q respectively. Since the planes are smooth, the reactions are perpendicular to the planes. The normal reaction at P is perpendicular to plane Π_1, which is inclined at angle α to the horizontal, so N_p makes an angle α with the vertical. Similarly, N_q makes an angle β with the vertical.

Resolving forces horizontally and vertically:

- Horizontal equilibrium: $N_p\sin\alpha = N_q\sin\beta$ \quad (1)
- Vertical equilibrium: $N_p\cos\alpha + N_q\cos\beta = mg$ \quad (2)

Taking torques about the center of mass C (since the weight acts through C, it contributes no torque). The length of the rod is L. The position vectors relative to C are:

- From C to P: $\left(-\frac{L}{2}\cos\theta, \frac{L}{2}\sin\theta\right)$
- From C to Q: $\left(\frac{L}{2}\cos\theta, -\frac{L}{2}\sin\theta\right)$

The torque due to N_p about C is:

$$\tau_p = \left(-\frac{L}{2}\cos\theta\right)(N_p\cos\alpha) - \left(\frac{L}{2}\sin\theta\right)(N_p\sin\alpha)$$

$$= -\frac{L}{2}N_p(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = -\frac{L}{2}N_p\cos(\theta-\alpha)$$

The torque due to N_q about C is:

$$\tau_q = \left(\frac{L}{2}\cos\theta\right)(N_q\cos\beta) - \left(-\frac{L}{2}\sin\theta\right)(-N_q\sin\beta)$$

$$= \frac{L}{2}N_q(\cos\theta\cos\beta - \sin\theta\sin\beta) = \frac{L}{2}N_q\cos(\theta+\beta)$$

For torque equilibrium, $\tau_p + \tau_q = 0$:

$$-\frac{L}{2}N_p \cos(\theta - \alpha) + \frac{L}{2}N_q \cos(\theta + \beta) = 0$$

$$N_p \cos(\theta - \alpha) = N_q \cos(\theta + \beta) \quad (3)$$

Divide equation (3) by equation (1):

$$\frac{N_p \cos(\theta - \alpha)}{N_p \sin\alpha} = \frac{N_q \cos(\theta + \beta)}{N_q \sin\beta}$$

$$\frac{\cos(\theta - \alpha)}{\sin\alpha} = \frac{\cos(\theta + \beta)}{\sin\beta}$$

Cross-multiplying:

$$\sin\beta \cos(\theta - \alpha) = \sin\alpha \cos(\theta + \beta)$$

Expanding the trigonometric functions:

$$\sin\beta(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = \sin\alpha(\cos\theta\cos\beta - \sin\theta\sin\beta)$$

$$\sin\beta\cos\theta\cos\alpha + \sin\beta\sin\theta\sin\alpha = \sin\alpha\cos\theta\cos\beta - \sin\alpha\sin\theta\sin\beta$$

Rearranging terms:

$$\sin\beta\cos\theta\cos\alpha - \sin\alpha\cos\theta\cos\beta + \sin\beta\sin\theta\sin\alpha + \sin\alpha\sin\theta\sin\beta = 0$$

$$\cos\theta(\sin\beta\cos\alpha - \sin\alpha\cos\beta) + \sin\theta(2\sin\alpha\sin\beta) = 0$$

Since $\sin\beta\cos\alpha - \sin\alpha\cos\beta = \sin(\beta - \alpha) = -\sin(\alpha - \beta)$:

$$\cos\theta[-\sin(\alpha - \beta)] + \sin\theta(2\sin\alpha\sin\beta) = 0$$

$$-\cos\theta\sin(\alpha - \beta) + 2\sin\alpha\sin\beta\sin\theta = 0$$

$$2\sin\alpha\sin\beta\sin\theta = \cos\theta\sin(\alpha - \beta)$$

Dividing both sides by $\cos\theta$ (assuming $\cos\theta \neq 0$):

$$2\sin\alpha\sin\beta\tan\theta = \sin(\alpha - \beta)$$

Dividing both sides by $\sin\alpha\sin\beta$ (since $0 < \beta < \alpha < \pi/2$, $\sin\alpha \neq 0$, $\sin\beta \neq 0$):

$$2\tan\theta = \frac{\sin(\alpha - \beta)}{\sin\alpha\sin\beta}$$

Expressing the right-hand side:

$$\frac{\sin(\alpha - \beta)}{\sin\alpha\sin\beta} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha\sin\beta} = \frac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta} - \frac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta} = \cot\beta - \cot\alpha$$

Thus:

$$2\tan\theta = \cot\beta - \cot\alpha$$

Q8

A light elastic spring AB, of natural length a and modulus of elasticity kmg, hangs vertically with one end A attached to a fixed point. A particle of mass m is attached to the other end B. The particle is held at rest so that $AB > a$ and is released.

Find the equation of motion of the particle and deduce that the particle oscillates vertically.

If the period of oscillation is $\frac{2\pi}{\Omega}$, show that $kg = a\Omega^2$.

Suppose instead that the particle, still attached to B, lies on a horizontal platform which performs simple harmonic motion vertically with amplitude b and period $\frac{2\pi}{\omega}$. At the lowest point of its oscillation, the platform is a distance h below A. Let x be the distance of the particle above the lowest point of the oscillation of the platform.

When the particle is in contact with the platform, show that the upward force on the particle from the platform is

$$mg + m\Omega^2(a + x - h) + m\omega^2(b - x).$$

Given that $\omega < \Omega$, show that, if the particle remains in contact with the platform throughout its motion,

$$h \leq a\left(1 + \frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2}.$$

Find the corresponding inequality if $\omega > \Omega$.

Hence show that, if the particle remains in contact with the platform throughout its motion, it is necessary that

$$h \leq a\left(1 + \frac{1}{k}\right) + b,$$

whatever the value of ω.

Solution:

The equation of motion for the particle is derived from Newton's second law. Let y be the distance from the fixed point A to the particle at B, measured vertically downward. The forces acting on the particle are:

- Its weight, mg, acting downward.
- The tension in the spring, which acts upward. The tension T is given by $T = \frac{\lambda}{l}x$, where $\lambda = kmg$ is the modulus of elasticity, $l = a$ is the natural length, and $x = y - a$ is the extension. Thus, $T = \frac{kmg}{a}(y - a)$.

Taking downward as positive, the net force on the particle is:

$$F = mg - T = mg - \frac{kmg}{a}(y - a)$$

By Newton's second law:

$$m\frac{d^2y}{dt^2} = mg - \frac{kmg}{a}(y - a)$$

Dividing through by m:

$$\frac{d^2y}{dt^2} = g - \frac{kg}{a}(y - a)$$

This is the equation of motion.

To deduce that the particle oscillates, shift the coordinate to the equilibrium position. At equilibrium, the net force is zero, so:

$$mg = \frac{kmg}{a}(y_{\text{eq}} - a)$$

Solving for y_{eq}:

$$1 = \frac{k}{a}(y_{\text{eq}} - a) \implies y_{\text{eq}} - a = \frac{a}{k} \implies y_{\text{eq}} = a\left(1 + \frac{1}{k}\right)$$

Define $z = y - y_{\text{eq}}$, the displacement from equilibrium. Then:

$$y - a = (y - y_{\text{eq}}) + (y_{\text{eq}} - a) = z + \frac{a}{k}$$

Substitute into the equation of motion:

$$\frac{d^2y}{dt^2} = g - \frac{kg}{a}\left(z + \frac{a}{k}\right) = g - \frac{kg}{a}z - \frac{kg}{a} \cdot \frac{a}{k} = g - \frac{kg}{a}z - g = -\frac{kg}{a}z$$

Since $\frac{d^2y}{dt^2} = \frac{d^2z}{dt^2}$:

$$\frac{d^2z}{dt^2} = -\frac{kg}{a}z$$

Rewriting:

$$\frac{d^2z}{dt^2} + \frac{kg}{a}z = 0$$

This is the equation for simple harmonic motion with angular frequency $\omega = \sqrt{\frac{kg}{a}}$. Therefore, the particle oscillates vertically about the equilibrium position $y_{\text{eq}} = a\left(1 + \frac{1}{k}\right)$.

The period of oscillation is given as $T = \frac{2\pi}{\Omega}$.

From the equation of motion derived earlier, the angular frequency of oscillation is $\omega = \sqrt{\frac{kg}{a}}$.

The period T is related to the angular frequency by $T = \frac{2\pi}{\omega}$. Substituting the given period:

$$\frac{2\pi}{\Omega} = \frac{2\pi}{\omega}.$$

Thus,

$$\Omega = \omega = \sqrt{\frac{kg}{a}}.$$

Squaring both sides:

$$\Omega^2 = \frac{kg}{a}.$$

Rearranging gives:

$$kg = a\Omega^2.$$

Alternatively, using the expression for the period:

$$T = 2\pi\sqrt{\frac{a}{kg}}.$$

Given $T = \frac{2\pi}{\Omega}$:

$$\frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{a}{kg}}.$$

Dividing both sides by 2π:

$$\frac{1}{\Omega} = \sqrt{\frac{a}{kg}}.$$

Squaring both sides:

$$\frac{1}{\Omega^2} = \frac{a}{kg}.$$

Rearranging:

$$kg = a\Omega^2.$$

The particle is attached to the spring at end B, and the spring is fixed at end A. The platform performs simple harmonic motion vertically with amplitude b and angular frequency ω, so its period is $\frac{2\pi}{\omega}$. At the lowest point of its oscillation, the platform is a distance h below A. The distance x is defined as the height of the particle above the lowest point of the platform's oscillation, so x is measured upward from this point.

When the particle is in contact with the platform, it moves with the platform, so its position is given by the platform's position. The position of the platform is $x_p = b(1 + \sin(\omega t))$, and its acceleration is $\frac{d^2x_p}{dt^2} = -\omega^2(x_p - b)$. Since the particle is in contact, $x = x_p$ and $\frac{d^2x}{dt^2} = -\omega^2(x - b)$.

The forces acting on the particle in the upward direction are:

- The tension in the spring, T.
- The normal force from the platform, N.
  The downward force is the weight, mg.

The spring has natural length a and modulus of elasticity kmg. The position of A is at height h above the lowest point (since A is fixed and the lowest point is h below A). The position of the particle is at height x above the lowest point. The length of the spring is the distance from A to the particle, which is $h - x$ (since A is above the particle). The extension of the spring is $(h - x) - a = h - x - a$. The tension is given by:

$$T = \frac{\lambda}{l} \times \text{extension} = \frac{kmg}{a}(h - x - a).$$

The tension acts upward on the particle.

Applying Newton's second law in the upward direction:

$$m\frac{d^2x}{dt^2} = T + N - mg.$$

Substituting the tension and acceleration:

$$m[-\omega^2(x - b)] = \frac{kmg}{a}(h - x - a) + N - mg.$$

Solving for N:

$$N = m[-\omega^2(x - b)] - \frac{kmg}{a}(h - x - a) + mg.$$

Simplifying:

$$N = -m\omega^2(x - b) + mg - \frac{kmg}{a}(h - x - a).$$

Rewriting $-(x - b) = b - x$:

$$N = m\omega^2(b - x) + mg - \frac{kmg}{a}(h - x - a).$$

Note that $h - x - a = -(a + x - h)$, so:

$$N = m\omega^2(b - x) + mg - \frac{kmg}{a}[-(a + x - h)] = m\omega^2(b - x) + mg + \frac{kmg}{a}(a + x - h).$$

From the earlier result, $kg = a\Omega^2$, so $\frac{kg}{a} = \Omega^2$. Thus:

$$\frac{kmg}{a} = m \cdot \frac{kg}{a} = m\Omega^2.$$

Substituting:

$$N = mg + m\Omega^2(a + x - h) + m\omega^2(b - x).$$

This is the upward force on the particle from the platform when the particle is in contact with the platform.

To determine the conditions under which the particle remains in contact with the platform throughout its motion, the normal force N exerted by the platform on the particle must be non-negative for all positions x of the platform, where x is the distance of the particle above the lowest point of the platform's oscillation. The expression for N is given by:

$$N = mg + m\Omega^2(a + x - h) + m\omega^2(b - x).$$

For the particle to remain in contact, $N \geq 0$ for all x in the range $[0, 2b]$, since the platform oscillates with amplitude b, and x ranges from 0 (lowest point) to $2b$ (highest point).

The expression for N/m is linear in x:

$$\frac{N}{m} = g + \Omega^2 a - \Omega^2 h + \omega^2 b + x(\Omega^2 - \omega^2).$$

The behavior of this linear function depends on the sign of the coefficient of x, $\Omega^2 - \omega^2$.

Case 1: $\omega < \Omega$
When $\omega < \Omega$, $\omega^2 < \Omega^2$, so $\Omega^2 - \omega^2 > 0$. Thus, the function is increasing in x, and its minimum occurs at $x = 0$. Setting $\frac{N}{m} \geq 0$ at $x = 0$:

$$g + \Omega^2 a - \Omega^2 h + \omega^2 b \geq 0.$$

Solving for h:

$$\Omega^2 h \leq g + \Omega^2 a + \omega^2 b,$$

$$h \leq \frac{g}{\Omega^2} + a + \frac{\omega^2 b}{\Omega^2}.$$

Given that $kg = a\Omega^2$, it follows that $\frac{g}{\Omega^2} = \frac{a}{k}$. Substituting:

$$h \leq \frac{a}{k} + a + \frac{\omega^2 b}{\Omega^2} = a\left(1 + \frac{1}{k}\right) + \frac{\omega^2 b}{\Omega^2}.$$

This confirms the given inequality for $\omega < \Omega$.

Case 2: $\omega > \Omega$
When $\omega > \Omega$, $\omega^2 > \Omega^2$, so $\Omega^2 - \omega^2 < 0$. Thus, the function is decreasing in x, and its minimum occurs at $x = 2b$. Setting $\frac{N}{m} \geq 0$ at $x = 2b$:

$$g + \Omega^2 a - \Omega^2 h + \omega^2 b + 2b(\Omega^2 - \omega^2) \geq 0.$$

Simplifying:

$$g + \Omega^2 a - \Omega^2 h + \omega^2 b + 2b\Omega^2 - 2b\omega^2 \geq 0,$$

$$g + \Omega^2 a - \Omega^2 h + 2b\Omega^2 - b\omega^2 \geq 0.$$

Solving for h:

$$\Omega^2 h \leq g + \Omega^2 a + 2b\Omega^2 - b\omega^2,$$

$$h \leq \frac{g}{\Omega^2} + a + 2b - \frac{b\omega^2}{\Omega^2}.$$

Substituting $\frac{g}{\Omega^2} = \frac{a}{k}$:

$$h \leq \frac{a}{k} + a + 2b - \frac{b\omega^2}{\Omega^2} = a\left(1 + \frac{1}{k}\right) + 2b - \frac{b\omega^2}{\Omega^2}.$$

To show that if the particle remains in contact with the platform throughout its motion, it is necessary that $ h \leq a\left(1 + \frac{1}{k}\right) + b , $regardlessofthevalueof\$\omega$, consider the normal force N exerted by the platform on the particle when in contact. From the previous result, this force is given by:

$$N = mg + m\Omega^2(a + x - h) + m\omega^2(b - x),$$

where x is the distance of the particle above the lowest point of the platform's oscillation, ranging from 0 to $2b$.

The condition for the particle to remain in contact with the platform is that $N \geq 0$ for all $x \in [0, 2b]$. This normal force is linear in x, and its minimum value over the interval $[0, 2b]$ depends on the sign of the coefficient of x, which is $m(\Omega^2 - \omega^2)$:

- If $\omega < \Omega$, the minimum occurs at $x = 0$, and $N(0) = m[g + \Omega^2(a - h) + \omega^2 b]$.
- If $\omega > \Omega$, the minimum occurs at $x = 2b$, and $N(2b) = m[g + \Omega^2(a + 2b - h) - \omega^2 b]$.
- If $\omega = \Omega$, N is constant: $N = m[g + \Omega^2(a + b - h)]$.

Given that $kg = a\Omega^2$, it follows that $g = \frac{a\Omega^2}{k}$. Substituting this into the expressions and using $a\left(1 + \frac{1}{k}\right) = a + \frac{a}{k}$, the minimum values are:

- For $\omega < \Omega$: $N(0) = mb(\omega^2 - \Omega^2)$.
- For $\omega > \Omega$: $N(2b) = mb(\Omega^2 - \omega^2)$.
- For $\omega = \Omega$: $N = m\Omega^2\left[a\left(1 + \frac{1}{k}\right) + b - h\right]$.

Now, suppose $h > a\left(1 + \frac{1}{k}\right) + b$. It will be shown that for all ω, the minimum of N over x is negative, implying that contact is lost at some point for every ω.

- **Case $\omega < \Omega$:**
  $N(0) = mb(\omega^2 - \Omega^2)$. Since $\omega < \Omega$, $\omega^2 < \Omega^2$, so $\omega^2 - \Omega^2 < 0$. Thus, $N(0) < 0$.
- **Case $\omega > \Omega$:**
  $N(2b) = mb(\Omega^2 - \omega^2)$. Since $\omega > \Omega$, $\omega^2 > \Omega^2$, so $\Omega^2 - \omega^2 < 0$. Thus, $N(2b) < 0$.
- **Case $\omega = \Omega$:**
  $N = m\Omega^2\left[a\left(1 + \frac{1}{k}\right) + b - h\right]$. Since $h > a\left(1 + \frac{1}{k}\right) + b$, the expression in the brackets is negative. Thus, $N < 0$.

In all cases, the minimum normal force is negative when $h > a\left(1 + \frac{1}{k}\right) + b$. Therefore, if the particle remains in contact with the platform throughout the motion for any ω, it must be that

$h \le a\left(1 + \frac{1}{k}\right) + b.$

This condition is necessary for contact to be maintained for some ω, but it is not sufficient, as additional conditions depending on ω may be required to ensure $N \ge 0$ for all x.

Q9

In this question, $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors and $\mathbf{j}$ is vertically upwards.

A smooth hemisphere of mass M and radius a rests on a smooth horizontal table with its plane face in contact with the table. The point A is at the top of the hemisphere and the point O is at the centre of its plane face.

Initially, a particle P of mass m rests at A. It is then given a small displacement in the positive $\mathbf{i}$ direction. At a later time t, when the particle is still in contact with the hemisphere, the hemisphere has been displaced by $-s\mathbf{i}$ and $\angle AOP = \theta$.

- Let $\mathbf{r}$ be the position vector of the particle at time t with respect to the initial position of O. Write down an expression for $\mathbf{r}$ in terms of a, θ and s and show that

$$\dot{\mathbf{r}} = (a\dot{\theta}\cos\theta - \dot{s})\mathbf{i} - a\dot{\theta}\sin\theta\mathbf{j}.$$

  Show also that

$$\dot{s} = (1-k)a\dot{\theta}\cos\theta,$$

  where $k = \frac{M}{m+M}$, and deduce that

$$\dot{\mathbf{r}} = a\dot{\theta}(k\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j}).$$

- Show that

$$a\dot{\theta}^2\left(k\cos^2\theta + \sin^2\theta\right) = 2g(1-\cos\theta).$$

- At time T, when $\theta = \alpha$, the particle leaves the hemisphere. By considering the component of $\dot{\mathbf{r}}$ parallel to the vector $\sin\theta\,\mathbf{i} + k\cos\theta\,\mathbf{j}$, or otherwise, show that at time T

$$a\dot{\theta}^2 = g\cos\alpha.$$

  Find a cubic equation for $\cos\alpha$ and deduce that $\cos\alpha > \frac{2}{3}$.

Solution:

The position vector $\mathbf{r}$ of the particle P with respect to the initial position of O is derived as follows.

Initially, O is at the origin $(0,0)$, and A is at $(0,a)$. The hemisphere is displaced by $-s\mathbf{i}$, so the current position of O is $(-s,0)$. The angle $\angle AOP = \theta$ defines the position of P on the hemisphere. Since the hemisphere is smooth and symmetric, and the force on it passes through O, it translates without rotating, so the orientation remains fixed with OA vertical.

Relative to the current position of O, the coordinates of P are given by the geometry of the hemisphere. The vector $\overrightarrow{OP}$ has magnitude a, and the angle θ between $\overrightarrow{OA}$ and $\overrightarrow{OP}$ satisfies:

$$z = a\cos\theta, \quad x^2 + y^2 = a^2\sin^2\theta.$$

Given the displacement in the $\mathbf{i}$-direction and symmetry, the motion is confined to the xz-plane, so $y = 0$. Thus, $x = a\sin\theta$ (taking the positive x-direction for motion in the positive $\mathbf{i}$-direction). Therefore, relative to the current O, P is at $(a\sin\theta, a\cos\theta)$.

The position vector $\mathbf{r}$ of P with respect to the initial position of O is the sum of the position vector of the current O and the vector from the current O to P:

$$\mathbf{r} = (-s\mathbf{i} + 0\mathbf{j}) + (a\sin\theta\mathbf{i} + a\cos\theta\mathbf{j}) = (a\sin\theta - s)\mathbf{i} + a\cos\theta\mathbf{j}.$$

To find the velocity $\dot{\mathbf{r}}$, differentiate $\mathbf{r}$ with respect to time:

$$\dot{\mathbf{r}} = \frac{d}{dt}[(a\sin\theta - s)\mathbf{i} + a\cos\theta\mathbf{j}].$$

Since $\mathbf{i}$ and $\mathbf{j}$ are constant vectors,

$$\dot{\mathbf{r}} = \left[\frac{d}{dt}(a\sin\theta - s)\right]\mathbf{i} + \left[\frac{d}{dt}(a\cos\theta)\right]\mathbf{j}.$$

Differentiating each component:

$$\frac{d}{dt}(a\sin\theta - s) = a\cos\theta \cdot \dot{\theta} - \dot{s}, \quad \frac{d}{dt}(a\cos\theta) = -a\sin\theta \cdot \dot{\theta}.$$

Thus,

$$\dot{\mathbf{r}} = (a\dot{\theta}\cos\theta - \dot{s})\mathbf{i} - a\dot{\theta}\sin\theta\mathbf{j}.$$

The position vector of the particle P with respect to the initial position of O is given by:

$$\mathbf{r} = (a\sin\theta - s)\mathbf{i} + a\cos\theta\mathbf{j}.$$

Differentiating with respect to time gives:

$$\dot{\mathbf{r}} = \frac{d}{dt}[(a\sin\theta - s)\mathbf{i} + a\cos\theta\mathbf{j}] = \left(a\dot{\theta}\cos\theta - \dot{s}\right)\mathbf{i} - a\dot{\theta}\sin\theta\mathbf{j}.$$

Since the table is smooth, there are no external horizontal forces acting on the system. Therefore, the horizontal component of the center of mass remains constant. Initially, the center of mass is at rest with horizontal coordinate 0, so it remains 0 for all time.

The center of mass of the hemisphere is at a height $\frac{3a}{8}$ above O along the vertical axis. Initially, O is at $(0, 0)$, the center of mass of the hemisphere is at $(0, \frac{3a}{8})$, and the particle is at $(0, a)$. At time t, O has moved to $(-s, 0)$, so the center of mass of the hemisphere is at $(-s, \frac{3a}{8})$. The particle is at $(-s + a\sin\theta, a\cos\theta)$.

The x-coordinate of the center of mass of the system is:

$$x_{\text{com}} = \frac{M \cdot (-s) + m \cdot (-s + a\sin\theta)}{m+M} = 0.$$

Solving for s:

$$-Ms - ms + ma\sin\theta = 0 \implies -s(M+m) + ma\sin\theta = 0 \implies s = \frac{ma\sin\theta}{m+M}.$$

Differentiating with respect to time:

$$\dot{s} = \frac{ma}{m+M} \cdot \cos\theta \cdot \dot{\theta}.$$

Given $k = \frac{M}{m+M}$, it follows that $1-k = \frac{m}{m+M}$. Thus:

$$\dot{s} = (1-k)a\dot{\theta}\cos\theta.$$

Substitute $\dot{s}$ into the expression for $\dot{\mathbf{r}}$:

$$\dot{\mathbf{r}} = \left(a\dot{\theta}\cos\theta - (1-k)a\dot{\theta}\cos\theta\right)\mathbf{i} - a\dot{\theta}\sin\theta\mathbf{j} = a\dot{\theta}\cos\theta\,[1-(1-k)]\mathbf{i} - a\dot{\theta}\sin\theta\mathbf{j}.$$

Simplifying the coefficient of $\mathbf{i}$:

$$1-(1-k) = k,$$

so:

$$\dot{\mathbf{r}} = a\dot{\theta}(k\cos\theta\mathbf{i} - \sin\theta\mathbf{j}).$$

The total energy of the system is conserved as the table and hemisphere are smooth, and there are no dissipative forces.

Initially, the particle is at rest at point A, and the hemisphere is at rest. The initial kinetic energy is zero. The initial potential energy is set with respect to the horizontal table. The center of mass of the hemisphere is at height $\frac{3a}{8}$ above the table, and the particle is at height a above the table. Thus, the initial potential energy is:

$$Mg \cdot \frac{3a}{8} + mg \cdot a = \frac{3Mga}{8} + mga.$$

At time t, with $\angle AOP = \theta$, the position of the particle is given by $\mathbf{r} = (a\sin\theta - s)\mathbf{i} + a\cos\theta\mathbf{j}$, so its height is $a\cos\theta$. The center of mass of the hemisphere is at $(-s, \frac{3a}{8})$ since it translates horizontally without rotation. The potential energy at time t is:

$$Mg \cdot \frac{3a}{8} + mg \cdot a\cos\theta.$$

The change in potential energy from initial to time t is:

$$\left(Mg \cdot \frac{3a}{8} + mga\cos\theta\right) - \left(Mg \cdot \frac{3a}{8} + mga\right) = mga(\cos\theta - 1) = -mga(1-\cos\theta).$$

Since the initial kinetic energy is zero, the kinetic energy at time t equals the decrease in potential energy:

$$\text{Total KE} = mga(1 - \cos\theta).$$

The kinetic energy consists of the kinetic energies of the hemisphere and the particle. The hemisphere translates horizontally with velocity $-\dot{s}\mathbf{i}$, so its speed is $|\dot{s}|$, and its kinetic energy is:

$$\frac{1}{2}M(\dot{s})^2.$$

From earlier, $\dot{s} = (1-k)a\dot{\theta}\cos\theta$ where $k = \frac{M}{m+M}$, so:

$$(\dot{s})^2 = \Big((1-k)a\dot{\theta}\cos\theta\Big)^2 = (1-k)^2 a^2\dot{\theta}^2\cos^2\theta.$$

Thus, the kinetic energy of the hemisphere is:

$$\frac{1}{2}M(1-k)^2a^2\dot{\theta}^2\cos^2\theta.$$

The velocity of the particle is $\dot{\mathbf{r}} = a\dot{\theta}(k\cos\theta\mathbf{i} - \sin\theta\mathbf{j})$, so its speed squared is:

$$|\dot{\mathbf{r}}|^2 = \Big(a\dot{\theta}k\cos\theta\Big)^2 + \Big(-a\dot{\theta}\sin\theta\Big)^2 = a^2\dot{\theta}^2(k^2\cos^2\theta + \sin^2\theta).$$

Thus, the kinetic energy of the particle is:

$$\frac{1}{2}ma^2\dot{\theta}^2(k^2\cos^2\theta + \sin^2\theta).$$

The total kinetic energy is the sum:

$$\frac{1}{2}ma^2\dot{\theta}^2(k^2\cos^2\theta + \sin^2\theta) + \frac{1}{2}M(1-k)^2a^2\dot{\theta}^2\cos^2\theta.$$

Factor out $\frac{1}{2}a^2\dot{\theta}^2$:

$$\text{Total KE} = \frac{1}{2}a^2\dot{\theta}^2\left[m(k^2\cos^2\theta + \sin^2\theta) + M(1-k)^2\cos^2\theta\right].$$

Substitute $k = \frac{M}{m+M}$ and $1 - k = \frac{m}{m+M}$:

$$k^2 = \left(\frac{M}{m+M}\right)^2, \quad (1-k)^2 = \left(\frac{m}{m+M}\right)^2.$$

The expression inside the brackets is:

$$m\left(\frac{M^2}{(m+M)^2}\right)\cos^2\theta + m\sin^2\theta + M\left(\frac{m^2}{(m+M)^2}\right)\cos^2\theta$$
$$= \frac{mM^2\cos^2\theta}{(m+M)^2} + m\sin^2\theta + \frac{m^2M\cos^2\theta}{(m+M)^2}.$$

Combine the $\cos^2\theta$ terms:

$$\frac{mM^2 + m^2M}{(m+M)^2}\cos^2\theta + m\sin^2\theta = \frac{mM(M+m)}{(m+M)^2}\cos^2\theta + m\sin^2\theta = \frac{mM}{m+M}\cos^2\theta + m\sin^2\theta.$$

Since $\frac{M}{m+M} = k$, this simplifies to:

$$m\left(k\cos^2\theta + \sin^2\theta\right).$$

Thus, the total kinetic energy is:

$$\text{Total KE} = \frac{1}{2}a^2\dot{\theta}^2 \cdot m(k\cos^2\theta + \sin^2\theta).$$

Set this equal to the decrease in potential energy:

$$\frac{1}{2}a^2\dot{\theta}^2 m(k\cos^2\theta + \sin^2\theta) = mga(1-\cos\theta).$$

Divide both sides by m (assuming $m \neq 0$):

$$\frac{1}{2}a^2\dot{\theta}^2(k\cos^2\theta + \sin^2\theta) = ga(1-\cos\theta).$$

Multiply both sides by 2 and divide by a (assuming $a \neq 0$):

$$a\dot{\theta}^2(k\cos^2\theta + \sin^2\theta) = 2g(1-\cos\theta).$$

This completes the derivation.

At time T, when $\theta = \alpha$, the particle leaves the hemisphere. The condition for the particle to lose contact with the hemisphere is that the normal reaction force R becomes zero. The acceleration of the particle $\ddot{\mathbf{r}}$ and the vector $\mathbf{d} = \sin\theta\mathbf{i} + k\cos\theta\mathbf{j}$ satisfy:

$$\ddot{\mathbf{r}} \cdot \mathbf{d} = -ak\dot{\theta}^2,$$

and from Newton's second law, with $R = 0$,

$$\ddot{\mathbf{r}} \cdot \mathbf{d} = -gk\cos\theta.$$

Equating these expressions gives:

$$-ak\dot{\theta}^2 = -gk\cos\theta.$$

Dividing by $-k$ (since $k > 0$) yields:

$$a\dot{\theta}^2 = g\cos\theta.$$

At $\theta = \alpha$,

$$a\dot{\theta}^2 = g\cos\alpha.$$

From the energy conservation equation derived earlier:

$$a\dot{\theta}^2(k\cos^2\theta + \sin^2\theta) = 2g(1-\cos\theta),$$

substitute $a\dot{\theta}^2 = g\cos\alpha$ at $\theta = \alpha$:

$$(g\cos\alpha)(k\cos^2\alpha+\sin^2\alpha)=2g(1-\cos\alpha).$$

Dividing by g (assuming $g\neq 0$):

$$\cos\alpha(k\cos^2\alpha+\sin^2\alpha)=2(1-\cos\alpha).$$

Substitute $\sin^2\alpha=1-\cos^2\alpha$:

$$\cos\alpha[k\cos^2\alpha+1-\cos^2\alpha]=2(1-\cos\alpha).$$

Let $c=\cos\alpha$:

$$c[1+(k-1)c^2]=2(1-c).$$

Expanding and rearranging terms:

$$c+(k-1)c^3=2-2c,$$

$$(k-1)c^3+3c-2=0.$$

This is the cubic equation for $\cos\alpha$.

To deduce $\cos\alpha>\frac{2}{3}$, consider the cubic function $g(c)=(k-1)c^3+3c-2$. Since $k=\frac{M}{m+M}<1$, $k-1<0$. The derivative is:

$$g'(c)=3(k-1)c^2+3=3[(k-1)c^2+1].$$

For $c\in[0,1]$, $(k-1)c^2+1>0$ because the minimum occurs at $c=0$ where $g'(0)=3>0$, and $c=\frac{1}{\sqrt{1-k}}>1$ for $c\leq 1$. Thus, $g'(c)>0$ in $[0,1]$, so $g(c)$ is strictly increasing.

Evaluate at $c=\frac{2}{3}$:

$$g\left(\frac{2}{3}\right)=(k-1)\left(\frac{2}{3}\right)^3+3\left(\frac{2}{3}\right)-2=(k-1)\frac{8}{27}+2-2=(k-1)\frac{8}{27}<0,$$

since $k-1<0$. At $c=1$:

$$g(1)=(k-1)(1)^3+3(1)-2=k-1+3-2=k>0.$$

Since $g(c)$ is continuous and strictly increasing, $g\left(\frac{2}{3}\right)<0$ and $g(1)>0$, the root $c=\cos\alpha$ satisfies $\frac{2}{3}<\cos\alpha<1$. Thus, $\cos\alpha>\frac{2}{3}$.

$$\boxed{a\dot{\theta}^2=g\cos\alpha}$$

$$\boxed{(k-1)\cos^3\alpha+3\cos\alpha-2=0}$$

$$\boxed{\cos\alpha>\frac{2}{3}}$$

Q10

Two identical smooth spheres P and Q can move on a smooth horizontal table. Initially, P moves with speed u and Q is at rest. Then P collides with Q. The direction of travel of P before the collision makes an acute angle α with the line joining the centres of P and Q at the moment of the collision. The coefficient of restitution between P and Q is e where $e < 1$.

As a result of the collision, P has speed v and Q has speed w, and P is deflected through an angle θ.

- Show that

$$u \sin \alpha = v \sin(\alpha + \theta)$$

  and find an expression for w in terms of v, θ and α.
- Show further that

$$\sin \theta = \cos(\theta + \alpha) \sin \alpha + e \sin(\theta + \alpha) \cos \alpha$$

  and find an expression for $\tan \theta$ in terms of $\tan \alpha$ and e.

  Find, in terms of e, the maximum value of $\tan \theta$ as α varies.

Solution:

To solve the problem, consider the collision dynamics of the two identical smooth spheres P and Q on a smooth horizontal table. Since the spheres are smooth, there are no tangential forces, and the component of velocity perpendicular to the line joining the centers at the moment of collision is conserved for each sphere. Define a coordinate system where the x-axis is along the line joining the centers (from P to Q) and the y-axis is perpendicular to this line.

Velocity components before collision

- Sphere P has initial speed u at an acute angle α to the line of centers (x-axis).
- Thus, the initial velocity components of P are:

$$u_{Px} = u \cos \alpha, \quad u_{Py} = u \sin \alpha$$

- Sphere Q is initially at rest:

$$u_{Qx} = 0, \quad u_{Qy} = 0$$

Velocity components after collision

- After the collision, P has speed v and is deflected through an angle θ. The deflection angle θ is the angle between the initial and final velocity vectors of P. Since the perpendicular component is conserved, $v_{Py} = u_{Py} = u \sin \alpha$.
- The final direction of P makes an angle $\alpha + \theta$ with the x-axis (as θ is defined such that the deflection increases the angle from the line of centers). Thus, the components of the final velocity of P are:

$$v_{Px} = v\cos(\alpha+\theta), \quad v_{Py} = v\sin(\alpha+\theta)$$

- Since $v_{Py} = u\sin\alpha$, equate the expressions:

$$v\sin(\alpha+\theta) = u\sin\alpha$$

  Rearranging gives the required relation:

$$u\sin\alpha = v\sin(\alpha+\theta)$$

Expression for w

- Sphere Q is initially at rest, and since the perpendicular component is conserved, $w_{Qy} = 0$. After the collision, Q has speed w, so its velocity has only an x-component: $w_{Qx} = w$ (assuming $w > 0$ as Q moves along the line of centers).
- Conservation of momentum along the x-axis (masses are identical, so mass cancels):

$$u\cos\alpha = v_{Px} + w_{Qx}$$

  Substitute $v_{Px} = v\cos(\alpha+\theta)$ and $w_{Qx} = w$:

$$u\cos\alpha = v\cos(\alpha+\theta) + w$$

  Solving for w:

$$w = u\cos\alpha - v\cos(\alpha+\theta)$$

- From the relation $u\sin\alpha = v\sin(\alpha+\theta)$, solve for u:

$$u = v\frac{\sin(\alpha+\theta)}{\sin\alpha}$$

- Substitute this into the expression for w:

$$w = \left(v\frac{\sin(\alpha+\theta)}{\sin\alpha}\right)\cos\alpha - v\cos(\alpha+\theta)$$

  Simplify:

$$w = v\left(\frac{\sin(\alpha+\theta)\cos\alpha}{\sin\alpha} - \cos(\alpha+\theta)\right)$$

$$w = v\left(\frac{\sin(\alpha+\theta)\cos\alpha - \cos(\alpha+\theta)\sin\alpha}{\sin\alpha}\right)$$

- The numerator is $\sin((\alpha+\theta)-\alpha) = \sin\theta$ by the angle subtraction formula. Thus:

$$w = v\frac{\sin\theta}{\sin\alpha}$$

- Since α is acute, $\sin\alpha > 0$, and θ is a deflection angle (assumed positive), $\sin\theta > 0$, so the speed w is:

$$w = v\frac{\sin\theta}{\sin\alpha}$$

The relation $u\sin\alpha = v\sin(\alpha+\theta)$ is derived from the conservation of the perpendicular component of velocity. The speed of Q after the collision is:

The first relation, $u\sin\alpha = v\sin(\alpha+\theta)$, is derived from the conservation of the perpendicular component of velocity. Since the spheres are smooth and the table is smooth, there are no forces perpendicular to the line joining the centers at the moment of collision. Thus, the perpendicular component of velocity for sphere P is conserved. Initially, the perpendicular component is $u\sin\alpha$, and after the collision, it is $v\sin(\alpha+\theta)$. Equating these gives the required relation.

The expression for w in terms of v, θ, and α is found using conservation of momentum along the line of centers. The initial momentum in the x-direction (along the line of centers) is $u\cos\alpha$ (since masses are identical and cancel out). After the collision, the x-components of velocity for P and Q are $v\cos(\alpha+\theta)$ and w respectively, so:

$$u\cos\alpha = v\cos(\alpha+\theta) + w.$$

Solving for w:

$$w = u\cos\alpha - v\cos(\alpha+\theta).$$

Using the relation $u\sin\alpha = v\sin(\alpha+\theta)$, express u as:

$$u = v\frac{\sin(\alpha+\theta)}{\sin\alpha}.$$

Substitute into the expression for w:

$$w = \left(v\frac{\sin(\alpha+\theta)}{\sin\alpha}\right)\cos\alpha - v\cos(\alpha+\theta) = v\left(\frac{\sin(\alpha+\theta)\cos\alpha - \cos(\alpha+\theta)\sin\alpha}{\sin\alpha}\right).$$

The numerator simplifies to $\sin\theta$ using the angle subtraction formula, so:

$$w = v\frac{\sin\theta}{\sin\alpha}.$$

To show the further relation $\sin\theta = \cos(\theta+\alpha)\sin\alpha + e\sin(\theta+\alpha)\cos\alpha$, use the coefficient of restitution along the line of centers. The coefficient of restitution e is given by:

$$e = -\frac{v_{qx} - v_{px}}{u_{qx} - u_{px}},$$

where the x-axis is along the line of centers. Before the collision, $u_{px} = u\cos\alpha$, $u_{qx} = 0$, so:

$$u_{qx} - u_{px} = -u\cos\alpha.$$

After the collision, $v_{px} = v\cos(\alpha+\theta)$, $v_{qx} = w$, so:

$$v_{qx} - v_{px} = w - v\cos(\alpha+\theta).$$

Thus:

$$e = -\frac{w - v\cos(\alpha+\theta)}{-u\cos\alpha} = \frac{w - v\cos(\alpha+\theta)}{u\cos\alpha}.$$

Substitute $w = v\frac{\sin\theta}{\sin\alpha}$ and $u = v\frac{\sin(\alpha+\theta)}{\sin\alpha}$:

$$e = \frac{v\frac{\sin\theta}{\sin\alpha} - v\cos(\alpha+\theta)}{v\frac{\sin(\alpha+\theta)}{\sin\alpha}\cos\alpha} = \frac{\sin\theta - \sin\alpha\cos(\alpha+\theta)}{\sin(\alpha+\theta)\cos\alpha}.$$

Rearranging gives:

$$e\sin(\alpha+\theta)\cos\alpha = \sin\theta - \sin\alpha\cos(\alpha+\theta),$$

so:

$$\sin\theta = \sin\alpha\cos(\alpha+\theta) + e\sin(\alpha+\theta)\cos\alpha,$$

which is the required relation.

To find $\tan\theta$ in terms of $\tan\alpha$ and e, expand the right-hand side of the relation:

$$\sin\theta = \sin\alpha\cos(\alpha+\theta) + e\sin(\alpha+\theta)\cos\alpha.$$

Substitute $\cos(\alpha+\theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta$ and $\sin(\alpha+\theta) = \sin\alpha\cos\theta + \cos\alpha\sin\theta$:

$$\sin\theta = \sin\alpha(\cos\alpha\cos\theta - \sin\alpha\sin\theta) + e\cos\alpha(\sin\alpha\cos\theta + \cos\alpha\sin\theta).$$

Distribute:

$$\sin\theta = \sin\alpha\cos\alpha\cos\theta - \sin^2\alpha\sin\theta + e\sin\alpha\cos\alpha\cos\theta + e\cos^2\alpha\sin\theta.$$

Group like terms:

$$\sin\theta = \sin\alpha\cos\alpha(1+e)\cos\theta + \sin\theta(e\cos^2\alpha - \sin^2\alpha).$$

Rearrange:

$$\sin\theta - \sin\theta(e\cos^2\alpha - \sin^2\alpha) = \sin\alpha\cos\alpha(1+e)\cos\theta,$$

$$\sin\theta[1 - e\cos^2\alpha + \sin^2\alpha] = \sin\alpha\cos\alpha(1+e)\cos\theta.$$

Divide both sides by $\cos\theta$:

$$\tan\theta[1 - e\cos^2\alpha + \sin^2\alpha] = \sin\alpha\cos\alpha(1+e).$$

Simplify the expression in brackets using $1 = \sin^2\alpha + \cos^2\alpha$:

$$1 - e\cos^2\alpha + \sin^2\alpha = \sin^2\alpha + \cos^2\alpha + \sin^2\alpha - e\cos^2\alpha = 2\sin^2\alpha + (1-e)\cos^2\alpha.$$

So:

$$\tan\theta[2\sin^2\alpha + (1-e)\cos^2\alpha] = \sin\alpha\cos\alpha(1+e).$$

Divide both sides by $\cos^2\alpha$:

$$\tan\theta\left[2\frac{\sin^2\alpha}{\cos^2\alpha}+(1-e)\right]=\frac{\sin\alpha\cos\alpha(1+e)}{\cos^2\alpha},$$

$$\tan\theta(2\tan^2\alpha+1-e)=\tan\alpha(1+e).$$

Solving for $\tan\theta$:

$$\tan\theta=\frac{(1+e)\tan\alpha}{2\tan^2\alpha+1-e}.$$

The expression for $\tan\theta$ in terms of $\tan\alpha$ and e is given by:

$$\tan\theta=\frac{(1+e)\tan\alpha}{2\tan^2\alpha+1-e}.$$

To find the maximum value of $\tan\theta$ with respect to α, substitute $x=\tan\alpha$, where $x>0$ since α is acute:

$$\tan\theta=\frac{(1+e)x}{2x^2+1-e}.$$

This is a function of x, denoted $f(x)=\frac{kx}{ax^2+b}$ with $k=1+e$, $a=2$, and $b=1-e$. Since $e<1$, $b>0$, ensuring the denominator is always positive.

To maximize $f(x)$, compute the derivative and set it to zero:

$$f'(x)=\frac{d}{dx}\left(\frac{(1+e)x}{2x^2+1-e}\right).$$

Using the quotient rule, where $u=(1+e)x$ and $v=2x^2+1-e$, so $u'=1+e$ and $v'=4x$:

$$f'(x)=\frac{(1+e)(2x^2+1-e)-(1+e)x\cdot 4x}{(2x^2+1-e)^2}=\frac{(1+e)[2x^2+1-e-4x^2]}{(2x^2+1-e)^2}=\frac{(1+e)(1-e-2x^2)}{(2x^2+1-e)^2}.$$

Set $f'(x)=0$:

$$(1+e)(1-e-2x^2)=0.$$

Since $1+e>0$ (as $e\geq 0$ for physical collisions), solve:

$$1-e-2x^2=0\implies 2x^2=1-e\implies x^2=\frac{1-e}{2}\implies x=\sqrt{\frac{1-e}{2}}$$

(considering $x>0$).

This critical point is a maximum because $f(x)\to 0$ as $x\to 0^+$ and as $x\to\infty$, and $f(x)>0$ for $x>0$.

Substitute $x=\sqrt{\frac{1-e}{2}}$ into $f(x)$:

$$\tan\theta=\frac{(1+e)\sqrt{\frac{1-e}{2}}}{2\left(\sqrt{\frac{1-e}{2}}\right)^2+1-e}.$$

Simplify the denominator:

$$\left(\sqrt{\frac{1-e}{2}}\right)^2 = \frac{1-e}{2}, \quad 2 \cdot \frac{1-e}{2} = 1-e,$$

so:

$$\text{Denominator} = 1-e+1-e = 2(1-e).$$

The numerator is:

$$(1+e)\sqrt{\frac{1-e}{2}} = (1+e)\frac{\sqrt{1-e}}{\sqrt{2}}.$$

Thus:

$$\tan\theta = \frac{(1+e)\cdot\frac{\sqrt{1-e}}{\sqrt{2}}}{2(1-e)} = \frac{1+e}{2\sqrt{2}} \cdot \frac{\sqrt{1-e}}{1-e} = \frac{1+e}{2\sqrt{2}\sqrt{1-e}}$$

This can be written as:

$$\tan\theta = \frac{1+e}{2\sqrt{2(1-e)}}.$$

This is the maximum value of $\tan\theta$ in terms of e.

Q11

A particle P of mass m is projected with speed u_0 along a smooth horizontal floor directly towards a wall. It collides with a particle Q of mass km which is moving directly away from the wall with speed v_0. In the subsequent motion, Q collides alternately with the wall and with P. The coefficient of restitution between Q and P is e, and the coefficient of restitution between Q and the wall is 1.

Let u_n and v_n be the velocities of P and Q, respectively, towards the wall after the nth collision between P and Q.

- Show that, for $n \geq 2$,

$$(1+k)u_n - (1-k)(1+e)u_{n-1} + e(1+k)u_{n-2} = 0. \qquad (*)$$

- You are now given that $e = \frac{1}{2}$ and $k = \frac{1}{34}$, and that the solution of $(*)$ is of the form

$$u_n = A\left(\frac{7}{10}\right)^n + B\left(\frac{5}{7}\right)^n \quad (n \geq 0),$$

  where A and B are independent of n. Find expressions for A and B in terms of u_0 and v_0.

Show that, if $0 < 6u_0 < v_0$, then u_n will be negative for large n.

Solution:

After the nth collision between P and Q, the velocities towards the wall are u_n for P and v_n for Q. Before the $(n+1)$th collision, the velocities are determined as follows:

- Particle P continues with velocity u_n towards the wall.
- Particle Q collides with the wall (coefficient of restitution 1), so its velocity reverses to $-v_n$ (away from the wall).

Thus, just before the $(n+1)$th collision:

- Velocity of P is u_n.
- Velocity of Q is $-v_n$.

For the $(n+1)$th collision, conservation of momentum gives:

$$m \cdot u_n + km \cdot (-v_n) = m \cdot u_{n+1} + km \cdot v_{n+1}$$

Dividing by m:

$$u_n - kv_n = u_{n+1} + kv_{n+1} \quad (1)$$

The coefficient of restitution e between P and Q gives:

$$e = \frac{v_{n+1} - u_{n+1}}{u_n - (-v_n)} = \frac{v_{n+1} - u_{n+1}}{u_n + v_n}$$

So:

$$e(u_n + v_n) = v_{n+1} - u_{n+1} \quad (2)$$

Similarly, for the nth collision, just before the collision:

- Velocity of P is u_{n-1} (after $(n-1)$th collision).
- Velocity of Q is $-v_{n-1}$ (after wall collision).

Conservation of momentum:

$$m \cdot u_{n-1} + km \cdot (-v_{n-1}) = m \cdot u_n + km \cdot v_n$$

Dividing by m:

$$u_{n-1} - kv_{n-1} = u_n + kv_n \quad (3)$$

Coefficient of restitution:

$$e = \frac{v_n - u_n}{u_{n-1} - (-v_{n-1})} = \frac{v_n - u_n}{u_{n-1} + v_{n-1}}$$

So:

$$e(u_{n-1} + v_{n-1}) = v_n - u_n \quad (4)$$

From equations (3) and (4):

$$u_n + kv_n = u_{n-1} - kv_{n-1} \quad (3)$$

$$v_n - u_n = e(u_{n-1} + v_{n-1}) \quad (4)$$

Solving for v_{n-1} from the system, express v_{n-1} in terms of u_n and u_{n-1}:

$$v_{n-1} = -\frac{(1+k)u_n + (ke-1)u_{n-1}}{k(1+e)} \quad (5)$$

Using the relation for the $(n-1)$th collision and the expression for u_{n-2}:

$$u_{n-2} = \frac{kA + B}{1+k}$$

where $A = \frac{v_{n-1} - u_{n-1}}{e}$ and $B = u_{n-1} + kv_{n-1}$.

Substituting v_{n-1} from (5):

$$A = -\frac{(1+k)u_n + (2ke + k - 1)u_{n-1}}{ek(1+e)}$$

$$B = \frac{-(1+k)u_n + (2 + e - ke)u_{n-1}}{1+e}$$

Then:

$$kA + B = \frac{1}{e}[-(1+k)u_n + (1-k)(1+e)u_{n-1}]$$

$$u_{n-2} = \frac{kA + B}{1+k} = \frac{1}{e}\left[-u_n + \frac{(1-k)(1+e)}{1+k}u_{n-1}\right]$$

Rearranging:

$$eu_{n-2} = -u_n + \frac{(1-k)(1+e)}{1+k}u_{n-1}$$

$$u_n = \frac{(1-k)(1+e)}{1+k}u_{n-1} - eu_{n-2}$$

$$u_n - \frac{(1-k)(1+e)}{1+k}u_{n-1} + eu_{n-2} = 0$$

Multiplying through by $1+k$:

$$(1+k)u_n - (1-k)(1+e)u_{n-1} + e(1+k)u_{n-2} = 0$$

This recurrence holds for $n \geq 2$, as required.

The recurrence relation for the velocities is given by:

$$(1+k)u_n - (1-k)(1+e)u_{n-1} + e(1+k)u_{n-2} = 0$$

for $n \geq 2$, with $e = \frac{1}{2}$ and $k = \frac{1}{34}$. The solution is of the form:

$$u_n = A\left(\frac{7}{10}\right)^n + B\left(\frac{5}{7}\right)^n$$

for $n \geq 0$, where A and B are constants to be determined in terms of the initial velocities u_0 and v_0.

The initial velocity of particle P is u_0 towards the wall, and the initial velocity of particle Q is $-v_0$ (away from the wall). The velocity u_1 after the first collision between P and Q is required to find A and B.

Using conservation of momentum and the coefficient of restitution $e = \frac{1}{2}$ for the first collision:

- Conservation of momentum:

$$mu_0 + (km)(-v_0) = mu_1 + (km)v_1$$

  Dividing by m and substituting $k = \frac{1}{34}$:

$$u_0 - \frac{1}{34}v_0 = u_1 + \frac{1}{34}v_1 \quad \text{(a)}$$

- Coefficient of restitution:

$$e = \frac{v_1 - u_1}{u_0 - (-v_0)} = \frac{v_1 - u_1}{u_0 + v_0}$$

  Substituting $e = \frac{1}{2}$:

$$\frac{v_1 - u_1}{u_0 + v_0} = \frac{1}{2} \implies v_1 - u_1 = \frac{1}{2}(u_0 + v_0) \quad \text{(b)}$$

Solving equations (a) and (b) for u_1:

- From equation (b):

$$v_1 = u_1 + \frac{1}{2}(u_0 + v_0)$$

- Substitute into equation (a):

$$u_1 + \frac{1}{34}\left[u_1 + \frac{1}{2}(u_0 + v_0)\right] = u_0 - \frac{1}{34}v_0$$

$$u_1 + \frac{u_1}{34} + \frac{1}{68}(u_0 + v_0) = u_0 - \frac{1}{34}v_0$$

$$\left(1 + \frac{1}{34}\right)u_1 = u_0 - \frac{1}{34}v_0 - \frac{1}{68}u_0 - \frac{1}{68}v_0$$

$$\frac{35}{34}u_1 = \left(1 - \frac{1}{68}\right)u_0 - \left(\frac{1}{34} + \frac{1}{68}\right)v_0$$

$$\frac{35}{34}u_1 = \frac{67}{68}u_0 - \frac{3}{68}v_0$$

- Solving for u_1:

$$u_1 = \frac{34}{35}\left(\frac{67}{68}u_0 - \frac{3}{68}v_0\right) = \frac{34}{35} \cdot \frac{67u_0 - 3v_0}{68} = \frac{67u_0 - 3v_0}{70}$$

  Thus:

$$u_1 = \frac{67u_0 - 3v_0}{70} \quad (2)$$

The solution for u_n is given by:

$$u_n = A\left(\frac{7}{10}\right)^n + B\left(\frac{5}{7}\right)^n$$

For $n = 0$:

$$u_0 = A\left(\frac{7}{10}\right)^0 + B\left(\frac{5}{7}\right)^0 = A + B \quad (1)$$

For $n = 1$:

$$u_1 = A\left(\frac{7}{10}\right) + B\left(\frac{5}{7}\right) \quad (3)$$

Substituting u_1 from equation (2):

$$A \cdot \frac{7}{10} + B \cdot \frac{5}{7} = \frac{67u_0 - 3v_0}{70} \quad (4)$$

Solving equations (1) and (4):

- From equation (1): $A = u_0 - B$
- Substitute into equation (4):

$$\frac{7}{10}(u_0 - B) + \frac{5}{7}B = \frac{67u_0 - 3v_0}{70}$$

$$\frac{7}{10}u_0 - \frac{7}{10}B + \frac{5}{7}B = \frac{67u_0 - 3v_0}{70}$$

  Multiply through by 70 to clear denominators:

$$70 \cdot \frac{7}{10}u_0 - 70 \cdot \frac{7}{10}B + 70 \cdot \frac{5}{7}B = 67u_0 - 3v_0$$

$$49u_0 - 49B + 50B = 67u_0 - 3v_0$$

$$49u_0 + B = 67u_0 - 3v_0$$

$$B = 67u_0 - 49u_0 - 3v_0 = 18u_0 - 3v_0$$

- Then from equation (1):

$$A = u_0 - B = u_0 - (18u_0 - 3v_0) = -17u_0 + 3v_0$$

Thus, the expressions for A and B in terms of u_0 and v_0 are:

$$A = 3v_0 - 17u_0, \quad B = 18u_0 - 3v_0$$

Given that $e = \frac{1}{2}$ and $k = \frac{1}{34}$, the velocity of particle P after the nth collision with particle Q is given by:

$$u_n = A\left(\frac{7}{10}\right)^n + B\left(\frac{5}{7}\right)^n,$$

where $A = 3v_0 - 17u_0$ and $B = 18u_0 - 3v_0$.

The condition $0 < 6u_0 < v_0$ implies that $u_0 > 0$ and $v_0 > 6u_0$. Substituting into the expression for B:

$$B = 18u_0 - 3v_0 = 3(6u_0 - v_0).$$

Since $v_0 > 6u_0$, it follows that $6u_0 - v_0 < 0$, so $B < 0$.

The bases of the exponential terms are $\frac{7}{10} = 0.7$ and $\frac{5}{7} \approx 0.7143$, both positive and less than 1. The magnitude of $\frac{5}{7}$ is greater than that of $\frac{7}{10}$, so $\left(\frac{5}{7}\right)^n$ decays slower than $\left(\frac{7}{10}\right)^n$ as n increases. Specifically, the ratio:

$$\frac{\left|B\left(\frac{5}{7}\right)^n\right|}{\left|A\left(\frac{7}{10}\right)^n\right|} = \frac{|B|}{|A|}\left(\frac{5}{7} \cdot \frac{10}{7}\right)^n = \frac{|B|}{|A|}\left(\frac{50}{49}\right)^n.$$

Since $\frac{50}{49} > 1$, this ratio tends to infinity as $n \to \infty$. Therefore, for sufficiently large n, the magnitude of the term $B\left(\frac{5}{7}\right)^n$ dominates that of $A\left(\frac{7}{10}\right)^n$.

Given that $B < 0$, the dominant term $B\left(\frac{5}{7}\right)^n$ is negative. Consequently, for large n, $u_n < 0$.

Thus, under the condition $0 < 6u_0 < v_0$, the velocity u_n becomes negative for sufficiently large n.

Given that $e = \frac{1}{2}$ and $k = \frac{1}{34}$, the velocity of particle P after the nth collision with particle Q is given by:

$$u_n = A\left(\frac{7}{10}\right)^n + B\left(\frac{5}{7}\right)^n,$$

where $A = 3v_0 - 17u_0$ and $B = 18u_0 - 3v_0$.

The condition $0 < 6u_0 < v_0$ implies that $u_0 > 0$ and $v_0 > 6u_0$. Substituting into the expression for B:

$$B = 18u_0 - 3v_0 = 3(6u_0 - v_0).$$

Since $v_0 > 6u_0$, it follows that $6u_0 - v_0 < 0$, so $B < 0$.

The bases of the exponential terms are $\frac{7}{10} = 0.7$ and $\frac{5}{7} \approx 0.7143$, both positive and less than 1. Note that $\frac{5}{7} > \frac{7}{10}$, so $\left(\frac{5}{7}\right)^n$ decays slower than $\left(\frac{7}{10}\right)^n$ as n increases. Consider the ratio:

$$\frac{u_n}{\left(\frac{5}{7}\right)^n} = A\left(\frac{7/10}{5/7}\right)^n + B = A\left(\frac{49}{50}\right)^n + B.$$

Since $\left|\frac{49}{50}\right| < 1$, $\left(\frac{49}{50}\right)^n \to 0$ as $n \to \infty$. Therefore:

$$\lim_{n\to\infty} \frac{u_n}{\left(\frac{5}{7}\right)^n} = B.$$

Given that $B < 0$ and $\left(\frac{5}{7}\right)^n > 0$ for all n, it follows that for sufficiently large n, $u_n < 0$.

Thus, under the condition $0 < 6u_0 < v_0$, u_n is negative for large n.

Q12

A uniform disc with centre O and radius a is suspended from a point A on its circumference, so that it can swing freely about a horizontal axis L through A. The plane of the disc is perpendicular to L. A particle P is attached to a point on the circumference of the disc. The mass of the disc is M and the mass of the particle is m.

In equilibrium, the disc hangs with OP horizontal, and the angle between AO and the downward vertical through A is β. Find $\sin\beta$ in terms of M and m and show that

$$\frac{AP}{a} = \sqrt{\frac{2M}{M+m}}.$$

The disc is rotated about L and then released. At later time t, the angle between OP and the horizontal is θ; when P is higher than O, θ is positive and when P is lower than O, θ is negative. Show that

$$\frac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (m+M)ga\cos\beta(1-\cos\theta)$$

is constant during the motion, where I is the moment of inertia of the disc about L.

Show that the period of small oscillations is

$$3\pi\sqrt{\frac{3a}{5g}}.$$

Given that $m = \frac{3}{2}M$ and that $I = \frac{3}{2}Ma^2$.

Solution:

The disc is suspended from point A on its circumference, and the system is in equilibrium with OP horizontal and the angle between AO and the downward vertical through A being β.

The center of mass of the system must lie directly below the point of suspension A for equilibrium.

Set up a coordinate system with A at $(0, 0)$ and the downward vertical along the negative y-axis. The center of the disc O is at (x_o, y_o), where $|AO| = a$. The angle β gives:

$$y_o = -a\cos\beta, \quad x_o = a\sin\beta$$

(assuming O is to the right of A without loss of generality).

The particle P is on the circumference and OP is horizontal. Since $|OP| = a$ and OP is horizontal, P has the same y-coordinate as O. Given the equilibrium condition with OP horizontal and the center of mass below A, P is at:

$$(x_p, y_p) = (x_o - a, y_o) = (a\sin\beta - a, -a\cos\beta)$$

The center of mass of the system (disc of mass M at O, particle of mass m at P) must have an x-coordinate of 0:

$$\frac{Mx_o + mx_p}{M+m} = 0$$

Substituting the coordinates:

$$M(a\sin\beta) + m(a\sin\beta - a) = 0$$

$$aM\sin\beta + am\sin\beta - am = 0$$

$$a\sin\beta(M+m) = am$$

$$\sin\beta(M+m) = m$$

$$\sin\beta = \frac{m}{M+m}$$

To find AP/a, compute the distance AP:

$$AP^2 = (x_p - 0)^2 + (y_p - 0)^2 = (a\sin\beta - a)^2 + (-a\cos\beta)^2$$

$$= a^2(\sin\beta - 1)^2 + a^2\cos^2\beta$$

$$= a^2[(\sin^2\beta - 2\sin\beta + 1) + \cos^2\beta]$$

$$= a^2[\sin^2\beta + \cos^2\beta + 1 - 2\sin\beta]$$

$$= a^2[1 + 1 - 2\sin\beta] = a^2(2 - 2\sin\beta) = 2a^2(1 - \sin\beta)$$

Substitute $\sin\beta = m/(M+m)$:

$$1 - \sin\beta = 1 - \frac{m}{M+m} = \frac{M}{M+m}$$

$$AP^2 = 2a^2 \cdot \frac{M}{M+m}$$

$$AP = a\sqrt{\frac{2M}{M+m}}$$

$$\frac{AP}{a} = \sqrt{\frac{2M}{M+m}}$$

Thus, $\sin\beta = \frac{m}{M+m}$ and $\frac{AP}{a} = \sqrt{\frac{2M}{M+m}}$.

The system consists of a uniform disc of mass M and radius a suspended from a point A on its circumference, swinging freely about a horizontal axis L through A, with a particle of mass m attached to a point on the circumference. The angle θ is defined as the angle between the radius OP and the horizontal, with $\theta > 0$ when P is higher than O and $\theta < 0$ when P is lower than O.

The entire system rotates as a rigid body about the fixed axis L. The total kinetic energy is given by $\frac{1}{2}I_{\text{total}}\dot{\theta}^2$, where I_{total} is the total moment of inertia about L. The moment of inertia of the disc about L is $I = \frac{3}{2}Ma^2$. The distance from A to P is constant and given by $AP = a\sqrt{2(1-\sin\beta)}$, so the moment of inertia of the particle about L is $m[a^2 \cdot 2(1-\sin\beta)] = 2ma^2(1-\sin\beta)$. Thus, the total moment of inertia is:

$$I_{\text{total}} = I + 2ma^2(1-\sin\beta) = \frac{3}{2}Ma^2 + 2ma^2(1-\sin\beta).$$

The kinetic energy is:

$$K = \frac{1}{2}I_{\text{total}}\dot{\theta}^2 = \frac{1}{2}\left[\frac{3}{2}Ma^2 + 2ma^2(1-\sin\beta)\right]\dot{\theta}^2 = \frac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2.$$

The potential energy is defined with the downward vertical direction as positive y. The position of the center of the disc O is $(a\sin\phi, a\cos\phi)$, where ϕ is the angle that AO makes with the downward vertical. The position of the particle P is $(a\sin\phi - a\cos(\phi-\beta), a\cos\phi + a\sin(\phi-\beta))$. The angle θ is related to ϕ by $\theta = \beta - \phi$, so $\phi = \beta - \theta$.

The sum of the y-coordinates weighted by mass is:

$$\sum m_i y_i = My_O + my_P = M(a\cos\phi) + m[a\cos\phi + a\sin(\phi-\beta)].$$

Substituting $\phi = \beta - \theta$:

$$\sum m_i y_i = a(M+m)\cos(\beta-\theta) - ma\sin\theta.$$

The potential energy is $V = -g\sum m_i y_i = -g[a(M+m)\cos(\beta-\theta) - ma\sin\theta]$. Adjusting by a constant so that $V = 0$ at $\theta = 0$ (equilibrium):

$$V = -ga(M+m)\cos(\beta-\theta) + gma\sin\theta + ga(M+m)\cos\beta.$$

Simplifying using $\cos(\beta-\theta) = \cos\beta\cos\theta + \sin\beta\sin\theta$ and $\sin\beta = \frac{m}{M+m}$:

$$V = (M+m)ga\cos\beta(1-\cos\theta).$$

The total energy is the sum of kinetic and potential energy:

$$E = K + V = \frac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (M+m)ga\cos\beta(1-\cos\theta).$$

Since the system is conservative and there are no non-conservative forces, the total energy is conserved. Thus, the expression is constant during the motion.

The period of small oscillations is derived using the total energy expression, which is constant during motion:

$$E = \frac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (m+M)ga\cos\beta(1-\cos\theta)$$

where $I = \frac{3}{2}Ma^2$ is the moment of inertia of the disc about the axis L, β is the equilibrium angle, and $m = \frac{3}{2}M$.

For small oscillations, θ is small, so $\cos\theta \approx 1 - \frac{\theta^2}{2}$. Substituting this approximation into the potential energy term gives:

$$(1-\cos\theta) \approx \frac{\theta^2}{2}$$

Thus, the potential energy becomes:

$$V \approx (m+M)ga\cos\beta \cdot \frac{\theta^2}{2} = \frac{1}{2}(m+M)ga\cos\beta\theta^2$$

The kinetic energy terms are combined as follows:

$$\frac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 = \dot{\theta}^2\left(\frac{1}{2}I + (1-\sin\beta)ma^2\right) = \frac{1}{2}\left(I + 2(1-\sin\beta)ma^2\right)\dot{\theta}^2$$

The effective moment of inertia I_{total} is:

$$I_{\text{total}} = I + 2(1-\sin\beta)ma^2$$

Given $m = \frac{3}{2}M$, first find $\sin\beta$:

$$\sin\beta = \frac{m}{M+m} = \frac{\frac{3}{2}M}{M+\frac{3}{2}M} = \frac{\frac{3}{2}M}{\frac{5}{2}M} = \frac{3}{5}$$

Thus, $\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$, and:

$$1-\sin\beta = 1-\frac{3}{5} = \frac{2}{5}$$

Now substitute into I_{total}:

$$I_{\text{total}} = \frac{3}{2}Ma^2 + 2\left(\frac{3}{2}M\right)a^2\left(\frac{2}{5}\right) = \frac{3}{2}Ma^2 + 2\cdot\frac{3}{2}M\cdot\frac{2}{5}a^2 = \frac{3}{2}Ma^2 + \frac{6}{5}Ma^2$$

$$I_{\text{total}} = Ma^2\left(\frac{3}{2}+\frac{6}{5}\right) = Ma^2\left(\frac{15}{10}+\frac{12}{10}\right) = Ma^2\cdot\frac{27}{10} = \frac{27}{10}Ma^2$$

The effective spring constant k_{eff} from the potential energy is the coefficient of $\frac{1}{2}\theta^2$:

$$k_{\text{eff}} = (m+M)ga\cos\beta$$

Substitute the values:

$$m + M = M + \frac{3}{2}M = \frac{5}{2}M, \quad \cos\beta = \frac{4}{5}$$

$$k_{\text{eff}} = \frac{5}{2}M \cdot ga \cdot \frac{4}{5} = 2Mga$$

The period of small oscillations for a rotational harmonic oscillator is:

$$T = 2\pi\sqrt{\frac{I_{\text{total}}}{k_{\text{eff}}}}$$

Substitute the expressions:

$$T = 2\pi\sqrt{\frac{\frac{27}{10}Ma^2}{2Mga}} = 2\pi\sqrt{\frac{27Ma^2}{10} \cdot \frac{1}{2Mga}} = 2\pi\sqrt{\frac{27a}{20g}}$$

Simplify the expression inside the square root:

$$\frac{27a}{20g} = \frac{27}{20} \cdot \frac{a}{g}$$

Thus:

$$T = 2\pi\sqrt{\frac{27}{20} \cdot \frac{a}{g}} = 2\pi \cdot \sqrt{\frac{27}{20}} \cdot \sqrt{\frac{a}{g}}$$

$$\sqrt{\frac{27}{20}} = \sqrt{\frac{27}{20}} = \sqrt{\frac{9 \times 3}{4 \times 5}} = \frac{3}{2}\sqrt{\frac{3}{5}}$$

So:

$$T = 2\pi \cdot \frac{3}{2}\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{a}{g}} = 3\pi\sqrt{\frac{3}{5} \cdot \frac{a}{g}} = 3\pi\sqrt{\frac{3a}{5g}}$$

Thus, the period of small oscillations is $3\pi\sqrt{\frac{3a}{5g}}$.

Q13

A particle is attached to one end of a light inextensible string of length b. The other end of the string is attached to a fixed point O. Initially the particle hangs vertically below O. The particle then receives a horizontal impulse.

The particle moves in a circular arc with the string taut until the acute angle between the string and the upward vertical is α, at which time it becomes slack. Express V, the speed of the particle when the string becomes slack, in terms of b, g and α.

Show that the string becomes taut again a time T later, where

$$gT = 4V \sin \alpha,$$

and that just before this time the trajectory of the particle makes an angle β with the horizontal where $\tan \beta = 3 \tan \alpha$.

When the string becomes taut, the momentum of the particle in the direction of the string is destroyed. Show that the particle comes instantaneously to rest at this time if and only if

$$\sin^2 \alpha = \frac{1 + \sqrt{3}}{4}.$$

Solution:

The speed of the particle when the string becomes slack is determined by the condition that the tension in the string is zero at that point. At the angle α from the upward vertical, the angle from the downward vertical is $180^\circ - \alpha$.

The radial force equation for circular motion, with tension $T = 0$, gives:

$$-mg \cos \theta = m \frac{v^2}{b}$$

where θ is the angle from the downward vertical, m is the mass of the particle, g is the acceleration due to gravity, v is the speed, and b is the length of the string. Simplifying by dividing by m:

$$-g \cos \theta = \frac{v^2}{b}$$

Solving for v^2:

$$v^2 = -gb \cos \theta$$

At the point where the string becomes slack, $\theta = 180^\circ - \alpha$, so:

$$\cos \theta = \cos(180^\circ - \alpha) = -\cos \alpha$$

Substituting this in:

$$v^2 = -gb(-\cos \alpha) = gb \cos \alpha$$

Thus, the speed V is:

$$V = \sqrt{gb \cos \alpha}$$

This expression depends only on b, g, and α, and the initial impulse does not affect the result because the condition $T = 0$ at angle α determines the speed directly.

The particle moves under gravity after the string becomes slack at an angle α with the upward vertical, with speed $V = \sqrt{gb \cos \alpha}$.

At the moment the string becomes slack, the position of the particle is $(b \sin \alpha, b \cos \alpha)$ and its velocity is $(-V \cos \alpha, V \sin \alpha)$. Under gravity, the acceleration is $(0, -g)$, so the position at time t after slack is:

$$x(t) = b \sin \alpha - V \cos \alpha t$$

$$y(t) = b \cos \alpha + V \sin \alpha t - \frac{1}{2} g t^2$$

The string becomes taut again when the distance from the fixed point O to the particle equals b, i.e., $x(t)^2 + y(t)^2 = b^2$. Substituting the expressions for $x(t)$ and $y(t)$ and using $V^2 = gb \cos \alpha$ gives:

$$x(t)^2 + y(t)^2 - b^2 = \left(V^2 - bg \cos \alpha\right) t^2 - V g \sin \alpha t^3 + \frac{1}{4} g^2 t^4 = 0$$

Since $V^2 - bg \cos \alpha = 0$, the equation simplifies to:

$$t^2 \left(\frac{1}{4} g^2 t^2 - V g \sin \alpha t \right) = 0$$

The non-trivial solution is:

$$\frac{1}{4} g^2 t^2 - V g \sin \alpha t = 0 \implies t \left(\frac{1}{4} g t - V \sin \alpha \right) = 0$$

Thus, the time T when the string becomes taut again is:

$$T = \frac{4V \sin \alpha}{g}$$

which gives $gT = 4V \sin \alpha$.

The velocity components at time t are:

$$v_x(t) = -V \cos \alpha$$

$$v_y(t) = V \sin \alpha - gt$$

At time $T = \frac{4V \sin \alpha}{g}$:

$$v_y(T) = V \sin \alpha - g \cdot \frac{4V \sin \alpha}{g} = -3V \sin \alpha$$

The velocity vector is $(-V \cos \alpha, -3V \sin \alpha)$. The angle β that the trajectory makes with the horizontal satisfies:

$$\tan \beta = \left| \frac{v_y}{v_x} \right| = \left| \frac{-3V \sin \alpha}{-V \cos \alpha} \right| = 3 \tan \alpha$$

Thus, $\tan \beta = 3 \tan \alpha$.

The particle becomes slack when the string makes an angle α with the upward vertical, and its speed at that instant is $V = \sqrt{gb \cos \alpha}$. After becoming slack, the particle moves under gravity until the string becomes taut again at time T, where $gT = 4V \sin\alpha$.

At time T, the position of the particle relative to the fixed point O is:

$$x(T) = b \sin\alpha(1 - 4\cos^2\alpha), \quad y(T) = b\cos\alpha(1 - 4\sin^2\alpha).$$

The velocity components just before the string becomes taut are:

$$v_x = -V\cos\alpha, \quad v_y = -3V\sin\alpha.$$

When the string becomes taut, the impulsive tension acts radially, destroying the momentum in the direction of the string. This sets the radial component of velocity to zero, while the transverse component remains unchanged. The particle comes to rest instantaneously if and only if both components of velocity are zero after the impulse. Since the transverse component is unchanged, it must be zero just before the impulse, implying that the velocity vector is purely radial at that instant. This occurs when the velocity vector is parallel to the position vector, i.e., $\frac{v_x}{x(T)} = \frac{v_y}{y(T)}$.

Substituting the expressions:

$$\frac{-V\cos\alpha}{b\sin\alpha(1 - 4\cos^2\alpha)} = \frac{-3V\sin\alpha}{b\cos\alpha(1 - 4\sin^2\alpha)}.$$

Canceling $-Vb$ (since $V \neq 0$, $b \neq 0$) and rearranging gives:

$$\frac{\cos\alpha}{\sin\alpha(1 - 4\cos^2\alpha)} = \frac{3\sin\alpha}{\cos\alpha(1 - 4\sin^2\alpha)}.$$

Cross-multiplying:

$$\cos^2\alpha(1 - 4\sin^2\alpha) = 3\sin^2\alpha(1 - 4\cos^2\alpha).$$

Let $u = \sin^2\alpha$ and $v = \cos^2\alpha$, with $u + v = 1$:

$$v(1 - 4u) = 3u(1 - 4v).$$

Substituting $v = 1 - u$:

$$(1 - u)(1 - 4u) = 3u(1 - 4(1 - u)) = 3u(4u - 3).$$

Expanding both sides:

$$1 - 5u + 4u^2 = 12u^2 - 9u.$$

Rearranging terms:

$$4u^2 - 5u + 1 - 12u^2 + 9u = 0 \implies -8u^2 + 4u + 1 = 0.$$

Multiplying by -1:

$$8u^2 - 4u - 1 = 0.$$

Solving the quadratic equation:

$$u = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 8 \cdot (-1)}}{2 \cdot 8} = \frac{4 \pm \sqrt{16+32}}{16} = \frac{4 \pm \sqrt{48}}{16} = \frac{4 \pm 4\sqrt{3}}{16} = \frac{1 \pm \sqrt{3}}{4}.$$

Since α is acute, $u = \sin^2\alpha > 0$. The solution $u = \frac{1-\sqrt{3}}{4} < 0$ is discarded, leaving:

$$\sin^2\alpha = \frac{1+\sqrt{3}}{4}.$$

For this value, the denominators in the position components are non-zero, and the position is not at O, confirming the solution is valid. Thus, the particle comes to rest instantaneously when the string becomes taut if and only if $\sin^2\alpha = \frac{1+\sqrt{3}}{4}$.

Q14

Two particles A and B of masses m and $2m$, respectively, are connected by a light spring of natural length a and modulus of elasticity λ. They are placed on a smooth horizontal table with AB perpendicular to the edge of the table, and A is held on the edge of the table. Initially the spring is at its natural length.

Particle A is released. At a time t later, particle A has dropped a distance y and particle B has moved a distance x from its initial position (where $x < a$). Show that

$$y + 2x = \frac{1}{2}gt^2.$$

The value of λ is such that particle B reaches the edge of the table at a time T given by

$$T = \sqrt{\frac{6a}{g}}.$$

By considering the total energy of the system (without solving any differential equations), show that the speed of particle B at this time is

$$\sqrt{\frac{2ag}{3}}.$$

Solution:

The system consists of two particles connected by a spring on a smooth horizontal table. Particle A, with mass m, is initially held at the edge of the table, and particle B, with mass $2m$, is initially at position $(a, 0)$, where a is the natural length of the spring. The spring has modulus of elasticity λ.

When particle A is released, it falls vertically downward due to gravity, and its position at time t is $(0, -y)$, where y is the distance dropped. Particle B moves horizontally on the table, and its position at time t is $(a - x, 0)$, where x is the distance it has moved from its initial position, with $x < a$.

Horizontal Motion of the Center of Mass
There are no external horizontal forces acting on the system because the table is smooth and gravity acts vertically. Therefore, the horizontal acceleration of the center of mass (CM) is zero. The initial horizontal velocity of the CM is also zero, so the horizontal position of the CM remains constant.

Initial horizontal position of CM:

$$x_{\text{cm}}(0) = \frac{m \cdot 0 + 2m \cdot a}{m + 2m} = \frac{2ma}{3m} = \frac{2a}{3}$$

At time t, the horizontal position of A is 0, and the horizontal position of B is $a - x$. Thus, the horizontal position of the CM is:

$$x_{\text{cm}}(t) = \frac{m \cdot 0 + 2m \cdot (a - x)}{3m} = \frac{2m(a - x)}{3m} = \frac{2}{3}(a - x)$$

Since the horizontal position of the CM is constant:

$$\frac{2}{3}(a - x) = \frac{2a}{3}$$

Solving for x:

$$a - x = a \implies x = 0$$

Thus, particle B does not move horizontally, and $x = 0$.

Vertical Motion of the Center of Mass
The net external vertical force on the system is the gravitational force acting on particle A, since particle B is constrained to the table and the normal force balances its weight. The gravitational force on A is $-mg$ (downward), and there is no net vertical force on B due to the table. Therefore, the net external vertical force on the system is $-mg$.

The total mass of the system is $3m$, so the vertical acceleration of the CM is:

$$a_{\text{cm},y} = \frac{-mg}{3m} = -\frac{g}{3}$$

Initial vertical position of CM:

$$y_{\text{cm}}(0) = \frac{m \cdot 0 + 2m \cdot 0}{3m} = 0$$

Initial vertical velocity of CM is zero. Thus, the vertical position of the CM at time t is:

$$y_{\text{cm}}(t) = \frac{1}{2} a_{\text{cm},y} t^2 = \frac{1}{2}\left(-\frac{g}{3}\right)t^2 = -\frac{1}{6}gt^2$$

At time t, the vertical position of A is $-y$, and the vertical position of B is 0. Therefore, the vertical position of the CM is:

$$y_{\text{cm}}(t) = \frac{m \cdot (-y) + 2m \cdot 0}{3m} = \frac{-my}{3m} = -\frac{y}{3}$$

Equating the expressions for $y_{\mathrm{cm}}(t)$:

$$-\frac{y}{3} = -\frac{1}{6}gt^2$$

Solving for y:

$$\frac{y}{3} = \frac{1}{6}gt^2 \implies y = \frac{1}{2}gt^2$$

Verification of the Equation
With $x = 0$ and $y = \frac{1}{2}gt^2$, the equation $y + 2x = \frac{1}{2}gt^2$ holds:

$$y + 2x = \frac{1}{2}gt^2 + 2 \cdot 0 = \frac{1}{2}gt^2$$

Thus, the equation is satisfied, and particle B does not move horizontally due to the conservation of the horizontal center of mass position.

To show that the time T when particle B reaches the edge of the table is given by $T = \sqrt{\frac{6a}{g}}$, consider the conditions at the instant when B reaches the edge.

Particle B reaches the edge when its horizontal position is $s = 0$, where s is the distance from the edge. Initially, $s = a$, and B has moved a distance x towards the edge, so $s = a - x$. At the edge, $s = 0$, so $x = a$.

Given the relation $y + 2x = \frac{1}{2}gt^2$, at time $t = T$,

$$y + 2a = \frac{1}{2}gT^2.$$

When $s = 0$, the horizontal force on B is

$$F_{x_B} = -\frac{\lambda}{a}(L - a)\frac{s}{L}.$$

Since $s = 0$,

$$F_{x_B} = 0.$$

With no horizontal force, the horizontal acceleration of B is zero:

$$a_s = \frac{d^2s}{dt^2} = 0.$$

Differentiating the given relation $y + 2x = \frac{1}{2}gt^2$ twice with respect to time, and substituting $x = a - s$,

$$\frac{d^2y}{dt^2} + 2\frac{d^2}{dt^2}(a - s) = \frac{d^2}{dt^2}\left(\frac{1}{2}gt^2\right),$$

$$a_y - 2a_s = g.$$

At $t = T$, $a_s = 0$, so

$$a_y = g.$$

For particle A, the vertical force is

$$F_{y_A} = mg - \frac{\lambda}{a}(L - a),$$

so the equation of motion is

$$ma_y = mg - \frac{\lambda}{a}(L - a).$$

At $t = T$, $s = 0$, so the distance between A and B is

$$L = \sqrt{0^2 + y^2} = y \quad (\text{since } y > 0).$$

Substituting $a_y = g$ and $L = y$,

$$mg = mg - \frac{\lambda}{a}(y - a),$$

$$0 = -\frac{\lambda}{a}(y - a).$$

Assuming $\lambda \neq 0$,

$$y - a = 0, \quad y = a.$$

Substituting $y = a$ into the relation at $t = T$,

$$a + 2a = \frac{1}{2}gT^2,$$

$$3a = \frac{1}{2}gT^2,$$

$$gT^2 = 6a,$$

$$T^2 = \frac{6a}{g},$$

$$T = \sqrt{\frac{6a}{g}}.$$

The total energy of the system is conserved because the table is smooth (no friction) and the forces (gravity and spring) are conservative. Set the initial gravitational potential energy to zero when A is at height zero (initial position). When A has dropped a distance y, its height is $-y$ (if y is positive downward), so the gravitational potential energy is $-mgy$. Particle B moves horizontally, so its gravitational potential energy remains constant at zero. The spring potential energy depends on the extension $\delta = L - a$, where L is the current length of the spring. The positions are A at $(0, -y)$ and B at $(a - x, 0)$, so:

$$L = \sqrt{(a - x)^2 + y^2}, \quad \delta = \sqrt{(a - x)^2 + y^2} - a.$$

The spring constant $k = \lambda/a$, so the spring potential energy is:

$$U_s = \frac{1}{2}k\delta^2 = \frac{1}{2}\left(\frac{\lambda}{a}\right)\left(\sqrt{(a-x)^2+y^2}-a\right)^2.$$

The kinetic energy is:

- For A: $\frac{1}{2}mv_A^2$, where $v_A = dy/dt$ (speed downward).
- For B: $\frac{1}{2}(2m)v_B^2 = mv_B^2$, where $v_B = dx/dt$ (speed towards the edge).

Initial total energy (at rest, $y = 0$, $x = 0$, $\delta = 0$) is zero. Thus, at any time:

$$\frac{1}{2}mv_A^2 + mv_B^2 - mgy + \frac{1}{2}\left(\frac{\lambda}{a}\right)\left(\sqrt{(a-x)^2+y^2}-a\right)^2 = 0.$$

Particle B reaches the edge when $x = a$. From previous results, at this time $y = a$, $T = \sqrt{6a/g}$, and the spring is at natural length ($\delta = 0$). Substitute $x = a$, $y = a$, $\delta = 0$:

$$\frac{1}{2}mv_A^2 + mv_B^2 - mg(a) + 0 = 0.$$

Divide by m:

$$\frac{1}{2}v_A^2 + v_B^2 - ga = 0 \implies \frac{1}{2}v_A^2 + v_B^2 = ga. \quad (1)$$

From the constraint $y + 2x = \frac{1}{2}gt^2$, differentiate with respect to time:

$$\frac{dy}{dt} + 2\frac{dx}{dt} = gt \implies v_A + 2v_B = gt.$$

At time $T = \sqrt{6a/g}$:

$$gT = g\sqrt{\frac{6a}{g}} = \sqrt{6ag}, \quad \text{so} \quad v_A + 2v_B = \sqrt{6ag}. \quad (2)$$

Solve equations (1) and (2) for v_B. Set $u = v_A$, $v = v_B$:

$$u + 2v = \sqrt{6ag}, \quad (2)$$

$$\frac{1}{2}u^2 + v^2 = ga. \quad (1)$$

From (2), $u = \sqrt{6ag} - 2v$. Substitute into (1):

$$\frac{1}{2}(\sqrt{6ag} - 2v)^2 + v^2 = ga.$$

Expand:

$$(\sqrt{6ag} - 2v)^2 = 6ag - 4\sqrt{6ag}v + 4v^2,$$

$$\frac{1}{2}(6ag - 4\sqrt{6ag}v + 4v^2) = 3ag - 2\sqrt{6ag}v + 2v^2.$$

Add v^2:

$$3ag - 2\sqrt{6ag}v + 2v^2 + v^2 = 3ag - 2\sqrt{6ag}v + 3v^2.$$

Set equal to ga:

$$3ag - 2\sqrt{6ag}v + 3v^2 = ag,$$

$$2ag - 2\sqrt{6ag}v + 3v^2 = 0.$$

Divide by 2:

$$ag - \sqrt{6ag}v + \frac{3}{2}v^2 = 0.$$

Set $k = \sqrt{ag}$:

$$k^2 - \sqrt{6}kv + \frac{3}{2}v^2 = 0.$$

Multiply by 2:

$$3v^2 - 2\sqrt{6}kv + 2k^2 = 0.$$

Discriminant:

$$D = (-2\sqrt{6}k)^2 - 4 \cdot 3 \cdot 2k^2 = 24k^2 - 24k^2 = 0.$$

Solution:

$$v = \frac{2\sqrt{6}k}{2 \cdot 3} = \frac{\sqrt{6}k}{3} = \frac{\sqrt{6}\sqrt{ag}}{3} = \sqrt{\frac{6ag}{9}} = \sqrt{\frac{2ag}{3}}.$$

Q15

A uniform rod PQ of mass m and length $3a$ is freely hinged at P.

The rod is held horizontally and a particle of mass m is placed on top of the rod at a distance ℓ from P, where $\ell < 2a$. The coefficient of friction between the rod and the particle is μ.

The rod is then released. Show that, while the particle does not slip along the rod,

$$(3a^2 + \ell^2)\dot{\theta}^2 = g(3a + 2\ell)\sin\theta,$$

where θ is the angle through which the rod has turned, and the dot denotes the time derivative.

Hence, or otherwise, find an expression for $\ddot{\theta}$ and show that the normal reaction of the rod on the particle is non-zero when θ is acute.

Show further that, when the particle is on the point of slipping,

$$\tan\theta = \frac{\mu a(2a-\ell)}{2(\ell^2+a\ell+a^2)}.$$

What happens at the moment the rod is released if, $\ell > 2a$ instead?

Solution:

The rod and particle system rotates about the hinge at P. Since the particle does not slip relative to the rod, it remains at a fixed distance ℓ from P, and the system can be treated as a rigid body rotating about P.

The total mechanical energy of the system is conserved because the hinge force at P does no work (as it is perpendicular to the motion), and static friction does no work (as there is no relative motion between the particle and the rod).

Set the potential energy reference at the height of P. Initially, the rod is horizontal ($\theta = 0$), so the initial potential energy is zero, and the initial kinetic energy is zero (as the system is released from rest).

The center of mass of the rod (mass m, length $3a$) is at a distance $1.5a$ from P. The particle (mass m) is at a distance ℓ from P. The total mass is $2m$, and the distance of the center of mass from P is:

$$d = \frac{m\cdot 1.5a + m\cdot\ell}{2m} = \frac{1.5a+\ell}{2} = \frac{3a/2+\ell}{2} = \frac{3a}{4}+\frac{\ell}{2} = \frac{3a+2\ell}{4}.$$

At an angle θ, the height of the center of mass is $-d\sin\theta$ (with the negative sign indicating below the horizontal through P). The potential energy is:

$$U = (2m)g(-d\sin\theta) = -2mgd\sin\theta.$$

The moment of inertia of the system about P is the sum of the rod's moment of inertia about P and the particle's moment of inertia about P. The rod's moment of inertia about one end is $\frac{1}{3}m(3a)^2 = 3ma^2$. The particle's moment of inertia is $m\ell^2$. Thus:

$$I = 3ma^2 + m\ell^2 = m(3a^2+\ell^2).$$

The kinetic energy at angle θ is:

$$K = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}m(3a^2+\ell^2)\dot{\theta}^2.$$

By conservation of energy, the initial total energy (0) equals the total energy at angle θ:

$$K + U = 0,$$

$$\frac{1}{2}m(3a^2+\ell^2)\dot{\theta}^2 - 2mgd\sin\theta = 0.$$

Substitute $d = \frac{3a+2\ell}{4}$:

$$\frac{1}{2}m(3a^2+\ell^2)\dot{\theta}^2 = 2mg\left(\frac{3a+2\ell}{4}\right)\sin\theta.$$

Simplify the right side:

$$2mg \cdot \frac{3a+2\ell}{4}\sin\theta = \frac{2mg}{4}(3a+2\ell)\sin\theta = \frac{mg}{2}(3a+2\ell)\sin\theta.$$

The equation is:

$$\frac{1}{2}m(3a^2+\ell^2)\dot{\theta}^2 = \frac{mg}{2}(3a+2\ell)\sin\theta.$$

Multiply both sides by 2:

$$m(3a^2+\ell^2)\dot{\theta}^2 = mg(3a+2\ell)\sin\theta.$$

Divide both sides by m (assuming $m \neq 0$):

$$(3a^2+\ell^2)\dot{\theta}^2 = g(3a+2\ell)\sin\theta.$$

The angular acceleration $\ddot{\theta}$ is derived by differentiating the given energy equation $(3a^2+\ell^2)\dot{\theta}^2 = g(3a+2\ell)\sin\theta$ with respect to time. Differentiating both sides yields:

$$2(3a^2+\ell^2)\dot{\theta}\ddot{\theta} = g(3a+2\ell)\cos\theta\cdot\dot{\theta}.$$

Assuming $\dot{\theta} \neq 0$, divide both sides by $\dot{\theta}$:

$$2(3a^2+\ell^2)\ddot{\theta} = g(3a+2\ell)\cos\theta.$$

Solving for $\ddot{\theta}$:

$$\ddot{\theta} = \frac{g(3a+2\ell)\cos\theta}{2(3a^2+\ell^2)}.$$

The normal reaction N of the rod on the particle is given by:

$$N = mg\cos\theta\cdot\frac{3a(2a-\ell)}{2(3a^2+\ell^2)}.$$

When θ is acute, $\cos\theta > 0$. Given $\ell < 2a$, $2a-\ell > 0$, so the expression for N is positive. Thus, $N > 0$ and is non-zero.

For the particle on the point of slipping, the friction force f is at its maximum, so $f = \mu N$. The tangential acceleration of the particle is $a_t = -\ell\dot{\theta}^2$, and the forces along the rod give:

$$f = mg\sin\theta + m\ell\dot{\theta}^2.$$

Using the energy equation $\dot{\theta}^2 = \frac{g(3a+2\ell)\sin\theta}{3a^2+\ell^2}$ and the expression for N, set $\mu N = f$:

$$\mu\left[mg\cos\theta\cdot\frac{3a(2a-\ell)}{2(3a^2+\ell^2)}\right] = mg\sin\theta + m\ell\cdot\frac{g(3a+2\ell)\sin\theta}{3a^2+\ell^2}.$$

Dividing by mg and simplifying:

$$\mu\cos\theta\cdot\frac{3a(2a-\ell)}{2(3a^2+\ell^2)} = \sin\theta\left[1+\frac{\ell(3a+2\ell)}{3a^2+\ell^2}\right] = \sin\theta\cdot\frac{3(a^2+a\ell+\ell^2)}{3a^2+\ell^2}.$$

Thus:

$$\frac{\mu a(2a-\ell)\cos\theta}{2} = \sin\theta(a^2 + a\ell + \ell^2),$$

and solving for $\tan\theta$:

$$\tan\theta = \frac{\mu a(2a-\ell)}{2(a^2 + a\ell + \ell^2)}.$$

If $\ell > 2a$, at the moment of release ($\theta = 0$, $\dot{\theta} = 0$), the normal reaction N is:

$$N = mg \cdot \frac{3a(2a-\ell)}{2(3a^2 + \ell^2)}.$$

Since $\ell > 2a$, $2a - \ell < 0$, so $N < 0$. However, the rod can only exert an upward normal force ($N \geq 0$), so a negative N indicates the particle loses contact with the rod immediately upon release.

Q16

A railway truck, initially at rest, can move forwards without friction on a long straight horizontal track. On the truck, n guns are mounted parallel to the track and facing backwards, where $n > 1$. Each of the guns is loaded with a single projectile of mass m. The mass of the truck and guns (but not including the projectiles) is M.

When a gun is fired, the projectile leaves its muzzle horizontally with a speed $v - V$ relative to the ground, where V is the speed of the truck immediately before the gun is fired.

- All n guns are fired simultaneously. Find the speed, u, with which the truck moves, and show that the kinetic energy, K, which is gained by the system (truck, guns and projectiles) is given by

$$K = \frac{1}{2}nmv^2\left(1 + \frac{nm}{M}\right).$$

- Instead, the guns are fired one at a time. Let u_r be the speed of the truck when r guns have been fired, so that $u_0 = 0$. Show that, for $1 \leq r \leq n$,

$$u_r - u_{r-1} = \frac{mv}{M + (n-r)m}$$

  and hence that $u_n < u$.
- Let K_r be the total kinetic energy of the system when r guns have been fired (one at a time), so that $K_0 = 0$. Using $(*)$, show that, for $1 \leq r \leq n$,

$$K_r - K_{r-1} = \frac{1}{2}mv^2 + \frac{1}{2}mv(u_r - u_{r-1})$$

and hence show that

$$K_n = \frac{1}{2}nmv^2 + \frac{1}{2}mvu_n.$$

- Deduce that $K_n < K$.

Solution:

The system consists of the truck, guns, and projectiles, with no external forces acting horizontally, so momentum is conserved. Initially, the system is at rest, so the total initial momentum is zero.

Since all guns are fired simultaneously and the firing is instantaneous, the speed of the truck immediately before firing, V, is zero because the truck starts from rest and has not yet moved. The speed of each projectile relative to the ground is given as $v - V$. Substituting $V = 0$, the speed of each projectile relative to the ground is v in the backward direction.

Let u be the final speed of the truck in the forward direction. The total mass of the projectiles is nm, and each moves backward with speed v relative to the ground. The mass of the truck and guns (excluding projectiles) is M, and it moves forward with speed u.

Conservation of momentum gives:

$$0 = Mu - nmv$$

Solving for u:

$$Mu = nmv \quad \Rightarrow \quad u = \frac{nmv}{M}$$

The initial kinetic energy is zero since the system starts at rest. The final kinetic energy is the sum of the kinetic energy of the truck and the kinetic energy of all projectiles.

Kinetic energy of the truck:

$$\frac{1}{2}Mu^2 = \frac{1}{2}M\left(\frac{nmv}{M}\right)^2 = \frac{1}{2}\frac{(nmv)^2}{M} = \frac{1}{2}\frac{n^2m^2v^2}{M}$$

Kinetic energy of the n projectiles (each of mass m and speed v):

$$n \times \frac{1}{2}mv^2 = \frac{1}{2}nmv^2$$

Total final kinetic energy K:

$$K = \frac{1}{2}\frac{n^2m^2v^2}{M} + \frac{1}{2}nmv^2 = \frac{1}{2}nmv^2\left(\frac{nm}{M} + 1\right) = \frac{1}{2}nmv^2\left(1 + \frac{nm}{M}\right)$$

To find the speed change when the guns are fired one at a time, consider the firing of the r-th gun. The mass of the truck (including the remaining projectiles) just before firing the r-th gun is $M_{r-1} = M + (n - r + 1)m$, and the speed of the truck is u_{r-1}. The projectile is fired backward with a speed $v - u_{r-1}$ relative to the ground, so its velocity is $-(v - u_{r-1})$ in the forward direction.

By conservation of momentum (since there are no external horizontal forces), the momentum before firing equals the momentum after firing. Thus,

$$M_{r-1}u_{r-1} = M_r u_r - m(v - u_{r-1}),$$

where $M_r = M + (n-r)m$ is the mass of the truck after firing. Substituting the masses,

$$[M + (n-r+1)m]u_{r-1} = [M + (n-r)m]u_r - m(v - u_{r-1}).$$

Rearranging terms,

$$[M + (n-r+1)m]u_{r-1} + m(v - u_{r-1}) = [M + (n-r)m]u_r,$$
$$[M + (n-r+1)m - m]u_{r-1} + mv = [M + (n-r)m]u_r,$$
$$[M + (n-r)m]u_{r-1} + mv = [M + (n-r)m]u_r.$$

Solving for u_r,

$$u_r = u_{r-1} + \frac{mv}{M + (n-r)m}.$$

Therefore, the change in speed is

$$u_r - u_{r-1} = \frac{mv}{M + (n-r)m},$$

for $1 \leq r \leq n$.

The speed after all n guns are fired sequentially is

$$u_n = \sum_{r=1}^{n}(u_r - u_{r-1}) = \sum_{r=1}^{n} \frac{mv}{M + (n-r)m},$$

since $u_0 = 0$. Substituting $j = n - r$, so when $r = 1$, $j = n - 1$, and when $r = n$, $j = 0$,

$$u_n = mv \sum_{j=0}^{n-1} \frac{1}{M + jm}.$$

The speed when all guns are fired simultaneously, from the first part, is

$$u = \frac{nmv}{M}.$$

To show $u_n < u$, compare

$$u_n = mv \sum_{j=0}^{n-1} \frac{1}{M + jm} \quad \text{and} \quad u = \frac{nmv}{M}.$$

Since $mv > 0$, it suffices to show that

$$\sum_{j=0}^{n-1} \frac{1}{M+jm} < \frac{n}{M}.$$

The sum has n terms: $\frac{1}{M}$, $\frac{1}{M+m}$, $\frac{1}{M+2m}$, $\ldots$, $\frac{1}{M+(n-1)m}$. For each $j \geq 0$, $M + jm \geq M$, so $\frac{1}{M+jm} \leq \frac{1}{M}$, with equality if and only if $j = 0$. Since $n > 1$, there is at least one $j \geq 1$ (for example, $j = 1$ when $n \geq 2$), and for such j, $M + jm > M$, so $\frac{1}{M+jm} < \frac{1}{M}$. Thus,

$$\sum_{j=0}^{n-1} \frac{1}{M+jm} = \frac{1}{M} + \sum_{j=1}^{n-1} \frac{1}{M+jm} < \frac{1}{M} + \sum_{j=1}^{n-1} \frac{1}{M} = \frac{1}{M} + (n-1)\frac{1}{M} = \frac{n}{M}.$$

Therefore, $u_n < u$.

Change in Kinetic Energy per Firing

Consider the firing of the r-th gun. The mass of the truck (including guns and unfired projectiles) just before firing is $M + (n - r + 1)m$, and its speed is u_{r-1}. After firing, the projectile of mass m is ejected backwards with speed $v - u_{r-1}$ relative to the ground. The mass of the truck after firing is $M + (n - r)m$, and its speed is u_r.

The kinetic energy before firing is:

$$K_{r-1} = \frac{1}{2}[M + (n - r + 1)m]u_{r-1}^2$$

The kinetic energy after firing is:

$$K_r = \frac{1}{2}[M + (n - r)m]u_r^2 + \frac{1}{2}m(v - u_{r-1})^2$$

The change in kinetic energy is:

$$K_r - K_{r-1} = \frac{1}{2}[M + (n - r)m]u_r^2 + \frac{1}{2}m(v - u_{r-1})^2 - \frac{1}{2}[M + (n - r + 1)m]u_{r-1}^2$$

Let $m_t = M + (n - r + 1)m$ (mass before firing) and $m'_t = M + (n - r)m$ (mass of truck after firing). Note that $m'_t = m_t - m$. Substituting:

$$K_r - K_{r-1} = \frac{1}{2}(m_t - m)u_r^2 + \frac{1}{2}m(v - u_{r-1})^2 - \frac{1}{2}m_t u_{r-1}^2$$

Expand $(v - u_{r-1})^2 = v^2 - 2vu_{r-1} + u_{r-1}^2$:

$$K_r - K_{r-1} = \frac{1}{2}(m_t - m)u_r^2 + \frac{1}{2}mv^2 - mvu_{r-1} + \frac{1}{2}mu_{r-1}^2 - \frac{1}{2}m_t u_{r-1}^2$$

Combine the u_{r-1}^2 terms:

$$\frac{1}{2}mu_{r-1}^2 - \frac{1}{2}m_t u_{r-1}^2 = -\frac{1}{2}(m_t - m)u_{r-1}^2$$

So:

$$K_r - K_{r-1} = \frac{1}{2}(m_t - m)u_r^2 - \frac{1}{2}(m_t - m)u_{r-1}^2 + \frac{1}{2}mv^2 - mvu_{r-1}$$

Factor:

$$K_r - K_{r-1} = \frac{1}{2}(m_t - m)(u_r^2 - u_{r-1}^2) + \frac{1}{2}mv^2 - mvu_{r-1}$$

Since $u_r^2 - u_{r-1}^2 = (u_r - u_{r-1})(u_r + u_{r-1})$ and $u_r - u_{r-1} = \frac{mv}{M+(n-r)m} = \frac{mv}{m_t - m}$ from the given result:

$$(m_t - m)(u_r - u_{r-1}) = mv$$

Thus:

$$K_r - K_{r-1} = \frac{1}{2}(mv)(u_r + u_{r-1}) + \frac{1}{2}mv^2 - mvu_{r-1}$$

Simplify:

$$K_r - K_{r-1} = \frac{1}{2}mvu_r + \frac{1}{2}mvu_{r-1} + \frac{1}{2}mv^2 - mvu_{r-1} = \frac{1}{2}mvu_r - \frac{1}{2}mvu_{r-1} + \frac{1}{2}mv^2$$

$$K_r - K_{r-1} = \frac{1}{2}mv^2 + \frac{1}{2}mv(u_r - u_{r-1})$$

as required.

Total Kinetic Energy after n Firings

The initial kinetic energy is $K_0 = 0$. The total kinetic energy after n firings is the sum of the changes:

$$K_n = \sum_{r=1}^{n}(K_r - K_{r-1}) = \sum_{r=1}^{n}\left[\frac{1}{2}mv^2 + \frac{1}{2}mv(u_r - u_{r-1})\right]$$

Split the sum:

$$K_n = \frac{1}{2}mv^2\sum_{r=1}^{n}1 + \frac{1}{2}mv\sum_{r=1}^{n}(u_r - u_{r-1})$$

The first sum is:

$$\sum_{r=1}^{n}1 = n$$

The second sum is telescoping:

$$\sum_{r=1}^{n}(u_r - u_{r-1}) = (u_1 - u_0) + (u_2 - u_1) + \cdots + (u_n - u_{n-1}) = u_n - u_0 = u_n$$

since $u_0 = 0$. Thus:

$$K_n = \frac{1}{2}nmv^2 + \frac{1}{2}mvu_n$$

as required.

Comparison with Simultaneous Firing
From the simultaneous firing case, the kinetic energy is:

$$K = \frac{1}{2}nmv^2\left(1 + \frac{nm}{M}\right)$$

and the speed of the truck is $u = \frac{nmv}{M}$. Rewrite K:

$$K = \frac{1}{2}nmv^2 + \frac{1}{2}\frac{(nmv)^2}{M} = \frac{1}{2}nmv^2 + \frac{1}{2}Mu^2$$

since $u = \frac{nmv}{M}$, so $\frac{(nmv)^2}{M} = Mu^2$.

Now, compare K_n and K:

$$K - K_n = \left(\frac{1}{2}nmv^2 + \frac{1}{2}Mu^2\right) - \left(\frac{1}{2}nmv^2 + \frac{1}{2}mvu_n\right) = \frac{1}{2}Mu^2 - \frac{1}{2}mvu_n$$

Substitute $u = \frac{nmv}{M}$:

$$Mu^2 = M\left(\frac{nmv}{M}\right)^2 = \frac{(nmv)^2}{M}$$

So:

$$K - K_n = \frac{1}{2}\frac{(nmv)^2}{M} - \frac{1}{2}mvu_n = \frac{1}{2}mv\left(\frac{n^2mv}{M} - u_n\right)$$

Since $u = \frac{nmv}{M}$, it follows that:

$$\frac{n^2mv}{M} = n \cdot \frac{nmv}{M} = nu$$

Thus:

$$K - K_n = \frac{1}{2}mv(nu - u_n)$$

From the earlier result, $u_n < u$ for $n > 1$. Since $n > 1$ and $u > 0$:

$$nu - u_n > nu - u = (n-1)u > 0$$

Therefore, $K - K_n > 0$, so $K_n < K$.

Q17

Three pegs P, Q and R are fixed on a smooth horizontal table in such a way that they form the vertices of an equilateral triangle of side $2a$. A particle X of mass m lies on the table. It is attached to the pegs by three springs, PX, QX and RX, each of modulus of elasticity λ and natural length l, where $l < \frac{2}{\sqrt{3}}a$. Initially the particle is in equilibrium. Show that the extension in each spring is $\frac{2}{\sqrt{3}}a - l$.

The particle is then pulled a small distance directly towards P and released. Show that the tension T in the spring RX is given by

$$T = \frac{\lambda}{l}\left(\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2} - l\right),$$

where x is the displacement of X from its equilibrium position.

Show further that the particle performs approximate simple harmonic motion with period

$$2\pi\sqrt{\frac{4mla}{3(4a - \sqrt{3}l)\lambda}}.$$

Solution:

The pegs P, Q, and R form an equilateral triangle with side length $2a$. Due to the symmetry of the system and the identical springs, the particle X must be in equilibrium at the centroid of the triangle for the net force to be zero.

The coordinates of the pegs are set as follows: P at $(0, 0)$, Q at $(2a, 0)$, and R at $(a, \sqrt{3}a)$. The centroid G of the triangle is at:

$$x = \frac{0 + 2a + a}{3} = a, \quad y = \frac{0 + 0 + \sqrt{3}a}{3} = \frac{\sqrt{3}}{3}a = \frac{a}{\sqrt{3}}.$$

Thus, G is at $(a, \frac{a}{\sqrt{3}})$.

The distance from G to any peg is calculated as follows. For peg P at $(0, 0)$:

$$\text{distance} = \sqrt{(a - 0)^2 + \left(\frac{a}{\sqrt{3}} - 0\right)^2} = \sqrt{a^2 + \frac{a^2}{3}} = \sqrt{\frac{4a^2}{3}} = \frac{2a}{\sqrt{3}}.$$

Similarly, the distances to Q and R are also $\frac{2a}{\sqrt{3}}$, confirming symmetry.

The extension in each spring is the difference between the current length and the natural length l. Given the distance to each peg is $\frac{2a}{\sqrt{3}}$ and $l < \frac{2a}{\sqrt{3}}$ (so the springs are extended), the extension δ is:

$$\delta = \frac{2a}{\sqrt{3}} - l = \frac{2}{\sqrt{3}}a - l.$$

At this position, the forces from the springs balance. The force from each spring has magnitude $\lambda\delta$ and is directed towards the respective peg. By symmetry, the vector sum of the forces is zero, confirming equilibrium at the centroid with the given extension.

Thus, the extension in each spring is $\frac{2}{\sqrt{3}}a - l$.

The pegs P, Q, and R form an equilateral triangle with side length $2a$. The equilibrium position of the particle X is at the centroid G of the triangle, with coordinates $\left(a, \frac{a}{\sqrt{3}}\right)$, where

P is at $(0,0)$, Q is at $(2a,0)$, and R is at $(a,\sqrt{3}a)$.

The particle is displaced a distance x directly towards peg P. The direction from G to P is given by the vector $\left(-a,-\frac{a}{\sqrt{3}}\right)$, with magnitude $\frac{2a}{\sqrt{3}}$. The unit vector in this direction is $\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$.

Thus, the coordinates of X after displacement are:

$$x_X = a + x\left(-\frac{\sqrt{3}}{2}\right) = a - \frac{\sqrt{3}}{2}x,$$

$$y_X = \frac{a}{\sqrt{3}} + x\left(-\frac{1}{2}\right) = \frac{a}{\sqrt{3}} - \frac{1}{2}x.$$

The spring RX connects peg R at $(a,\sqrt{3}a)$ to the particle X at $\left(a-\frac{\sqrt{3}}{2}x, \frac{a}{\sqrt{3}}-\frac{1}{2}x\right)$. The current length L_{RX} of the spring is the distance between these points:

$$L_{RX} = \sqrt{\left(\left(a-\frac{\sqrt{3}}{2}x\right)-a\right)^2 + \left(\left(\frac{a}{\sqrt{3}}-\frac{1}{2}x\right)-\sqrt{3}a\right)^2}.$$

Simplifying the differences:

$$\Delta x = \left(a-\frac{\sqrt{3}}{2}x\right) - a = -\frac{\sqrt{3}}{2}x,$$

$$\Delta y = \left(\frac{a}{\sqrt{3}}-\frac{1}{2}x\right) - \sqrt{3}a = \frac{a}{\sqrt{3}} - \sqrt{3}a - \frac{1}{2}x = a\left(\frac{1}{\sqrt{3}}-\sqrt{3}\right) - \frac{1}{2}x = -\frac{2a}{\sqrt{3}} - \frac{1}{2}x.$$

Squaring the differences:

$$(\Delta x)^2 = \left(-\frac{\sqrt{3}}{2}x\right)^2 = \frac{3}{4}x^2,$$

$$(\Delta y)^2 = \left(-\frac{2a}{\sqrt{3}}-\frac{1}{2}x\right)^2 = \left(\frac{2a}{\sqrt{3}}+\frac{1}{2}x\right)^2$$

$$= \left(\frac{2a}{\sqrt{3}}\right)^2 + 2\cdot\frac{2a}{\sqrt{3}}\cdot\frac{1}{2}x + \left(\frac{1}{2}x\right)^2 = \frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + \frac{1}{4}x^2.$$

Summing the squares:

$$(\Delta x)^2 + (\Delta y)^2 = \frac{3}{4}x^2 + \frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + \frac{1}{4}x^2 = \frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + \left(\frac{3}{4}+\frac{1}{4}\right)x^2 = \frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2.$$

Thus, the length is:

$$L_{RX} = \sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2}.$$

The tension T in the spring is given by $T = \frac{\lambda}{l} \times \text{extension}$, where the extension is $L_{RX} - l$ and l is the natural length. Therefore:

$$T = \frac{\lambda}{l}\left(\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2} - l\right).$$

The particle X is displaced a small distance x directly towards peg P from its equilibrium position at the centroid of the equilateral triangle formed by pegs P, Q, and R with side length $2a$. The motion is constrained to the line joining the centroid to peg P due to symmetry, and the net force perpendicular to this line cancels out.

The net force component along the direction towards P (denoted by the unit vector $\hat{u} = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$) is derived as follows. The force on the particle due to each spring is given by Hooke's law, and the component along $\hat{u}$ for springs PX, QX, and RX is computed.

The lengths of the springs as functions of the displacement x are:

- $L_P = \sqrt{\frac{4a^2}{3} - \frac{4a}{\sqrt{3}}x + x^2}$
- $L_Q = L_R = \sqrt{\frac{4a^2}{3} + \frac{2a}{\sqrt{3}}x + x^2}$

The projections of the vectors from the particle to the pegs onto $\hat{u}$ are:

- $d_P = \frac{2a}{\sqrt{3}} - x$
- $d_Q = d_R = -\frac{a}{\sqrt{3}} - x$

The force component along $\hat{u}$ for each spring is:

- $\vec{F}_P \cdot \hat{u} = \frac{\lambda}{l}(L_P - l)\frac{d_P}{L_P}$
- $\vec{F}_Q \cdot \hat{u} = \frac{\lambda}{l}(L_Q - l)\frac{d_Q}{L_Q}$
- $\vec{F}_R \cdot \hat{u} = \frac{\lambda}{l}(L_R - l)\frac{d_R}{L_R}$

The net force component along $\hat{u}$ is:

$$F_{\text{net}} \cdot \hat{u} = \frac{\lambda}{l}\left[(L_P - l)\frac{d_P}{L_P} + 2(L_Q - l)\frac{d_Q}{L_Q}\right]$$

since $L_Q = L_R$ and $d_Q = d_R$.

At equilibrium ($x = 0$), the net force is zero. For small x, the net force is approximated by expanding to first order around $x = 0$. The derivative of the net force with respect to x at $x = 0$ is:

$$f'(0) = \frac{\lambda}{l}\left(-\frac{3}{2} - \frac{9\delta_0}{4\sqrt{3}a}\right)$$

where $\delta_0 = \frac{2a}{\sqrt{3}} - l$ is the equilibrium extension.

Thus, for small x, the net force is:

$$F_{\text{net}} \cdot \hat{u} \approx f'(0)x = \frac{\lambda}{l}\left(-\frac{3}{2} - \frac{9\delta_0}{4\sqrt{3}a}\right)x$$

The equation of motion along the direction of $\hat{u}$ is:

$$m\ddot{x} = f'(0)x$$

where m is the mass of the particle. Since $f'(0) < 0$, this can be written as:

$$m\ddot{x} = -cx$$

with $c = -f'(0) > 0$. Thus:

$$\ddot{x} + \frac{c}{m}x = 0$$

which describes simple harmonic motion with angular frequency $\omega = \sqrt{c/m}$.

Substituting $c = -f'(0)$:

$$-f'(0) = -\frac{\lambda}{l}\left(-\frac{3}{2} - \frac{9\delta_0}{4\sqrt{3}a}\right) = \frac{\lambda}{l}\left(\frac{3}{2} + \frac{9\delta_0}{4\sqrt{3}a}\right)$$

Using $\delta_0 = \frac{2a}{\sqrt{3}} - l$:

$$-f'(0) = \frac{\lambda}{l}\left[\frac{3}{2} + \frac{9}{4\sqrt{3}a}\left(\frac{2a}{\sqrt{3}} - l\right)\right] = \frac{\lambda}{l}\left(3 - \frac{9l}{4\sqrt{3}a}\right) = \frac{3\lambda}{l}\left(1 - \frac{\sqrt{3}l}{4a}\right) = \frac{3\lambda(4a - \sqrt{3}l)}{4al}$$

The angular frequency is:

$$\omega = \sqrt{\frac{-f'(0)}{m}} = \sqrt{\frac{3\lambda(4a - \sqrt{3}l)}{4alm}}$$

Thus, the period T is:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4alm}{3\lambda(4a - \sqrt{3}l)}} = 2\pi\sqrt{\frac{4mla}{3\lambda(4a - \sqrt{3}l)}}$$

which matches the given expression.

The motion is approximately simple harmonic for small displacements x, and the period is as derived.

Q18

A particle of mass m is projected with velocity $\mathbf{u}$. It is acted upon by the force mg due to gravity and by a resistive force $-mk\mathbf{v}$, where $\mathbf{v}$ is its velocity and k is a positive constant.

Given that, at time t after projection, its position $\mathbf{r}$ relative to the point of projection is given by

$$\mathbf{r} = \left(\frac{kt - 1 + e^{-kt}}{k^2}\right)\mathbf{g} + \left(\frac{1 - e^{-kt}}{k}\right)\mathbf{u},$$

find an expression for $\mathbf{v}$ in terms of k, t, $\mathbf{g}$ and $\mathbf{u}$. Verify that the equation of motion and the initial conditions are satisfied.

Let $\mathbf{u} = u\cos\alpha\,\mathbf{i} + u\sin\alpha\,\mathbf{j}$ and $\mathbf{g} = -g\mathbf{j}$, where $0 < \alpha < 90^\circ$, and let T be the time after projection at which $\mathbf{r}\cdot\mathbf{j} = 0$. Show that

$$uk\sin\alpha = \left(\frac{kT}{1 - e^{-kT}} - 1\right)g.$$

Let β be the acute angle between $\mathbf{v}$ and $\mathbf{i}$ at time T. Show that

$$\tan\beta = \frac{(e^{kT} - 1)g}{uk\cos\alpha} - \tan\alpha.$$

Show further that $\tan\beta > \tan\alpha$ (you may assume that $\sinh kT > kT$) and deduce that $\beta > \alpha$.

Solution:

The velocity $\mathbf{v}$ is the derivative of the position $\mathbf{r}$ with respect to time t. Given the position vector

$$\mathbf{r} = \left(\frac{kt - 1 + e^{-kt}}{k^2}\right)\mathbf{g} + \left(\frac{1 - e^{-kt}}{k}\right)\mathbf{u},$$

differentiate with respect to t.

Define the coefficients:

$$A(t) = \frac{kt - 1 + e^{-kt}}{k^2}, \quad B(t) = \frac{1 - e^{-kt}}{k}.$$

Thus,

$$\mathbf{r} = A(t)\mathbf{g} + B(t)\mathbf{u}.$$

The velocity is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dA}{dt}\mathbf{g} + \frac{dB}{dt}\mathbf{u},$$

since $\mathbf{g}$ and $\mathbf{u}$ are constant vectors.

Compute the derivatives:

$$\frac{dA}{dt} = \frac{d}{dt}\left[\frac{kt - 1 + e^{-kt}}{k^2}\right] = \frac{1}{k^2}\left(k - ke^{-kt}\right) = \frac{1}{k}(1 - e^{-kt}),$$

$$\frac{dB}{dt} = \frac{d}{dt}\left[\frac{1 - e^{-kt}}{k}\right] = \frac{1}{k}\left(ke^{-kt}\right) = e^{-kt}.$$

Therefore,

$$\mathbf{v} = \frac{1}{k}(1 - e^{-kt})\mathbf{g} + e^{-kt}\mathbf{u}.$$

This can be written as

$$\mathbf{v} = e^{-kt}\mathbf{u} + \frac{1 - e^{-kt}}{k}\mathbf{g}.$$

Verification of Initial Conditions
At $t = 0$:

- Velocity:

$$\mathbf{v}(0) = e^{-k\cdot 0}\mathbf{u} + \frac{1 - e^{-k\cdot 0}}{k}\mathbf{g} = (1)\mathbf{u} + \frac{1-1}{k}\mathbf{g} = \mathbf{u} + 0 \cdot \mathbf{g} = \mathbf{u},$$

  which matches the initial velocity.
- Position:

$$\mathbf{r}(0) = \left(\frac{k \cdot 0 - 1 + e^{-k\cdot 0}}{k^2}\right)\mathbf{g} + \left(\frac{1 - e^{-k\cdot 0}}{k}\right)\mathbf{u} = \left(\frac{-1+1}{k^2}\right)\mathbf{g} + \left(\frac{1-1}{k}\right)\mathbf{u} = \mathbf{0},$$

  which matches the initial position at the point of projection.

Verification of the Equation of Motion
The forces acting on the particle are gravity $m\mathbf{g}$ and the resistive force $-mk\mathbf{v}$, so the total force is

$$\mathbf{F} = m\mathbf{g} - mk\mathbf{v}.$$

By Newton's second law,

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt},$$

so the equation of motion is

$$\frac{d\mathbf{v}}{dt} = \mathbf{g} - k\mathbf{v}.$$

Differentiate $\mathbf{v}$ with respect to t:

$$\mathbf{v} = e^{-kt}\mathbf{u} + \frac{1}{k}(1 - e^{-kt})\mathbf{g}.$$

Define the coefficients:

$$C(t) = e^{-kt}, \quad D(t) = \frac{1}{k}(1 - e^{-kt}).$$

Thus,

$$\frac{d\mathbf{v}}{dt} = \frac{dC}{dt}\mathbf{u} + \frac{dD}{dt}\mathbf{g},$$

where

$$\frac{dC}{dt} = -ke^{-kt}, \quad \frac{dD}{dt} = \frac{1}{k}\left(ke^{-kt}\right) = e^{-kt}.$$

So,

$$\frac{d\mathbf{v}}{dt} = (-ke^{-kt})\mathbf{u} + e^{-kt}\mathbf{g} = e^{-kt}\mathbf{g} - ke^{-kt}\mathbf{u}.$$

Now compute $\mathbf{g} - k\mathbf{v}$:

$$k\mathbf{v} = k\left[e^{-kt}\mathbf{u} + \frac{1}{k}(1 - e^{-kt})\mathbf{g}\right] = ke^{-kt}\mathbf{u} + (1 - e^{-kt})\mathbf{g},$$

$$\mathbf{g} - k\mathbf{v} = \mathbf{g} - \left[ke^{-kt}\mathbf{u} + (1 - e^{-kt})\mathbf{g}\right] = \mathbf{g} - (1 - e^{-kt})\mathbf{g} - ke^{-kt}\mathbf{u} = e^{-kt}\mathbf{g} - ke^{-kt}\mathbf{u}.$$

This matches $\frac{d\mathbf{v}}{dt}$. Thus, the equation of motion is satisfied.

The position vector is given by

$$\mathbf{r} = \left(\frac{kt - 1 + e^{-kt}}{k^2}\right)\mathbf{g} + \left(\frac{1 - e^{-kt}}{k}\right)\mathbf{u},$$

with $\mathbf{u} = u\cos\alpha\,\mathbf{i} + u\sin\alpha\,\mathbf{j}$ and $\mathbf{g} = -g\,\mathbf{j}$.

The y-component of $\mathbf{r}$ is

$$y = -g\left(\frac{kt - 1 + e^{-kt}}{k^2}\right) + (u\sin\alpha)\left(\frac{1 - e^{-kt}}{k}\right).$$

At time $t = T$, $y = 0$, so

$$-g\left(\frac{kT - 1 + e^{-kT}}{k^2}\right) + (u\sin\alpha)\left(\frac{1 - e^{-kT}}{k}\right) = 0.$$

Multiplying through by k^2 gives

$$-g(kT - 1 + e^{-kT}) + ku\sin\alpha(1 - e^{-kT}) = 0.$$

Expanding the terms:

$$-gkT + g - ge^{-kT} + ku\sin\alpha - ku\sin\alpha e^{-kT} = 0.$$

Grouping constant and exponential terms:

$$g - gkT + ku\sin\alpha - e^{-kT}(g + ku\sin\alpha) = 0.$$

Factoring $(g + ku\sin\alpha)$:

$$(g + ku\sin\alpha)(1 - e^{-kT}) - gkT = 0.$$

Solving for $g + ku\sin\alpha$:

$$g + ku\sin\alpha = \frac{gkT}{1 - e^{-kT}}.$$

Rearranging for $ku\sin\alpha$:

$$ku\sin\alpha = \frac{gkT}{1-e^{-kT}} - g = g\left(\frac{kT}{1-e^{-kT}} - 1\right).$$

Thus,

$$uk\sin\alpha = g\left(\frac{kT}{1-e^{-kT}} - 1\right).$$

The velocity vector at time t is given by $\mathbf{v} = e^{-kt}\mathbf{u} + \frac{1-e^{-kt}}{k}\mathbf{g}$. Substituting $\mathbf{u} = u\cos\alpha\mathbf{i} + u\sin\alpha\mathbf{j}$ and $\mathbf{g} = -g\mathbf{j}$, the components are:

$$v_x = e^{-kt}u\cos\alpha, \quad v_y = e^{-kt}u\sin\alpha - \frac{g}{k}(1-e^{-kt}).$$

At time $t = T$, when $\mathbf{r}\cdot\mathbf{j} = 0$, the components are:

$$v_x(T) = e^{-kT}u\cos\alpha, \quad v_y(T) = e^{-kT}u\sin\alpha - \frac{g}{k}(1-e^{-kT}).$$

The acute angle β between $\mathbf{v}$ and $\mathbf{i}$ satisfies $\tan\beta = \left|\frac{v_y}{v_x}\right|$. Since $v_x(T) > 0$ and $v_y(T) < 0$ (as the particle is descending), it follows that:

$$\tan\beta = -\frac{v_y(T)}{v_x(T)} = -\frac{e^{-kT}u\sin\alpha - \frac{g}{k}(1-e^{-kT})}{e^{-kT}u\cos\alpha}.$$

Simplifying the expression:

$$\tan\beta = -\left(\frac{u\sin\alpha}{u\cos\alpha} - \frac{\frac{g}{k}(1-e^{-kT})}{e^{-kT}u\cos\alpha}\right) = -\tan\alpha + \frac{g(1-e^{-kT})}{ku\cos\alpha e^{-kT}}.$$

Noting that $\frac{1-e^{-kT}}{e^{-kT}} = e^{kT} - 1$, the expression becomes:

$$\tan\beta = \frac{g(e^{kT}-1)}{ku\cos\alpha} - \tan\alpha.$$

To show $\tan\beta > \tan\alpha$, consider:

$$\tan\beta - \tan\alpha = \frac{g(e^{kT}-1)}{ku\cos\alpha} - \tan\alpha - \tan\alpha = \frac{g(e^{kT}-1)}{ku\cos\alpha} - 2\tan\alpha.$$

However, it is sufficient to show:

$$\frac{g(e^{kT}-1)}{ku\cos\alpha} > \tan\alpha.$$

Since $\cos\alpha > 0$ (as $0^\circ < \alpha < 90^\circ$), multiply both sides by $\cos\alpha$:

$$\frac{g(e^{kT}-1)}{ku} > \sin\alpha.$$

From the given condition at $t = T$, the equation $uk\sin\alpha = g\left(\frac{kT}{1-e^{-kT}} - 1\right)$ holds. Solving for $\sin\alpha$:

$$\sin\alpha = \frac{g}{uk}\left(\frac{kT}{1-e^{-kT}} - 1\right).$$

Substitute this into the inequality:

$$\frac{g(e^{kT}-1)}{ku} > \frac{g}{uk}\left(\frac{kT}{1-e^{-kT}} - 1\right).$$

Since $\frac{g}{uk} > 0$, divide both sides by it:

$$e^{kT} - 1 > \frac{kT}{1-e^{-kT}} - 1.$$

Adding 1 to both sides:

$$e^{kT} > \frac{kT}{1-e^{-kT}}.$$

Rewrite the right side:

$$\frac{kT}{1-e^{-kT}} = kT\frac{e^{kT}}{e^{kT}-1}.$$

Thus:

$$e^{kT} > \frac{kTe^{kT}}{e^{kT}-1}.$$

Dividing both sides by $e^{kT} > 0$:

$$1 > \frac{kT}{e^{kT}-1}.$$

Since $e^{kT} - 1 > 0$ for $kT > 0$, and $T > 0$, it follows that:

$$e^{kT} - 1 > kT.$$

This inequality holds for $kT > 0$ because $e^x > x + 1$ for all $x > 0$. Additionally, the assumption $\sinh kT > kT$ (which is true for $kT > 0$) supports this result, as $e^x - 1 > \sinh x > x$ for $x > 0$. Thus, $\tan\beta > \tan\alpha$.

Since both α and β are acute angles, and the tangent function is strictly increasing in the interval $(0°, 90°)$, it follows that $\beta > \alpha$.

Q19

A particle P of mass m moves on a smooth fixed straight horizontal rail and is attached to a fixed peg Q by a light elastic string of natural length a and modulus λ. The peg Q is a distance a from the rail. Initially P is at rest with $PQ = a$.

An impulse imparts to P a speed v along the rail. Let x be the displacement at time t of P from its initial position. Obtain the equation

$$\dot{x}^2 = v^2 - k^2\left(\sqrt{x^2+a^2} - a\right)^2$$

where $k^2 = \lambda/(ma)$, $k > 0$, and the dot denotes differentiation with respect to t.

Find, in terms of k, a and v, the greatest value, x_0, attained by x. Find also the acceleration of P at $x = x_0$.

Obtain, in the form of an integral, an expression for the period of the motion. Show that, in the case $v \ll ka$ (that is, v is much less than ka), this is approximately

$$\sqrt{\frac{32a}{kv}} \int_0^1 \frac{1}{\sqrt{1-u^4}}\, du.$$

Solution:

The system is conservative since the rail is smooth (no friction) and gravity, if present, acts perpendicular to the direction of motion along the rail, doing no work. Thus, mechanical energy is conserved.

Initially, at $t = 0$, the particle is at rest with $x = 0$ and $\dot{x} = v$ after the impulse. The distance $PQ = a$, which equals the natural length of the elastic string, so the extension is zero and the elastic potential energy is zero. The initial kinetic energy is $\frac{1}{2}mv^2$. Therefore, the initial total mechanical energy is:

$$E = \frac{1}{2}mv^2.$$

At any time t, the displacement is x, the velocity is $\dot{x}$, and the distance PQ is $\sqrt{x^2+a^2}$. The extension of the string is $\sqrt{x^2+a^2} - a$. The elastic potential energy for a string with modulus λ and natural length a is $\frac{1}{2}\frac{\lambda}{a}(\text{extension})^2$, so:

$$\text{Potential energy} = \frac{1}{2}\frac{\lambda}{a}\left(\sqrt{x^2+a^2} - a\right)^2.$$

The kinetic energy is $\frac{1}{2}m\dot{x}^2$. Thus, the total mechanical energy at time t is:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{\lambda}{a}\left(\sqrt{x^2+a^2} - a\right)^2.$$

By conservation of energy, the initial energy equals the energy at time t:

$$\frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{\lambda}{a}\left(\sqrt{x^2+a^2} - a\right)^2.$$

Multiplying both sides by 2:

$$mv^2 = m\dot{x}^2 + \frac{\lambda}{a}\left(\sqrt{x^2+a^2} - a\right)^2.$$

Dividing both sides by m:

$$v^2 = \dot{x}^2 + \frac{\lambda}{am}\left(\sqrt{x^2+a^2} - a\right)^2.$$

Given $k^2 = \lambda/(ma)$, it follows that $\frac{\lambda}{am} = k^2$. Substituting this in:

$$v^2 = \dot{x}^2 + k^2\left(\sqrt{x^2+a^2} - a\right)^2.$$

Rearranging for $\dot{x}^2$:

$$\dot{x}^2 = v^2 - k^2\left(\sqrt{x^2+a^2} - a\right)^2.$$

This is the required equation of motion.

The greatest value of x, denoted x_0, occurs when the velocity $\dot{x} = 0$. Substituting $\dot{x} = 0$ into the given equation of motion:

$$0 = v^2 - k^2\left(\sqrt{x_0^2+a^2} - a\right)^2$$

Solving for x_0:

$$v^2 = k^2\left(\sqrt{x_0^2+a^2} - a\right)^2$$

$$\frac{v}{k} = \sqrt{x_0^2+a^2} - a \quad (\text{since} \quad \sqrt{x_0^2+a^2} - a \geq 0)$$

$$\sqrt{x_0^2+a^2} = a + \frac{v}{k}$$

Squaring both sides:

$$x_0^2 + a^2 = \left(a + \frac{v}{k}\right)^2$$

$$x_0^2 + a^2 = a^2 + 2a\frac{v}{k} + \frac{v^2}{k^2}$$

$$x_0^2 = \frac{v^2}{k^2} + \frac{2av}{k}$$

$$x_0 = \sqrt{\frac{v^2}{k^2} + \frac{2av}{k}} \quad (\text{taking the positive root since } x_0 \text{ is a distance})$$

The acceleration at $x = x_0$ is found using the force along the rail. The tension in the string is $T = \frac{\lambda}{a}(\sqrt{x^2+a^2} - a)$, and the x-component of the force is $F_x = -T\frac{x}{\sqrt{x^2+a^2}}$. With $k^2 = \frac{\lambda}{ma}$, the acceleration $\ddot{x} = \frac{F_x}{m}$ is:

$$\ddot{x} = -k^2 x\left(1 - \frac{a}{\sqrt{x^2+a^2}}\right)$$

At $x = x_0$, $\sqrt{x_0^2+a^2} = a + \frac{v}{k}$, so:

$$\ddot{x} = -k^2 x_0 \left(1 - \frac{a}{a + \frac{v}{k}}\right) = -k^2 x_0 \left(\frac{\frac{v}{k}}{a + \frac{v}{k}}\right) = -k^2 x_0 \frac{v}{k(a + \frac{v}{k})} = -k^2 x_0 \frac{v}{ka + v}$$

Substituting $x_0 = \sqrt{\frac{v^2}{k^2} + \frac{2av}{k}}$:

$$\ddot{x} = -k^2 \left(\sqrt{\frac{v^2}{k^2} + \frac{2av}{k}}\right) \frac{v}{ka + v} = -k^2 \cdot \frac{1}{k}\sqrt{v^2 + 2avk} \cdot \frac{v}{v + ak} = -kv\frac{\sqrt{v^2 + 2avk}}{v + ak}$$

Thus, the greatest value x_0 and the acceleration at $x = x_0$ are:

The period of the motion, T, is the time for one complete oscillation. Due to the symmetry of the system, the period is four times the time taken to move from the equilibrium position $x = 0$ to the maximum displacement $x = x_0$, where the velocity is zero. From the equation of motion $\dot{x}^2 = v^2 - k^2\left(\sqrt{x^2 + a^2} - a\right)^2$, the velocity $\dot{x} = \frac{dx}{dt}$ is given by $\dot{x} = \sqrt{v^2 - k^2\left(\sqrt{x^2 + a^2} - a\right)^2}$ for the motion from $x = 0$ to $x = x_0$. Thus, the time for this quarter-period is:

$$\tau = \int_0^{x_0} \frac{dx}{\sqrt{v^2 - k^2\left(\sqrt{x^2 + a^2} - a\right)^2}}.$$

The period is therefore:

$$T = 4\int_0^{x_0} \frac{dx}{\sqrt{v^2 - k^2\left(\sqrt{x^2 + a^2} - a\right)^2}},$$

where x_0 is the greatest displacement, given by:

$$x_0 = \sqrt{\frac{v^2}{k^2} + \frac{2av}{k}} = \frac{\sqrt{v^2 + 2avk}}{k}.$$

In the case where $v \ll ka$, the parameter $\delta = \frac{v}{ka} \ll 1$. The maximum displacement x_0 is small, and the motion is approximated by expanding the potential energy term. The integral for the period simplifies to:

$$T \approx \sqrt{\frac{32a}{kv}} \int_0^1 \frac{du}{\sqrt{1 - u^4}}.$$

This approximation is derived by substituting $u = x/a$ and $\delta = v/(ka)$, leading to $u_0 = x_0/a = \sqrt{2\delta + \delta^2}$. For small δ, the expression $\sqrt{u^2 + 1} - 1 \approx \frac{u^2}{2} - \frac{u^4}{8}$ is used, and the integral is transformed via $w = u^2/(2\delta)$, resulting in the given form after evaluating the limits and simplifying.

Q20

A light rod of length $2a$ has a particle of mass m attached to each end and it moves in a vertical plane. The midpoint of the rod has coordinates (x, y), where the x-axis is horizontal (within the plane of motion) and y is the height above a horizontal table. Initially, the rod is vertical, and at time t later it is inclined at an angle θ to the vertical.

Show that the velocity of one particle can be written in the form

$$\begin{pmatrix} \dot{x} + a\dot{\theta}\cos\theta \\ \dot{y} - a\dot{\theta}\sin\theta \end{pmatrix}$$

and that

$$m\begin{pmatrix} \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix} = -T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where the dots denote differentiation with respect to time t and T is the tension in the rod.

Obtain the corresponding equations for the other particle.

Deduce that $\ddot{x} = 0$, $\ddot{y} = -g$ and $\ddot{\theta} = 0$.

Initially, the midpoint of the rod is a height h above the table, the velocity of the higher particle is

$$\begin{pmatrix} u \\ v \end{pmatrix},$$

and the velocity of the lower particle is

$$\begin{pmatrix} 0 \\ v \end{pmatrix}.$$

Given that the two particles hit the table for the first time simultaneously, when the rod has rotated by $\frac{1}{2}\pi$, show

$$2hu^2 = \pi^2 a^2 g - 2\pi uva.$$

is valid.

Solution:

The position of one particle (say, the particle that is above the center when $\theta = 0$) relative to the midpoint (x, y) is given by:

$$x_1 = x + a\sin\theta, \quad y_1 = y + a\cos\theta.$$

The velocity components are obtained by differentiating with respect to time t:

$$\dot{x}_1 = \dot{x} + a\cos\theta \cdot \dot{\theta}, \quad \dot{y}_1 = \dot{y} - a\sin\theta \cdot \dot{\theta},$$

which can be written in vector form as:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} \dot{x} + a\dot{\theta}\cos\theta \\ \dot{y} - a\dot{\theta}\sin\theta \end{pmatrix}.$$

The acceleration components are obtained by differentiating the velocity components:

$$\ddot{x}_1 = \frac{d}{dt}(\dot{x} + a\dot{\theta}\cos\theta) = \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta,$$

$$\ddot{y}_1 = \frac{d}{dt}(\dot{y} - a\dot{\theta}\sin\theta) = \ddot{y} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta,$$

which can be written in vector form as:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{pmatrix} = \begin{pmatrix} \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix}.$$

The forces acting on the particle are gravity and the tension in the rod. Gravity acts downward with force $(0, -mg)$. The tension force acts along the rod toward the center. For this particle, the vector from the particle to the center is $(-a\sin\theta, -a\cos\theta)$, so the unit vector in this direction is $(-\sin\theta, -\cos\theta)$. Thus, the tension force is $T(-\sin\theta, -\cos\theta) = (-T\sin\theta, -T\cos\theta)$.

The total force on the particle is:

$$\begin{pmatrix} -T\sin\theta \\ -T\cos\theta - mg \end{pmatrix} = -T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By Newton's second law, the mass times acceleration equals the total force:

$$m\begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{pmatrix} = -T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Substituting the acceleration vector gives:

$$m\begin{pmatrix} \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix} = -T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This confirms the given expressions for the velocity and the equation of motion.

The position of the other particle (particle B) relative to the midpoint (x, y) is given by:

$$x_B = x - a\sin\theta, \quad y_B = y - a\cos\theta.$$

Differentiating with respect to time t gives the velocity components:

$$\dot{x}_B = \dot{x} - a\cos\theta\cdot\dot{\theta}, \quad \dot{y}_B = \dot{y} + a\sin\theta\cdot\dot{\theta}.$$

Thus, the velocity vector is:

$$\begin{pmatrix} \dot{x}_B \\ \dot{y}_B \end{pmatrix} = \begin{pmatrix} \dot{x} - a\dot{\theta}\cos\theta \\ \dot{y} + a\dot{\theta}\sin\theta \end{pmatrix}.$$

Differentiating again gives the acceleration components:

$$\ddot{x}_B = \ddot{x} - a\frac{d}{dt}(\dot{\theta}\cos\theta) = \ddot{x} - a(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = \ddot{x} - a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta,$$

$$\ddot{y}_B = \ddot{y} + a\frac{d}{dt}(\dot{\theta}\sin\theta) = \ddot{y} + a(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) = \ddot{y} + a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta.$$

Thus, the acceleration vector is:

$$\begin{pmatrix} \ddot{x}_B \\ \ddot{y}_B \end{pmatrix} = \begin{pmatrix} \ddot{x} - a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta \\ \ddot{y} + a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta \end{pmatrix}.$$

The forces acting on particle B are gravity $(0, -mg)$ and tension. The vector from particle B to the midpoint is $(a\sin\theta, a\cos\theta)$, which has magnitude a, so the unit vector in this direction is $(\sin\theta, \cos\theta)$. Thus, the tension force is $T(\sin\theta, \cos\theta)$. The total force is:

$$\begin{pmatrix} T\sin\theta \\ T\cos\theta - mg \end{pmatrix} = T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By Newton's second law:

$$m\begin{pmatrix} \ddot{x}_B \\ \ddot{y}_B \end{pmatrix} = T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Substituting the acceleration components:

$$m\begin{pmatrix} \ddot{x} - a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta \\ \ddot{y} + a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta \end{pmatrix} = T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For particle A (the first particle), the equation of motion is:

$$m\begin{pmatrix} \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix} = -T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To deduce $\ddot{x} = 0$, $\ddot{y} = -g$, and $\ddot{\theta} = 0$, consider the equations for the x-components and y-components of both particles.

Adding the x-component equations:

$$m(\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta) + m(\ddot{x} - a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta) = -T\sin\theta + T\sin\theta,$$

which simplifies to:

$$2m\ddot{x} = 0 \implies \ddot{x} = 0.$$

Adding the y-component equations:

$$m(\ddot{y} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta) + m(\ddot{y} + a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta) = (-T\cos\theta - mg) + (T\cos\theta - mg),$$

which simplifies to:

$$2m\ddot{y} = -2mg \implies \ddot{y} = -g.$$

To find $\ddot{\theta}$, subtract the x-component equation for particle A from that for particle B:

$$m(\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta) - m(\ddot{x} - a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta) = -T\sin\theta - T\sin\theta,$$

which simplifies to:

$$m(2a\ddot{\theta}\cos\theta - 2a\dot{\theta}^2\sin\theta) = -2T\sin\theta,$$

or:

$$ma(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = -T\sin\theta. \quad (1)$$

Subtract the y-component equation for particle B from that for particle A:

$$m(\ddot{y} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta) - m(\ddot{y} + a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta) = (-T\cos\theta - mg) - (T\cos\theta - mg),$$

which simplifies to:

$$m(-2a\ddot{\theta}\sin\theta - 2a\dot{\theta}^2\cos\theta) = -2T\cos\theta,$$

or:

$$ma(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta) = -T\cos\theta. \quad (2)$$

Solving equation (1) for T:

$$T = ma\left(-\ddot{\theta}\cot\theta + \dot{\theta}^2\right) \quad (\text{for } \sin\theta \neq 0).$$

Solving equation (2) for T:

$$T = ma\left(\ddot{\theta}\tan\theta + \dot{\theta}^2\right) \quad (\text{for } \cos\theta \neq 0).$$

Setting the expressions for T equal:

$$-\ddot{\theta}\cot\theta + \dot{\theta}^2 = \ddot{\theta}\tan\theta + \dot{\theta}^2,$$

which simplifies to:

$$-\ddot{\theta}\cot\theta = \ddot{\theta}\tan\theta.$$

Rearranging:

$$\ddot{\theta}(-\cot\theta - \tan\theta) = 0.$$

Since $-\cot\theta - \tan\theta = -\frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = -\frac{1}{\sin\theta\cos\theta} \neq 0$ for all θ where defined, it follows that:

$$\ddot{\theta} = 0.$$

Thus, $\ddot{x} = 0$, $\ddot{y} = -g$, and $\ddot{\theta} = 0$.

The center of mass of the rod moves with constant horizontal velocity and accelerates downward with acceleration g, while the rod rotates with constant angular velocity. The initial conditions are that the rod is vertical ($\theta = 0$) with the midpoint at height h, and the velocities of the particles are given.

The velocity of the center of mass initially is $\begin{pmatrix} u/2 \\ v \end{pmatrix}$, since the masses are equal. Thus, $\dot{x} = u/2$ (constant) and $\dot{y} = v - gt$. The angular velocity $\dot{\theta} = \omega$ is constant, and from the initial velocities, $\omega = u/(2a)$.

The angular displacement is $\theta = \omega t = \frac{u}{2a}t$. The particles hit the table simultaneously for the first time when $\theta = \pi/2$, so:

$$\frac{u}{2a}t = \frac{\pi}{2} \implies t = \frac{\pi a}{u}.$$

At this time, both particles are at the same height as the center of mass, so for simultaneous impact, the height of the center of mass must be zero:

$$y(t) = h + vt - \frac{1}{2}gt^2 = 0.$$

Substituting $t = \pi a/u$:

$$h + v\left(\frac{\pi a}{u}\right) - \frac{1}{2}g\left(\frac{\pi a}{u}\right)^2 = 0.$$

Multiplying through by u^2:

$$hu^2 + v\pi a u - \frac{1}{2}g\pi^2 a^2 = 0.$$

Rearranging terms:

$$hu^2 = -v\pi a u + \frac{1}{2}g\pi^2 a^2.$$

Multiplying through by 2:

$$2hu^2 = -2v\pi a u + g\pi^2 a^2 = \pi^2 a^2 g - 2\pi u v a.$$

Thus, the required equation is obtained.

Q21

Let $I_1 < I_2 < I_3$ be the three principal moments of inertia of a rigid body that rotates freely with angular velocity $\boldsymbol{\omega}$ according to the Euler equations

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3, \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1, \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2, \end{aligned}$$

where the components ω_1, ω_2 and ω_3 of the angular velocity are taken with respect to the principal axes of inertia.

(a) Write down expressions for the energy E and the total angular momentum squared L^2, and prove that these are conserved using the Euler equations.

(b) Show that if $L^2 = 2EI_2$ there exist solutions in which the angular velocity is directed along the second principal axis, i.e., ω_1 and ω_3 are zero. What are the possible values for ω_2? Use linearisation to analyse the stability of these solutions.

(c) Still working under the condition $L^2 = 2EI_2$, use your expressions from part (a) to express ω_1 and ω_3 in terms of E and L^2, and hence obtain a first-order differential equation for ω_2. Integrate this equation and show that $\omega_2(t) = \mu \tanh(\lambda t)$ for some constants μ, λ which you should find. Briefly comment on the relation of this solution to your answer to part (b).

Solution:

Part (a)

The rotational kinetic energy E and the square of the total angular momentum L^2 for a rigid body rotating freely about its center of mass, with principal moments of inertia $I_1 < I_2 < I_3$ and angular velocity components $\omega_1, \omega_2, \omega_3$ along the principal axes, are given by:

$$E = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

$$L^2 = (I_1\omega_1)^2 + (I_2\omega_2)^2 + (I_3\omega_3)^2$$

These quantities are conserved, meaning their time derivatives are zero. The conservation is proven using the Euler equations:

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3, \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1, \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2. \end{aligned}$$

Conservation of Energy
The time derivative of E is:

$$\dot{E} = \frac{d}{dt}\left(\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2\right) = I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3,$$

since the moments of inertia are constant. Substituting the expressions for $\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3$ from the Euler equations:

$$\dot{\omega}_1 = \frac{(I_2 - I_3)\omega_2\omega_3}{I_1}, \quad \dot{\omega}_2 = \frac{(I_3 - I_1)\omega_3\omega_1}{I_2}, \quad \dot{\omega}_3 = \frac{(I_1 - I_2)\omega_1\omega_2}{I_3}.$$

Thus,

$$\dot{E} = I_1\omega_1\left(\frac{(I_2 - I_3)\omega_2\omega_3}{I_1}\right) + I_2\omega_2\left(\frac{(I_3 - I_1)\omega_3\omega_1}{I_2}\right) + I_3\omega_3\left(\frac{(I_1 - I_2)\omega_1\omega_2}{I_3}\right).$$

Simplifying:

$$\dot{E} = (I_2 - I_3)\omega_1\omega_2\omega_3 + (I_3 - I_1)\omega_1\omega_2\omega_3 + (I_1 - I_2)\omega_1\omega_2\omega_3$$

$$= \omega_1\omega_2\omega_3 \left[(I_2 - I_3) + (I_3 - I_1) + (I_1 - I_2)\right].$$

The expression in the brackets is:

$$(I_2 - I_3) + (I_3 - I_1) + (I_1 - I_2) = I_2 - I_3 + I_3 - I_1 + I_1 - I_2 = 0.$$

Therefore, $\dot{E} = \omega_1\omega_2\omega_3 \cdot 0 = 0$, so E is conserved.

Conservation of L^2
The time derivative of L^2 is:

$$\frac{d}{dt}(L^2) = \frac{d}{dt}\left((I_1\omega_1)^2 + (I_2\omega_2)^2 + (I_3\omega_3)^2\right) = 2I_1^2\omega_1\dot{\omega}_1 + 2I_2^2\omega_2\dot{\omega}_2 + 2I_3^2\omega_3\dot{\omega}_3,$$

since the moments of inertia are constant. Substituting the expressions for $\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3$:

$$\frac{d}{dt}(L^2) = 2\left[I_1^2\omega_1\left(\frac{(I_2 - I_3)\omega_2\omega_3}{I_1}\right) + I_2^2\omega_2\left(\frac{(I_3 - I_1)\omega_3\omega_1}{I_2}\right) + I_3^2\omega_3\left(\frac{(I_1 - I_2)\omega_1\omega_2}{I_3}\right)\right].$$

Simplifying:

$$\begin{aligned}\frac{d}{dt}(L^2) &= 2\left[I_1(I_2 - I_3)\omega_1\omega_2\omega_3 + I_2(I_3 - I_1)\omega_1\omega_2\omega_3 + I_3(I_1 - I_2)\omega_1\omega_2\omega_3\right] \\ &= 2\omega_1\omega_2\omega_3\left[I_1(I_2 - I_3) + I_2(I_3 - I_1) + I_3(I_1 - I_2)\right].\end{aligned}$$

The expression in the brackets is:

$$I_1(I_2 - I_3) + I_2(I_3 - I_1) + I_3(I_1 - I_2)$$

$$= I_1I_2 - I_1I_3 + I_2I_3 - I_2I_1 + I_3I_1 - I_3I_2 = (I_1I_2 - I_1I_2) + (-I_1I_3 + I_1I_3) + (I_2I_3 - I_2I_3) = 0.$$

Therefore, $\frac{d}{dt}(L^2) = 2\omega_1\omega_2\omega_3 \cdot 0 = 0$, so L^2 is conserved.

For part (b), consider the condition $L^2 = 2EI_2$, where E is the energy and L^2 is the square of the total angular momentum, given by:

$$E = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2, \quad L^2 = (I_1\omega_1)^2 + (I_2\omega_2)^2 + (I_3\omega_3)^2.$$

Assume a solution where the angular velocity is directed along the second principal axis, so $\omega_1 = 0$ and $\omega_3 = 0$. Then $\boldsymbol{\omega} = (0, \omega_2, 0)$, and substituting into the expressions for E and L^2 gives:

$$E = \frac{1}{2}I_2\omega_2^2, \quad L^2 = (I_2\omega_2)^2.$$

The condition $L^2 = 2EI_2$ becomes:

$$(I_2\omega_2)^2 = 2\left(\frac{1}{2}I_2\omega_2^2\right)I_2 = I_2^2\omega_2^2,$$

which holds identically for any ω_2. Substituting $\omega_1 = 0$, $\omega_3 = 0$, and $\omega_2 = \Omega$ (a constant) into the Euler equations:

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3, \quad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1, \quad I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2,$$

yields:

$$I_1 \cdot 0 = (I_2 - I_3)\Omega \cdot 0 = 0, \quad I_2 \cdot 0 = (I_3 - I_1) \cdot 0 \cdot 0 = 0, \quad I_3 \cdot 0 = (I_1 - I_2) \cdot 0 \cdot \Omega = 0,$$

which are satisfied. Thus, solutions exist with $\omega_1 = 0$, $\omega_3 = 0$, and $\omega_2 = \Omega$ for any constant Ω. The possible values for ω_2 are all real numbers.

To analyze stability, consider small perturbations around the equilibrium $\boldsymbol{\omega} = (0, \Omega, 0)$. Set:

$$\omega_1 = x, \quad \omega_2 = \Omega + y, \quad \omega_3 = z,$$

where x, y, and z are small. Linearizing the Euler equations by neglecting higher-order terms gives:

$$\begin{aligned} I_1 \dot{x} &= (I_2 - I_3)(\Omega + y)z \approx (I_2 - I_3)\Omega z, \\ I_2 \dot{y} &= (I_3 - I_1)zx \approx 0, \\ I_3 \dot{z} &= (I_1 - I_2)x(\Omega + y) \approx (I_1 - I_2)\Omega x. \end{aligned}$$

The second equation implies $\dot{y} = 0$, so y is constant. The first and third equations form the system:

$$\dot{x} = \frac{(I_2 - I_3)\Omega}{I_1} z, \quad \dot{z} = \frac{(I_1 - I_2)\Omega}{I_3} x.$$

The characteristic equation for the matrix $\begin{pmatrix} 0 & \frac{(I_2 - I_3)\Omega}{I_1} \\ \frac{(I_1 - I_2)\Omega}{I_3} & 0 \end{pmatrix}$ is:

$$\lambda^2 - \left(\frac{(I_2 - I_3)\Omega}{I_1} \right) \left(\frac{(I_1 - I_2)\Omega}{I_3} \right) = 0,$$

so:

$$\lambda^2 = \frac{(I_2 - I_3)(I_1 - I_2)\Omega^2}{I_1 I_3}.$$

Given $I_1 < I_2 < I_3$, it follows that $I_2 - I_3 < 0$ and $I_1 - I_2 < 0$, so $(I_2 - I_3)(I_1 - I_2) > 0$. With $\Omega^2 > 0$ (assuming non-zero rotation), $I_1 > 0$, and $I_3 > 0$, the right-hand side is positive. Thus, $\lambda^2 > 0$, so $\lambda = \pm\sqrt{\text{positive constant}}$, yielding one positive real eigenvalue. Therefore, the equilibrium is unstable for $\Omega \neq 0$.

Part (c)

Given the condition $L^2 = 2EI_2$, the expressions for energy E and the square of the total angular momentum L^2 are:

$$E = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2, \quad L^2 = (I_1 \omega_1)^2 + (I_2 \omega_2)^2 + (I_3 \omega_3)^2.$$

Under the condition $L^2 = 2EI_2$, substituting the expressions yields:

$$I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2 = I_2 (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2).$$

Rearranging terms gives:

$$(I_1^2 - I_1 I_2)\omega_1^2 + (I_3^2 - I_2 I_3)\omega_3^2 = 0.$$

Since $I_1 < I_2 < I_3$, this simplifies to:

$$I_1(I_1 - I_2)\omega_1^2 = -I_3(I_3 - I_2)\omega_3^2,$$

or equivalently:

$$\omega_1^2 = \frac{I_3(I_3 - I_2)}{I_1(I_2 - I_1)}\omega_3^2.$$

Let $k = \frac{I_3(I_3-I_2)}{I_1(I_2-I_1)} > 0$, so $\omega_1^2 = k\omega_3^2$. Using the energy expression:

$$E = \frac{1}{2}I_1(k\omega_3^2) + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}(I_1 k + I_3)\omega_3^2 + \frac{1}{2}I_2\omega_2^2.$$

Substituting k and simplifying:

$$I_1 k + I_3 = I_3\frac{I_3 - I_1}{I_2 - I_1},$$

so:

$$E = \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\frac{I_3 - I_1}{I_2 - I_1}\omega_3^2.$$

Let $C = I_3\frac{I_3-I_1}{I_2-I_1} > 0$, then:

$$E = \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}C\omega_3^2.$$

Solving for ω_3^2:

$$\omega_3^2 = \frac{2}{C}\left(E - \frac{1}{2}I_2\omega_2^2\right).$$

From the Euler equations, the equation for $\dot{\omega}_2$ is:

$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1,$$

so:

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_1\omega_3.$$

Since $\omega_1^2 = k\omega_3^2$, then $\omega_1\omega_3 = \pm\sqrt{k}\omega_3^2$ (the sign depends on initial conditions). Substituting:

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}(\pm\sqrt{k})\omega_3^2 = D\omega_3^2,$$

where $D = \frac{I_3-I_1}{I_2}(\pm\sqrt{k})$. Substituting ω_3^2:

$$\dot{\omega}_2 = D\left[\frac{2}{C}\left(E - \frac{1}{2}I_2\omega_2^2\right)\right] = D\left(\frac{2E}{C} - \frac{I_2}{C}\omega_2^2\right).$$

Let $P = \frac{2E}{C}$ and $Q = \frac{I_2}{C}$, so:

$$\dot{\omega}_2 = D(P - Q\omega_2^2).$$

This is a first-order differential equation for ω_2. Separating variables:

$$\frac{d\omega_2}{P - Q\omega_2^2} = Ddt.$$

Integrating both sides:

$$\int \frac{d\omega_2}{P - Q\omega_2^2} = \int Ddt.$$

The left side is:

$$\int \frac{d\omega_2}{P(1 - \frac{Q}{P}\omega_2^2)} = \frac{1}{P}\int \frac{d\omega_2}{1 - \alpha^2\omega_2^2}, \quad \alpha = \sqrt{\frac{Q}{P}}.$$

The integral is:

$$\int \frac{d\omega_2}{1 - \alpha^2\omega_2^2} = \frac{1}{\alpha}\tanh^{-1}(\alpha\omega_2),$$

so:

$$\frac{1}{P} \cdot \frac{1}{\alpha}\tanh^{-1}(\alpha\omega_2) = Dt + \text{const}.$$

Solving for ω_2:

$$\alpha\omega_2 = \tanh(P\alpha Dt + \phi),$$

where ϕ is a constant. With the initial condition $\omega_2(0) = 0$, $\phi = 0$, so:

$$\omega_2(t) = \frac{1}{\alpha}\tanh(P\alpha Dt).$$

Substituting $\alpha = \sqrt{\frac{Q}{P}} = \sqrt{\frac{I_2}{2E}}$, so:

$$\mu = \frac{1}{\alpha} = \sqrt{\frac{2E}{I_2}}.$$

Now, $P\alpha D = \frac{2E}{C} \cdot \sqrt{\frac{I_2}{2E}} \cdot D$. Substituting $D = \frac{I_3 - I_1}{I_2}(\pm\sqrt{k})$ and $k = \frac{I_3(I_3 - I_2)}{I_1(I_2 - I_1)}$, and simplifying, the expression for λ is:

$$\lambda = |P\alpha D| = \sqrt{\frac{2E(I_2 - I_1)(I_3 - I_2)}{I_1 I_2 I_3}}.$$

The sign of D affects the sign of λ, but since $\tanh$ is odd, $\omega_2(t) = \mu\tanh(\lambda t)$ covers the solution by taking $\lambda > 0$. Thus:

$$\omega_2(t) = \sqrt{\frac{2E}{I_2}} \tanh\left(\sqrt{\frac{2E(I_2 - I_1)(I_3 - I_2)}{I_1 I_2 I_3}} t \right).$$

The constants are:

$$\mu = \sqrt{\frac{2E}{I_2}}, \quad \lambda = \sqrt{\frac{2E(I_2 - I_1)(I_3 - I_2)}{I_1 I_2 I_3}}.$$

Relation to part (b):
In part (b), solutions exist where the angular velocity is directed along the second principal axis, i.e., $\omega_1 = 0$, $\omega_3 = 0$, and $\omega_2 = \Omega$ for any constant Ω, and these solutions are unstable for $\Omega \neq 0$. The solution here, $\omega_2(t) = \mu \tanh(\lambda t)$, asymptotically approaches these constant solutions as $t \to \pm\infty$: as $t \to \infty$, $\omega_2(t) \to \mu = \sqrt{2E/I_2}$, and as $t \to -\infty$, $\omega_2(t) \to -\mu = -\sqrt{2E/I_2}$. These are exactly the values of Ω for the constant rotations about the second axis. Thus, this solution describes a heteroclinic orbit connecting the two unstable equilibria found in part (b).