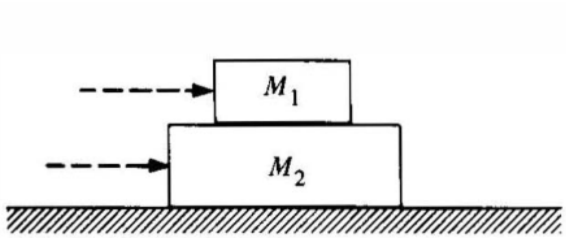
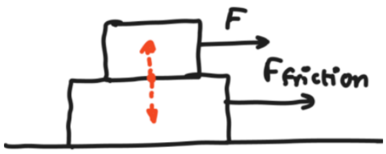
1. A block of mass M_1 rests on a block of mass M_2 which lies on a frictionless table. The coefficient of friction between the blocks is μ. What is the maximum horizontal force which can be applied to the blocks for them to accelerate without slipping on one another if the force is applied to (a) block 1, and (b) block 2?



(a) When F is applied to M_1, frictional force causes M_2 to accelerate with M_1

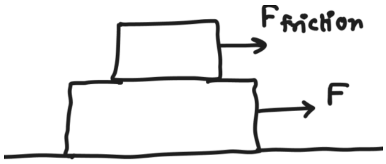


$$a = \frac{F}{M_1 + M_2}$$

$$F_{\text{Friction}} = \mu M_1 g = M_2 \frac{F}{M_1 + M_2}$$

$$F = \frac{(\mu M_1 g)(M_1 + M_2)}{M_2} \tag{1}$$

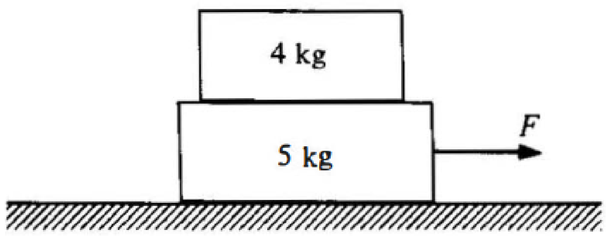
(b) When F is applied to M_2, the frictional force causes M_1 to accelerate with M_2



$$F_{\text{Friction}} = \mu M_1 g = M_1 \frac{F}{M_1 + M_2}$$

$$F = \mu g(M_1 + M_2) \tag{2}$$

2. A 4-kg block rests on top of a 5-kg block, which rests on a frictionless table. The coefficient of friction between the two blocks is such that the blocks start to slip when the horizontal force F applied to the lower block is $27N$. Suppose that a horizontal force is now applied only to the upper block. What is its maximum value for the blocks to slide without slipping relative to each other?



When $F = 27N$, we can use the case in (2)

$$27 = \mu g(9) \Rightarrow \mu = \frac{3}{g}$$

For F' applied on the 4 kg block, we can use (1)

$$F' = \frac{(4\mu g) \times 9}{5} = \frac{\left(4 \times \frac{3}{g} g\right) \times 9}{5} = \frac{4}{5} \times 27 = 21.6N$$

3. Find the radius of the orbit of a synchronous satellite which circles the earth. (A synchronous satellite goes around the earth once every 24 h, so that its position appears stationary with respect to a ground station.) The simplest way to find the answer and give your results is by expressing all distances in terms of the earth's radius.

We know the radius of orbit remains constant ($\ddot{r} = \dot{r} = 0$) and the angular velocity is constant ($\ddot{\theta} = 0$). Thus, the acceleration in polar coordinates is:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}} = -r\dot{\theta}^2\hat{\mathbf{r}}$$

$$r\dot{\theta}^2 = \frac{GM_e}{r^2}$$

$$r^3 = \tfrac{GM_e}{\omega^2} \quad M_e = \tfrac{4}{3}\pi R_e^3 \rho$$

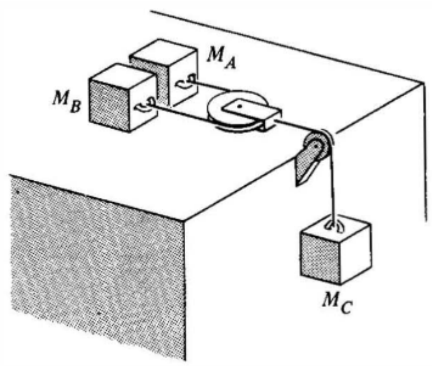
$$r = \left(\frac{G\rho}{\omega^2}\right)^{\frac{1}{3}} R_e$$

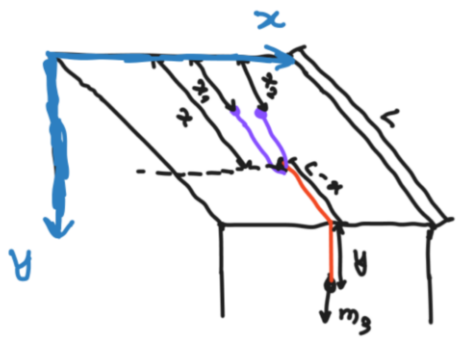
$$\tfrac{d\theta}{dt} = \omega \Rightarrow \int_0^{2\pi} d\theta = \int_0^T \omega \cdot dt \quad \omega = \tfrac{2\pi}{T}$$

$$T = 86400s \Rightarrow \omega = \frac{2\pi}{86400}$$

$$r = \left(\frac{86400 \times G \times \rho}{2\pi}\right)^{\frac{1}{3}} R_e$$

4. Two masses, A and B, lie on a frictionless table. They are attached to either end of a light rope of length l which passes around a pulley of negligible mass. The pulley is attached to a rope connected to a hanging mass, C. Find the acceleration of each mass. (You can check whether or not your answer is reasonable by considering special cases—for instance, the cases $M_A = 0$, or $M_A = M_B = M_C$.)





Constraints:

$$l_1 = (x - x_1) + (x - x_2) = 2x - (x_1 + x_2)$$

$$2\ddot{x} = \ddot{x}_1 + \ddot{x}_2$$

$$l_2 = (L - x) + y$$

$$\ddot{x} = \ddot{y}$$

Equation of motion:

$$T_A = T_B = T$$

$$\ddot{x}_2 = \frac{m_A}{m_B}\ddot{x}_1$$

$$T = m_A\ddot{x}_1$$

$$T = m_B\ddot{x}_2$$

The tension in C, T_C arises from $m_A\ddot{x}_1 + m_B\ddot{x}_2$. Thus, $T_C = 2T$.

$$\frac{m_C}{2}(\ddot{x}_1 + \ddot{x}_2) = m_C g - 2m_A\ddot{x}_1$$

$$\frac{m_C}{2}\left(\ddot{x}_1 + \frac{m_A}{m_B}\ddot{x}_1\right) + 2m_A\ddot{x}_1 = m_C g$$

$$\ddot{x}_1\left(\frac{m_C}{2} + \frac{m_A m_C}{2m_B} + 2m_A\right) = m_C g$$

$$\ddot{x}_1\left(\frac{m_B m_C + m_A m_C + 4m_A m_B}{2m_B}\right) = m_C g$$

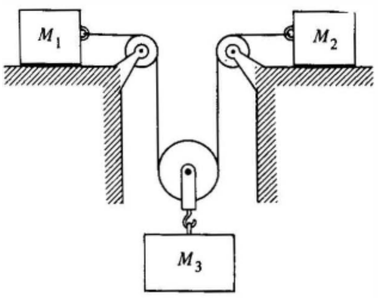
$$\ddot{x}_1 = \frac{2m_B m_C g}{m_B m_C + m_A m_C + 4m_A m_B}$$

Similarly, for $\ddot{x}_2$,

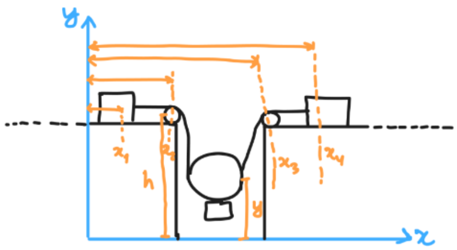
$$\ddot{x}_2 = \frac{2m_A m_C g}{m_B m_C + m_A m_C + 4m_A m_B}$$

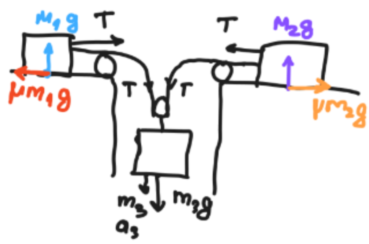
$$\ddot{y} = \ddot{x} = \frac{m_B m_C g}{m_B m_C + m_A m_C + 4m_A m_B} + \frac{m_A m_C g}{m_B m_C + m_A m_C + 4m_A m_B} = \frac{m_C(m_A + m_B)g}{m_B m_C + m_A m_C + 4m_A m_B}$$

5. The system on the right above uses massless pulleys and rope. The coefficient of friction between the masses and horizontal surfaces is μ. Assume that M_1 and M_2 are sliding. Gravity is directed downward. Find the tension in the rope, T.



Scaling it to an inertial coordinate system:





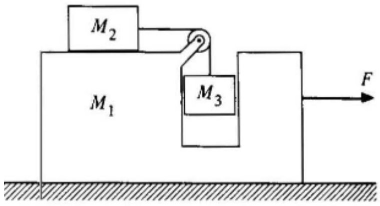
$$(x_2 - x_1) + (x_4 - x_3) + 2(h - y) = l$$
$$-\ddot{x}_1 + \ddot{x}_4 - 2\ddot{y} = 0$$

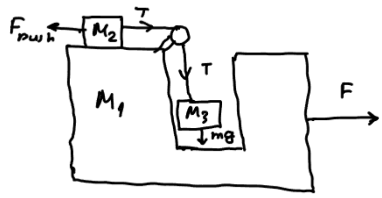
$$-T + \mu m_2 g = m_2 \ddot{x}_4$$
$$T - \mu m_1 g = m_1 \ddot{x}_1$$
$$-m_3 g + 2T = m_3 \ddot{y}$$

For convenient convention, we rename $\ddot{x}_4$ to $\ddot{x}_2$, the acceleration of the M_2. Thus, $\ddot{x}_2 = \ddot{x}_1 + 2\ddot{y}$

$$\begin{aligned}
-\frac{T}{m_2} + \mu g &= \ddot{x}_2, && (3)\\
\frac{T}{m_1} - \mu g &= \ddot{x}_1 \quad \Longrightarrow \quad \ddot{x}_2 - 2\ddot{y} = \frac{T}{m_1} - \mu g, && (4)\\
-\frac{T}{m_1} + \mu g - 2\ddot{y} &= \frac{T}{m_2} - \mu g, && (5)\\
2\ddot{y} &= -\frac{T}{m_1} - \frac{T}{m_2} + 2\mu g, && (6)\\
-2g + \frac{4T}{m_3} &= -\frac{T}{m_1} - \frac{T}{m_2} + 2\mu g, && (7)\\
\frac{4T}{m_3} + \frac{T}{m_1} + \frac{T}{m_2} &= 2g\,(\mu + 1), && (8)\\
\left(\frac{4}{m_3} + \frac{1}{m_1} + \frac{1}{m_2}\right)T &= 2g\,(\mu + 1), && (9)\\
T &= \frac{2g\,(\mu+1)}{\frac{4}{m_3} + \frac{1}{m_1} + \frac{1}{m_2}} = \frac{g\,(\mu+1)}{\frac{2}{m_3} + \frac{1}{2m_1} + \frac{1}{2m_2}}. && (10)
\end{aligned}$$

6. A "Pedagogical Machine" is illustrated in the sketch below. All surfaces are frictionless. What force F must be applied to M_1 to keep M_3 from rising or falling?

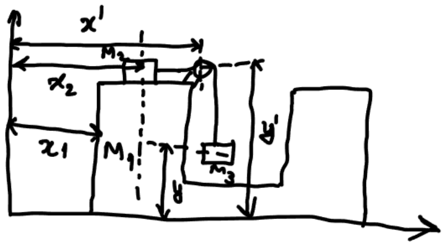




$$
\begin{aligned}
a &= \frac{F}{M_1 + M_2 + M_3}, && (11)\\
F_{\text{push}} &= T = M_3\, g, && (12)\\
\implies \quad M_2\, \frac{F}{M_1 + M_2 + M_3} &= M_3\, g, && (13)\\
\implies \quad F &= \frac{M_3\, g\, (M_1 + M_2 + M_3)}{M_2}. && (14)
\end{aligned}
$$

7. Consider the "Pedagogical Machine" of the last problem in the case where F is zero. Find the acceleration of M_1.

   Introduce an inertial coordinate system:



$$
\begin{aligned}
& x' - (x_1 + x_2) + (y' - y) = L \quad \implies \quad \ddot{x}' - \ddot{x}_1 - \ddot{x}_2 - \ddot{y} = 0, && (15)\\
& \ddot{x}' = \ddot{x}_1 \quad \implies \quad \ddot{x}_2 = -\, \ddot{y}. && (16)
\end{aligned}
$$

Vertical:

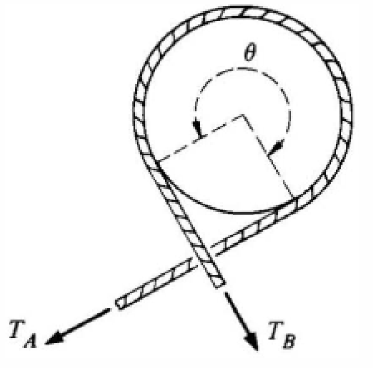
$$
\begin{aligned}
& T - M_3 g = M_3\, \ddot{y}, && (17)\\
& M_3 g - T = M_3\, \ddot{x}_2 \quad \implies \quad \ddot{x}_2 = \frac{M_3 g - T}{M_3}. && (18)
\end{aligned}
$$

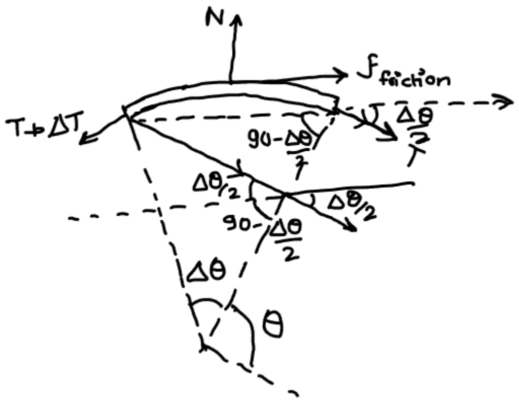
But from the first vertical: $T = M_3\, \ddot{y} + M_3 g = -\, M_3\, \ddot{x}_2 + M_3 g,$ (19)

Horizontal:

$$
\begin{aligned}
& M_1 \ddot{x}_1 + M_3 \ddot{x}_1 + M_2 \ddot{x}_1 + M_2 \ddot{x}_2 = 0, && (20)\\
& \implies M_1 \ddot{x}_1 + M_3 \ddot{x}_1 + M_2 \ddot{x}_1 + M_2 \left(\tfrac{M_3 g - M_2 \ddot{x}_1}{M_2 + M_3} \right) = 0, && (21)\\
& \implies \left(M_1 M_2 + M_1 M_3 + 2\, M_2 M_3 + M_3^2 \right) \ddot{x}_1 = -\, M_2 M_3\, g, && (22)\\
& \ddot{x}_1 = -\, \frac{M_2\, M_3\, g}{M_1 M_2 + M_1 M_3 + 2\, M_2 M_3 + M_3^2}. && (23)
\end{aligned}
$$

8. A device called a capstan is used aboard ships in order to control a rope which is under great tension. The rope is wrapped around a fixed drum, usually for several turns. The load on the rope pulls it with a force T_A, and the sailor holds it with a much smaller force T_B. Can you show that $T_B = T_A e^{-\mu\theta}$, where μ is the coefficient of friction and θ is the total angle subtended by the rope on the drum?





$$\text{Horizontally:} \quad T\cos\frac{\Delta\theta}{2} \;+\; f_{\text{friction}} \;-\; (T+\Delta T)\cos\frac{\Delta\theta}{2} = 0, \tag{24}$$

$$\cos\frac{\Delta\theta}{2} \to 1 \quad (\Delta\theta \to 0) \quad\Longrightarrow\quad T + f_{\text{friction}} - T - \Delta T = 0 \;\Longrightarrow\; \Delta T = f_{\text{friction}}, \tag{25}$$

$$\text{Vertically:} \quad N = T\sin\frac{\Delta\theta}{2} + (T+\Delta T)\sin\frac{\Delta\theta}{2} \tag{26}$$

$$= T\sin\frac{\Delta\theta}{2} + T\sin\frac{\Delta\theta}{2} + \Delta T\,\sin\frac{\Delta\theta}{2}, \tag{27}$$

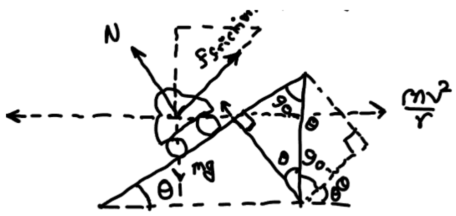
$$\sin\frac{\Delta\theta}{2} \to \frac{\Delta\theta}{2} \quad (\Delta\theta \to 0) \quad\Longrightarrow\quad N = T\,\Delta\theta + \tfrac{1}{2}\,\Delta T\,\Delta\theta, \tag{28}$$

$$\Delta T\,\Delta\theta \approx 0 \quad\Longrightarrow\quad N \approx T\,\Delta\theta, \tag{29}$$

$$\frac{\Delta T}{\mu} = T\,\Delta\theta \quad\Longrightarrow\quad \int_{T_B}^{T_A} \frac{1}{T}\,dT = \mu\int_0^{\theta} d\theta, \tag{30}$$

$$\ln|T_A| - \ln|T_B| = \mu\,\theta \quad\Longrightarrow\quad T_A = T_B\,e^{\mu\theta} \quad\text{or}\quad T_B = T_A\,e^{-\mu\theta}\,. \tag{31}$$

9. An automobile enters a turn whose radius is R. The road is banked at angle θ, and the coefficient of friction between wheels and road is μ. Find the maximum and minimum speeds for the car to stay on the road without skidding sideways.



$$v_r = 0, \qquad\qquad v_\theta = r\,\dot\theta \;\Longrightarrow\; v = r\,\dot\theta, \tag{32}$$

$$a_r = -\,r\,\dot\theta^2 = -\,\frac{v^2}{r}, \qquad\qquad F_r = m\,a_r = -\,\frac{m\,v^2}{r}. \tag{33}$$

If $\dfrac{m\,v^2}{r} > N\sin\theta$, friction acts *down* the slope; if $\dfrac{m\,v^2}{r} < N\sin\theta$, friction acts *up* the slope.

Case 1 (friction up the slope)

$$\frac{m\,v^2}{r} + f\cos\theta = N\sin\theta, \tag{34}$$

$$N\cos\theta + f\sin\theta = m\,g, \tag{35}$$

$$f = \mu\,N \quad\Longrightarrow\quad N(\cos\theta + \mu\sin\theta) = m\,g \Longrightarrow N = \frac{m\,g}{\cos\theta + \mu\sin\theta}, \tag{36}$$

$$\frac{m\,v^2}{r} = N(\sin\theta - \mu\cos\theta) = \frac{m\,g\,(\sin\theta - \mu\cos\theta)}{\cos\theta + \mu\sin\theta}, \tag{37}$$

$$v = \sqrt{\frac{g\,r\,(\sin\theta - \mu\cos\theta)}{\cos\theta + \mu\sin\theta}}\,. \tag{38}$$

Case 2 (friction down the slope)

$$f\cos\theta + N\sin\theta = \frac{m\,v^2}{r}, \tag{39}$$

$$m\,g + f\sin\theta = N\cos\theta, \tag{40}$$

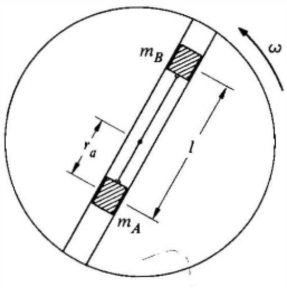
$$f = \mu\,N \quad\Longrightarrow\quad N(\cos\theta - \mu\sin\theta) = m\,g \Longrightarrow N = \frac{m\,g}{\cos\theta - \mu\sin\theta}, \tag{41}$$

$$\frac{m\,v^2}{r} = N(\sin\theta + \mu\cos\theta) = \frac{m\,g\,(\sin\theta + \mu\cos\theta)}{\cos\theta - \mu\sin\theta}, \tag{42}$$

$$v = \sqrt{\frac{g\,r\,(\sin\theta + \mu\cos\theta)}{\cos\theta - \mu\sin\theta}}\,. \tag{43}$$

10. A disk rotates with constant angular velocity ω, as shown. Two masses, m_A and m_B, slide without friction in a groove passing through the center of the disk. They are connected by a light string of length l, and are initially held in position by a catch, with mass m_A at distance r_A from the center. Neglect gravity. At $t = 0$ the catch is removed and the masses are free to slide.

    Find $\ddot{r}_A$ immediately after the catch is removed in terms of m_A, m_B, l, r_A, and ω.



$$a_{A,\hat{\theta}} = 0, \qquad a_{A,\hat{r}} = \ddot{r}_A - r_A\,\dot{\theta}^2, \tag{44}$$

Radial equilibrium for A and B: $\quad -T = m_A\big(\ddot{r}_A - r_A\,\omega^2\big), \quad -T = m_B\big(\ddot{r}_B - r_B\,\omega^2\big), \qquad (45)$

$$\Longrightarrow \quad m_A\big(\ddot{r}_A - r_A\,\omega^2\big) = m_B\big(\ddot{r}_B - r_B\,\omega^2\big), \tag{46}$$

$$\text{But } r_B = L - r_A, \quad \ddot{r}_B = -\,\ddot{r}_A, \tag{47}$$

$$\Longrightarrow \quad (m_A + m_B)\,\ddot{r}_A = m_A\,r_A\,\omega^2 \;-\; m_B\,(L - r_A)\,\omega^2 \tag{48}$$

$$= \Big[(m_A + m_B)\,r_A - m_B\,L\Big]\omega^2, \tag{49}$$

$$\ddot{r}_A = r_A\,\omega^2 \;-\; \frac{m_B\,L\,\omega^2}{m_A + m_B}\,. \tag{50}$$