A04

Q1

(1-Q1) Let $\mathbf{r}(t) = \langle \cos e^t, \sin e^t, e^t \rangle$

- Compute the unit tangent vector $\mathbf{T}(t)$
- Compute $\mathbf{T}'(t)$

Solution:
The position vector is given by $\mathbf{r}(t) = \langle \cos(e^t), \sin(e^t), e^t \rangle$.

The derivative is found by differentiating each component with respect to t. For the x-component: $x(t) = \cos(e^t)$. Using the chain rule, let $u = e^t$, so:

$$\frac{d}{dt}\cos(u) = -\sin(u) \cdot \frac{du}{dt} = -\sin(e^t) \cdot e^t$$

For the y-component: $y(t) = \sin(e^t)$. Using the chain rule:

$$\frac{d}{dt}\sin(u) = \cos(u) \cdot \frac{du}{dt} = \cos(e^t) \cdot e^t$$

For the z-component: $z(t) = e^t$. The derivative is e^t.

Thus,

$$\mathbf{r}'(t) = \langle -\sin(e^t)e^t, \cos(e^t)e^t, e^t \rangle = e^t \langle -\sin(e^t), \cos(e^t), 1 \rangle.$$

The magnitude is given by:

$$|\mathbf{r}'(t)| = \sqrt{(-\sin(e^t)e^t)^2 + (\cos(e^t)e^t)^2 + (e^t)^2}.$$

Simplify inside the square root:

$$\left(-\sin(e^t)e^t\right)^2 + \left(\cos(e^t)e^t\right)^2 + \left(e^t\right)^2 = e^{2t}\sin^2(e^t) + e^{2t}\cos^2(e^t) + e^{2t} = e^{2t}\left(\sin^2(e^t) + \cos^2(e^t) + 1\right).$$

Since $\sin^2(\theta) + \cos^2(\theta) = 1$,

$$\sin^2(e^t) + \cos^2(e^t) + 1 = 1 + 1 = 2.$$

Thus,

$$|\mathbf{r}'(t)| = \sqrt{e^{2t} \cdot 2} = \sqrt{2e^{2t}} = \sqrt{2} \cdot e^t.$$

The unit tangent vector is defined as:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{e^t \langle -\sin(e^t), \cos(e^t), 1 \rangle}{\sqrt{2}e^t} = \frac{1}{\sqrt{2}} \langle -\sin(e^t), \cos(e^t), 1 \rangle.$$

In component form,

$$\mathbf{T}(t) = \left\langle -\frac{\sin(e^t)}{\sqrt{2}}, \frac{\cos(e^t)}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

Differentiate each component of $\mathbf{T}(t)$ with respect to t.

For the x-component: $T_x(t) = -\frac{1}{\sqrt{2}}\sin(e^t)$. Using the chain rule:

$$\frac{d}{dt}\sin(e^t) = \cos(e^t) \cdot e^t$$

so

$$\frac{d}{dt}T_x = -\frac{1}{\sqrt{2}} \cdot \cos(e^t) \cdot e^t = -\frac{e^t \cos(e^t)}{\sqrt{2}}.$$

For the y-component: $T_y(t) = \frac{1}{\sqrt{2}}\cos(e^t)$. Using the chain rule:

$$\frac{d}{dt}\cos(e^t) = -\sin(e^t) \cdot e^t$$

so

$$\frac{d}{dt}T_y = \frac{1}{\sqrt{2}} \cdot \left(-\sin(e^t) \cdot e^t\right) = -\frac{e^t \sin(e^t)}{\sqrt{2}}.$$

For the z-component: $T_z(t) = \frac{1}{\sqrt{2}}$. This is constant, so its derivative is 0.

Thus,

$$\mathbf{T}'(t) = \left\langle -\frac{e^t \cos(e^t)}{\sqrt{2}}, -\frac{e^t \sin(e^t)}{\sqrt{2}}, 0 \right\rangle.$$

This can also be written as:

$$\mathbf{T}'(t) = -\frac{e^t}{\sqrt{2}} \langle \cos(e^t), \sin(e^t), 0 \rangle.$$

$$\boxed{\mathbf{T}(t) = \left\langle -\frac{\sin\left(\mathrm{e}^t\right)}{\sqrt{2}}, \frac{\cos\left(\mathrm{e}^t\right)}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle}$$

$$\boxed{\mathbf{T}'(t) = \left\langle -\frac{\mathrm{e}^t \cos\left(\mathrm{e}^t\right)}{\sqrt{2}}, -\frac{\mathrm{e}^t \sin\left(\mathrm{e}^t\right)}{\sqrt{2}}, 0 \right\rangle}$$

Q2

(1-Q5) Let G be the solid 3-D cone bounded by the lateral surface given by $z = 2\sqrt{x^2 + y^2}$ and by the plane $z = 2$. Assume the density is given by $\rho(x, y, z) = z$.

- Find the mass of G using cylindrical coordinates.
- Set up the calculation for $\bar{z}$ using cylindrical coordinates.
- Set up the calculation for $\bar{z}$ using spherical coordinates.

Solution:

The surface $z = 2\sqrt{x^2+y^2}$ becomes $z = 2r$ in cylindrical coordinates. The plane $z = 2$ intersects the cone when $2r = 2$, so $r = 1$. Thus, the solid is defined for $0 \le r \le 1$, $0 \le \theta \le 2\pi$, and for each (r, θ), z ranges from the cone $z = 2r$ to the plane $z = 2$.

The mass is given by the triple integral:

$$m = \iiint_G \rho dV = \iiint_G z dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=2r}^{2} z \cdot r dz dr d\theta.$$

Since the integrand does not depend on θ, the integral can be separated:

$$m = \int_0^{2\pi} d\theta \int_0^1 r dr \int_{2r}^{2} z dz.$$

First, evaluate the innermost integral with respect to z:

$$\int_{2r}^{2} z dz = \left[\frac{1}{2}z^2\right]_{2r}^{2} = \frac{1}{2}(2)^2 - \frac{1}{2}(2r)^2 = \frac{1}{2} \cdot 4 - \frac{1}{2} \cdot 4r^2 = 2 - 2r^2.$$

Substitute this result into the integral:

$$m = \int_0^{2\pi} d\theta \int_0^1 r(2 - 2r^2) dr = \int_0^{2\pi} d\theta \int_0^1 (2r - 2r^3) dr.$$

Next, evaluate the integral with respect to r:

$$\int_0^1 (2r - 2r^3) dr = \left[r^2 - \frac{1}{2}r^4\right]_0^1 = \left(1^2 - \frac{1}{2} \cdot 1^4\right) - (0) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Substitute this result into the integral:

$$m = \int_0^{2\pi} \frac{1}{2} d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \frac{1}{2} \cdot 2\pi = \pi.$$

Thus, the mass of the solid G is π.

To compute the z-coordinate of the center of mass, $\overline{z}$, for the solid cone G with density $\rho(x, y, z) = z$, use the formula:

$$\overline{z} = \frac{1}{\text{mass}} \iiint_G z\rho dV.$$

The mass of G has been previously calculated as π. Substituting the density $\rho(x, y, z) = z$ gives:

$$\overline{z} = \frac{1}{\pi} \iiint_G z \cdot z dV = \frac{1}{\pi} \iiint_G z^2 dV.$$

In cylindrical coordinates, the solid G is bounded by the cone $z = 2r$ and the plane $z = 2$, with $0 \le r \le 1$, $0 \le \theta \le 2\pi$, and for each (r, θ), z ranges from $2r$ to 2. The volume element is $dV = r dz dr d\theta$.

Thus, the triple integral is:

$$\iiint_G z^2 dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=2r}^{2} z^2 \cdot r dz dr d\theta.$$

Therefore, the setup for $\overline{z}$ is:

$$\boxed{\overline{z} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \int_{2r}^2 z^2 r dz dr d\theta}$$

In spherical coordinates, the solid G is bounded by the cone $z = 2\sqrt{x^2 + y^2}$ and the plane $z = 2$. The conversion to spherical coordinates is given by:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi,$$

with the volume element $dV = \rho^2 \sin\phi d\rho d\phi d\theta$.

The cone surface $z = 2\sqrt{x^2 + y^2}$ simplifies to $\rho \cos\phi = 2\rho \sin\phi$, which gives $\tan\phi = \frac{1}{2}$. Thus, ϕ ranges from 0 to $\phi_0 = \arctan(1/2)$. The plane $z = 2$ corresponds to $\rho \cos\phi = 2$, so $\rho = \frac{2}{\cos\phi}$. For each fixed ϕ and θ, ρ ranges from 0 to $\frac{2}{\cos\phi}$, and θ ranges from 0 to 2π.

The integrand z^2 in spherical coordinates is:

$$z^2 = (\rho \cos\phi)^2 = \rho^2 \cos^2\phi.$$

Thus, the integrand $z^2 dV$ becomes:

$$z^2 dV = \rho^2 \cos^2\phi \cdot \rho^2 \sin\phi d\rho d\phi d\theta = \rho^4 \cos^2\phi \sin\phi d\rho d\phi d\theta.$$

The triple integral for $\iiint_G z^2 dV$ is:

$$\iiint_G z^2 dV = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\arctan(1/2)} \int_{\rho=0}^{2/\cos\phi} \rho^4 \cos^2\phi \sin\phi d\rho d\phi d\theta.$$

Therefore, the setup for $\overline{z}$ is:

$$\boxed{\overline{z} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\arctan(1/2)} \int_0^{\frac{2}{\cos\phi}} \rho^4 \cos^2\phi \sin\phi d\rho d\phi d\theta}$$

Q3

(1-Q10) Find the area of the ellipse,

$$(2x + 5y - 3)^2 + (3x - 7y + 8)^2 = 1$$

Solution:

The equation $(2x + 5y - 3)^2 + (3x - 7y + 8)^2 = 1$ represents an ellipse. To find its area, define the transformations:

$$u = 2x + 5y - 3, \quad v = 3x - 7y + 8.$$

The equation becomes $u^2 + v^2 = 1$, which is a circle of radius 1 in the uv-plane with area $\pi \cdot 1^2 = \pi$.

The Jacobian matrix of the transformation from (x, y) to (u, v) is:

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ \\ 3 & -7 \end{pmatrix}.$$

The determinant of J is:

$$\det(J) = (2)(-7) - (5)(3) = -14 - 15 = -29.$$

The absolute value is $|\det(J)| = 29$.

The area element in the xy-plane relates to the area element in the uv-plane by:

$$dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \frac{1}{|\det(J)|} dudv = \frac{1}{29} dudv,$$

since $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{|\det(J)|}$.

The area in the xy-plane is the integral over the region where $u^2 + v^2 \leq 1$:

$$\text{Area} = \iint_{u^2+v^2\leq 1} dxdy = \iint_{u^2+v^2\leq 1} \frac{1}{29} dudv.$$

The integral is the area of the disk in the uv-plane:

$$\iint_{u^2+v^2\leq 1} dudv = \pi.$$

Thus,

$$\text{Area} = \frac{1}{29} \cdot \pi = \frac{\pi}{29}.$$

The constants in the transformations do not affect the area, as translations preserve area. The area is therefore $\frac{\pi}{29}$.

Q4

(Q-21) Let C be the portion of the cylinder $x^2 + y^2 \leq 1$ lying in the first octant ($x \geq 0, y \geq 0, z \geq 0$) and below the plane $z = 1$. Set up a triple integral in cylindrical coordinates which gives the moment of inertia of C about the z-axis; assume the density to be $\delta = 1$.

Solution:

The moment of inertia about the z-axis for a solid with constant density $\delta = 1$ is given by the triple integral:

$$I_z = \iiint_C (x^2 + y^2) dV,$$

where C is the region defined by $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$, and $z \leq 1$.

The transformations in cylindrical coordinates are:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

with the volume element $dV = r dr d\theta dz$. Additionally, $x^2 + y^2 = r^2$, so the integrand becomes r^2.

The region C is bounded as follows:

- The cylinder $x^2 + y^2 \leq 1$ corresponds to $r \leq 1$.
- The first octant ($x \geq 0$, $y \geq 0$) corresponds to $0 \leq \theta \leq \pi/2$.
- The conditions $z \geq 0$ and $z \leq 1$ correspond to $0 \leq z \leq 1$.

Thus, the limits of integration are:

- r from 0 to 1,
- θ from 0 to $\pi/2$,
- z from 0 to 1.

The integrand is r^2, and the volume element is $dV = r dr d\theta dz$, so the product is $r^2 \cdot r = r^3$. Therefore, the moment of inertia integral is:

$$\boxed{I_z = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \int_{z=0}^{1} r^3 dz dr d\theta}$$
