Filter Design Assignment

EE338 - 2023
Elliptic Filter Design Report

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Contents

1 Student Details

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2 Jacobian Elliptic Function

The elliptic filters are characterised by **Jacobian Elliptic Function**, which is defined as follows: The elliptic function $w = sn(z, k)$ is defined indirectly through the elliptic integral:

$$z = \int_0^{\phi} \frac{1}{\sqrt{1 - k^2 sin^2\theta}} \, d\theta \tag{1}$$

Substituting $w = sin\phi$ in the above expression,

$$z = \int_0^{w} \frac{1}{\sqrt{(1 - k^2t^2)(1 - t^2)}} \, dt \tag{2}$$

We define following functions based on the Jacobian elliptic functions which will be used in filter design:

$$w = cn(z, k) = cos\phi(z, k) \tag{3}$$

$$w = sn(z, k) = sin\phi(z, k) \tag{4}$$

$$w = dn(z, k) = \sqrt{1 - k^2 sn^2(z, k)} \tag{5}$$

$$w = cd(z, k) = \frac{cn(z, k)}{dn(z, k)} \tag{6}$$

Next, we define the complete elliptic integral K which will be used in our filter design:

$$K = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 sin^2\theta}} \, d\theta \tag{7}$$

and it's complementary elliptic integral which is defined for k' = $\sqrt{1 - k^2}$

$$K' = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k'^2 sin^2\theta}} \, d\theta \tag{8}$$

$$K' = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - (1 - k^2) sin^2\theta}} \, d\theta \tag{9}$$

Now that we have defined the functions required for elliptic filter design, we move on to finding the specifications and design our elliptic bandpass and bandstop filters.

3 Elliptic Bandpass Filter

3.1 Un-normailzed Discrete time specifications

- 1. m = 23
  2. q(m) = [2.3] = 2
  3. r(m) = 23 - 10*2 = 3
  4. BL(m) = 20 + 3*2 + 11*3 = 59
  5. BH(m) = 59 + 75 = 134
- Passband = 59 kHz to 134 kHz.
- Transition Band Width = 5 kHz.
- Stopband = 0 to 54 kHz and 139 to 300 kHz.
- Tolerance = 0.15.
- Nature of Passband = Equiripple.
- Nature of Stopband = Equiripple.

3.2 Normailzed discrete filter specifications

Sampling frequency = $600kHz$

$$\omega = \frac{\Omega * 2\pi}{\Omega_{sampling}} \tag{10}$$

- Passband = 0.1967π to 0.4467π .
- Transition Band Width = 0.0167π .
- Stopband = 0 to 0.18 and 0.4633π to π .
- Tolerance = 0.15.
- Nature of Passband = Equiripple.
- Nature of Stopband = Equiripple.

3.3 Converting to Analog Low-pass filter

We have the following transformation for converting into analog low-pass filter:

$$\Omega = tan(\frac{\omega}{2}) \tag{11}$$

Using the above Bilinear Transformation, the passband and stopband specifications are updated as follows:

- Passband = 0.3191 to 0.8451 .

- Transition Band = 0.2905 to 0.3191 and 0.8451 to 0.8910 .
- Stopband = 0 to 0.2905 and 0.8910 to ∞ .
- Tolerance = 0.15.
- Nature of Passband = Equiripple.
- Nature of Stopband = Equiripple.

3.4 Frequency Transformation for Band-Pass Filter

The frequency transformation for converting a bandpass filter to low pass filter are as follows:

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{12}$$

$$B = \Omega_{p2} - \Omega_{p1} \tag{13}$$

$$\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} \tag{14}$$

According to the above transformation, we can tabulate the updated value of passband and stopband edges:

Ω	Ω_L
0^+	$-\infty$
$0.3191(\Omega_{p1})$	-1
$0.2905(\Omega_{s1})$	-1.2127
$0.5193(\Omega_0)$	0
$0.8910(\Omega_{s2})$	1.1185
$0.8451(\Omega_{p2})$	1
∞	∞

- Passband edge = 1 (Ω_{lp})
- Stopband edge = 1.1185 (Ω_{ls})
- Tolerance = 0.15.
- Nature of Passband = Equiripple.
- Nature of Stopband = Equiripple.

3.5 Parameters for elliptic filter design

The general form of a transfer function of a low pass filter is :

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon_p^2 F_N^2(\omega)}, \omega = \frac{\Omega}{\Omega_p} \tag{15}$$

In case of elliptic filter, the function $F_N(\omega)$ is given by :

$$F_N(\omega) = cd(NuK_1, k_1), \omega = cd(uK, k) \tag{16}$$

where,

$$\epsilon = \sqrt{\frac{2\delta - \delta^2}{1 - 2\delta + \delta^2}} \tag{17}$$

$$k_1 = \frac{\epsilon}{\sqrt{\frac{1}{\delta^2} - 1}} \tag{18}$$

$$k_1' = \sqrt{1 - k_1^2} \tag{19}$$

$$k = \frac{1}{\Omega_{sL}} \tag{20}$$

$$k_1 = \sqrt{1 - k^2} \tag{21}$$

The *degree equation* for elliptic filter is:

$$N = \frac{KK_1'}{K'K_1} \tag{22}$$

To get the minimum order, we take the ceiling of the above mentioned degree equation.

We substitute our filter specification in the above mentioned equations to get the parameters for our elliptic bandpass filter.

parameters	values
ϵ	0.6197
k	0.8571
K	2.5508
k’	0.4480
K’	1.1096
k_1	0.0940
K_1	1.6098
k_1'	0.9956
K_1'	1.5332
N	3

3.6 Poles and Zeros

To find the poles and zeros of the transfer function, we set $N = 2L + r$, where $r \in (0, 1)$. Hence for our filter, since N is 3, we get L = 1 and r = 1.
Next we define $\zeta_i = cd(u_i K, k)$, where $u_i = \frac{2i-1}{N}$ where i = 1,2,.....L
Since L = 1, we get $\frac{1}{N} = 0.3333$ and $\zeta = cd(uK, k) = 0.9257$.

The zeros of the transfer function is given by

$$z = \frac{j}{k\zeta} \tag{23}$$

Plugging in the values, we get $z = 7.9193j$. The other zero is the conjugate of the previously found zero.

For finding the poles of the transfer function, we define

$$\nu_0 = \frac{-j}{NK_1} sn^{-1}(\frac{j}{\epsilon}, k_1) \tag{24}$$

The expression for polesi is given by :

$$p = jcd((u - j\nu_0)K, k) \tag{25}$$

If N is odd, we get an additional pole on the real axis, given by

$$p_0 = jcd((1 - j\nu_0)K, k) \tag{26}$$

Finally, we get three poles of given specifications, which are as follows: $p_0 = -3.9154$
$p_1 = -0.7246 + j6.2431$
$p_2 = -.07246 - j6.2431$

3.7 Analog Lowpass Transfer Function

In the previous section, we found the poles and zeros of our transfer function. From the poles and zeros, we construct the transfer function, which is given by:

$$H_a(s) = H_0 \frac{1}{(1 - s/p_0)} \frac{(1 - s/z)(1 - s/\bar{z})}{(1 - s/p_1)(1 - s/\bar{p_1})} \tag{27}$$

$$H_a(s) = \frac{0.85s^2 + 53.3077}{s^3 + 5.3647s^2 + 45.1753s + 154.6604} \tag{28}$$

Next, we apply the transformation

$$s_L = \frac{\Omega_0^2 + s^2}{Bs}$$

to get the Bandpass transfer function.

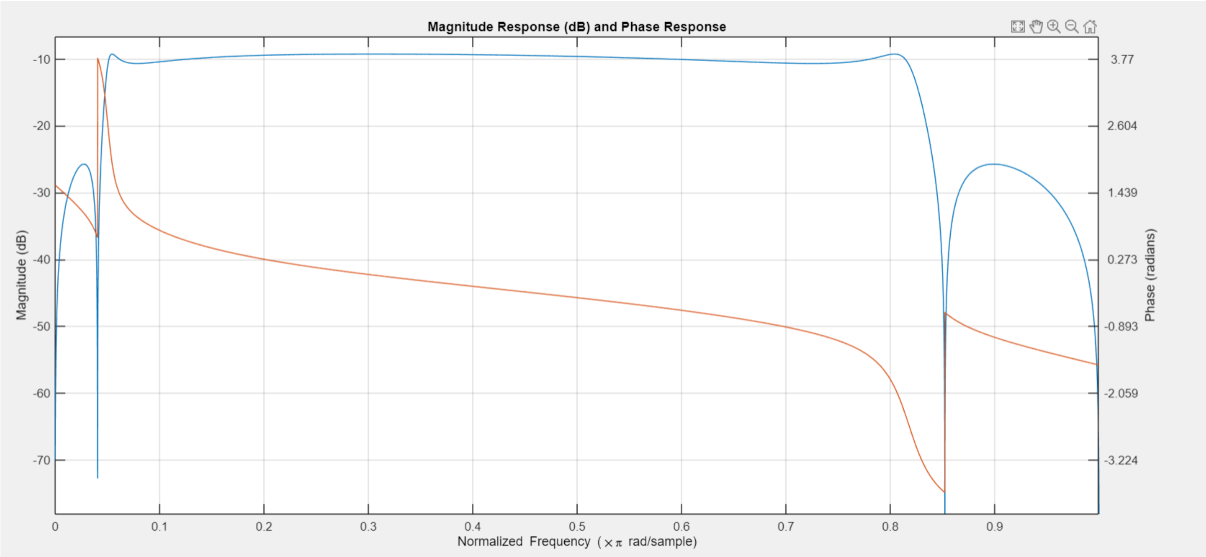
After applying the transformation, the transfer function for analog bandpass filter becomes:

$$H_{bpf}(s) = \frac{0.4470s^5 + 7.9958s^3 + 0.0325s}{s^6 + 2.8214s^5 + 13.3044s^4 + 24.0202s^3 + 3.5881s^2 + 0.2052s + 0.0196} \tag{29}$$

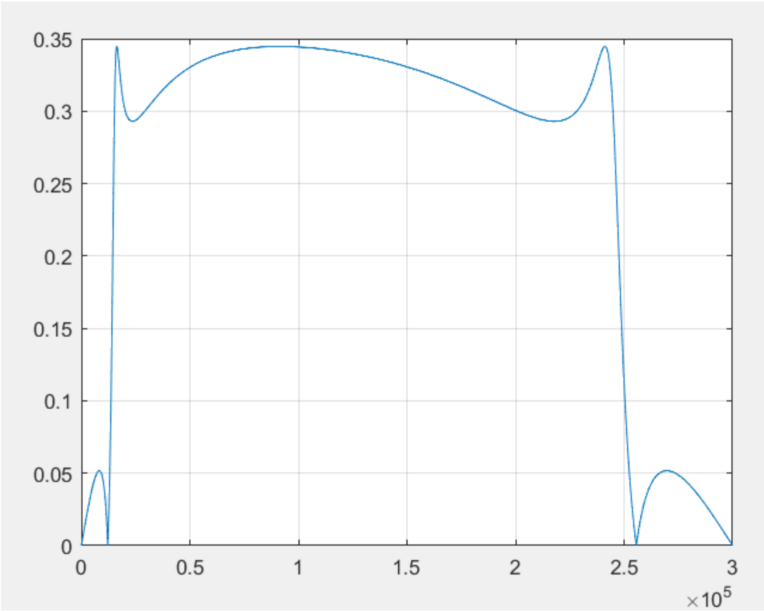
Then we apply the bilinear transformation, to obtain the discrete domain transfer function

$$H_{bpf}(z) = \frac{0.1885z^{-6} - 0.0369z^{-5} - 0.4802z^{-4} + 0.4802z^{-2} + 0.0369z^{-1} - 0.1885}{z^{-6} - 0.7958z^{-5} - 1.3018z^{-4} + 0.4283z^{-3} + 1.2307z^{-2} - 0.3303z^{-1} - 0.2032} \tag{30}$$

3.8 Magnitude and Phase response



Frequency Response



Magnitude plot

4 Elliptic Bandstop Filter

4.1 Un-normailzed Discrete time specifications

- 1. m = 23
  2. q(m) = [2.3] = 2
  3. r(m) = 23 - 10*2 = 3
  4. BL(m) = 20 + 3*2 + 11*3 = 59
  5. BH(m) = 59 + 40 = 99
- Stopband = 59 kHz to 99 kHz.
- Transition Band Width = 5 kHz.
- Passband = 0 to 54 kHz and 104 to 212.5 kHz.
- Tolerance = 0.15.
- Nature of Passband and Stopband : Both equiripple.

4.2 Normailzed discrete filter specifications

Sampling frequency = $425kHz$

$$\omega = \frac{\Omega * 2\pi}{\Omega_{sampling}} \tag{31}$$

- Stopband = 0.27π to 0.46π .
- Transition Band Width = 0.0235π .
- Passband = 0 to 0.25 and 0.48π to π .
- Tolerance = 0.15.
- Nature of Passband and Stopband : Both equiripple.

4.3 Converting to Analog Low-pass filter

We have the following transformation for converting into analog low-pass filter:

$$\Omega = tan(\tfrac{\omega}{2})$$

Using the above Bilinear Transformation, the passband and stopband specifications are updated as follows:

- Stopband = 0.451 to 0.88 .
- Transition Band = 0.414 to 0.4509 and 0.88 to 0.939 .
- Passband = 0 to 0.414 and 0.939 to ∞ .
- Tolerance = 0.15.
- Nature of Passband and Stopband : Both equiripple.

4.4 Frequency Transformation for Band-Stop Filter

The frequency transformation for converting a bandpass filter to low pass filter are as follows:

$$\Omega_L = \frac{B\Omega}{\Omega_0^2 - \Omega^2} \tag{32}$$

$$B = \Omega_{p2} - \Omega_{p1} \tag{33}$$

$$\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} \tag{34}$$

According to the above transformation, we can tabulate the updated value of stopband and passband edges:

Ω	Ω_L
0^+	0^+
$0.414(\Omega_{p1})$	1.00
$0.451(\Omega_{s1})$	1.28
$0.623^-(\Omega_0)$	$-\infty$
$0.623^+(\Omega_0)$	$+\infty$
$0.88(\Omega_{s2})$	-1.19
$0.939(\Omega_{p2})$	-0.99
∞	0^-

- Passband edge = 1 (Ω_{lp})
- Stopband edge = 1.19 (Ω_{ls})
- Tolerance = 0.15.
- Nature of Passband and Stopband : Both equiripple.

4.5 Parameters for elliptic filter design

In the previous section on parameters for elliptic bandpass function, I had already discussed the formulae for finding the parameters. Using the same formulae here as well, we get the parameters as follows:

4.6 Poles and Zeros

For the given specifications for bandstop filter, we get the order to be 2 and hence L = 1 and r = 0.
The zeros of the transfer function is given by

$$z = \frac{j}{k\zeta} \tag{35}$$

parameters	values
ϵ	0.6197
k	0.5606
K	2.2898
k'	0.5811
K'	1.1691
k_1	0.0940
K_1	1.6098
k_1'	0.9956
K_1'	1.5332
N	2

Plugging in the values, we get $z = 15.1550j$. The other zero is the conjugate of the previously found zero.

For finding the poles of the transfer function, we define

$$\nu_0 = \frac{-j}{NK_1} sn^{-1}(\frac{j}{\epsilon}, k_1) \tag{36}$$

The expression for polesi is given by :

$$p = jcd((u - j\nu_0)K, k) \tag{37}$$

as N is even, we won't get the additional pole on the X-axis.

Finally, we get two poles of given specifications, which are as follows: $p_1 = -2.6696 + j5.7796$ $p_2 = -2.6696 - j5.7796$

4.7 Analog Lowpass Transfer Function

In the previous section, we found the poles and zeros of our transfer function. From the poles and zeros, we construct the transfer function, which is given by:

$$H_a(s) = \frac{(1 - s/z)(1 - s/\bar{z})}{(1 - s/p_1)(1 - s/\bar{p_1})} \tag{38}$$

$$H_a(s) = \frac{s^2 + 229.6739}{s^2 + 5.3392s + 40.5307} \tag{39}$$

Next, we apply the transformation

$$s_L = \frac{Bs}{\Omega_0^2 + s^2}$$

to get the Bandpass transfer function.

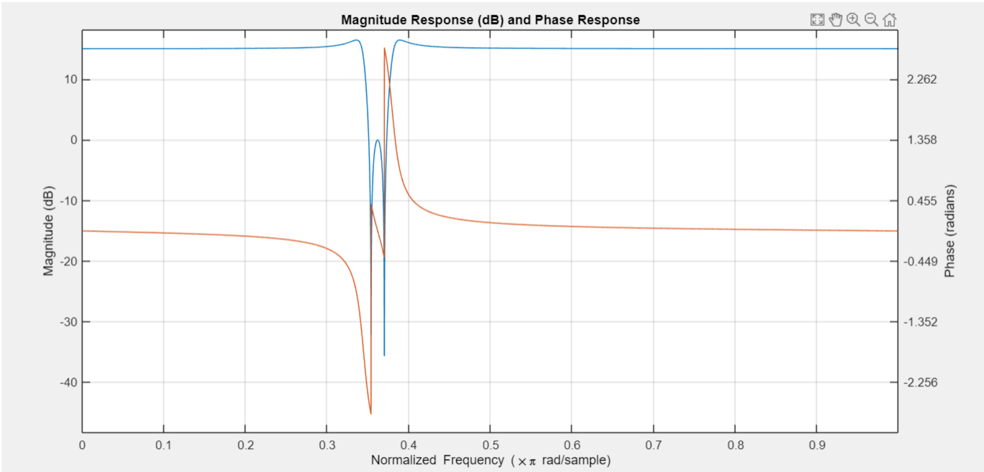
After applying the transformation, the transfer function for analog bandpass filter becomes:

$$H_{bpf}(s) = \frac{5.6669s^4 + 4.6314s^2 + 0.9433}{s^4 + 0.0719s^3 + 0.8234s^2 + 0.0293s + 0.1665} \tag{40}$$

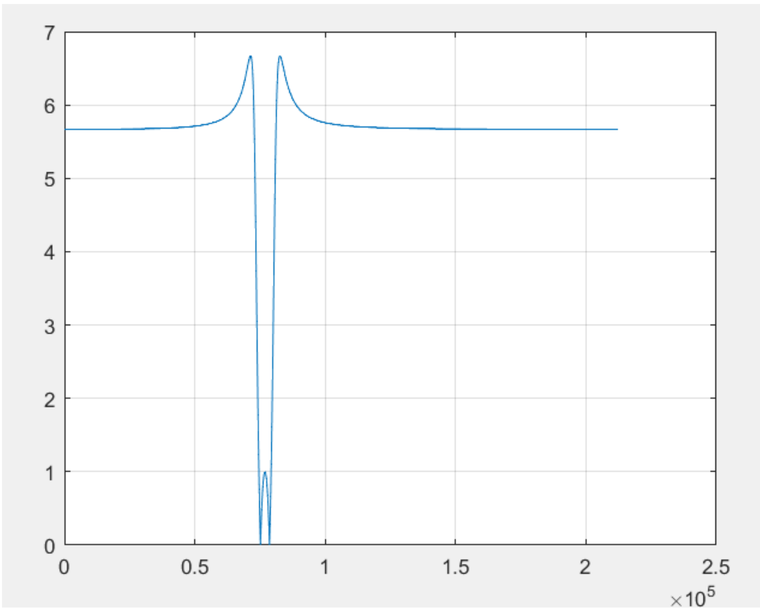
Then we apply the bilinear transformation, to obtain the discrete domain transfer function

$$H_{bpf}(z) = \frac{5.3761z^{-4} - 9.0355z^{-3} + 14.5371z^{-2} - 9.0355z^{-1} + 5.3761}{z^{-4} - 1.6352z^{-3} + 2.5596z^{-2} - 1.5538z^{-1} - 0.9032} \tag{41}$$

4.8 Magnitude and Phase response



Frequency Response



Magnitude plot

5 Comparison between Elliptic and Chebyschev/ Butterworth Filters

When we compare the order of Elliptic, Chebyschev and Butterworth filter, we observe that the order of Butterworth filter is highest while it is lowest in the case of elliptic filter. However, when we compare the phase response of the three filters, we observe that the phase response of Butterworth filter is most linear among all the three, while it is least linear for Elliptic Filter. Based on the above two observations, we can state that there exist a trade-off between order and the linear phase response. If we try to reduce the order of the filter, the poorer linear phase response we get and vice versa.