A03

Q1

(1-Q3) In this problem, S is the surface given by the quarter of the right-circular cylinder centered on the z-axis, of radius 2 and height 4, which lies in the first octant. The field $\mathbf{F}(x, y, z) = x\mathbf{i}$.

- Compute the flux integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

- (Use the normal which points 'outward' from S, i.e., on the side away from the z-axis.)
- Let G be the 3D solid in the first octant given by the interior of the quarter-cylinder defined above. Use the divergence theorem to compute the flux of the field $\mathbf{F} = x\mathbf{i}$ out of the region G.
- The boundary surface of$G$is comprised of S together with four other faces. What is the flux outward through these four faces, and why? Use the answers to parts (b) and (c).

Solution:

The surface S is the lateral surface of a quarter cylinder in the first octant, given by $x^2 + y^2 = 4$, $x \geq 0$, $y \geq 0$, and $0 \leq z \leq 4$, with radius 2 and height 4.

The vector field is $\mathbf{F}(x, y, z) = x\mathbf{i}$. The outward-pointing unit normal vector$\mathbf{n}$(away from the$z$-axis) for the cylinder is $\mathbf{n} = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$, since $\sqrt{x^2 + y^2} = 2$ on the cylinder.

The dot product is:

$$\mathbf{F} \cdot \mathbf{n} = (x\mathbf{i}) \cdot \left(\frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}\right) = x \cdot \frac{x}{2} = \frac{x^2}{2}.$$

Parametrize the surface using θ and z, with $x = 2\cos\theta$, $y = 2\sin\theta$, $z = z$, where $\theta \in [0, \pi/2]$ and $z \in [0, 4]$. The position vector is $\mathbf{r}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$.

The partial derivatives are:

$$\frac{\partial \mathbf{r}}{\partial \theta} = (-2\sin\theta, 2\cos\theta, 0), \quad \frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1).$$

The cross product is:

$$\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \mathbf{i}(2\cos\theta \cdot 1 - 0 \cdot 0) - \mathbf{j}(-2\sin\theta \cdot 1 - 0 \cdot 0) + \mathbf{k}(-2\sin\theta \cdot 0 - 2\cos\theta \cdot 0) = (2\cos\theta, 2\sin\theta, 0).$$

The magnitude is:

$$\left| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} \right| = \sqrt{(2\cos\theta)^2 + (2\sin\theta)^2} = \sqrt{4\cos^2\theta + 4\sin^2\theta} = \sqrt{4} = 2.$$

Thus, the surface element is $dS = 2d\theta dz$.

The dot product in terms of θ is:

$$\mathbf{F} \cdot \mathbf{n} = \frac{x^2}{2} = \frac{(2\cos\theta)^2}{2} = \frac{4\cos^2\theta}{2} = 2\cos^2\theta.$$

The flux integral is:

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \int_0^4 \int_0^{\pi/2} (2\cos^2\theta) \cdot 2d\theta dz = \int_0^4 \int_0^{\pi/2} 4\cos^2\theta d\theta dz.$$

Factor out constants:

$$4 \int_0^4 dz \int_0^{\pi/2} \cos^2\theta d\theta.$$

First, $\int_0^4 dz = 4$.

Next, $\int_0^{\pi/2} \cos^2\theta d\theta$:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \int_0^{\pi/2} \cos^2\theta d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}.$$

Thus:

$$4 \cdot 4 \cdot \frac{\pi}{4} = 4 \cdot \pi = 4\pi.$$

The flux integral is 4π.

This result is confirmed by the divergence theorem. Close the surface by adding the bottom disk at $z = 0$ and the top disk at $z = 4$, both in the first octant. The outward normal for the bottom is $-\mathbf{k}$ and for the top is $\mathbf{k}$. Since $\mathbf{F}$ has no z-component, $\mathbf{F} \cdot \mathbf{n} = 0$ on both disks. Thus, the flux through the closed surface equals the flux through$S$.

The divergence of $\mathbf{F}$ is:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = 1.$$

The volume$V$is the quarter cylinder: $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq 4$, $0 \leq z \leq 4$. The volume is:

$$\text{Volume} = \frac{1}{4}\pi r^2 h = \frac{1}{4}\pi(2)^2 \cdot 4 = \frac{1}{4}\pi \cdot 4 \cdot 4 = 4\pi.$$

The flux through the closed surface is:

$$\iiint_V \nabla \cdot \mathbf{F} dV = \iiint_V 1 dV = 4\pi.$$

Since the flux through the top and bottom disks is zero, the flux through S is 4π.

The solid G is the quarter-cylinder in the first octant bounded by $x^2 + y^2 = 4$, $x \geq 0$, $y \geq 0$, and $0 \leq z \leq 4$. The vector field is $\mathbf{F} = x\mathbf{i}$.

The divergence theorem states that the flux of $\mathbf{F}$ out of the closed surface bounding G is equal to the triple integral of the divergence of $\mathbf{F}$ over G:

$$\iint_{\partial G} \mathbf{F} \cdot \mathbf{n} dS = \iiint_G (\nabla \cdot \mathbf{F}) dV,$$

where ∂G is the boundary of G with outward-pointing normal $\mathbf{n}$.

Compute the divergence of $\mathbf{F}$:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = 1.$$

Thus, the flux is:

$$\iiint_G 1 dV,$$

which is the volume of G.

G is a quarter-cylinder of radius 2 and height 4. The volume of a full cylinder of radius r and height h is $\pi r^2 h$. Here, $r = 2$ and $h = 4$, so the volume of the full cylinder is:

$$\pi(2)^2 \cdot 4 = 16\pi.$$

Since G is a quarter of this cylinder, its volume is:

$$\frac{1}{4} \cdot 16\pi = 4\pi.$$

Therefore, the flux of $\mathbf{F}$ out of G is 4π.

The solid G is a quarter-cylinder in the first octant with radius 2 and height 4, bounded by the surfaces $x^2 + y^2 = 4$ (for $x \geq 0$, $y \geq 0$, $0 \leq z \leq 4$), $z = 0$, $z = 4$, $x = 0$, and $y = 0$. The boundary consists of five faces:

- S: the lateral surface $x^2 + y^2 = 4$, $x \geq 0$, $y \geq 0$, $0 \leq z \leq 4$.

    - Bottom face:$z = 0,\ x^2 + y^2 \leq 4,\ x \geq 0,\ y \geq 0.$
    - Top face:$z = 4,\ x^2 + y^2 \leq 4,\ x \geq 0,\ y \geq 0.$
- $x = 0$ face: $x = 0,\ 0 \leq y \leq 2,\ 0 \leq z \leq 4.$
- $y = 0$ face: $y = 0,\ 0 \leq x \leq 2,\ 0 \leq z \leq 4.$

The vector field is $\mathbf{F} = x\mathbf{i}$.

From part (b), the flux of $\mathbf{F}$ outward through S is 4π. From part (c), using the divergence theorem, the total outward flux through the entire boundary of G is 4π. The total outward flux is the sum of the fluxes through all five faces. Therefore:

$$\text{Flux through } S + \text{Flux through the other four faces} = 4\pi.$$

Substituting the known flux through S:

$$4\pi + \text{Flux through the other four faces} = 4\pi,$$

which implies that the flux through the other four faces is 0.

This result is consistent with direct computation of the flux through each of the four faces:

- **Bottom face ($z = 0$):** The outward normal is$-\mathbf{k}$. Then $\mathbf{F} \cdot (-\mathbf{k}) = (x\mathbf{i}) \cdot (-\mathbf{k}) = 0$, so the flux is 0.
- **Top face ($z = 4$):** The outward normal is $\mathbf{k}$. Then $\mathbf{F} \cdot \mathbf{k} = (x\mathbf{i}) \cdot \mathbf{k} = 0$, so the flux is 0.
- $x = 0$ **face:** The outward normal is $-\mathbf{i}$. On this face, $x = 0$, so $\mathbf{F} = \mathbf{0}$. Then$\mathbf{F} \cdot (-\mathbf{i}) = \mathbf{0} \cdot (-\mathbf{i}) = 0$, so the flux is 0.
- $y = 0$ **face:** The outward normal is$-\mathbf{j}$. Then $\mathbf{F} \cdot (-\mathbf{j}) = (x\mathbf{i}) \cdot (-\mathbf{j}) = 0$, since $\mathbf{i}$ and $\mathbf{j}$ are orthogonal.

Thus, the flux through each of the four faces is0, and the total flux through these faces is0.

Q2

(1-Q4) $\mathbf{F}(x, y, z) = (yz)\mathbf{i} + (-xz)\mathbf{j} + \mathbf{k}$. Let S be the portion of the surface of the paraboloid

$$z = 4 - x^2 - y^2$$

- which lies above the first octant; and let C be the closed curve $C = C_1 + C_2 + C_3$, where the curves C_1, C_2, and C_3 are the three curves formed by intersecting S with the xy, yz, and xz planes respectively (so that C is the boundary of S). Orient C so that it is traversed counterclockwise when seen from above in the first octant.
- Use Stokes' Theorem to compute

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

by using the surface integral over the capping surface S.

- Set up and evaluate the loop integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly by parametrizing each piece of the curve C and then adding up the three line integrals.

Solution:

To compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' Theorem, where $\mathbf{F}(x, y, z) = yz\mathbf{i} - xz\mathbf{j} + \mathbf{k}$ and C is the boundary of the surface S (the portion of the paraboloid $z = 4 - x^2 - y^2$ in the first octant), oriented counterclockwise when viewed from above in the first octant, Stokes' Theorem states that:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Here, S is oriented with the upward-pointing normal vector, consistent with the counterclockwise orientation of C when viewed from above.

First, compute the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & 1 \end{vmatrix} = \left(\frac{\partial(1)}{\partial y} - \frac{\partial(-xz)}{\partial z}, \frac{\partial(yz)}{\partial z} - \frac{\partial(1)}{\partial x}, \frac{\partial(-xz)}{\partial x} - \frac{\partial(yz)}{\partial y} \right).$$

Evaluating each component:
- $\frac{\partial(1)}{\partial y} - \frac{\partial(-xz)}{\partial z} = 0 - (-x) = x,$

- $\frac{\partial(yz)}{\partial z} - \frac{\partial(1)}{\partial x} = y - 0 = y,$
- $\frac{\partial(-xz)}{\partial x} - \frac{\partial(yz)}{\partial y} = -z - z = -2z.$

Thus, $\nabla \times \mathbf{F} = (x, y, -2z)$.

Next, parametrize the surface S. Since S is given by $z = 4 - x^2 - y^2$, use x and y as parameters:

$$\mathbf{r}(x, y) = (x, y, 4 - x^2 - y^2), \quad \text{where } (x, y) \in D,$$

and D is the projection of$S$onto the xy-plane, which is the quarter disk $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq 4$.

The surface element $d\mathbf{S}$ is given by $\mathbf{r}_x \times \mathbf{r}_y dxdy$. Compute:

$$\mathbf{r}_x = (1, 0, -2x), \quad \mathbf{r}_y = (0, 1, -2y),$$

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}$$

$$= \mathbf{i}(0 \cdot (-2y) - (-2x) \cdot 1) - \mathbf{j}(1 \cdot (-2y) - (-2x) \cdot 0) + \mathbf{k}(1 \cdot 1 - 0 \cdot 0) = (2x, 2y, 1).$$

The normal vector $(2x, 2y, 1)$ has a positive z-component, confirming it is upward-pointing. Thus,

$$d\mathbf{S} = (2x, 2y, 1)dxdy.$$

Now, compute the dot product:

$$(\nabla \times \mathbf{F}) \cdot d\mathbf{S} = (x, y, -2z) \cdot (2x, 2y, 1) = 2x^2 + 2y^2 - 2z.$$

Substitute $z = 4 - x^2 - y^2$:

$$2x^2 + 2y^2 - 2(4 - x^2 - y^2) = 2x^2 + 2y^2 - 8 + 2x^2 + 2y^2 = 4x^2 + 4y^2 - 8.$$

The surface integral becomes:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_D (4x^2 + 4y^2 - 8) dx dy,$$

where D is the quarter disk $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq 4$.

To evaluate this integral, use polar coordinates. Set $x = r\cos\theta$, $y = r\sin\theta$, so $dxdy = rdrd\theta$, and $x^2 + y^2 = r^2$. The region D corresponds to $0 \leq r \leq 2$, $0 \leq \theta \leq \pi/2$. The integrand is:

$$4x^2 + 4y^2 - 8 = 4r^2 - 8.$$

Thus, the integral is:

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{2} (4r^2 - 8) r dr d\theta = \int_0^{\pi/2} \int_0^2 (4r^3 - 8r) dr d\theta.$$

First, evaluate the inner integral:

$$\int_0^2 (4r^3 - 8r) dr = \left[r^4 - 4r^2\right]_0^2 = (16 - 16) - (0) = 0.$$

Then, evaluate the outer integral:

$$\int_0^{\pi/2} 0 d\theta = 0.$$

Therefore, the surface integral is 0, and by Stokes' Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

To compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly, parametrize each piece of the curve$C = C_1 + C_2 + C_3$, where $C_1$is the intersection of the paraboloid $z = 4 - x^2 - y^2$ with the xy-plane ($z = 0$), C_2 with the yz-plane ($x = 0$), and C_3 with the xz-plane ($y = 0$), all in the first octant. The curve is oriented counterclockwise when viewed from above in the first octant, so the traversal is from $(2, 0, 0)$ to $(0, 2, 0)$ along C_1, then to $(0, 0, 4)$ along C_2, and finally back to $(2, 0, 0)$ along C_3.

The vector field is $\mathbf{F}(x, y, z) = (yz)\mathbf{i} + (-xz)\mathbf{j} + \mathbf{k}$.

- C_1 lies in the xy-plane ($z = 0$) and is the quarter-circle $x^2 + y^2 = 4$ from $(2, 0, 0)$ to $(0, 2, 0)$.
- Parametrize using $t \in [0, \pi/2]$:

$$\mathbf{r}_1(t) = (2\cos t, 2\sin t, 0)$$

- Derivative:

$$\mathbf{r}_1'(t) = (-2\sin t, 2\cos t, 0)$$

- Vector field along C_1 (since $z = 0$):

$$\mathbf{F}(\mathbf{r}_1(t)) = ((2\sin t)(0), -(2\cos t)(0), 1) = (0, 0, 1)$$

- Dot product:

$$\mathbf{F} \cdot \mathbf{r}_1'(t) = (0, 0, 1) \cdot (-2\sin t, 2\cos t, 0) = 0$$

- Line integral:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} 0 dt = 0$$

- C_2 lies in the yz-plane ($x = 0$) with $z = 4 - y^2$ from $(0, 2, 0)$ to $(0, 0, 4)$.
- Parametrize using $t \in [0, 2]$ (so $y = 2 - t$, $z = 4 - (2 - t)^2 = 4t - t^2$):

$$\mathbf{r}_2(t) = (0, 2 - t, 4t - t^2)$$

- Derivative:

$$\mathbf{r}_2'(t) = (0, -1, 4 - 2t)$$

- Vector field along C_2 (since $x = 0$):

$$\mathbf{F}(\mathbf{r}_2(t)) = ((2 - t)(4t - t^2), -(0)(4t - t^2), 1) = ((2 - t)(4t - t^2), 0, 1)$$

- Dot product:

$$\mathbf{F} \cdot \mathbf{r}_2'(t) = ((2-t)(4t-t^2)) \cdot 0 + 0 \cdot (-1) + 1 \cdot (4-2t) = 4-2t$$

- Line integral:

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (4-2t)dt = \left[4t - t^2\right]_0^2 = (8-4) - 0 = 4$$

- C_3 lies in the xz-plane ($y=0$) with $z = 4 - x^2$ from $(0,0,4)$ to $(2,0,0)$.
- Parametrize using $t \in [0,2]$:

$$\mathbf{r}_3(t) = (t, 0, 4-t^2)$$

- Derivative:

$$\mathbf{r}_3'(t) = (1, 0, -2t)$$

- Vector field along C_3 (since $y=0$):

$$\mathbf{F}(\mathbf{r}_3(t)) = ((0)(4-t^2), -t(4-t^2), 1) = (0, -4t+t^3, 1)$$

- Dot product:

$$\mathbf{F} \cdot \mathbf{r}_3'(t) = 0 \cdot 1 + (-4t+t^3) \cdot 0 + 1 \cdot (-2t) = -2t$$

- Line integral:

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 -2t dt = \left[-t^2\right]_0^2 = -4$$

Sum the line integrals over each segment:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 0 + 4 + (-4) = 0$$

The line integral is 0, which matches the result obtained using Stokes' Theorem.

Q3

(1-Q2) Let $\mathbf{F}(x, y, z) = (y + y^2 z)\hat{\mathbf{i}} + (x - z + 2xyz)\hat{\mathbf{j}} + (-y + xy^2)\hat{\mathbf{k}}$

- Show that $\mathbf{F}(x, y, z)$ is a gradient field.
- Find a potential function $f(x, y, z)$ for $\mathbf{F}(x, y, z)$ such that $\nabla f = \mathbf{F}$.
- Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is a straight line from $(2, 2, 1)$ to $(1, -1, 2)$.

Solution:

To show that $\mathbf{F}(x, y, z) = (y + y^2 z)\hat{\mathbf{i}} + (x - z + 2xyz)\hat{\mathbf{j}} + (-y + xy^2)\hat{\mathbf{k}}$ is a gradient field, it suffices to show that $\mathbf{F}$ is conservative, which requires that the curl of $\mathbf{F}$ is zero, i.e., $\nabla \times \mathbf{F} = \mathbf{0}$. The domain is $\mathbb{R}^3$, which is simply connected, so a zero curl implies that $\mathbf{F}$ is a gradient field.

Define the components of $\mathbf{F}$ as:

$$P = y + y^2 z, \quad Q = x - z + 2xyz, \quad R = -y + xy^2.$$

The curl $\nabla \times \mathbf{F}$ is given by:

$$\nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}.$$

Compute the necessary partial derivatives:

- $\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}(-y + xy^2) = -1 + 2xy,$
- $\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}(x - z + 2xyz) = -1 + 2xy,$
- $\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}(-y + xy^2) = y^2,$
- $\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}(y + y^2 z) = y^2,$
- $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x - z + 2xyz) = 1 + 2yz,$
- $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y + y^2 z) = 1 + 2yz.$

Now substitute into the curl expression:

- $\hat{\mathbf{i}}$-component: $\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = (-1 + 2xy) - (-1 + 2xy) = 0,$
- $\hat{\mathbf{j}}$-component: $-\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = -(y^2 - y^2) = 0,$
- $\hat{\mathbf{k}}$-component: $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (1 + 2yz) - (1 + 2yz) = 0.$

Since all components are zero, $\nabla \times \mathbf{F} = \mathbf{0}$. Therefore, $\mathbf{F}$ is conservative and hence a gradient field.

To find a potential function $f(x, y, z)$ such that $\nabla f = \mathbf{F}(x, y, z) = (y + y^2 z)\hat{\mathbf{i}} + (x - z + 2xyz)\hat{\mathbf{j}} + (-y + xy^2)\hat{\mathbf{k}}$, integrate the components of $\mathbf{F}$ step by step, ensuring consistency with all partial derivatives.

Start with the x-component:

$$\frac{\partial f}{\partial x} = y + y^2 z.$$

Integrate with respect to x, treating y and z as constants:

$$f(x, y, z) = \int (y + y^2 z) dx = xy + xy^2 z + g(y, z),$$

where $g(y, z)$ is an arbitrary function of y and z.

Next, use the y-component:

$$\frac{\partial f}{\partial y} = x - z + 2xyz.$$

Compute the partial derivative of f with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy + xy^2 z + g(y, z)) = x + 2xyz + \frac{\partial g}{\partial y}.$$

Set this equal to the given y-component:

$$x + 2xyz + \frac{\partial g}{\partial y} = x - z + 2xyz.$$

Simplify to find:

$$\frac{\partial g}{\partial y} = -z.$$

Integrate with respect to y, treating z as constant:

$$g(y, z) = \int (-z) dy = -yz + h(z),$$

where $h(z)$ is an arbitrary function of z. Substitute back into f:

$$f(x, y, z) = xy + xy^2 z - yz + h(z).$$

Finally, use the z-component:

$$\frac{\partial f}{\partial z} = -y + xy^2.$$

Compute the partial derivative of f with respect to z:

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(xy + xy^2 z - yz + h(z)) = xy^2 - y + h'(z).$$

Set this equal to the given z-component:

$$xy^2 - y + h'(z) = -y + xy^2.$$

Simplify to find:

$$h'(z) = 0.$$

Thus, $h(z)$ is a constant, denoted C. The potential function is:

$$f(x, y, z) = xy + xy^2 z - yz + C.$$

Since potential functions are defined up to an additive constant, set $C = 0$ for simplicity:

$$f(x, y, z) = xy + xy^2 z - yz.$$

The vector field $\mathbf{F}(x, y, z) = (y + y^2 z)\hat{\mathbf{i}} + (x - z + 2xyz)\hat{\mathbf{j}} + (-y + xy^2)\hat{\mathbf{k}}$ is conservative, as previously established by showing that its curl is zero. A potential function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$ is given by $f(x, y, z) = xy + xy^2 z - yz$.

To compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the straight line path C from $(2, 2, 1)$ to $(1, -1, 2)$, the fundamental theorem for line integrals is applied. This theorem states that for a conservative vector field, the line integral depends only on the endpoints of the path and is given by the difference in the potential function evaluated at these points:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, -1, 2) - f(2, 2, 1).$$

Evaluate the potential function at the endpoints:

- At $(1, -1, 2)$:

$$f(1, -1, 2) = (1)(-1) + (1)(-1)^2(2) - (-1)(2) = -1 + (1)(1)(2) - (-2) = -1 + 2 + 2 = 3.$$

- At $(2, 2, 1)$:

$$f(2, 2, 1) = (2)(2) + (2)(2)^2(1) - (2)(1) = 4 + (2)(4)(1) - 2 = 4 + 8 - 2 = 10.$$

Compute the difference:

$$f(1, -1, 2) - f(2, 2, 1) = 3 - 10 = -7.$$

Q4

(1-Q6) Evaluate the following single integral by converting it to an equivalent double integral:

$$\int_0^\infty \frac{e^{-x}-e^{-ax}}{x} \cdot dx$$

Solution:

To evaluate the integral $\int_0^\infty \frac{e^{-x}-e^{-ax}}{x} dx$, convert it to an equivalent double integral. Note that for$a > 0$, the integrand can be expressed as:

$$\frac{e^{-x}-e^{-ax}}{x} = \int_1^a e^{-ux} du.$$

This holds for all $a > 0$, as the limits account for the direction of integration (e.g., if $a < 1$, the integral from 1 to a is negative, matching the sign of the integrand). Substituting this into the original integral gives:

$$\int_0^\infty \frac{e^{-x}-e^{-ax}}{x} dx = \int_0^\infty \left(\int_1^a e^{-ux} du \right) dx.$$

Since the integrand e^{-ux} is continuous and non-negative for $x > 0$ and $u > 0$, and the integral converges absolutely for $a > 0$, Fubini's theorem allows switching the order of integration:

$$\int_0^\infty \int_1^a e^{-ux} du dx = \int_1^a \int_0^\infty e^{-ux} dx du.$$

Now evaluate the inner integral with respect to x. For fixed $u > 0$,

$$\int_0^\infty e^{-ux} dx = \left[-\frac{1}{u} e^{-ux} \right]_0^\infty = 0 - \left(-\frac{1}{u} \right) = \frac{1}{u}.$$

Substitute this result back into the outer integral:

$$\int_1^a \frac{1}{u} du = [\ln u]_1^a = \ln a - \ln 1 = \ln a,$$

Q5

(1-Q7) Show that the average distance of the points of a disk of radius a to its center is $2a/3$.

Solution:

To find the average distance of the points in a disk of radius a to its center, consider the disk centered at the origin. The area of the disk is πa^2.

The distance from a point (x, y) to the center is $r = \sqrt{x^2 + y^2}$. The average distance is given by the integral of the distance over the disk divided by the area of the disk:

$$\bar{r} = \frac{1}{\pi a^2} \iint_{\text{disk}} r dA.$$

Using polar coordinates, where r is the radial distance and θ is the angle, the area element is $dA = r dr d\theta$. The distance to the center is r, and the disk is described by $0 \le r \le a$ and $0 \le \theta \le 2\pi$. Thus, the integral becomes:

$$\iint_{\text{disk}} r dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} r \cdot r dr d\theta = \int_{0}^{2\pi} \int_{0}^{a} r^2 dr d\theta.$$

First, evaluate the inner integral with respect to r:

$$\int_0^a r^2 dr = \left[\frac{r^3}{3} \right]_0^a = \frac{a^3}{3}.$$

Next, evaluate the outer integral with respect to θ:

$$\int_0^{2\pi} d\theta = 2\pi.$$

So, the double integral is:

$$2\pi \cdot \frac{a^3}{3} = \frac{2\pi a^3}{3}.$$

Now, divide by the area πa^2:

$$\bar{r} = \frac{1}{\pi a^2} \cdot \frac{2\pi a^3}{3} = \frac{2\pi a^3}{3\pi a^2} = \frac{2a}{3}.$$

Thus, the average distance is $\frac{2a}{3}$.

Q6

(1-Q8) In general, the moment of inertia around an axis (a line) $L$is,

$$I_L = \iint_R dist(\cdot, L)^2 \delta \cdot dA$$

The collection of lines parallel to the y-axis have the form $x = a$. Let $I = I_y$ be the usual moment of inertia around the y-axis,

$$I = \iint_R x^2 \delta \cdot dA$$

Let $\bar{I}$ be the moment of inertia around the axis $x = \bar{x}$, where $(\bar{x}, \bar{y})$ is the center of mass. Show that

$$I = \bar{I} + M\bar{x}^2$$

Solution:

The moment of inertia around the y-axis (x = 0) is given by:

$$I = \iint_R x^2 \delta dA.$$

The moment of inertia around the parallel axis through the center of mass$(\bar{x}, \bar{y})$, which is the line $x = \bar{x}$, is given by:

$$\bar{I} = \iint_R (x - \bar{x})^2 \delta dA.$$

The center of mass $\bar{x}$ and the total mass M are defined as:

$$\bar{x} = \frac{1}{M} \iint_R x\delta dA, \quad M = \iint_R \delta dA,$$

so that:

$$\iint_R x\delta dA = M\bar{x}.$$

Expand the expression for $\bar{I}$:

$$\bar{I} = \iint_R (x - \bar{x})^2 \delta dA = \iint_R (x^2 - 2x\bar{x} + \bar{x}^2)\delta dA.$$

Distribute the integral:

$$\bar{I} = \iint_R x^2 \delta dA - 2\bar{x} \iint_R x\delta dA + \bar{x}^2 \iint_R \delta dA.$$

Substitute the known expressions:

$$\bar{I} = I - 2\bar{x}(M\bar{x}) + \bar{x}^2 M = I - 2M\bar{x}^2 + M\bar{x}^2 = I - M\bar{x}^2.$$

Rearrange to solve for$I$:

$$I = \bar{I} + M\bar{x}^2.$$

Q7

(1-Q11) Consider the vector field $\vec{F} = (x^2y + \frac{1}{3}y^3)\hat{\imath}$, and let C be the portion of the graph $y = f(x)$ running from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ (assume that $x_1 < x_2$, and f takes positive values). Show that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is equal to the polar moment of inertia of the region R lying below C and above the x-axis (with density $\delta = 1$).

Solution:

The vector field is given by $\vec{F} = \left(x^2y + \frac{1}{3}y^3\right)\hat{\imath}$, so the line integral along the curve C parameterized by $y = f(x)$ from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ is:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \left(x^2y + \frac{1}{3}y^3\right) dx,$$

since the $\hat{\jmath}$-component is zero. Substituting $y = f(x)$, the integral becomes:

$$\int_{x_1}^{x_2} \left(x^2 f(x) + \frac{1}{3}[f(x)]^3\right) dx.$$

The region$R$is bounded below by the x-axis ($y = 0$), above by the curve $y = f(x)$, and between $x = x_1$ and $x = x_2$. The polar moment of inertia about the origin, with density $\delta = 1$, is:

$$\iint_R (x^2 + y^2) dA.$$

This double integral can be expressed as an iterated integral:

$$\iint_R (x^2 + y^2) dA = \int_{x_1}^{x_2} \int_0^{f(x)} (x^2 + y^2) dy dx.$$

Evaluating the inner integral with respect to$y$:

$$\int_0^{f(x)} (x^2 + y^2) dy = \left[x^2 y + \frac{1}{3}y^3\right]_0^{f(x)} = x^2 f(x) + \frac{1}{3}[f(x)]^3.$$

Thus, the double integral is:

$$\int_{x_1}^{x_2} \left(x^2 f(x) + \frac{1}{3}[f(x)]^3\right) dx.$$

This expression is identical to the line integral:

$$\int_{x_1}^{x_2} \left(x^2 f(x) + \frac{1}{3}[f(x)]^3\right) dx.$$

Q8

(1-Q12) Consider the vector field

$$\vec{F} = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$$

- Show that $\vec{F}$ is the gradient of the polar function $\theta(x, y) = \arctan(y/x)$ over the right half-plane $x > 0$.
- Suppose that C is a smooth curve in the right half-plane $x > 0$ joining two points $A : (x_1, y_1)$and $B : (x_2, y_2)$. Express $\int_C \vec{F} \cdot d\vec{r}$ in terms of the polar coordinates (r_1, θ_1) and (r_2, θ_2) of A and B.
- Compute directly from the definition the line integrals $\int_{C_1} \vec{F} \cdot d\vec{r}$ and $\int_{C_2} \vec{F} \cdot d\vec{r}$ where C_1 is the upper half of the unit circle running from $(1, 0)$ to $(-1, 0)$ and C_2 is the lower half of the unit circle, also going from $(1, 0)$ to $(-1, 0)$.

Solution:

To show that the vector field $\vec{F} = \frac{-y\hat{i}+x\hat{j}}{x^2+y^2}$ is the gradient of the polar function $\theta(x, y) = \arctan(y/x)$over the right half-plane $x > 0$, compute the gradient of $\theta(x, y)$ and verify that it matches $\vec{F}$.

The gradient of $\theta(x, y)$ is given by:

$$\nabla\theta = \frac{\partial\theta}{\partial x}\hat{i} + \frac{\partial\theta}{\partial y}\hat{j}.$$

First, compute $\frac{\partial\theta}{\partial x}$. Let $u = y/x$, so $\theta = \arctan(u)$. Using the chain rule:

$$\frac{\partial\theta}{\partial x} = \frac{d\arctan(u)}{du} \cdot \frac{\partial u}{\partial x} = \frac{1}{1+u^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right).$$

Since$y$is treated as constant with respect to$x$,

$$\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = y \cdot (-x^{-2}) = -\frac{y}{x^2}.$$

Substituting $u = y/x$:

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\cdot\left(-\frac{y}{x^2}\right) = \frac{1}{\frac{x^2+y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = \frac{x^2}{x^2+y^2}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}.$$

Next, compute $\frac{\partial\theta}{\partial y}$:

$$\frac{\partial\theta}{\partial y} = \frac{d\arctan(u)}{du}\cdot\frac{\partial u}{\partial y} = \frac{1}{1+u^2}\cdot\frac{\partial}{\partial y}\left(\frac{y}{x}\right).$$

Since$x$is treated as constant with respect to y,

$$\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x}.$$

Substituting $u = y/x$:

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{1}{\frac{x^2+y^2}{x^2}}\cdot\frac{1}{x} = \frac{x^2}{x^2+y^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}.$$

Thus, the gradient is:

$$\nabla\theta = -\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}.$$

The given vector field is:

$$\vec{F} = \frac{-y\hat{i}+x\hat{j}}{x^2+y^2} = -\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}.$$

Since $\nabla\theta = \vec{F}$ for$x > 0$, $\vec{F}$ is the gradient of $\theta(x,y) = \arctan(y/x)$ over the right half-plane.

The vector field $\vec{F} = \frac{-y\hat{i}+x\hat{j}}{x^2+y^2}$ is conservative in the right half-plane $x > 0$, as it is the gradient of the potential function $\theta(x,y) = \arctan(y/x)$. Specifically, $\nabla\theta = \vec{F}$.

Since$\vec{F}$is conservative, the line integral $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints of the curve$C$, which are $A : (x_1, y_1)$ and $B : (x_2, y_2)$. The value of the line integral is given by the difference in the potential function evaluated at the endpoints:

$$\int_C \vec{F} \cdot d\vec{r} = \theta(B) - \theta(A).$$

In polar coordinates, the angle$\theta$is defined as $\theta = \arctan(y/x)$for$x > 0$. The polar coordinates of A and B are (r_1, θ_1) and (r_2, θ_2), respectively, where $\theta_1 = \arctan(y_1/x_1)$ and $\theta_2 = \arctan(y_2/x_2)$.

Since the curve C lies entirely in the right half-plane $x > 0$, $\theta(x, y)$ is well-defined and smooth, and the angles θ_1 and θ_2 are both in the interval $(-\pi/2, \pi/2)$. Therefore,

$$\theta(A) = \theta_1, \quad \theta(B) = \theta_2.$$

Substituting these into the expression for the line integral gives:

$$\int_C \vec{F} \cdot d\vec{r} = \theta_2 - \theta_1.$$

The vector field is $\vec{F} = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$.

For C_1 (upper half of the unit circle from $(1, 0)$ to $(-1, 0)$):

- Parameterize C_1 as $\vec{r}(t) = (\cos t, \sin t)$ for $t \in [0, \pi]$.
- Then $\vec{r}'(t) = (-\sin t, \cos t)$.
- On the unit circle, $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so $\vec{F}(\vec{r}(t)) = (-\sin t, \cos t)$.
- The dot product is:

$$\vec{F} \cdot \vec{r}' = (-\sin t)(-\sin t) + (\cos t)(\cos t) = \sin^2 t + \cos^2 t = 1.$$

- The line integral is:

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^\pi 1 dt = [t]_0^\pi = \pi.$$

For C_2 (lower half of the unit circle from $(1, 0)$ to $(-1, 0)$):

- Parameterize C_2 as $\vec{r}(t) = (\cos t, -\sin t)$ for $t \in [0, \pi]$.
- Then $\vec{r}'(t) = (-\sin t, -\cos t)$.
- On the unit circle, $x^2 + y^2 = \cos^2 t + (-\sin t)^2 = 1$, so $\vec{F}(\vec{r}(t)) = (-(-\sin t), \cos t) = (\sin t, \cos t)$.
- The dot product is:

$$\vec{F} \cdot \vec{r}' = (\sin t)(-\sin t) + (\cos t)(-\cos t) = -\sin^2 t - \cos^2 t = -1.$$

- The line integral is:

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^\pi -1 dt = [-t]_0^\pi = -\pi.$$

Q9

(1-Q14) Show that a constant force field does zero work on a particle that winds uniformly$w$times around the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution:

Parameterize the ellipse. Let $\vec{r}(t) = (a \cos t, b \sin t)$for $t \in [0, 2\pi w]$. Then, $d\vec{r} = (-a \sin t, b \cos t)dt$.

The line integral for work is:

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi w} (c_1(-a \sin t) + c_2(b \cos t))dt = -ac_1 \int_0^{2\pi w} \sin t dt + bc_2 \int_0^{2\pi w} \cos t dt.$$

Evaluate the integrals:

$$\int_0^{2\pi w} \sin t dt = [-\cos t]_0^{2\pi w} = -\cos(2\pi w) - (-\cos 0) = -\cos(2\pi w) + 1,$$

$$\int_0^{2\pi w} \cos t dt = [\sin t]_0^{2\pi w} = \sin(2\pi w) - \sin 0.$$

Since w is an integer, $\cos(2\pi w) = \cos(0) = 1$ and $\sin(2\pi w) = \sin(0) = 0$. Thus:

$$-\cos(2\pi w) + 1 = -1 + 1 = 0, \quad \sin(2\pi w) - 0 = 0 - 0 = 0.$$

Therefore, the work done is:

$$-ac_1 \cdot 0 + bc_2 \cdot 0 = 0.$$

Thus, the work done by the constant force field is zero for any integer$w$.

Q10

(1-Q17)

- Let $f(x, y, z) = 1/\rho = (x^2 + y^2 + z^2)^{-1/2}$. Calculate $\vec{F} = \nabla f$.
- Evaluate the flux of $\vec{F}$ over the sphere of radius a centered at the origin.
- Show that $div(\vec{F}) = 0$. Does this violate the divergence theorem?

Solution:

The function is given by $f(x, y, z) = \frac{1}{\rho} = (x^2 + y^2 + z^2)^{-1/2}$, where $\rho = \sqrt{x^2 + y^2 + z^2}$.

The gradient $\vec{F} = \nabla f$ is computed as follows:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

Compute the partial derivative with respect to$x$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{-1/2}\right) = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x = -x(x^2+y^2+z^2)^{-3/2} = -\frac{x}{\rho^3}.$$

Similarly, for y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left((x^2+y^2+z^2)^{-1/2}\right) = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2y = -y(x^2+y^2+z^2)^{-3/2} = -\frac{y}{\rho^3}.$$

And for z:

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left((x^2+y^2+z^2)^{-1/2}\right) = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2z = -z(x^2+y^2+z^2)^{-3/2} = -\frac{z}{\rho^3}.$$

Thus,

$$\vec{F} = \left(-\frac{x}{\rho^3}, -\frac{y}{\rho^3}, -\frac{z}{\rho^3}\right) = -\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\rho^3}.$$

The vector field is given by $\vec{F} = \nabla f = -\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\rho^3}$, where $\rho = \sqrt{x^2+y^2+z^2}$. This can be expressed as $\vec{F} = -\frac{\vec{r}}{r^3}$, with $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and $r = |\vec{r}|$.

The flux of $\vec{F}$ over the sphere of radius a centered at the origin is given by the surface integral $\iint_S \vec{F} \cdot d\vec{S}$, where S is the sphere $x^2+y^2+z^2 = a^2$.

On the sphere, $r = a$, so $\vec{F} = -\frac{\vec{r}}{a^3}$. The outward-pointing unit normal vector is $\hat{n} = \frac{\vec{r}}{r} = \frac{\vec{r}}{a}$, and the area element is $d\vec{S} = \hat{n}dS = \frac{\vec{r}}{a}dS$.

The dot product is:

$$\vec{F} \cdot d\vec{S} = \left(-\frac{\vec{r}}{a^3}\right) \cdot \left(\frac{\vec{r}}{a}dS\right) = -\frac{1}{a^4}(\vec{r}\cdot\vec{r})dS.$$

Since $\vec{r}\cdot\vec{r} = r^2 = a^2$,

$$\vec{F} \cdot d\vec{S} = -\frac{1}{a^4} \cdot a^2 dS = -\frac{1}{a^2} dS.$$

The flux is:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S -\frac{1}{a^2} dS = -\frac{1}{a^2} \iint_S dS.$$

The surface area of the sphere is $\iint_S dS = 4\pi a^2$, so:

$$\iint_S \vec{F} \cdot d\vec{S} = -\frac{1}{a^2} \cdot 4\pi a^2 = -4\pi.$$

Alternatively, using the divergence theorem, $\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{F} dV$, where V is the volume enclosed by the sphere. The divergence of $\vec{F}$ is:

$$\nabla \cdot \vec{F} = \nabla \cdot \left(-\frac{\vec{r}}{r^3} \right) = -4\pi\delta(\vec{r}),$$

where $\delta(\vec{r})$ is the three-dimensional Dirac delta function. The integral over V is:

$$\iiint_V -4\pi\delta(\vec{r}) dV = -4\pi,$$

since the origin is inside the sphere, confirming the result.

The flux is independent of the radius a.

Q11

(1-Q30) Show that the average straight-line distance to a fixed point on the surface of a sphere of radius a is $4a/3$.

Solution:

Any point Q on the sphere can be represented in spherical coordinates as (a, θ, ϕ), where θ is the polar angle (from the positive z-axis) and ϕ is the azimuthal angle. The Cartesian coordinates of Q are $(a \sin\theta \cos\phi, a \sin\theta \sin\phi, a \cos\theta)$.

The straight-line distance d from P to Q is given by:

$$d = \sqrt{(0 - a\sin\theta\cos\phi)^2 + (0 - a\sin\theta\sin\phi)^2 + (a - a\cos\theta)^2}.$$

Simplifying the expression inside the square root:

$$d = a\sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + (1 - \cos\theta)^2}$$
$$= a\sqrt{\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta} = a\sqrt{2 - 2\cos\theta} = a\sqrt{2(1 - \cos\theta)}.$$

Using the trigonometric identity $1 - \cos\theta = 2\sin^2(\theta/2)$:

$$d = a\sqrt{2 \cdot 2\sin^2(\theta/2)} = a\sqrt{4\sin^2(\theta/2)} = 2a\sin(\theta/2),$$

since $\sin(\theta/2) \geq 0$ for $\theta \in [0, \pi]$.

The average distance $\bar{d}$ is the integral of d over the sphere divided by the surface area of the sphere, which is $4\pi a^2$. The surface area element in spherical coordinates is $dA = a^2 \sin\theta d\theta d\phi$. Thus:

$$\bar{d} = \frac{1}{4\pi a^2} \iint_S d\,dA = \frac{1}{4\pi a^2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} 2a\sin(\theta/2) \cdot a^2 \sin\theta d\theta d\phi.$$

Separating the integrals:

$$\bar{d} = \frac{1}{4\pi a^2} \int_0^{2\pi} d\phi \int_0^{\pi} 2a^3 \sin(\theta/2)\sin\theta d\theta = \frac{1}{4\pi a^2} \cdot 2\pi \cdot 2a^3 \int_0^{\pi} \sin(\theta/2)\sin\theta d\theta$$
$$= a\int_0^{\pi} \sin(\theta/2)\sin\theta d\theta.$$

Using the identity $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$:

$$\sin(\theta/2)\sin\theta = \sin(\theta/2)\cdot 2\sin(\theta/2)\cos(\theta/2) = 2\sin^2(\theta/2)\cos(\theta/2).$$

Substituting into the integral:

$$\int_0^\pi \sin(\theta/2)\sin\theta d\theta = \int_0^\pi 2\sin^2(\theta/2)\cos(\theta/2)d\theta.$$

Use the substitution $u = \sin(\theta/2)$, so $du = \frac{1}{2}\cos(\theta/2)d\theta$ and $\cos(\theta/2)d\theta = 2du$. When $\theta = 0$, $u = 0$; when $\theta = \pi$, $u = 1$:

$$\int_0^\pi 2\sin^2(\theta/2)\cos(\theta/2)d\theta = \int_0^1 2u^2\cdot 2du = \int_0^1 4u^2 du = 4\left[\frac{u^3}{3}\right]_0^1 = 4\cdot\frac{1}{3} = \frac{4}{3}.$$

Thus:

$$\bar{d} = a\cdot\frac{4}{3} = \frac{4a}{3}.$$

Q12

(1-Q33) The Laplacian of a function of three variables is defined by

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz}.$$

Suppose that the simple closed surface S is the iso-surface of some smooth function $f(x,y,z)$, that is, the set of points in 3-space satisfying $f(x,y,z) = c$ for some constant c.

Use the Divergence Theorem to show that if G is the interior of S, then

$$\iint_S |\nabla f|\, dS = \pm\iiint_G \nabla^2 f\, dV.$$

Solution:

Set $\vec{F} = \nabla f$. Then, the divergence of $\vec{F}$ is $\nabla\cdot\vec{F} = \nabla\cdot(\nabla f) = \nabla^2 f$.

By the Divergence Theorem, for the outward-pointing unit normal $\hat{n}$ on S,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_G \nabla \cdot \vec{F} dV = \iiint_G \nabla^2 f dV,$$

where $d\vec{S} = \hat{n} dS$.

Since S is an iso-surface of f, the gradient ∇f is normal to S. The magnitude $|\nabla f|$ is positive, and $\nabla f = |\nabla f| \hat{n}_f$, where $\hat{n}_f$ is the unit normal in the direction of ∇f. The outward unit normal $\hat{n}$ may align with or oppose ∇f, so

$$\nabla f \cdot \hat{n} = \begin{cases} |\nabla f| & \text{if } \nabla f \text{ points outward,} \\ -|\nabla f| & \text{if } \nabla f \text{ points inward.} \end{cases}$$

Thus,

$$\vec{F} \cdot d\vec{S} = \nabla f \cdot \hat{n} dS = \pm |\nabla f| dS,$$

where the sign depends on the direction of ∇f relative to $\hat{n}$.

Substituting into the Divergence Theorem result,

$$\iint_S \pm |\nabla f| dS = \iiint_G \nabla^2 f dV.$$

Rearranging gives

$$\iint_S |\nabla f| dS = \pm \iiint_G \nabla^2 f dV,$$

as required. The $\pm$ accounts for whether ∇f points outward or inward relative to G.

Q13

(4-Q14) Determine the surface area of the surface given by

$$z = \frac{2}{3}(x^{3/2} + y^{3/2}),$$

over the square $0 \le x \le 1, 0 \le y \le 1$.

Solution:

The surface is given by $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ over the region $0 \le x \le 1$, $0 \le y \le 1$.

The surface area A for a surface $z = f(x, y)$ is given by:

$$A = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy,$$

where R is the region $[0, 1] \times [0, 1]$.

First, compute the partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2}{3}x^{3/2} + \frac{2}{3}y^{3/2}\right) = \frac{2}{3} \cdot \frac{3}{2}x^{1/2} = x^{1/2} = \sqrt{x},$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2}{3}x^{3/2} + \frac{2}{3}y^{3/2}\right) = \frac{2}{3} \cdot \frac{3}{2}y^{1/2} = y^{1/2} = \sqrt{y}.$$

Then,

$$\left(\frac{\partial z}{\partial x}\right)^2 = (\sqrt{x})^2 = x, \quad \left(\frac{\partial z}{\partial y}\right)^2 = (\sqrt{y})^2 = y.$$

Thus,

$$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + x + y.$$

The surface area integral is:

$$A = \iint_R \sqrt{1+x+y}dxdy = \int_{y=0}^{1}\int_{x=0}^{1} \sqrt{1+x+y}dxdy.$$

Compute the inner integral with respect to x:

$$\int_{x=0}^{1} \sqrt{1+x+y}dx.$$

Substitute $u = 1+x+y$, so $du = dx$. When $x = 0$, $u = 1+y$; when $x = 1$, $u = 2+y$. Then,

$$\int_{x=0}^{1} \sqrt{1+x+y}dx = \int_{u=1+y}^{2+y} u^{1/2}du = \left[\frac{2}{3}u^{3/2}\right]_{1+y}^{2+y} = \frac{2}{3}\left[(2+y)^{3/2} - (1+y)^{3/2}\right].$$

The double integral becomes:

$$A = \int_{y=0}^{1} \frac{2}{3}\left[(2+y)^{3/2} - (1+y)^{3/2}\right]dy = \frac{2}{3}\int_{0}^{1} \left[(2+y)^{3/2} - (1+y)^{3/2}\right]dy.$$

Now compute the integral:

$$\int_{0}^{1} (2+y)^{3/2}dy - \int_{0}^{1} (1+y)^{3/2}dy.$$

For the first integral, substitute $v = 2+y$, $dv = dy$; when $y = 0$, $v = 2$; when $y = 1$, $v = 3$:

$$\int_{2}^{3} v^{3/2}dv = \left[\frac{2}{5}v^{5/2}\right]_{2}^{3} = \frac{2}{5}\left(3^{5/2} - 2^{5/2}\right).$$

For the second integral, substitute $w = 1+y$, $dw = dy$; when $y = 0$, $w = 1$; when $y = 1$, $w = 2$:

$$\int_{1}^{2} w^{3/2}dw = \left[\frac{2}{5}w^{5/2}\right]_{1}^{2} = \frac{2}{5}\left(2^{5/2} - 1^{5/2}\right) = \frac{2}{5}\left(2^{5/2} - 1\right).$$

Thus,

$$\int_0^1 \left[(2+y)^{3/2} - (1+y)^{3/2}\right] dy = \frac{2}{5}\left(3^{5/2} - 2^{5/2}\right) - \frac{2}{5}\left(2^{5/2} - 1\right)$$

$$= \frac{2}{5}\left(3^{5/2} - 2^{5/2} - 2^{5/2} + 1\right) = \frac{2}{5}\left(3^{5/2} - 2 \cdot 2^{5/2} + 1\right).$$

Simplify the exponents:

$$3^{5/2} = 3^{2+1/2} = 3^2 \cdot 3^{1/2} = 9\sqrt{3}, \quad 2^{5/2} = 2^{2+1/2} = 2^2 \cdot 2^{1/2} = 4\sqrt{2},$$

so

$$2 \cdot 2^{5/2} = 2 \cdot 4\sqrt{2} = 8\sqrt{2}.$$

Thus,

$$\int_0^1 \left[(2+y)^{3/2} - (1+y)^{3/2}\right] dy = \frac{2}{5}\left(9\sqrt{3} - 8\sqrt{2} + 1\right).$$

Now substitute back:

$$A = \frac{2}{3} \cdot \frac{2}{5}\left(9\sqrt{3} - 8\sqrt{2} + 1\right) = \frac{4}{15}\left(1 + 9\sqrt{3} - 8\sqrt{2}\right).$$

Therefore, the surface area is $\frac{4}{15}\left(1 + 9\sqrt{3} - 8\sqrt{2}\right)$.
