Multivariable Calculus: The Problem Set

1 General Problems

1. Let $\mathbf{r}(t) = \langle \cos e^t, \sin e^t, e^t \rangle$
   (a) Compute the unit tangent vector $\mathbf{T}(t)$
   (b) Compute $\mathbf{T}'(t)$
2. Let $\mathbf{F}(x, y, z) = (y + y^2 z)\hat{\mathbf{i}} + (x - z + 2xyz)\hat{\mathbf{j}} + (-y + xy^2)\hat{\mathbf{k}}$
   (a) Show that $\mathbf{F}(x, y, z)$ is a gradient field.
   (b) Find a potential function $f(x, y, z)$ for $\mathbf{F}(x, y, z)$ such that $\nabla f = \mathbf{F}$.
   (c) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is a straight line from $(2, 2, 1)$ to $(1, -1, 2)$.
3. In this problem, S is the surface given by the quarter of the right-circular cylinder centered on the z-axis, of radius 2 and height 4, which lies in the first octant. The field $\mathbf{F}(x, y, z) = x\mathbf{i}$.
   (a) Sketch the surface S and the field $\mathbf{F}$.
   (Suggestion: use a coordinate system with y pointing out of the paper.)
   (b) Compute the flux integral
   $$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$
   (Use the normal which points 'outward' from S, i.e., on the side away from the z-axis.)
   (c) Let G be the 3D solid in the first octant given by the interior of the quarter-cylinder defined above. Use the divergence theorem to compute the flux of the field $\mathbf{F} = x\mathbf{i}$ out of the region G.
   (d) The boundary surface of G is comprised of S together with four other faces. What is the flux outward through these four faces, and why? Use the answers to parts (b) and (c), and also verify using the sketch of part (a).
4. $\mathbf{F}(x, y, z) = (yz)\mathbf{i} + (-xz)\mathbf{j} + \mathbf{k}$. Let S be the portion of the surface of the paraboloid
   $$z = 4 - x^2 - y^2$$
   which lies above the first octant; and let C be the closed curve $C = C_1 + C_2 + C_3$, where the curves C_1, C_2, and C_3 are the three curves formed by intersecting S with the xy, yz, and xz planes respectively (so that C is the boundary of S). Orient C so that it is traversed counterclockwise when seen from above in the first octant.
   (a) Use Stokes' Theorem to compute
   $$\oint_C \mathbf{F} \cdot d\mathbf{r}$$
   by using the surface integral over the capping surface S.
   (b) Set up and evaluate the loop integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly by parametrizing each piece of the curve C and then adding up the three line integrals.
5. Let G be the solid 3-D cone bounded by the lateral surface given by $z = 2\sqrt{x^2 + y^2}$ and by the plane $z = 2$.
   (a) Find the mass of G using cylindrical coordinates.

(b) Set up the calculation for $\overline{z}$ using cylindrical coordinates.
(c) Set up the calculation for $\overline{z}$ using spherical coordinates

6. Evaluate

$$\int_0^\infty \frac{e^{-x} - e^{-ax}}{x} \cdot dx$$

7. Show that the average distance of the points of a disk of radius a to its center is $2a/3$.

8. In general, the moment of inertia around an axis (a line) L is,

$$I_L = \iint_R dist(\cdot, L)^2 \delta \cdot dA$$

The collection of lines parallel to the y-axis have the form $x = a$. Let $I = I_y$ be the usual moment of inertia around the y-axis,

$$I = \iint_R x^2 \delta \cdot dA$$

Let $\bar{I}$ be the moment of inertia around the axis $x = \bar{x}$, where $(\bar{x}, \bar{y})$ is the center of mass. Show that

$$I = \bar{I} + M\bar{x}^2$$

9. Find the average area of an inscribed triangle in the unit circle.

10. Find the area of the ellipse,

$$(2x + 5y - 3)^2 + (3x - 7y + 8)^2 = 1$$

11. Consider the vector field $\vec{F} = (x^2 y + \frac{1}{3}y^3)\hat{\imath}$, and let C be the portion of the graph $y = f(x)$ running from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ (assume that $x_1 < x_2$, and f takes positive values). Show that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is equal to the polar moment of inertia of the region R lying below C and above the x-axis (with density $\delta = 1$).

12. Consider the vector field

$$\vec{F} = \frac{-y\hat{\imath} + x\hat{\jmath}}{x^2 + y^2}$$

(a) Show that $\vec{F}$ is the gradient of the polar function $\theta(x, y) = \arctan(y/x)$ over the right half-plane $x > 0$.
(b) Suppose that C is a smooth curve in the right half-plane $x > 0$ joining two points $A : (x_1, y_1)$ and $B : (x_2, y_2)$. Express $\int_C \vec{F} \cdot d\vec{r}$ in terms of the polar coordinates (r_1, θ_1) and (r_2, θ_2) of A and B.
(c) Compute directly from the definition the line integrals $\int_{C_1} \vec{F} \cdot d\vec{r}$ and $\int_{C_2} \vec{F} \cdot d\vec{r}$ where C_1 is the upper half of the unit circle running from $(1, 0)$ to $(-1, 0)$ and C_2 is the lower half of the unit circle, also going from $(1, 0)$ to $(-1, 0)$.

13. (a) Calculate the curl of $\vec{F} = r^n(x\hat{\imath} + y\hat{\jmath})$.
(b) Whenever possible, find a potential g such that $\vec{F} = \nabla g$.

14. Show that a constant force field does zero work on a particle that winds uniformly w times around the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

15. Find the flux of the vector field $\vec{F} = \frac{x}{x^2+y^2}\hat{\imath} + \frac{y}{x^2+y^2}\hat{\jmath}$ outwards through any circle centered at $(1, 0)$ of radius $r \neq 1$.

16. Evaluate the surface integral $\iint_\Sigma \vec{F} \cdot \hat{n} \cdot dS$ where Σ is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation and $\vec{F} = x\hat{\imath} + y\hat{\jmath} + z^4\hat{k}$.

17. (a) Let $f(x, y, z) = 1/\rho = (x^2 + y^2 + z^2)^{-1/2}$. Calculate $\vec{F} = \nabla f$.
    (b) Evaluate the flux of $\vec{F}$ over the sphere of radius a centered at the origin.
    (c) Show that $div(\vec{F}) = 0$. Does this violate the divergence theorem?

18. (a) Show that the vector field

$$\vec{F} = (e^x yz)\hat{\imath} + (e^x z + 2yz)\hat{\jmath} + (e^x y + y^2 + 1)\hat{k}$$

    is conservative.
    (b) Find the potential of $\vec{F}$.
    (c) Show that the vector field

$$\vec{G} = y\hat{\imath} + x\hat{\jmath} + y\hat{z}$$

    is not conservative.

19. Let S be the part of the spherical surface $x^2 + y^2 + z^2 = 4$, lying in $x^2 + y^2 < 1$, which is to say outside the cylinder of radius one with axis the z-axis.
    (a) Compute the flux outward through S of the vector field $\vec{F} = y\hat{\imath} - x\hat{\jmath} + z\hat{k}$.
    (b) Show that the flux of this vector field through any part of the cylindrical surface is zero.
    (c) Using the divergence theorem applied to $\vec{F}$, compute the volume of the region between S and the cylinder.

20. Use the divergence theorem to compute the flux of $\vec{F} = \hat{\imath} + \hat{\jmath} + \hat{k}$ outwards across the closed surface $x^4 + y^4 + z^4 = 1$.

21. Let C be the portion of the cylinder $x^2 + y^2 \leq 1$ lying in the first octant $(x \geq 0, y \geq 0, z \geq 0)$ and below the plane $z = 1$. Set up a triple integral in cylindrical coordinates which gives the moment of inertia of C about the z-axis; assume the density to be $\delta = 1$.

22. Let S be the part of the spherical surface $x^2 + y^2 + z^2 = 2$ lying in $z > 1$. Orient S upwards and give its bounding circle, C, lying in $z = 1$ the compatible orientation.
    (a) Parametrize C and use the parametrization to evaluate

$$I = \oint xz \cdot dx + y \cdot (dy + dz)$$

    (b) Compute the curl of the vector field $\mathbf{F} = xz\hat{\mathbf{i}} + yxz\hat{\mathbf{j}} + y\hat{\mathbf{k}}$
    (c) Write down a flux integral through S which can be computed using the value of I.

23. Use the divergence theorem to compute the flux of $\mathbf{F} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ outwards across the closed surface $x^4 + y^4 + z^4 = 1$.

24. Consider the surface S given by the equation

$$z = (x^2 + y^2 + z^2)^2$$

    (a) Show that S lies in the upper half space $z > 0$.
    (b) Write out the equation for the surface in spherical polar coordinates.
    (c) Using the equation obtained in part b), give an iterated integral, with explicit integrand and limits of integration, which gives the volume of the region inside this surface.

25. Let S be the part of the surface $z = xy$ where $x^2 + y^2 < 1$. Compute the flux of $\mathbf{F} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ upward across S by reducing the surface integral to a double integral over the disk $x^2 + y^2 < 1$.

26. Let

$$\mathbf{F}(x, y, z) = \left(\frac{-z}{x^2 + z^2}\right)\mathbf{i} + (y)\,\mathbf{j} + \left(\frac{x}{x^2 + z^2}\right)\mathbf{k}$$

    defined for all points (x, y, z) in 3-space not on the y-axis (that is, all points for which $x^2 + z^2 > 0$).

(a) By direct computation, show that $\nabla \times \mathbf{F} = \mathbf{0}$ for all points not on the y-axis.

(b) By direct computation, show that $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ where C_1 is the closed curve defined by $x^2 + y^2 = 1, z = 1$.

(c) Can you use Stokes' Theorem and the fact (from part (a)) that $\nabla \times \mathbf{F} = \mathbf{0}$ to conclude that $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$ when C_2 is the closed curve defined by $x^2 + z^2 = 1, y = 0$? Why/why not?

(d) Compute $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$ and see what happens.

27. Suppose the field $\mathbf{F}$ in the previous problem is replaced by the field

$$\mathbf{G}(x, y, z) = \left(\frac{x}{x^2 + y^2 + z^2}\right)\mathbf{i} + \left(\frac{y}{x^2 + y^2 + z^2}\right)\mathbf{j} + \left(\frac{z}{x^2 + y^2 + z^2}\right)\mathbf{k}$$

defined for all points $(x, y, z) \neq (0, 0, 0)$.

(a) Show that $\nabla \times \mathbf{G} = 0$ for all points $(x, y, z) \neq (0, 0, 0)$.

(b) Can you use Stokes' Theorem in this case and the fact that $\nabla \times \mathbf{G} = 0$ for all points $(x, y, z) \neq (0, 0, 0)$ to conclude that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all simple closed curves C which do not pass through the origin?

(c) Explain the difference between these two cases in terms of the connectedness type of the domains of definition of the two fields.

28. (a) Show that this flow $\mathbf{F}$ satisfies the *equation of continuity*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{F} = 0.$$

In general, for non-steady flows, the divergence $\nabla \cdot \mathbf{F}$ of the field $\mathbf{F}(x, y, z, t)$ is defined using differentiation

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

with respect to the *space* variables only. For this reason, at any fixed time t, the Divergence Theorem holds, and the physical interpretations of the divergence $\nabla \cdot \mathbf{F}$ of the field and of the flux of $\mathbf{F}$ through a surface are the same for the case of steady flows.

By the same token, Stokes' Theorem holds at each fixed time t, and hence the physical interpretation of the curl $\nabla \times \mathbf{F}$ is the same for the case of steady flows.

(b) Referring to the facts about non-steady flows given above: what is the net outward flux of $\mathbf{F}$ through any simple closed surface, and why?

29. (a) Compute the curl $\nabla \times \mathbf{F}$ of $\mathbf{F}$.

(b) Referring to the facts about non-steady flows given above: at any fixed time t and position (x, y, z), what is the axial direction perpendicular to the plane in which the flow produces the maximum angular velocity, and what is the maximum angular velocity? Does it vary at the different points (x, y, z) in space?

(c) Show that at any time t:

(i) the velocity vectors $\mathbf{v}(x, y, z, t)$ of the flow lie in the plane through the origin with normal $\mathbf{n}_t = \langle \cos t, \sin t, 0 \rangle$ (which we'll call $\mathcal{P}_t$); and

(ii) the velocity vectors $\mathbf{v}(x, y, z, t)$ are perpendicular to the radial vectors $\mathbf{r} = \langle x, y, z \rangle$.

Combining the result of part (b) with (i): what is the plane of greatest spin for this flow?

(d) Use the results of part (c) to give a sketch of some representative velocity vectors in the plane $\mathcal{P}_t$ at a fixed time t. (Take t to be about $\frac{\pi}{4}$, and first sketch $\mathcal{P}_t$ in 3-space; then for the purposes of sketching the velocity field, you can take $\mathcal{P}_t$ to be the plane of the paper.)

(e) Putting together the results above, describe in general terms the pattern of the motion of the flow in 3-space over time.

30. Show that the average straight-line distance to a fixed point on the surface of a sphere of radius a is $4a/3$.

31. (a) Consider a solid in the shape of an ice-cream cone. It is bounded above by (part of) a sphere of radius a centered at the origin. It is bounded below by the cone with vertex at the origin, vertex angle $2\phi_0$, and slant height a.

    Find the gravitational force on a unit test mass placed at the origin. (Assume density $\delta = 1$).

    (b) Continuing with the solid: take $a = \sqrt{2}$ and the vertex angle to be $\frac{\pi}{2}$ (so $\phi_0 = \frac{\pi}{4}$). Let $\mathbf{F} = z\mathbf{k}$.

    i. Let T be the horizontal disk with boundary the intersection of the sphere and the cone. Compute directly the upward flux of $\mathbf{F}$ through T.

    ii. Let U be the boundary of the conical lower surface, S the upper spherical cap of the solid. Use the divergence theorem and part (a) to compute the upward flux of $\mathbf{F}$ through U and S. (You will need to be careful with signs.)

    iii. Set up, but do not compute, the integral for the flux of $\mathbf{F}$ through U. Write the integral in cylindrical coordinates.

32. Let $f(x, y, z) = \frac{1}{\rho}$.

    (a) Compute $\mathbf{F} = \nabla f$ and show that $\nabla \cdot \mathbf{F} = 0$.

    (b) Find the outward flux of $\mathbf{F}$ through the sphere of radius a centered at the origin. Why does this not contradict the divergence theorem?

    (c) Use the extended divergence theorem to show that the flux of $\mathbf{F}$ through any closed surface surrounding the origin is -4π.

33. The Laplacian of a function of three variables is defined by

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz}.$$

Suppose that the simple closed surface S is the iso-surface of some smooth function $f(x, y, z)$, that is, the set of points in 3-space satisfying $f(x, y, z) = c$ for some constant c.

Use the Divergence Theorem to show that if G is the interior of S, then

$$\iint_S |\nabla f| \, dS = \pm \iiint_G \nabla^2 f \, dV.$$

34. Let $\mathcal{G}$ be the solid region in 3-space which lies inside the surface

$$x^2 + (y - 1)^2 = 1,$$

above $z = 0$, and below the surface

$$z = \sqrt{x^2 + y^2}.$$

    (a) Find the volume of $\mathcal{G}$.

    (b) Find the z-coordinate of the centroid of $\mathcal{G}$.

35.

$$\iiint\limits_G f(x, y, z) \, dV = \int_0^2 \int_0^{2x} \int_0^{\sqrt{4-x^2}} f(x, y, z) \, dz \, dy \, dx.$$

    (a) Sketch the 3-D solid region G.

    (b) Rewrite $\iiint\limits_G f(x, y, z) \, dV$ as a triple integral with the correct limits of integration in the order:

$$\iiint f(x, y, z) \, dy \, dx \, dz$$

    (c) Reorder the integral as:

$$\iiint f(x, y, z) \, dx \, dz \, dy$$

    (d) Take $f = 1$ (constant) and verify that you get the same answer (which is the volume of G) from:

    (i) The iterated integrals in the given order.

    (ii) The iterated integrals you found in part (b).

36. (a) Find the curve of intersection of the surfaces

$$z = x^2 - y^2 \quad \text{and} \quad z = 2 + (x - y)^2$$

in parametric form.

(b) Find the angle of intersection of these two surfaces at the point $(2, 1, 3)$. (The angle of intersection of two surfaces is defined to be the angle made by their tangent planes.)

(c) Check that the tangent vector to the curve of intersection found in part (a) at the point $(2, 1, 3)$ lies in (i.e., is parallel to) the tangent plane of each of the two surfaces.

37. Suppose that three non-negative numbers are restricted by the condition that the sum of their squares is equal to 27. Using critical point analysis, with 2^{nd} derivative and/or boundary tests as needed, find the maximum and minimum values of the sum of their cubes.

38. Let $\mathbf{F} = (ax^2y + y^3 + 1)\mathbf{i} + (2x^3 + bxy^2 + 2)\mathbf{j}$ be a vector field, where a and b are constants.

(a) Find the values of a and b for which $\mathbf{F}$ is conservative.

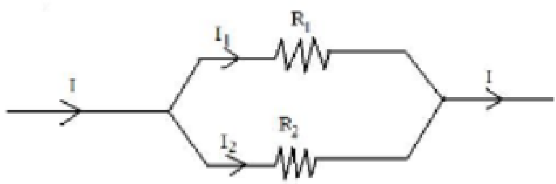
(b) For these values of a and b, find $f(x, y)$ such that $\mathbf{F} = \nabla f$.

(c) Still using the values of a and b from part (a), compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

along the curve C such that $x = e^t \cos t$, $y = e^t \sin t$, $0 \leq t \leq \pi$.

39. A current I flowing over a resistor R results in an energy loss (in the form of heat/light) equal to I^2R per second. It turns out that, in a sense, "electricity prefers to flow in the way that minimizes energy loss to resistance". For example, when an electric current comes to a fork, it will divide itself up in such a way that a large portion of the current flows where the resistance is low and a small portion flows where the resistance is high (you might think all the electricity would flow where the resistance is low but the energy loss is proportional to I^2 so it is better to spread the current around). Suppose we have the following situation where a current I comes to a pair of resistors in parallel:



(a) Determine the choice of currents that will minimize the loss in energy and hence determine the currents flowing along the two paths.

(b) Suppose instead we had three resistors in parallel. In terms of R_1, R_2, and R_3 determine the values of I_1, I_2, and I_3 which minimize the loss in energy.

40. Using the usual rectangular and polar coordinates, let w be the area of the right triangle in the first quadrant having its vertices at $(0, 0)$, $(x, 0)$ and (x, y). Using the equation expressing w in terms of x and y and the equations expressing y in terms of x and θ, calculate the two partial derivatives

$$\left(\frac{\partial w}{\partial x}\right)_\theta \quad \text{and} \quad \left(\frac{\partial w}{\partial \theta}\right)_x$$

in three different ways.

(a) Directly, by first expressing w in terms of the independent variables x and θ.

(b) By using the chain rule – for example

$$\left(\frac{\partial w}{\partial x}\right)_\theta = w_x \left(\frac{\partial x}{\partial x}\right)_\theta + w_y \left(\frac{\partial y}{\partial x}\right)_\theta,$$

where w_x and w_y are the formal partial derivatives.

(c) By using differentials.

(d) Using the triangle picture and geometric intuition, estimate

$$\left(\frac{\Delta w}{\Delta x}\right)_\theta \quad \text{and} \quad \left(\frac{\Delta w}{\Delta \theta}\right)_x$$

and show they agree with the two corresponding partial derivatives.

41. A ladybug is climbing on a Volkswagen Bug (= VW). In its starting position, the surface of the VW is represented by the unit semicircle

$$x^2 + y^2 = 1, \quad y \geq 0$$

in the xy-plane. The road is represented as the x-axis. At time $t = 0$, the ladybug starts at the front bumper, $(1, 0)$, and walks counterclockwise around the VW at unit speed relative to the VW. At the same time, the VW moves to the right at speed 10.

(a) Find the parametric formula for the trajectory of the ladybug, and find its position when it reaches the rear bumper. (At $t = 0$, the rear bumper is at $(-1, 0)$.)

(b) Compute the speed of the bug, and find where it is largest and smallest. Hint: It is easier to work with the square of the speed.

42. Find the point on the plane

$$2x + y - z = 6$$

which is closest to the origin, by using Lagrange multipliers.

2 Vectors

1. Consider the following time-parametrized curve:

$$\mathbf{r}(t) = \left(\cos\left(\frac{\pi}{4}t\right), (t-5)^2\right)$$

   List the three points $(-1/\sqrt{2}, 0), (1, 25), (0, 25)$ in chronological order.

2. At what points in the xy-plane does the curve $(\sin t, t^2)$ cross itself? What is the difference in t between the first time the curve crosses through a point, and the last?

3. Find the specified parametrization of the first quadrant part of the circle $x^2 + y^2 = a^2$.

   (a) In terms of the y coordinate.

   (b) In terms of the angle between the tangent line and the positive x-axis.

   (c) In terms of the arc length from $(0, a)$.

4. The curve C is defined to be the intersection of the hyperboloid

$$x^2 - \frac{1}{4}y^2 + 3z^2 = 1$$

   and the plane

$$x + y + z = 0.$$

   When y is very close to 0, and z is negative, find an expression giving z in terms of y.

5. A particle traces out a curve in space, so that its position at time t is

$$\mathbf{r}(t) = e^{-t}\hat{\imath} + \frac{1}{t}\hat{\jmath} + (t-1)^2(t-3)^2\hat{k}$$

   for $t > 0$.

   Let the positive z-axis point vertically upwards, as usual. When is the particle moving upwards, and when is it moving downwards? Is it moving faster at time $t = 1$ or at time $t = 3$?

6. Let

$$\mathbf{r}(t) = \left(t^2, 3, \frac{1}{3}t^3\right)$$

   Find the unit tangent vector to this parametrized curve at $t = 1$, pointing in the direction of increasing t.

   Find the arc length of the curve from (a) between the points $(0, 3, 0)$ and $(1, 3, -\frac{1}{3})$.

7. A particle's position at time t is given by $\mathbf{r}(t) = (t + \sin t, \cos t)$. What is the magnitude of the acceleration of the particle at time t?

8. A curve in $\mathbb{R}^3$ is given by the vector equation $\mathbf{r}(t) = \left(2t\cos t, 2t\sin t, \frac{t^3}{3}\right)$.

   (a) Find the length of the curve between $t = 0$ and $t = 2$.

   (b) Find the parametric equations of the tangent line to the curve at $t = \pi$.

9. Let $\mathbf{r}(t) = (3\cos t, 3\sin t, 4t)$ be the position vector of a particle as a function of time $t \geq 0$.

   (a) Find the velocity of the particle as a function of time t.

   (b) Find the arclength of its path between $t = 1$ and $t = 2$.

10. Consider the curve

$$\mathbf{r}(t) = \frac{1}{3}\cos^3 t\hat{\imath} + \frac{1}{3}\sin^3 t\hat{\jmath} + \sin^3 t\hat{k}$$

   (a) Compute the arc length of the curve from $t = 0$ to $t = \frac{\pi}{2}$.

   (b) Compute the arc length of the curve from $t = 0$ to $t = \pi$.

11. A particle moves along the curve C of intersection of the surfaces $z^2 = 12y$ and $18x = yz$ in the upward direction. When the particle is at $(1, 3, 6)$ its velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are given by

$$\mathbf{v} = 6\hat{\imath} + 12\hat{\jmath} + 12\hat{k}, \quad \mathbf{a} = 27\hat{\imath} + 30\hat{\jmath} + 6\hat{k}.$$

(a) Write a vector parametric equation for C using $u = \frac{z}{6}$ as a parameter.

(b) Find the length of C from $(0, 0, 0)$ to $(1, 3, 6)$.

12. A camera mounted to a pole can swivel around in a full circle. It is tracking an object whose position at time t seconds is $x(t)$ metres east of the pole, and $y(t)$ metres north of the pole.

In order to always be pointing directly at the object, how fast should the camera be programmed to rotate at time t? (Give your answer in terms of $x(t)$ and $y(t)$ and their derivatives, in the units rad/sec.)

13. A projectile falling under the influence of gravity and slowed by air resistance proportional to its speed has position satisfying

$$\frac{d^2r}{dt^2} = -g\mathbf{k} - \alpha\frac{dr}{dt}$$

where α is a positive constant. If $r = r_0$ and $\frac{dr}{dt} = v_0$ at time $t = 0$, find $r(t)$. (Hint: Define $u(t) = e^{\alpha t}\frac{dr}{dt}$ and substitute $\frac{d}{dt}(u(t)) = e^{-\alpha t}u(t)$ into the given differential equation to find a differential equation for u.)

14. At time $t = 0$ a particle has position and velocity vectors $\mathbf{r}(0) = (-1, 0, 0)$ and $\mathbf{v}(0) = (0, -1, 1)$. At time t, the particle has acceleration vector

$$\mathbf{a}(t) = \langle \cos t, \sin t, 0 \rangle$$

(a) Find the position of the particle after t seconds.

(b) Show that the velocity and acceleration of the particle are always perpendicular for every t.

(c) Find the equation of the tangent line to the particle's path at $t = -\pi/2$.

(d) True or False: None of the lines tangent to the path of the particle pass through $(0, 0, 0)$. Justify your answer.

15. The position of a particle at time t (measured in seconds) is given by

$$\mathbf{r}(t) = t\cos\left(\frac{\pi t}{2}\right)\hat{\imath} + t\sin\left(\frac{\pi t}{2}\right)\hat{\jmath} + t\hat{k}$$

(a) Show that the path of the particle lies on the cone $z^2 = x^2 + y^2$.

(b) Find the velocity vector and the speed at time t.

(c) Suppose that at time $t = 1$ the particle flies off the path on a line L in the direction tangent to the path. Find the equation of the line L.

(d) How long does it take for the particle to hit the plane $x = -1$ after it started moving along the straight line L?

16. (a) The curve $\mathbf{r}_1(t) = (1+t, t^2, t^3)$ and $\mathbf{r}_2(t) = (\cos t, \sin t, t)$ intersect at the point $P(1, 0, 0)$. Find the angle of intersection between the curves at the point P.

(b) Find the distance between the line of intersection of the planes $x + y = 2$ and $2x - z = 4$ and the line $\mathbf{r}(t) = (t, -1 + 2t, 1 + 3t)$.

3 Differenciation

1. Let $f(x)$ and $g(x)$ be two functions of x satisfying $f''(7) = -2$ and $g''(-4) = -1$. If $z = h(s,t) = f(2s+3t) + g(s-6t)$ is a function of s and t, find the value of $\frac{\partial^2 z}{\partial t^2}$ when $s = 2$ and $t = 1$.

2. Suppose that $w = f(xz, yz)$, where f is a differentiable function. Show that
$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = z\frac{\partial w}{\partial z}.$$

3. Suppose $z = f(x,y)$ has continuous second order partial derivatives, and $x = r\cos t, y = r\sin t$. Express the following partial derivatives in terms r, t, and partial derivatives of f.
   (a) $\frac{\partial z}{\partial t}$
   (b) $\frac{\partial^2 z}{\partial t^2}$

4. Let $z = f(x,y)$, where $f(x,y)$ has continuous second-order partial derivatives, and
$$f_x(2,1) = 5, \quad f_y(2,1) = -2, \quad f_{xx}(2,1) = 2, \quad f_{xy}(2,1) = 1, \quad f_{yy}(2,1) = -4.$$
Find $\frac{d^2}{dt^2} z(x(t), y(t))$ when $x(t) = 2t^2$, $y(t) = t^3$ and $t = 1$.

5. Assume that the function $F(x,y,z)$ satisfies the equation
$$\frac{\partial F}{\partial z} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$$
and the mixed partial derivatives $\frac{\partial^2 F}{\partial x \partial y}$ and $\frac{\partial^2 F}{\partial y \partial x}$ are equal. Let A be some constant and let
$$G(\gamma, t) = F(\gamma + s, \gamma - s, At).$$
Find the value of A such that
$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial \gamma^2} + \frac{\partial^2 G}{\partial s^2}.$$

6. Let $f(x)$ be a differentiable function, and suppose it is given that $f'(0) = 10$. Let $g(s,t) = f(as-bt)$, where a and b are constants. Evaluate $\frac{\partial g}{\partial s}$ at the point $(s,t) = (b,a)$, that is, find $\frac{\partial g}{\partial s}\big|_{(b,a)}$.

7. Let $f(u,v)$ be a differentiable function of two variables, and let z be a differentiable function of x and y defined implicitly by $f(xz, yz) = 0$. Show that
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -z.$$

8. The wave equation
$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$
arises in many models involving wave-like phenomena. Let $u(x,t)$ and $v(\xi,\eta)$ be related by the change of variables
$$u(x,t) = v(\xi(x,t), \eta(x,t))$$
$$\xi(x,t) = x - ct$$
$$\eta(x,t) = x + ct$$
   (a) Show that $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$ if and only if $\frac{\partial^2 v}{\partial \xi \partial \eta} = 0$.
   (b) Show that $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$ if and only if $u(x,t) = F(x-ct) + G(x+ct)$ for some functions F and G.
   (c) Interpret $F(x-ct) + G(x+ct)$ in terms of travelling waves.

9. Evaluate

(a) $\frac{\partial y}{\partial z}$ if $e^{yz} - x^2 z \ln y = \pi$

(b) $\frac{dy}{dx}$ if $F(x, y, x^2 - y^2) = 0$

(c) $\left(\frac{\partial y}{\partial x}\right)_u$ if $xyw = 1$ and $x + y + u + v = 0$

10. Find all horizontal planes that are tangent to the surface with equation

$$z = xye^{-(x^2+y^2)/2}$$

What are the largest and smallest values of z on this surface?

11. Let S be the surface

$$xy - 2x + yz + x^2 + y^2 + z^3 = 7$$

(a) Find the tangent plane and normal line to the surface S at the point $(0, 2, 1)$.

(b) The equation defining S implicitly defines z as a function of x and y. Find expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Evaluate $\frac{\partial z}{\partial y}$ at $(x, y, z) = (0, 2, 1)$.

(c) Find an expression for $\frac{\partial^2 z}{\partial x \partial y}$.

12. (a) Find a vector perpendicular at the point $(1, 1, 3)$ to the surface with equation $x^2 + z^2 = 10$.

(b) Find a vector tangent at the same point to the curve of intersection of the surface in part (a) with surface $y^2 + z^2 = 10$.

(c) Find parametric equations for the line tangent to that curve at that point.

13. Let P be the point where the curve

$$\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, \quad (0 \le t < \infty)$$

intersects the surface

$$z^3 + xyz - 2 = 0$$

Find the (acute) angle between the curve and the surface at P.

14. Find the distance from the point $(1, 1, 0)$ to the circular paraboloid with equation $z = x^2 + y^2$.

15. Suppose it is known that the direction of the fastest increase of the function $f(x, y)$ at the origin is given by the vector $\langle 1, 2\rangle$. Find a unit vector $\mathbf{u}$ that is tangent to the level curve of $f(x, y)$ that passes through the origin.

16. Let the pressure P and temperature T at a point (x, y, z) be

$$P(x, y, z) = \frac{x^2 + 2y^2}{1 + z^2}, \quad T(x, y, z) = 5 + xy - z^2$$

(a) If the position of an airplane at time t is

$$(x(t), y(t), z(t)) = (2t, t^2 - 1, \cos t)$$

find $\frac{d}{dt}(PT)^2$ at time $t = 0$ as observed from the airplane.

(b) In which direction should a bird at the point $(0, -1, 1)$ fly if it wants to keep both P and T constant? (Give one possible direction vector. It does not need to be a unit vector.)

(c) An ant crawls on the surface $z^3 + zx + y^2 = 2$. When the ant is at the point $(0, -1, 1)$, in which direction should it go for maximum increase of the temperature $T = 5 + xy - z^2$? Your answer should be a vector $\langle a, b, c\rangle$, not necessarily of unit length.

17. Find all saddle points, local minima and local maxima of the function

$$f(x, y) = x^3 + x^2 - 2xy + y^2 - x.$$

18. For the surface
$$z = f(x, y) = x^3 + xy^2 - 3x^2 - 4y^2 + 4$$
Find and classify all critical points of $f(x, y)$.

19. Find the maximum and minimum values of $f(x, y) = xy - x^3y^2$ when (x, y) runs over the square $0 \le x \le 1, 0 \le y \le 1$.

20. The temperature at all points in the disc $x^2 + y^2 \le 1$ is given by $T(x, y) = (x + y)e^{-x^2-y^2}$. Find the maximum and minimum temperatures at points of the disc.

21. Find the maximum and minimum values of the function $f(x, y, z) = x + y - z$ on the sphere $x^2 + y^2 + z^2 = 1$.

22. Find a, b, and c so that the volume $\frac{4\pi}{3}abc$ of an ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
passing through the point $(1, 2, 1)$ is as small as possible.

23. Use the Method of Lagrange Multipliers to find the minimum value of $z = x^2 + y^2$ subject to $x^2y = 1$. At which point or points does the minimum occur?

24. Use the Method of Lagrange Multipliers to find the radius of the base and the height of a right circular cylinder of maximum volume which can be fit inside the unit sphere $x^2 + y^2 + z^2 = 1$.

25. Use the method of Lagrange Multipliers to find the maximum and minimum values of
$$f(x, y) = xy$$
subject to the constraint
$$x^2 + 2y^2 = 1.$$

26. Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to the constraint $x^4 + y^4 = 1$.

27. Use Lagrange multipliers to find the points on the sphere $z^2 + x^2 + y^2 - 2y - 10 = 0$ closest to and farthest from the point $(1, -2, 1)$.

28. Use Lagrange multipliers to find the maximum and minimum values of the function
$$f(x, y, z) = x^2 + y^2 - \frac{1}{2}z^2$$
on the curve of intersection of the plane $x + 2y + z = 10$ and the paraboloid $x^2 + y^2 - z = 0$.

29. Find the point $P = (x, y, z)$ (with x, y, and $z > 0$) on the surface $x^3y^2z = 6\sqrt{3}$ that is closest to the origin.

30. Find the maximum value of $f(x, y, z) = xyz$ on the ellipsoid
$$g(x, y, z) = x^2 + xy + y^2 + 3z^2 = 9$$
Specify all points at which this maximum value occurs.

4 Integration

1. Use polar coordinates to evaluate each of the following integrals.

   (a) $\iint\limits_S (x+y)\,dx\,dy$ where S is the region in the first quadrant lying inside the disc $x^2+y^2 \le a^2$ and under the line $y=\sqrt{3}x$.

   (b) $\iint\limits_S x\,dx\,dy$, where S is the disc segment $x^2+y^2 \le 2$, $x \ge 1$.

   (c) $\iint\limits_T (x^2+y^2)\,dx\,dy$ where T is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

   (d) $\iint\limits_{x^2+y^2\le 1} \ln(x^2+y^2)\,dx\,dy$

2. Find the volume lying inside the sphere $x^2+y^2+z^2=2$ and above the paraboloid $z=x^2+y^2$.

3. Let $a>0$. Find the volume lying inside the cylinder $x^2+(y-a)^2=a^2$ and between the upper and lower halves of the cone $z^2=x^2+y^2$.

4. Let $a>0$. Find the volume common to the cylinders $x^2+y^2 \le 2ax$ and $z^2 \le 2ax$.

5. Consider the region E in 3-dimensions specified by the inequalities

$$x^2+y^2 \le 2y \quad \text{and} \quad 0 \le z \le \sqrt{x^2+y^2}.$$

6. Use polar coordinates to evaluate each of the following integrals.

   (a) $\iint\limits_S (x+y)\,dx\,dy$ where S is the region in the first quadrant lying inside the disc $x^2+y^2 \le a^2$ and under the line $y=\sqrt{3}x$.

   (b) $\iint\limits_S x\,dx\,dy$, where S is the disc segment $x^2+y^2 \le 2$, $x \ge 1$.

   (c) $\iint\limits_T (x^2+y^2)\,dx\,dy$ where T is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

   (d) $\iint\limits_{x^2+y^2\le 1} \ln(x^2+y^2)\,dx\,dy$

7. Find the volume lying inside the sphere $x^2+y^2+z^2=2$ and above the paraboloid $z=x^2+y^2$.

8. Let $a>0$. Find the volume lying inside the cylinder $x^2+(y-a)^2=a^2$ and between the upper and lower halves of the cone $z^2=x^2+y^2$.

9. Let $a>0$. Find the volume common to the cylinders $x^2+y^2 \le 2ax$ and $z^2 \le 2ax$.

10. Consider the region E in 3-dimensions specified by the inequalities

$$x^2+y^2 \le 2y \quad \text{and} \quad 0 \le z \le \sqrt{x^2+y^2}.$$

11. Find the area of the part of the surface $z=y^{3/2}$ that lies above $0 \le x, y \le 1$.

12. Find the surface area of the part of the paraboloid $z=a^2-x^2-y^2$ which lies above the xy-plane.

13. Find the area of the portion of the cone $z^2=x^2+y^2$ lying between the planes $z=2$ and $z=3$.

14. Determine the surface area of the surface given by

$$z=\frac{2}{3}(x^{3/2}+y^{3/2}),$$

    over the square $0 \le x \le 1, 0 \le y \le 1$.

15. (a) To find the surface area of the surface $z=f(x,y)$ above the region D, we integrate

$$\iint\limits_D F(x,y)\,dA.$$

    What is $F(x,y)$?

(b) Consider a "Death Star," a ball of radius 2 centred at the origin with another ball of radius 2 centred at $(0, 0, 2\sqrt{3})$ cut out of it. The diagram below shows the slice where $y = 0$.

16. Evaluate

$$\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{-\sqrt{1-x^2}}\int_{1+\sqrt{1-x^2-y^2}}^{1-\sqrt{1-x^2-y^2}} (x^2 + y^2 + z^2)^{5/2}\, dz\, dy\, dx$$

by changing to spherical coordinates.

17. Evaluate the volume of a circular cylinder of radius a and height h by means of an integral in spherical coordinates.

18. Let B denote the region inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $x^2 + y^2 = z^2$. Compute the moment of inertia

$$\iiint_B z^2\, dV.$$

19. (a) Evaluate

$$\iiint_\Omega z\, dV$$

where Ω is the three-dimensional region in the first octant $x \geq 0, y \geq 0, z \geq 0$, occupying the inside of the sphere $x^2 + y^2 + z^2 = 1$.

(b) Use the result in part (a) to quickly determine the centroid of a hemispherical ball given by $z \geq 0, x^2 + y^2 + z^2 \leq 1$.

20. The density of hydrogen gas in a region of space is given by the formula

$$\rho(x, y, z) = \frac{z + 2x^2}{1 + x^2 + y^2}$$

(a) At $(1, 0, -1)$, in which direction is the density of hydrogen increasing most rapidly?

(b) You are in a spacecraft at the origin. Suppose the spacecraft flies in the direction of $\langle 0, 0, 1\rangle$. It has a disc of radius 1, centred on the spacecraft and deployed perpendicular to the direction of travel, to catch hydrogen. How much hydrogen has been collected by the time that the spacecraft has traveled a distance 2?

You may use the fact that

$$\int_0^{2\pi} \cos^2\theta\, d\theta = \pi.$$

21. A torus of mass M is generated by rotating a circle of radius a about an axis in its plane at distance b from the centre $(b > a)$. The torus has constant density. Find the moment of inertia about the axis of rotation. By definition the moment of inertia is

$$\iiint r^2\, dm$$

where dm is the mass of an infinitesimal piece of the solid and r is its distance from the axis.

22. Consider the top half of a ball of radius 2 centred at the origin. Suppose that the ball has variable density equal to $9z$ units of mass per unit volume.

(a) Set up a triple integral giving the mass of this half-ball.

(b) Find out what fraction of that mass lies inside the cone

$$z = \sqrt{x^2 + y^2}.$$