CS663 - Assignment 1

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Solution 1

Let the images taken with pixel sizes 0.5 x 0.5, 0.25 x 0.25, and 0.5 x 0.25 be labeled as I_1, I_2, and I_3, respectively. We will align the images with the **scaling motion model**.

The relationship between I_1 and I_2 can be represented as:

$$I_2 = I_1 \times \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, each coordinate value (both x and y) in I_1 is scaled by 0.5 to get aligned.

Similarly, The relationship between I_1 and I_3 can be represented as:

$$I_3 = I_1 \times \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, the abscissas of I_1 are scaled by 0.5 whereas the ordinate remains the same.

Solution 2

Let $\mathbf{u_{ij}} = \begin{pmatrix} x_{ij} \\ y_{ij} \end{pmatrix}$

Now, we know the following relationships between the images:

$$I_2 = I_1 + u_{12}$$
$$I_3 = I_2 + u_{23}$$
$$I_3 = I_1 + u_{13}$$

Expanding these equations in terms of the coordinates:

$$x_2 = x_1 + x_{12}, \quad y_2 = y_1 + y_{12}$$
$$x_3 = x_2 + x_{23}, \quad y_3 = y_2 + y_{23}$$
$$x_3 = x_1 + x_{13}, \quad y_3 = y_1 + y_{13}$$

Substituting x_2 and y_2 from the first equation into the second:

$$x_3 = (x_1 + x_{12}) + x_{23}, \quad y_3 = (y_1 + y_{12}) + y_{23}$$
$$x_3 = x_1 + x_{12} + x_{23}, \quad y_3 = y_1 + y_{12} + y_{23}$$

Comparing this with the third equation, we obtain:

$$x_1 + x_{12} + x_{23} = x_1 + x_{13}$$

$$y_1 + y_{12} + y_{23} = y_1 + y_{13}$$

Therefore, the translational vectors satisfy:

$$x_{13} = x_{12} + x_{23}$$

$$y_{13} = y_{12} + y_{23}$$

This means that the translational vector u_{13} between I_1 and I_3 can be obtained as the sum of the translational vectors u_{12} and u_{23} between I_1, I_2, and I_3, respectively.

$$\therefore \mathbf{u_{13}} = \mathbf{u_{12}} + \mathbf{u_{23}}$$

However, in practice, this relationship may **not hold**. Issues such as the **field of view** or **overlap** can affect the alignment. When aligning images, we only consider the overlapping region and ignore the non-overlapping areas.

For example, consider the case where $I_1 = I_3$. Suppose I_1 and I_2 have the same dimensions (i.e., the same number of pixels), but $I_1 \neq I_2$. In this case, there will be regions without overlap, which will appear as blackened areas when attempting to align the images. Consequently, when translating the aligned image back to the original, some information may be lost, as the blackened regions cannot be correctly aligned with their original positions.

Solution 3

Let graph coordinates be represented as $\begin{pmatrix} x_g \\ y_g \\ 1 \end{pmatrix}$ and MATLAB coordinates be represented as $\begin{pmatrix} x_m \\ y_m \\ 1 \end{pmatrix}$ After observing the graph, it is easy to see that applying **translation** and **scaling** motion models will suffice to align MATLAB coordinates with the graph's coordinates. Hence,

$$\begin{pmatrix} x_g \\ y_g \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_x & 0 & 0 \\ 0 & c_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_m \\ y_m \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x_g \\ y_g \\ 1 \end{pmatrix} = \begin{pmatrix} c_x & 0 & t_x \\ 0 & c_y & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_m \\ y_m \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x_g \\ y_g \\ 1 \end{pmatrix} = \begin{pmatrix} c_x \cdot x_m + t_x \\ c_y \cdot y_m + t_y \\ 1 \end{pmatrix}$$

Note: Translation and Scaling are not commutative. Since we are solving for c_x, c_y, t_x, and t_y, this fact does not matter. We first scaled the image and then translated it. If we translate first and then scale, the only difference will be in the translation vector (by a scaling factor of the scaling vector).

Now, there are 4 unknown variables. By the **control point approach**, we require >= 2 known points to evaluate these unknown variables.

Using the *impixelinfo* library in MATLAB, we obtained two points on graph with their corresponding MATLAB co-ordinates:

$$\begin{pmatrix} 0 \\ 635 \\ 1 \end{pmatrix} = \begin{pmatrix} c_x \cdot 565 + t_x \\ c_y \cdot 1518 + t_y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -10 \\ 630 \\ 1 \end{pmatrix} = \begin{pmatrix} c_x \cdot 400 + t_x \\ c_y \cdot 1435 + t_y \\ 1 \end{pmatrix}$$

Extracting the equations:

$$\begin{aligned} 0 &= c_x \cdot 565 + t_x \\ 635 &= c_y \cdot 1518 + t_y \end{aligned}$$

$$\begin{aligned} -10 &= c_x \cdot 400 + t_x \\ 630 &= c_y \cdot 1435 + t_y \end{aligned}$$

Solve for c_x and t_x:
Subtract the first equation from the second:

$$\begin{aligned} -10 - 0 &= (c_x \cdot 400 + t_x) - (c_x \cdot 565 + t_x) \\ -10 &= c_x \cdot (400 - 565) \\ -10 &= c_x \cdot (-165) \\ c_x &= \frac{10}{165} \approx 0.0606 \end{aligned}$$

Substitute $c_x = 0.0606$ back into the first equation to find t_x:

$$\begin{aligned} 0 &= 0.0606 \cdot 565 + t_x \\ t_x &= -0.0606 \cdot 565 \\ t_x &= -34.2 \end{aligned}$$

Solve for c_y and t_y:
Subtract the second equation from the fifth:

$$\begin{aligned} 630 - 635 &= (c_y \cdot 1435 + t_y) - (c_y \cdot 1518 + t_y) \\ -5 &= c_y \cdot (1435 - 1518) \\ -5 &= c_y \cdot (-83) \\ c_y &= \frac{5}{83} \approx 0.0602 \end{aligned}$$

Substitute $c_y = 0.0602$ back into the second equation to find t_y:

$$\begin{aligned} 635 &= 0.0602 \cdot 1518 + t_y \\ t_y &= 635 - (0.0602 \cdot 1518) \\ t_y &= 635 - 91.2 \\ t_y &= 543.8 \end{aligned}$$

$$\therefore \begin{pmatrix} x_g \\ y_g \\ 1 \end{pmatrix} = \begin{pmatrix} 0.606 \cdot x_m - 34.2 \\ 0.602 \cdot y_m + 543.8 \\ 1 \end{pmatrix}$$

Solution 4

Let there be n pairs of control points: $(x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)$ in Image 1 and $(X_1, Y_1), (X_2, Y_2) \cdots (X_n, Y_n)$ in Image 2.
For each pair (X_i, Y_i) and (x_i, y_i), we know that:

$$X_i = ax_i^2 + by_i^2 + cx_iy_i + dx_i + ey_i + f$$

$$Y_i = Ax_i^2 + By_i^2 + Cx_iy_i + Dx_i + Ey_i + F$$

Writing it in matrices and vector format we have:

$$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^2 & y_n^2 & x_ny_n & x_n & y_n & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}$$

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^2 & y_n^2 & x_ny_n & x_n & y_n & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix}$$

$$\begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \\ \vdots & \vdots \\ X_n & Y_n \end{pmatrix} = \begin{pmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^2 & y_n^2 & x_ny_n & x_n & y_n & 1 \end{pmatrix} \begin{pmatrix} a & A \\ b & B \\ c & C \\ d & D \\ e & E \\ f & F \end{pmatrix}$$

We know the solution to the matrix equations $p = HX$ is $X = (H^T H)^{-1} H^T p$ where H is $a \times b$, X is $b \times r$ and p is $a \times r$ and $a > b$.

Here, $a = n$, $b = 6$, and $r = 2$. Also,

$$p = \begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \\ \vdots & \vdots \\ X_n & Y_n \end{pmatrix}$$

$$H = \begin{pmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^2 & y_n^2 & x_ny_n & x_n & y_n & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} a & A \\ b & B \\ c & C \\ d & D \\ e & E \\ f & F \end{pmatrix}$$

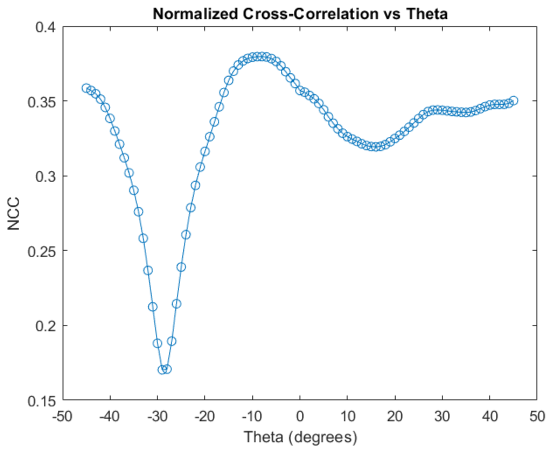
Hence, our solution is :

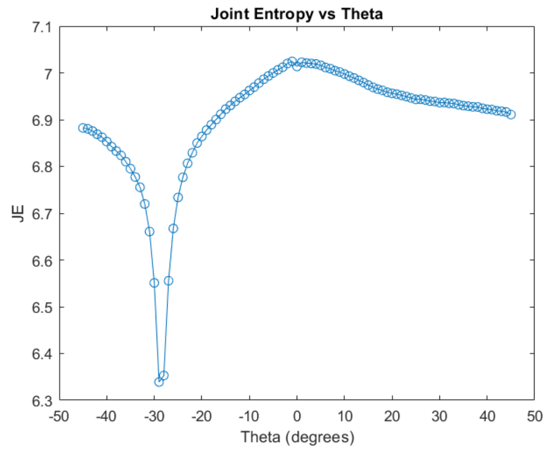
$$\mathbf{X} = (\mathbf{H^T H})^{-1} \mathbf{H^T p}$$

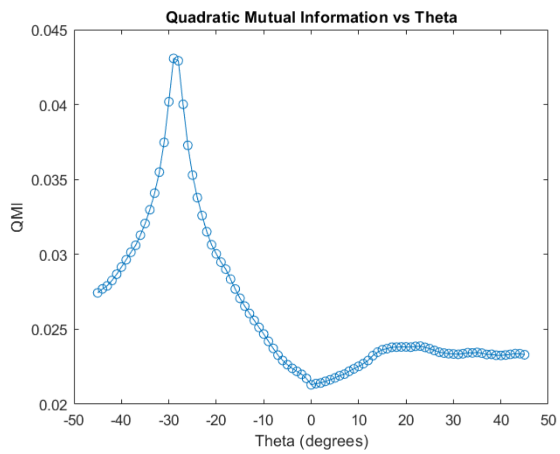
Note: There are 12 unknown parameters. Hence the number of control parameters $n \geq \frac{12}{2} = 6$.

Solution 5

5.c) Measures of Dependence Plots







5.d) Optimal Rotation Analysis

NCC

The optimal angle for this measure is the maxima of the graph which is $-8°$. The actual answer was $-28.5°$ hence, this measure is not suitable here. This might be because of the fact the relationship between intensities of the two images is not linear.

However, it was quite intriguing that the minima of this plot occurs very close to the optimal angle. On further analysing we found that the correlation between the "brain" part of both images was overshadowed by the strong positively correlation of the black background in both the images. It turns out that the images (without the background) have a negative correlation. Hence for most of the analysis, high positive correlation between the backgrounds keep the value of NCC high. But as theta aproaches optimal angle, the negative correlation of "brain" parts decreases the correlation (hence the minima).

To verify this further we manually changed backgrounds of one of the images to white (manually) and got a maxima at an optimal angle (This part is not included in the code submitted with the question).

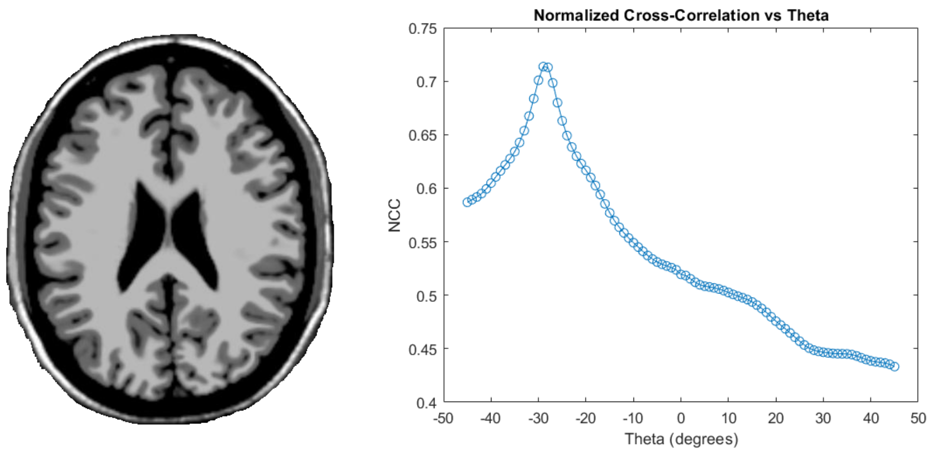


Image 1 with bg removed and NCC vs Theta graph w.r.t. new image

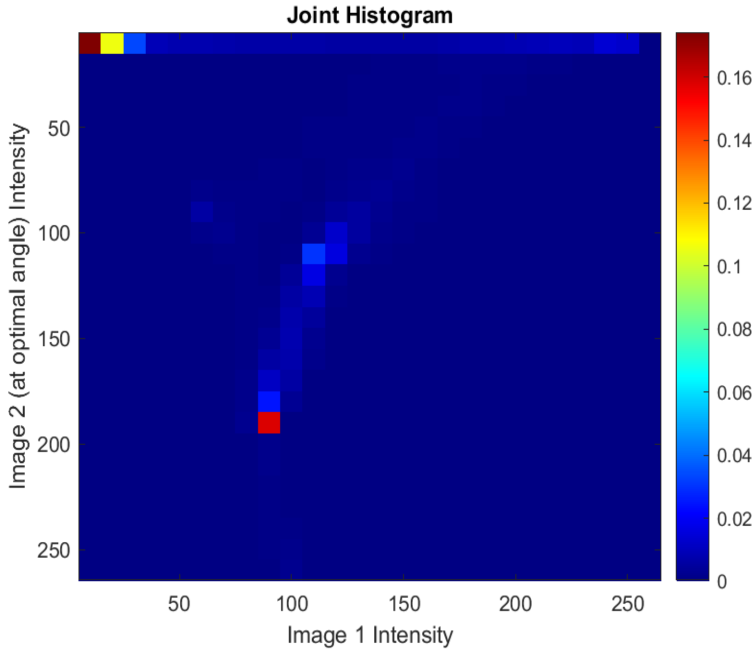
JE

The optimal angle for this measure is the minima of the graph which is $-29°$. The actual answer was $-28.5°$. Hence this is a good measure for image alignment for this question.

QMI

The optimal angle for this measure is the maxima of the graph which is $-29°$. The actual answer was $-28.5°$. Hence this is a good measure for image alignment for this question.

5.e) Joint Histogram



5.f) Intuition Behind QMI

QMI is defined as

$$\sum_{i_1}\sum_{i_2}(p_{I_1I_2}(i_1,i_2) - p_{I_1}(i_1)p_{I_1}(i_1))^2$$

Two events X and Y are considered independent if

$$P(X,Y) = P(X) \cdot P(Y)$$

Let X be the event that a given pixel in I_1 is in bin i_1 and Y be the event that a given pixel in I_2 is in bin i_2. Here, P(X) is $p_{I_1}(i_1)$, P(Y) is $p_{I_2}(i_2)$ and P(X,Y) becomes $p_{I_1I_2}(i_1,i_2)$. Hence, the value of QMI will be low if the two images are statistically independent and it will be high if they are statistically dependent and share some mutual dependence.
If two images are similar, then there will be some mutual dependence in the intensity values of the images Hence the value of Quadratic Mutual Information will be higher.

Solution 6

On selecting 12 physically corresponding points the following value of the transformation matrix was found

$$T = \begin{pmatrix} 1.0452 & -0.0061 & 30.4099 \\ -0.0146 & 1.0079 & 21.2444 \\ 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

6.c) Nearest Neighbour Interpolation

Image 1

Image 2

Transformed Image

6.d) Bilinear Interpolation

Image 1

Image 2

Transformed Image

6.e) Collinear Control Points

For affine transformation, the following equation holds,

$$G = TF$$

where,

$$F = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ y_{11} & y_{12} & \cdots & y_{1k} \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$G = \begin{pmatrix} x_{21} & x_{22} & \cdots & x_{2k} \\ y_{21} & y_{22} & \cdots & y_{2k} \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

note in this case k = 12. To compute the transformation matrix, we multiply both sides by F^T.

$$\begin{aligned} GF^T &= TFF^T \\ T &= GF^T(FF^T)^{-1} \end{aligned}$$

Here $F^T(FF^T)^{-1}$ is also called pseudo-inverse. Clearly $(FF^T)^{-1}$ must be non-singular.

Now if the points in the first image belong to the same line, then

$$y_{1i} = mx_{1i} + c$$

$$F = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ mx_{11}+c & mx_{12}+c & \cdots & mx_{1k}+c \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$\begin{aligned} FF^T &= \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ mx_{11}+c & mx_{12}+c & \cdots & mx_{1k}+c \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_{11} & mx_{11}+c & 1 \\ x_{12} & mx_{12}+c & 1 \\ \vdots & \vdots & \vdots \\ x_{1k} & mx_{1k}+c & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sum {x_{1i}}^2 & \sum m{x_{1i}}^2 + cx_{1i} & \sum x_{1i} \\ \sum x_{1i}(mx_{1i}+c) & \sum mx_{1i}(mx_{1i}+c) + c(mx_{1i}+c) & \sum mx_{1i}+c \\ \sum x_{1i} & \sum mx_{1i}+c & \sum 1 \end{pmatrix} \end{aligned}$$

Note each summation in above matrix is $\sum_{i=1}^{k}$. Now note for FF^T,

$$C_2 - mC_1 + cC_3 = 0$$

Hence $\det(FF^T) = 0$. Hence we will not be able to find transformation matrix using the above method. Hence affine transformation cannot be estimated if all points are chosen on a line.