A02

Q1

(1-Q42) Find the point on the plane

$$2x + y - z = 6$$

which is closest to the origin, by using Lagrange multipliers.

Solution:

To find the point on the plane $2x + y - z = 6$ closest to the origin using Lagrange multipliers, minimize the square of the distance to the origin, $f(x, y, z) = x^2 + y^2 + z^2$, subject to the constraint $g(x, y, z) = 2x + y - z - 6 = 0$.

The gradients are:

$$\nabla f = (2x, 2y, 2z), \quad \nabla g = (2, 1, -1).$$

Set $\nabla f = \lambda \nabla g$:

$$2x = 2\lambda, \quad 2y = \lambda, \quad 2z = -\lambda.$$

Solving for x, y, and z:

$$x = \lambda, \quad y = \frac{\lambda}{2}, \quad z = -\frac{\lambda}{2}.$$

Substitute into the constraint $2x + y - z = 6$:

$$2(\lambda) + \left(\frac{\lambda}{2}\right) - \left(-\frac{\lambda}{2}\right) = 2\lambda + \frac{\lambda}{2} + \frac{\lambda}{2} = 2\lambda + \lambda = 3\lambda = 6.$$

Thus, $\lambda = 2$. Then:

$$x = 2, \quad y = \frac{2}{2} = 1, \quad z = -\frac{2}{2} = -1.$$

The point $(2, 1, -1)$ lies on the plane since $2(2) + 1 - (-1) = 4 + 1 + 1 = 6$.

The distance to the origin is $\sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$, which is consistent with the distance formula for a plane.

Thus, the point on the plane closest to the origin is $(2, 1, -1)$.

$$\boxed{(2,\ 1,\ -1)}$$

Q2

(3-Q5) Assume that the function $F(x, y, z)$ satisfies the equation

$$\frac{\partial F}{\partial z} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$$

and the mixed partial derivatives $\frac{\partial^2 F}{\partial x \partial y}$ and $\frac{\partial^2 F}{\partial y \partial x}$ are equal. Let A be some constant and let

$$G(\gamma, t) = F(\gamma + s, \gamma - s, At).$$

Find the value of A such that

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial \gamma^2} + \frac{\partial^2 G}{\partial s^2}.$$

Solution:

The function $F(x, y, z)$ satisfies the partial differential equation

$$\frac{\partial F}{\partial z} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2},$$

with equal mixed partial derivatives $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$.

The function $G(\gamma, s, t)$ is defined as

$$G(\gamma, s, t) = F(\gamma + s, \gamma - s, At),$$

where $x = \gamma + s$, $y = \gamma - s$, and $z = At$. The goal is to find the constant A such that

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial \gamma^2} + \frac{\partial^2 G}{\partial s^2}.$$

Using the chain rule, the partial derivatives of G are computed as follows:

- The partial derivative of G with respect to t is

$$\frac{\partial G}{\partial t} = A \frac{\partial F}{\partial z}.$$

- The second partial derivative of G with respect to γ is

$$\frac{\partial^2 G}{\partial \gamma^2} = \frac{\partial^2 F}{\partial x^2} + 2 \frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2}.$$

- The second partial derivative of G with respect to s is

$$\frac{\partial^2 G}{\partial s^2} = \frac{\partial^2 F}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2}.$$

Adding the second derivatives with respect to γ and s gives

$$\frac{\partial^2 G}{\partial \gamma^2} + \frac{\partial^2 G}{\partial s^2} = \left(\frac{\partial^2 F}{\partial x^2} + 2 \frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2} \right) + \left(\frac{\partial^2 F}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2} \right) = 2 \frac{\partial^2 F}{\partial x^2} + 2 \frac{\partial^2 F}{\partial y^2}.$$

The equation to satisfy is

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial \gamma^2} + \frac{\partial^2 G}{\partial s^2},$$

so substituting the expressions yields

$$A\frac{\partial F}{\partial z} = 2\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right).$$

Given that F satisfies $\frac{\partial F}{\partial z} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$, the right-hand side simplifies to $2\frac{\partial F}{\partial z}$. Thus,

$$A\frac{\partial F}{\partial z} = 2\frac{\partial F}{\partial z}.$$

For this to hold, $A = 2$.

Q3

(3-Q8) The wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

arises in many models involving wave-like phenomena. Let $u(x,t)$ and $v(\xi,\eta)$ be related by the change of variables

$$u(x,t) = v(\xi(x,t), \eta(x,t))$$

$$\xi(x,t) = x - ct$$

$$\eta(x,t) = x + ct$$

- Questions:
  - Show that $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$ if and only if $\frac{\partial^2 v}{\partial \xi \partial \eta} = 0$.
    - Show that $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$ if and only if $u(x,t) = F(x-ct) + G(x+ct)$ for some functions F and G.

Solution:

To show that the wave equation holds if and only if $\frac{\partial^2 v}{\partial\xi\partial\eta} = 0$, express the second partial derivatives of u with respect to x and t in terms of the derivatives of v with respect to ξ and η.

First, compute the first partial derivatives of u:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial v}{\partial \xi}(1) + \frac{\partial v}{\partial \eta}(1) = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta},$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial v}{\partial \xi}(-c) + \frac{\partial v}{\partial \eta}(c) = -c\frac{\partial v}{\partial \xi} + c\frac{\partial v}{\partial \eta}.$$

Next, compute the second partial derivatives. For $\frac{\partial^2 u}{\partial x^2}$:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\right).$$

Applying the chain rule:

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial \xi}\right) = \frac{\partial^2 v}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 v}{\partial \xi\partial\eta}\frac{\partial \eta}{\partial x} = \frac{\partial^2 v}{\partial \xi^2}(1) + \frac{\partial^2 v}{\partial \xi\partial\eta}(1) = \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \xi\partial\eta},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial \eta}\right) = \frac{\partial^2 v}{\partial \eta\partial\xi}\frac{\partial \xi}{\partial x} + \frac{\partial^2 v}{\partial \eta^2}\frac{\partial \eta}{\partial x} = \frac{\partial^2 v}{\partial \xi\partial\eta}(1) + \frac{\partial^2 v}{\partial \eta^2}(1) = \frac{\partial^2 v}{\partial \xi\partial\eta} + \frac{\partial^2 v}{\partial \eta^2},$$

assuming equal mixed partial derivatives. Thus,

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \xi\partial\eta}\right) + \left(\frac{\partial^2 v}{\partial \xi\partial\eta} + \frac{\partial^2 v}{\partial \eta^2}\right) = \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi\partial\eta} + \frac{\partial^2 v}{\partial \eta^2}.$$

For $\frac{\partial^2 u}{\partial t^2}$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(-c\frac{\partial v}{\partial \xi} + c\frac{\partial v}{\partial \eta}\right) = c\frac{\partial}{\partial t}\left(-\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\right).$$

Applying the chain rule:

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial \xi}\right) = \frac{\partial^2 v}{\partial \xi^2}\frac{\partial \xi}{\partial t} + \frac{\partial^2 v}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial^2 v}{\partial \xi^2}(-c) + \frac{\partial^2 v}{\partial \xi \partial \eta}(c) = -c\frac{\partial^2 v}{\partial \xi^2} + c\frac{\partial^2 v}{\partial \xi \partial \eta},$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial \eta}\right) = \frac{\partial^2 v}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial^2 v}{\partial \eta^2}\frac{\partial \eta}{\partial t} = \frac{\partial^2 v}{\partial \xi \partial \eta}(-c) + \frac{\partial^2 v}{\partial \eta^2}(c) = -c\frac{\partial^2 v}{\partial \xi \partial \eta} + c\frac{\partial^2 v}{\partial \eta^2}.$$

Thus,

$$\frac{\partial}{\partial t}\left(-\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\right) = -\left(-c\frac{\partial^2 v}{\partial \xi^2} + c\frac{\partial^2 v}{\partial \xi \partial \eta}\right) + \left(-c\frac{\partial^2 v}{\partial \xi \partial \eta} + c\frac{\partial^2 v}{\partial \eta^2}\right)$$

$$= c\frac{\partial^2 v}{\partial \xi^2} - c\frac{\partial^2 v}{\partial \xi \partial \eta} - c\frac{\partial^2 v}{\partial \xi \partial \eta} + c\frac{\partial^2 v}{\partial \eta^2} = c\left(\frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}\right),$$

and

$$\frac{\partial^2 u}{\partial t^2} = c \cdot c\left(\frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}\right) = c^2\left(\frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}\right).$$

Substitute into the wave equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$= \left(\frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}\right) - \frac{1}{c^2}\left(c^2\left(\frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}\right)\right).$$

Simplify:

$$= \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2} - \left(\frac{\partial^2 v}{\partial \xi^2} - 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2}\right)$$

$$= \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial^2 v}{\partial \eta^2} - \frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta} - \frac{\partial^2 v}{\partial \eta^2} = 4\frac{\partial^2 v}{\partial \xi \partial \eta}.$$

The wave equation requires:

$$4\frac{\partial^2 v}{\partial\xi\partial\eta} = 0,$$

which implies:

$$\frac{\partial^2 v}{\partial\xi\partial\eta} = 0.$$

Conversely, if $\frac{\partial^2 v}{\partial\xi\partial\eta} = 0$, then the expression is zero, satisfying the wave equation. Therefore, the wave equation holds if and only if $\frac{\partial^2 v}{\partial\xi\partial\eta} = 0$.

The wave equation is given by

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0.$$

Consider the change of variables $\xi = x - ct$ and $\eta = x + ct$, and define $u(x,t) = v(\xi,\eta)$. From the previous result, the wave equation holds if and only if

$$\frac{\partial^2 v}{\partial\xi\partial\eta} = 0.$$

Integrating $\frac{\partial^2 v}{\partial\xi\partial\eta} = 0$ with respect to ξ gives

$$\frac{\partial v}{\partial\xi} = f(\xi),$$

where f is a function of ξ only. Integrating again with respect to ξ yields

$$v(\xi,\eta) = \int f(\xi)d\xi + g(\eta),$$

where g is a function of η only. Let $F(\xi) = \int f(\xi)d\xi$ and $G(\eta) = g(\eta)$, so

$$v(\xi,\eta) = F(\xi) + G(\eta).$$

Substituting back $\xi = x - ct$ and $\eta = x + ct$,

$$u(x,t) = F(x-ct) + G(x+ct).$$

Conversely, assume $u(x,t) = F(x-ct) + G(x+ct)$ for some functions F and G. Compute the partial derivatives:

$$\frac{\partial u}{\partial x} = F'(x-ct) + G'(x+ct), \quad \frac{\partial u}{\partial t} = -cF'(x-ct) + cG'(x+ct),$$

where the prime denotes the derivative with respect to the argument. The second derivatives are

$$\frac{\partial^2 u}{\partial x^2} = F''(x-ct) + G''(x+ct),$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left[-cF'(x-ct) + cG'(x+ct)\right]$$

$$= -c \cdot (-cF''(x-ct)) + c \cdot (cG''(x+ct)) = c^2F''(x-ct) + c^2G''(x+ct).$$

Substitute into the wave equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$= \left[F''(x-ct) + G''(x+ct)\right] - \frac{1}{c^2}\left[c^2F''(x-ct) + c^2G''(x+ct)\right]$$

$$= F''(x-ct) + G''(x+ct) - F''(x-ct) - G''(x+ct) = 0.$$

Thus, $u(x,t) = F(x-ct) + G(x+ct)$ satisfies the wave equation.

Q4

(3-Q10) Find all horizontal planes that are tangent to the surface with equation

$$z = xye^{-(x^2+y^2)/2}$$

What are the largest and smallest values of z on this surface?

The surface is given by $z = xye^{-(x^2+y^2)/2}$.

Solution:

A horizontal tangent plane occurs where the partial derivatives of z with respect to x and y are both zero. The partial derivatives are:

$$\frac{\partial z}{\partial x} = y(1-x^2)e^{-(x^2+y^2)/2}, \quad \frac{\partial z}{\partial y} = x(1-y^2)e^{-(x^2+y^2)/2}.$$

Since the exponential factor is never zero, setting the partial derivatives to zero gives:

$$y(1-x^2) = 0, \quad x(1-y^2) = 0.$$

Solving this system:

- If $y = 0$, then the second equation implies $x = 0$, so $(x, y) = (0, 0)$.
- If $x = 1$, then the second equation implies $1 - y^2 = 0$, so $y = \pm 1$, giving points $(1, 1)$ and $(1, -1)$.
- If $x = -1$, then the second equation implies $1 - y^2 = 0$, so $y = \pm 1$, giving points $(-1, 1)$ and $(-1, -1)$.

The corresponding z-values are:

- At $(0, 0)$: $z = 0 \cdot 0 \cdot e^0 = 0$.
- At $(1, 1)$: $z = 1 \cdot 1 \cdot e^{-(1+1)/2} = e^{-1} = \frac{1}{e}$.
- At $(1, -1)$: $z = 1 \cdot (-1) \cdot e^{-(1+1)/2} = -e^{-1} = -\frac{1}{e}$.
- At $(-1, 1)$: $z = (-1) \cdot 1 \cdot e^{-(1+1)/2} = -e^{-1} = -\frac{1}{e}$.
- At $(-1, -1)$: $z = (-1) \cdot (-1) \cdot e^{-(1+1)/2} = e^{-1} = \frac{1}{e}$.

Thus, the horizontal tangent planes are at $z = -\frac{1}{e}$, $z = 0$, and $z = \frac{1}{e}$.

To find the largest and smallest values of z on the surface, note that as $|x|$ or $|y|$ approaches infinity, $z \to 0$ because the exponential decay dominates. The critical points yield $z = \frac{1}{e}$, 0, and $-\frac{1}{e}$. Considering $|z| = |xy|e^{-(x^2+y^2)/2}$, and using $|xy| \leq \frac{x^2+y^2}{2}$ (with equality when $|x| = |y|$), it follows that:

$$|z| \leq \frac{x^2+y^2}{2} e^{-(x^2+y^2)/2}.$$

Set $u = \frac{x^2+y^2}{2}$, so $|z| \leq ue^{-u}$. The maximum of ue^{-u} occurs at $u = 1$, where the value is $e^{-1} = \frac{1}{e}$, achieved when $x^2 + y^2 = 2$ and $|x| = |y|$, i.e., at points like $(1, 1)$, $(-1, -1)$ (for maximum $z = \frac{1}{e}$) and $(1, -1)$, $(-1, 1)$ (for minimum $z = -\frac{1}{e}$). Thus, the maximum z is $\frac{1}{e}$ and the minimum z is $-\frac{1}{e}$.

The horizontal tangent planes are $z = -\frac{1}{e}$, $z = 0$, and $z = \frac{1}{e}$. The largest value of z is $\frac{1}{e}$, and the smallest value is $-\frac{1}{e}$.

Q5

(3-Q11) Let S be the surface

$$xy - 2x + yz + x^2 + y^2 + z^3 = 7$$

- Find the tangent plane and normal line to the surface S at the point $(0, 2, 1)$.
- The equation defining S implicitly defines z as a function of x and y. Find expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Evaluate $\frac{\partial z}{\partial y}$ at $(x, y, z) = (0, 2, 1)$.
- Find an expression for $\frac{\partial^2 z}{\partial x \partial y}$.

Solution:

The surface is given by the equation $xy - 2x + yz + x^2 + y^2 + z^3 = 7$. Define the function $F(x, y, z) = xy - 2x + yz + x^2 + y^2 + z^3$, so the surface is the level set $F(x, y, z) = 7$.

The gradient of F is:

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$$

Compute the partial derivatives:

- $\frac{\partial F}{\partial x} = 2x + y - 2$
- $\frac{\partial F}{\partial y} = x + 2y + z$
- $\frac{\partial F}{\partial z} = y + 3z^2$

At the point $(0, 2, 1)$:

- $\frac{\partial F}{\partial x} = 2(0) + 2 - 2 = 0$
- $\frac{\partial F}{\partial y} = 0 + 2(2) + 1 = 5$
- $\frac{\partial F}{\partial z} = 2 + 3(1)^2 = 5$

Thus, the gradient (normal vector) at $(0, 2, 1)$ is $(0, 5, 5)$. This can be simplified to $(0, 1, 1)$ by dividing by 5, as the direction is the same.

Tangent Plane:
The equation of the tangent plane at (x_0, y_0, z_0) is given by:

$$\nabla F \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Using the normal vector $(0, 5, 5)$ and the point $(0, 2, 1)$:

$$0 \cdot (x - 0) + 5 \cdot (y - 2) + 5 \cdot (z - 1) = 0$$

Simplify:

$$5(y - 2) + 5(z - 1) = 0 \implies y - 2 + z - 1 = 0 \implies y + z = 3$$

Normal Line:
The normal line passes through $(0, 2, 1)$ in the direction of the normal vector $(0, 1, 1)$. The parametric equations are:

$$x = 0 + 0 \cdot t = 0, \quad y = 2 + 1 \cdot t, \quad z = 1 + 1 \cdot t$$

for $t \in \mathbb{R}$. Thus:

$$x = 0, \quad y = 2 + t, \quad z = 1 + t$$

The point $(0, 2, 1)$ satisfies the surface equation:

$$(0)(2) - 2(0) + (2)(1) + (0)^2 + (2)^2 + (1)^3 = 0 - 0 + 2 + 0 + 4 + 1 = 7$$

The tangent plane and normal line are as derived.

The surface S is defined by the equation $xy - 2x + yz + x^2 + y^2 + z^3 = 7$, which implicitly defines z as a function of x and y.

To find $\frac{\partial z}{\partial x}$, differentiate the equation with respect to x, treating y as constant and z as a function of x:

$$\frac{\partial}{\partial x}(xy - 2x + yz + x^2 + y^2 + z^3) = 0$$

This gives:

$$y - 2 + y\frac{\partial z}{\partial x} + 2x + 3z^2\frac{\partial z}{\partial x} = 0$$

Solving for $\frac{\partial z}{\partial x}$:

$$(y + 3z^2)\frac{\partial z}{\partial x} = -(2x + y - 2)$$

$$\frac{\partial z}{\partial x} = -\frac{2x + y - 2}{y + 3z^2}$$

To find $\frac{\partial z}{\partial y}$, differentiate the equation with respect to y, treating x as constant and z as a function of y:

$$\frac{\partial}{\partial y}(xy - 2x + yz + x^2 + y^2 + z^3) = 0$$

This gives:

$$x + z + y\frac{\partial z}{\partial y} + 2y + 3z^2\frac{\partial z}{\partial y} = 0$$

Solving for $\frac{\partial z}{\partial y}$:

$$(y + 3z^2)\frac{\partial z}{\partial y} = -(x + 2y + z)$$

$$\frac{\partial z}{\partial y} = -\frac{x + 2y + z}{y + 3z^2}$$

Evaluate $\frac{\partial z}{\partial y}$ at the point $(x, y, z) = (0, 2, 1)$:

$$\frac{\partial z}{\partial y} = -\frac{0 + 2(2) + 1}{2 + 3(1)^2} = -\frac{5}{5} = -1$$

Thus, the expressions are:

$$\frac{\partial z}{\partial x} = -\frac{2x + y - 2}{y + 3z^2}, \quad \frac{\partial z}{\partial y} = -\frac{x + 2y + z}{y + 3z^2}$$

and at $(0, 2, 1)$, $\frac{\partial z}{\partial y} = -1$.
The surface S is defined by the equation $xy - 2x + yz + x^2 + y^2 + z^3 = 7$, which implicitly defines z as a function of x and y.

The first partial derivatives of z with respect to x and y are given by:

$$\frac{\partial z}{\partial x} = -\frac{2x + y - 2}{y + 3z^2}, \quad \frac{\partial z}{\partial y} = -\frac{x + 2y + z}{y + 3z^2}.$$

To find the mixed partial derivative $\frac{\partial^2 z}{\partial x \partial y}$, differentiate $\frac{\partial z}{\partial x}$ with respect to y. Set:

$$A = 2x + y - 2, \quad B = y + 3z^2, \quad D = x + 2y + z.$$

Then:

$$\frac{\partial z}{\partial x} = -\frac{A}{B}.$$

Differentiating with respect to y and applying the quotient rule, while noting that z is a function of x and y, gives:

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{\frac{\partial}{\partial y}(A) \cdot B - A \cdot \frac{\partial}{\partial y}(B)}{B^2}.$$

Compute the partial derivatives:

$$\frac{\partial A}{\partial y} = 1, \quad \frac{\partial B}{\partial y} = 1 + 6z\frac{\partial z}{\partial y}.$$

Substitute $\frac{\partial z}{\partial y} = -\frac{D}{B}$:

$$\frac{\partial B}{\partial y} = 1 + 6z\left(-\frac{D}{B}\right) = 1 - \frac{6zD}{B}.$$

Now:

$$\frac{\partial}{\partial y}(A) \cdot B = 1 \cdot B = B,$$

$$A \cdot \frac{\partial}{\partial y}(B) = A\left(1 - \frac{6zD}{B}\right) = A - \frac{6AzD}{B}.$$

So:

$$\frac{\partial}{\partial y}(A) \cdot B - A \cdot \frac{\partial}{\partial y}(B) = B - \left(A - \frac{6AzD}{B}\right) = B - A + \frac{6AzD}{B}.$$

Thus:

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{B - A + \frac{6AzD}{B}}{B^2} = -\frac{B - A}{B^2} - \frac{6AzD}{B^3}.$$

Simplify $B - A$:

$$B - A = (y + 3z^2) - (2x + y - 2) = 3z^2 - 2x + 2.$$

Substitute A, D, and B:

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{3z^2 - 2x + 2}{B^2} - \frac{6(2x + y - 2)z(x + 2y + z)}{B^3}.$$

Since $B = y + 3z^2$, write as a single fraction with denominator B^3:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-B(B - A) - 6AzD}{B^3} = \frac{-(y + 3z^2)(3z^2 - 2x + 2) - 6(2x + y - 2)z(x + 2y + z)}{(y + 3z^2)^3}.$$

Q6

(3-Q16) Let the pressure P and temperature T at a point (x, y, z) be

$$P(x, y, z) = \frac{x^2 + 2y^2}{1 + z^2}, \quad T(x, y, z) = 5 + xy - z^2$$

- If the position of an airplane at time t is

$$(x(t), y(t), z(t)) = (2t, t^2 - 1, \cos t)$$

- find $\frac{d}{dt}(PT)^2$ at time $t = 0$ as observed from the airplane.
- In which direction should a bird at the point $(0, -1, 1)$ fly if it wants to keep both P and T constant? (Give one possible direction vector. It does not need to be a unit vector.)
- An ant crawls on the surface $z^3 + zx + y^2 = 2$. When the ant is at the point $(0, -1, 1)$, in which direction should it go for maximum increase of the temperature $T = 5 + xy - z^2$? Your answer should be a vector $\langle a, b, c \rangle$, not necessarily of unit length.

Solution:

The pressure P and temperature T are given by:

$$P(x, y, z) = \frac{x^2 + 2y^2}{1 + z^2}, \quad T(x, y, z) = 5 + xy - z^2.$$

The position of the airplane at time t is:

$$(x(t), y(t), z(t)) = (2t, t^2 - 1, \cos t).$$

Define $S = (PT)^2$. The derivative $\frac{dS}{dt}$ is required at $t = 0$.

At $t = 0$:

$$x(0) = 0, \quad y(0) = -1, \quad z(0) = 1.$$

The derivatives of the position components at $t = 0$ are:

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 2t = 0, \quad \frac{dz}{dt} = -\sin t = 0.$$

Evaluate P and T at $(x, y, z) = (0, -1, 1)$:

$$P(0, -1, 1) = \frac{0^2 + 2(-1)^2}{1 + 1^2} = \frac{2}{2} = 1, \quad T(0, -1, 1) = 5 + (0)(-1) - 1^2 = 4.$$

The partial derivatives of P and T at $(0, -1, 1)$ are:

$$\frac{\partial P}{\partial x} = \frac{2x}{1+z^2} = 0, \quad \frac{\partial P}{\partial y} = \frac{4y}{1+z^2} = -2, \quad \frac{\partial P}{\partial z} = \frac{-2z(x^2+2y^2)}{(1+z^2)^2} = -1,$$

$$\frac{\partial T}{\partial x} = y = -1, \quad \frac{\partial T}{\partial y} = x = 0, \quad \frac{\partial T}{\partial z} = -2z = -2.$$

Now compute $\frac{dP}{dt}$ and $\frac{dT}{dt}$ at $t = 0$:

$$\frac{dP}{dt} = \frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial P}{\partial y}\frac{dy}{dt} + \frac{\partial P}{\partial z}\frac{dz}{dt} = (0)(2) + (-2)(0) + (-1)(0) = 0,$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} = (-1)(2) + (0)(0) + (-2)(0) = -2.$$

Set $U = PT$, so $S = U^2$. Then:

$$\frac{dS}{dt} = 2U\frac{dU}{dt}, \quad \frac{dU}{dt} = P\frac{dT}{dt} + T\frac{dP}{dt}.$$

At $t = 0$, $U = (1)(4) = 4$, and:

$$\frac{dU}{dt} = (1)(-2) + (4)(0) = -2.$$

Thus:

$$\frac{dS}{dt} = 2(4)(-2) = -16.$$

The derivative $\frac{d}{dt}(PT)^2$ at $t = 0$ is -16.

To determine the direction in which the bird should fly to keep both pressure P and temperature T constant at the point $(0, -1, 1)$, the direction vector must be perpendicular to the gradients of both P and T. This ensures that the directional derivatives of both functions are zero in that direction, meaning the functions remain constant along the path.

The pressure and temperature functions are:

$$P(x, y, z) = \frac{x^2 + 2y^2}{1 + z^2}, \quad T(x, y, z) = 5 + xy - z^2.$$

The gradient of P is:

$$\nabla P = \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z} \right) = \left(\frac{2x}{1 + z^2}, \frac{4y}{1 + z^2}, -\frac{2z(x^2 + 2y^2)}{(1 + z^2)^2} \right).$$

The gradient of T is:

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right) = (y, x, -2z).$$

Evaluating at $(x, y, z) = (0, -1, 1)$:

$$\nabla P = \left(\frac{2 \cdot 0}{1 + 1^2}, \frac{4 \cdot (-1)}{1 + 1}, -\frac{2 \cdot 1 \cdot (0^2 + 2 \cdot (-1)^2)}{(1 + 1)^2} \right) = (0, -2, -1),$$

$$\nabla T = (-1, 0, -2).$$

The direction vector is given by the cross product $\nabla P \times \nabla T$:

$$\nabla P \times \nabla T = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & -1 \\ -1 & 0 & -2 \end{vmatrix}$$

$$= \mathbf{i}((-2) \cdot (-2) - (-1) \cdot 0) - \mathbf{j}(0 \cdot (-2) - (-1) \cdot (-1)) + \mathbf{k}(0 \cdot 0 - (-2) \cdot (-1)).$$

Computing the components:

- $\mathbf{i}$-component: $(-2) \cdot (-2) - (-1) \cdot 0 = 4 - 0 = 4$,
- $\mathbf{j}$-component: $-(0 \cdot (-2) - (-1) \cdot (-1)) = -(0 - 1) = -(-1) = 1$,
- $\mathbf{k}$-component: $0 \cdot 0 - (-2) \cdot (-1) = 0 - 2 = -2$.

Thus, the cross product is $(4, 1, -2)$.

This vector is perpendicular to both gradients, as verified by the dot products:

$$(4, 1, -2) \cdot (0, -2, -1) = 4 \cdot 0 + 1 \cdot (-2) + (-2) \cdot (-1) = 0 - 2 + 2 = 0,$$

$$(4, 1, -2) \cdot (-1, 0, -2) = 4 \cdot (-1) + 1 \cdot 0 + (-2) \cdot (-2) = -4 + 0 + 4 = 0.$$

Therefore, the direction vector $(4, 1, -2)$ ensures that both P and T remain constant at the point $(0, -1, 1)$.

The ant is constrained to the surface $z^3 + zx + y^2 = 2$, and the goal is to maximize the increase of the temperature function $T(x, y, z) = 5 + xy - z^2$ at the point $(0, -1, 1)$.

The direction of steepest ascent for T on the surface is given by the component of the gradient of T that is tangent to the surface. The gradient of T is:

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right) = (y, x, -2z).$$

At the point $(0, -1, 1)$:

$$\nabla T = (-1, 0, -2).$$

The surface is defined by $g(x, y, z) = z^3 + zx + y^2 - 2 = 0$. The gradient of g is:

$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) = (z, 2y, 3z^2 + x).$$

At the point $(0, -1, 1)$:

$$\nabla g = (1, -2, 3).$$

The tangential component of ∇T is found by subtracting the projection of ∇T onto ∇g:

$$\text{proj}_{\nabla g} \nabla T = \frac{\nabla T \cdot \nabla g}{\|\nabla g\|^2} \nabla g.$$

First, compute the dot product:

$$\nabla T \cdot \nabla g = (-1)(1) + (0)(-2) + (-2)(3) = -1 - 6 = -7.$$

Then, $\|\nabla g\|^2 = 1^2 + (-2)^2 + 3^2 = 1 + 4 + 9 = 14$, so:

$$\frac{\nabla T \cdot \nabla g}{\|\nabla g\|^2} = \frac{-7}{14} = -\frac{1}{2}.$$

The projection is:

$$\text{proj}_{\nabla g} \nabla T = -\frac{1}{2}(1, -2, 3) = \left(-\frac{1}{2}, 1, -\frac{3}{2}\right).$$

The tangential component is:

$$\nabla T - \text{proj}_{\nabla g} \nabla T = (-1, 0, -2) - \left(-\frac{1}{2}, 1, -\frac{3}{2}\right) = \left(-1 + \frac{1}{2}, 0 - 1, -2 + \frac{3}{2}\right) = \left(-\frac{1}{2}, -1, -\frac{1}{2}\right).$$

To avoid fractions, multiply by -2 to get:

$$\langle -1, -2, -1 \rangle.$$

This vector is tangent to the surface, as verified by the dot product with ∇g:

$$(1, -2, 3) \cdot (-1, -2, -1) = 1(-1) + (-2)(-2) + 3(-1) = -1 + 4 - 3 = 0.$$

The directional derivative of T in this direction is positive, confirming it is the direction of maximum increase:

$$\nabla T \cdot \langle -1, -2, -1 \rangle = (-1)(-1) + (0)(-2) + (-2)(-1) = 1 + 0 + 2 = 3 > 0.$$

Thus, the direction for maximum increase of temperature is $\langle -1, -2, -1 \rangle$.

Q7

(3-Q17) Find all saddle points, local minima and local maxima of the function

$$f(x, y) = x^3 + x^2 - 2xy + y^2 - x.$$

Solution:

The critical points of the function $f(x, y) = x^3 + x^2 - 2xy + y^2 - x$ are found by setting the first partial derivatives to zero.

The partial derivative with respect to x is:

$$f_x = 3x^2 + 2x - 2y - 1$$

The partial derivative with respect to y is:

$$f_y = -2x + 2y$$

Setting $f_y = 0$ gives:

$$-2x + 2y = 0 \implies y = x$$

Substituting $y = x$ into $f_x = 0$:

$$3x^2 + 2x - 2(x) - 1 = 0 \implies 3x^2 - 1 = 0 \implies x^2 = \frac{1}{3} \implies x = \pm\frac{\sqrt{3}}{3}$$

Thus, the critical points are $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ and $\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$.

To classify these critical points, the second derivative test is used. The second partial derivatives are:

$$f_{xx} = 6x + 2, \quad f_{yy} = 2, \quad f_{xy} = -2$$

The discriminant is:

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (6x+2)(2) - (-2)^2 = 12x + 4 - 4 = 12x$$

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- At $\left( \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right)$:
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$$x = \frac{\sqrt{3}}{3} > 0, \quad D = 12 \cdot \frac{\sqrt{3}}{3} = 4\sqrt{3} > 0$$

$$f_{xx} = 6 \cdot \frac{\sqrt{3}}{3} + 2 = 2\sqrt{3} + 2 > 0$$

Since $D > 0$ and $f_{xx} > 0$, this is a local minimum.
- At $\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$:

$$x = -\frac{\sqrt{3}}{3} < 0, \quad D = 12 \cdot \left(-\frac{\sqrt{3}}{3}\right) = -4\sqrt{3} < 0$$

Since $D < 0$, this is a saddle point.

There are no other critical points, as $f_y = 0$ implies $y = x$, and substituting into $f_x = 0$ gives only the solutions above. The function has no local maxima, as the discriminant is negative at one critical point and positive with a positive second derivative at the other, and no other critical points exist.

Thus, the saddle point is at $\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$ and the local minimum is at $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.

Q8

(3-19) Find the maximum and minimum values of $f(x, y) = xy - x^3y^2$ when (x, y) runs over the square $0 \leq x \leq 1, 0 \leq y \leq 1$.

Solution:

The function is $f(x, y) = xy - x^3y^2$ defined over the square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

The minimum value of $f(x, y)$ is 0, achieved along the boundaries $x = 0$ and $y = 0$, and at the point $(1, 1)$. This is because:

- On $x = 0$, $f(0, y) = 0$ for all $y \in [0, 1]$.
- On $y = 0$, $f(x, 0) = 0$ for all $x \in [0, 1]$.
- At $(1, 1)$, $f(1, 1) = 1 \cdot 1 - 1^3 \cdot 1^2 = 0$.
- For all other points in the square, $f(x, y) = xy(1 - x^2y) \geq 0$ since $x \geq 0$, $y \geq 0$, and $1 - x^2y \geq 0$ (with equality only at $(1, 1)$).

The maximum value of $f(x, y)$ is $\frac{2\sqrt{3}}{9}$, achieved at the point $\left(\frac{\sqrt{3}}{3}, 1\right)$. This is determined as follows:

- The only critical point in the interior is $(0, 0)$, where $f(0, 0) = 0$.
- On the boundary $x = 1$, $f(1, y) = y - y^2$, which has a maximum value of $\frac{1}{4}$ at $y = \frac{1}{2}$.
- On the boundary $y = 1$, $f(x, 1) = x - x^3$, which has a maximum value of $\frac{2\sqrt{3}}{9}$ at $x = \frac{\sqrt{3}}{3}$.
- On the boundaries $x = 0$ and $y = 0$, $f = 0$.
- For each fixed $x \in (0, 1]$, the maximum over $y \in [0, 1]$ is given by:

$$m(x) = \begin{cases} x - x^3 & \text{if } 0 < x \leq \frac{\sqrt{2}}{2}, \\ \\ \frac{1}{4x} & \text{if } \frac{\sqrt{2}}{2} < x \leq 1. \end{cases}$$

- Maximizing $m(x)$ over $[0, 1]$ shows that the global maximum is $\frac{2\sqrt{3}}{9}$ at $x = \frac{\sqrt{3}}{3}$ (which is less than $\frac{\sqrt{2}}{2} \approx 0.707$), with $y = 1$.

The value $\frac{2\sqrt{3}}{9} \approx 0.3849$ is greater than the maximum values on other boundaries, and no interior point yields a higher value.

Thus, the minimum value is 0 and the maximum value is $\frac{2\sqrt{3}}{9}$.

Q9

(3-Q20) The temperature at all points in the disc $x^2 + y^2 \leq 1$ is given by $T(x, y) = (x + y)e^{-x^2-y^2}$. Find the maximum and minimum temperatures at points of the disc.

Solution

The temperature function is $T(x,y) = (x+y)e^{-x^2-y^2}$ defined on the closed disk $x^2+y^2 \leq 1$. To find the maximum and minimum temperatures, critical points inside the disk ($x^2+y^2<1$) and on the boundary ($x^2+y^2=1$) must be evaluated.

The partial derivatives are:

$$\frac{\partial T}{\partial x} = e^{-x^2-y^2}[1-2x(x+y)], \quad \frac{\partial T}{\partial y} = e^{-x^2-y^2}[1-2y(x+y)].$$

Setting both to zero and noting that $e^{-x^2-y^2} \neq 0$, the equations are:

$$1-2x(x+y)=0, \quad 1-2y(x+y)=0.$$

Solving, if $x+y \neq 0$, then $x=y$. Substituting $x=y$ gives:

$$1-2x(2x) = 1-4x^2 = 0 \implies x^2 = \frac{1}{4} \implies x = \pm\frac{1}{2}, \quad y = \pm\frac{1}{2}.$$

The critical points are $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, both satisfying $x^2+y^2 = \frac{1}{2} < 1$, so they are inside the disk. Evaluating T:

- At $\left(\frac{1}{2}, \frac{1}{2}\right)$: $T = \left(\frac{1}{2}+\frac{1}{2}\right)e^{-\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2} = 1 \cdot e^{-1/2} = e^{-1/2}$.
- At $\left(-\frac{1}{2}, -\frac{1}{2}\right)$: $T = \left(-\frac{1}{2}-\frac{1}{2}\right)e^{-\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2} = (-1) \cdot e^{-1/2} = -e^{-1/2}$.

The boundary is $x^2+y^2=1$. This is an equality constraint, so the method of Lagrange multipliers is appropriate. Define the constraint $g(x,y) = x^2+y^2-1 = 0$. Solve $\nabla T = \lambda \nabla g$:

$$\nabla g = (2x, 2y).$$

The gradient of T is:

$$\nabla T = \left(e^{-x^2-y^2}[1-2x(x+y)], e^{-x^2-y^2}[1-2y(x+y)]\right).$$

On the boundary, $x^2+y^2=1$, so $e^{-x^2-y^2} = e^{-1}$. The equations are:

$$e^{-1}[1-2x(x+y)] = \lambda \cdot 2x, \quad e^{-1}[1-2y(x+y)] = \lambda \cdot 2y.$$

Rearranging:

$$1-2x(x+y)=2x\lambda e, \quad 1-2y(x+y)=2y\lambda e.$$

Set $k=\lambda e$, so:

$$1-2xu=2xk, \quad 1-2yu=2yk,$$

where $u=x+y$. Assuming $x\neq 0$ and $y\neq 0$, equate the expressions for k:

$$\frac{1-2xu}{2x}=\frac{1-2yu}{2y} \implies \frac{1}{2x}-u=\frac{1}{2y}-u \implies \frac{1}{2x}=\frac{1}{2y} \implies x=y.$$

With $x=y$ and $x^2+y^2=1$:

$$2x^2=1 \implies x=\pm\frac{\sqrt{2}}{2}, \quad y=\pm\frac{\sqrt{2}}{2}.$$

- At $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$: $T=\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)e^{-1}=\sqrt{2}\cdot e^{-1}$.
- At $\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$: $T=\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right)e^{-1}=-\sqrt{2}\cdot e^{-1}$.

If $x=0$, then $y^2=1$, so $y=\pm 1$:

- At $(0,1)$: $T=(0+1)e^{-1}=e^{-1}$.
- At $(0,-1)$: $T=(0-1)e^{-1}=-e^{-1}$.

If $y=0$, then $x^2=1$, so $x=\pm 1$:

- At $(1,0)$: $T=(1+0)e^{-1}=e^{-1}$.
- At $(-1,0)$: $T=(-1+0)e^{-1}=-e^{-1}$.

However, for $x=0$ or $y=0$, the Lagrange multiplier equations lead to contradictions (e.g., for $x=0$, $y=1$, the equation $e^{-1}\cdot 1=\lambda\cdot 0$ implies $e^{-1}=0$, which is false). Thus, the only solutions to the Lagrange multiplier equations are when $x=y$, giving the points $\left(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2}\right)$.

The extreme values on the boundary are $\sqrt{2}/e$ and $-\sqrt{2}/e$, since $|\sqrt{2}/e| > |1/e|$ and these values occur at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Q10

(3-Q29) Find the point $P = (x, y, z)$ (with x, y, and $z > 0$) on the surface $x^3y^2z = 6\sqrt{3}$ that is closest to the origin.

Solution

To find the point $P = (x, y, z)$ with $x, y, z > 0$ on the surface $x^3y^2z = 6\sqrt{3}$ that is closest to the origin, minimize the square of the distance from the origin, $s = x^2 + y^2 + z^2$, subject to the constraint $g(x, y, z) = x^3y^2z - 6\sqrt{3} = 0$.

Using the method of Lagrange multipliers, solve $\nabla s = \lambda \nabla g$, where:

$$\nabla s = (2x, 2y, 2z), \quad \nabla g = (3x^2y^2z, 2x^3yz, x^3y^2).$$

This gives the system of equations:

$$2x = \lambda \cdot 3x^2y^2z, \tag{1}$$

$$2y = \lambda \cdot 2x^3yz, \tag{2}$$

$$2z = \lambda \cdot x^3y^2, \tag{3}$$

with the constraint:

$$x^3y^2z = 6\sqrt{3}. \tag{4}$$

Since $x, y, z > 0$, equations (1), (2), and (3) can be manipulated. Dividing equation (1) by equation (2):

$$\frac{2x}{2y} = \frac{\lambda \cdot 3x^2y^2z}{\lambda \cdot 2x^3yz} \implies \frac{x}{y} = \frac{3y}{2x} \implies 2x^2 = 3y^2 \implies y^2 = \frac{2}{3}x^2 \implies y = \sqrt{\frac{2}{3}}x.$$

Dividing equation (1) by equation (3):

$$\frac{2x}{2z} = \frac{\lambda \cdot 3x^2y^2z}{\lambda \cdot x^3y^2} \implies \frac{x}{z} = \frac{3z}{x} \implies x^2 = 3z^2 \implies z^2 = \frac{1}{3}x^2 \implies z = \frac{1}{\sqrt{3}}x.$$

Substitute $y = \sqrt{\frac{2}{3}}x$ and $z = \frac{1}{\sqrt{3}}x$ into the constraint (4):

$$x^3\left(\sqrt{\frac{2}{3}}x\right)^2\left(\frac{1}{\sqrt{3}}x\right) = 6\sqrt{3}.$$

Simplifying:

$$x^3 \cdot \frac{2}{3}x^2 \cdot \frac{1}{\sqrt{3}}x = x^6 \cdot \frac{2}{3\sqrt{3}} = 6\sqrt{3}.$$

Solving for x^6:

$$x^6 \cdot \frac{2}{3\sqrt{3}} = 6\sqrt{3} \implies 2x^6 = 6\sqrt{3} \cdot 3\sqrt{3} \implies 2x^6$$

$$= 6 \cdot 3 \cdot 3 \implies 2x^6 = 54 \implies x^6 = 27 \implies x = 27^{1/6} = \sqrt{3}.$$

Then:

$$y = \sqrt{\frac{2}{3}} \cdot \sqrt{3} = \sqrt{2}, \quad z = \frac{1}{\sqrt{3}} \cdot \sqrt{3} = 1.$$

Thus, the point is $(\sqrt{3}, \sqrt{2}, 1)$.

Q11

(3-Q30) Find the maximum value of $f(x, y, z) = xyz$ on the ellipsoid

$$g(x, y, z) = x^2 + xy + y^2 + 3z^2 = 9$$

Specify all points at which this maximum value occurs.

Solution:

To find the maximum value of $f(x, y, z) = xyz$ on the ellipsoid $g(x, y, z) = x^2 + xy + y^2 + 3z^2 = 9$, the method of Lagrange multipliers is used. The gradients are:

$$\nabla f = (yz, xz, xy), \quad \nabla g = (2x + y, x + 2y, 6z).$$

The equations are:

1. $yz = \lambda(2x + y)$
2. $xz = \lambda(x + 2y)$
3. $xy = \lambda(6z)$
4. $x^2 + xy + y^2 + 3z^2 = 9$

Assuming $x, y, z \neq 0$, solving these equations yields critical points. Dividing the first equation by the second gives:

$$\frac{y}{x} = \frac{2x + y}{x + 2y}.$$

Setting $k = y/x$, the equation becomes $k = (2 + k)/(1 + 2k)$, leading to $k^2 = 1$, so $k = 1$ or $k = -1$.

- **Case 1:** $y = x$ Substituting into the equations and constraint gives $z = 3\lambda$, $x^2 = 18\lambda^2$, and $x^2 + z^2 = 3$. Solving yields $\lambda = \pm 1/3$:
  - $\lambda = 1/3$: $z = 1$, $x^2 = 2$, so $x = y = \sqrt{2}$ or $x = y = -\sqrt{2}$, and $f = (\sqrt{2})(\sqrt{2})(1) = 2$ or $f = (-\sqrt{2})(-\sqrt{2})(1) = 2$.
  - $\lambda = -1/3$: $z = -1$, $x^2 = 2$, so $x = y = \sqrt{2}$ or $x = y = -\sqrt{2}$, and $f = (\sqrt{2})(\sqrt{2})(-1) = -2$ or $f = (-\sqrt{2})(-\sqrt{2})(-1) = -2$.
- **Case 2:** $y = -x$ Substituting gives $\lambda = -z$, $x^2 = 6z^2$, and $x^2 + 3z^2 = 9$. Solving yields $z = \pm 1$:
  - $z = 1$: $x^2 = 6$, so $x = \sqrt{6}, y = -\sqrt{6}$ or $x = -\sqrt{6}, y = \sqrt{6}$, and $f = (\sqrt{6})(-\sqrt{6})(1) = -6$ or $f = (-\sqrt{6})(\sqrt{6})(1) = -6$.
  - $z = -1$: $x^2 = 6$, so $x = \sqrt{6}, y = -\sqrt{6}$ or $x = -\sqrt{6}, y = \sqrt{6}$, and $f = (\sqrt{6})(-\sqrt{6})(-1) = 6$ or $f = (-\sqrt{6})(\sqrt{6})(-1) = 6$.

Cases where any variable is zero yield $f = 0$, which is less than 6. The ellipsoid is compact, so the maximum exists. The critical values are compared, and the maximum value of f is 6, occurring at the points where $z = -1$ and $y = -x$ with $x^2 = 6$.

The points are $(\sqrt{6}, -\sqrt{6}, -1)$ and $(-\sqrt{6}, \sqrt{6}, -1)$. At both points, $f = 6$ and the constraint is satisfied.

The maximum occurs at the points $\left(\sqrt{6},\ -\sqrt{6},\ -1\right)$ and $\left(-\sqrt{6},\ \sqrt{6},\ -1\right)$.