A_M_PS3

Q1

(6.3) A ring of mass M and radius R lies on its side on a frictionless table. It is pivoted to the table at its rim. A bug of mass m walks around the ring with speed v, starting at the pivot. What is the rotational velocity of the ring when the bug is halfway around, and back at the pivot?

Solution:

The ring is pivoted at a point on its rim, and the system is on a frictionless table, so there are no external torques about the pivot point. Thus, angular momentum about the pivot is conserved. Initially, the system is at rest, so the total angular momentum is zero.

Let ω denote the angular velocity of the ring, and let θ be the angle that the bug has traversed along the ring from the pivot, with $\theta = 0$ at the pivot. The bug walks with constant speed v relative to the ring, so its angular speed relative to the ring is $\omega_{\text{rel}} = v/R$.

Angular momentum conservation about the pivot gives:

$$2MR^2\omega + mR^2(1 - \cos\theta)(2\omega - \omega_{\text{rel}}) = 0$$

Solving for ω:

$$\omega = \frac{m(1 - \cos\theta)\omega_{\text{rel}}}{2[M + m(1 - \cos\theta)]}$$

Substituting $\omega_{\text{rel}} = v/R$:

$$\omega = \frac{m(1 - \cos\theta)(v/R)}{2[M + m(1 - \cos\theta)]}$$

When the bug is halfway around:
At $\theta = \pi$ (half the circumference), $\cos\pi = -1$, so $1 - \cos\pi = 2$:

$$\omega = \frac{m \cdot 2 \cdot (v/R)}{2[M + m \cdot 2]} = \frac{mv/R}{M + 2m} = \frac{mv}{R(M + 2m)}$$

The rotational velocity (angular velocity) of the ring is $\frac{mv}{R(M+2m)}$.

When the bug is back at the pivot:
At $\theta = 2\pi$, $\cos(2\pi) = 1$, so $1 - \cos(2\pi) = 0$:

$$\omega = \frac{m \cdot 0 \cdot (v/R)}{2[M + m \cdot 0]} = 0$$

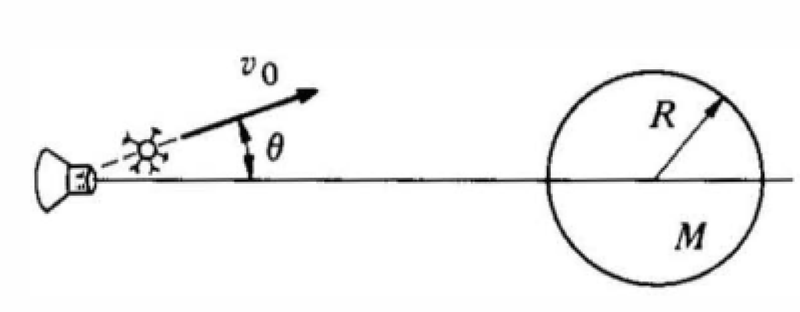
The rotational velocity of the ring is 0.

Thus:

- At halfway around ($\theta = \pi$), the rotational velocity is $\frac{mv}{R(M+2m)}$.
- At back at the pivot ($\theta = 2\pi$), the rotational velocity is 0.

Q2

(6.4) A spaceship is sent to investigate a planet of mass M and radius R. While hanging motionless in space at a distance $5R$ from the center of the planet, the ship fires an instrument package with speed v_0, as shown in the sketch. The package has mass m, which is much smaller than the mass of the spaceship. For what angle θ will the package just graze the surface of the planet?



Solution:

To determine the angle θ for which the instrument package just grazes the surface of the planet, consider the conservation of energy and angular momentum. The package is fired from a distance of $5R$ from the planet's center with an initial speed v_0 at an angle θ measured from the radial direction toward the planet (where $\theta = 0^\circ$ corresponds to firing directly toward the planet).

The condition for grazing the surface is that the periapsis of the orbit occurs at the planet's radius R, where the radial velocity is zero. At this point, the package has only tangential velocity.

- **Angular momentum conservation**: The specific angular momentum h is constant. At the initial point ($r_i = 5R$),

$$h = r_i v_\theta = 5R \cdot (v_0 \sin\theta),$$

  since the tangential component of the initial velocity is $v_0 \sin\theta$. At periapsis ($r_p = R$),

$$h = r_p v_\perp = R v_\perp,$$

  where $v_\perp$ is the tangential velocity at periapsis. Equating these,

$$5Rv_0 \sin\theta = Rv_\perp \implies v_\perp = 5v_0 \sin\theta.$$

- **Energy conservation**: The specific orbital energy ϵ is constant. At the initial point,

$$\epsilon = \frac{1}{2}v_0^2 - \frac{GM}{5R},$$

  where G is the gravitational constant and M is the planet's mass. At periapsis ($r_p = R$, radial velocity zero),

$$\epsilon = \frac{1}{2}v_\perp^2 - \frac{GM}{R}.$$

  Substituting $v_\perp = 5v_0 \sin\theta$,

$$\epsilon = \frac{1}{2}(5v_0 \sin\theta)^2 - \frac{GM}{R} = \frac{1}{2} \cdot 25v_0^2 \sin^2\theta - \frac{GM}{R}.$$

  Equating the expressions for ϵ,

$$\frac{1}{2}v_0^2 - \frac{GM}{5R} = \frac{25}{2}v_0^2 \sin^2\theta - \frac{GM}{R}.$$

Multiply through by 2 to eliminate the fractions:

$$v_0^2 - \frac{2GM}{5R} = 25v_0^2 \sin^2\theta - \frac{2GM}{R}.$$

Rearrange to isolate terms:

$$v_0^2 - 25v_0^2 \sin^2\theta - \frac{2GM}{5R} + \frac{2GM}{R} = 0.$$

Combine the GM/R terms:

$$-\frac{2GM}{5R} + \frac{2GM}{R} = \left(-\frac{2}{5} + 2\right)\frac{GM}{R} = \left(-\frac{2}{5} + \frac{10}{5}\right)\frac{GM}{R} = \frac{8}{5}\frac{GM}{R}.$$

Thus,

$$v_0^2(1 - 25\sin^2\theta) + \frac{8}{5}\frac{GM}{R} = 0.$$

Solve for $\sin^2\theta$:

$$v_0^2(1 - 25\sin^2\theta) = -\frac{8}{5}\frac{GM}{R},$$

$$25v_0^2 \sin^2\theta - v_0^2 = \frac{8}{5}\frac{GM}{R},$$

$$25\sin^2\theta = 1 + \frac{8}{5}\frac{GM}{Rv_0^2},$$

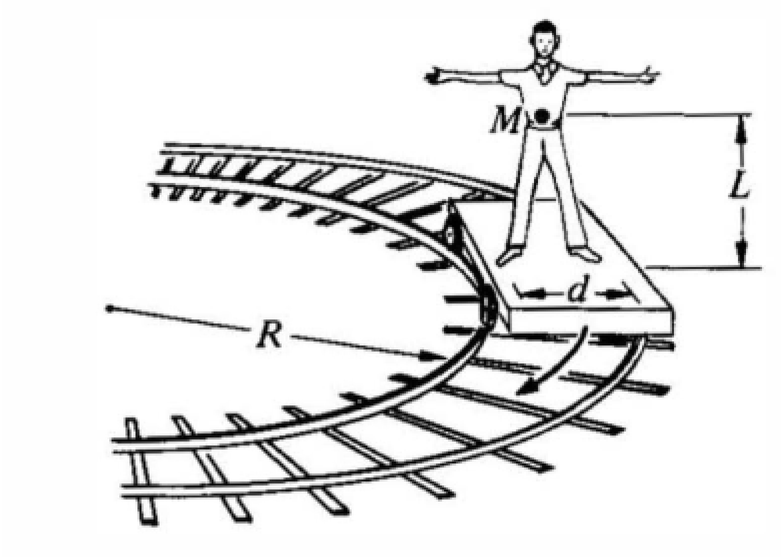
$$\sin^2\theta = \frac{1}{25}\left(1 + \frac{8}{5}\frac{GM}{Rv_0^2}\right).$$

Therefore,

$$\sin\theta = \frac{1}{5}\sqrt{1 + \frac{8}{5}\frac{GM}{Rv_0^2}}.$$

Q3

(6.6) A man of mass M stands on a railroad car which is rounding an unbanked turn of radius R at speed v. His center of mass is height L above the car, and his feet are distance d apart. The man is facing the direction of motion. How much weight is on each of his feet?



Solution:

The centrifugal force acts outward from the center of the turn, and since the man is facing the direction of motion, the line joining his feet is radial. The inner foot is closer to the center of the turn, and the outer foot is farther away.

The forces acting on the man are:

- Gravity: Mg downward at the center of mass.

- Centrifugal force: $\frac{Mv^2}{R}$ radially outward at the center of mass.
- Normal forces: N_{inner} and N_{outer} upward at the inner and outer feet, respectively.
- Friction forces: acting radially inward at the feet, but their individual magnitudes are not needed for determining the normal forces.

The center of mass is at height L above the car floor and midway between the feet, which are separated by distance d. The coordinate system is set with the origin at the midpoint between the feet on the car floor, the x-axis radial outward, and the y-axis vertical upward. The inner foot is at $x = -\frac{d}{2}$, the outer foot at $x = \frac{d}{2}$, and the center of mass at $(0, L)$.

For equilibrium in the rotating frame:

1. Sum of forces in the vertical direction (y-direction) is zero:

$$N_{\text{inner}} + N_{\text{outer}} - Mg = 0$$

$$N_{\text{inner}} + N_{\text{outer}} = Mg \quad (1)$$

2. Sum of forces in the radial direction (x-direction) is zero (with outward positive):

$$\frac{Mv^2}{R} - f_{\text{inner}} - f_{\text{outer}} = 0$$

$$f_{\text{inner}} + f_{\text{outer}} = \frac{Mv^2}{R} \quad (2)$$

   where f_{inner} and f_{outer} are the magnitudes of the friction forces on the inner and outer feet, respectively, acting inward.
3. Sum of torques about the center of mass is zero. The torque due to gravity and centrifugal force is zero since they act at the center of mass. The torques due to the normal and friction forces are:

   - Torque due to normal force on inner foot: $-\frac{d}{2}N_{\text{inner}}$
   - Torque due to normal force on outer foot: $\frac{d}{2}N_{\text{outer}}$
   - Torque due to friction on inner foot: $-Lf_{\text{inner}}$
   - Torque due to friction on outer foot: $-Lf_{\text{outer}}$

   Summing these torques:

$$-\frac{d}{2}N_{\text{inner}} + \frac{d}{2}N_{\text{outer}} - Lf_{\text{inner}} - Lf_{\text{outer}} = 0$$

$$\frac{d}{2}(N_{\text{outer}} - N_{\text{inner}}) - L(f_{\text{inner}} + f_{\text{outer}}) = 0 \quad (3)$$

Substituting equation (2) into equation (3):

$$\frac{d}{2}(N_{\text{outer}} - N_{\text{inner}}) - L\frac{Mv^2}{R} = 0$$

$$N_{\text{outer}} - N_{\text{inner}} = \frac{2LMv^2}{dR} \quad (4)$$

Solving equations (1) and (4) simultaneously:

Add equations (1) and (4):

$$(N_{\text{inner}} + N_{\text{outer}}) + (N_{\text{outer}} - N_{\text{inner}}) = Mg + \frac{2LMv^2}{dR}$$

$$2N_{\text{outer}} = Mg + \frac{2LMv^2}{dR}$$

$$N_{\text{outer}} = \frac{Mg}{2} + \frac{Mv^2L}{dR}$$

Subtract equation (4) from equation (1):

$$(N_{\text{inner}} + N_{\text{outer}}) - (N_{\text{outer}} - N_{\text{inner}}) = Mg - \frac{2LMv^2}{dR}$$

$$2N_{\text{inner}} = Mg - \frac{2LMv^2}{dR}$$

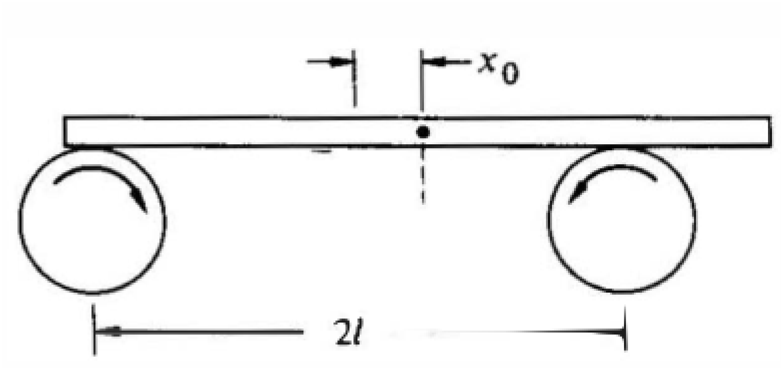
$$N_{\text{inner}} = \frac{Mg}{2} - \frac{Mv^2L}{dR}$$

The weight on each foot is the normal force exerted by the car on the foot. Thus:

- The weight on the inner foot is $\frac{Mg}{2} - \frac{Mv^2L}{dR}$.
- The weight on the outer foot is $\frac{Mg}{2} + \frac{Mv^2L}{dR}$.

Q4

(6.9) A heavy uniform bar of mass M rests on top of two identical rollers which are continuously turned rapidly in opposite directions, as shown. The centers of the rollers are a distance 2ℓ apart. The coefficient of friction between the bar and the roller surfaces is μ, a constant independent of the relative speed of the two surfaces. Initially the bar is held at rest with its center at distance x_0 from the midpoint of the rollers. At time $t = 0$ it is released. Find the subsequent motion of the bar.



Solution:

The rollers are turned rapidly in opposite directions such that the top surfaces move away from the midpoint. Specifically, the left roller's top surface moves to the left, and the right roller's top surface moves to the right. This setup ensures that the friction forces provide a restoring force toward the midpoint.

The normal forces on the bar due to the left and right rollers are given by:

$$N_l = Mg\frac{\ell - x}{2\ell}, \quad N_r = Mg\frac{\ell + x}{2\ell},$$

where M is the mass of the bar, g is the acceleration due to gravity, 2ℓ is the distance between the centers of the rollers, and x is the displacement of the center of the bar from the midpoint.

The friction forces are kinetic, with magnitude μN and direction opposite to the relative motion of the roller surfaces relative to the bar. Given the rapid rotation of the rollers, the relative velocity is dominated by the roller surface velocities. Thus:

- For the left roller, the surface moves left, so the friction force on the bar is to the right: $f_l = +\mu N_l$.
- For the right roller, the surface moves right, so the friction force on the bar is to the left: $f_r = -\mu N_r$.

The net force in the x-direction is:

$$F_x = f_l + f_r = \mu N_l - \mu N_r = \mu(N_l - N_r).$$

Substituting the expressions for N_l and N_r:

$$F_x = \mu \left[Mg\frac{\ell - x}{2\ell} - Mg\frac{\ell + x}{2\ell} \right] = \mu Mg\frac{(\ell - x) - (\ell + x)}{2\ell} = \mu Mg\frac{-2x}{2\ell} = -\frac{\mu Mg}{\ell}x.$$

The equation of motion is:

$$M\ddot{x} = F_x = -\frac{\mu Mg}{\ell}x,$$

which simplifies to:

$$\ddot{x} = -\frac{\mu g}{\ell}x.$$

This is the equation for simple harmonic motion, $\ddot{x} + \omega^2 x = 0$, where:

$$\omega^2 = \frac{\mu g}{\ell}, \quad \omega = \sqrt{\frac{\mu g}{\ell}}.$$

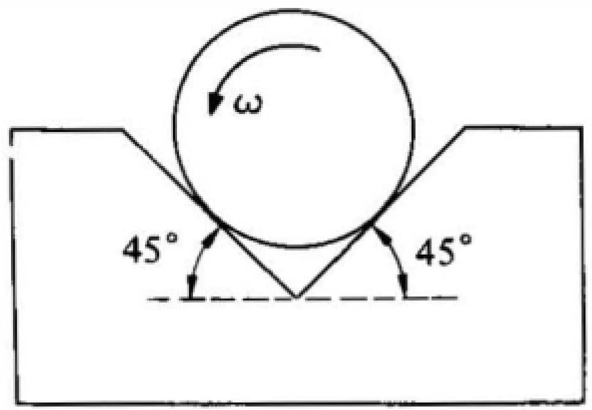
The initial conditions are $x(0) = x_0$ and $\dot{x}(0) = 0$. The solution is:

$$x(t) = x_0 \cos(\omega t) = x_0 \cos\left(t\sqrt{\frac{\mu g}{\ell}}\right).$$

The bar undergoes simple harmonic motion about the midpoint between the rollers with angular frequency $\omega = \sqrt{\mu g/\ell}$ and amplitude x_0.

Q5

(6.10) A cylinder of mass M and radius R is rotated in a uniform V groove with constant angular velocity ω. The coefficient of friction between the cylinder and each surface is μ. What torque must be applied to the cylinder to keep it rotating?



Solution:

To determine the torque required to keep a cylinder of mass M and radius R rotating at a constant angular velocity ω in a uniform V groove with a coefficient of friction μ, the forces and torques acting on the cylinder must be analyzed. The V groove is symmetric with an included angle of 90°, so each surface makes a 45° angle with the vertical. The center of the cylinder is stationary, so the net force and net torque about the center must be zero.

Coordinate System and Forces:
Set up a coordinate system with the apex of the V groove at $(0, 0)$ and the center of the cylinder at $(0, h)$, where $h = R\sqrt{2}$. The contact points are:

- Left contact: $\left(-\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}\right)$

- Right contact: $\left(\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}\right)$

The forces at each contact point are:

- Normal force $\vec{N}_1$ at the left contact, directed along $\hat{n}_1 = \frac{1}{\sqrt{2}}(1,1)$
- Normal force $\vec{N}_2$ at the right contact, directed along $\hat{n}_2 = \frac{1}{\sqrt{2}}(-1,1)$
- Friction force $\vec{f}_1$ at the left contact, opposing motion
- Friction force $\vec{f}_2$ at the right contact, opposing motion

For clockwise rotation ($\vec{\omega} = -\omega\hat{k}$), the velocities at the contact points are:

- Left contact velocity: $\vec{v}_1 = \frac{\omega R}{\sqrt{2}}(-1,1)$ (up the surface)
- Right contact velocity: $\vec{v}_2 = \frac{\omega R}{\sqrt{2}}(-1,-1)$ (down the surface)

The friction forces oppose the motion:

- $\vec{f}_1 = f_1\frac{1}{\sqrt{2}}(1,-1)$ (down the surface)
- $\vec{f}_2 = f_2\frac{1}{\sqrt{2}}(1,1)$ (up the surface)

Kinetic friction gives $f_1 = \mu N_1$ and $f_2 = \mu N_2$, where N_1 and N_2 are the magnitudes of the normal forces.

Force Balance Equations:
The net force in the x- and y-directions must be zero.

X-component:

$$\frac{1}{\sqrt{2}}(N_1 + f_1 - N_2 + f_2) = 0$$

Substitute $f_1 = \mu N_1$, $f_2 = \mu N_2$:

$$N_1(1+\mu) + N_2(\mu - 1) = 0$$

$$N_1(1+\mu) = N_2(1-\mu) \quad \text{(Equation A)}$$

Y-component (including gravity Mg downward):

$$-Mg + \frac{1}{\sqrt{2}}(N_1 - f_1 + N_2 + f_2) = 0$$

Substitute $f_1 = \mu N_1$, $f_2 = \mu N_2$:

$$-Mg + \frac{1}{\sqrt{2}}[N_1(1-\mu) + N_2(1+\mu)] = 0 \quad \text{(Equation B)}$$

Solving for Normal Forces:
From Equation A:

$$N_1 = N_2 \frac{1-\mu}{1+\mu}$$

Substitute into Equation B:

$$-Mg + \frac{1}{\sqrt{2}}\left[\left(N_2 \frac{1-\mu}{1+\mu}\right)(1-\mu) + N_2(1+\mu)\right] = 0$$

$$\frac{1}{\sqrt{2}} N_2 \left[\frac{(1-\mu)^2}{1+\mu} + (1+\mu)\right] = Mg$$

Simplify the expression in brackets:

$$\frac{(1-\mu)^2}{1+\mu} + (1+\mu) = \frac{1 - 2\mu + \mu^2 + 1 + 2\mu + \mu^2}{1+\mu} = \frac{2 + 2\mu^2}{1+\mu} = 2\frac{1+\mu^2}{1+\mu}$$

Thus:

$$\frac{1}{\sqrt{2}} N_2 \cdot 2\frac{1+\mu^2}{1+\mu} = Mg$$

$$\sqrt{2} N_2 \frac{1+\mu^2}{1+\mu} = Mg$$

$$N_2 = \frac{Mg}{\sqrt{2}} \frac{1+\mu}{1+\mu^2}$$

$$N_1 = \frac{1-\mu}{1+\mu} N_2 = \frac{Mg}{\sqrt{2}} \frac{1-\mu}{1+\mu^2}$$

Friction Forces:

$$f_1 = \mu N_1 = \mu \frac{Mg}{\sqrt{2}} \frac{1-\mu}{1+\mu^2}$$

$$f_2 = \mu N_2 = \mu \frac{Mg}{\sqrt{2}} \frac{1+\mu}{1+\mu^2}$$

Torque Due to Friction:

The torque about the center due to each friction force is calculated using $\vec{\tau} = \vec{r}_c \times \vec{f}$.

Left contact:

Position vector $\vec{r}_{c1} = \left(-\frac{R}{\sqrt{2}}, -\frac{R}{\sqrt{2}}\right)$, force $\vec{f_1} = f_1 \frac{1}{\sqrt{2}}(1, -1)$

$$\tau_1 = x_{c1} f_{y1} - y_{c1} f_{x1} = \left(-\frac{R}{\sqrt{2}}\right)\left(-\frac{f_1}{\sqrt{2}}\right) - \left(-\frac{R}{\sqrt{2}}\right)\left(\frac{f_1}{\sqrt{2}}\right) = \frac{Rf_1}{2} + \frac{Rf_1}{2} = Rf_1$$

Right contact:

Position vector $\vec{r}_{c2} = \left(\frac{R}{\sqrt{2}}, -\frac{R}{\sqrt{2}}\right)$, force $\vec{f_2} = f_2 \frac{1}{\sqrt{2}}(1, 1)$

$$\tau_2 = x_{c2} f_{y2} - y_{c2} f_{x2} = \left(\frac{R}{\sqrt{2}}\right)\left(\frac{f_2}{\sqrt{2}}\right) - \left(-\frac{R}{\sqrt{2}}\right)\left(\frac{f_2}{\sqrt{2}}\right) = \frac{Rf_2}{2} + \frac{Rf_2}{2} = Rf_2$$

Net friction torque:

$$\tau_{\text{friction}} = \tau_1 + \tau_2 = Rf_1 + Rf_2 = R(f_1 + f_2)$$

$$f_1 + f_2 = \mu(N_1 + N_2) = \mu \left[\frac{Mg}{\sqrt{2}} \frac{1-\mu}{1+\mu^2} + \frac{Mg}{\sqrt{2}} \frac{1+\mu}{1+\mu^2}\right] = \mu \frac{Mg}{\sqrt{2}} \frac{2}{1+\mu^2}$$

$$= \frac{2\mu Mg}{\sqrt{2}(1+\mu^2)} = \sqrt{2}\mu\frac{Mg}{1+\mu^2}$$

$$\tau_{\text{friction}} = R\cdot\sqrt{2}\mu\frac{Mg}{1+\mu^2} = \frac{\sqrt{2}\mu MgR}{1+\mu^2}$$

Applied Torque:
To maintain constant angular velocity, the applied torque must balance the friction torque:

$$\tau_{\text{applied}} = \tau_{\text{friction}} = \frac{\sqrt{2}\mu MgR}{1+\mu^2}$$

The torque is independent of ω because kinetic friction is constant for constant normal forces.

Q6

(6.11) A wheel is attached to a fixed shaft, and the system is free to rotate without friction. To measure the moment of inertia of the wheel-shaft system, a tape of negligible mass wrapped around the shaft is pulled with a known constant force F. When a length L of tape has unwound, the system is rotating with angular speed ω_0. Find the moment of inertia of the system, I_0.

Solution:

Consider the work done by the constant force F and the resulting rotational kinetic energy.

The force F is applied tangentially to the tape, which unwinds a length L. The work done by the force is given by the product of the force and the distance over which it acts. Since the tape unwinds a length L, the point of application of the force moves a distance L. Thus, the work done W is:

$$W = F\cdot L$$

This work is converted entirely into rotational kinetic energy of the system, as there is no friction and the tape has negligible mass. The rotational kinetic energy when the angular speed is ω_0 is:

$$K = \frac{1}{2} I_0 \omega_0^2$$

Assuming the system starts from rest (as no initial angular speed is specified), the initial kinetic energy is zero. Therefore, the work done equals the change in kinetic energy:

$$W = \Delta K = \frac{1}{2} I_0 \omega_0^2 - 0$$

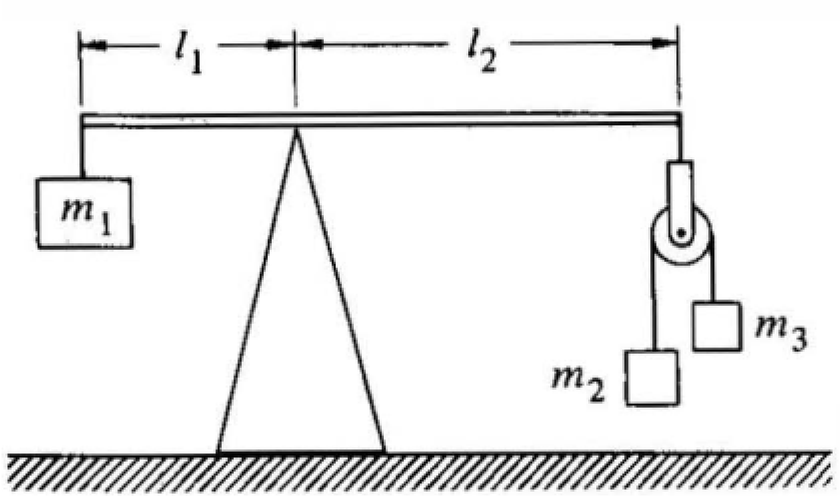
$$FL = \frac{1}{2} I_0 \omega_0^2$$

Solving for I_0:

$$I_0 = \frac{2FL}{\omega_0^2}$$

Q7

(6.12) A pivoted beam has a mass M_1 suspended from one end and an Atwood's machine suspended from the other (see sketch below). The frictionless pulley has negligible mass and dimension. Gravity is directed downward, and $M_2 > M_3$. Find a relation between M_1, M_2, M_3, l_1, and l_2 which will ensure that the beam has no tendency to rotate just after the masses are released.



Solution:

Since the acceleration of the left end of the beam is zero (due to zero angular acceleration and zero angular velocity), the acceleration of M_1 is also zero. Applying Newton's second law to M_1:

$$M_1 g - F_A = M_1 \cdot 0 \implies M_1 g - F_A = 0 \implies F_A = M_1 g.$$

Thus, F_A acts downward on the beam with magnitude $M_1 g$.

The Atwood's machine consists of masses M_2 and M_3 (with $M_2 > M_3$) connected by a string over a frictionless, massless pulley attached to the beam. The acceleration of the pulley (which is the same as the acceleration of the right end of the beam) is zero. Let a be the acceleration of M_2 downward relative to the pulley. Since the string is inextensible, the acceleration of M_3 relative to the pulley is a upward. With the pulley acceleration $a_p = 0$, the absolute accelerations (with downward as positive) are:

- Acceleration of M_2: $a_2 = a_p + a = a$,
- Acceleration of M_3: $a_3 = a_p - a = -a$.

Let T be the tension in the string. Applying Newton's second law to each mass:

- For M_2: $M_2 g - T = M_2 a_2 = M_2 a$,

- For M_3: $M_3 g - T = M_3 a_3 = M_3(-a) \implies M_3 g - T = -M_3 a$.

Solving the system of equations:

1. $M_2 g - T = M_2 a$,
2. $M_3 g - T = -M_3 a$.

Add equations (1) and (2):

$$(M_2 g - T) + (M_3 g - T) = M_2 a - M_3 a \implies M_2 g + M_3 g - 2T = a(M_2 - M_3).$$

From equation (2), solve for T:

$$T = M_3 g + M_3 a.$$

Substitute into equation (1):

$$M_2 g - (M_3 g + M_3 a) = M_2 a \implies M_2 g - M_3 g - M_3 a = M_2 a \implies g(M_2 - M_3) = a(M_2 + M_3).$$

Thus,

$$a = \frac{(M_2 - M_3)g}{M_2 + M_3}.$$

Now substitute back to find T:

$$T = M_3 g + M_3 \left(\frac{(M_2 - M_3)g}{M_2 + M_3} \right) = M_3 g \left(1 + \frac{M_2 - M_3}{M_2 + M_3} \right)$$
$$= M_3 g \left(\frac{M_2 + M_3 + M_2 - M_3}{M_2 + M_3} \right) = \frac{2 M_2 M_3 g}{M_2 + M_3}.$$

For a massless, frictionless pulley, the net force on the pulley is zero. The string exerts two downward forces of magnitude T each on the pulley, so the force exerted by the beam on the pulley is $2T$ upward. By Newton's third law, the force exerted by the pulley on the beam is $F_B = 2T$ downward. Thus,

$$F_B = 2 \times \frac{2M_2M_3g}{M_2+M_3} = \frac{4M_2M_3g}{M_2+M_3}.$$

The torque due to F_A about the pivot is clockwise with magnitude F_Al_1. The torque due to F_B about the pivot is counterclockwise with magnitude F_Bl_2. For no net torque:

$$F_Bl_2 = F_Al_1.$$

Substitute the expressions for F_A and F_B:

$$\left(\frac{4M_2M_3g}{M_2+M_3}\right)l_2 = (M_1g)l_1.$$

Divide both sides by g (assuming $g \neq 0$):

$$\frac{4M_2M_3l_2}{M_2+M_3} = M_1l_1.$$

Rearrange to standard form:

$$M_1l_1(M_2+M_3) = 4M_2M_3l_2.$$

Q8

(6.14) A uniform stick of mass M and length l is suspended horizontally with end B on the edge of a table, and the other end, A, is held by hand. Point A is suddenly released. At the instant after release:

- What is the torque about B?
- What is the angular acceleration about B?
- What is the vertical acceleration of the center of mass? \textit{Ans. } $3g/4$
- From part (c), find by inspection the vertical force at B. \textit{Ans. } $mg/4$

Solution:

The stick is uniform with mass M and length l. At the instant after end A is released, the stick is still horizontal, and the only forces acting on it are gravity and the normal force at point B.

- Gravity acts downward at the center of mass, located at a distance $l/2$ from point B.
- The normal force at point B acts upward at the pivot point itself, so it produces no torque about B.

The torque about point B is solely due to the gravitational force. The magnitude of this torque is given by the product of the gravitational force and the perpendicular distance from B to the line of action of the force. Since the stick is horizontal, the perpendicular distance is $l/2$.

Thus, the torque about B is:

$$\tau = (Mg) \times \left(\frac{l}{2}\right) = \frac{1}{2}Mgl$$

This torque causes a clockwise rotation about point B.

The torque about B is:

$$\tau = Mg \cdot \frac{l}{2} = \frac{1}{2}Mgl$$

The moment of inertia of the stick about an axis through one end (point B) is given by the formula for a uniform rod. Using the parallel axis theorem, where the moment of inertia about the center of mass is $\frac{1}{12}Ml^2$ and the distance from the center of mass to B is $l/2$:

$$I_B = \frac{1}{12}Ml^2 + M\left(\frac{l}{2}\right)^2 = \frac{1}{12}Ml^2 + \frac{1}{4}Ml^2 = \frac{1}{12}Ml^2 + \frac{3}{12}Ml^2 = \frac{4}{12}Ml^2 = \frac{1}{3}Ml^2$$

The angular acceleration α about B is related to the torque and moment of inertia by $\tau = I_B\alpha$. Solving for α:

$$\alpha = \frac{\tau}{I_B} = \frac{\frac{1}{2}Mgl}{\frac{1}{3}Ml^2} = \frac{1}{2}Mgl \cdot \frac{3}{Ml^2} = \frac{3}{2}\frac{g}{l}$$

The vertical acceleration of the center of mass is determined at the instant after end A is released, while the stick is still horizontal and pivoting about point B.

The center of mass is located at a distance $l/2$ from B. The angular acceleration about B is $\alpha = \frac{3g}{2l}$ in the clockwise direction. At the instant of release, the angular velocity $\omega = 0$, so the radial acceleration component is zero, and only the tangential acceleration is present.

The tangential acceleration of a point is given by $a_t = r\alpha$, where r is the distance from the pivot. For the center of mass, $r = l/2$:

$$a_t = \left(\frac{l}{2}\right)\alpha = \left(\frac{l}{2}\right)\left(\frac{3g}{2l}\right) = \frac{3g}{4}.$$

Since the stick is horizontal and the rotation is clockwise, the tangential acceleration at the center of mass is directed vertically downward. Taking upward as the positive direction, the vertical acceleration is:

$$a_{\text{cm},y} = -\frac{3g}{4}.$$

This result is confirmed using Newton's second law for the center of mass. The net vertical force is $N - Mg$, where N is the normal force at B (upward). The normal force is found to be $N = \frac{1}{4}Mg$ (as derived from the torque about the center of mass and the angular acceleration). Thus:

$$N - Mg = \frac{1}{4}Mg - Mg = -\frac{3}{4}Mg.$$

Applying Newton's second law:

$$Ma_{\text{cm},y} = -\frac{3}{4}Mg,$$

so:

$$a_{\text{cm},y} = -\frac{3g}{4}.$$

From part (c), the vertical acceleration of the center of mass is $a_y = -\frac{3g}{4}$, where the negative sign indicates downward acceleration.

The forces acting vertically on the stick are:

- Gravitational force: Mg downward.
- Vertical force at B: denoted as N_B upward.

Applying Newton's second law for the vertical motion of the center of mass:

$$N_B - Mg = Ma_y$$

Substitute $a_y = -\frac{3g}{4}$:

$$N_B - Mg = M\left(-\frac{3g}{4}\right) = -\frac{3}{4}Mg$$

Solving for N_B:

$$N_B = Mg - \frac{3}{4}Mg = \frac{1}{4}Mg$$

Q9

(6.15) A pendulum is made of two disks each of mass M and radius R separated by a massless rod. One of the disks is pivoted through its center by a small pin. The disks hang in the same plane and their centers are a distance l apart. Find the period for small oscillations.

Solution:

The pendulum consists of two disks, each of mass M and radius R, connected by a massless rod of length l between their centers. One disk is pivoted at its center, and the system oscillates in a plane.

For small oscillations, the period is derived using the formula for a physical pendulum: $T = 2\pi\sqrt{\frac{I}{mgd}}$, where I is the moment of inertia about the pivot point, m is the total mass, g is the acceleration due to gravity, and d is the distance from the pivot to the center of mass.

- Total mass: $m = 2M$.
- Center of mass: Since both disks have equal mass and are separated by distance l, the center of mass is midway between the centers. Thus, $d = l/2$.
- Moment of inertia about the pivot (center of the first disk):
  - The first disk, pivoted at its center, contributes $I_A = \frac{1}{2}MR^2$.

- The second disk, with its center at distance l from the pivot, contributes $I_B = \frac{1}{2}MR^2 + Ml^2$ (using the parallel axis theorem).
- Total moment of inertia: $I = I_A + I_B = \frac{1}{2}MR^2 + \frac{1}{2}MR^2 + Ml^2 = MR^2 + Ml^2 = M(l^2 + R^2)$.

Substitute into the period formula:

$$T = 2\pi\sqrt{\frac{M(l^2 + R^2)}{(2M)g(l/2)}} = 2\pi\sqrt{\frac{M(l^2 + R^2)}{Mgl}} = 2\pi\sqrt{\frac{l^2 + R^2}{gl}}.$$

Thus, the period for small oscillations is $2\pi\sqrt{\frac{l^2+R^2}{gl}}$.

Q10

(6.16) A physical pendulum is made of a uniform disk of mass M and radius R suspended from a rod of negligible mass. The distance from the pivot to the center of the disk is l. What value of l makes the period a minimum?

Solution:

The period T of a physical pendulum is given by the formula:

$$T = 2\pi\sqrt{\frac{I}{Mgd}}$$

where I is the moment of inertia about the pivot point, M is the mass of the disk, g is the acceleration due to gravity, and d is the distance from the pivot to the center of mass. For this pendulum, the center of mass is at the center of the disk, so $d = l$. The mass is M, so the period simplifies to:

$$T = 2\pi\sqrt{\frac{I}{Mgl}}$$

The moment of inertia about the center of mass for a uniform disk is $I_{\text{cm}} = \frac{1}{2}MR^2$. Using the parallel axis theorem, the moment of inertia about the pivot point, which is at a distance l from the center of mass, is:

$$I = I_{\text{cm}} + Ml^2 = \frac{1}{2}MR^2 + Ml^2$$

Substitute this into the period formula:

$$T = 2\pi\sqrt{\frac{\frac{1}{2}MR^2 + Ml^2}{Mgl}} = 2\pi\sqrt{\frac{\frac{1}{2}R^2 + l^2}{gl}}$$

since M cancels out.

To minimize T, minimize the expression inside the square root, as 2π and g are constants. Define:

$$k(l) = \frac{\frac{1}{2}R^2 + l^2}{l} = \frac{R^2}{2l} + l$$

so that:

$$T = 2\pi\sqrt{\frac{k(l)}{g}}$$

Minimizing T is equivalent to minimizing $k(l)$, since g is constant.

Take the derivative of $k(l)$ with respect to l:

$$\frac{dk}{dl} = -\frac{R^2}{2l^2} + 1$$

Set the derivative equal to zero to find critical points:

$$-\frac{R^2}{2l^2} + 1 = 0$$

$$1 = \frac{R^2}{2l^2}$$

$$2l^2 = R^2$$

$$l^2 = \frac{R^2}{2}$$

$$l = \frac{R}{\sqrt{2}} \quad \text{(taking the positive root since } l > 0\text{)}$$

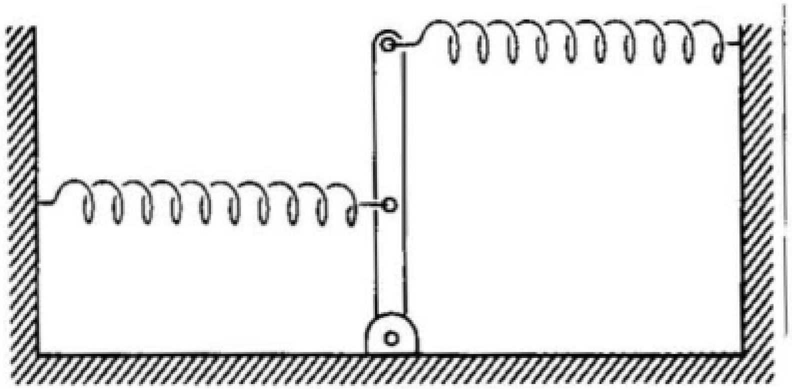
To confirm this is a minimum, check the second derivative:

$$\frac{d^2k}{dl^2} = \frac{d}{dl}\left(-\frac{R^2}{2l^2} + 1\right) = \frac{R^2}{l^3}$$

For $l > 0$, $\frac{d^2k}{dl^2} > 0$, so the critical point is a minimum.

Q11

(6.17) A rod of length l and mass m, pivoted at one end, is held by a spring at its midpoint and a spring at its far end, both pulling in opposite directions. The springs have spring constant k, and at equilibrium their pull is perpendicular to the rod. Find the frequency of small oscillations about the equilibrium position. See figure below.



Solution:

The rod of length l and mass m is pivoted at one end and held by two springs, each with spring constant k, attached at the midpoint and the far end. At equilibrium, the rod is horizontal, and the springs pull perpendicular to the rod in opposite directions. The spring at the midpoint is attached to a fixed point directly above it, and the spring at the far end is attached to a fixed point directly below it, ensuring the pulls are opposite and perpendicular at equilibrium.

For small angular displacements θ from equilibrium, the restoring torque is derived as follows:

- The change in length of the spring at the midpoint (distance $x_m = l/2$ from the pivot) is $\delta s_m \approx -\frac{l}{2}\theta$.
- The change in force magnitude for this spring is $\delta F_m = k\delta s_m \approx -k\frac{l}{2}\theta$.
- Since the force direction is approximately vertical and upward, the change in the y-component of the force is $\delta F_{y,m} \approx -k\frac{l}{2}\theta$.
- The change in torque about the pivot is $\delta\tau_m = x_m \delta F_{y,m} \approx \frac{l}{2}\left(-k\frac{l}{2}\theta\right) = -\frac{kl^2}{4}\theta$.
- The change in length of the spring at the far end (distance $x_a = l$ from the pivot) is $\delta s_a \approx l\theta$.
- The change in force magnitude for this spring is $\delta F_a = k\delta s_a \approx kl\theta$.
- Since the force direction is approximately vertical and downward, the change in the y-component of the force is $\delta F_{y,a} \approx -kl\theta$.
- The change in torque about the pivot is $\delta\tau_a = x_a \delta F_{y,a} \approx l(-kl\theta) = -kl^2\theta$.

The total change in torque is:

$$\delta\tau = \delta\tau_m + \delta\tau_a \approx -\frac{kl^2}{4}\theta - kl^2\theta = -\frac{5kl^2}{4}\theta.$$

Thus, the restoring torque is $\tau = -\kappa\theta$, where $\kappa = \frac{5kl^2}{4}$.

The moment of inertia of the rod about the pivot is $I = \frac{1}{3}ml^2$.

The equation of motion is $I\ddot{\theta} + \kappa\theta = 0$, which simplifies to:

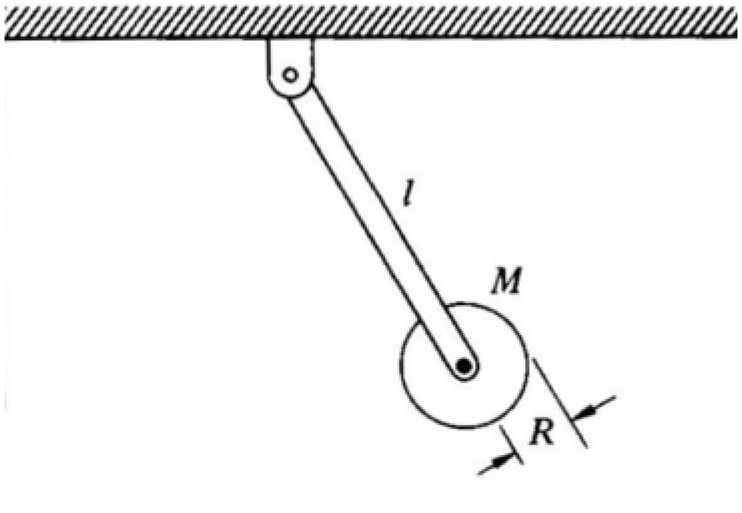
$$\ddot{\theta} + \frac{\kappa}{I}\theta = 0.$$

The angular frequency ω is given by:

$$\omega = \sqrt{\frac{\kappa}{I}} = \sqrt{\frac{\frac{5kl^2}{4}}{\frac{1}{3}ml^2}} = \sqrt{\frac{5kl^2}{4} \cdot \frac{3}{ml^2}} = \sqrt{\frac{15k}{4m}}.$$

Q12

(6.18) Find the period of a pendulum consisting of a disk of mass M and radius R fixed to the end of a rod of length l and mass m. How does the period change if the disk is mounted to the rod by a frictionless bearing so that it is perfectly free to spin? See figure below.



Solution:

The period of a physical pendulum depends on the moment of inertia about the pivot point and the distance to the center of mass. The pivot is at the top of the rod.

Case 1: Disk Fixed (Cannot Spin)
The disk is rigidly attached, so the system rotates as a single rigid body about the pivot.

- **Moment of inertia about pivot, I_{fixed}:**

- Rod: $I_{\text{rod}} = \frac{1}{3}ml^2$
- Disk: The moment of inertia includes the disk's moment about its own center plus the parallel axis contribution.
  $I_{\text{disk}} = \frac{1}{2}MR^2 + Ml^2$
  (where $\frac{1}{2}MR^2$ is the moment about the disk's center, and Ml^2 is due to the distance l from the pivot).
- Total:

$$I_{\text{fixed}} = I_{\text{rod}} + I_{\text{disk}} = \frac{1}{3}ml^2 + \frac{1}{2}MR^2 + Ml^2 = l^2\left(\frac{m}{3} + M\right) + \frac{1}{2}MR^2$$

- **Center of mass distance from pivot, d:**
  The center of mass of the rod is at $l/2$, and the center of mass of the disk is at l.

$$d = \frac{m \cdot \frac{l}{2} + M \cdot l}{m + M} = \frac{\frac{ml}{2} + Ml}{m + M} = l\frac{M + \frac{m}{2}}{m + M}$$

- **Total mass:** $m + M$
- **Period, T_{fixed}:**
  For a physical pendulum, $T = 2\pi\sqrt{\frac{I}{mgd}}$, where m is the total mass.

$$T_{\text{fixed}} = 2\pi\sqrt{\frac{l^2\left(\frac{m}{3} + M\right) + \frac{1}{2}MR^2}{gl\left(M + \frac{m}{2}\right)}}$$

Case 2: Disk Free to Spin (Frictionless Bearing)
The disk is mounted such that it does not rotate when the pendulum swings (due to conservation of angular momentum with no initial spin and no friction). Thus, the disk only translates, contributing no rotational inertia about its own center to the pendulum motion.

- **Effective moment of inertia about pivot, I_{free}:**
  - Rod: Still rotates about the pivot, so $I_{\text{rod}} = \frac{1}{3}ml^2$.
  - Disk: Only translational kinetic energy, equivalent to a point mass at distance l, so contribution is Ml^2.
  - Total:

$$I_{\text{free}} = \frac{1}{3}ml^2 + Ml^2 = l^2\left(\frac{m}{3} + M\right)$$

- **Center of mass distance, d:**
  Same as fixed case, since mass distribution is unchanged:

$$d = l\frac{M + \frac{m}{2}}{m + M}$$

- **Total mass:** $m + M$
- **Period, T_{free}:**

$$T_{\text{free}} = 2\pi\sqrt{\frac{l^2\left(\frac{m}{3} + M\right)}{gl\left(M + \frac{m}{2}\right)}} = 2\pi\sqrt{\frac{l\left(\frac{m}{3} + M\right)}{g\left(M + \frac{m}{2}\right)}}$$

Change in Period
When the disk is free to spin, the period decreases compared to the fixed disk case because the effective moment of inertia is smaller. Specifically:

- $I_{\text{free}} < I_{\text{fixed}}$ due to the absence of the $\frac{1}{2}MR^2$ term.
- Since $T \propto \sqrt{I}$ (with d and total mass constant), $T_{\text{free}} < T_{\text{fixed}}$.

The ratio of the periods is:

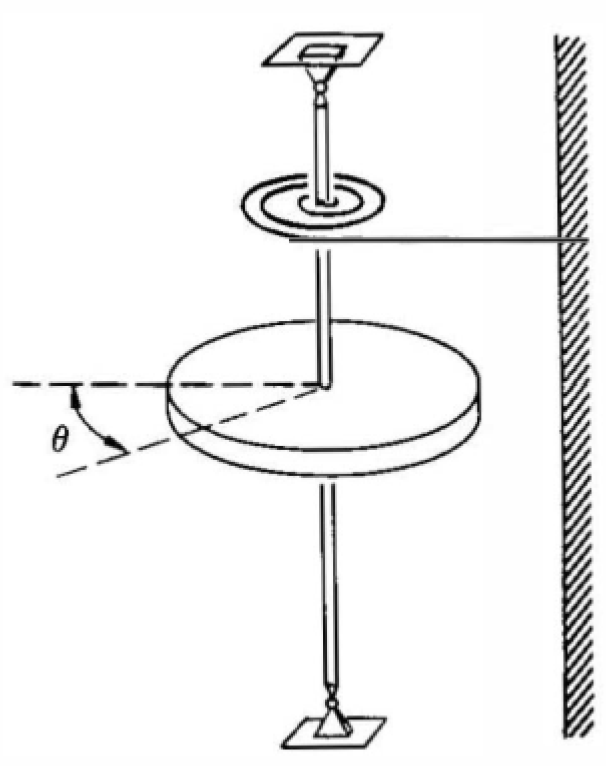
$$\frac{T_{\text{free}}}{T_{\text{fixed}}} = \sqrt{\frac{I_{\text{free}}}{I_{\text{fixed}}}} = \sqrt{\frac{l^2\left(\frac{m}{3} + M\right)}{l^2\left(\frac{m}{3} + M\right) + \frac{1}{2}MR^2}}$$

which is less than 1, confirming the decrease.

Q13

(6.19) A solid disk of mass M and radius R is on a vertical shaft. The shaft is attached to a coil spring which exerts a linear restoring torque of magnitude $C\theta$, where θ is the angle measured from the static equilibrium position and C is a constant. Neglect the mass of the shaft and the spring, and assume the bearings to be frictionless.

- Show that the disk can undergo simple harmonic motion, and find the frequency of the motion.
- Suppose that the disk is moving according to $\theta = \theta_0 \sin(\omega t)$, where ω is the frequency found in part (a). At time $t_1 = \pi/\omega$, a ring of sticky putty of mass M and radius R is dropped concentrically on the disk. Find:
  - The new frequency of the motion
  - The new amplitude of the motion



Solution:

The disk can undergo simple harmonic motion because the equation of motion derived from the torque and angular acceleration is that of a simple harmonic oscillator.

The restoring torque exerted by the spring is given by $\tau = -C\theta$, where C is the torque constant and θ is the angular displacement from equilibrium.

The moment of inertia I of a solid disk of mass M and radius R rotating about its central axis is $I = \frac{1}{2}MR^2$.

Using Newton's second law for rotational motion, $\tau = I\alpha$, where $\alpha = \frac{d^2\theta}{dt^2}$ is the angular acceleration:

$$-C\theta = I\frac{d^2\theta}{dt^2}$$

Rearranging terms:

$$I\frac{d^2\theta}{dt^2} + C\theta = 0$$

Dividing by I:

$$\frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$$

This equation has the form of the simple harmonic motion equation $\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$, where $\omega = \sqrt{\frac{C}{I}}$ is the angular frequency.

Substituting the moment of inertia:

$$\omega = \sqrt{\frac{C}{\frac{1}{2}MR^2}} = \sqrt{\frac{2C}{MR^2}}$$

The cyclic frequency f is related to the angular frequency by $f = \frac{\omega}{2\pi}$:

$$f = \frac{1}{2\pi}\sqrt{\frac{2C}{MR^2}}$$

The solid disk has a moment of inertia $I_{\text{disk}} = \frac{1}{2}MR^2$. When the ring of sticky putty of mass M and radius R is dropped concentrically onto the disk and sticks to it, the ring's moment of inertia about the central axis is $I_{\text{ring}} = MR^2$ (since it is a thin ring with all mass at radius R).

The total moment of inertia of the combined system (disk plus ring) is:

$$I_{\text{total}} = I_{\text{disk}} + I_{\text{ring}} = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2.$$

The restoring torque provided by the spring is still $\tau = -C\theta$, where C is the torsional spring constant and θ is the angular displacement from equilibrium. The equation of motion for the combined system is given by Newton's second law for rotation:

$$\tau = I_{\text{total}}\alpha,$$

where $\alpha = \frac{d^2\theta}{dt^2}$ is the angular acceleration. Substituting the torque:

$$-C\theta = I_{\text{total}}\frac{d^2\theta}{dt^2}.$$

Rearranging terms:

$$I_{\text{total}}\frac{d^2\theta}{dt^2} + C\theta = 0,$$

$$\frac{d^2\theta}{dt^2} + \frac{C}{I_{\text{total}}}\theta = 0.$$

This is the equation of simple harmonic motion, where the angular frequency ω_{new} is given by:

$$\omega_{\text{new}} = \sqrt{\frac{C}{I_{\text{total}}}}.$$

Substituting $I_{\text{total}} = \frac{3}{2}MR^2$:

$$\omega_{\text{new}} = \sqrt{\frac{C}{\frac{3}{2}MR^2}} = \sqrt{\frac{2C}{3MR^2}}.$$

The cyclic frequency f_{new} is related to the angular frequency by $f_{\text{new}} = \frac{\omega_{\text{new}}}{2\pi}$:

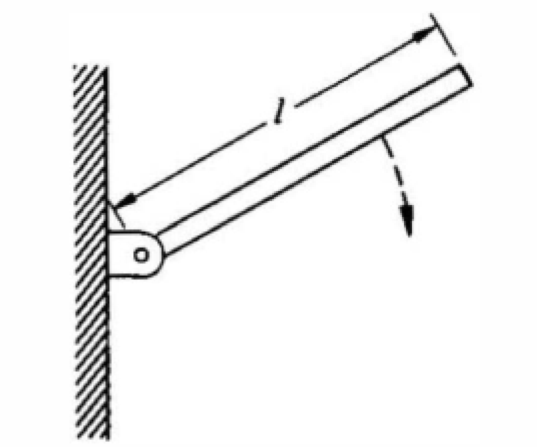
$$f_{\text{new}} = \frac{1}{2\pi}\sqrt{\frac{2C}{3MR^2}}.$$

The frequency depends only on the moment of inertia and the spring constant, which are unchanged after the collision, except for the increase in moment of inertia due to the added ring. The collision at $t_1 = \pi/\omega$ (where $\theta = 0$) conserves angular momentum but does not affect the frequency, as the frequency is a property of the system parameters.

Thus, the new frequency of the motion is $\frac{1}{2\pi}\sqrt{\frac{2C}{3MR^2}}$.

Q14

(6.20) A thin plank of mass M and length l is pivoted at one end (see figure below). The plank is released at 60° from the vertical. What is the magnitude and direction of the force on the pivot when the plank is horizontal?



Solution:

The plank is pivoted at one end and released from a position 60° from the upward vertical. The goal is to find the force on the pivot when the plank is horizontal. The mass of the plank is M and its length is l.

Conservation of Energy to Find Angular Velocity

The moment of inertia of a thin plank (modeled as a thin rod) about one end is $I = \frac{1}{3}Ml^2$. Set the potential energy reference at the pivot level, with the y-axis upward.

- Initial position (60° from upward vertical):
  - Height of center of mass (CM): $y_i = \frac{l}{2}\cos 60^\circ = \frac{l}{2} \cdot \frac{1}{2} = \frac{l}{4}$
  - Initial potential energy: $PE_i = Mgy_i = Mg \cdot \frac{l}{4} = \frac{Mgl}{4}$
  - Initial kinetic energy: $KE_i = 0$ (released from rest)
- Final position (horizontal, 90° from upward vertical):
  - Height of CM: $y_f = \frac{l}{2}\cos 90^\circ = 0$
  - Final potential energy: $PE_f = 0$
  - Final kinetic energy: $KE_f = \frac{1}{2}I\omega^2 = \frac{1}{2} \cdot \frac{1}{3}Ml^2\omega^2 = \frac{1}{6}Ml^2\omega^2$

By conservation of energy:

$$KE_i + PE_i = KE_f + PE_f$$

$$0 + \frac{Mgl}{4} = \frac{1}{6}Ml^2\omega^2 + 0$$

Solving for ω^2:

$$\frac{1}{6}Ml^2\omega^2 = \frac{Mgl}{4}$$

$$l^2\omega^2 = \frac{6Mgl}{4} = \frac{3Mgl}{2}$$

$$\omega^2 = \frac{3g}{2l}$$

Angular Acceleration at Horizontal Position

The torque about the pivot due to gravity when the plank is horizontal:

- Gravitational force: Mg downward

- Lever arm: $\frac{l}{2}$ (perpendicular distance from pivot to line of gravity)
- Torque (clockwise, taken as positive): $\tau = Mg \cdot \frac{l}{2} = \frac{Mgl}{2}$
- Moment of inertia: $I = \frac{1}{3}Ml^2$
- Torque equation: $\tau = I\alpha$

$$\frac{Mgl}{2} = \frac{1}{3}Ml^2\alpha$$

  Solving for α:

$$\alpha = \frac{\frac{Mgl}{2}}{\frac{1}{3}Ml^2} = \frac{Mgl}{2} \cdot \frac{3}{Ml^2} = \frac{3g}{2l}$$

Acceleration of Center of Mass

The CM is at a distance $r = \frac{l}{2}$ from the pivot.

- Radial acceleration (toward pivot): $a_r = \omega^2 r = \frac{3g}{2l} \cdot \frac{l}{2} = \frac{3g}{4}$
  - Direction: When horizontal and to the right of pivot, radial acceleration is leftward (negative x-direction).
  - Component: $a_{r,x} = -\frac{3g}{4}$, $a_{r,y} = 0$
- Tangential acceleration: $a_t = \alpha r = \frac{3g}{2l} \cdot \frac{l}{2} = \frac{3g}{4}$
  - Direction: For clockwise motion at the right side, tangential acceleration is downward (negative y-direction).
  - Component: $a_{t,x} = 0$, $a_{t,y} = -\frac{3g}{4}$
- Total acceleration of CM:

$$a_x = a_{r,x} + a_{t,x} = -\frac{3g}{4} + 0 = -\frac{3g}{4}$$

$$a_y = a_{r,y} + a_{t,y} = 0 + \left(-\frac{3g}{4}\right) = -\frac{3g}{4}$$

Step 4: Forces on the Plank and Newton's Second Law
Forces on the plank:

- Gravitational force: $(0, -Mg)$
- Pivot force on plank: (H, V), where H is horizontal component, V is vertical component

Newton's second law for CM:

$$\sum F_x = Ma_x, \quad \sum F_y = Ma_y$$

- x-direction: $H = M\left(-\frac{3g}{4}\right) = -\frac{3Mg}{4}$
- y-direction: $V - Mg = M\left(-\frac{3g}{4}\right)$

$$V = Mg - \frac{3Mg}{4} = \frac{Mg}{4}$$

Thus, force on plank by pivot: $H = -\frac{3Mg}{4}$ (leftward), $V = \frac{Mg}{4}$ (upward).

Force on the Pivot
By Newton's third law, the force on the pivot by the plank is the opposite:

- Horizontal component: $-H = \frac{3Mg}{4}$ (to the right)
- Vertical component: $-V = -\frac{Mg}{4}$ (downward)

Magnitude and Direction

- Magnitude:

$$|\vec{F}_{\text{pivot}}| = \sqrt{\left(\frac{3Mg}{4}\right)^2 + \left(\frac{Mg}{4}\right)^2} = \sqrt{\frac{9M^2g^2}{16} + \frac{M^2g^2}{16}} = \sqrt{\frac{10M^2g^2}{16}} = \frac{\sqrt{10}Mg}{4}$$

- Direction: The force is to the right and downward. The angle θ below the horizontal is given by:

$$\tan\theta = \frac{\text{vertical component}}{\text{horizontal component}} = \frac{\frac{Mg}{4}}{\frac{3Mg}{4}} = \frac{1}{3}, \quad \theta = \tan^{-1}\left(\frac{1}{3}\right)$$

The magnitude of the force on the pivot is $\frac{\sqrt{10}Mg}{4}$.
The direction is to the right and downward, with the horizontal component to the right and the vertical component downward, at an angle of $\tan^{-1}\left(\frac{1}{3}\right)$ below the horizontal.

Q15

(6.21) A cylinder of radius R and mass M rolls without slipping down a plane inclined at angle θ. The coefficient of friction is μ. What is the maximum value of θ for the cylinder to roll without slipping?

Solution:

The forces along the incline are:

- The component of gravity parallel to the incline: $Mg\sin\theta$ (down the incline).
- The friction force f (up the incline, opposing motion).

The force perpendicular to the incline is balanced by the normal force N:

$$N = Mg\cos\theta.$$

The maximum friction force is given by:

$$f_{\max} = \mu N = \mu Mg\cos\theta.$$

For rolling without slipping, the linear acceleration a of the center of mass and the angular acceleration α are related by:

$$a = R\alpha.$$

The equation of motion along the incline (Newton's second law) is:

$$Mg\sin\theta - f = Ma. \quad (1)$$

The torque about the center of mass due to friction is $\tau = fR$. For a solid cylinder, the moment of inertia about its central axis is $I = \frac{1}{2}MR^2$. The torque equation is:

$$\tau = I\alpha \implies fR = I\alpha. \quad (2)$$

Substituting $\alpha = a/R$ from the no-slip condition into equation (2):

$$fR = I\frac{a}{R} \implies f = \frac{Ia}{R^2}.$$

Using $I = \frac{1}{2}MR^2$:

$$f = \frac{\frac{1}{2}MR^2 \cdot a}{R^2} = \frac{1}{2}Ma. \quad (3)$$

Substitute equation (3) into equation (1):

$$Mg\sin\theta - \frac{1}{2}Ma = Ma.$$

Divide both sides by M:

$$g\sin\theta - \frac{1}{2}a = a \implies g\sin\theta = a + \frac{1}{2}a = \frac{3}{2}a.$$

Solve for a:

$$a = \frac{2}{3}g\sin\theta.$$

Now substitute back into equation (3) to find f:

$$f = \frac{1}{2}M\left(\frac{2}{3}g\sin\theta\right) = \frac{1}{3}Mg\sin\theta.$$

For no slipping, the friction force must not exceed the maximum static friction:

$$f \leq f_{\max} \implies \frac{1}{3} M g \sin\theta \leq \mu M g \cos\theta.$$

Divide both sides by Mg (assuming $M > 0$ and $g > 0$):

$$\frac{1}{3}\sin\theta \leq \mu\cos\theta.$$

Rearrange to:

$$\frac{\sin\theta}{\cos\theta} \leq 3\mu \implies \tan\theta \leq 3\mu.$$

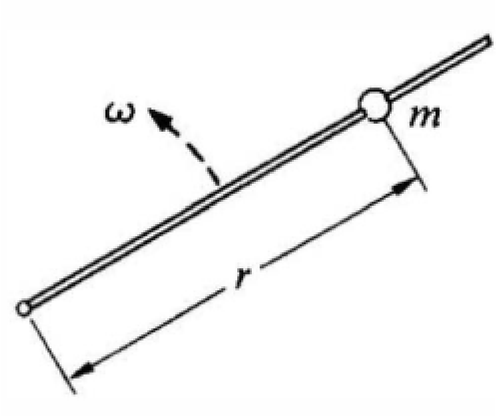
The maximum angle $\theta_{\max}$ occurs when equality holds, just before slipping begins:

$$\tan\theta_{\max} = 3\mu.$$

Q16

(6.22) A bead of mass m slides without friction on a rod that is made to rotate at a constant angular velocity ω. Neglect gravity.

- Show that $r = r_0 e^{\omega t}$ is a possible motion of the bead, where r_0 is the initial distance of the bead from the pivot.
- For the motion described previously, find the force exerted on the bead by the rod.
- For the motion described above, find the power exerted by the agency which is turning the rod and show by direct calculation that this power equals the rate of change of kinetic energy of the bead.



Solution:

Since gravity is neglected and there is no friction, the only force acting on the bead is the normal force from the rod. This normal force is perpendicular to the rod, which is along the radial direction, so it has no radial component. Therefore, the radial component of the net force on the bead is zero.

In polar coordinates, with the rod rotating at constant angular velocity ω, the angular position θ satisfies $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$. The acceleration of the bead has radial and tangential components:

- Radial acceleration: $\ddot{r} - r\dot{\theta}^2$
- Tangential acceleration: $r\ddot{\theta} + 2\dot{r}\dot{\theta}$

Substituting $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$:

- Radial acceleration: $\ddot{r} - r\omega^2$
- Tangential acceleration: $2\dot{r}\omega$

Since the radial component of the net force is zero, Newton's second law in the radial direction gives:

$$m(\ddot{r} - r\omega^2) = 0$$

Dividing by m:

$$\ddot{r} - r\omega^2 = 0$$

Thus, the equation of motion is:

$$\ddot{r} = \omega^2 r$$

This is a second-order linear homogeneous differential equation. The general solution is:

$$r(t) = Ae^{\omega t} + Be^{-\omega t}$$

where A and B are constants determined by initial conditions.

Now, consider the proposed motion $r = r_0 e^{\omega t}$. At $t = 0$, $r(0) = r_0 e^0 = r_0$, which matches the initial distance from the pivot.

To verify that this satisfies the equation of motion, compute the derivatives:

$$\dot{r} = \frac{d}{dt}(r_0 e^{\omega t}) = r_0 \omega e^{\omega t} = \omega r$$

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d}{dt}(\omega r) = \omega \dot{r} = \omega(\omega r) = \omega^2 r$$

Substituting into the equation of motion:

$$\ddot{r} = \omega^2 r$$

which holds true. Thus, $r = r_0 e^{\omega t}$ satisfies the differential equation.

For this motion to be possible, the initial radial velocity must be $\dot{r}(0) = \omega r_0$, as derived from $\dot{r} = \omega r$.

The force exerted on the bead by the rod is perpendicular to the rod due to the absence of friction. In polar coordinates, this force has no radial component and is purely tangential. The tangential force component is denoted as F_θ.

The acceleration of the bead in polar coordinates, with constant angular velocity ω, has a tangential component given by:

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

Since $\ddot{\theta} = 0$ and $\dot{\theta} = \omega$, this simplifies to:

$$a_\theta = 2\dot{r}\omega$$

Applying Newton's second law in the tangential direction:

$$F_\theta = ma_\theta = m \cdot 2\dot{r}\omega = 2m\omega\dot{r}$$

For the given motion $r = r_0 e^{\omega t}$, the radial velocity is:

$$\dot{r} = \frac{d}{dt}(r_0 e^{\omega t}) = r_0 \omega e^{\omega t} = \omega r$$

Substituting $\dot{r} = \omega r$ into the expression for F_θ:

$$F_\theta = 2m\omega \cdot (\omega r) = 2m\omega^2 r$$

Since $r = r_0 e^{\omega t}$, this can be written as:

$$F_\theta = 2m\omega^2 r_0 e^{\omega t}$$

This force is tangential and acts in the direction of the rotation because $\dot{r} > 0$ (the bead is moving outward) and $\omega > 0$, resulting in a positive tangential acceleration.

The agency turning the rod exerts a torque to maintain constant angular velocity ω. The force exerted by the rod on the bead is tangential and given by $F_\theta = 2m\omega^2 r_0 e^{\omega t}$ (from the previous part). By Newton's third law, the bead exerts an equal and opposite tangential force on the rod, $-F_\theta$. The torque τ that the agency must apply to counteract this force and maintain constant ω is:

$$\tau = r \cdot F_\theta = r \cdot (2m\omega^2 r) = 2m\omega^2 r^2$$

where $r = r_0 e^{\omega t}$. Substituting r:

$$\tau = 2m\omega^2 (r_0 e^{\omega t})^2 = 2m\omega^2 r_0^2 e^{2\omega t}$$

The power P exerted by the agency is the product of torque and angular velocity:

$$P = \tau\omega = (2m\omega^2 r_0^2 e^{2\omega t}) \cdot \omega = 2m\omega^3 r_0^2 e^{2\omega t}$$

The kinetic energy K of the bead in polar coordinates is:

$$K = \frac{1}{2}m(v_r^2 + v_\theta^2)$$

where the radial velocity $v_r = \dot{r}$ and the tangential velocity $v_\theta = r\dot{\theta} = r\omega$ (since $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$).

Given the motion $r = r_0 e^{\omega t}$:

- $\dot{r} = \omega r_0 e^{\omega t} = \omega r$
- $v_r = \omega r$
- $v_\theta = r\omega$

Substitute into the kinetic energy expression:

$$K = \frac{1}{2}m\left[(\omega r)^2 + (r\omega)^2\right] = \frac{1}{2}m(\omega^2 r^2 + \omega^2 r^2) = \frac{1}{2}m(2\omega^2 r^2) = m\omega^2 r^2$$

Substitute $r = r_0 e^{\omega t}$:

$$K = m\omega^2 (r_0 e^{\omega t})^2 = m\omega^2 r_0^2 e^{2\omega t}$$

The rate of change of kinetic energy is:

$$\frac{dK}{dt} = \frac{d}{dt}(m\omega^2 r_0^2 e^{2\omega t})$$

Since m, ω, and r_0 are constant:

$$\frac{dK}{dt} = m\omega^2 r_0^2 \cdot \frac{d}{dt}(e^{2\omega t}) = m\omega^2 r_0^2 \cdot (2\omega e^{2\omega t}) = 2m\omega^3 r_0^2 e^{2\omega t}$$

The power exerted by the agency is:

$$P = 2m\omega^3 r_0^2 e^{2\omega t}$$

The rate of change of kinetic energy is:

$$\frac{dK}{dt} = 2m\omega^3 r_0^2 e^{2\omega t}$$

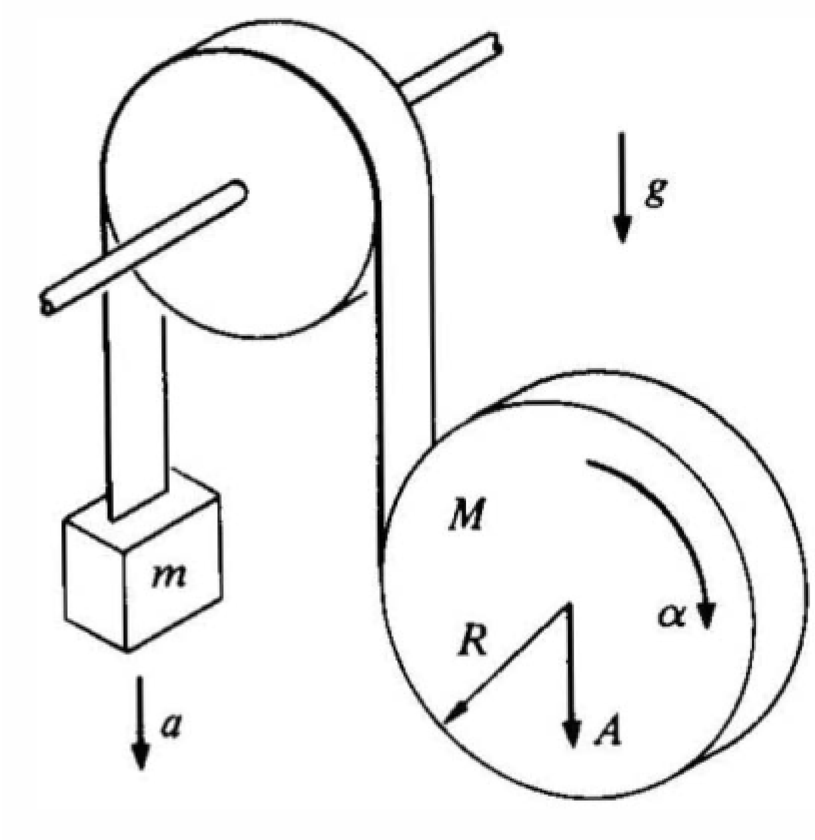
Thus:

$$P = \frac{dK}{dt}$$

Q17

(6.23) A disk of mass M and radius R unwinds from a tape wrapped around it (see figure below). The tape passes over a frictionless pulley, and a mass m is suspended from the other end. Assume that the disk drops vertically.

- Relate the accelerations of m and the disk, a and A, respectively, to the angular acceleration of the disk.
- Find a, A, and α.



Solution:

Define the following coordinates:

- Let y_c be the vertical position of the center of the disk, measured downward from a fixed reference.
- Let y_m be the vertical position of mass m, measured downward from the pulley.
- Let θ be the angle of rotation of the disk, with $\theta = 0$ corresponding to the initial position.

The amount of tape unwound from the disk is $s = R\theta$. The total length of the tape is constant. Let L_0 be the initial wrapped length on the disk, H be the initial height of the pulley above the reference, and R be the radius of the disk. The position of the point where the tape leaves the disk (point P at the top) is $y_p = y_c + R$.

The distance from the pulley to point P is $H - y_p = H - (y_c + R)$. The length of tape from the pulley to mass m is y_m. The wrapped length on the disk at any time is $L_0 - s = L_0 - R\theta$.

The total tape length is constant, so:

$$(L_0 - R\theta) + [H - (y_c + R)] + y_m = \text{constant}$$

The constant is determined from the initial conditions. At $t = 0$, assume $y_c = 0$, $y_m = 0$, and $\theta = 0$, so:

$$\text{constant} = L_0 + [H - (0 + R)] + 0 = L_0 + H - R$$

Thus:

$$(L_0 - R\theta) + H - y_c - R + y_m = L_0 + H - R$$

Simplify:

$$L_0 - R\theta + H - y_c - R + y_m = L_0 + H - R$$

$$-R\theta - y_c + y_m = 0$$

So:

$$y_m - y_c = R\theta \quad (1)$$

Differentiate equation (1) with respect to time:

$$\frac{dy_m}{dt} - \frac{dy_c}{dt} = R\frac{d\theta}{dt}$$

Let $v_m = \frac{dy_m}{dt}$ (velocity of mass m downward) and $v_c = \frac{dy_c}{dt}$ (velocity of disk center downward), and $\omega = \frac{d\theta}{dt}$ (angular velocity of disk). Thus:

$$v_m - v_c = R\omega \quad (2)$$

Differentiate equation (2) with respect to time to find accelerations:

$$\frac{dv_m}{dt} - \frac{dv_c}{dt} = R\frac{d\omega}{dt}$$

Let $a_m = \frac{dv_m}{dt}$ (acceleration of mass m downward), $A = \frac{dv_c}{dt}$ (acceleration of disk center downward), and $\alpha = \frac{d\omega}{dt}$ (angular acceleration of disk). Thus:

$$a_m - A = R\alpha \quad (3)$$

The acceleration a of mass m is defined as upward, so $a_m = -a$. The acceleration A of the disk is defined as downward. Substituting into equation (3):

$$(-a) - A = R\alpha$$

$$-a - A = R\alpha$$

Rearrange to:

$$a + A + R\alpha = 0$$

Equations of Motion

- **For mass m**:
  Forces acting on m are tension T upward and weight mg downward. The acceleration of m is a upward. Applying Newton's second law:

  $$T - mg = ma \quad (1)$$

- **For the disk (translational motion)**:
  Forces acting on the disk are weight Mg downward and tension T upward. The acceleration of the disk's center is A downward. Taking downward as positive:

  $$Mg - T = MA \quad (2)$$

- **For the disk (rotational motion)**:
  The tension acts tangentially at the rim, providing a torque. The torque is TR (clockwise, positive for unwinding). Applying $\tau = I\alpha$:

  $$TR = \left(\frac{1}{2}MR^2\right)\alpha$$

  Solving for T:

$$T = \frac{1}{2}MR\alpha \quad (3)$$

Kinematic Constraint

The tape is unwound from the disk, and the acceleration of the tape at the disk is related to the angular acceleration. The kinematic constraint used is that the acceleration of mass m upward is related to the angular acceleration by:

$$a = -R\alpha \quad (4)$$

This assumes that the acceleration of the tape is solely due to the unwinding of the disk, which is a common approximation in such problems.

Solving the System of Equations

Substitute the expression for T from equation (3) into equations (1) and (2), and use the constraint (4).

From equation (4):

$$a = -R\alpha$$

Substitute $a = -R\alpha$ and $T = \frac{1}{2}MR\alpha$ into equation (1):

$$\frac{1}{2}MR\alpha - mg = m(-R\alpha)$$

$$\frac{1}{2}MR\alpha - mg = -mR\alpha$$

Bring all terms to one side:

$$\frac{1}{2}MR\alpha + mR\alpha = mg$$

Factor out $R\alpha$:

$$R\alpha\left(\frac{M}{2} + m\right) = mg$$

Solve for $R\alpha$:

$$R\alpha = \frac{mg}{\frac{M}{2} + m} = \frac{mg}{\frac{M+2m}{2}} = \frac{2mg}{M + 2m}$$

Thus,

$$\alpha = \frac{2mg}{R(M + 2m)} = \frac{2mg}{R(2m + M)} \quad (since\$M + 2m = 2m + M\$)$$

Now find a:

$$a = -R\alpha = -R \cdot \frac{2mg}{R(2m + M)} = -\frac{2mg}{2m + M}$$

To find A, use equation (2) and the expression for T:

$$Mg - T = MA$$

Substitute $T = \frac{1}{2}MR\alpha$:

$$Mg - \frac{1}{2}MR\alpha = MA$$

Divide both sides by M:

$$g - \frac{1}{2}R\alpha = A$$

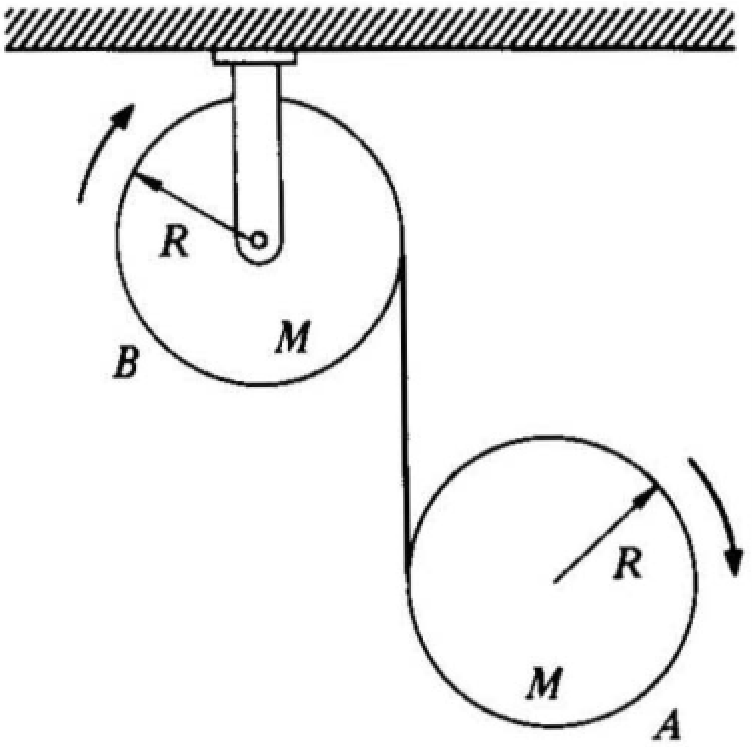
Substitute $\alpha = \frac{2mg}{R(2m+M)}$:

$$A = g - \frac{1}{2}R \cdot \frac{2mg}{R(2m + M)} = g - \frac{mg}{2m + M}$$

$$A = g\left(1 - \frac{m}{2m + M}\right) = g\left(\frac{2m + M - m}{2m + M}\right) = g\left(\frac{m + M}{2m + M}\right) = \frac{g(M + m)}{2m + M}$$

Q18

(6.24) Drum A of mass M and radius R is suspended from a drum B, also of mass M and radius R, which is free to rotate about its axis (see sketch below). The suspension is in the form of a massless metal tape wound around the outside of each drum, and free to unwind, as shown. Gravity is directed downward. Both drums are initially at rest. Find the initial acceleration of drum A, assuming that it moves straight down.



Solution:

The initial acceleration of drum A is found by analyzing the forces and torques acting on the system and applying the constraint from the massless, inextensible tape.

Define Variables and Constraints

- Let y be the downward displacement of the center of drum A from its initial position, with $y = 0$ at rest.
- Let θ_A and θ_B be the rotation angles of drums A and B respectively, with $\theta_A = 0$ and $\theta_B = 0$ at rest.
- The tape constraint relates the displacement and rotations: $y = R\theta_B + R\theta_A$.
- Differentiating twice with respect to time gives the acceleration constraint:

$$\frac{d^2y}{dt^2} = R\frac{d^2\theta_A}{dt^2} + R\frac{d^2\theta_B}{dt^2}$$

  Let $a_A = \frac{d^2y}{dt^2}$ (acceleration of A), $\alpha_A = \frac{d^2\theta_A}{dt^2}$ (angular acceleration of A), and $\alpha_B = \frac{d^2\theta_B}{dt^2}$ (angular acceleration of B). Thus:

$$a_A = R\alpha_A + R\alpha_B$$

Equations of Motion

For drum A (mass M, radius R):

- Linear motion (downward positive):

$$Ma_A = Mg - T$$

  where T is the tension in the tape.
- Rotational motion: The moment of inertia about its central axis is $I = \frac{1}{2}MR^2$. The tension T acts upward at the top tangent point, causing unwinding (clockwise rotation, positive for α_A):

$$I\alpha_A = TR$$

  Substituting I:

$$\frac{1}{2}MR^2\alpha_A = TR \implies \alpha_A = \frac{2T}{MR}$$

For drum B (mass M, radius R):

- Rotational motion: Fixed axis, no translation. The tension T acts downward at the bottom tangent point, causing unwinding (positive for α_B):

$$I\alpha_B = TR$$

Substituting I:

$$\frac{1}{2}MR^2\alpha_B = TR \implies \alpha_B = \frac{2T}{MR}$$

Thus, $\alpha_A = \alpha_B$.

Solve for Acceleration

From the constraint and the equality of angular accelerations:

$$a_A = R\alpha_A + R\alpha_B = R\alpha_A + R\alpha_A = 2R\alpha_A$$

Substitute $\alpha_A = \frac{2T}{MR}$:

$$a_A = 2R\left(\frac{2T}{MR}\right) = \frac{4T}{M}$$

From the linear motion equation of A:

$$Ma_A = Mg - T$$

Substitute $a_A = \frac{4T}{M}$:

$$M\left(\frac{4T}{M}\right) = Mg - T \implies 4T = Mg - T \implies 5T = Mg \implies T = \frac{Mg}{5}$$

Now substitute T back into the expression for a_A:

$$a_A = \frac{4}{M}\left(\frac{Mg}{5}\right) = \frac{4}{5}g$$

Thus, the initial acceleration of drum A is $\frac{4}{5}g$ downward.

Q19

(6.25) A marble of mass M and radius R is rolled up a plane of angle θ. If the initial velocity of the marble is v_0, what is the distance l it travels up the plane before it begins to roll back down?

Solution:

The marble is initially rolling without slipping up the incline with initial velocity v_0. The distance l it travels before stopping and beginning to roll back down is found using energy conservation, as friction does no work in pure rolling motion.

The initial kinetic energy consists of translational and rotational components. For a solid sphere (marble), the moment of inertia about its center is $I = \frac{2}{5}MR^2$.

- Initial translational kinetic energy: $\frac{1}{2}Mv_0^2$
- Initial rotational kinetic energy: $\frac{1}{2}I\omega_0^2$, where ω_0 is the initial angular velocity. Since the marble is rolling without slipping, $\omega_0 = \frac{v_0}{R}$. Thus,

$$\frac{1}{2}I\omega_0^2 = \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v_0}{R}\right)^2 = \frac{1}{2}\cdot\frac{2}{5}Mv_0^2 = \frac{1}{5}Mv_0^2$$

- Total initial kinetic energy:

$$\frac{1}{2}Mv_0^2 + \frac{1}{5}Mv_0^2 = \frac{5}{10}Mv_0^2 + \frac{2}{10}Mv_0^2 = \frac{7}{10}Mv_0^2$$

At the highest point, the marble stops (kinetic energy is zero) and has gained potential energy. The height gained is $h = l\sin\theta$, where l is the distance along the incline. The potential energy is $Mgh = Mgl\sin\theta$.

By conservation of energy:

$$\text{Initial kinetic energy} = \text{Potential energy at highest point}$$

$$\frac{7}{10}Mv_0^2 = Mgl\sin\theta$$

Solving for l:

$$\frac{7}{10}v_0^2 = gl\sin\theta$$

$$l = \frac{7v_0^2}{10g\sin\theta}$$

This result assumes that the marble rolls without slipping throughout the motion, which requires that the coefficient of static friction μ_s satisfies $\mu_s \geq \frac{2}{7}\tan\theta$ to prevent slipping. For a marble on a typical incline, this condition is reasonable.

Q20

(6.26) A uniform sphere of mass M and radius R and a uniform cylinder of mass M and radius R are released simultaneously from rest at the top of an inclined plane. Which body reaches the bottom first if they both roll without slipping?

Solution:

To determine which body reaches the bottom of the inclined plane first, consider the motion of both the sphere and the cylinder as they roll without slipping. The key factor is their linear acceleration down the incline, which depends on their moments of inertia. The body with the greater acceleration will reach the bottom first, as the distance and initial velocity are the same for both.

For an object rolling down an incline with angle θ, the linear acceleration a can be derived using Newton's second law and the no-slip condition.

- The component of gravitational force down the incline is $Mg\sin\theta$.
- The frictional force f opposes motion but provides the torque for rotation.
- The net force down the incline is $Mg\sin\theta - f = Ma$, where a is the linear acceleration.
- The torque about the center of mass is $fR = I\alpha$, where I is the moment of inertia, α is the angular acceleration, and R is the radius.
- Since there is no slipping, $\alpha = a/R$, so $fR = I(a/R)$ or $f = Ia/R^2$.
- Substitute f into the net force equation:

$$Mg\sin\theta - \frac{Ia}{R^2} = Ma$$

- Solve for a:

$$Mg\sin\theta = Ma + \frac{Ia}{R^2} = a\left(M + \frac{I}{R^2}\right)$$

$$a = \frac{Mg\sin\theta}{M + \frac{I}{R^2}} = \frac{g\sin\theta}{1 + \frac{I}{MR^2}}$$

- Define $k = I/(MR^2)$, so:

$$a = \frac{g\sin\theta}{1+k}$$

  The acceleration is constant, and a smaller k results in a larger acceleration.
- For a **uniform solid sphere** about its center, $I = \frac{2}{5}MR^2$, so:

$$k_{\text{sphere}} = \frac{I}{MR^2} = \frac{\frac{2}{5}MR^2}{MR^2} = \frac{2}{5} = 0.4$$

- For a **uniform solid cylinder** (disk) about its central axis, $I = \frac{1}{2}MR^2$, so:

$$k_{\text{cylinder}} = \frac{I}{MR^2} = \frac{\frac{1}{2}MR^2}{MR^2} = \frac{1}{2} = 0.5$$

  Since $k_{\text{sphere}} = 0.4 < 0.5 = k_{\text{cylinder}}$, the sphere has a smaller k.
- Acceleration for the sphere:

$$a_{\text{sphere}} = \frac{g\sin\theta}{1 + k_{\text{sphere}}} = \frac{g\sin\theta}{1+0.4} = \frac{g\sin\theta}{1.4}$$

- Acceleration for the cylinder:

$$a_{\text{cylinder}} = \frac{g\sin\theta}{1 + k_{\text{cylinder}}} = \frac{g\sin\theta}{1+0.5} = \frac{g\sin\theta}{1.5}$$

Since $1/1.4 > 1/1.5$, it follows that $a_{\text{sphere}} > a_{\text{cylinder}}$.

The time t to travel a distance s down the incline from rest is given by:

$$s = \frac{1}{2}at^2 \implies t = \sqrt{\frac{2s}{a}}$$

Since s is the same for both bodies, $t \propto 1/\sqrt{a}$. Therefore, a larger acceleration results in a smaller time. Because $a_{\text{sphere}} > a_{\text{cylinder}}$, we have $t_{\text{sphere}} < t_{\text{cylinder}}$.

Thus, the sphere reaches the bottom first.

The result can be confirmed using energy conservation. At the top, potential energy is Mgh. At the bottom, kinetic energy is translational plus rotational:

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

With no slipping, $\omega = v/R$, so:

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2 = \frac{1}{2}Mv^2 + \frac{1}{2}\frac{I}{R^2}v^2 = \frac{1}{2}Mv^2(1+k)$$

Solving for v^2:

$$v^2 = \frac{2gh}{1+k}$$

The velocity at the bottom is greater for smaller k. For the sphere, $k = 0.4$, so $v^2_{\text{sphere}} = 2gh/1.4$. For the cylinder, $k = 0.5$, so $v^2_{\text{cylinder}} = 2gh/1.5$. Thus, $v_{\text{sphere}} > v_{\text{cylinder}}$.

With constant acceleration, the average velocity is $v/2$, and time $t = 2s/v$. Since $v_{\text{sphere}} > v_{\text{cylinder}}$, $t_{\text{sphere}} < t_{\text{cylinder}}$, confirming the sphere arrives first.

Q21

(6.27) A Yo-Yo of mass M has an axle of radius b and a spool of radius R. Its moment of inertia can be taken to be $MR^2/2$. The Yo-Yo is placed upright on a table and the string is pulled with a horizontal force F as shown. The coefficient of friction between the Yo-Yo and the table is μ. What is the maximum value of F for which the Yo-Yo will roll without slipping?

Solution:

The yo-yo is placed upright on a table, with the string pulled horizontally from the bottom of the axle. The forces acting on the yo-yo are gravity Mg downward at the center of mass, the normal force N upward at the point of contact with the table, the friction force f horizontal at the point of contact, and the applied force F horizontal to the right at the point of application on the axle.

The center of mass is at height R above the table, and the point of application of F is at height $R - b$ above the table. The moment of inertia about the center of mass is $I = \frac{1}{2}MR^2$.

Equations of Motion

1. **Translation in the x-direction (horizontal):**
   The net force in the x-direction is $F - f = Ma_{\text{cm}}$, where a_{cm} is the acceleration of the center of mass, and f is the friction force (taken positive to the left for the initial assumption, but direction will be determined).

$$F - f = Ma_{\text{cm}} \quad (1)$$

2. **Translation in the y-direction (vertical):**
   Since there is no vertical acceleration, the net force in the y-direction is zero. The forces are gravity Mg downward and the normal force N upward. The applied force F is horizontal, so it has no y-component.

$$N - Mg = 0 \implies N = Mg \quad (2)$$

3. **Rotation about the center of mass:**
   The net torque about the center of mass is $\tau_{\text{net}} = I\alpha$, where α is the angular acceleration (defined positive for clockwise rotation).
   - Applied force F: applied at a relative position $(0, -b)$ from the center of mass. The torque is bF clockwise, so $\tau_F = bF$.

- Friction force f: applied at the point of contact, relative position $(0, -R)$ from the center of mass. The torque is $-Rf$ if f is taken positive to the left (counterclockwise), but to be consistent, the torque is expressed as Rf for friction force magnitude with direction handled by sign. The net torque is:

$$\tau_{\text{net}} = bF - Rf = \frac{1}{2}MR^2\alpha \quad (3)$$

4. **Rolling without slipping condition:**
For rolling without slipping on the outer radius R, the relationship between linear and angular acceleration is:

$$a_{\text{cm}} = R\alpha \quad (4)$$

Solving the Equations
Substitute $\alpha = a_{\text{cm}}/R$ from (4) into (3):

$$bF - Rf = \frac{1}{2}MR^2\left(\frac{a_{\text{cm}}}{R}\right) = \frac{1}{2}MRa_{\text{cm}} \quad (3a)$$

From (1), solve for a_{cm}:

$$a_{\text{cm}} = \frac{F - f}{M} \quad (1a)$$

Substitute (1a) into (3a):

$$bF - Rf = \frac{1}{2}MR\left(\frac{F - f}{M}\right) = \frac{1}{2}R(F - f)$$

Rearrange terms:

$$bF - Rf = \frac{1}{2}RF - \frac{1}{2}Rf$$

$$bF - \frac{1}{2}RF - Rf + \frac{1}{2}Rf = 0$$

$$F\left(b-\frac{R}{2}\right)+f\left(-\frac{R}{2}\right)=0$$

$$F\left(b-\frac{R}{2}\right)=f\frac{R}{2}$$

$$f=F\left(\frac{2b}{R}-1\right) \quad (5)$$

This is the friction force required for rolling without slipping. The magnitude of friction must satisfy $|f| \leq \mu N=\mu M g$ to prevent slipping. At the maximum force F before slipping occurs, $|f|=\mu M g$. Therefore:

$$\left|F\left(\frac{2b}{R}-1\right)\right|=\mu M g$$

$$|F|\left|\frac{2b}{R}-1\right|=\mu M g$$

Since $F>0$:

$$F\left|\frac{2b}{R}-1\right|=\mu M g$$

$$F=\frac{\mu M g}{\left|\frac{2b}{R}-1\right|}$$

Simplify the denominator:

$$\left|\frac{2b}{R}-1\right|=\left|\frac{2b-R}{R}\right|=\frac{|2b-R|}{R}$$

$$F=\frac{\mu M g}{\frac{|2b-R|}{R}}=\frac{\mu M g R}{|2b-R|}$$

This expression gives the maximum force F for rolling without slipping. The absolute value accounts for the direction of friction depending on whether $b<R/2$ or $b>R/2$, and the normal force $N=Mg$ remains constant since there are no vertical components from the applied force.

Q22

(6.28) The Yo-Yo of the previous problem is pulled so that the string makes an angle θ with the horizontal. For what value of θ does the Yo-Yo have no tendency to rotate?

Solution:

The Yo-Yo has no tendency to rotate when the net torque about its center of mass is zero, which occurs when the applied force F produces no torque. The force F is applied at the point of string attachment, which is at a distance b below the center of mass, with coordinates $(0, -b)$ if the center of mass is at $(0, 0)$.

The torque due to F about the center of mass is given by:

$$\tau_F = bF\cos\theta$$

where θ is the angle the string makes with the horizontal. This torque is zero when:

$$bF\cos\theta = 0$$

Assuming $F \neq 0$ and $b \neq 0$, this requires:

$$\cos\theta = 0$$

$$\theta = 90^\circ$$

At $\theta = 90^\circ$, the force F is vertical. Since the line of action of a vertical force applied at $(0, -b)$ passes through the center of mass at $(0, 0)$, the torque is zero. Consequently, there is no tendency for the Yo-Yo to rotate, regardless of the magnitude of F, as long as other forces (such as friction) do not produce a net torque.

Q23

(6.29) A Yo-Yo of mass M has an axle of radius b and a spool of radius R. Its moment of inertia can be taken to be $MR^2/2$ and the thickness of the string can be neglected. The Yo-Yo is released from rest.

- What is the tension in the cord as the Yo-Yo descends and as it ascends?
- The center of the Yo-Yo descends distance h before the string is fully unwound. Assuming that it reverses direction with uniform spin velocity, find the average force on the string while the Yo-Yo turns around.

Solution:

The tension in the cord depends on whether the yo-yo is descending or ascending, and the expressions are derived from the equations of motion considering the forces and torques involved.

As the Yo-Yo Descends:

- The net force downward is $Mg - T = Ma$, where a is the linear acceleration downward.
- The torque about the center is $\tau = Tb$, and with moment of inertia $I = \frac{1}{2}MR^2$, the torque equation is $Tb = I\alpha$.
- The no-slip condition relates linear and angular acceleration: $\alpha = \frac{a}{b}$.
- Substituting α gives $Tb = I\frac{a}{b}$, so $T = \frac{Ia}{b^2} = \frac{\frac{1}{2}MR^2a}{b^2}$.
- Combining with the force equation: $Mg - \frac{\frac{1}{2}MR^2a}{b^2} = Ma$.
- Solving for a: $a = g\frac{2b^2}{2b^2+R^2}$.
- Substituting into the tension expression: $T = \frac{\frac{1}{2}MR^2}{b^2} \cdot g\frac{2b^2}{2b^2+R^2} = Mg\frac{R^2}{R^2+2b^2}$.

Thus, the tension as the yo-yo descends is:

$$T_{\text{desc}} = Mg\frac{R^2}{R^2 + 2b^2}$$

As the Yo-Yo Ascends:

- The net force upward is $T - Mg = Ma$, where a is the linear acceleration upward.
- The torque about the center is $\tau = -Tb$ (negative for winding torque), and $\tau = I\alpha$.

- The no-slip condition gives $\alpha = -\frac{a}{b}$.
- The torque equation is $-Tb = I\left(-\frac{a}{b}\right)$, simplifying to $Tb = I\frac{a}{b}$, so $T = \frac{Ia}{b^2} = \frac{\frac{1}{2}MR^2 a}{b^2}$.
- Combining with the force equation: $\frac{\frac{1}{2}MR^2 a}{b^2} - Mg = Ma$.
- Solving for a: $a = g\frac{2b^2}{R^2 - 2b^2}$ (valid when $R^2 > 2b^2$ for positive acceleration or deceleration).
- Substituting into the tension expression: $T = \frac{\frac{1}{2}MR^2}{b^2} \cdot g\frac{2b^2}{R^2 - 2b^2} = Mg\frac{R^2}{R^2 - 2b^2}$.

Thus, the tension as the yo-yo ascends is:

$$T_{\text{asc}} = Mg\frac{R^2}{R^2 - 2b^2}$$

The average force on the string during the turnaround phase, where the yo-yo reverses direction with uniform spin velocity, is the average tension in the string. This force is derived by considering the impulse required to reverse the linear velocity of the center of mass while keeping the angular velocity constant, and assuming a characteristic turnaround time based on the rotation by 1 radian.

From the descent phase, the angular velocity at the bottom, when the string is fully unwound, is given by:

$$\omega_{\text{desc}} = 2\sqrt{\frac{hg}{2b^2 + R^2}}$$

The change in linear velocity of the center of mass is from $-b\omega_{\text{desc}}$ downward to $+b\omega_{\text{desc}}$ upward, resulting in a change of $2b\omega_{\text{desc}}$. The impulse required for this change is:

$$\text{Impulse} = M \cdot 2b\omega_{\text{desc}} = 2Mb\omega_{\text{desc}}$$

The characteristic turnaround time, based on rotation by 1 radian at constant angular velocity, is:

$$\Delta t = \frac{1}{\omega_{\text{desc}}}$$

The average net force (accounting for gravity) is:

$$\langle F_{\text{net}} \rangle = \frac{\text{Impulse}}{\Delta t} = \frac{2Mb\omega_{\text{desc}}}{1/\omega_{\text{desc}}} = 2Mb\omega_{\text{desc}}^2$$

The average tension is the average net force plus the weight, as gravity is constant:

$$\langle T\rangle = \langle F_{\text{net}}\rangle + Mg = 2Mb\omega_{\text{desc}}^2 + Mg$$

Substituting $\omega_{\text{desc}}^2 = 4\frac{hg}{2b^2+R^2}$:

$$2Mb\omega_{\text{desc}}^2 = 2Mb \cdot 4\frac{hg}{2b^2+R^2} = 8Mb\frac{hg}{2b^2+R^2}$$

$$\langle T\rangle = 8Mb\frac{hg}{2b^2+R^2} + Mg = Mg\left(1 + \frac{8bh}{2b^2+R^2}\right)$$

Q24

(6.30) A bowling ball is thrown down the alley with speed v_0. Initially it slides without rolling, but due to friction it begins to roll. Show that its speed when it rolls without sliding is $\frac{5}{7}v_0$.

Solution:

To determine the speed of the bowling ball when it rolls without sliding, consider the initial and final conditions, and the role of friction.

The ball has mass m, radius r, and moment of inertia about its center $I = \frac{2}{5}mr^2$ (for a solid sphere). Initially, it slides with linear velocity v_0 and angular velocity $\omega_0 = 0$. Kinetic friction acts opposite to the direction of motion with magnitude $f_k = \mu_k N = \mu_k mg$, where μ_k is the coefficient of kinetic friction and $N = mg$ is the normal force.

The friction force causes two effects:

- It decelerates the linear motion.
- It provides a torque that accelerates the rotational motion.

Define the direction such that the linear velocity v is positive to the right, and the angular velocity ω is positive clockwise (so that rolling without slipping to the right satisfies $v = r\omega$).

The equations of motion are derived from Newton's second law and the torque equation:

1. **Linear motion**:
   The friction force opposes motion, so:

$$-f_k = m\frac{dv}{dt}$$

   Let $a = \frac{dv}{dt}$ be the linear acceleration. Then:

$$a = -\frac{f_k}{m}$$

2. **Rotational motion**:
   The torque due to friction about the center is $\tau = f_k r$ (since the force acts at the point of contact and tends to rotate the ball clockwise). The torque equation is:

$$\tau = I\alpha$$

   where $\alpha = \frac{d\omega}{dt}$ is the angular acceleration. Substituting $I = \frac{2}{5}mr^2$:

$$f_k r = \left(\frac{2}{5}mr^2\right)\alpha$$

   Solving for α:

$$\alpha = \frac{f_k r}{\frac{2}{5}mr^2} = \frac{5f_k}{2mr}$$

Since f_k is constant during sliding, both a and α are constant. Integrate to find the linear and angular velocities as functions of time, with initial conditions $v(0) = v_0$ and $\omega(0) = 0$:

- **Linear velocity**:

$$v(t) = v_0 + at = v_0 - \frac{f_k}{m}t$$

- **Angular velocity**:

$$\omega(t) = \omega_0 + \alpha t = 0 + \frac{5f_k}{2mr}t = \frac{5f_k}{2mr}t$$

Rolling without slipping begins when $v(t) = r\omega(t)$. Set up the equation:

$$v_0 - \frac{f_k}{m}t = r\left(\frac{5f_k}{2mr}t\right)$$

Simplify the right side:

$$r \cdot \frac{5f_k}{2mr}t = \frac{5f_k}{2m}t$$

So the equation is:

$$v_0 - \frac{f_k}{m}t = \frac{5f_k}{2m}t$$

Solve for t. First, isolate terms with t:

$$v_0 = \frac{f_k}{m}t + \frac{5f_k}{2m}t$$

$$v_0 = \frac{f_k}{m}t\left(1 + \frac{5}{2}\right) = \frac{f_k}{m}t\left(\frac{2}{2} + \frac{5}{2}\right) = \frac{f_k}{m}t\left(\frac{7}{2}\right)$$

Thus:

$$v_0 = \frac{7f_k}{2m}t$$

Solving for t:

$$t = \frac{2mv_0}{7f_k}$$

Now, find the linear velocity at this time:

$$v(t) = v_0 - \frac{f_k}{m}t$$

Substitute t:

$$v(t) = v_0 - \frac{f_k}{m}\left(\frac{2mv_0}{7f_k}\right) = v_0 - \frac{2v_0}{7} = \frac{7v_0}{7} - \frac{2v_0}{7} = \frac{5v_0}{7}$$

Q25

(6.31) A cylinder of radius R spins with angular velocity ω_0. When the cylinder is gently laid on a plane, it skids for a short time and eventually rolls without slipping. What is the final angular velocity, ω_f?

Solution:

The cylinder is initially spinning with angular velocity ω_0 and no linear velocity ($v_0 = 0$) when gently laid on the plane. Due to slipping, kinetic friction acts to oppose the relative motion. The friction force f_k accelerates the center of mass linearly and decelerates the rotation until the rolling without slipping condition $v = \omega R$ is achieved.

The friction force $f_k = \mu_k N = \mu_k mg$, where μ_k is the coefficient of kinetic friction, m is the mass, and g is the acceleration due to gravity. This force causes a linear acceleration $a = \frac{f_k}{m} = \mu_k g$ in the direction of motion. For rotation, the torque about the center of mass is $\tau = -f_k R$ (negative because it opposes the initial clockwise rotation), leading to angular acceleration $\alpha = \frac{\tau}{I}$, where $I = \frac{1}{2}mR^2$ is the moment of inertia for a solid cylinder about its central axis. Thus,

$$\alpha = \frac{-f_k R}{I} = \frac{-f_k R}{\frac{1}{2}mR^2} = \frac{-2f_k}{mR} = \frac{-2\mu_k g}{R}.$$

The equations of motion are:

- Linear velocity: $v(t) = v_0 + at = 0 + (\mu_k g)t = \mu_k g t$
- Angular velocity: $\omega(t) = \omega_0 + \alpha t = \omega_0 + \left(\frac{-2\mu_k g}{R}\right)t$

At the time t when rolling without slipping begins, $v(t) = \omega(t)R$:

$$\mu_k g t = \left[\omega_0 - \frac{2\mu_k g}{R}t\right]R$$

Simplify the right side:

$$\mu_k g t = \omega_0 R - 2\mu_k g t$$

Rearrange terms:

$$\mu_k g t + 2\mu_k g t = \omega_0 R$$

$$3\mu_k g t = \omega_0 R$$

$$t = \frac{\omega_0 R}{3\mu_k g}$$

Substitute t into the expression for $\omega(t)$ to find the final angular velocity ω_f:

$$\omega_f = \omega_0 - \frac{2\mu_k g}{R}\left(\frac{\omega_0 R}{3\mu_k g}\right)$$

Simplify:

$$\omega_f = \omega_0 - \frac{2\mu_k g}{R} \cdot \frac{\omega_0 R}{3\mu_k g} = \omega_0 - \frac{2}{3}\omega_0 = \frac{1}{3}\omega_0$$

Thus, the final angular velocity is $\omega_f = \frac{1}{3}\omega_0$.

Alternatively, using conservation of angular momentum about the point of contact (where torque is zero due to friction acting at that point):

- Initial angular momentum: $L_i = I\omega_0$ (since $v_0 = 0$)
- Final angular momentum: $L_f = (I + mR^2)\omega_f$ (for rolling without slipping about the point of contact)

Set $L_i = L_f$:

$$I\omega_0 = (I + mR^2)\omega_f$$

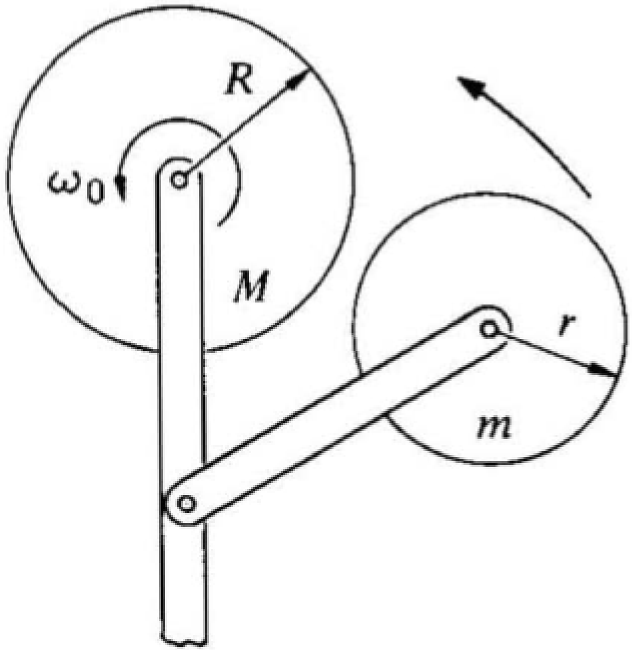
$$\omega_f = \frac{I\omega_0}{I + mR^2}$$

For a solid cylinder, $I = \frac{1}{2}mR^2$:

$$\omega_f = \frac{\frac{1}{2}mR^2\omega_0}{\frac{1}{2}mR^2 + mR^2} = \frac{\frac{1}{2}mR^2\omega_0}{\frac{3}{2}mR^2} = \frac{1}{2} \cdot \frac{2}{3}\omega_0 = \frac{1}{3}\omega_0$$

Q26

(6.32) A solid rubber wheel of radius R and mass M rotates with angular velocity ω_0 about a frictionless pivot (see sketch). A second rubber wheel of radius r and mass m, also mounted on a frictionless pivot, is brought into contact with it. What is the final angular velocity of the first wheel?



Solution:

The final angular velocity of the first wheel is found using conservation of angular momentum and the no-slip condition at the point of contact.

Consider the system of both wheels. Since the pivots are frictionless, there are no external torques, and angular momentum is conserved about the pivot of the first wheel.

The initial angular momentum about the pivot of the first wheel is due only to the first wheel, as the second wheel is initially at rest:

$$L_i = I_1\omega_0$$

where $I_1 = \frac{1}{2}MR^2$ is the moment of inertia of the first wheel.

After the wheels come into contact and reach a no-slip condition, the final angular momentum is:

$$L_f = I_1\omega_1 + I_2\omega_2$$

where $I_2 = \frac{1}{2}mr^2$ is the moment of inertia of the second wheel, ω_1 is the final angular velocity of the first wheel, and ω_2 is the final angular velocity of the second wheel.

Conservation of angular momentum gives:

$$I_1\omega_0 = I_1\omega_1 + I_2\omega_2$$

The no-slip condition requires that the tangential velocities at the point of contact are equal in magnitude but opposite in direction. Assuming the wheels are positioned such that the first wheel is on the left and the second on the right, if ω_1 is clockwise (positive), then ω_2 must be counterclockwise (negative) for no slipping. The condition is:

$$|\omega_1|R = |\omega_2|r \quad \Rightarrow \quad \omega_2 = -\frac{R}{r}\omega_1$$

Substitute ω_2 into the angular momentum equation:

$$I_1\omega_0 = I_1\omega_1 + I_2\left(-\frac{R}{r}\omega_1\right)$$

$$I_1\omega_0 = \omega_1\left(I_1 - I_2\frac{R}{r}\right)$$

Solve for ω_1:

$$\omega_1 = \frac{I_1\omega_0}{I_1 - I_2\frac{R}{r}}$$

Substitute the moments of inertia:

$$I_1 = \frac{1}{2}MR^2, \quad I_2 = \frac{1}{2}mr^2$$

$$\omega_1 = \frac{\frac{1}{2}MR^2\omega_0}{\frac{1}{2}MR^2 - \frac{1}{2}mr^2 \cdot \frac{R}{r}} = \frac{\frac{1}{2}MR^2\omega_0}{\frac{1}{2}MR^2 - \frac{1}{2}mrR}$$

$$\omega_1 = \frac{MR^2\omega_0}{MR^2 - mrR} = \frac{MR^2\omega_0}{R(MR - mr)} = \frac{MR\omega_0}{MR - mr}$$

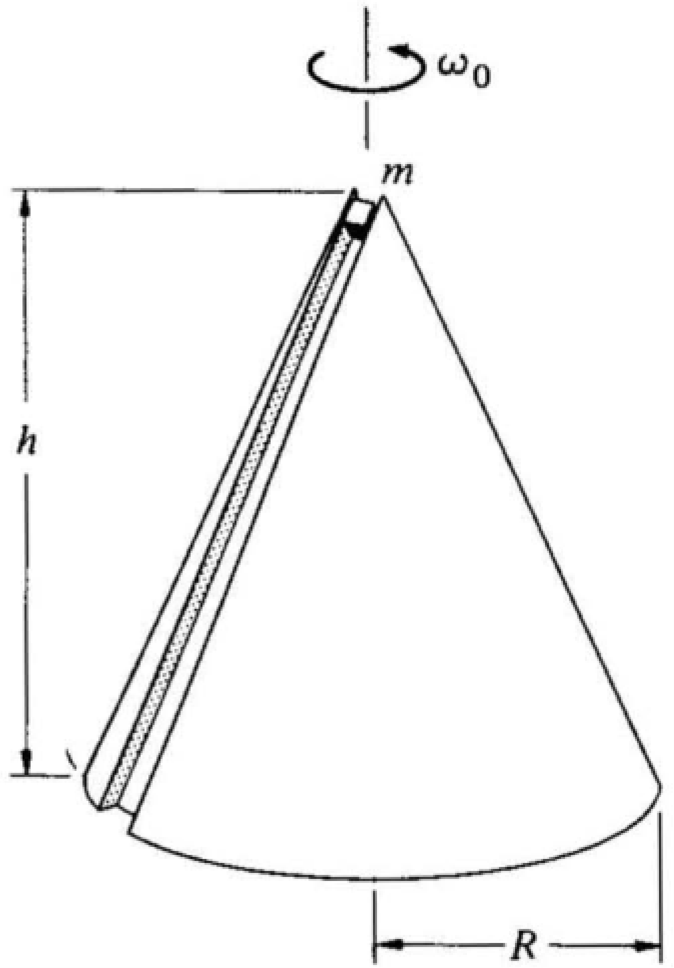
Thus, the final angular velocity of the first wheel is:

$$\omega_1 = \omega_0 \frac{MR}{MR - mr}$$

Q27

(6.33) A cone of height h and base radius R is free to rotate about a fixed vertical axis. It has a thin groove cut in the surface. The cone is set rotating freely with angular speed ω_0, and a small block of mass m is released in the top of the frictionless groove and allowed to slide under gravity. Assume that the block stays in the groove. Take the moment of inertia of the cone about the vertical axis to be I_0.

- What is the angular velocity of the cone when the block reaches the bottom?
- Find the speed of the block in inertial space when it reaches the bottom.



Solution:

The angular momentum about the fixed vertical axis is conserved because there are no external torques acting on the system. Gravity acts vertically and parallel to the axis, resulting in zero torque about this axis. The forces between the block and the groove are internal to the system, so they do not contribute to external torques.

Initially, the cone rotates with angular speed ω_0, and the block is at the top of the groove, where its distance from the axis is R. The initial angular momentum is the sum of the angular momentum of the cone and the block:

$$L_i = I_0\omega_0 + mR^2\omega_0 = (I_0 + mR^2)\omega_0$$

When the block reaches the bottom (the apex of the cone), its distance from the axis is $r = 0$. At this point, the angular momentum of the block is zero because $r = 0$. Thus, the final angular momentum is solely due to the cone:

$$L_f = I_0\omega_f$$

By conservation of angular momentum:

$$L_i = L_f$$

$$(I_0 + mR^2)\omega_0 = I_0\omega_f$$

Solving for ω_f:

$$\omega_f = \frac{I_0 + mR^2}{I_0}\omega_0 = \omega_0\left(1 + \frac{mR^2}{I_0}\right)$$

The angular velocity of the cone when the block reaches the bottom is $\omega_f = \omega_0\left(1 + \frac{mR^2}{I_0}\right)$, as determined from conservation of angular momentum about the vertical axis, since no external torques act on the system.

To find the speed of the block in inertial space when it reaches the bottom, conservation of energy is applied. The total mechanical energy is conserved because the groove is frictionless, gravity is conservative, and no dissipative forces act on the system.

The initial total energy, when the block is at the top (base of the cone), consists of the kinetic energy of the cone, the kinetic energy of the block, and the gravitational potential energy. The cone has moment of inertia I_0 and initial angular speed ω_0, so its kinetic energy is $\frac{1}{2}I_0\omega_0^2$. The block is released from rest relative to the cone at a distance R from the axis, so its initial speed is $R\omega_0$ and its kinetic energy is $\frac{1}{2}m(R\omega_0)^2$. The potential energy, with the reference point at the bottom (apex), is mgh, where h is the height of the cone. Thus, the initial total energy is:

$$E_i = \frac{1}{2}I_0\omega_0^2 + \frac{1}{2}mR^2\omega_0^2 + mgh.$$

At the bottom, the block is at the apex ($s = 0$), where the radial distance from the axis is zero. The cone has angular velocity ω_f, so its kinetic energy is $\frac{1}{2}I_0\omega_f^2$. The block's speed in inertial space is the magnitude of its velocity along the groove. At $s = 0$, the velocity of the block is purely along the groove, and its speed is $|\dot{s}|$. Thus, its kinetic energy is $\frac{1}{2}m\dot{s}^2$. The potential energy is zero. The final total energy is:

$$E_f = \frac{1}{2}I_0\omega_f^2 + \frac{1}{2}m\dot{s}^2.$$

Using conservation of energy, $E_i = E_f$:

$$\frac{1}{2}I_0\omega_0^2 + \frac{1}{2}mR^2\omega_0^2 + mgh = \frac{1}{2}I_0\omega_f^2 + \frac{1}{2}m\dot{s}^2.$$

Substitute $\omega_f = \omega_0\left(1 + \frac{mR^2}{I_0}\right)$:

$$\frac{1}{2}I_0\omega_0^2 + \frac{1}{2}mR^2\omega_0^2 + mgh = \frac{1}{2}I_0\left[\omega_0^2\left(1 + \frac{mR^2}{I_0}\right)^2\right] + \frac{1}{2}m\dot{s}^2.$$

Simplify the right side:

$$\frac{1}{2}I_0\omega_0^2\left(1 + \frac{mR^2}{I_0}\right)^2 = \frac{1}{2}I_0\omega_0^2\left(1 + \frac{2mR^2}{I_0} + \frac{m^2R^4}{I_0^2}\right) = \frac{1}{2}\omega_0^2\left(I_0 + 2mR^2 + \frac{m^2R^4}{I_0}\right).$$

The equation becomes:

$$\frac{1}{2}I_0\omega_0^2 + \frac{1}{2}mR^2\omega_0^2 + mgh = \frac{1}{2}\omega_0^2\left(I_0 + 2mR^2 + \frac{m^2R^4}{I_0}\right) + \frac{1}{2}m\dot{s}^2.$$

Rearrange to solve for $\dot{s}^2$:

$$mgh = \frac{1}{2}\omega_0^2\left(I_0 + 2mR^2 + \frac{m^2R^4}{I_0}\right) - \frac{1}{2}I_0\omega_0^2 - \frac{1}{2}mR^2\omega_0^2 + \frac{1}{2}m\dot{s}^2.$$

Simplify the ω_0^2 terms:

$$\frac{1}{2}\omega_0^2\left(I_0 + 2mR^2 + \frac{m^2R^4}{I_0} - I_0 - mR^2\right) = \frac{1}{2}\omega_0^2\left(mR^2 + \frac{m^2R^4}{I_0}\right).$$

Thus:

$$mgh = \frac{1}{2}\omega_0^2\left(mR^2 + \frac{m^2R^4}{I_0}\right) + \frac{1}{2}m\dot{s}^2.$$

Divide by m and multiply by 2:

$$2gh = \omega_0^2\left(R^2 + \frac{mR^4}{I_0}\right) + \dot{s}^2.$$

Solve for $\dot{s}^2$:

$$\dot{s}^2 = 2gh - \omega_0^2R^2\left(1 + \frac{mR^2}{I_0}\right).$$

The speed of the block is the magnitude of its velocity, so at the bottom, where $s = 0$, the speed is:

$$v = |\dot{s}| = \sqrt{2gh - \omega_0^2R^2\left(1 + \frac{mR^2}{I_0}\right)}.$$

Q28

(6.34) A marble of radius b rolls back and forth in a shallow dish of radius R. Find the frequency of small oscillations.

Solution:

To find the frequency of small oscillations for a marble of radius b rolling back and forth in a shallow dish of radius R, without using Lagrangian mechanics, we use Newton's laws and consider the forces and torques acting on the marble. The dish is spherical, but for small oscillations, it is approximated as parabolic with height $y = \frac{x^2}{2R}$, where x is the horizontal distance from the center.

The center of the marble has horizontal coordinate x_c and vertical coordinate $y_c = b + \frac{x_c^2}{2R}$. The forces acting on the marble are:

- Gravity mg downward.
- Normal force N perpendicular to the dish.

- Friction force f parallel to the dish.

The slope of the dish at the contact point is $\frac{dy}{dx} = \frac{x}{R}$. For small oscillations, $x_c \ll R$, so the angle θ that the normal makes with the vertical is small, and $\sin\theta \approx \tan\theta = \frac{x_c}{R}$, $\cos\theta \approx 1$.

Equations of Motion:

1. **Vertical Force Balance**:
   The vertical acceleration is negligible for small oscillations. The vertical forces are:

$$N\cos\theta - f\sin\theta - mg = 0.$$

   Using $\cos\theta \approx 1$, $\sin\theta \approx \frac{x_c}{R}$, and neglecting the small term $f\frac{x_c}{R}$ (second order in x_c):

$$N - mg = 0 \implies N \approx mg.$$

2. **Horizontal Force Balance**:
   The horizontal force equation is:

$$m\ddot{x}_c = -f\cos\theta - N\sin\theta.$$

   Using $\cos\theta \approx 1$, $\sin\theta \approx \frac{x_c}{R}$, and $N \approx mg$:

$$m\ddot{x}_c = -f - mg\frac{x_c}{R}.$$

3. **Torque Equation**:
   The friction force provides torque about the center of mass. For a solid sphere, the moment of inertia is $I = \frac{2}{5}mb^2$. The torque is $\tau = fb$ (since friction acts at the bottom and the lever arm is b), and it causes angular acceleration $\dot{\omega}$. By Newton's second law for rotation:

$$\tau = I\dot{\omega} \implies fb = \left(\frac{2}{5}mb^2\right)\dot{\omega}.$$

   For rolling without slipping, $\omega = \frac{\dot{x}_c}{b}$, so $\dot{\omega} = \frac{\ddot{x}_c}{b}$. Substituting:

$$fb = \frac{2}{5}mb^2 \cdot \frac{\ddot{x}_c}{b} \implies f = \frac{2}{5}m\ddot{x}_c.$$

Combining Equations:
Substitute $f = \frac{2}{5}m\ddot{x}_c$ into the horizontal force equation:

$$m\ddot{x}_c = -\left(\frac{2}{5}m\ddot{x}_c\right) - mg\frac{x_c}{R}.$$

Rearrange to:

$$m\ddot{x}_c + \frac{2}{5}m\ddot{x}_c + mg\frac{x_c}{R} = 0,$$

$$\frac{7}{5}m\ddot{x}_c + mg\frac{x_c}{R} = 0.$$

Divide by m:

$$\frac{7}{5}\ddot{x}_c + g\frac{x_c}{R} = 0,$$

$$\ddot{x}_c + \frac{5g}{7R}x_c = 0.$$

This is the equation of simple harmonic motion, $\ddot{x}_c + \omega^2 x_c = 0$, where the angular frequency ω is:

$$\omega = \sqrt{\frac{5g}{7R}}.$$

ALTERNATIVE (using Lagrangian mechanics)
The marble of radius b rolls without slipping in a shallow spherical dish of radius R. For small oscillations, the dish can be approximated as parabolic, with height $y = \frac{x^2}{2R}$, where x is the horizontal distance from the center.

The position of the marble's center of mass is denoted by x_c (horizontal coordinate). The height of the center of mass is approximately $y_c = b + \frac{x_c^2}{2R}$, ignoring the small horizontal offset due to curvature since the dish is shallow and $b << R$.

The gravitational potential energy, ignoring constants, is:

$$U = \frac{1}{2}mg\frac{x_c^2}{R}$$

The kinetic energy has translational and rotational components. The translational kinetic energy is $\frac{1}{2}mv_c^2$, where v_c is the speed of the center of mass. For small x_c, $v_c^2 \approx \dot{x}_c^2$. The rotational kinetic energy is $\frac{1}{2}I\omega^2$, where $I = \frac{2}{5}mb^2$ for a solid sphere. From rolling without slipping, $\omega = \frac{|\dot{x}_c|}{b}$, so $\omega^2 = \frac{\dot{x}_c^2}{b^2}$. Thus:

$$\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{2}{5}mb^2\right)\left(\frac{\dot{x}_c^2}{b^2}\right) = \frac{1}{5}m\dot{x}_c^2$$

The total kinetic energy is:

$$T = \frac{1}{2}m\dot{x}_c^2 + \frac{1}{5}m\dot{x}_c^2 = \frac{7}{10}m\dot{x}_c^2$$

The Lagrangian is:

$$L = T - U = \frac{7}{10}m\dot{x}_c^2 - \frac{1}{2}mg\frac{x_c^2}{R}$$

The Euler-Lagrange equation gives the equation of motion. Compute:

$$\frac{\partial L}{\partial \dot{x}_c} = \frac{7}{5}m\dot{x}_c, \quad \frac{\partial L}{\partial x_c} = -mg\frac{x_c}{R}$$

$$\frac{d}{dt}\left(\frac{7}{5}m\dot{x}_c\right) = -mg\frac{x_c}{R}$$

$$\frac{7}{5}m\ddot{x}_c = -\frac{mg}{R}x_c$$

Divide by m:

$$\frac{7}{5}\ddot{x}_c = -\frac{g}{R}x_c$$

$$\ddot{x}_c + \frac{5}{7}\frac{g}{R}x_c = 0$$

This is the equation for simple harmonic motion with angular frequency:

$$\omega = \sqrt{\frac{5}{7}\frac{g}{R}}$$

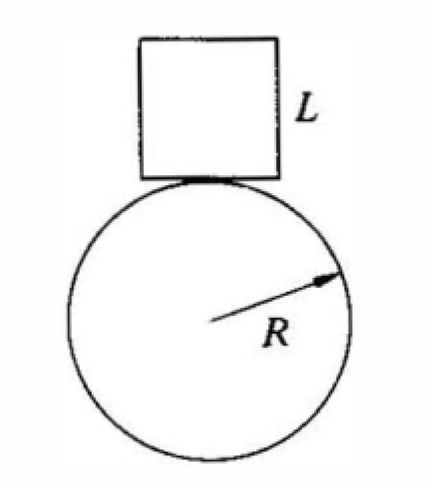
Thus, the angular frequency of small oscillations is $\sqrt{\frac{5g}{7R}}$.

The frequency (cyclic frequency) is:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{5g}{7R}}$$

Q29

(6.35) A cubical block of side L rests on a fixed cylindrical drum of radius R. Find the largest value of L for which the block is stable. See figure below.



Solution:

The cubical block of side length L rests on a fixed cylindrical drum of radius R. For the block to be stable in the upright position, it must be able to rest on the drum without immediately tipping over. The stability depends on whether the block can be placed in equilibrium on the drum and remain stable under small disturbances.

Consider the block in the upright position with its bottom face horizontal and tangent to the top of the cylinder at the point $(0, R)$ in a coordinate system where the center of the cylinder is at $(0, 0)$. The bottom face of the block extends from $x = -L/2$ to $x = L/2$ at height $y = R$. The cylinder has the equation $x^2 + y^2 = R^2$, so the height of the cylinder at any x is $y = \sqrt{R^2 - x^2}$.

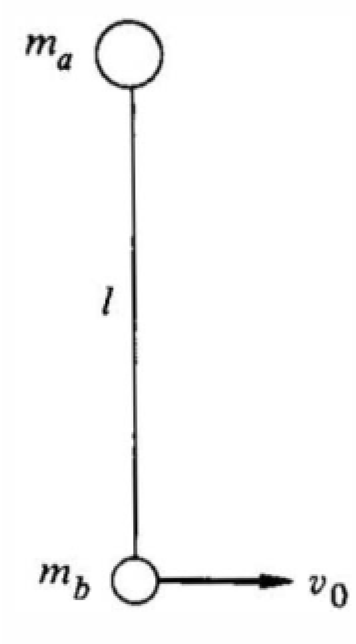
For the block to rest on the cylinder, the entire bottom face must be above or tangent to the cylinder. At the corners of the bottom face, $x = \pm L/2$, the height of the cylinder is $\sqrt{R^2 - (L/2)^2}$. For the corners to be supported, the x-coordinates must satisfy $|x| \leq R$, meaning $L/2 \leq R$, or $L \leq 2R$. If $L > 2R$, the corners at $x = \pm L/2$ lie outside the cylinder (since $|L/2| > R$), and the cylinder does not support these points, causing the block to tip over immediately.

When $L \leq 2R$, the block can be placed in equilibrium at the top of the cylinder. Analysis of the torque for small angular displacements θ shows that the torque is restoring, indicating local stability. Specifically, the center of mass is at height $R + L/2$ above the center of the cylinder, and the center of curvature of the cylinder at the contact point is at depth R below the contact point. For stability, the height of the center of mass above the contact point, which is $L/2$, must be less than or equal to the depth of the center of curvature, R. This gives $L/2 \leq R$, or $L \leq 2R$, consistent with the placement condition.

The largest L for which the block is stable is therefore $L = 2R$. At this value, the corners are at the widest points of the cylinder ($x = \pm R$, $y = R$), and the block is supported, with local stability maintained under small disturbances.

Q30

(6.36) Two masses m_A and m_B are connected by a string of length l and lie on a frictionless table. The system is twirled and released with m_A instantaneously at rest and m_B moving with instantaneous velocity v_0 at right angles to the line of centers, as shown below. Find the subsequent motion of the system and the tension in the string.



Solution:

The system consists of two masses, m_A and m_B, connected by a string of fixed length l, lying on a frictionless table. Initially, at $t = 0$, mass m_A is at rest at position $(0, 0)$, and mass m_B is at position $(l, 0)$ with velocity $(0, v_0)$ perpendicular to the line of centers.

Center of Mass Motion

The total mass is $M = m_A + m_B$. The initial velocity of the center of mass (CM) is found from the velocities of the masses:

$$\vec{v}_{\text{cm}} = \frac{m_A\vec{v}_A + m_B\vec{v}_B}{M} = \frac{m_A(0,0) + m_B(0,v_0)}{M} = \left(0, \frac{m_B v_0}{M}\right).$$

Since there are no external forces, the CM moves with constant velocity $\vec{v}_{\text{cm}} = (0, v_{\text{cm}})$, where $v_{\text{cm}} = \frac{m_B v_0}{M}$.

The initial position of the CM is:

$$\vec{r}_{\text{cm}}(0) = \frac{m_A(0,0) + m_B(l,0)}{M} = \left(\frac{m_B l}{M}, 0\right).$$

Thus, the position of the CM at time t is:

$$\vec{r}_{\text{cm}}(t) = \left(\frac{m_B l}{M}, v_{\text{cm}} t\right) = \left(\frac{m_B l}{M}, \frac{m_B v_0}{M} t\right).$$

Motion Relative to Center of Mass

In the CM frame, the masses rotate about the CM with constant angular velocity. The distances from the CM to each mass are constant:

$$r_A = \frac{m_B l}{M}, \quad r_B = \frac{m_A l}{M}.$$

The initial angular momentum about the CM is conserved. At $t = 0$, the velocity of m_A relative to CM is:

$$\vec{v}_{A,\text{rel}} = (0,0) - \left(0, \frac{m_B v_0}{M}\right) = \left(0, -\frac{m_B v_0}{M}\right),$$

and the velocity of m_B relative to CM is:

$$\vec{v}_{B,\text{rel}} = (0, v_0) - \left(0, \frac{m_B v_0}{M}\right) = \left(0, \frac{m_A v_0}{M}\right).$$

The angular momentum about the CM is:

$$L = m_A |\vec{r}_A \times \vec{v}_{A,\text{rel}}| + m_B |\vec{r}_B \times \vec{v}_{B,\text{rel}}|,$$

where $\vec{r}_A = \left(-\frac{m_B l}{M}, 0\right)$ and $\vec{r}_B = \left(\frac{m_A l}{M}, 0\right)$. Thus,

$$L = m_A \left(\frac{m_B l}{M} \cdot \frac{m_B v_0}{M}\right) + m_B \left(\frac{m_A l}{M} \cdot \frac{m_A v_0}{M}\right) = \frac{m_A m_B^2 l v_0}{M^2} + \frac{m_A^2 m_B l v_0}{M^2} = \frac{m_A m_B l v_0}{M}.$$

The moment of inertia about the CM is:

$$I_{\text{cm}} = m_A r_A^2 + m_B r_B^2 = m_A \left(\frac{m_B l}{M}\right)^2 + m_B \left(\frac{m_A l}{M}\right)^2 = \frac{m_A m_B^2 l^2}{M^2} + \frac{m_B m_A^2 l^2}{M^2} = \frac{m_A m_B l^2}{M}.$$

The angular velocity ω satisfies $L = I_{\text{cm}} \omega$, so:

$$\omega = \frac{L}{I_{\text{cm}}} = \frac{\frac{m_A m_B l v_0}{M}}{\frac{m_A m_B l^2}{M}} = \frac{v_0}{l}.$$

The rotation is counterclockwise. At $t = 0$, mass m_A is at angle π and mass m_B at angle 0 relative to the positive x-axis in the CM frame. Thus, the angular positions at time t are:

$$\theta_A(t) = \pi + \omega t, \quad \theta_B(t) = \omega t.$$

The positions relative to CM are:

- For m_A: $\vec{r}'_A(t) = (-r_A \cos(\omega t), -r_A \sin(\omega t)) = \left(-\frac{m_B l}{M}\cos(\omega t), -\frac{m_B l}{M}\sin(\omega t)\right)$
- For m_B: $\vec{r}'_B(t) = (r_B \cos(\omega t), r_B \sin(\omega t)) = \left(\frac{m_A l}{M}\cos(\omega t), \frac{m_A l}{M}\sin(\omega t)\right)$

Positions in Lab Frame
The position in the lab frame is $\vec{r} = \vec{r}_{\text{cm}} + \vec{r}'$.

For mass m_A:

$$x_A(t) = \frac{m_B l}{M} - \frac{m_B l}{M}\cos(\omega t) = \frac{m_B l}{M}(1 - \cos(\omega t))$$

$$y_A(t) = \frac{m_B v_0}{M}t - \frac{m_B l}{M}\sin(\omega t) = \frac{m_B}{M}(v_0 t - l\sin(\omega t))$$

For mass m_B:

$$x_B(t) = \frac{m_B l}{M} + \frac{m_A l}{M}\cos(\omega t) = \frac{l}{M}(m_B + m_A \cos(\omega t))$$

$$y_B(t) = \frac{m_B v_0}{M}t + \frac{m_A l}{M}\sin(\omega t) = \frac{1}{M}(m_B v_0 t + m_A l \sin(\omega t))$$

with $\omega = v_0/l$.

Tension in the String
The tension provides the centripetal force for circular motion about the CM. For mass m_A:

$$T = m_A\omega^2 r_A = m_A\left(\frac{v_0}{l}\right)^2 \frac{m_B l}{M} = \frac{m_A m_B v_0^2}{lM}.$$

Similarly, for mass m_B:

$$T = m_B\omega^2 r_B = m_B\left(\frac{v_0}{l}\right)^2 \frac{m_A l}{M} = \frac{m_A m_B v_0^2}{lM}.$$

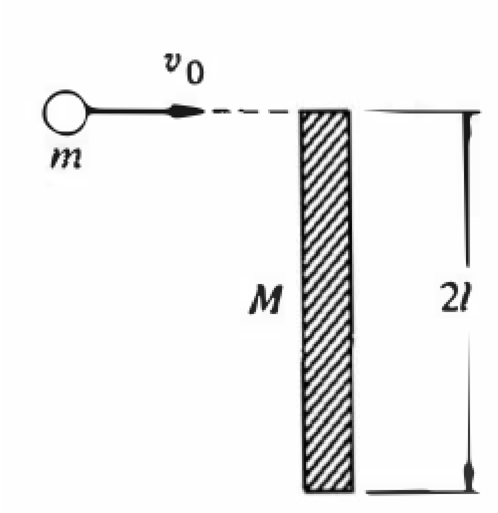
Thus, the tension is constant and given by:

$$T = \frac{m_A m_B v_0^2}{l(m_A + m_B)}.$$

Q31

(6.37)

- A plank of length $2l$ and mass M lies on a frictionless plane. A ball of mass m and speed v_0 strikes its end as shown. Find the final velocity of the ball, v_f, assuming that mechanical energy is conserved and that v_f is along the original line of motion.
- Find v_f assuming that the stick is pivoted at the lower end.



Solution:

The plank of length $2l$ and mass M is initially at rest on a frictionless plane. A ball of mass m and initial speed v_0 strikes one end of the plank perpendicularly to its length. After the elastic collision, the final velocity of the ball, v_f, is along the original line of motion (perpendicular to the plank). Mechanical energy is conserved, and the collision is elastic.

Conservation Laws:

1. **Conservation of Linear Momentum (y-direction):**
   Initial momentum: mv_0 (plank at rest).
   Final momentum: $mv_f + MV$, where V is the velocity of the plank's center of mass (COM) in the y-direction.

$$mv_0 = mv_f + MV \tag{1}$$

2. **Conservation of Energy:**
   Initial kinetic energy: $\frac{1}{2}mv_0^2$.
   Final kinetic energy:

   - Ball: $\frac{1}{2}mv_f^2$
   - Plank (translational): $\frac{1}{2}MV^2$

- Plank (rotational): $\frac{1}{2}I\omega^2$, where I is the moment of inertia about the COM and ω is the angular velocity.

For a uniform plank (rod) of length $2l$ and mass M, the moment of inertia about the COM is:

$$I = \frac{1}{12}M(2l)^2 = \frac{1}{3}Ml^2$$

Rotational kinetic energy: $\frac{1}{2}\left(\frac{1}{3}Ml^2\right)\omega^2 = \frac{1}{6}Ml^2\omega^2$.

Energy conservation:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}MV^2 + \frac{1}{6}Ml^2\omega^2 \tag{2}$$

3. **Conservation of Angular Momentum (about the point of impact, $(l, 0)$):**
   Initial angular momentum: 0 (ball at point of impact with velocity perpendicular, plank at rest).
   Final angular momentum:
   - Angular momentum of the plank about the point of impact: $I\omega - MlV$, where $I\omega$ is the angular momentum about the COM and $-MlV$ is the contribution from the linear motion.
   - Ball: at the point of impact, so angular momentum is 0.

   $$I\omega - MlV = 0 \tag{3}$$

   Substituting $I = \frac{1}{3}Ml^2$:

   $$\frac{1}{3}Ml^2\omega - MlV = 0$$

   Simplifying (assuming $M \neq 0$):

   $$\frac{1}{3}l\omega - V = 0 \implies V = \frac{1}{3}l\omega \tag{3}$$

Solving the Equations:
From equation (3):

$$V = \frac{1}{3} l\omega \tag{4}$$

Substitute equation (4) into equation (1):

$$mv_0 = mv_f + M\left(\frac{1}{3} l\omega\right) \implies m(v_0 - v_f) = \frac{Ml\omega}{3} \implies \omega = \frac{3m(v_0 - v_f)}{Ml} \tag{5}$$

Substitute equations (4) and (5) into the energy equation (2). First, multiply equation (2) by 2 to simplify:

$$mv_0^2 = mv_f^2 + MV^2 + \frac{1}{3} Ml^2\omega^2$$

Substitute $V = \frac{1}{3} l\omega$:

$$V^2 = \left(\frac{1}{3} l\omega\right)^2 = \frac{1}{9} l^2\omega^2$$

Now substitute into the energy equation:

$$mv_0^2 = mv_f^2 + M\left(\frac{1}{9} l^2\omega^2\right) + \frac{1}{3} Ml^2\omega^2 = mv_f^2 + Ml^2\omega^2\left(\frac{1}{9} + \frac{1}{3}\right) = mv_f^2 + Ml^2\omega^2\left(\frac{1}{9} + \frac{3}{9}\right) = mv_f^2 + \frac{4}{9} Ml^2\omega^2 \tag{6}$$

Substitute ω from equation (5):

$$\omega = \frac{3m(v_0 - v_f)}{Ml} \implies \omega^2 = \left(\frac{3m(v_0 - v_f)}{Ml}\right)^2 = \frac{9m^2(v_0 - v_f)^2}{M^2 l^2}$$

Now plug into equation (6):

$$mv_0^2 = mv_f^2 + \frac{4}{9} Ml^2\left(\frac{9m^2(v_0 - v_f)^2}{M^2 l^2}\right) = mv_f^2 + \frac{4}{9} \cdot \frac{9m^2(v_0 - v_f)^2}{M} = mv_f^2 + \frac{4m^2(v_0 - v_f)^2}{M}$$

Multiply both sides by M to eliminate the denominator:

$$Mmv_0^2 = Mmv_f^2 + 4m^2(v_0 - v_f)^2$$

Divide both sides by m (assuming $m \neq 0$):

$$Mv_0^2 = Mv_f^2 + 4m(v_0 - v_f)^2 \tag{7}$$

Expand the right-hand side:

$$4m(v_0 - v_f)^2 = 4m(v_0^2 - 2v_0v_f + v_f^2) = 4mv_0^2 - 8mv_0v_f + 4mv_f^2$$

Substitute into equation (7):

$$Mv_0^2 = Mv_f^2 + 4mv_0^2 - 8mv_0v_f + 4mv_f^2$$

Bring all terms to the left:

$$Mv_0^2 - Mv_f^2 - 4mv_0^2 + 8mv_0v_f - 4mv_f^2 = 0$$

Group like terms:

$$(M - 4m)v_0^2 + 8mv_0v_f - (M + 4m)v_f^2 = 0$$

This is a quadratic equation in v_f. Let $u = v_f$:

$$-(M + 4m)u^2 + 8mv_0u + (M - 4m)v_0^2 = 0$$

Multiply by -1:

$$(M + 4m)u^2 - 8mv_0u - (M - 4m)v_0^2 = 0$$

Solve using the quadratic formula:

$$u = \frac{8mv_0 \pm \sqrt{(-8mv_0)^2 - 4(M + 4m)[-(M - 4m)v_0^2]}}{2(M + 4m)}$$

Simplify the discriminant:

$$(-8mv_0)^2 = 64m^2v_0^2$$

$$-4(M+4m)[-(M-4m)v_0^2] = 4(M+4m)(M-4m)v_0^2 = 4(M^2-16m^2)v_0^2$$

$$\text{Discriminant} = 64m^2v_0^2 + 4(M^2-16m^2)v_0^2 = 64m^2v_0^2 + 4M^2v_0^2 - 64m^2v_0^2 = 4M^2v_0^2$$

$$\sqrt{\text{Discriminant}} = \sqrt{4M^2v_0^2} = 2Mv_0 \quad (\text{since } v_0 > 0)$$

Thus:

$$u = \frac{8mv_0 \pm 2Mv_0}{2(M+4m)} = \frac{4mv_0 \pm Mv_0}{M+4m} = v_0\frac{4m \pm M}{4m+M}$$

The solutions are:

$$u = v_0\frac{4m+M}{4m+M} = v_0 \quad \text{and} \quad u = v_0\frac{4m-M}{4m+M}$$

The solution $u = v_0$ corresponds to no collision (ball passes through without interacting), which is unphysical. Thus, the valid solution is:

$$v_f = v_0\frac{4m-M}{4m+M}$$

For part two, mechanical energy is conserved, and the final velocity v_f of the ball is along the original line of motion. Since the plank is pivoted, it can only rotate about the fixed pivot, and linear momentum is not conserved due to the pivot force. However, angular momentum about the pivot and mechanical energy are conserved.

The moment of inertia of the plank about the pivot is $I = \frac{1}{3}M(2l)^2 = \frac{4}{3}Ml^2$.

Assume the ball approaches with velocity v_0 in the negative y-direction (towards the plank). The initial angular momentum about the pivot is:

$$L_{\text{initial}} = -2lmv_0 \quad (\text{negative due to direction}).$$

After the collision, let the ball have final velocity v_f (the y-component) and the plank have angular velocity ω. The final angular momentum is:

$$L_{\text{final}} = 2lmv_f + I\omega = 2lmv_f + \frac{4}{3}Ml^2\omega.$$

Conservation of angular momentum gives:

$$-2lmv_0 = 2lmv_f + \frac{4}{3}Ml^2\omega.$$

Dividing by l:

$$-2mv_0 = 2mv_f + \frac{4}{3}Ml\omega.$$

Multiplying by 3:

$$-6mv_0 = 6mv_f + 4Ml\omega.$$

Dividing by 2:

$$-3mv_0 = 3mv_f + 2Ml\omega. \quad (1)$$

Conservation of Energy
Initial kinetic energy (plank at rest):

$$KE_{\text{initial}} = \frac{1}{2}mv_0^2.$$

Final kinetic energy (ball and rotating plank):

$$KE_{\text{final}} = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}\left(\frac{4}{3}Ml^2\right)\omega^2 = \frac{1}{2}mv_f^2 + \frac{2}{3}Ml^2\omega^2.$$

Conservation of energy gives:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 + \frac{2}{3}Ml^2\omega^2.$$

Multiplying by 6:

$$3mv_0^2 = 3mv_f^2 + 4Ml^2\omega^2. \quad (2)$$

Solving the Equations

From equation (1):

$$2Ml\omega = -3mv_0 - 3mv_f,$$

$$\omega = \frac{-3m(v_0 + v_f)}{2Ml}. \quad (3)$$

Substitute equation (3) into equation (2):

$$\omega^2 = \left(\frac{-3m(v_0 + v_f)}{2Ml}\right)^2 = \frac{9m^2(v_0 + v_f)^2}{4M^2l^2}.$$

So equation (2) becomes:

$$3mv_0^2 = 3mv_f^2 + 4Ml^2 \cdot \frac{9m^2(v_0 + v_f)^2}{4M^2l^2} = 3mv_f^2 + \frac{9m^2(v_0 + v_f)^2}{M}.$$

Divide by m ($m \neq 0$):

$$3v_0^2 = 3v_f^2 + \frac{9m(v_0 + v_f)^2}{M}.$$

Multiply by M:

$$3Mv_0^2 = 3Mv_f^2 + 9m(v_0 + v_f)^2.$$

Divide by 3:

$$Mv_0^2 = Mv_f^2 + 3m(v_0 + v_f)^2.$$

Expand:

$$Mv_0^2 = Mv_f^2 + 3m(v_0^2 + 2v_0v_f + v_f^2) = Mv_f^2 + 3mv_0^2 + 6mv_0v_f + 3mv_f^2,$$

$$Mv_0^2 - Mv_f^2 - 3mv_0^2 - 6mv_0v_f - 3mv_f^2 = 0,$$

$$(M - 3m)v_0^2 - (M + 3m)v_f^2 - 6mv_0v_f = 0.$$

This is a quadratic in v_f:

$$(M + 3m)v_f^2 + 6mv_0v_f - (M - 3m)v_0^2 = 0. \quad (4)$$

Solve using the quadratic formula:

$$v_f = \frac{-6mv_0 \pm \sqrt{(6mv_0)^2 - 4(M + 3m)[-(M - 3m)v_0^2]}}{2(M + 3m)}.$$

The discriminant is:

$$D = 36m^2v_0^2 + 4(M + 3m)(M - 3m)v_0^2 = 36m^2v_0^2 + 4(M^2 - 9m^2)v_0^2$$
$$= 36m^2v_0^2 + 4M^2v_0^2 - 36m^2v_0^2 = 4M^2v_0^2,$$
$$\sqrt{D} = 2Mv_0 \quad (v_0 > 0).$$

Thus:

$$v_f = \frac{-6mv_0 \pm 2Mv_0}{2(M + 3m)} = v_0\frac{-3m \pm M}{M + 3m}.$$

The solutions are:

$$v_f = v_0\frac{M - 3m}{M + 3m}, \quad v_f = v_0\frac{-3m - M}{M + 3m} = -v_0.$$

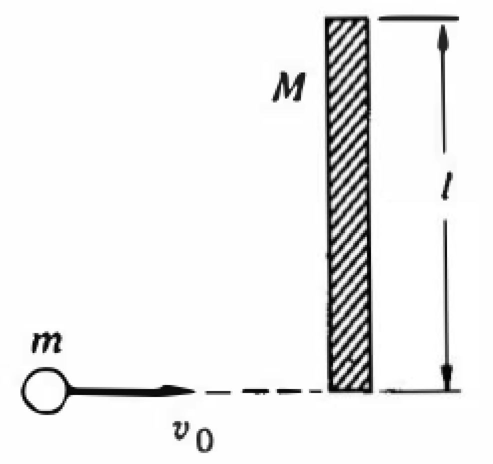
The solution $v_f = -v_0$ corresponds to no collision (ball passes through without interacting), which is unphysical. The physical solution is:

$$v_f = v_0\frac{M - 3m}{M + 3m}.$$

Q32

(6.39) A boy of mass m runs on ice with velocity v_0 and steps on the end of a plank of length l and mass M which is perpendicular to his path.

- Describe quantitatively the motion of the system after the boy is on the plank. Neglect friction with the ice.
- One point on the plank is at rest immediately after the collision. Where is it?



Solution:

Motion of the System After the Boy Steps on the Plank
The boy of mass m runs with initial velocity v_0 in the x-direction and steps onto one end of a plank of mass M and length l, initially at rest and perpendicular to his path. Friction with the ice is neglected, so no external forces act on the system. The motion consists of translation of the center of mass (CM) and rotation about the CM.

1. **Center of Mass (CM) Velocity**:
   Conservation of linear momentum in the x-direction gives:

$$(M+m)V_{\text{cm}} = mv_0$$

$$V_{\text{cm}} = \frac{m}{M+m}v_0$$

   The CM moves with constant velocity V_{cm} in the x-direction.

2. **Position of the CM**:
   The plank's CM is initially at $(0,0)$, and the boy steps onto the end at $(0,l/2)$. The y-coordinate of the system's CM is:

$$y_{\text{cm}} = \frac{M \cdot 0 + m \cdot (l/2)}{M+m} = \frac{ml}{2(M+m)}$$

   The CM is at $(0, y_{\text{cm}})$.
3. **Angular Momentum About the CM**:
   The initial angular momentum about the CM is conserved (no external torques). The boy's position relative to the CM is:

$$\vec{r}_{\text{boy}} = \left(0, \frac{l}{2} - y_{\text{cm}}\right) = \left(0, \frac{Ml}{2(M+m)}\right)$$

The initial angular momentum (in the z-direction) is:

$$L_z = -\left(\frac{Ml}{2(M+m)}\right)(mv_0) = -\frac{mMlv_0}{2(M+m)}$$

The negative sign indicates clockwise rotation.

4. **Moment of Inertia About the CM**:
   - **Plank's contribution**: The moment of inertia about its own CM is $\frac{1}{12}Ml^2$. The distance from the plank's CM to the system's CM is $y_{\text{cm}} = \frac{ml}{2(M+m)}$. Using the parallel axis theorem:

$$I_{\text{plank}} = \frac{1}{12}Ml^2 + M\left(\frac{ml}{2(M+m)}\right)^2 = \frac{1}{12}Ml^2 + \frac{Mm^2l^2}{4(M+m)^2}$$

   - **Boy's contribution**: The boy is a point mass at a distance $\frac{Ml}{2(M+m)}$ from the system's CM:

$$I_{\text{boy}} = m\left(\frac{Ml}{2(M+m)}\right)^2 = \frac{mM^2l^2}{4(M+m)^2}$$

   - **Total moment of inertia**:

$$I_{\text{tot}} = I_{\text{plank}} + I_{\text{boy}} = \frac{1}{12}Ml^2 + \frac{Mm^2l^2}{4(M+m)^2} + \frac{mM^2l^2}{4(M+m)^2}$$

Combining terms:

$$I_{\text{tot}} = \frac{1}{12}Ml^2 + \frac{mMl^2}{4(M+m)} = \frac{M(M+4m)l^2}{12(M+m)}$$

5. **Angular Velocity**:
   The angular velocity ω about the CM is:

$$\omega = \frac{L_z}{I_{\text{tot}}} = \frac{-\frac{mMlv_0}{2(M+m)}}{\frac{M(M+4m)l^2}{12(M+m)}} = -\frac{6mv_0}{l(M+4m)}$$

   The system rotates clockwise with constant angular velocity ω.

Point on the Plank at Rest Immediately After Collision

The velocity of any point on the plank is the sum of the CM velocity and the rotational velocity about the CM. For a point at initial position $(0, y)$ (relative to the plank's center), its position relative to the system's CM is $(0, y - y_{\text{cm}})$. The x-component of velocity is:

$$v_x = V_{\text{cm}} - \omega(y - y_{\text{cm}})$$

The point is at rest if $v_x = 0$:

$$V_{\text{cm}} - \omega(y - y_{\text{cm}}) = 0$$

Solving for y:

$$y - y_{\text{cm}} = \frac{V_{\text{cm}}}{\omega} = \frac{\frac{m}{M+m}v_0}{-\frac{6mv_0}{l(M+4m)}} = -\frac{l(M+4m)}{6(M+m)}$$

The absolute y-coordinate is:

$$y = y_{\text{cm}} + \left(-\frac{l(M+4m)}{6(M+m)}\right) = \frac{ml}{2(M+m)} - \frac{l(M+4m)}{6(M+m)}$$

Simplifying:

$$y = \frac{3ml - l(M+4m)}{6(M+m)} = \frac{-Ml - ml}{6(M+m)} = -\frac{l(M+m)}{6(M+m)} = -\frac{l}{6}$$

This point is at $y = -l/6$ relative to the plank's center. The boy stepped onto the end at $y = l/2$. The distance from this end to the point at rest is:

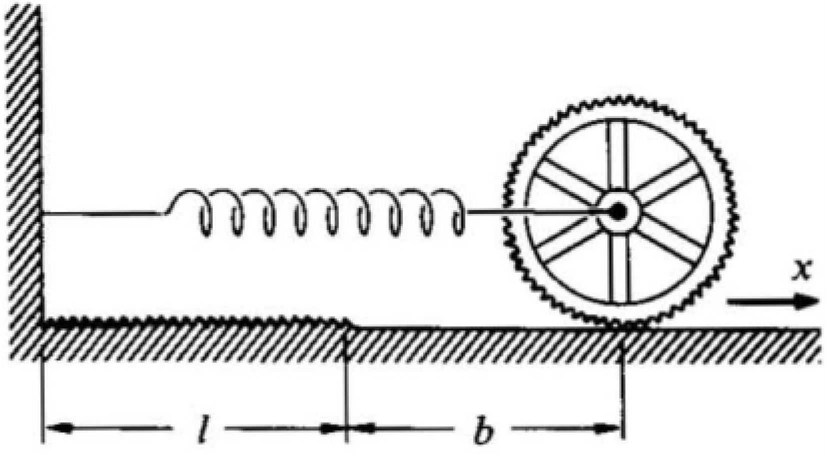
$$\left|\frac{l}{2} - \left(-\frac{l}{6}\right)\right| = \frac{l}{2} + \frac{l}{6} = \frac{2l}{3}$$

Thus, the point at rest is at a distance $\frac{2l}{3}$ from the end where the boy stepped, along the plank toward the other end.

Q33

(6.40) A wheel with fine teeth is attached to the end of a spring with constant k and unstretched length l. For $x > l$, the wheel slips freely on the surface, but for $x < l$ the teeth mesh with the teeth on the ground so that it cannot slip. Assume that all the mass of the wheel is in its rim.

- The wheel is pulled to $x = l + b$ and released. How close will it come to the wall on its first trip?
- How far out will it go as it leaves the wall?
- What happens when the wheel next hits the gear track?



Solution

At $x = l$, which occurs when $\cos(\omega t) = 0$ (i.e., $\omega t = \pi/2$), the velocity is $\frac{dx}{dt} = -b\omega = -b\sqrt{\frac{k}{m}}$, and the angular velocity is zero since there was no rotation during sliding.

At $x = l$, the teeth mesh, enforcing rolling without slipping. Due to the sudden engagement, there is an impulsive force that adjusts the velocities to satisfy the no-slip condition while conserving angular momentum about the contact point. Just before meshing, the angular momentum about the contact point is $mv_i r$ clockwise, where $v_i = -b\sqrt{\frac{k}{m}}$. Just after meshing, the wheel rotates about the contact point with moment of inertia $I_P = 2mr^2$ (since $I_{\text{cm}} = mr^2$ for mass in the rim, and parallel axis theorem applies). Conservation of angular momentum gives:

$$mv_i r = 2mr^2\Omega \implies \Omega = \frac{v_i}{2r}$$

where Ω is the angular velocity about the contact point. The center of mass velocity after meshing is:

$$v_f = \Omega r = \frac{v_i}{2} = -\frac{1}{2}b\sqrt{\frac{k}{m}}$$

and the angular velocity about the center is $\omega_f = \Omega = \frac{v_i}{2r} = -\frac{1}{2}b\frac{\sqrt{k/m}}{r}$.

For $x < l$, the wheel rolls without slipping. The spring force is $F_{\text{spring}} = -k(x-l) = k(l-x)$ (directed away from the wall). The equation of motion for rolling without slipping is derived from Newton's second law and the no-slip constraint $a_{\text{cm}} = -\alpha r$, where $a_{\text{cm}} = \frac{d^2x}{dt^2}$ and α is angular acceleration. Solving yields:

$$2m\frac{d^2x}{dt^2} = -k(x-l)$$

for $x < l$. Defining $y = x - l$, the equation becomes:

$$\frac{d^2y}{dt^2} = -\frac{k}{2m}y$$

which is simple harmonic motion with angular frequency $\omega_{\text{roll}} = \sqrt{\frac{k}{2m}}$, centered at $y = 0$ (i.e., $x = l$).

At the meshing point ($y = 0$), the initial velocity for this motion is $\frac{dy}{dt} = v_f = -\frac{1}{2}b\sqrt{\frac{k}{m}}$. The general solution is $y(t) = A\sin(\omega_{\text{roll}}t) + B\cos(\omega_{\text{roll}}t)$. Applying initial conditions:

- At $t = 0$, $y = 0$: $0 = B$, so $y(t) = A\sin(\omega_{\text{roll}}t)$.
- At $t = 0$, $\frac{dy}{dt} = v_f$: $A\omega_{\text{roll}} = v_f$, so $A = \frac{v_f}{\omega_{\text{roll}}} = \frac{-\frac{1}{2}b\sqrt{\frac{k}{m}}}{\sqrt{\frac{k}{2m}}} = -\frac{1}{2}b\sqrt{2} = -\frac{b}{\sqrt{2}}$.

Thus, $y(t) = -\frac{b}{\sqrt{2}}\sin(\omega_{\text{roll}}t)$. The minimum y occurs when $\sin(\omega_{\text{roll}}t) = 1$, so $y_{\min} = -\frac{b}{\sqrt{2}}$. Therefore, the minimum x is:

$$x_{\min} = l + y_{\min} = l - \frac{b}{\sqrt{2}}$$

This is the closest point to the wall on the first trip, as it occurs during the initial motion toward the wall after meshing.

The wheel is released from $x = l + b$ with initial velocity zero. For $x > l$, the wheel slips freely, and the motion is simple harmonic with angular frequency $\omega = \sqrt{\frac{k}{m}}$. The position is given by $x(t) = l + b\cos(\omega t)$, and the velocity by $\dot{x}(t) = -b\omega\sin(\omega t)$.

At $x = l$, which occurs when $\cos(\omega t) = 0$ (i.e., $\omega t = \pi/2$), the velocity is $\dot{x} = -b\omega = -b\sqrt{\frac{k}{m}}$. At this point, the teeth mesh, and due to the impulsive force, the no-slip condition is enforced. Conservation of angular momentum about the contact point gives:

$$mv_ir = I_P\Omega$$

where $v_i = b\sqrt{\frac{k}{m}}$, $I_P = 2mr^2$ (using the parallel axis theorem, since $I_{\text{cm}} = mr^2$), and Ω is the angular velocity about the contact point. Solving for Ω:

$$m\left(b\sqrt{\frac{k}{m}}\right)r = 2mr^2\Omega \implies \Omega = \frac{b\sqrt{\frac{k}{m}}}{2r}$$

The center of mass velocity after meshing is:

$$v_f = \Omega r = \frac{b\sqrt{\frac{k}{m}}}{2}$$

For $x < l$, the wheel rolls without slipping. The equation of motion is derived from Newton's second law and the no-slip constraint:

$$m\ddot{x} = -k(x-l), \quad I_{\text{cm}}\alpha = -rk(x-l)$$

with $\alpha = -\frac{\ddot{x}}{r}$ (since $a_{\text{cm}} = \alpha r$ and $\ddot{x} < 0$ for motion toward the wall). Substituting $I_{\text{cm}} = mr^2$:

$$m\ddot{x} = -k(x-l), \quad mr^2\left(-\frac{\ddot{x}}{r}\right) = -rk(x-l) \implies -mr\ddot{x} = -rk(x-l)$$

Combining equations:

$$m\ddot{x} + k(x-l) = 0, \quad -mr\ddot{x} + rk(x-l) = 0$$

Adding these:

$$2m\ddot{x} = -2k(x-l) \implies \ddot{x} = -\frac{k}{2m}(x-l)$$

With $y = x - l$, the equation is $\ddot{y} = -\frac{k}{2m}y$, which is simple harmonic with angular frequency $\omega_{\text{roll}} = \sqrt{\frac{k}{2m}}$. The solution is $y(t) = A\sin(\omega_{\text{roll}}t) + B\cos(\omega_{\text{roll}}t)$. Initial conditions at $t = 0$ (when $y = 0$):

- $y(0) = 0 \implies B = 0$, so $y(t) = A\sin(\omega_{\text{roll}}t)$.

- $\dot{y}(0) = v_f = -\frac{b}{2}\sqrt{\frac{k}{m}}$ (negative as motion is toward the wall), so:

$$\dot{y}(0) = A\omega_{\text{roll}} = -\frac{b}{2}\sqrt{\frac{k}{m}} \implies A = \frac{-\frac{b}{2}\sqrt{\frac{k}{m}}}{\sqrt{\frac{k}{2m}}} = -\frac{b}{2}\sqrt{2} = -\frac{b}{\sqrt{2}}$$

Thus, $y(t) = -\frac{b}{\sqrt{2}}\sin(\omega_{\text{roll}}t)$. The minimum y occurs when $\sin(\omega_{\text{roll}}t) = 1$, so $y_{\text{min}} = -\frac{b}{\sqrt{2}}$ and $x_{\text{min}} = l - \frac{b}{\sqrt{2}}$.

The wheel then moves back toward $x = l$. At $y = 0$ (i.e., $x = l$), which occurs when $\omega_{\text{roll}}t = \pi$:

$$\dot{y} = -\frac{b}{\sqrt{2}}\omega_{\text{roll}}\cos(\omega_{\text{roll}}t) = -\frac{b}{\sqrt{2}}\sqrt{\frac{k}{2m}}\cos(\pi) = -\frac{b}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\sqrt{\frac{k}{m}} \cdot (-1) = \frac{b}{2}\sqrt{\frac{k}{m}}$$

So the velocity as it leaves the wall (crosses $x = l$ from below) is $\dot{x} = \frac{b}{2}\sqrt{\frac{k}{m}}$ (positive, moving away from the wall).

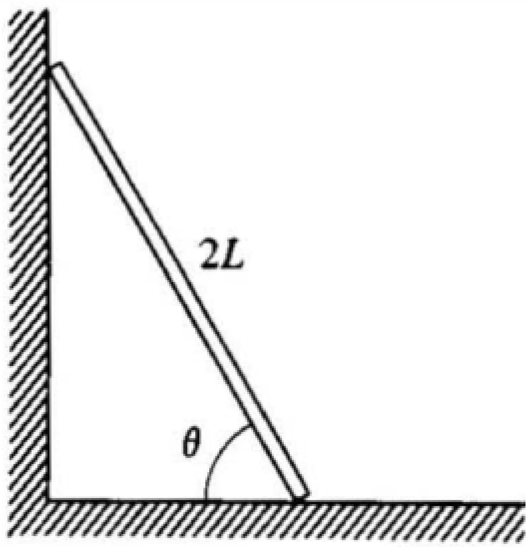
For $x > l$, the teeth disengage, and the wheel slips freely. The center of mass motion is simple harmonic with $\omega_{\text{slip}} = \sqrt{\frac{k}{m}}$, independent of rotation (since no friction). At $x = l$, the velocity is $\dot{x} = \frac{b}{2}\sqrt{\frac{k}{m}} = \frac{b}{2}\omega_{\text{slip}}$. The solution is $y(t) = A\sin(\omega_{\text{slip}}t)$ (with $y = x - l$), and:

$$\dot{y}(0) = A\omega_{\text{slip}} = \frac{b}{2}\omega_{\text{slip}} \implies A = \frac{b}{2}$$

Thus, $y(t) = \frac{b}{2}\sin(\omega_{\text{slip}}t)$, and the maximum y is $\frac{b}{2}$. The maximum distance from the wall is $x_{\text{max}} = l + \frac{b}{2}$.

Q34

(6.41) A plank of length $2L$ leans against a wall. It starts to slip downward without friction. Show that the top of the plank loses contact with the wall when it is at two-thirds of its initial height. *Hint: Only a single variable is needed to describe the system. Note the motion of the center of mass.*



Solution:

To determine when the top of the plank loses contact with the wall, consider the plank of length $2L$ leaning against a wall with no friction. The system is described using the angle θ between the plank and the vertical, where $\theta = 0$ corresponds to the plank being vertical. The initial angle is θ_i, so the initial height of the top is $y_i = 2L\cos\theta_i$.

The center of mass (CM) of the plank, being uniform, is at the midpoint. The coordinates of the CM are:

$$x_{\text{cm}} = L\sin\theta, \quad y_{\text{cm}} = L\cos\theta.$$

The forces acting on the plank are:

- Gravity mg downward at the CM.
- Normal force from the wall N_w horizontal (in the $+x$ direction).
- Normal force from the floor N_f vertical (in the $+y$ direction).

The plank starts from rest, and energy is conserved since there is no friction. The initial potential energy, with the floor as the reference ($y = 0$), is:

$$U_i = mgy_{\text{cm},i} = mgL\cos\theta_i.$$

The initial kinetic energy is zero. At angle θ, the potential energy is:

$$U = mgL\cos\theta.$$

The kinetic energy includes both translational and rotational components. The moment of inertia about the CM for a thin rod of length $2L$ is:

$$I = \frac{1}{12}m(2L)^2 = \frac{1}{3}mL^2.$$

The velocity of the CM is found from:

$$v_{\mathrm{cm},x} = \frac{dx_{\mathrm{cm}}}{dt} = L\cos\theta\,\omega, \quad v_{\mathrm{cm},y} = \frac{dy_{\mathrm{cm}}}{dt} = -L\sin\theta\,\omega,$$

where $\omega = d\theta/dt$. Thus,

$$v_{\mathrm{cm}}^2 = v_{\mathrm{cm},x}^2 + v_{\mathrm{cm},y}^2 = (L\cos\theta\,\omega)^2 + (-L\sin\theta\,\omega)^2 = L^2\omega^2(\cos^2\theta + \sin^2\theta) = L^2\omega^2.$$

The kinetic energy is:

$$K = \frac{1}{2}mv_{\mathrm{cm}}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m(L^2\omega^2) + \frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega^2 = \frac{1}{2}mL^2\omega^2 + \frac{1}{6}mL^2\omega^2 = \frac{2}{3}mL^2\omega^2.$$

Energy conservation gives:

$$mgL\cos\theta_i = mgL\cos\theta + \frac{2}{3}mL^2\omega^2.$$

Dividing by mL:

$$g\cos\theta_i = g\cos\theta + \frac{2}{3}L\omega^2,$$

so

$$L\omega^2 = \frac{3}{2}g(\cos\theta_i - \cos\theta). \quad (1)$$

The top loses contact with the wall when the normal force $N_w = 0$. From Newton's second law in the horizontal direction:

$$ma_{\mathrm{cm},x} = N_w.$$

The acceleration of the CM in the x-direction is:

$$a_{\mathrm{cm},x} = \frac{d^2 x_{\mathrm{cm}}}{dt^2} = \frac{d}{dt}(L\cos\theta\,\omega) = L\left(-\sin\theta\,\omega^2 + \cos\theta\,\alpha\right),$$

where $\alpha = d\omega/dt$. Setting $N_w = 0$ implies $a_{\mathrm{cm},x} = 0$:

$$L\left(-\sin\theta\,\omega^2 + \cos\theta\,\alpha\right) = 0,$$

so

$$-\sin\theta\,\omega^2 + \cos\theta\,\alpha = 0,$$

$$\alpha = \tan\theta\,\omega^2. \quad (2)$$

The angular acceleration α can be found from the energy equation. Differentiating equation (1) with respect to θ:

$$\frac{d}{d\theta}(L\omega^2) = L\cdot 2\omega\frac{d\omega}{d\theta} = \frac{d}{d\theta}\left[\frac{3}{2}g(\cos\theta_i - \cos\theta)\right] = \frac{3}{2}g\sin\theta.$$

Thus,

$$2L\omega\frac{d\omega}{d\theta} = \frac{3}{2}g\sin\theta,$$

$$\frac{d\omega}{d\theta} = \frac{3g\sin\theta}{4L\omega}.$$

Then,

$$\alpha = \frac{d\omega}{dt} = \omega\frac{d\omega}{d\theta} = \omega\cdot\frac{3g\sin\theta}{4L\omega} = \frac{3g\sin\theta}{4L}. \quad (3)$$

Setting equations (2) and (3) equal:

$$\tan\theta\,\omega^2 = \frac{3g\sin\theta}{4L}.$$

Since $\sin\theta \neq 0$ for $0 < \theta < 90^\circ$, divide both sides by $\sin\theta$:

$$\frac{1}{\cos\theta}\omega^2 = \frac{3g}{4L},$$

$$\omega^2 = \frac{3g}{4L}\cos\theta. \quad (4)$$

From energy (equation (1)):

$$L\omega^2 = \frac{3}{2}g(\cos\theta_i - \cos\theta).$$

Substitute equation (4):

$$L\left(\frac{3g}{4L}\cos\theta\right) = \frac{3}{2}g(\cos\theta_i - \cos\theta),$$

$$\frac{3g}{4}\cos\theta = \frac{3}{2}g(\cos\theta_i - \cos\theta).$$

Divide both sides by $3g/2$:

$$\frac{\frac{3g}{4}\cos\theta}{\frac{3g}{2}} = \cos\theta_i - \cos\theta,$$

$$\frac{1}{2}\cdot\frac{2}{1}\cos\theta\cdot\frac{1}{\cos\theta} \quad \text{(simplify)}:$$

$$\frac{1}{2}\cdot 2\cdot\frac{\cos\theta}{\cos\theta} \quad \text{is incorrect. Instead:}$$

$$\frac{\frac{3g}{4}\cos\theta}{\frac{3g}{2}} = \frac{3g}{4}\cos\theta\cdot\frac{2}{3g} = \frac{1}{2}\cos\theta\cdot\frac{2}{1}\cdot\frac{1}{2} \quad \text{no:}$$

$$\frac{a}{b} = a\cdot\frac{1}{b}, \quad \frac{1}{\frac{3g}{2}} = \frac{2}{3g}, \quad \text{so}$$

$$\frac{3g}{4}\cos\theta \cdot \frac{2}{3g} = \frac{1}{2}\cos\theta \cdot 1 \quad (\text{since } 3g/4 \cdot 2/3g = (3g \cdot 2)/(4 \cdot 3g) = 6g/12g = 1/2),$$

so

$$\frac{1}{2}\cos\theta = \cos\theta_i - \cos\theta.$$

Then:

$$\frac{1}{2}\cos\theta + \cos\theta = \cos\theta_i,$$

$$\frac{3}{2}\cos\theta = \cos\theta_i,$$

$$\cos\theta = \frac{2}{3}\cos\theta_i.$$

The height of the top when contact is lost is:

$$y = 2L\cos\theta = 2L \cdot \frac{2}{3}\cos\theta_i = \frac{4}{3}L\cos\theta_i.$$

The initial height is $y_i = 2L\cos\theta_i$, so:

$$y = \frac{4}{3}L\cos\theta_i = \frac{2}{3} \cdot (2L\cos\theta_i) = \frac{2}{3}y_i.$$

Thus, the top of the plank loses contact with the wall when it is at two-thirds of its initial height.
