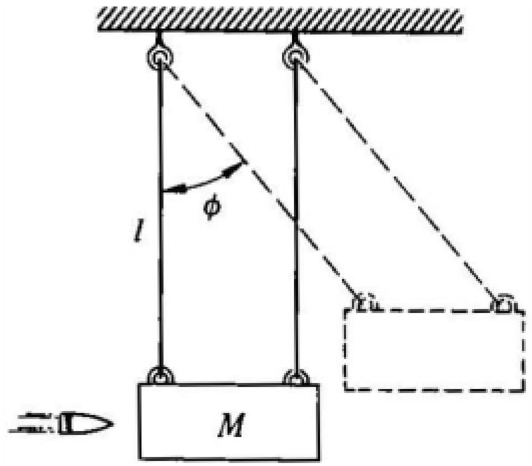
A_M_PS2

Q1

(4.3) A simple way to measure the speed of a bullet is with a ballistic pendulum. As illustrated, this consists of a wooden block of mass M into which the bullet is shot. The block is suspended from cables of length l, and the impact of the bullet causes it to swing through a maximum angle ϕ, as shown. The initial speed of the bullet is v, and its mass is m.

- How fast is the block moving immediately after the bullet comes to rest? (Assume that this happens quickly.)
- Show how to find the velocity of the bullet by measuring m, M, l, and ϕ.



Solution

The collision between the bullet and the block is perfectly inelastic, as the bullet embeds into the block and they move together. Since there are no external horizontal forces during the collision, momentum is conserved horizontally.

- Initial momentum: $m \cdot v + M \cdot 0 = mv$ (block is initially at rest).

- Final momentum: $(M + m) \cdot u$, where u is the speed of the block (with embedded bullet) immediately after the collision.

$$mv = (M + m)u$$

Solving for u:

$$u = \frac{mv}{M + m}$$

After the collision, the block (with embedded bullet) swings as a pendulum. At the maximum angle ϕ, the system momentarily comes to rest. Energy is conserved during the swing, as non-conservative forces (like air resistance) are negligible, and the tension in the cable does no work (perpendicular to motion).

- Initial kinetic energy (right after collision): $\frac{1}{2}(M + m)u^2$.
- Potential energy at maximum height: $(M + m)gh$, where h is the height gained.

The height h is determined from the geometry of the pendulum. The vertical rise from the lowest point to the maximum angle is:

$$h = l - l \cos\phi = l(1 - \cos\phi)$$

Using conservation of energy:

$$\frac{1}{2}(M + m)u^2 = (M + m)gl(1 - \cos\phi)$$

Cancel $(M + m)$ from both sides:

$$\frac{1}{2}u^2 = gl(1 - \cos\phi)$$

Thus:

$$u^2 = 2gl(1 - \cos\phi)$$

Substitute the expression for u from Step 1 into the energy equation from Step 2:

$$\left(\frac{mv}{M+m}\right)^2 = 2gl(1-\cos\phi)$$

Solve for v^2:

$$\frac{m^2v^2}{(M+m)^2} = 2gl(1-\cos\phi)$$

$$v^2 = \frac{(M+m)^2}{m^2}\cdot 2gl(1-\cos\phi)$$

Take the square root to solve for v (considering the positive root since velocity is positive):

$$v = \frac{M+m}{m}\sqrt{2gl(1-\cos\phi)}$$

Using the trigonometric identity $1-\cos\phi = 2\sin^2(\phi/2)$, simplify:

$$v = \frac{M+m}{m}\sqrt{2gl\cdot 2\sin^2(\phi/2)} = \frac{M+m}{m}\sqrt{4gl\sin^2(\phi/2)} = \frac{M+m}{m}\cdot 2\sin\left(\frac{\phi}{2}\right)\sqrt{gl}$$

$$v = \frac{2(M+m)\sin\left(\frac{\phi}{2}\right)\sqrt{gl}}{m}$$

Q2

(4.13) A commonly used potential energy function to describe the interaction between two atoms is the Lennard-Jones 6,12 potential

$$U = \varepsilon\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^6\right].$$

- Show that the radius at the potential minimum is r_0, and that the depth of the potential well is ε.
- Find the frequency of small oscillations about equilibrium for 2 identical atoms of mass m bound to each other by the Lennard-Jones interaction.

Solution

To show that the radius at the potential minimum is r_0 and the depth of the potential well is ε, consider the Lennard-Jones potential:

$$U = \varepsilon\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right].$$

The minimum occurs where the first derivative of U with respect to r is zero. Set $s = \frac{r_0}{r}$, so the potential becomes:

$$U = \varepsilon\left[s^{12} - 2s^{6}\right].$$

The first derivative with respect to r is:

$$\frac{dU}{dr} = \frac{dU}{ds} \cdot \frac{ds}{dr},$$

where:

$$\frac{dU}{ds} = \varepsilon\left[12s^{11} - 12s^{5}\right] = 12\varepsilon s^{5}(s^{6} - 1),$$

and:

$$\frac{ds}{dr} = \frac{d}{dr}\left(\frac{r_0}{r}\right) = -\frac{r_0}{r^2}.$$

Thus:

$$\frac{dU}{dr} = 12\varepsilon s^{5}(s^{6} - 1) \cdot \left(-\frac{r_0}{r^2}\right) = -12\varepsilon s^{5}(s^{6} - 1)\frac{r_0}{r^2}.$$

Setting $\frac{dU}{dr} = 0$:

$$-12\varepsilon s^{5}(s^{6} - 1)\frac{r_0}{r^2} = 0.$$

Since $\varepsilon \neq 0$, $r_0 \neq 0$, and $r \neq 0$, this simplifies to:

$$s^{5}(s^{6} - 1) = 0.$$

The solutions are $s^5 = 0$ or $s^6 - 1 = 0$. The case $s^5 = 0$ implies $s = 0$, so $r \to \infty$, which corresponds to $U = 0$ and is not a minimum. The case $s^6 - 1 = 0$ implies $s^6 = 1$, so $s = 1$ (since $s > 0$). Thus, $s = 1$ corresponds to $r = r_0$.

To confirm this is a minimum, evaluate the second derivative at $r = r_0$. First, express $\frac{dU}{dr}$ as:

$$\frac{dU}{dr} = \varepsilon \left[12s^{11} - 12s^5\right] \left(-\frac{r_0}{r^2}\right),$$

where $s = \frac{r_0}{r}$. The second derivative is:

$$\frac{d^2U}{dr^2} = \frac{d}{dr}\left(\frac{dU}{dr}\right).$$

At $r = r_0$, $s = 1$. Define:

$$f(s) = 12s^{11} - 12s^5, \quad g(r) = -\frac{r_0}{r^2}.$$

Then:

$$\frac{dU}{dr} = \varepsilon f(s) g(r),$$

and:

$$\frac{d^2U}{dr^2} = \varepsilon \left[f'(s)\frac{ds}{dr}g(r) + f(s)g'(r)\right],$$

where:

$$f'(s) = 132s^{10} - 60s^4, \quad \frac{ds}{dr} = -\frac{r_0}{r^2}, \quad g'(r) = \frac{2r_0}{r^3}.$$

At $s = 1$, $r = r_0$:

$$f(1) = 12(1)^{11} - 12(1)^5 = 0, \quad f'(1) = 132(1)^{10} - 60(1)^4 = 72,$$

$$\left.\frac{ds}{dr}\right|_{r=r_0} = -\frac{r_0}{r_0^2} = -\frac{1}{r_0}, \quad g(r_0) = -\frac{r_0}{r_0^2} = -\frac{1}{r_0}, \quad g'(r_0) = \frac{2r_0}{r_0^3} = \frac{2}{r_0^2}.$$

Thus:

$$\left.\frac{d^2U}{dr^2}\right|_{r=r_0} = \varepsilon\left[(72)\left(-\frac{1}{r_0}\right)\left(-\frac{1}{r_0}\right) + (0)\left(\frac{2}{r_0^2}\right)\right] = \varepsilon\left[72 \cdot \frac{1}{r_0^2}\right] = \frac{72\varepsilon}{r_0^2}.$$

Since $\varepsilon > 0$ and $r_0 > 0$, $\frac{d^2U}{dr^2} > 0$, confirming a minimum at $r = r_0$.

At $r = r_0$, the potential energy is:

$$U(r_0) = \varepsilon\left[\left(\frac{r_0}{r_0}\right)^{12} - 2\left(\frac{r_0}{r_0}\right)^{6}\right] = \varepsilon\,[1-2] = -\varepsilon.$$

As $r \to \infty$, $U \to 0$. The depth of the potential well is the difference from the asymptotic value to the minimum, so:

$$\mathrm{depth} = |U(\infty) - U(r_0)| = |0 - (-\varepsilon)| = \varepsilon.$$

Thus, the radius at the potential minimum is r_0, and the depth of the potential well is ε.

The Lennard-Jones potential is given by $U = \varepsilon\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right]$.

For two identical atoms of mass m, the system can be reduced to a one-body problem with reduced mass μ. Since the masses are identical, $\mu = \frac{m \cdot m}{m+m} = \frac{m}{2}$.

The frequency of small oscillations about the equilibrium position $r = r_0$ is determined by the effective spring constant k, which is the second derivative of the potential energy with respect to r evaluated at r_0:

$$k = \left.\frac{d^2U}{dr^2}\right|_{r=r_0}.$$

From the previous result, $\left.\frac{d^2U}{dr^2}\right|_{r=r_0} = \frac{72\varepsilon}{r_0^2}$.

The angular frequency ω of small oscillations is given by:

$$\omega = \sqrt{\frac{k}{\mu}}.$$

Substituting $k = \frac{72\varepsilon}{r_0^2}$ and $\mu = \frac{m}{2}$:

$$\omega = \sqrt{\frac{\frac{72\varepsilon}{r_0^2}}{\frac{m}{2}}} = \sqrt{\frac{72\varepsilon}{r_0^2} \cdot \frac{2}{m}} = \sqrt{\frac{144\varepsilon}{mr_0^2}} = 12\sqrt{\frac{\varepsilon}{mr_0^2}}.$$

Thus, the angular frequency of small oscillations about the equilibrium is $12\sqrt{\frac{\varepsilon}{mr_0^2}}$.

Q3

(4.29) A 'superball' of mass m bounces back and forth between two surfaces with speed v_0. Gravity is neglected and the collisions are perfectly elastic.

- Find the average force F on each wall.

$\textbf{Ans.} \quad F = \dfrac{mv_0^2}{l}$

- If one surface is slowly moved toward the other with speed $V \ll v_0$, the bounce rate will increase due to the shorter distance between collisions, and because the ball's speed increases when it bounces from the moving surface. Find F in terms of the separation of the surfaces, x. *Hint: Find the average rate at which the ball's speed increases as the surface moves.*

$\textbf{Ans.} \quad F = \dfrac{mv_0^2}{l}\left(\dfrac{l}{x}\right)^3$

- Show that the work needed to push the surface from l to x equals the gain in kinetic energy of the ball. (This problem illustrates the mechanism which causes a gas to heat up as it is compressed.)

Solution:

The average force on each wall is derived from the impulse per collision and the time between collisions.

Consider two parallel walls separated by a distance L. The ball moves with constant speed v_0 due to perfectly elastic collisions and no gravity. When the ball collides with a wall, it reverses direction, resulting in a change in momentum of magnitude $2mv_0$ for the ball. By Newton's third law, the impulse imparted to the wall per collision is also $2mv_0$ in magnitude.

The time between consecutive collisions with the same wall is the time for the ball to travel to the opposite wall and back, which is the round-trip time. The distance for one way is L, so the time for one way is L/v_0. The round-trip time is therefore $2L/v_0$.

The average force on a wall is the impulse per collision divided by the time between collisions:

$$F = \frac{\text{impulse per collision}}{\text{time between collisions}} = \frac{2mv_0}{2L/v_0} = \frac{2mv_0 \cdot v_0}{2L} = \frac{mv_0^2}{L}.$$

Thus, the magnitude of the average force on each wall is $\frac{mv_0^2}{L}$, where L is the distance between the two walls. The direction of the force depends on the wall: for the left wall, the force is to the left, and for the right wall, the force is to the right, but the magnitude is the same for both.

The distance L is not specified in the problem, so the expression $\frac{mv_0^2}{L}$ is the general form for the average force, with L being the separation between the surfaces.

The average force on each wall depends on the separation x between the surfaces and the initial conditions. Due to the slow movement of one surface toward the other with speed $V \ll v_0$, the ball's speed increases as the separation decreases, and the adiabatic invariant $vx = v_0 L$ is conserved, where v_0 is the initial speed and L is the initial separation.

Thus, the speed of the ball at any separation x is given by $v = \frac{v_0 L}{x}$.

The average force on each wall is derived from the impulse per collision and the time between collisions. For both walls, the magnitude of the average force is:

$$F = \frac{mv^2}{x}$$

Substituting $v = \frac{v_0 L}{x}$:

$$F = \frac{m}{x}\left(\frac{v_0 L}{x}\right)^2 = \frac{m v_0^2 L^2}{x^3}$$

This expression holds for both walls, as the adiabatic invariant ensures that the speed at collision with either wall satisfies $v_{\text{coll}} x = v_0 L$, and the force calculation yields the same result for each wall. The approximation is valid since $V \ll v_0$, and higher-order terms in V/v are negligible.

The force increases as x decreases, consistent with the increase in the ball's kinetic energy and collision frequency.

To show that the work needed to push the surface from separation l to separation x (where $x < l$) equals the gain in kinetic energy of the ball, consider the setup where one surface is fixed and the other is moved slowly toward it with speed $V \ll v_0$. The ball's speed increases due to the adiabatic invariant $vx = v_0 l$, where v_0 is the initial speed and l is the initial separation. Thus, at any separation x, the speed of the ball is $v = \frac{v_0 l}{x}$.

The average force on the moving surface (wall) due to the ball's collisions is given by $F(s) = \frac{m v_0^2 l^2}{s^3}$, where s is the current separation. The force exerted by the ball on the moving wall is in the positive x-direction (to the right, assuming the wall moves left). For the wall to move at constant velocity (since $V \ll v_0$ and the wall is assumed massless), the external agent must apply a force equal in magnitude but opposite in direction to balance the force from the ball. Thus, the force exerted by the external agent on the wall is:

$$\vec{F}_{\text{ext}} = -\frac{m v_0^2 l^2}{s^3} \hat{i},$$

where $\hat{i}$ is the unit vector in the positive x-direction.

The displacement of the wall is in the negative x-direction as it moves from $s = l$ to $s = x$. The differential displacement vector is $d\vec{s} = ds\hat{i}$, with $ds < 0$ during the motion. The work done by the external agent is the integral of $\vec{F}_{\text{ext}} \cdot d\vec{s}$:

$$W = \int_l^x \vec{F}_{\text{ext}} \cdot d\vec{s} = \int_l^x \left(-\frac{m v_0^2 l^2}{s^3}\hat{i}\right) \cdot (ds\hat{i}) = -\int_l^x \frac{m v_0^2 l^2}{s^3} ds.$$

Evaluating the integral:

$$W = -m v_0^2 l^2 \int_l^x s^{-3} ds = -m v_0^2 l^2 \left[-\frac{1}{2} s^{-2}\right]_l^x = -m v_0^2 l^2 \left(-\frac{1}{2}\right)\left(\frac{1}{x^2} - \frac{1}{l^2}\right) = \frac{1}{2} m v_0^2 l^2 \left(\frac{1}{x^2} - \frac{1}{l^2}\right).$$

Simplifying:

$$W = \frac{1}{2}mv_0^2 l^2 \cdot \frac{1}{x^2} - \frac{1}{2}mv_0^2 l^2 \cdot \frac{1}{l^2} = \frac{1}{2}mv_0^2 \frac{l^2}{x^2} - \frac{1}{2}mv_0^2.$$

The kinetic energy of the ball at separation x is:

$$K_f = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{v_0 l}{x}\right)^2 = \frac{1}{2}mv_0^2 \frac{l^2}{x^2}.$$

The initial kinetic energy at separation l is:

$$K_i = \frac{1}{2}mv_0^2.$$

The gain in kinetic energy is:

$$\Delta K = K_f - K_i = \frac{1}{2}mv_0^2 \frac{l^2}{x^2} - \frac{1}{2}mv_0^2 = \frac{1}{2}mv_0^2 \left(\frac{l^2}{x^2} - 1\right).$$

Comparing the work done and the gain in kinetic energy:

$$W = \frac{1}{2}mv_0^2 \left(\frac{l^2}{x^2} - 1\right) = \Delta K.$$

Thus, the work needed to push the surface from separation l to separation x equals the gain in kinetic energy of the ball.

Q4

(4.30) A particle of mass m and velocity v_0 collides elastically with a particle of mass M initially at rest and is scattered through angle Θ in the center of mass system.

- Find the final velocity of m in the laboratory system.

- Find the fractional loss of kinetic energy of m.

Solution:

The final velocity of the particle of mass m in the laboratory system is found by transforming the velocity from the center of mass (CM) system back to the laboratory system.

The velocity of the center of mass in the laboratory system is given by:

$$\vec{V}_{\text{cm}} = \frac{mv_0}{m+M}\hat{i}$$

where the initial velocity of mass m is v_0 along the x-axis, and mass M is initially at rest.

In the CM system, after an elastic collision, the speed of mass m remains unchanged and is:

$$u_1 = \frac{Mv_0}{m+M}$$

Given that the particle is scattered at an angle Θ in the CM system, its velocity in the CM system is:

$$\vec{v}_{1f,\text{cm}} = u_1(\cos\Theta\hat{i} + \sin\Theta\hat{j}) = \frac{Mv_0}{m+M}(\cos\Theta\hat{i} + \sin\Theta\hat{j})$$

The final velocity in the laboratory system is obtained by adding the CM velocity:

$$\vec{v}_{1f,\text{lab}} = \vec{v}_{1f,\text{cm}} + \vec{V}_{\text{cm}} = \frac{Mv_0}{m+M}(\cos\Theta\hat{i} + \sin\Theta\hat{j}) + \frac{mv_0}{m+M}\hat{i}$$

Combining components:

$$v_x = v_0\frac{m + M\cos\Theta}{m+M}, \quad v_y = v_0\frac{M\sin\Theta}{m+M}$$

The magnitude of the final velocity (speed) is:

$$v_f = \sqrt{v_x^2 + v_y^2} = v_0 \sqrt{\left(\frac{m + M\cos\Theta}{m+M}\right)^2 + \left(\frac{M\sin\Theta}{m+M}\right)^2}$$

Simplifying the expression inside the square root:

$$\left(\frac{m + M\cos\Theta}{m+M}\right)^2 + \left(\frac{M\sin\Theta}{m+M}\right)^2 = \frac{(m + M\cos\Theta)^2 + (M\sin\Theta)^2}{(m+M)^2}$$

$$= \frac{m^2 + 2mM\cos\Theta + M^2\cos^2\Theta + M^2\sin^2\Theta}{(m+M)^2} = \frac{m^2 + 2mM\cos\Theta + M^2}{(m+M)^2}$$

Thus:

$$v_f = v_0 \sqrt{\frac{m^2 + M^2 + 2mM\cos\Theta}{(m+M)^2}} = v_0 \frac{\sqrt{m^2 + M^2 + 2mM\cos\Theta}}{m+M}$$

This expression gives the speed (magnitude of velocity) of mass m in the laboratory system after the collision.

The fractional loss of kinetic energy of the particle of mass m is the ratio of the change in its kinetic energy to its initial kinetic energy.

The initial kinetic energy of m is $\frac{1}{2}mv_0^2$.

From the center of mass analysis, the magnitude of the final velocity of m in the laboratory system is $v_f = v_0 \frac{\sqrt{m^2+M^2+2mM\cos\Theta}}{m+M}$. Thus, the final kinetic energy is:

$$KE_f = \frac{1}{2}mv_f^2 = \frac{1}{2}m\left(v_0 \frac{\sqrt{m^2 + M^2 + 2mM\cos\Theta}}{m+M}\right)^2 = \frac{1}{2}mv_0^2 \frac{m^2 + M^2 + 2mM\cos\Theta}{(m+M)^2}.$$

The change in kinetic energy is:

$$KE_i - KE_f = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_0^2 \frac{m^2 + M^2 + 2mM\cos\Theta}{(m+M)^2} = \frac{1}{2}mv_0^2 \left(1 - \frac{m^2 + M^2 + 2mM\cos\Theta}{(m+M)^2}\right).$$

Simplifying the expression inside the parentheses:

$$1-\frac{m^2+M^2+2mM\cos\Theta}{(m+M)^2}=\frac{(m+M)^2-(m^2+M^2+2mM\cos\Theta)}{(m+M)^2}$$

$$=\frac{m^2+2mM+M^2-m^2-M^2-2mM\cos\Theta}{(m+M)^2}=\frac{2mM(1-\cos\Theta)}{(m+M)^2}.$$

Thus:

$$KE_i-KE_f=\frac{1}{2}mv_0^2\cdot\frac{2mM(1-\cos\Theta)}{(m+M)^2}=\frac{1}{2}mv_0^2\cdot\frac{2mM(1-\cos\Theta)}{(m+M)^2}.$$

The fractional loss is:

$$\frac{KE_i-KE_f}{KE_i}=\frac{\frac{1}{2}mv_0^2\cdot\frac{2mM(1-\cos\Theta)}{(m+M)^2}}{\frac{1}{2}mv_0^2}=\frac{2mM(1-\cos\Theta)}{(m+M)^2}.$$

Q5

(4.10) A block of mass M on a horizontal frictionless table is connected to a spring (spring constant k), as shown. The block is set in motion so that it oscillates about its equilibrium point with a certain amplitude A_0. The period of motion is $T_0=2\pi\sqrt{M/k}$.

- A lump of sticky putty of mass m is dropped onto the block. The putty sticks without bouncing. The putty hits M at the instant when the velocity of M is zero. Find:
  - The new period
  - The new amplitude
  - The change in the mechanical energy of the system
- Repeat part (a), but this time assume that the sticky putty hits M at the instant when M has its maximum velocity.

Solution:

The putty is dropped onto the block when the block is at a point of zero velocity, which occurs at the amplitude position $x = A_0$ (either maximum extension or compression). At this instant, the horizontal velocity of both the block and the putty is zero, assuming the putty is dropped with zero horizontal velocity.

- **New period**:

  The period of oscillation for a spring-mass system is given by $T = 2\pi\sqrt{\frac{M}{k}}$, where M is the mass and k is the spring constant. After the putty sticks, the total mass becomes $M + m$. Thus, the new period is:

$$T' = 2\pi\sqrt{\frac{M+m}{k}} = T_0\sqrt{\frac{M+m}{M}} = T_0\sqrt{1+\frac{m}{M}}$$

- **New amplitude**:
  At the instant the putty sticks, the position is $x = A_0$ and the velocity is zero. Since the velocity is zero at this point, it becomes the turning point for the new system. The equilibrium position remains unchanged because the spring is horizontal and gravity does not affect the equilibrium. Therefore, the new amplitude is the same as the original amplitude:

$$A' = A_0$$

- **Change in mechanical energy**:
  Before the putty is added, the mechanical energy of the system is entirely spring potential energy at the amplitude: $E_0 = \frac{1}{2}kA_0^2$. After the putty sticks, at the same position $x = A_0$, the velocity is still zero, so the mechanical energy is $E' = \frac{1}{2}kA_0^2$. Since there is no relative horizontal velocity between the block and the putty at impact, no kinetic energy is dissipated in the horizontal direction (the collision is inelastic but with zero relative velocity). The vertical motion of the putty does not affect the horizontal oscillator, and any energy dissipation in the vertical direction is irrelevant to the spring-block system's horizontal mechanical energy. Thus, the mechanical energy remains unchanged:

$$\Delta E = E' - E_0 = 0$$

When the sticky putty hits the block at the instant when the block has its maximum velocity, the block is at the equilibrium position ($x = 0$). The putty is dropped with zero horizontal velocity and sticks to the block in an inelastic collision. Conservation of momentum applies, but mechanical energy is not conserved during the collision.

New Period:

The period of oscillation for a spring-mass system depends on the total mass and the spring constant. After the putty sticks, the total mass is $M + m$, and the spring constant k remains unchanged. The new period is:

$$T' = 2\pi\sqrt{\frac{M+m}{k}} = T_0\sqrt{1 + \frac{m}{M}}$$

where $T_0 = 2\pi\sqrt{M/k}$ is the original period.

New Amplitude:

At the instant of collision, the block is at the equilibrium position with maximum velocity $v_{\max}$. For the original system, $v_{\max} = \sqrt{\frac{k}{M}}A_0$.

- Before collision:
  - Block mass M: velocity $v_{\max}$
  - Putty mass m: horizontal velocity 0
  - Total momentum: $Mv_{\max}$
- After collision (inelastic, sticks together):
  - Combined mass: $M + m$
  - Let v' be the new velocity.
  - Conservation of momentum: $Mv_{\max} = (M + m)v'$
  - Thus, $v' = \frac{M}{M+m}v_{\max} = \frac{M}{M+m}\sqrt{\frac{k}{M}}A_0$

The new system (mass $M + m$, spring constant k) has its equilibrium position unchanged. The mechanical energy is conserved after the collision. At the equilibrium position (just after collision), the total energy is kinetic:

$$E' = \frac{1}{2}(M + m)(v')^2$$

This equals the maximum potential energy at amplitude A':

$$\frac{1}{2}k(A')^2 = \frac{1}{2}(M+m)(v')^2$$

Solving for A':

$$A' = v'\sqrt{\frac{M+m}{k}} = \left(\frac{M}{M+m}\sqrt{\frac{k}{M}}A_0\right)\sqrt{\frac{M+m}{k}} = A_0\sqrt{\frac{M}{M+m}}$$

The original mechanical energy of the system (block and spring) is $E_0 = \frac{1}{2}kA_0^2$.

- Just before collision:
  - The block is at equilibrium, so spring potential energy is 0.
  - Kinetic energy of block: $\frac{1}{2}Mv_{\max}^2 = \frac{1}{2}M\left(\frac{k}{M}A_0^2\right) = \frac{1}{2}kA_0^2 = E_0$
  - Putty has zero horizontal velocity, so its kinetic energy is 0.
  - Total mechanical energy just before collision: E_0
- Just after collision:
  - Combined mass at equilibrium, so spring potential energy is 0.
  - Kinetic energy: $\frac{1}{2}(M+m)(v')^2 = \frac{1}{2}(M+m)\left(\frac{M}{M+m}\sqrt{\frac{k}{M}}A_0\right)^2 = \frac{1}{2}(M+m)\left(\frac{M^2}{(M+m)^2}\cdot\frac{k}{M}A_0^2\right) = \frac{1}{2}kA_0^2\cdot\frac{M}{M+m} = E_0\frac{M}{M+m}$
  - Thus, the new mechanical energy is $E' = E_0\frac{M}{M+m}$

The change in mechanical energy is:

$$\Delta E = E' - E_0 = E_0\frac{M}{M+m} - E_0 = E_0\left(\frac{M}{M+m} - 1\right) = E_0\left(\frac{M-M-m}{M+m}\right) = -\frac{m}{M+m}E_0$$

where $E_0 = \frac{1}{2}kA_0^2$.

- New period: $T' = T_0\sqrt{1+\frac{m}{M}}$
- New amplitude: $A' = A_0\sqrt{\frac{M}{M+m}}$

- Change in mechanical energy: $\Delta E = -\dfrac{m}{M+m}E_0$

Q6

(3.4) An instrument-carrying projectile accidentally explodes at the top of its trajectory. The horizontal distance between the launch point and the point of explosion is L. The projectile breaks into two pieces which fly apart horizontally. The larger piece has three times the mass of the smaller piece. To the surprise of the scientist in charge, the smaller piece returns to earth at the launching station. How far away does the larger piece land? Neglect air resistance and effects due to the earth's curvature.

Solution:

After the explosion, both pieces have only horizontal velocities, as they "fly apart horizontally," meaning no vertical velocity components are imparted. Thus, they begin to fall under gravity from the height H of the explosion point.

The smaller piece returns to the launch point, which is at $x = 0$. The explosion occurs at $x = L$. The time t for both pieces to fall to the ground from height H is given by $t = \sqrt{\frac{2H}{g}}$, but this time also equals the time to reach the top of the trajectory, $t = \frac{v_y}{g}$, where v_y is the initial vertical component of velocity.

The horizontal distance to the explosion point is $L = v_x t$. For the smaller piece to land at $x = 0$, its horizontal displacement is $-L$. Therefore, its horizontal velocity u_s after the explosion is:

$$u_s = \frac{-L}{t} = -v_x,$$

since $L = v_x t$.

Conservation of momentum in the horizontal direction (no external horizontal forces) gives:

$$4mv_x = mu_s + 3mu_l,$$

where u_l is the horizontal velocity of the larger piece after the explosion. Substituting $u_s = -v_x$:

$$\begin{aligned}4mv_x &= m(-v_x) + 3mu_l,\\4mv_x &= -mv_x + 3mu_l,\\5mv_x &= 3mu_l,\\u_l &= \frac{5}{3}v_x.\end{aligned}$$

The horizontal displacement of the larger piece during time t is:

$$\Delta x_l = u_l t = \frac{5}{3}v_x t = \frac{5}{3}L,$$

since $v_x t = L$. Therefore, the landing position D of the larger piece relative to the launch point is:

$$D = L + \Delta x_l = L + \frac{5}{3}L = \frac{8}{3}L.$$

Alternatively, using the center of mass concept: the center of mass continues with the original horizontal velocity v_x and lands at the original landing point, which is at $x = 2L$ (since the range is twice the distance to the apex for a symmetric trajectory). The center of mass position at landing is:

$$x_{\text{cm}} = \frac{m \cdot 0 + 3m \cdot D}{4m} = \frac{3D}{4}.$$

Setting this equal to $2L$:

$$\frac{3D}{4} = 2L,$$

$$D = \frac{8}{3}L.$$

Q7

(3.11) Material is blown into cart A from cart B at a rate b kilograms per second. The material leaves the chute vertically downward, so that it has the same horizontal velocity as cart B, u. At the moment of interest, cart A has mass M and velocity v, as shown. Find dv/dt, the instantaneous acceleration of A.

Solution

The acceleration of cart A is determined by considering the variable mass system and the momentum transfer due to the incoming material.

The material is blown into cart A at a rate b kg/s with horizontal velocity u, while cart A has mass M and velocity v at the moment of interest. There are no external horizontal forces acting on cart A.

The equation of motion for a system gaining mass is given by:

$$F_{\text{ext}} = m\frac{dv}{dt} + (v - v_{\text{in}})\frac{dm}{dt}$$

where F_{ext} is the net external force, m is the mass, v is the velocity, v_{in} is the velocity of the incoming mass, and $\frac{dm}{dt}$ is the rate of mass change.

Since there are no external horizontal forces, $F_{\text{ext}} = 0$. The velocity of the incoming mass is $v_{\text{in}} = u$, and the mass of cart A is increasing, so $\frac{dm}{dt} = b$. Substituting these values gives:

$$0 = M\frac{dv}{dt} + (v - u)b$$

Solving for $\frac{dv}{dt}$:

$$M\frac{dv}{dt} = -(v - u)b = (u - v)b$$

$$\frac{dv}{dt} = \frac{b(u - v)}{M}$$

Q8

(3.13) A ski tow consists of a long belt of rope around two pulleys, one at the bottom of a slope and the other at the top. The pulleys are driven by a husky electric motor so that the rope moves at a steady speed of 1.5 m/s. The pulleys are separated by a distance of 100 m, and the angle of the slope is 20°. Skiers take hold of the rope and are pulled up to the top, where they release the rope and glide off. If a skier of mass 70 kg takes the tow every 5 s on the average, what is the average force required to pull the rope? Neglect friction between the skis and the snow.

Solution:

The average force required to pull the rope is determined by considering the average number of skiers being pulled at any given time and the force required to pull each skier at a constant speed up the slope, neglecting friction.

The time for one skier to travel the length of the slope is calculated from the distance and rope speed:

$$t = \frac{\text{distance}}{\text{speed}} = \frac{100\text{m}}{1.5\text{m/s}} = \frac{200}{3}\text{s} \approx 66.667\text{s}.$$

Skiers are added every 5 seconds on average, so the average number of skiers on the rope is:

$$N_{\text{avg}} = \frac{t}{\Delta t} = \frac{200/3}{5} = \frac{200}{15} = \frac{40}{3} \approx 13.333.$$

The force required to pull one skier at constant speed up the slope, overcoming the component of gravity parallel to the slope, is:

$$f = mg\sin\theta,$$

where $m = 70\text{kg}$, $g = 9.8\text{m/s}^2$, and $\theta = 20^\circ$ so $\sin 20^\circ \approx 0.3420$.

$$f = 70 \times 9.8 \times 0.3420 = 234.612\text{N}.$$

The average force to pull the rope is then the product of the average number of skiers and the force per skier:

$$F_{\text{avg}} = N_{\text{avg}} \times f = \frac{40}{3} \times 234.612 = \frac{40 \times 234.612}{3} = \frac{9384.48}{3} = 3128.16\text{N}.$$

Rounding to three significant figures, the average force is 3130N.

Q9

(3.14) N men, each with mass m, stand on a railway flatcar of mass M. They jump off one end of the flatcar with velocity u relative to the car. The car rolls in the opposite direction without friction.

- What is the final velocity of the flatcar if all the men jump at the same time?
- What is the final velocity of the flatcar if they jump off one at a time? (The answer can be left in the form of a sum of terms.)

Solution:

The final velocity of the flatcar is determined using conservation of momentum, as there is no friction. The initial total momentum is zero since the system is at rest.

Case 1: All men jump at the same time
When all N men jump simultaneously with velocity u relative to the flatcar in the positive direction, the absolute velocity of each man is $v_c + u$, where v_c is the velocity of the flatcar. The total momentum after the jump is:

$$Nm(v_c + u) + Mv_c = 0$$

Solving for v_c:

$$v_c(Nm + M) = -Nmu$$

$$v_c = -\frac{Nmu}{Nm + M}$$

The negative sign indicates that the flatcar moves in the opposite direction to the jump.

Case 2: Men jump off one at a time
When the men jump sequentially, the velocity of the flatcar changes after each jump. After k jumps, the velocity of the flatcar (with the remaining $N - k$ men) is v_k. The recurrence relation is:

$$v_k = v_{k-1} - \frac{mu}{M + (N - k + 1)m}, \quad k = 1, 2, \ldots, N$$

with initial condition $v_0 = 0$. The final velocity after all N jumps is:

$$v_N = \sum_{k=1}^{N} (v_k - v_{k-1}) = \sum_{k=1}^{N} \left(-\frac{mu}{M + (N - k + 1)m} \right)$$

Substituting $j = N - k + 1$, the sum becomes:

$$v_N = -mu \sum_{j=1}^{N} \frac{1}{M + jm}$$

Thus, the final velocity is:

$$v_c = -mu \sum_{k=1}^{N} \frac{1}{M + km}$$

Q10

(3.16) Water shoots out of a fire hydrant having nozzle diameter D with nozzle speed V_o. What is the reaction force on the hydrant?

Solution:

The reaction force on the fire hydrant is due to the change in momentum of the water as it exits the nozzle. The mass flow rate of water through the nozzle is given by $\dot{m} = \rho A V_o$, where ρ is the density of water, A is the cross-sectional area of the nozzle, and V_o is the nozzle speed.

The cross-sectional area A is related to the nozzle diameter D by $A = \pi(D/2)^2 = \pi D^2/4$.

The rate of change of momentum of the water is $\dot{m} V_o$, which corresponds to the force exerted by the water on the hydrant. By Newton's third law, the reaction force on the hydrant is equal in magnitude but opposite in direction to this force. Therefore, the magnitude of the reaction force is:

$$F = \dot{m}V_o = \left(\rho \cdot \frac{\pi D^2}{4} \cdot V_o\right)V_o = \frac{\pi}{4}\rho D^2 V_o^2$$

This expression assumes that the water enters the hydrant with negligible velocity compared to the exit velocity V_o, which is typical for such problems. The reaction force acts in the direction opposite to the jet flow.

Q11

(3.17) An inverted garbage can of weight W is suspended in air by water from a geyser. The water shoots up from the ground with a speed v_0, at a constant rate dm/dt. The problem is to find the maximum height at which the garbage can rides. What assumption must be fulfilled for the maximum height to be reached?

Solution:

The maximum height h at which the inverted garbage can rides is determined by the balance between the upward force from the water jet and the weight of the can. The water jet is emitted from the geyser with speed v_0 at a constant mass flow rate $\frac{dm}{dt}$.

The upward force exerted on the can by the water jet at height h is given by the rate of change of momentum of the water. The speed of the water at height h, accounting for gravity, is $v = \sqrt{v_0^2 - 2gh}$. The mass flow rate $\frac{dm}{dt}$ is constant, as given. When the water hits the can, it is assumed to come to rest vertically relative to the can, resulting in a change in vertical momentum. The upward force on the can is therefore $\frac{dm}{dt}v = \frac{dm}{dt}\sqrt{v_0^2 - 2gh}$.

For the can to be suspended at height h, this upward force must equal the weight W:

$$\frac{dm}{dt}\sqrt{v_0^2 - 2gh} = W$$

Solving for h:

$$\sqrt{v_0^2 - 2gh} = \frac{W}{\frac{dm}{dt}}$$

$$v_0^2 - 2gh = \left(\frac{W}{\frac{dm}{dt}}\right)^2$$

$$2gh = v_0^2 - \left(\frac{W}{\frac{dm}{dt}}\right)^2$$

$$h = \frac{v_0^2 - \left(\frac{W}{\frac{dm}{dt}}\right)^2}{2g}$$

This height is the maximum at which the can can ride in equilibrium. For this height to be physically achievable (i.e., positive and real), the initial thrust from the water jet must exceed the weight of the can. Specifically, $\frac{dm}{dt}v_0 > W$. If $\frac{dm}{dt}v_0 \leq W$, the force at ground level is insufficient to lift the can, and no positive height of suspension exists.

Q12

(3.18) A raindrop of initial mass M_0 starts falling from rest under the influence of gravity. Assume that the drop gains mass from the cloud at a rate proportional to the product of its instantaneous mass and its instantaneous velocity:

$$\frac{dM}{dt} = kMV,$$

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where $k$ is a constant. Show that the speed of the drop eventually becomes effectively constant, and give an
expression for the terminal speed. Neglect air resistance.
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Solution:

The equation of motion for the raindrop is derived from Newton's second law for variable mass systems. The external force is gravity, given by $F_{\text{ext}} = Mg$, where M is the instantaneous mass and g is the acceleration due to gravity. The mass is gained from a stationary cloud, so the velocity of the added mass relative to the ground is zero. The rate of change of mass is given as $\frac{dM}{dt} = kMV$, where k is a constant and V is the instantaneous velocity.

The general equation for variable mass systems is:

$$M\frac{dV}{dt} = F_{\text{ext}} + (u - V)\frac{dM}{dt},$$

where u is the velocity of the added mass. Since the cloud is stationary, $u = 0$. Substituting the external force and mass rate:

$$M\frac{dV}{dt} = Mg + (0 - V)\frac{dM}{dt} = Mg - V\frac{dM}{dt}.$$

Using $\frac{dM}{dt} = kMV$:

$$M\frac{dV}{dt} = Mg - V(kMV) = Mg - kMV^2.$$

Dividing by M (assuming $M > 0$):

$$\frac{dV}{dt} = g - kV^2.$$

The terminal speed occurs when acceleration is zero, i.e., $\frac{dV}{dt} = 0$:

$$g - kV^2 = 0 \implies V^2 = \frac{g}{k} \implies V = \sqrt{\frac{g}{k}}.$$

Since the raindrop is falling, the speed is taken as positive.

To show that the speed approaches this terminal value, solve the differential equation:

$$\frac{dV}{dt} = g - kV^2 = k\left(\frac{g}{k} - V^2\right).$$

Let $a = \sqrt{\frac{g}{k}}$, so:

$$\frac{dV}{dt} = k(a^2 - V^2).$$

Separating variables:

$$\int \frac{dV}{a^2 - V^2} = \int k dt.$$

The integral on the left is:

$$\int \frac{dV}{a^2 - V^2} = \frac{1}{a} \tanh^{-1}\left(\frac{V}{a}\right) + C.$$

Thus:

$$\frac{1}{a} \tanh^{-1}\left(\frac{V}{a}\right) = kt + C.$$

At $t = 0$, $V = 0$:

$$\frac{1}{a} \tanh^{-1}(0) = C \implies C = 0.$$

So:

$$\frac{1}{a} \tanh^{-1}\left(\frac{V}{a}\right) = kt \implies \frac{V}{a} = \tanh(akt).$$

Substituting $a = \sqrt{\frac{g}{k}}$ and $ak = \sqrt{gk}$:

$$V = \sqrt{\frac{g}{k}} \tanh\left(\sqrt{gk} t\right).$$

As $t \to \infty$, $\tanh(\sqrt{gk} t) \to 1$, so:

$$V \to \sqrt{\frac{g}{k}}.$$

The speed approaches the terminal speed asymptotically and becomes effectively constant for large t.

Q13

(3.20) A rocket ascends from rest in a uniform gravitational field by ejecting exhaust with constant speed u. Assume that the rate at which mass is expelled is given by $\frac{dm}{dt} = \gamma m$, where m is the instantaneous mass of the rocket and γ is a constant, and that the rocket is retarded by air resistance with a force mbv, where b is a constant. Find the velocity of the rocket as a function of time.

Solution:

The rocket ascends under the influence of gravity, air resistance, and thrust from ejecting exhaust. The rate of change of mass is given by $\frac{dm}{dt} = \gamma m$, where $\gamma < 0$ since the mass decreases. The exhaust is ejected with constant speed u relative to the rocket, and air resistance is given by mbv, where b is a constant.

The equation of motion for the rocket is derived from Newton's second law. The net external force includes gravity ($-mg$) and air resistance ($-mbv$), while the thrust force is due to the ejection of mass. The thrust force is given by $\dot{m}u_{\mathrm{rel}}, y$, where $u_{\mathrm{rel}}, y = -u$ (since the exhaust is ejected downward and upward is positive). Substituting $\dot{m} = \gamma m$, the thrust force is $\gamma m(-u) = -u\gamma m$.

The equation of motion is:

$$m\frac{dv}{dt} = -mg - mbv - u\gamma m.$$

Dividing by m (assuming $m > 0$):

$$\frac{dv}{dt} = -g - bv - u\gamma.$$

Rearranging gives:

$$\frac{dv}{dt} + bv = -(g + u\gamma).$$

This is a linear first-order differential equation. The integrating factor is e^{bt}, so:

$$e^{bt}\frac{dv}{dt} + be^{bt}v = -(g + u\gamma)e^{bt},$$

which simplifies to:

$$\frac{d}{dt}(ve^{bt}) = -(g + u\gamma)e^{bt}.$$

Integrating both sides:

$$ve^{bt} = -(g + u\gamma)\int e^{bt}dt = -\frac{g + u\gamma}{b}e^{bt} + C,$$

where C is the constant of integration. Thus:

$$v(t) = -\frac{g + u\gamma}{b} + Ce^{-bt}.$$

Using the initial condition $v(0) = 0$:

$$0 = -\frac{g + u\gamma}{b} + C \implies C = \frac{g + u\gamma}{b}.$$

Substituting C:

$$v(t) = -\frac{g + u\gamma}{b} + \frac{g + u\gamma}{b}e^{-bt} = \frac{g + u\gamma}{b}(e^{-bt} - 1).$$

The velocity as a function of time is:

$$v(t) = \frac{g + u\gamma}{b}\left(e^{-bt} - 1\right).$$

This expression is valid for $b > 0$. As $t \to \infty$, the velocity approaches the terminal velocity $v_{\text{terminal}} = -\frac{g+u\gamma}{b}$, which is positive if $g + u\gamma < 0$ (i.e., the initial thrust exceeds the weight).