A04

Q1

(1-Q1) Let $\mathbf{r}(t) = \langle \cos e^t, \sin e^t, e^t \rangle$

- Compute the unit tangent vector $\mathbf{T}(t)$
- Compute $\mathbf{T}'(t)$

Solution:
The position vector is given by $\mathbf{r}(t) = \langle \cos(e^t), \sin(e^t), e^t \rangle$.

The derivative is found by differentiating each component with respect to t. For the x-component: $x(t) = \cos(e^t)$. Using the chain rule, let $u = e^t$, so:

$$\frac{d}{dt}\cos(u) = -\sin(u) \cdot \frac{du}{dt} = -\sin(e^t) \cdot e^t$$

For the y-component: $y(t) = \sin(e^t)$. Using the chain rule:

$$\frac{d}{dt}\sin(u) = \cos(u) \cdot \frac{du}{dt} = \cos(e^t) \cdot e^t$$

For the z-component: $z(t) = e^t$. The derivative is e^t.

Thus,

$$\mathbf{r}'(t) = \langle -\sin(e^t)e^t, \cos(e^t)e^t, e^t \rangle = e^t \langle -\sin(e^t), \cos(e^t), 1 \rangle.$$

The magnitude is given by:

$$|\mathbf{r}'(t)| = \sqrt{(-\sin(e^t)e^t)^2 + (\cos(e^t)e^t)^2 + (e^t)^2}.$$

Simplify inside the square root:

$$\big(-\sin(e^t)e^t\big)^2 + \big(\cos(e^t)e^t\big)^2 + \big(e^t\big)^2 = e^{2t}\sin^2(e^t) + e^{2t}\cos^2(e^t) + e^{2t} = e^{2t}\left(\sin^2(e^t) + \cos^2(e^t) + 1\right).$$

Since $\sin^2(\theta) + \cos^2(\theta) = 1$,

$$\sin^2(e^t) + \cos^2(e^t) + 1 = 1 + 1 = 2.$$

Thus,

$$|\mathbf{r}'(t)| = \sqrt{e^{2t} \cdot 2} = \sqrt{2e^{2t}} = \sqrt{2} \cdot e^t.$$

The unit tangent vector is defined as:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{e^t\langle -\sin(e^t), \cos(e^t), 1\rangle}{\sqrt{2}e^t} = \frac{1}{\sqrt{2}}\langle -\sin(e^t), \cos(e^t), 1\rangle.$$

In component form,

$$\mathbf{T}(t) = \left\langle -\frac{\sin(e^t)}{\sqrt{2}}, \frac{\cos(e^t)}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

Differentiate each component of $\mathbf{T}(t)$ with respect to t.

For the x-component: $T_x(t) = -\frac{1}{\sqrt{2}}\sin(e^t)$. Using the chain rule:

$$\frac{d}{dt}\sin(e^t) = \cos(e^t) \cdot e^t$$

so

$$\frac{d}{dt}T_x = -\frac{1}{\sqrt{2}} \cdot \cos(e^t) \cdot e^t = -\frac{e^t\cos(e^t)}{\sqrt{2}}.$$

For the y-component: $T_y(t) = \frac{1}{\sqrt{2}}\cos(e^t)$. Using the chain rule:

$$\frac{d}{dt}\cos(e^t) = -\sin(e^t)\cdot e^t$$

so

$$\frac{d}{dt}T_y = \frac{1}{\sqrt{2}}\cdot\left(-\sin(e^t)\cdot e^t\right) = -\frac{e^t\sin(e^t)}{\sqrt{2}}.$$

For the z-component: $T_z(t) = \frac{1}{\sqrt{2}}$. This is constant, so its derivative is 0.

Thus,

$$\mathbf{T}'(t) = \left\langle -\frac{e^t\cos(e^t)}{\sqrt{2}}, -\frac{e^t\sin(e^t)}{\sqrt{2}}, 0\right\rangle.$$

This can also be written as:

$$\mathbf{T}'(t) = -\frac{e^t}{\sqrt{2}}\langle\cos(e^t), \sin(e^t), 0\rangle.$$

$$\boxed{\mathbf{T}(t) = \left\langle -\frac{\sin\left(\mathrm{e}^t\right)}{\sqrt{2}}, \frac{\cos\left(\mathrm{e}^t\right)}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle}$$

$$\boxed{\mathbf{T}'(t) = \left\langle -\frac{\mathrm{e}^t\cos\left(\mathrm{e}^t\right)}{\sqrt{2}}, -\frac{\mathrm{e}^t\sin\left(\mathrm{e}^t\right)}{\sqrt{2}}, 0\right\rangle}$$

Q2

(1-Q5) Let G be the solid 3-D cone bounded by the lateral surface given by $z = 2\sqrt{x^2+y^2}$ and by the plane $z = 2$. Assume the density is given by $\rho(x,y,z) = z$.

- Find the mass of G using cylindrical coordinates.

- Set up the calculation for $\overline{z}$ using cylindrical coordinates.
- Set up the calculation for $\overline{z}$ using spherical coordinates.

Solution:

The surface $z = 2\sqrt{x^2 + y^2}$ becomes $z = 2r$ in cylindrical coordinates. The plane $z = 2$ intersects the cone when $2r = 2$, so $r = 1$. Thus, the solid is defined for $0 \le r \le 1$, $0 \le \theta \le 2\pi$, and for each (r, θ), z ranges from the cone $z = 2r$ to the plane $z = 2$.

The mass is given by the triple integral:

$$m = \iiint_G \rho dV = \iiint_G z dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=2r}^{2} z \cdot r dz dr d\theta.$$

Since the integrand does not depend on θ, the integral can be separated:

$$m = \int_0^{2\pi} d\theta \int_0^1 r dr \int_{2r}^{2} z dz.$$

First, evaluate the innermost integral with respect to z:

$$\int_{2r}^{2} z dz = \left[\frac{1}{2} z^2\right]_{2r}^{2} = \frac{1}{2}(2)^2 - \frac{1}{2}(2r)^2 = \frac{1}{2} \cdot 4 - \frac{1}{2} \cdot 4r^2 = 2 - 2r^2.$$

Substitute this result into the integral:

$$m = \int_0^{2\pi} d\theta \int_0^1 r(2 - 2r^2) dr = \int_0^{2\pi} d\theta \int_0^1 (2r - 2r^3) dr.$$

Next, evaluate the integral with respect to r:

$$\int_0^1 (2r - 2r^3) dr = \left[r^2 - \frac{1}{2} r^4\right]_0^1 = \left(1^2 - \frac{1}{2} \cdot 1^4\right) - (0) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Substitute this result into the integral:

$$m = \int_0^{2\pi} \frac{1}{2} d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \frac{1}{2} \cdot 2\pi = \pi.$$

Thus, the mass of the solid G is π.

To compute the z-coordinate of the center of mass, $\overline{z}$, for the solid cone G with density $\rho(x, y, z) = z$, use the formula:

$$\overline{z} = \frac{1}{\text{mass}} \iiint_G z\rho dV.$$

The mass of G has been previously calculated as π. Substituting the density $\rho(x, y, z) = z$ gives:

$$\overline{z} = \frac{1}{\pi} \iiint_G z \cdot z dV = \frac{1}{\pi} \iiint_G z^2 dV.$$

In cylindrical coordinates, the solid G is bounded by the cone $z = 2r$ and the plane $z = 2$, with $0 \le r \le 1$, $0 \le \theta \le 2\pi$, and for each (r, θ), z ranges from $2r$ to 2. The volume element is $dV = rdzdrd\theta$.

Thus, the triple integral is:

$$\iiint_G z^2 dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=2r}^{2} z^2 \cdot rdzdrd\theta.$$

Therefore, the setup for $\overline{z}$ is:

$$\boxed{\overline{z} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \int_{2r}^2 z^2 rdzdrd\theta}$$

In spherical coordinates, the solid G is bounded by the cone $z = 2\sqrt{x^2 + y^2}$ and the plane $z = 2$. The conversion to spherical coordinates is given by:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi,$$

with the volume element $dV = \rho^2 \sin\phi d\rho d\phi d\theta$.

The cone surface $z = 2\sqrt{x^2+y^2}$ simplifies to $\rho\cos\phi = 2\rho\sin\phi$, which gives $\tan\phi = \frac{1}{2}$. Thus, ϕ ranges from 0 to $\phi_0 = \arctan(1/2)$. The plane $z = 2$ corresponds to $\rho\cos\phi = 2$, so $\rho = \frac{2}{\cos\phi}$. For each fixed ϕ and θ, ρ ranges from 0 to $\frac{2}{\cos\phi}$, and θ ranges from 0 to 2π.

The integrand z^2 in spherical coordinates is:

$$z^2 = (\rho\cos\phi)^2 = \rho^2\cos^2\phi.$$

Thus, the integrand z^2dV becomes:

$$z^2dV = \rho^2\cos^2\phi \cdot \rho^2\sin\phi d\rho d\phi d\theta = \rho^4\cos^2\phi\sin\phi d\rho d\phi d\theta.$$

The triple integral for $\iiint_G z^2dV$ is:

$$\iiint_G z^2dV = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\arctan(1/2)}\int_{\rho=0}^{2/\cos\phi}\rho^4\cos^2\phi\sin\phi d\rho d\phi d\theta.$$

Therefore, the setup for $\overline{z}$ is:

$$\boxed{\overline{z} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\arctan(1/2)}\int_0^{\frac{2}{\cos\phi}}\rho^4\cos^2\phi\sin\phi d\rho d\phi d\theta}$$

Q3

(1-Q10) Find the area of the ellipse,

$$(2x+5y-3)^2 + (3x-7y+8)^2 = 1$$

Solution:

The equation $(2x+5y-3)^2 + (3x-7y+8)^2 = 1$ represents an ellipse. To find its area, define the transformations:

$$u = 2x+5y-3, \quad v = 3x-7y+8.$$

The equation becomes $u^2 + v^2 = 1$, which is a circle of radius 1 in the uv-plane with area $\pi \cdot 1^2 = \pi$.

The Jacobian matrix of the transformation from (x, y) to (u, v) is:

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ \\ 3 & -7 \end{pmatrix}.$$

The determinant of J is:

$$\det(J) = (2)(-7) - (5)(3) = -14 - 15 = -29.$$

The absolute value is $|\det(J)| = 29$.

The area element in the xy-plane relates to the area element in the uv-plane by:

$$dxdy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv = \frac{1}{|\det(J)|} dudv = \frac{1}{29} dudv,$$

since $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{|\det(J)|}$.

The area in the xy-plane is the integral over the region where $u^2 + v^2 \leq 1$:

$$\text{Area} = \iint_{u^2+v^2 \leq 1} dxdy = \iint_{u^2+v^2 \leq 1} \frac{1}{29} dudv.$$

The integral is the area of the disk in the uv-plane:

$$\iint_{u^2+v^2 \leq 1} dudv = \pi.$$

Thus,

$$\text{Area} = \frac{1}{29} \cdot \pi = \frac{\pi}{29}.$$

The constants in the transformations do not affect the area, as translations preserve area. The area is therefore $\frac{\pi}{29}$.

Q4

(Q-21) Let C be the portion of the cylinder $x^2 + y^2 \leq 1$ lying in the first octant $(x \geq 0, y \geq 0, z \geq 0)$ and below the plane $z = 1$. Set up a triple integral in cylindrical coordinates which gives the moment of inertia of C about the z-axis; assume the density to be $\delta = 1$.

Solution:

The moment of inertia about the z-axis for a solid with constant density $\delta = 1$ is given by the triple integral:

$$I_z = \iiint_C (x^2 + y^2) dV,$$

where C is the region defined by $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$, and $z \leq 1$.

The transformations in cylindrical coordinates are:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

with the volume element $dV = r dr d\theta dz$. Additionally, $x^2 + y^2 = r^2$, so the integrand becomes r^2.

The region C is bounded as follows:

- The cylinder $x^2 + y^2 \leq 1$ corresponds to $r \leq 1$.
- The first octant $(x \geq 0, y \geq 0)$ corresponds to $0 \leq \theta \leq \pi/2$.
- The conditions $z \geq 0$ and $z \leq 1$ correspond to $0 \leq z \leq 1$.

Thus, the limits of integration are:

- r from 0 to 1,
- θ from 0 to $\pi/2$,
- z from 0 to 1.

The integrand is r^2, and the volume element is $dV = rdrd\theta dz$, so the product is $r^2 \cdot r = r^3$. Therefore, the moment of inertia integral is:

$$\boxed{I_z = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \int_{z=0}^{1} r^3 dzdrd\theta}$$

Q5

Define the curvature κ of a curve γ in $\mathbb{R}^3$ and describe its geometric significance.
Determine the curvature for the curve

$$\gamma(t) = (\cos(2t), \sqrt{5}\, t, \sin(2t)), \quad t \in [0, \pi]$$

as a function of its arclength (starting from the initial point $\gamma(0)$). What is the integral of the curvature over the curve? Without performing any further computations, write down the integral of the curvature for the curve

$$\tilde{\gamma}(t) = (\cos(2t), 0, \sin(2t)), \quad t \in [0, \pi]$$

and explain why your result is different from the result for γ.

Solution

The curvature κ of a curve γ in $\mathbb{R}^3$ is defined as the magnitude of the derivative of the unit tangent vector with respect to arc length. Specifically, if $\mathbf{T}$ is the unit tangent vector and s is the arc length parameter, then

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|.$$

Geometrically, the curvature measures the rate at which the curve bends at a given point. It quantifies how sharply the curve deviates from a straight line. The curvature is the reciprocal of the radius of the osculating circle at that point, which is the circle that best approximates the curve locally.

For the curve $\gamma(t) = (\cos(2t), \sqrt{5}\, t, \sin(2t))$ with $t \in [0, \pi]$, the curvature is constant. The speed $\|\gamma'(t)\|$ is computed as follows:

$$\boldsymbol{\gamma}'(t) = (-2\sin(2t), \sqrt{5}, 2\cos(2t)),$$

$$\|\boldsymbol{\gamma}'(t)\| = \sqrt{(-2\sin(2t))^2 + (\sqrt{5})^2 + (2\cos(2t))^2}$$

$$= \sqrt{4\sin^2(2t) + 5 + 4\cos^2(2t)} = \sqrt{4(\sin^2(2t) + \cos^2(2t)) + 5} = \sqrt{4+5} = \sqrt{9} = 3.$$

The arc length s starting from $\boldsymbol{\gamma}(0)$ is given by

$$s(t) = \int_0^t \|\boldsymbol{\gamma}'(\tau)\| d\tau = \int_0^t 3 d\tau = 3t,$$

so $t = s/3$.

The curvature $\kappa(t)$ is found using the formula $\kappa(t) = \|\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)\| / \|\boldsymbol{\gamma}'(t)\|^3$. First,

$$\boldsymbol{\gamma}''(t) = (-4\cos(2t), 0, -4\sin(2t)).$$

The cross product is

$$\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin(2t) & \sqrt{5} & 2\cos(2t) \\ -4\cos(2t) & 0 & -4\sin(2t) \end{vmatrix} = (-4\sqrt{5}\sin(2t), -8, 4\sqrt{5}\cos(2t)),$$

and its magnitude is

$$\|\boldsymbol{\gamma}'(t) \times \boldsymbol{\gamma}''(t)\| = \sqrt{(-4\sqrt{5}\sin(2t))^2 + (-8)^2 + (4\sqrt{5}\cos(2t))^2}$$

$$= \sqrt{80\sin^2(2t) + 64 + 80\cos^2(2t)} = \sqrt{80(\sin^2(2t) + \cos^2(2t)) + 64} = \sqrt{80+64} = \sqrt{144} = 12.$$

Thus,

$$\kappa(t) = \frac{12}{3^3} = \frac{12}{27} = \frac{4}{9}.$$

Since the curvature is constant and independent of t, and $s = 3t$, it follows that $\kappa(s) = 4/9$ for all s.

The curve is a helix with radius 1 and pitch parameter $\sqrt{5}/2$, and helices have constant curvature. The arc length ranges from $s = 0$ at $t = 0$ to $s = 3\pi$ at $t = \pi$, but the curvature remains constant throughout.

Thus, the curvature as a function of arc length is $\kappa(s) = \frac{4}{9}$.

The curvature κ of a curve γ in $\mathbb{R}^3$ is defined as $\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$, where $\mathbf{T}$ is the unit tangent vector and s is the arc length parameter. Geometrically, curvature measures the rate at which the curve bends at a given point, quantifying how sharply it deviates from a straight line. It is the reciprocal of the radius of the osculating circle at that point, which is the circle that best approximates the curve locally.

For the curve $\gamma(t) = (\cos(2t), \sqrt{5}\,t, \sin(2t))$, $t \in [0, \pi]$, the integral of the curvature with respect to arc length is $\frac{4\pi}{3}$.

For the curve $\tilde{\gamma}(t) = (\cos(2t), 0, \sin(2t))$, $t \in [0, \pi]$, the integral of the curvature with respect to arc length is 2π.