A01

Q1

(1-Q36)

- Find the curve of intersection of the surfaces $z = x^2 - y^2 \quad \text{and} \quad z = 2 + (x-y)^2$ in parametric form.
- Find the angle of intersection of these two surfaces at the point $(2, 1, 3)$. (The angle of intersection of two surfaces is defined to be the angle made by their tangent planes.)
- Check that the tangent vector to the curve of intersection found in part (a) at the point $(2, 1, 3)$ lies in (i.e., is parallel to) the tangent plane of each of the two surfaces.

Solution:

The surfaces are given by $z = x^2 - y^2$ and $z = 2 + (x-y)^2$. Setting the expressions for z equal gives:

$$x^2 - y^2 = 2 + (x-y)^2.$$

Expanding the right side:

$$(x-y)^2 = x^2 - 2xy + y^2,$$

so:

$$x^2 - y^2 = 2 + x^2 - 2xy + y^2.$$

Rearranging terms:

$$x^2 - y^2 - x^2 + 2xy - y^2 - 2 = 0 \implies -2y^2 + 2xy - 2 = 0.$$

Multiplying by -1:

$$2y^2 - 2xy + 2 = 0.$$

Dividing by 2:

$$y^2 - xy + 1 = 0.$$

Solving for x in terms of y:

$$x = y + \frac{1}{y}, \quad y \neq 0.$$

Substituting into the first surface equation to find z:

$$z = \left(y + \frac{1}{y}\right)^2 - y^2 = y^2 + 2 \cdot y \cdot \frac{1}{y} + \left(\frac{1}{y}\right)^2 - y^2 = 2 + \frac{1}{y^2}.$$

Using $y = t$ as the parameter, where $t \neq 0$, the parametric equations are:

$$x = t + \frac{1}{t}, \quad y = t, \quad z = 2 + \frac{1}{t^2}.$$

This parameterization is valid for all real $t \neq 0$, and it satisfies both surface equations. The curve is defined for $|x| \geq 2$, which is consistent with the parameterization.

The parametric form of the curve of intersection is:

$$\boxed{\begin{cases} x = t + \dfrac{1}{t} \\ y = t \\ z = 2 + \dfrac{1}{t^2} \end{cases} \quad \text{for} \quad t \in \mathbb{R} \setminus 0}$$

The angle of intersection of the two surfaces $z = x^2 - y^2$ and $z = 2 + (x - y)^2$ at the point $(2, 1, 3)$ is defined as the angle between their tangent planes at that point. This angle is found using the normal vectors to the surfaces.

Define the surfaces as level surfaces:

- First surface: $F(x, y, z) = x^2 - y^2 - z = 0$

- Second surface: $G(x, y, z) = (x - y)^2 - z + 2 = 0$

The gradient vectors are normal to the surfaces:

- $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = (2x, -2y, -1)$
- $\nabla G = \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z}\right) = (2(x - y), -2(x - y), -1)$

At the point $(2, 1, 3)$:

- $\nabla F = (2 \cdot 2, -2 \cdot 1, -1) = (4, -2, -1)$
- $\nabla G = (2(2 - 1), -2(2 - 1), -1) = (2, -2, -1)$

The angle θ between the normal vectors is given by:

$$\cos\theta = \frac{\nabla F \cdot \nabla G}{|\nabla F||\nabla G|}$$

where the dot product and magnitudes are:

- $\nabla F \cdot \nabla G = 4 \cdot 2 + (-2) \cdot (-2) + (-1) \cdot (-1) = 8 + 4 + 1 = 13$
- $|\nabla F| = \sqrt{4^2 + (-2)^2 + (-1)^2} = \sqrt{16 + 4 + 1} = \sqrt{21}$
- $|\nabla G| = \sqrt{2^2 + (-2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$

Thus,

$$\cos\theta = \frac{13}{\sqrt{21} \cdot 3} = \frac{13}{3\sqrt{21}}$$

Rationalize the denominator:

$$\frac{13}{3\sqrt{21}} = \frac{13}{3\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}} = \frac{13\sqrt{21}}{63}$$

Since the dot product is positive, θ is acute, and the angle between the planes is θ. Therefore, the angle of intersection satisfies:

$$\boxed{\cos\theta = \frac{13\sqrt{21}}{63}}$$

The parametric equations for the curve of intersection are $x = t + \frac{1}{t}$, $y = t$, $z = 2 + \frac{1}{t^2}$ for $t \neq 0$. At the point $(2, 1, 3)$, $t = 1$.

The tangent vector to the curve is found by differentiating the parametric equations with respect to t:

$$\frac{dx}{dt} = 1 - \frac{1}{t^2}, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = -2\frac{1}{t^3}.$$

At $t = 1$:

$$\frac{dx}{dt} = 1 - \frac{1}{1^2} = 0, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = -2\frac{1}{1^3} = -2.$$

Thus, the tangent vector is $\langle 0, 1, -2 \rangle$.

For the surface $z = x^2 - y^2$, define $F(x, y, z) = x^2 - y^2 - z$. The gradient is $\nabla F = \langle 2x, -2y, -1 \rangle$. At $(2, 1, 3)$:

$$\nabla F = \langle 2 \cdot 2, -2 \cdot 1, -1 \rangle = \langle 4, -2, -1 \rangle.$$

A vector lies in the tangent plane if it is perpendicular to the normal vector, i.e., $\nabla F \cdot \langle 0, 1, -2 \rangle = 0$. Compute:

$$\langle 4, -2, -1 \rangle \cdot \langle 0, 1, -2 \rangle = 4 \cdot 0 + (-2) \cdot 1 + (-1) \cdot (-2) = 0 - 2 + 2 = 0.$$

The dot product is zero, so the tangent vector lies in the tangent plane of the first surface.

For the surface $z = 2 + (x - y)^2$, define $G(x, y, z) = (x - y)^2 - z + 2$. The gradient is $\nabla G = \langle 2(x - y), -2(x - y), -1 \rangle$. At $(2, 1, 3)$:

$$\nabla G = \langle 2(2 - 1), -2(2 - 1), -1 \rangle = \langle 2, -2, -1 \rangle.$$

Check if $\nabla G \cdot \langle 0, 1, -2 \rangle = 0$:

$$\langle 2, -2, -1 \rangle \cdot \langle 0, 1, -2 \rangle = 2 \cdot 0 + (-2) \cdot 1 + (-1) \cdot (-2) = 0 - 2 + 2 = 0.$$

The dot product is zero, so the tangent vector lies in the tangent plane of the second surface.

Thus, the tangent vector $\langle 0, 1, -2 \rangle$ is parallel to the tangent plane of each surface at the point $(2, 1, 3)$.

Q2

(2-Q3) Find the specified parametrization of the first quadrant part of the circle $x^2 + y^2 = a^2$:
- In terms of the y coordinate.
- In terms of the angle between the tangent line and the positive x-axis.
- In terms of the arc length from $(0, a)$.

To parametrize the first quadrant part of the circle $x^2 + y^2 = a^2$ in terms of the y-coordinate, let the parameter t represent the y-coordinate. Since the circle is in the first quadrant, $x \geq 0$ and $y \geq 0$, and the radius is a.

From the equation $x^2 + y^2 = a^2$, solve for x:

$$x = \sqrt{a^2 - y^2}$$

because x is non-negative in the first quadrant.

Set $y = t$, so:

$$x(t) = \sqrt{a^2 - t^2}, \quad y(t) = t$$

The parameter t ranges from 0 to a, corresponding to the points from $(a, 0)$ to $(0, a)$:
- When $t = 0$, $x = \sqrt{a^2 - 0^2} = a$, so the point is $(a, 0)$.
- When $t = a$, $x = \sqrt{a^2 - a^2} = 0$, so the point is $(0, a)$.

Thus, the parametric equations are:

$$x(t) = \sqrt{a^2 - t^2}, \quad y(t) = t, \quad \text{for} \quad 0 \leq t \leq a$$

In vector form, the parametrization is:

$$\vec{r}(t) = \left\langle \sqrt{a^2 - t^2}, t \right\rangle, \quad 0 \leq t \leq a$$

$$\boxed{\begin{array}{c} x = \sqrt{a^2 - t^2} \\ \\ y = t \\ \\ \text{for } 0 \le t \le a \end{array}}$$

To parametrize the first quadrant part of the circle $x^2 + y^2 = a^2$ in terms of the angle between the tangent line and the positive x-axis, let t be this angle. The parametric equations are derived as follows.

The slope of the tangent line to the circle at any point is given by the derivative. For $x^2 + y^2 = a^2$, implicit differentiation yields:

$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}.$$

Since the slope is also $\tan t$, where t is the angle with the positive x-axis, we have:

$$\tan t = -\frac{x}{y}.$$

Solving for x and y using the circle equation and the relation above:

$$x = -y \tan t.$$

Substituting into $x^2 + y^2 = a^2$:

$$(-y \tan t)^2 + y^2 = a^2 \implies y^2 \tan^2 t + y^2 = a^2 \implies y^2(\tan^2 t + 1) = a^2.$$

Since $\tan^2 t + 1 = \sec^2 t$, and $y > 0$ in the first quadrant:

$$y^2 \sec^2 t = a^2 \implies y^2 = a^2 \cos^2 t \implies y = a \cos t,$$

as $\cos t > 0$ for $t \in (-\pi/2, 0)$. Then:

$$x = -y \tan t = -a \cos t \cdot \frac{\sin t}{\cos t} = -a \sin t.$$

As the point moves from $(a, 0)$ to $(0, a)$ in the first quadrant, the tangent angle t ranges from $-\pi/2$ to 0:

- At $t = -\pi/2$, $x = -a\sin(-\pi/2) = -a(-1) = a$, $y = a\cos(-\pi/2) = a \cdot 0 = 0$.
- At $t = 0$, $x = -a\sin 0 = 0$, $y = a\cos 0 = a$.

Thus, the parametric equations are:

$$\boxed{x = -a\sin t \quad ; \quad y = a\cos t \quad ; \quad -\frac{\pi}{2} \leq t \leq 0}$$

To parametrize the first quadrant part of the circle $x^2 + y^2 = a^2$ in terms of the arc length s from the point $(0, a)$, consider the standard parametric equations for a circle of radius a:

$$x = a\cos\theta, \quad y = a\sin\theta,$$

where θ is the angle from the positive x-axis, measured counterclockwise.

The point $(0, a)$ corresponds to $\theta = \pi/2$, and the point $(a, 0)$ corresponds to $\theta = 0$. Moving from $(0, a)$ to $(a, 0)$ in the first quadrant involves decreasing θ from $\pi/2$ to 0, which is a clockwise traversal.

The arc length s from $(0, a)$ to a point at angle θ is given by the radius multiplied by the absolute angle traversed. Since θ decreases from $\pi/2$ to 0, the angle traversed is $\pi/2 - \theta$. Thus,

$$s = a\left(\frac{\pi}{2} - \theta\right).$$

Solving for θ,

$$\frac{s}{a} = \frac{\pi}{2} - \theta \implies \theta = \frac{\pi}{2} - \frac{s}{a}.$$

Substitute θ into the parametric equations:

$$x = a\cos\left(\frac{\pi}{2} - \frac{s}{a}\right), \quad y = a\sin\left(\frac{\pi}{2} - \frac{s}{a}\right).$$

Using the identities $\cos(\pi/2 - \alpha) = \sin\alpha$ and $\sin(\pi/2 - \alpha) = \cos\alpha$, with $\alpha = s/a$,

$$x = a\sin\left(\frac{s}{a}\right), \quad y = a\cos\left(\frac{s}{a}\right).$$

The arc length s ranges from 0 at $(0,a)$ to $\pi a/2$ at $(a,0)$. For s in $[0,\pi a/2]$, s/a ranges from 0 to $\pi/2$, where both $\sin$ and $\cos$ are non-negative, ensuring the curve lies in the first quadrant.

Thus, the parametric equations in terms of the arc length s from $(0,a)$ are:

$$\boxed{x = a\sin\left(\frac{s}{a}\right) \quad ; \quad y = a\cos\left(\frac{s}{a}\right) \quad ; \quad 0 \leq s \leq \frac{\pi a}{2}}$$

Q3

(2-Q4) The curve C is defined to be the intersection of the hyperboloid

$$x^2 - \frac{1}{4}y^2 + 3z^2 = 1$$

and the plane

$$x + y + z = 0.$$

When y is very close to 0, and z is negative, find an expression giving z in terms of y.

Solution:

The curve C is the intersection of the hyperboloid $x^2 - \frac{1}{4}y^2 + 3z^2 = 1$ and the plane $x + y + z = 0$. To express z in terms of y when y is close to 0 and z is negative, solve the system of equations.

From the plane equation, solve for x:

$$x = -y - z.$$

Substitute this into the hyperboloid equation:

$$(-y - z)^2 - \frac{1}{4}y^2 + 3z^2 = 1.$$

Expand and simplify:

$$y^2 + 2yz + z^2 - \frac{1}{4}y^2 + 3z^2 = 1,$$

$$\left(y^2 - \frac{1}{4}y^2\right) + 2yz + (z^2 + 3z^2) = 1,$$

$$\frac{3}{4}y^2 + 2yz + 4z^2 = 1.$$

Rearrange into a quadratic equation in z:

$$4z^2 + 2yz + \frac{3}{4}y^2 - 1 = 0.$$

Multiply through by 4 to eliminate the fraction:

$$16z^2 + 8yz + 3y^2 - 4 = 0.$$

Solve for z using the quadratic formula, where $a = 16$, $b = 8y$, and $c = 3y^2 - 4$:

$$z = \frac{-8y \pm \sqrt{(8y)^2 - 4(16)(3y^2 - 4)}}{32}.$$

Compute the discriminant:

$$(8y)^2 - 4(16)(3y^2 - 4) = 64y^2 - 64(3y^2 - 4) = 64y^2 - 192y^2 + 256 = -128y^2 + 256 = 128(2 - y^2).$$

Thus,

$$z = \frac{-8y \pm \sqrt{128(2 - y^2)}}{32} = \frac{-8y \pm 8\sqrt{2}\sqrt{2 - y^2}}{32} = \frac{-y \pm \sqrt{2}\sqrt{2 - y^2}}{4}.$$

At $y = 0$, the solutions are $z = \pm\frac{1}{2}$. Since z is negative when y is close to 0, take the negative sign:

$$z = \frac{-y - \sqrt{2}\sqrt{2 - y^2}}{4}.$$

Simplify the expression under the square root:

$$\sqrt{2}\sqrt{2-y^2} = \sqrt{4-2y^2},$$

so

$$z = \frac{-y-\sqrt{4-2y^2}}{4}.$$

This expression is valid for $|y| < \sqrt{2}$, and when y is close to 0, z is negative.

Q4

(2-Q5) A particle traces out a curve in space, so that its position at time t is

$$\mathbf{r}(t) = e^{-t}\hat{\imath} + \frac{1}{t}\hat{\jmath} + (t-1)^2(t-3)^2\hat{k}$$

for $t > 0$. Let the positive z-axis point vertically upwards, as usual. When is the particle moving upwards, and when is it moving downwards? Is it moving faster at time $t = 1$ or at time $t = 3$?

Solution:

The velocity vector is found by differentiating the position vector $\mathbf{r}(t) = e^{-t}\hat{\imath} + \frac{1}{t}\hat{\jmath} + (t-1)^2(t-3)^2\hat{k}$ with respect to time t. The derivative of the z-component is $\frac{d}{dt}[(t-1)^2(t-3)^2] = 4(t-1)(t-2)(t-3)$. Thus, the velocity vector is:

$$\mathbf{v}(t) = -e^{-t}\hat{\imath} - \frac{1}{t^2}\hat{\jmath} + 4(t-1)(t-2)(t-3)\hat{k}.$$

The z-component of velocity is $v_z(t) = 4(t-1)(t-2)(t-3)$. The sign of $v_z(t)$ determines the direction of motion in the vertical direction (since the positive z-axis points upwards).
- The particle is moving upwards when $v_z(t) > 0$.
- The particle is moving downwards when $v_z(t) < 0$.

The roots of $v_z(t) = 0$ are at $t = 1$, $t = 2$, and $t = 3$. Analyzing the sign of $v_z(t)$ in the intervals $(0, 1)$, $(1, 2)$, $(2, 3)$, and $(3, \infty)$:
- For $t \in (0, 1)$, $v_z(t) < 0$ (downwards).
- For $t \in (1, 2)$, $v_z(t) > 0$ (upwards).
- For $t \in (2, 3)$, $v_z(t) < 0$ (downwards).
- For $t \in (3, \infty)$, $v_z(t) > 0$ (upwards).

At $t = 1$, $t = 2$, and $t = 3$, $v_z(t) = 0$, so the particle is not moving upwards or downwards at these instants.

The speed is the magnitude of the velocity vector, $|\mathbf{v}(t)| = \sqrt{v_x^2 + v_y^2 + v_z^2}$. At $t = 1$ and $t = 3$, $v_z = 0$, so:
- At $t = 1$, $\mathbf{v}(1) = \langle -e^{-1}, -1, 0 \rangle$, so speed is $\sqrt{(-e^{-1})^2 + (-1)^2} = \sqrt{e^{-2} + 1}$.
- At $t = 3$, $\mathbf{v}(3) = \langle -e^{-3}, -\frac{1}{9}, 0 \rangle$, so speed is $\sqrt{(-e^{-3})^2 + (-\frac{1}{9})^2} = \sqrt{e^{-6} + \frac{1}{81}}$.

Comparing the squares of the speeds:

$$(\sqrt{e^{-2} + 1})^2 = e^{-2} + 1, \quad (\sqrt{e^{-6} + \frac{1}{81}})^2 = e^{-6} + \frac{1}{81}.$$

The difference is:

$$(e^{-2} + 1) - (e^{-6} + \frac{1}{81}) = e^{-2} - e^{-6} + 1 - \frac{1}{81} = e^{-6}(e^4 - 1) + \frac{80}{81}.$$

Since $e^4 > 1$ and $\frac{80}{81} > 0$, the difference is positive, so the speed at $t = 1$ is greater than at $t = 3$.

Thus:

- The particle is moving upwards for $t \in (1, 2) \cup (3, \infty)$.
- The particle is moving downwards for $t \in (0, 1) \cup (2, 3)$.
- The particle is moving faster at $t = 1$ than at $t = 3$.

Q5

(2-Q6) Let

$$\mathbf{r}(t) = \left(t^2, 3, \frac{1}{3}t^3\right)$$

- Find the unit tangent vector to this parametrized curve at $t = 1$, pointing in the direction of increasing t.
- Find the arc length of the curve from the previous question between the points $(0, 3, 0)$ and $(1, 3, -\frac{1}{3})$.

Solution:

The position vector is given by $\mathbf{r}(t) = \left(t^2, 3, \frac{1}{3}t^3\right)$.

To find the unit tangent vector at $t = 1$, first compute the derivative of $\mathbf{r}(t)$ with respect to t:

$$\mathbf{r}'(t) = \frac{d}{dt}\left(t^2, 3, \frac{1}{3}t^3\right) = \left(2t, 0, t^2\right).$$

Evaluate $\mathbf{r}'(t)$ at $t = 1$:

$$\mathbf{r}'(1) = (2 \cdot 1, 0, 1^2) = (2, 0, 1).$$

The magnitude of $\mathbf{r}'(1)$ is:

$$\|\mathbf{r}'(1)\| = \sqrt{2^2 + 0^2 + 1^2} = \sqrt{4 + 1} = \sqrt{5}.$$

The unit tangent vector is the derivative vector divided by its magnitude:

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{(2, 0, 1)}{\sqrt{5}} = \left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right).$$

This vector points in the direction of increasing t since it is derived from the derivative with respect to t.

$$\boxed{\left(\frac{2}{\sqrt{5}},\ 0,\ \frac{1}{\sqrt{5}}\right)}$$

The curve is given by $\mathbf{r}(t) = (t^2, 3, \frac{1}{3}t^3)$. The points of interest are $(0, 3, 0)$ and $(1, 3, -\frac{1}{3})$.

- At $t = 0$, $\mathbf{r}(0) = (0, 3, 0)$.
- At $t = -1$, $\mathbf{r}(-1) = (1, 3, -\frac{1}{3})$.

The arc length is computed from $t = -1$ to $t = 0$, as this corresponds to the segment between the given points.

The velocity vector is the derivative of $\mathbf{r}(t)$:

$$\mathbf{r}'(t) = \left(\frac{d}{dt}(t^2), \frac{d}{dt}(3), \frac{d}{dt}\left(\frac{1}{3}t^3\right)\right) = (2t, 0, t^2).$$

The speed is the magnitude of the velocity vector:

$$\|\mathbf{r}'(t)\| = \sqrt{(2t)^2 + 0^2 + (t^2)^2} = \sqrt{4t^2 + t^4} = \sqrt{t^4 + 4t^2} = \sqrt{t^2(t^2 + 4)} = |t|\sqrt{t^2 + 4}.$$

For $t \in [-1, 0]$, t is negative, so $|t| = -t$. Thus:

$$\|\mathbf{r}'(t)\| = -t\sqrt{t^2 + 4}.$$

The arc length s is given by:

$$s = \int_{-1}^{0} \|\mathbf{r}'(t)\| dt = \int_{-1}^{0} (-t)\sqrt{t^2 + 4} dt.$$

Use the substitution $u = t^2 + 4$. Then $du = 2tdt$, so $tdt = \frac{1}{2}du$ and $-tdt = -\frac{1}{2}du$.

- When $t = -1$, $u = (-1)^2 + 4 = 5$.
- When $t = 0$, $u = 0^2 + 4 = 4$.

The integral becomes:

$$s = \int_{5}^{4} -\frac{1}{2}\sqrt{u} du.$$

Switching the limits of integration to make it increasing:

$$s = -\frac{1}{2}\int_5^4 u^{1/2}du = -\frac{1}{2}\left(-\int_4^5 u^{1/2}du\right) = \frac{1}{2}\int_4^5 u^{1/2}du.$$

Integrate $u^{1/2}$:

$$\int u^{1/2}du = \frac{2}{3}u^{3/2}.$$

Thus:

$$s = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_4^5 = \frac{1}{3}\left[u^{3/2}\right]_4^5 = \frac{1}{3}\left(5^{3/2} - 4^{3/2}\right).$$

Compute the values:

- $5^{3/2} = \sqrt{5^3} = \sqrt{125} = \sqrt{25 \cdot 5} = 5\sqrt{5}$,
- $4^{3/2} = \sqrt{4^3} = \sqrt{64} = 8$.

So:

$$s = \frac{1}{3}(5\sqrt{5} - 8).$$

The arc length between the points $(0, 3, 0)$ and $(1, 3, -\frac{1}{3})$ is $\frac{5\sqrt{5}-8}{3}$.

$$\boxed{\frac{5\sqrt{5} - 8}{3}}$$

Q6

(2-Q8) A curve in $\mathbb{R}^3$ is given by the vector equation $\mathbf{r}(t) = \left(2t\cos t, 2t\sin t, \frac{t^3}{3}\right)$.

- Find the length of the curve between $t = 0$ and $t = 2$.

- Find the parametric equations of the tangent line to the curve at $t = \pi$.

Solution:

The position vector is given by $\mathbf{r}(t) = (2t\cos t, 2t\sin t, \frac{t^3}{3})$.

To find the arc length from $t = 0$ to $t = 2$, compute the derivative $\mathbf{r}'(t)$:

- $x(t) = 2t\cos t$, so $x'(t) = 2\cos t - 2t\sin t$
- $y(t) = 2t\sin t$, so $y'(t) = 2\sin t + 2t\cos t$
- $z(t) = \frac{t^3}{3}$, so $z'(t) = t^2$

Thus, $\mathbf{r}'(t) = (2\cos t - 2t\sin t, 2\sin t + 2t\cos t, t^2)$.

The magnitude of $\mathbf{r}'(t)$ is:

$$\|\mathbf{r}'(t)\| = \sqrt{(2\cos t - 2t\sin t)^2 + (2\sin t + 2t\cos t)^2 + (t^2)^2}$$

Factor out the 2 in the first two components:

$$\|\mathbf{r}'(t)\| = \sqrt{4(\cos t - t\sin t)^2 + 4(\sin t + t\cos t)^2 + t^4}$$

Expand the squares:

$$(\cos t - t\sin t)^2 = \cos^2 t - 2t\cos t\sin t + t^2\sin^2 t$$

$$(\sin t + t\cos t)^2 = \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t$$

Add them:

$$(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 = \cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t$$

The terms $-2t\cos t\sin t$ and $2t\sin t\cos t$ cancel, leaving:

$$\cos^2 t + \sin^2 t + t^2\sin^2 t + t^2\cos^2 t = 1 + t^2(\sin^2 t + \cos^2 t) = 1 + t^2$$

So:

$$4(\cos t - t\sin t)^2 + 4(\sin t + t\cos t)^2 = 4(1+t^2)$$

Thus:

$$\|\mathbf{r}'(t)\| = \sqrt{4(1+t^2)+t^4} = \sqrt{4+4t^2+t^4} = \sqrt{(t^2+2)^2}$$

Since $t^2+2>0$ for all real t, $\sqrt{(t^2+2)^2} = |t^2+2| = t^2+2$.

The arc length is:

$$L = \int_0^2 \|\mathbf{r}'(t)\| dt = \int_0^2 (t^2+2) dt$$

Integrate:

$$\int (t^2+2) dt = \frac{t^3}{3} + 2t$$

Evaluate from 0 to 2:

$$\left[\frac{2^3}{3} + 2\cdot 2\right] - \left[\frac{0^3}{3} + 2\cdot 0\right] = \frac{8}{3} + 4 = \frac{8}{3} + \frac{12}{3} = \frac{20}{3}$$

The length of the curve is $\frac{20}{3}$.

The curve is given by the vector equation $\mathbf{r}(t) = \left(2t\cos t, 2t\sin t, \frac{t^3}{3}\right)$.

To find the tangent line at $t=\pi$, first compute the point on the curve at $t=\pi$:

$$\mathbf{r}(\pi) = \left(2\pi\cos\pi, 2\pi\sin\pi, \frac{\pi^3}{3}\right) = \left(2\pi\cdot(-1), 2\pi\cdot 0, \frac{\pi^3}{3}\right) = \left(-2\pi, 0, \frac{\pi^3}{3}\right).$$

Next, compute the derivative $\mathbf{r}'(t)$, which gives the direction vector of the tangent line:

$$\mathbf{r}'(t) = \left(\frac{d}{dt}(2t\cos t), \frac{d}{dt}(2t\sin t), \frac{d}{dt}\left(\frac{t^3}{3}\right)\right).$$

Differentiate each component:

- For the x-component: $\frac{d}{dt}(2t\cos t) = 2\left(t \cdot (-\sin t) + \cos t \cdot 1\right) = 2(-\sin t + \cos t)$.
- For the y-component: $\frac{d}{dt}(2t\sin t) = 2\left(t \cdot \cos t + \sin t \cdot 1\right) = 2(t\cos t + \sin t)$.
- For the z-component: $\frac{d}{dt}\left(\frac{t^3}{3}\right) = t^2$.

Thus,

$$\mathbf{r}'(t) = \big(2(-t\sin t + \cos t), 2(t\cos t + \sin t), t^2\big).$$

Evaluate at $t = \pi$:

$$\mathbf{r}'(\pi) = \big(2(-\pi\sin\pi + \cos\pi), 2(\pi\cos\pi + \sin\pi), \pi^2\big).$$

Substitute $\sin\pi = 0$ and $\cos\pi = -1$:

- x-component: $2(-\pi \cdot 0 + (-1)) = 2(-1) = -2$.
- y-component: $2(\pi \cdot (-1) + 0) = 2(-\pi) = -2\pi$.
- z-component: π^2.

So,

$$\mathbf{r}'(\pi) = (-2, -2\pi, \pi^2).$$

The parametric equations of the tangent line at the point $(-2\pi, 0, \frac{\pi^3}{3})$ with direction vector $(-2, -2\pi, \pi^2)$ are given by:

$$\begin{cases} x = -2\pi + s \cdot (-2) = -2\pi - 2s, \\ y = 0 + s \cdot (-2\pi) = -2\pi s, \\ z = \frac{\pi^3}{3} + s \cdot \pi^2 = \frac{\pi^3}{3} + \pi^2 s, \end{cases}$$

where s is a real parameter.

Q7

(2-Q15) The position of a particle at time t (measured in seconds) is given by

$$\mathbf{r}(t) = t\cos\left(\frac{\pi t}{2}\right)\hat{\imath} + t\sin\left(\frac{\pi t}{2}\right)\hat{\jmath} + t\hat{k}$$

- Show that the path of the particle lies on the cone $z^2 = x^2 + y^2$.
- Find the velocity vector and the speed at time t.
- Suppose that at time $t = 1$ the particle flies off the path on a line L in the direction tangent to the path. Find the equation of the line L.
- How long does it take for the particle to hit the plane $x = -1$ after it started moving along the straight line L?

Solution:

The position vector of the particle is given by:

$$\mathbf{r}(t) = t\cos\left(\frac{\pi t}{2}\right)\hat{\imath} + t\sin\left(\frac{\pi t}{2}\right)\hat{\jmath} + t\hat{k}$$

This corresponds to the parametric equations:

$$x(t) = t\cos\left(\frac{\pi t}{2}\right), \quad y(t) = t\sin\left(\frac{\pi t}{2}\right), \quad z(t) = t$$

To show that the path lies on the cone $z^2 = x^2 + y^2$, compute $x(t)^2 + y(t)^2$:

$$x(t)^2 + y(t)^2 = \left[t\cos\left(\frac{\pi t}{2}\right)\right]^2 + \left[t\sin\left(\frac{\pi t}{2}\right)\right]^2 = t^2\cos^2\left(\frac{\pi t}{2}\right) + t^2\sin^2\left(\frac{\pi t}{2}\right)$$

Factor out t^2:

$$x(t)^2 + y(t)^2 = t^2 \left(\cos^2\left(\frac{\pi t}{2}\right) + \sin^2\left(\frac{\pi t}{2}\right) \right)$$

Using the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ with $\theta = \frac{\pi t}{2}$:

$$x(t)^2 + y(t)^2 = t^2 \cdot 1 = t^2$$

Now, compute $z(t)^2$:

$$z(t)^2 = t^2$$

Thus, for all t:

$$z(t)^2 = t^2 = x(t)^2 + y(t)^2$$

Therefore, the path satisfies $z^2 = x^2 + y^2$ for all t, meaning the path lies on the cone $z^2 = x^2 + y^2$.

The velocity vector $\mathbf{v}(t)$ is the derivative of the position vector $\mathbf{r}(t)$ with respect to time t.

Given:

$$\mathbf{r}(t) = t\cos\left(\frac{\pi t}{2}\right)\hat{\imath} + t\sin\left(\frac{\pi t}{2}\right)\hat{\jmath} + t\hat{k}$$

The components are:
- $x(t) = t\cos\left(\frac{\pi t}{2}\right)$
- $y(t) = t\sin\left(\frac{\pi t}{2}\right)$
- $z(t) = t$

Differentiating each component with respect to t:
- For $x(t)$:

$$\frac{dx}{dt} = \frac{d}{dt}\left[t\cos\left(\frac{\pi t}{2}\right)\right] = \cos\left(\frac{\pi t}{2}\right) - \frac{\pi t}{2}\sin\left(\frac{\pi t}{2}\right)$$

using the product rule and chain rule.
- For $y(t)$:

$$\frac{dy}{dt} = \frac{d}{dt}\left[t\sin\left(\frac{\pi t}{2}\right)\right] = \sin\left(\frac{\pi t}{2}\right) + \frac{\pi t}{2}\cos\left(\frac{\pi t}{2}\right)$$

using the product rule and chain rule.
- For $z(t)$:

$$\frac{dz}{dt} = \frac{d}{dt}[t] = 1$$

Thus, the velocity vector is:

$$\mathbf{v}(t) = \left[\cos\left(\frac{\pi t}{2}\right) - \frac{\pi t}{2}\sin\left(\frac{\pi t}{2}\right)\right]\hat{\imath} + \left[\sin\left(\frac{\pi t}{2}\right) + \frac{\pi t}{2}\cos\left(\frac{\pi t}{2}\right)\right]\hat{\jmath} + \hat{k}$$

The speed is the magnitude of the velocity vector:

$$|\mathbf{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Substitute the components:

$$|\mathbf{v}(t)| = \sqrt{\left[\cos\left(\frac{\pi t}{2}\right) - \frac{\pi t}{2}\sin\left(\frac{\pi t}{2}\right)\right]^2 + \left[\sin\left(\frac{\pi t}{2}\right) + \frac{\pi t}{2}\cos\left(\frac{\pi t}{2}\right)\right]^2 + (1)^2}$$

Simplify the expression inside the square root. Let $\theta = \frac{\pi t}{2}$, so:
- $\frac{dx}{dt} = \cos\theta - \theta\sin\theta$
- $\frac{dy}{dt} = \sin\theta + \theta\cos\theta$
- $\frac{dz}{dt} = 1$

Now compute:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2$$

$$= \cos^2\theta - 2\theta\cos\theta\sin\theta + \theta^2\sin^2\theta + \sin^2\theta + 2\theta\sin\theta\cos\theta + \theta^2\cos^2\theta$$

The cross terms cancel:

$$-2\theta\cos\theta\sin\theta + 2\theta\sin\theta\cos\theta = 0$$

So:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \cos^2\theta + \sin^2\theta + \theta^2\sin^2\theta + \theta^2\cos^2\theta = 1 + \theta^2(\sin^2\theta + \cos^2\theta) = 1 + \theta^2$$

Since $\theta = \frac{\pi t}{2}$:

$$\theta^2 = \left(\frac{\pi t}{2}\right)^2 = \frac{\pi^2 t^2}{4}$$

Thus:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 + \frac{\pi^2 t^2}{4}$$

Add the z-component:

$$\left(\frac{dz}{dt}\right)^2 = 1^2 = 1$$

So:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 1 + \frac{\pi^2 t^2}{4} + 1 = 2 + \frac{\pi^2 t^2}{4}$$

Therefore:

$$|\mathbf{v}(t)| = \sqrt{2 + \frac{\pi^2 t^2}{4}} = \sqrt{\frac{8 + \pi^2 t^2}{4}} = \frac{\sqrt{8 + \pi^2 t^2}}{2}$$

The velocity vector and speed are:

$$\boxed{\mathbf{v}(t) = \left(\cos\left(\frac{\pi t}{2}\right) - \frac{\pi t}{2}\sin\left(\frac{\pi t}{2}\right)\right)\hat{\imath} + \left(\sin\left(\frac{\pi t}{2}\right) + \frac{\pi t}{2}\cos\left(\frac{\pi t}{2}\right)\right)\hat{\jmath} + \hat{k}}$$

$$\boxed{\text{speed} = \frac{\sqrt{8 + \pi^2 t^2}}{2}}$$

The position of the particle at time $t = 1$ is found by substituting $t = 1$ into the position vector $\mathbf{r}(t)$:

$$x(1) = 1 \cdot \cos\left(\frac{\pi \cdot 1}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$y(1) = 1 \cdot \sin\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$z(1) = 1$$

Thus, the position is $(0, 1, 1)$.

The velocity vector is the derivative of the position vector. From previous calculations, the velocity vector is:

$$\mathbf{v}(t) = \left[\cos\left(\frac{\pi t}{2}\right) - \frac{\pi t}{2}\sin\left(\frac{\pi t}{2}\right)\right]\hat{\imath} + \left[\sin\left(\frac{\pi t}{2}\right) + \frac{\pi t}{2}\cos\left(\frac{\pi t}{2}\right)\right]\hat{\jmath} + \hat{k}$$

Substituting $t = 1$:

$$\frac{\pi t}{2} = \frac{\pi}{2}, \quad \cos\left(\frac{\pi}{2}\right) = 0, \quad \sin\left(\frac{\pi}{2}\right) = 1$$

$$v_x(1) = 0 - \frac{\pi}{2} \cdot 1 = -\frac{\pi}{2}$$

$$v_y(1) = 1 + \frac{\pi}{2} \cdot 0 = 1$$

$$v_z(1) = 1$$

Thus, the velocity vector at $t = 1$ is $\langle -\frac{\pi}{2}, 1, 1 \rangle$.

The line L is tangent to the path at $t = 1$ and passes through the point $(0, 1, 1)$ in the direction of the velocity vector $\langle -\frac{\pi}{2}, 1, 1 \rangle$. Parametric equations for the line are given by:

$$x = x_0 + s \cdot v_x, \quad y = y_0 + s \cdot v_y, \quad z = z_0 + s \cdot v_z$$

where $(x_0, y_0, z_0) = (0, 1, 1)$ and s is a real parameter. Substituting the components:

$$x = 0 + s \cdot \left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}s$$

$$y = 1 + s \cdot 1 = 1 + s$$

$$z = 1 + s \cdot 1 = 1 + s$$

Thus, the parametric equations for the line L are:

$$\boxed{x = -\frac{\pi}{2}s \quad ; \quad y = 1 + s \quad ; \quad z = 1 + s}$$

The particle flies off the path at time $t = 1$ and moves along the line L with constant velocity. The position at $t = 1$ is $(0, 1, 1)$, and the velocity vector at this point is $\langle -\frac{\pi}{2}, 1, 1 \rangle$.

The position as a function of time for $t \geq 1$ is given by:

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{\pi}{2} \\ 1 \\ 1 \end{pmatrix} (t - 1)$$

So the components are:

$$x(t) = -\frac{\pi}{2}(t - 1), \quad y(t) = 1 + (t - 1), \quad z(t) = 1 + (t - 1).$$

The particle hits the plane $x = -1$. Set $x(t) = -1$:

$$-\frac{\pi}{2}(t - 1) = -1.$$

Solving for t:

$$\frac{\pi}{2}(t-1) = 1 \implies t-1 = \frac{2}{\pi}.$$

The time taken after the particle starts moving along the line L (at $t=1$) is $t-1 = \frac{2}{\pi}$ seconds.

Thus, the time taken to hit the plane $x=-1$ is $\frac{2}{\pi}$ seconds.

Q8

(2-Q16)

- The curve $\mathbf{r}_1(t) = (1+t, t^2, t^3)$ and $\mathbf{r}_2(t) = (\cos t, \sin t, t)$ intersect at the point $P(1,0,0)$. Find the angle of intersection between the curves at the point P.
- Find the distance between the line of intersection of the planes $x+y=2$ and $2x-z=4$ and the line $\mathbf{r}(t) = (t, -1+2t, 1+3t)$.

Solution:

The curves are given by $\mathbf{r}_1(t) = (1+t, t^2, t^3)$ and $\mathbf{r}_2(t) = (\cos t, \sin t, t)$, and they intersect at the point $P(1,0,0)$.

To find the angle of intersection at P, compute the tangent vectors to each curve at this point.

For $\mathbf{r}_1(t)$:

- The derivative is $\mathbf{r}_1'(t) = (1, 2t, 3t^2)$.
- The curve passes through $P(1,0,0)$ when $t=0$ (since $1+0=1$, $0^2=0$, $0^3=0$).
- Thus, the tangent vector at P is $\mathbf{r}_1'(0) = (1,0,0)$.

For $\mathbf{r}_2(t)$:

- The derivative is $\mathbf{r}_2'(t) = (-\sin t, \cos t, 1)$.

- The curve passes through $P(1,0,0)$ when $t=0$ (since $\cos 0 = 1$, $\sin 0 = 0$, $0 = 0$).
- Thus, the tangent vector at P is $\mathbf{r}_2'(0) = (0,1,1)$.

The angle θ between the curves is the angle between their tangent vectors, given by the dot product formula:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\|\|\mathbf{v}_2\| \cos\theta$$

where $\mathbf{v}_1 = (1,0,0)$ and $\mathbf{v}_2 = (0,1,1)$.

Compute the dot product:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (1)(0) + (0)(1) + (0)(1) = 0$$

Compute the magnitudes:

$$\|\mathbf{v}_1\| = \sqrt{1^2+0^2+0^2} = 1, \quad \|\mathbf{v}_2\| = \sqrt{0^2+1^2+1^2} = \sqrt{2}$$

Substitute into the formula:

$$0 = (1)(\sqrt{2})\cos\theta \implies \cos\theta = 0 \implies \theta = \frac{\pi}{2} \text{ radians}$$

Thus, the angle of intersection between the curves at P is $\frac{\pi}{2}$ radians.

The normal vector to the plane $x+y=2$ is $\mathbf{n_1} = \langle 1,1,0\rangle$. The normal vector to the plane $2x-z=4$ is $\mathbf{n_2} = \langle 2,0,-1\rangle$.

The direction vector $\mathbf{D}$ of the line of intersection is the cross product of $\mathbf{n_1}$ and $\mathbf{n_2}$:

$$\mathbf{D} = \mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix} = \mathbf{i}(1\cdot(-1) - 0\cdot 0) - \mathbf{j}(1\cdot(-1) - 0\cdot 2) + \mathbf{k}(1\cdot 0 - 1\cdot 2) = \langle -1, 1, -2\rangle.$$

A point on the line of intersection is found by solving the system of equations. Setting $z=0$:
- $2x - 0 = 4$ gives $x = 2$,
- $2 + y = 2$ gives $y = 0$.

Thus, point $A = (2, 0, 0)$.

Parametric equations for the line are:

$$x = 2 - s, \quad y = s, \quad z = -2s, \quad s \in \mathbb{R}.$$

The line $\mathbf{r}(t) = (t, -1 + 2t, 1 + 3t)$ has direction vector $\mathbf{E} = \langle 1, 2, 3 \rangle$. A point on this line is found by setting $t = 0$, so $B = (0, -1, 1)$.

The direction vectors $\mathbf{D} = \langle -1, 1, -2 \rangle$ and $\mathbf{E} = \langle 1, 2, 3 \rangle$ are not parallel since there is no scalar k such that $k\langle 1, 2, 3 \rangle = \langle -1, 1, -2 \rangle$.

To check for intersection, set the parametric equations equal:

$$(2 - s, s, -2s) = (t, -1 + 2t, 1 + 3t).$$

Solving:
- $2 - s = t$,
- $s = -1 + 2t$,
- $-2s = 1 + 3t$.

Substituting $t = 2 - s$ into the second equation: $s = -1 + 2(2 - s) = 3 - 2s$, so $s = 1$. Then $t = 2 - 1 = 1$.

Check the third equation: $-2(1) = -2 \neq 1 + 3(1) = 4$. No solution, so the lines do not intersect.

Thus, the lines are skew.

Vector $\overrightarrow{AB} = B - A = \langle 0 - 2, -1 - 0, 1 - 0 \rangle = \langle -2, -1, 1 \rangle$. Cross product $\mathbf{D} \times \mathbf{E}$:

$$\mathbf{D} \times \mathbf{E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -2 \\ 1 & 2 & 3 \end{vmatrix} = \mathbf{i}(1 \cdot 3 - (-2) \cdot 2) - \mathbf{j}(-1 \cdot 3 - (-2) \cdot 1) + \mathbf{k}(-1 \cdot 2 - 1 \cdot 1) = \langle 7, 1, -3 \rangle.$$

Magnitude of $\mathbf{D} \times \mathbf{E}$:

$$\|\mathbf{D} \times \mathbf{E}\| = \sqrt{7^2 + 1^2 + (-3)^2} = \sqrt{49 + 1 + 9} = \sqrt{59}.$$

Dot product $\overrightarrow{AB} \cdot (\mathbf{D} \times \mathbf{E}) = \langle -2, -1, 1 \rangle \cdot \langle 7, 1, -3 \rangle = (-2)(7) + (-1)(1) + (1)(-3) = -14 - 1 - 3 = -18$. Absolute value: $|-18| = 18$. Distance:

$$\text{Distance} = \frac{18}{\sqrt{59}} = \frac{18\sqrt{59}}{59}.$$

Q9

A ladybug is climbing on a Volkswagen Bug (= VW). In its starting position, the surface of the VW is represented by the unit semicircle

$$x^2 + y^2 = 1, \quad y \geq 0$$

in the xy-plane. The road is represented as the x-axis. At time $t = 0$, the ladybug starts at the front bumper, $(1, 0)$, and walks counterclockwise around the VW at unit speed relative to the VW. At the same time, the VW moves to the right at speed 10.

- Find the parametric formula for the trajectory of the ladybug, and find its position when it reaches the rear bumper. (At $t = 0$, the rear bumper is at $(-1, 0)$.)
- Compute the speed of the bug, and find where it is largest and smallest. Hint: It is easier to work with the square of the speed.

Solution:

The Volkswagen Bug is represented by the unit semicircle $x^2 + y^2 = 1$, $y \geq 0$, with the road as the x-axis. At time $t = 0$, the center of the semicircle is at $(0, 0)$. The VW moves to the right at speed 10, so at time t, the center is at $(10t, 0)$.

The ladybug starts at the front bumper $(1, 0)$ relative to the center and walks counterclockwise at unit speed relative to the VW. The radius of the semicircle is 1, so the angular speed is $\omega = \text{speed}/\text{radius} = 1/1 = 1$ radian per second. At time t, the angular position relative to the center is $\theta(t) = t$ radians. The position relative to the center is $(\cos t, \sin t)$.

In the fixed coordinate system, the position of the ladybug is the sum of the position of the center and the relative position:

- $x(t) = 10t + \cos t$

- $y(t) = \sin t$

The rear bumper is at $(-1, 0)$ relative to the center at $t = 0$. The ladybug reaches the rear bumper when $\theta = t = \pi$ radians (since the arc length from front to rear is π units, and speed is 1 unit per second). At $t = \pi$:

- $x(\pi) = 10\pi + \cos\pi = 10\pi + (-1) = 10\pi - 1$
- $y(\pi) = \sin\pi = 0$

Thus, the position when the ladybug reaches the rear bumper is $(10\pi - 1, 0)$.

The parametric equations for the trajectory are given for $0 \le t \le \pi$.

$$\boxed{x(t) = 10t + \cos t \quad ; \quad y(t) = \sin t}$$

$$\boxed{(10\pi - 1,\ 0)}$$

The parametric equations for the trajectory of the ladybug are given by:

$$x(t) = 10t + \cos t, \quad y(t) = \sin t$$

for $0 \le t \le \pi$.

The velocity vector is obtained by differentiating the position vector with respect to time:

$$\frac{dx}{dt} = 10 - \sin t, \quad \frac{dy}{dt} = \cos t.$$

Thus, the velocity vector is $(10 - \sin t, \cos t)$.

The speed is the magnitude of the velocity vector:

$$s(t) = \sqrt{(10 - \sin t)^2 + \cos^2 t}.$$

Simplifying the expression inside the square root:

$$(10 - \sin t)^2 + \cos^2 t = 100 - 20\sin t + \sin^2 t + \cos^2 t = 100 - 20\sin t + 1 = 101 - 20\sin t,$$

since $\sin^2 t + \cos^2 t = 1$. Therefore,

$$s(t) = \sqrt{101 - 20\sin t}.$$

To find where the speed is largest and smallest, consider the square of the speed, $u(t) = s(t)^2 = 101 - 20\sin t$, as the square root is an increasing function and the extrema occur at the same points. The domain is $t \in [0, \pi]$, and $\sin t$ ranges from 0 to 1 in this interval, with $\sin t = 0$ at $t = 0$ and $t = \pi$, and $\sin t = 1$ at $t = \pi/2$.

Since the coefficient of $\sin t$ is negative, $u(t)$ is minimized when $\sin t$ is maximized and maximized when $\sin t$ is minimized:

- Minimum $u(t) = 101 - 20 \cdot 1 = 81$ at $t = \pi/2$.
- Maximum $u(t) = 101 - 20 \cdot 0 = 101$ at $t = 0$ and $t = \pi$.

Thus, the minimum speed is $s(t) = \sqrt{81} = 9$ and the maximum speed is $s(t) = \sqrt{101}$.

The positions corresponding to these times are:

- At $t = \pi/2$: $x(\pi/2) = 10 \cdot (\pi/2) + \cos(\pi/2) = 5\pi + 0 = 5\pi$, $y(\pi/2) = \sin(\pi/2) = 1$, so the point is $(5\pi, 1)$.
- At $t = 0$: $x(0) = 10 \cdot 0 + \cos 0 = 1$, $y(0) = \sin 0 = 0$, so the point is $(1, 0)$.
- At $t = \pi$: $x(\pi) = 10\pi + \cos\pi = 10\pi - 1$, $y(\pi) = \sin\pi = 0$, so the point is $(10\pi - 1, 0)$.

The critical points are confirmed by the derivative of $u(t)$: $u'(t) = -20\cos t$, which is zero at $t = \pi/2$ (minimum by second derivative test), and the maximum occurs at the endpoints $t = 0$ and $t = \pi$.

Therefore, the minimum speed of 9 occurs at the point $(5\pi, 1)$, and the maximum speed of $\sqrt{101}$ occurs at the points $(1, 0)$ and $(10\pi - 1, 0)$.

$$\boxed{\begin{array}{c}\text{Minimum speed: } 9 \\ \text{at } (5\pi,\ 1) \\ \text{Maximum speed: } \sqrt{101} \\ \text{at } (1,\ 0) \text{ and } (10\pi - 1,\ 0)\end{array}}$$

Q10

(3-Q13) Let P be the point where the curve

$$\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, \quad (0 \leq t < \infty)$$

intersects the surface

$$z^3 + xyz - 2 = 0$$

Find the (acute) angle between the curve and the surface at P.

Solution:

The curve is given by $\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$ for $0 \leq t < \infty$, and the surface is defined by $z^3 + xyz - 2 = 0$. To find the intersection point P, substitute the parametric equations $x = t^3$, $y = t$, $z = t^2$ into the surface equation:

$$(t^2)^3 + (t^3)(t)(t^2) - 2 = t^6 + t^6 - 2 = 2t^6 - 2 = 0.$$

Solving $2t^6 = 2$ gives $t^6 = 1$, so $t = 1$ or $t = -1$. Since $t \geq 0$, $t = 1$. At $t = 1$, $x = 1^3 = 1$, $y = 1$, $z = 1^2 = 1$, so $P = (1, 1, 1)$.

The tangent vector to the curve at P is the derivative of $\mathbf{r}(t)$ evaluated at $t = 1$:

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + \mathbf{j} + 2t\mathbf{k}, \quad \mathbf{r}'(1) = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} = (3, 1, 2).$$

The normal vector to the surface at P is the gradient of $F(x, y, z) = z^3 + xyz - 2$:

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) = (yz, xz, 3z^2 + xy).$$

At $(1, 1, 1)$:

$$\nabla F = (1 \cdot 1, 1 \cdot 1, 3(1)^2 + 1 \cdot 1) = (1, 1, 4).$$

The angle ϕ between the curve and the surface is the acute angle between the tangent vector and the tangent plane, given by:

$$\sin\phi = \frac{|\mathbf{T}\cdot\mathbf{n}|}{|\mathbf{T}||\mathbf{n}|},$$

where $\mathbf{T} = (3,1,2)$ and $\mathbf{n} = (1,1,4)$. Compute the dot product and magnitudes:

$$\mathbf{T}\cdot\mathbf{n} = 3\cdot 1 + 1\cdot 1 + 2\cdot 4 = 3 + 1 + 8 = 12,$$

$$|\mathbf{T}| = \sqrt{3^2+1^2+2^2} = \sqrt{14}, \quad |\mathbf{n}| = \sqrt{1^2+1^2+4^2} = \sqrt{18} = 3\sqrt{2}.$$

Thus,

$$|\mathbf{T}||\mathbf{n}| = \sqrt{14}\cdot 3\sqrt{2} = 3\sqrt{28} = 3\cdot 2\sqrt{7} = 6\sqrt{7},$$

$$\sin\phi = \frac{|12|}{6\sqrt{7}} = \frac{12}{6\sqrt{7}} = \frac{2}{\sqrt{7}} = \frac{2\sqrt{7}}{7}.$$

The acute angle ϕ satisfies $\sin\phi = \frac{2\sqrt{7}}{7}$, so:

$$\phi = \arcsin\left(\frac{2\sqrt{7}}{7}\right).$$
