Let $f(t)$ be a function for all +ve values of t then the Laplace transform of $f(t)$ denoted by $L\{f(t)\}$ or $\bar{F}(s)$

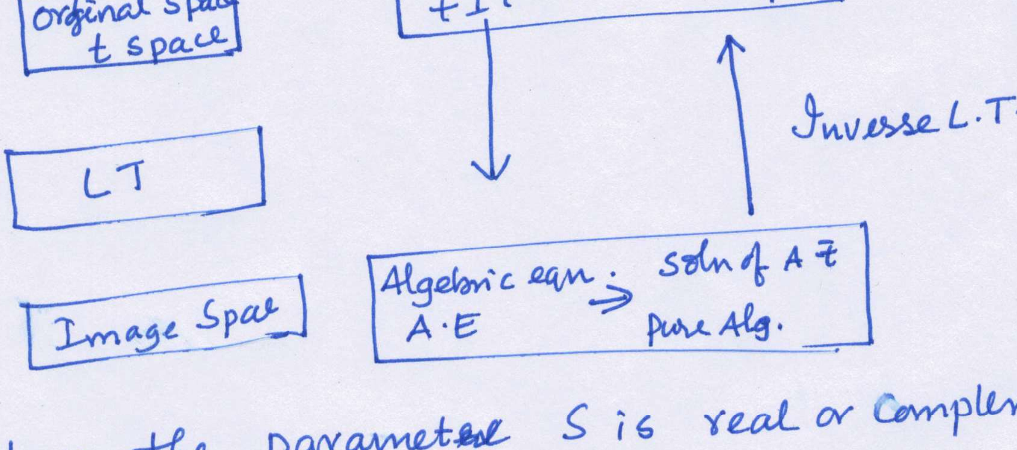
$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s). \quad \text{—— (1)}$$

$$s > 0.$$

L is known as a Laplace transform operator.

The orginal given function as determining function depends on it, while the new function depends on only s because the improper integral on the R.H.S of (1) is integrated w.r.t. t.

$F(s)$ is known as the Laplace transform of t or simply transforms, in which $f(t)$ is given and $F(s)$ is to be determined.



Here the parameter S is real or Complex.

The relation (1) can be expressed

$$f(t) = L^{-1}\{F(s)\}.$$

In such a case the function $f(t)$ is called Inverse Laplace transform of $F(s)$.

Find the Laplace transform of $f(t) = k$.

Where k is a constant $t > 0$.

$\bar{e}^{\infty} = 0$

$\bar{e}^{0} = \infty$.

$$L\{f(t)\} = \int_0^\infty \bar{e}^{st} \cdot k \, dt = \left[\frac{-k}{s} \bar{e}^{st} \right]_0^\infty$$

$$L\{k\} = \frac{k}{s}$$

Note L.T does not exists for $s \leq 0$.

if $k = 0$ $L\{0\} = 0$

for $k=1$ $L\{1\} = 1/s$.

Sufficient Conditions for the existence of the Laplace transforms function.

L.T does not satisfying for all functions.

If it exists, it is uniquely determined.

The following conditions are to be satisfied

let ① $f(t)$ be given function if

1. $f(t)$ a piecewise continuous on every finite interval.
2. $f(t)$ satisfies the following inequality

$$|f(t)| \leq b e^{at} \quad \forall \, t \geq 0$$

a and b are constant.

$L\{f(t)\}$ exists satisfies second condition is called exponential order.

for example $\cosh t < e^t \quad \forall t > 0.$

$t^n < \lfloor n \, e^t \quad \lfloor n = 0, 1, \ldots \; t > 0.$

$e^{t^2} > b e^{at}$ wherever may be and $b \leq 0$.

A function $f(t)$ is said to be of exponential order a as $t \to \infty$ if $\exists$ a $M > 0$

Such that $|f(t)| < Me^{at} \quad \forall t > 0.$

i.e, $\lim_{t \to \infty} e^{-st} f(t) =$ finite.

or equallently

$|e^{-at} f(t)| \leq M \quad \forall t \geq 0.$

e^{t^2} not of exponential order since

$|e^{-at} e^{t^2}| = e^{t^2 - at}.$

Which can be made larger than by any given constant by increasing t indefinitely

In other words $f(t)$ is of exponential order do not grew faster than e^{st}

Since $\lim_{t \to \infty} \frac{t^2}{e^{3t}} =$ finite $\qquad f(t) = t^2$

$$\lim_{t \to \infty} e^{-3t} t^2 = \lim_{t \to \infty} t^2 / e^{3t}$$

$$= \lim_{t \to \infty} \frac{2t}{3e^{3t}} = \lim_{t \to \infty} \frac{2}{9e^{3t}} = \frac{2}{\infty} = 0$$

finite

exponential order.

let $f(t) = t^3$.

$$\lim_{t\to\infty} e^{-at} e^{t^3} = \lim_{t\to\infty} e^{-at+t^3} = \lim_{t\to\infty} e^{t^3-at} = e^{\infty} = \infty$$

not a finite

So it is not an exponential order.

1. $$L\{1\} = \int_0^\infty e^{-st} \cdot 1\, ds = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \left[\frac{e^{-\infty}}{-s} - \frac{e^{-0}}{-s}\right]$$

$$= \frac{1}{s}.$$

2. $$L\{t\} = \int_0^\infty e^{-st} \cdot t\, dt = \left[t \cdot \frac{e^{-st}}{-s}\right]_0^\infty - 1\left[\frac{e^{-st}}{s^2}\right]_0^\infty$$

$$= \frac{1}{s^2}.$$

3. $$L\{t^2\} = \int_0^\infty e^{-st} \cdot t^2\, dt = \left[t^2 \frac{e^{-st}}{-s}\right]_0^\infty - 2\int_0^\infty t\, \frac{e^{-st}}{s^2} dt\, dt$$

$$= 0 - \left[2t \cdot \frac{e^{-st}}{-s^2}\right]_0^\infty - \left[\int_0^\infty \frac{e^{-st}}{-s^2} dt\right]$$

$$= 2\left(\frac{e^{-st}}{s^3}\right)_0^\infty = 2/s^3.$$

$$L\{t^3\} = 6/s^4$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$ n is a +ve integer.

If $n > 0$ then the gamma function is defined as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma(n+1) = n\Gamma n$$

$$\Gamma 1 = 1.$$

$$\Gamma(n+1) = n!$$

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}.$$

1. $L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} dt$.

$= \int_0^\infty e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$

$= \frac{1}{s-a}$.

2. Illy $L\{e^{-at}\} = \frac{1}{s+a}$.

3. $L\{\sinh at\} = L\left\{ \frac{e^{at} - e^{-at}}{2} \right\} = \frac{1}{2} L\{e^{at}\} - L\{e^{-at}\}$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a} \right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2} \right]$

$= \frac{a}{s^2-a^2}$

4. $L\{\cosh at\} = L\left\{ \frac{e^{at} + e^{-at}}{2} \right\} = \frac{1}{2} L\{e^{at} + e^{-at}\}$.

$= \frac{1}{2}\left\{ \frac{1}{s-a} + \frac{1}{s+a} \right\} = \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2} \right]$

$L\{\cosh at\} = \frac{s}{s^2-a^2}$

$\int_0^\infty e^{-st} t^n dt$ put $st = x$.

$t = x/s$

$dt = \frac{1}{s} dx$.

$\int_0^\infty e^{-x} \cdot (x/s)^n \frac{dx}{s} = \int_0^\infty e^{-x} \cdot \frac{x^n}{s^{n+1}} dx$.

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \cdot x^n dx = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$= \int_0^\infty e^{-st}\cdot \sin at\,.$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty.$$

$$= \frac{a}{s^2+a^2}$$

Illy

$$L\{\cos at\} = \frac{s}{s^2+a^2}.$$

Aliter.

$$L\{e^{at}\} = \frac{1}{s-a}.$$

$$L\{e^{iat}\} = \frac{1}{s-ia}\times\frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2}.$$

$$L\{\cos at + i\sin at\} = \frac{s}{s^2+a^2} + i\,\frac{a}{s^2+a^2}$$

Compare the terms.

$$L\{\cos at\} = \frac{s}{s^2+a^2} \;\;\&\;\; L\{\sin at\} = \frac{a}{s^2+a^2}.$$

$$L\left\{\frac{e^{-at}-1}{a}\right\} = \int_0^\infty e^{-st}\left(\frac{e^{-at}-1}{a}\right)dt$$

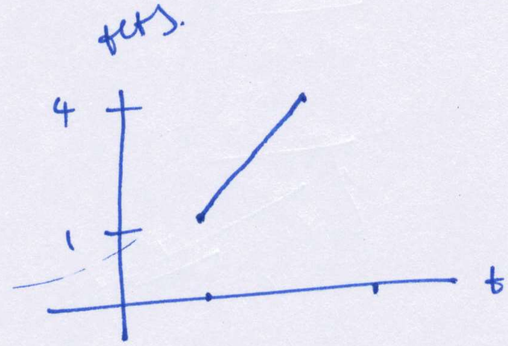
$$= \frac{1}{a}\int_0^\infty (e^{-st}e^{-at} - e^{-st})\,dt$$

$$= \frac{1}{a}\left[\frac{e^{-(s+a)t}}{-(s+a)} - \frac{e^{-st}}{-s}\right]_0^\infty.$$

$$= \frac{1}{a}\left[\frac{1}{s+a} - \frac{1}{s}\right] = \frac{1}{a}\left[\frac{s-s-a}{s(s+a)}\right] = -\frac{1}{s(s+a)}$$

Prob:

$$f(t) = 0 \quad 0 < t < 1$$
$$= t \quad 1 < t < 4$$
$$= 0 \quad t > 4$$



$$= \int_0^\infty e^{-st} f(t)$$

$$= \int_0^1 e^{-st} \cdot 0 \, dt + \int_1^4 e^{-st} \cdot t \, dt + \int_4^\infty e^{-st} \cdot 0$$

$$= \left[t \cdot \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_1^4$$

Linearity Property.

If $L\{f_1(t)\} = F_1(s) \quad L\{f_2(t)\} = F_2(s)$

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}$$
$$= c_1 F_1(s) + c_2 F_2(s).$$

$$L\{\cos^4 t\} = L\left\{\frac{1 + \cos 8ht}{2}\right\}.$$

Find the L-T of

$$f(t) = \begin{cases} 4 & 0 \le t < 2 \\ 0 & t \ge 2. \end{cases}$$

We know that $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^2 e^{-st} \cdot 4\,dt + \int_2^\infty e^{-st} \cdot 0\,dt$$

$$= 4\left[\frac{e^{-st}}{-s}\right]_0^2 = 4\left[\frac{e^{-2s}}{-s} + \frac{1}{s}\right]$$

$$= \frac{4}{s}\left(1 - e^{-2s}\right).$$

$$f(t) = \begin{cases} 2t+1 & 0 \le t \le 1. \\ 0 & t \ge 1. \end{cases}$$

We know that $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^1 e^{-st}(2t+1)\,dt + \int_1^\infty e^{-st} \cdot 0\,dt$$

$$= \left[(2t+1) \cdot \frac{e^{-st}}{-s} - 2 \cdot \frac{e^{-st}}{s^2}\right]_0^1$$

$$= \left[3 \cdot \frac{e^{-s}}{-s} - \frac{2e^{-s}}{s^2} - \frac{e^{-0}}{-s} + \frac{2}{s^2}\right]$$

$$= \frac{1}{s^2}\left[-se^{-s} - 2e^{-s} - 1 + 2\right]$$

$$= \frac{1}{s^2}\left[1 - 2e^{-s} - se^{-s}\right]$$

Prob: Find $L\{\sin 2t \cos 3t\}$

$$= L\left\{\frac{\sin 5t - \sin t}{2}\right\}.$$

$$= \frac{1}{2}\left\{L\{\sin 5t\} - L\{\sin t\}\right\}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right]$$

Prob: $L\{\sin^3 2t\}$.

We know $\sin 3t = 3\sin t - 4\sin^3 t.$

$$\sin 6t = 3\sin 2t - 4\sin^3 2t.$$

$$\sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$$

$$L\{\sin^3 2t\} = \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$$

$$= \frac{3}{4}\cdot\frac{2}{s^2+4} - \frac{1}{4}\cdot\frac{6}{s^2+36}$$

First Shifting Theorem:

If $L\{f(t)\} = F(s)$ then $L\{e^{at}f(t)\} = F(s-a)$.

$$L\{e^{at}f(t)\} = \int_0^\infty e^{-st}\cdot e^{at}f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt \quad s-a=u.$$

$$= \int_0^\infty e^{-ut} f(t)\,dt = F(u)$$

$$= F(s-a).$$

Illy $L\{e^{-at}f(t)\} = F(s+a)$.

$$L\{e^{at}t^n\} = \left.\frac{n!}{s^{n+1}}\right|_{s\to s-a} = \frac{n!}{(s-a)^{n+1}}.$$

Illy $$L\{e^{at}\sin bt\} = \left(\frac{b}{s^2+b^2}\right)_{s\to s-a} = \frac{b}{(s-a)^2+b^2}$$

$$L\{e^{at}\cos bt\} = \left(\frac{s}{s^2+b^2}\right)_{s\to s-a} = \frac{s}{(s-a)^2+b^2}$$

$$L\{e^{at}\sinh bt\} = \left(\frac{b}{s^2-b^2}\right)_{s\to s-a} = \frac{b}{(s-a)^2-b^2}$$

$$L\{e^{at}\cosh bt\} = \left(\frac{s}{s^2-b^2}\right)_{s\to s-a} = \frac{s-a}{(s-a)^2-b^2}.$$

Prob: $L\{e^{-3t}t^3\}$.

Let $f(t) = t^3$.

$L\{f(t)\} = \frac{3}{s^4}$ $\because L\{t^n\} = \frac{n!}{s^{n+1}}$

By F.S.Thm

Say

$L\{f(t)\} = f(s)$

$L\{e^{at}f(t)\} = F(s-a)$

$$L\{e^{-3t}t^3\} = \left(3/s^4\right)_{s\to s+3}$$

$$= \frac{3}{(s+3)^4}$$

Prob: $e^{-t}(3\cos 5t - 4\sin 5t)$.

Let $f(t) = 3\cos 5t - 4\sin 5t$

$L\{f(t)\} = 3\cdot\frac{s}{s^2+25} - 4\cdot\frac{5}{s^2+25}$ = Say $F(s)$.

By F.S. Thm. $L\{e^{at}f(t)\} = F(s-a)$ if $L\{f(t)\} = F(s)$

$\therefore = L\{e^{-t}(3\cos 5t - 4\sin 5t)\} = \left[\frac{3s}{s^2+25} - \frac{4s}{s^2+25}\right]_{s \to s+1}$

$$= \left[\frac{3(s+1) - 4(s+1)}{(s+1)^2+25}\right]$$

Prob: $(1+te^{-t})^3$, $e^{3t}\sin^2 t$, $e^{4t}\sin 2t\cos t$, $4e^{3t}\sin(2t+\alpha)$, $\cosh at \cos bt$.
$\cos^4 t$, $\cos^3 2t$,

Change Scale property

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a}F(s/a)$

By Def. $L\{f(at)\} = \int_0^\infty e^{-st}\cdot f(at)\,dt$

let $at = x$.
$a\,dt = dx$.
$t=0$ $x=0$
$t=\infty$ $x=\infty$

$$= \int_0^\infty e^{-(x/a)s}\cdot f(x)\,\frac{dx}{a}.$$

$$= \frac{1}{a}\int_0^\infty e^{-s/a\cdot x} f(x)\,dx = \frac{1}{a}F(s/a).$$

If $L\{f(t)\} = \frac{1}{s}\bar{e}^{1/s}$. Say $F(s)$.

find $L\{\bar{e}^{t} f(3t)\}$

By Change Scaleproperty $L\{f(t)\} = f(s)$

then $L\{f(at)\} = \frac{1}{a} F(s/a)$

$$F(3t) = \frac{1}{3} \cancel{F}(S/3)$$

$$= 1/3\left[\frac{1}{s/3} \cdot \bar{e}^{1/s/3}\right]$$

$$F(3t). = \left[\frac{1}{\cancel{3}} \cdot \frac{3}{s} \cdot \bar{e}^{3/s}\right] = \left[\frac{1}{s}\bar{e}^{3/s}\right]$$

$F(s) =$ Say

By First Shifting Theorem.

$L\{f(t)\} = f(a)$ then $L\{e^{at} f(t)\} = F(s-a)$

$$L\{\bar{e}^{t} f(3t)\} = \left[\frac{1}{s}\bar{e}^{3/s}\right]_{s \to s+1}$$

$$= \frac{1}{s+1} \cdot \bar{e}^{3/s+1}$$

Prob: $L\{f(t)\} = \frac{20-4s}{s^2-4s+20}$ fin $L\{f(3t)\}$.

Intial Value Theorem:

If $L\{f(t)\} = \bar{f}(s)$ then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s \cdot F(s)$.

Final Value Theorem:

If $L\{f(t)\} = F(s)$ then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s \cdot F(s)$.

Verify initial Value theorem for $(2t+3)^2$.

Let $f(t) = (2t+3)^2$

$\lim_{t\to 0} (2t+3)^2 = 9.$

$\lim_{s\to\infty} s \{L\{4t^2+9+12t\}\}$

$\lim_{s\to\infty} s \left[4 \cdot \frac{2!}{s^3} + \frac{9}{s} + \frac{12}{s^2}\right]$

$\lim_{s\to\infty} \left(\frac{4\cdot 2}{s^2} + 9\cancel{0} + \frac{12}{s}\right) = \underline{9}.$

$\therefore$ IVT. Verified

Prob: Verify the final value theorem for $t^3 e^{-2t} = f(t)$

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s.F(s).$$

$$\lim_{t \to \infty} t^3/e^{2t} \quad \frac{\infty}{\infty} \text{ form.}$$

$$\lim_{t \to \infty} \frac{3t^2}{2e^{2t}} \quad \text{again } \frac{\infty}{\infty} \text{ form}$$

$$\lim_{t \to \infty} \frac{6t}{4e^{2t}} \quad \text{"} \quad \text{"}$$

$$\lim_{t \to \infty} \frac{6}{8e^{2t}} = 0.$$

$$\lim_{s \to 0} s.F(s) = \lim_{s \to 0} s.\frac{6}{(s+2)^4}$$

$$= \lim_{s \to 0} . s.\frac{6}{(s+2)^4} = 0.$$

$L\{f(t)\} = f(s).$

$$L\{t^3\} = \frac{3!}{s^4}$$

$$L\{e^{-2t}t^3\} = \frac{6}{s^4}\Big|_{s \to s+2}$$

$$= \frac{6}{(s+2)^4} = F(s) \text{ say}$$

Hence Verified

Laplace transform of multiplication of t $\{t^n f(t)\}$.

If $f(t)$ is sectionally continuous and of exponential order and if $L\{f(t)\} = F(s)$

then $L\{t\, f(t)\} = -\bar{F}(s)$.

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt \cdot$$

diff w.r.t. s both sides.

$$F'(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty \partial/\partial s\, e^{-st} f(t)\, dt$$

$$F'(s) = \int_0^\infty e^{-st} \cdot -t\, f(t)\, dt$$

$$= -\int_0^\infty e^{-st} (t\, f(t))\, dt$$

$$-F'(s) = L\{t\, f(t)\}$$

If $f(t)$ is a function of piecewise continuous for all values of t and exponential order

$$L\{t^n f(t)\} = (-1)^n \underline{\underline{F^n(s)}} = (-1)^n \frac{d^n}{ds^n} F(s).$$

Prob: $L\{t\,e^{-2t}\} =$ let $f(t) = e^{-2t}$

$L\{f(t)\} = \frac{1}{s+2} = F(s)$ say

By multiplication of t

$L\{f(t)\} = F(s)$ then $L\{t^n f(t)\} = (-1)^n F^n(s)$.

$$L\{t e^{-2t}\} = (-1)^1 \cdot \frac{d}{ds} \cdot \frac{1}{s+2} = (-1) \cdot \frac{-1}{(s+2)^2}$$

$$= \frac{1}{(s+2)^2}.$$

Prob: $L\{t\cos at\}$ let $f(t) = \cos at$

$L\{f(t)\} = \frac{s}{s^2+a^2}$

$= F(s)$ say

$$L\{t\cos at\} = (-1)^n \frac{d}{ds} \frac{s}{s^2+a^2}$$

$$= -\cdot \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= -\left(\frac{s^2+a^2-s\cdot 2s}{(s^2+a^2)^2}\right)$$

$$= +\frac{s^2-a^2}{(s^2+a^2)^2}$$

Find $L\{t e^{-4t} \sin 3t\}$.

let $f(t) = \sin 3t$.

$$L\{f(t)\} = \frac{3}{s^2+9}. = F(s) \text{ say.}$$

$$L\{\cancel{E}\, t \sin 3t\} = \frac{-d}{ds} \cdot \frac{3}{s^2+9}.$$

By Multiplication of t.

If $L\{f(t)\} = F(s)$ then

$$L\{t f(t)\} = -F'(s)$$

$$= -\left[\frac{(s^2+9)\cdot 0 - 3 \cdot 2s}{(s^2+9)^2}\right]$$

$$= -\left(\frac{-6s}{(s^2+9)^2}\right) = F(s) \text{ say again}$$

By F.S. Theorem $L\{f(t)\} = F(s)$

$$L\{e^{at} f(t)\} = F(s-a).$$

$$L\{e^{-4t}\, t \sin 3t\} = \left.\frac{6s}{(s^2+9)^2}\right|_{s \to s+4}$$

$$= \frac{6(s+4)}{[(s+4)^2+9]}. = \frac{6(s+4)}{s^2+16+8s+9}$$

$$= \frac{6(s+4)}{s^2+8s+25}$$

Evaluate $\int_0^\infty t e^{-3t} \sin t \, dt$.

Let $f(t) = \sin t$.

$L\{f(t)\} = \frac{1}{s^2+1} = F(s)$ say.

$L\{t . \sin t\} = - F'(s)$. $\quad L\{f(t)\} = F(s)$

$L\{t f(t)\} = -F'(s)$

$= -\frac{d}{ds} \cdot \frac{1}{s^2+1}$

$= \frac{-(-2s)}{(s^2+1)^2} = \frac{2s}{(s^2+1)^2} = F(s)$ say

By Def. of L.T $\quad L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt$

$= \int_0^\infty e^{-3t} \cdot t \sin t \, dt = F(s) \Big|_{s=3}$.

$= \frac{2s}{(s^2+1)^2} \Big|_{s=3} = \frac{6}{100}$.

$= \frac{3}{50}$

Laplace transform of division by 't' (1/0).

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{F}(s)\,ds$.

Provided that integral exists.

i.e., $\lim_{t \to 0} \frac{f(t)}{t}$ exists

Given $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$.

Integrating both sides wrt s and the lts s to ∞

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds.$$

The order of integration in the double integral can be interchange since s and t are independent variables.

$$\therefore \int_s^\infty F(s)\,ds = \int_{t=0}^\infty dt \int_{s=s}^\infty e^{-st} f(t)\, \cancel{dt}\, ds.$$

$$= \int_{t=0}^\infty f(t)\,dt \int_s^\infty e^{-st}\, \cancel{f(t)}\, ds.$$

$$= \int_{t=0}^\infty \left(e^{-st} \cdot \frac{1}{-\cancel{t}}\right)_s^\infty f(t)\,dt.$$

$$= \cancel{\frac{1}{s}} \int_{t=0}^\infty e^{-st} \cdot \frac{f(t)}{t}\,dt$$

$$\int_s^\infty F(s)\,ds = L\left\{\frac{F(t)}{t}\right\}$$

Prob: $L\left\{\frac{\sin t}{t}\right\}$

Let $f(t) = \sin t$.

$$L\{f(t)\} = \frac{1}{s^2+1}.$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds.$$

$$= \left(\tan^{-1} s\right)_s^\infty = \tan^{-1}\infty - \tan^{-1} s.$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s.$$

Since $\tan^{-1} x + \cot^{-1} x = \pi/2$.

$$\cot^{-1} x = \pi/2 - \tan^{-1} x.$$

Prob: $\int_0^\infty e^{-t} \frac{\sin t}{t}\,dt = \pi/4$.

We know that $L\left\{\frac{f(t)}{t}\right\} = \cot^{-1} s$.

$$\int_0^\infty e^{-t} \cdot f(t)\,dt = \cot^{-1} s \Big|_{s=1}.$$

$$= \cot^{-1}(1) = \cot^{-1}(\cot \pi/4)$$

$$= \pi/4$$

$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$ $\qquad F(t) = e^{-at} - e^{-bt}$

$$L\{f(t)\} = \frac{1}{s+a} - \frac{1}{s+b} = F(s) \text{ say}$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds.$$

$$= \Big[\ln(s+a) - \ln(s+b)\Big]_s^\infty$$

$$= \cancel{\ln}\left[\ln\left(\frac{s+a}{s+b}\right)\right]_s^\infty.$$

$$= \lim_{s\to\infty} \ln\left(\frac{1+a/s}{1+b/s}\right) - \ln\left(\frac{s+a}{s+b}\right)$$

$s \to \infty$

$\frac{1}{s} \to 0.$

$$= \ln 1 - \ln\left(\frac{s+a}{s+b}\right).$$

$$= 0 - \ln\left(\frac{s+a}{s+b}\right) = \ln\left(\frac{s+a}{s+b}\right)^{-1}$$

$$= \ln\left(\frac{s+b}{s+a}\right)$$

$$L\left\{\frac{1-\cos at}{t}\right\}, \quad L\left\{\frac{1-\cos at}{t^2}\right\}.$$

Laplace transform of derivatives..

If $f(t)$ is continuous and exponential order and $f'(t)$ is sectionally continuous then the $L\{f'(t)\}$ is given by.

$$L\{f'(t)\} = s\,F(s) - f(0) \qquad \text{where } F(s) = L\{f(t)\}.$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty \left[e^{-st} f(t)\,dt\right]$$

$$= 0 - f(0) + s\,L\{f(t)\}.$$

$$= s\,L\{f(t)\} - f(0).$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$$

$$L\{\cos at\} = s/s^2+a^2$$

$$f(t) = \cos at \qquad f(0)=1.$$

$$f'(t) = -a\sin at \Rightarrow f'(0) = 0.$$

$$f''(t) = -a^2\cos at \Rightarrow f''(0) = 1.$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - s\,f(0) - f'(0)$$

$$L\{-a^2\cos at\} = s^2 L\{f(t)\} - s.1 - 0.$$

$$-a^2 L\{\cos at\} = s^2 L\{\cos at\} - s.$$

$$L\{\cos at\}(s^2+a^2) - s = 0.$$

$$L\{\cos at\} = s/s^2+a^2$$

Laplace transform ~~of~~ Second shifting Theorem.

$L\{f(t)\} = F(s)$ and.

$$g(t) = \begin{cases} f(t-a) & t > a. \\ 0 & t < a. \end{cases}$$

Then $L\{g(t)\} = e^{-as} F(s)$.

Find the L-T

$$G(t) = \begin{cases} \cos(t - \pi/3) & \text{if } t > \pi/3. \\ 0 & t < \pi/3. \end{cases}$$

Obviosly $f(t) = \cos t$.

$$L\{f(t)\} = \frac{s}{s^2+1} = F(s).$$

$$g(t) = \begin{cases} f(t-\pi/3) = \cos(t-\pi/3) & t > \pi/3 \\ 0 & t < \pi/3. \end{cases}$$

By the Second Shifting theorem.

$$L\{g(t)\} = e^{-\pi/3\, s} \cdot \frac{s}{s^2+1}$$

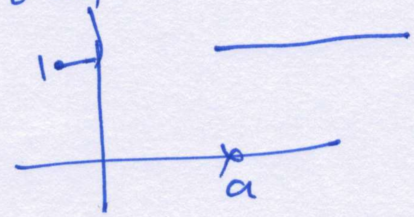
Laplace Transform of Integral.

$L\{f(t)\} = F(s)$ then $L\{\int_0^t f(t)\,dt\} = \frac{1}{s}F(s)$

Unit Step function.

The Unit Step function $u(t-a)$ defined as.

$$u(t-a) = \begin{cases} 1 & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$$



$$L\{u(t-a)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

$$= \int_0^a e^{-st}\,dt + \int_a^\infty e^{-st}\cdot 0\,dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_0^a = \frac{e^{-as}}{s} = \frac{1}{s}\cdot e^{-as}.$$

$L\{t\,u(t-2)\}$.

We know that $L\{u(t-2)\} = \frac{e^{-2s}}{s} = F(s)$ say.

$$L\{t\,u(t-2)\} = -\frac{d}{ds} L\{u(t-2)\} = -\frac{d}{ds}F(s).$$

$$= -\frac{d}{ds}\left(\frac{e^{-2s}}{s}\right) = -\left(\frac{s2e^{-2s} - e^{-2s}}{s^2}\right)$$

$$= \frac{2}{s}e^{-2s} + \frac{1}{s^2}e^{-2s}$$

If $f(t)$ is a periodic function with period T then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{[1-e^{-sT}]}.$$

By Def We have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$f(t) = f(t+T) = f(t+2T) = f(t+3T) \ldots = f(t+nT)$$

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_0^{2T} e^{-st} f(t)\,dt \quad \ldots \quad + \int_0^{nT} f(t) e^{-st}\,dt \ldots$$

put $t = u+T$ In the second integral

$t = 2u+2T$ " Third "

$\vdots$

Then

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \cdots$$

$$= \int_0^T e^{-st} f(t) + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du.$$

$$f(u) = f(u+T) = f(u+2T) \ldots$$

$$L\{f(t)\} = (1 - e^{-sT} + e^{-2sT} + \ldots \infty) \int_0^T e^{-st} f(t)\,dt.$$

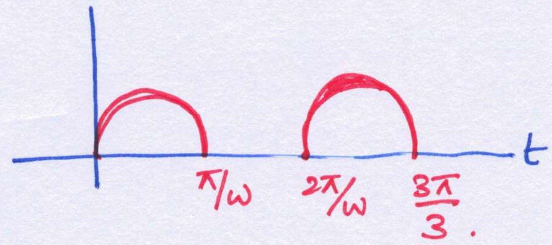
$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

Prob: $f(t) = \sin \omega t \quad 0 < t < \pi/\omega$

$\quad\quad = 0 \quad\quad \pi/\omega < t < 2\pi/\omega \cdot f(t).$

and periodic function of period $\frac{2\pi}{\omega}$.

which is half wave rectifier.



$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$$

$$= \frac{1}{1-e^{-s\frac{2\pi}{\omega}}} \int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} . \sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} . 0\,dt. \right.$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt \right].$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2+\omega^2} (s \sin\omega t - \omega \cos\omega t) \right]_0^{\pi/\omega}.$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2} . \omega + \frac{\omega}{s^2+\omega^2} \right]$$

$$= \frac{\omega}{s^2+\omega^2} \cdot \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left(e^{-\frac{\pi s}{\omega}} + 1 \right).$$

$$= \frac{\omega}{s^2+\omega^2} \cdot \frac{1}{\left(1^2 - \left(e^{-\frac{\pi s}{\omega}}\right)^2\right)} . \left(1 + e^{-\frac{\pi s}{\omega}}\right).$$

$$= \frac{\omega}{s^2+\omega^2} \cdot \frac{1}{\cancel{(1+e^{-\frac{\pi s}{\omega}})}(1-e^{-\frac{\pi s}{\omega}})} . \cancel{\left(1 + e^{-\frac{\pi s}{\omega}}\right)}$$

$$= \frac{\omega}{s^2+\omega^2} \cdot \frac{1}{1-e^{-\frac{\pi s}{\omega}}}$$

=.

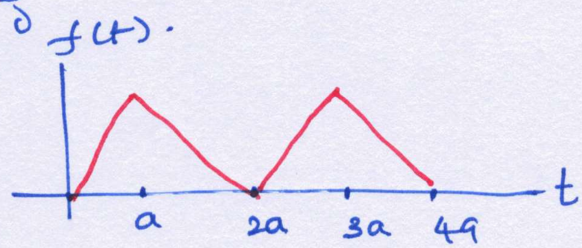
Prob: $f(t) = t \quad 0 < t < a$

$\quad = 2a - t \quad a < t < 2a$

$f(t+2a) = f(t).$

$T = 2a.$

$$\bar{F}(s) = L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$



$$= \frac{1}{1-e^{-2as}} \left[\int_0^{2a} e^{-st} f(t)\, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} e^{-st} \cdot t\, dt + \int_a^{2a} e^{-st} \cdot (2a-t)\, dt. \right.$$

$$= \frac{1}{1-e^{-2as}} \left[\left[\left(\frac{t\, e^{-st}}{-s} \right) - \left(\frac{e^{-st}}{s^2} \right) \right]_0^a + \left((2a-t) \cdot \frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right)_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{a\, e^{-as}}{-s} - \frac{e^{-as}}{s^2} + \frac{e^{-2as}}{s^2} + \frac{a\, e^{-as}}{s} - \frac{e^{-as}}{s^2} \right].$$

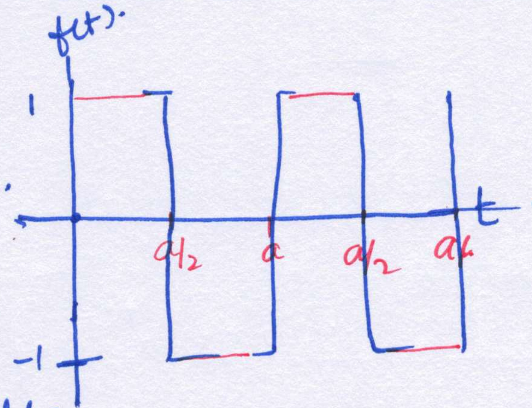
$$= \frac{1}{s^2(1-e^{-2as})} (1-e^{-as})^2.$$

$$= \frac{1}{s^2(1+e^{-as})(1-e^{-as})} (1-e^{-as})^2$$

$$= \frac{1}{s^2} \frac{(1-e^{-as})}{1+e^{-as}} = \frac{1}{s^2} \tanh \frac{as}{2}.$$

Prob: $f(t) = 1 \quad 0 < t < a/2$

$\qquad = -1 \quad a/2 < t < a.$



$T = a.$

$f(a+t) = f(t).$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}} \int_0^{a/2} e^{-st} \cdot 1\,dt + \int_{a/2}^{a} e^{-st} \cdot (-1)\,dt.$$

$$= \frac{1}{1-e^{-as}} \left[\frac{e^{-st}}{-s}\right]_0^{a/2} + \left[\frac{e^{-st}}{s}\right]_{a/2}^{a}.$$

$$= \frac{1}{1-e^{-as}} \left[-\frac{e^{-as/2}}{s} + \frac{1}{s}\right] + \left[\frac{e^{-as}}{s} - \frac{e^{-as/2}}{s}\right]$$

$$= \frac{1}{s(1-e^{-as})} \left[1 + e^{-as} - 2e^{-as/2}\right].$$

$$= \frac{1}{s(1-e^{-as})} \left[1 - e^{-as/2}\right)^2 = \frac{1}{s(1-e^{-as})} (1 - e^{-\frac{as}{2}})^2.$$

$$= \frac{1}{s(1^2-(e^{-\frac{as}{2}})^2)} \cdot (1 - e^{-\frac{as}{2}})^2.$$

$$= \frac{1 \cdot (1-e^{-as/2})^{\cancel{2}}}{s(1+e^{-as/2})\cancel{(1-e^{-as/2})}}$$

$$= \frac{1}{s} \frac{(1-e^{-as/2})}{1+e^{-as/2}}$$

Laplace transform of Integrals.

If $L\{f(t)\} = F(s)$ then $L\{\int_0^t f(t)\,dt\} = \frac{F(s)}{s}$.

Prob: $L\{\int_0^t \sin at\,dt\} = \frac{1}{s}\cdot\frac{a}{s^2+a^2}$.

Let $L\{\sin at\} = \frac{a}{s^2+a^2}$

Prob: Find the ILT of $\int_0^t \frac{\sin t}{t}\,dt$.

Let $f(t) = \sin t$.

$L\{f(t)\} = \frac{1}{s^2+1}$

$L\{\frac{\sin t}{t}\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \tan^{-1}s\Big]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}s = \pi/2 - \tan^{-1}s = \cot^{-1}s.$

$= \tan^{-1}(1/s). = F(s)$ say.

$L\{\int_0^t \frac{\sin t}{t}\,dt\} = \frac{1}{s}\tan^{-1}(1/s).$

Inverse Laplace Transform

If $L\{f(t)\} = F(s)$ then $L^{-1}\{F(s)\} = f(t)$.

L^{-1} is called the Inverse Laplace Transform operator but not reciprocal.

$$L\{e^{at}\} = \frac{1}{s-a} \Rightarrow L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$L\{e^{-at}\} = \frac{1}{s+a} \Rightarrow L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

$$L\{\sin at\} = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$$

$$L\{\cos at\} = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$L\{\sinh at\} = \frac{a}{s^2-a^2} \Rightarrow L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$$

$$L\{\cosh at\} = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at.$$

$$L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n.$$

Linearity Property

If $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ & $f_2(t)$

then $L^{-1}\{c_1 F_1(s) + c_2 F_2(s)\} = c_1 L^{-1}\{F_1(s)\} + c_2 L^{-1}\{F_2(s)\}$.

Find $L^{-1}\left\{\frac{1}{2s-5}\right\} = \frac{1}{2} L^{-1}\left\{\frac{1}{s-5/2}\right\}$.

$$= \frac{1}{2} e^{\frac{5t}{2}}$$

First Shifting Theorem:

$L^{-1}\{f(s)\} = f(t)$ then $L^{-1}\{F(s-a)\} = e^{at} f(t)$ i.e., $e^{at} L^{-1}\{F(s)\}$.

Prob: Find ILT of $\frac{3s+4}{s^2+16}$.

$$L^{-1}\left\{\frac{3s+4}{s^2+16}\right\} = L^{-1}\left\{\frac{3s}{s^2+16} + \frac{4}{s^2+16}\right\}$$

$$= 3 L^{-1}\left\{\frac{s}{s^2+16}\right\} + L^{-1}\left\{\frac{4}{s^2+16}\right\}$$

$$= 3.\cos 4t + \sin 4t.$$

Find $L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s+2}{(s-2)^2+9}\right\}$

$$= L^{-1}\left\{\frac{s-2+4}{(s-2)^2+9}\right\}$$

$$= L^{-1}\left\{\frac{(s-2)+4}{(s-2)^2+9}\right\} = F(s-2)$$

$$= e^{2t} L^{-1}\left\{\frac{s+4}{s^2+9}\right\} = e^{2t}\left\{L^{-1}\left\{\frac{s}{s^2+9}\right\} + 4 L^{-1}\left\{\frac{1}{s^2+9}\right\}\right.$$

$$= e^{2t}.\cos 3t + \frac{4}{3} e^{2t} \sin 3t$$

Change Scale property.

$L^{-1}\{f(s)\} = f(t)$ then $L^{-1}\{F(s/a)\} = a f(at)$.

I.L.T of Derivatives

$L^{-1}\{f(s)\} = f(t)$ then $L^{-1}\left\{\frac{d^n}{ds^n} f^n(s)\right\} = (-1)^n t^n f(t)$.

ILT of Multiplication ~~one~~ By . s.

$L^{-1}\{f(s)\} = f(t)$ then $L^{-1}\{s f(s)\} = f'(t)$.

where $f(0) = 0$.

ILT of Divison by s.

$L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t f(u)\,du$.

ILT of Second Shifting theorem.

$L^{-1}\{f(s)\} = f(t)$.

then $e^{-as} f(s) = G(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

then $L^{-1}\{G(t)\} = L^{-1}\{e^{-as} f(s)\} = F(t)$.

Prob: $L^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\}$. By Partial Fractions

let $\dfrac{3s+7}{s^2-2s-3} = \dfrac{3s+7}{(s+1)(s-3)} = \dfrac{A}{s+1} + \dfrac{B}{s-3}$

$A(s-3) + B(s+1) = 3s+7.$

if $s = 3$ $\quad 4B = 16 \Rightarrow B = 4.$

if $s = -1$ $\quad A(-4) = 4$

$A = -1.$

$\therefore \dfrac{3s+7}{s^2-2s-3} = \dfrac{-1}{s+1} + \dfrac{4}{s-3}$

$L^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\} = L^{-1}\left\{\dfrac{-1}{s+1} + \dfrac{4}{s-3}\right\}$

$= 4e^{3t} - e^{t}.$

Prob. $L^{-1}\left\{\dfrac{3s-2}{s^2-4s+20}\right\} = L^{-1}\left\{\dfrac{3s-2}{s^2-4s+4+16}\right\}$

$= L^{-1}\left\{\dfrac{3s-2}{(s-2)^2+16}\right\} = L^{-1}\left\{\dfrac{3(s-2)+4}{(s-2)^2+16}\right\} = F(s-2)$

$L^{-1}\{F(s-a)\} = e^{at} L^{-1}\{f(s)\}$ By FSTh

$= e^{2t} L^{-1}\left\{\dfrac{3s+4}{s^2+16}\right\} = e^{2t} L^{-1}\left\{\dfrac{3s}{s^2+16} + \dfrac{4}{s^2+16}\right\}$

$= e^{2t}\left\{3 L^{-1}\left\{\dfrac{s}{s^2+16}\right\} + L^{-1}\left\{\dfrac{4}{s^2+16}\right\}\right\}$

$= e^{2t}(3\cos 4t + \sin 4t)$

Find the $L^{-1}\left\{\frac{1-s}{(s+1)(s^2+4s+13)}\right\}$

Sol: By Partial fractions.

Let $\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+13}$.

$1-s = A(s^2+4s+13) + (Bs+C)(s+1)$.

Put $s=-1$. $\quad 2 = 10A \Rightarrow A = 1/5$.

Equating coeff of s^2 $\quad A+B=0$.

$B = -A = -\frac{1}{5}$.

Equating constant. $\quad 13A + C = 1$.

$C = 1 - 13A = 1 - 13\cdot\frac{1}{5} = -\frac{8}{5}$.

$\therefore \frac{1-s}{(s+1)(s^2+4s+13)} = \frac{1}{5(s+1)} + \frac{-1/5\, s - 8/5}{s^2+4s+13}$.

$L^{-1}\left\{\frac{1-s}{s+1 \cdot s^2+4s+13}\right\} = \frac{1}{5}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{5}L^{-1}\left\{\frac{s+8}{s^2+4s+13}\right\}$

$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left\{\frac{s+2+6}{(s+2)^2+9}\right\}$.

$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}L^{-1}\left\{\frac{s+6}{s^2+9}\right\}$

By FSThm $L^{-1}\{F(s-a)\} = e^{at}L^{-1}\{f(s)\}$.

$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cdot L^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{e^{-2t}}{5}L^{-1}\left\{\frac{3\cdot 2}{s^2+9}\right\}$

$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t - \frac{2}{5}e^{-2t}\sin 3t$.

$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}(\cos 3t - 2\sin 3t)$

Division Process of s.

If $L^{-1}\{f(s)\} = f(t)$ then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt$.

Find $L^{-1}\left\{\frac{1}{s(s+3)}\right\}$

Let $\frac{1}{s+3} = F(s)$ $\quad L^{-1}\{F(s)\} = e^{-3t} = F(t)$ say

$$L^{-1}\left\{\frac{1}{s(s+3)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s+3}\right\}.$$

$$= \int_0^t e^{-3t}\,dt = \left[\frac{e^{-3t}}{-3}\right]_0^t.$$

$$= \left(\frac{e^{-3t}}{-3} + \frac{1}{3}\right) = \frac{1}{3}\left(1 - e^{-3t}\right).$$

Prob: $L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$ $\quad$ Let $F(s) = \frac{1}{s^2+a^2}$

$$L^{-1}\{F(s)\} = \frac{1}{a}\sin at = F(s) \text{ say}$$

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a}\sin at\,dt = \frac{1}{a}\left(-\cos at\right)_0^t.$$

$$= \frac{1}{a^2}\left(-\cos at + 1\right) = \frac{1}{a^2}\left(1 - \cos at\right) = F(s) \text{ say}$$

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \frac{1}{a^2}\int_0^t (1 - \cos at)\,dt = \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)_0^t.$$

$$= \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right) = \frac{1}{a^2}\left(t - \frac{1}{a}\sin at\right)$$

$L\{t f(t)\} = (-1) F'(s).$

$L^{-1} F'(s) = -t f(t) = -t L^{-1}\{f(s)\}.$

Prob: Find $L^{-1}\{\tan^{-1}\left(\frac{s+3}{2}\right)\}.$

let $F(s) = \tan^{-1}\left(\frac{s+3}{2}\right)$

$F'(s) = \frac{d}{ds}\left(\tan^{-1}\left(\frac{s+3}{2}\right)\right) = \frac{1}{1+\left(\frac{s+3}{2}\right)^2} \cdot \frac{1}{2}$

$F'(s) = \frac{2}{(s+3)^2+2^2}.$

$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{2}{(s+3)^2+2^2}\right\}. = F(s+3)$ say

By ILT First Shifting theorem.

$L^{-1}\{F(s-a)\} = e^{at} L^{-1}\{F(s)\}.$

$= e^{-3t} L^{-1}\left\{\frac{2}{s^2+2^2}\right\}.$

$L^{-1}\{F'(s)\} = e^{-3t} \sin 2t.$

$-t f(t) = e^{-3t} \sin 2t.$

$f(t) = -\frac{e^{-3t} \sin 2t}{t}.$

Prob: Find $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}$.

let $F(s) = \log\left(\frac{s+1}{s-1}\right) = \ln(s+1) - \ln(s-1)$

$$F'(s) = \frac{1}{s+1} - \frac{1}{s-1}$$

Now $L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\}$.

$-t\,f(t) \qquad = e^{-t} - e^{t}$.

$\because \; L^{-1}\{F'(s)\} = -t\,f(t)$

$$f(t) = -\frac{e^{-t} - e^{t}}{t}$$

$$= \frac{e^{t} - e^{-t}}{t} = \frac{2\sinh t}{t}.$$

Second Shifting Theorem:

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{e^{-as}F(s)\} = G(t)$.

$$G(t) = \begin{cases} f(t-a), & t>a \\ 0 & t<a. \end{cases}$$

Find the inverse L.T of $\dfrac{e^{-3s}}{(s-4)^2}$.

let $F(s) = \dfrac{1}{(s-4)^2}$

$L^{-1}\{F(s)\} = L^{-1}\left\{\dfrac{1}{(s-4)^2}\right\}$.

By FSThm $L^{-1}\{F(s-a)\} = e^{at}L^{-1}\{f(s)\}$

$= e^{4t} L^{-1}\left\{\dfrac{1}{s^2}\right\}$

$= e^{4t} \cdot t \cdot = F(t)$ say

$L^{-1}\{e^{-3s}F(s)\} = G(t)$.

By Second Shifting theorem:

$$G(t) = \begin{cases} f(t-a) & \text{if } t>a \\ 0 & \text{if } t<a \end{cases}$$

$$= \begin{cases} e^{4(t-3)}(t-3) & \text{if } t>3 \\ 0 & \text{if } t<3 \end{cases}$$

(or) $= e^{4t} t \cdot H(t-3)$

$= e^{4t} t \cdot U(t-3)$.

Prob: Find the ILT of $\dfrac{e^{-\pi s}}{(s-1)^2}$

let $f(s) = \dfrac{1}{(s-1)^2} = F(s-1)$ say

$L^{-1}\{F(s)\} = e^{t} L^{-1}\{\frac{1}{s^2}\}$

By F S thm $L^{-1}\{F(s-a)\} = e^{at} L^{-1}\{F(s)\}$

$= e^{t} . t = f(t)$ say

Now $L^{-1}\left\{\dfrac{e^{-\pi s}}{(s-1)^2}\right\} = G(t)$

$G(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$

$= \begin{cases} \cancel{te^{t}}(t-\pi)e^{(t-\pi)} & \text{if } t > \pi \\ 0 & \text{if } t < \pi \end{cases}$

or $= te^{t} H(t-\pi)$ or $te^{t} U(t-\pi)$

Convolution Theorem:

$L^{-1}\{f(s)\} = f(t)$ $\quad L^{-1}\{g(s)\} = g(t)$ then

$\cancel{L\{f * g\}} \quad L^{-1}\{F(s)\, g(s)\} = f(t) * g(t)$

$L^{-1}\{F(s) \cdot G(s)\} = L^{-1}\{f(s)\} * L^{-1}\{g(s)\}.$

$= f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du.$

Prob: $L^{-1}\left\{\frac{1}{(s-1)^2}\right\}.$

Given $\frac{1}{(s-1)^2} = \frac{1}{(s-1)} \cdot \frac{1}{(s-1)}.$

$F(s) = \frac{1}{s-1} \qquad G(s) = \frac{1}{s-1}.$

$L^{-1}\{f(s)\} = e^t = f(t) \quad L^{-1}\{G(s)\} = e^t = g(t).$

then $f(u) = e^u \qquad g(t-u) = e^{t-u}$

then Convolution Theorem

$L^{-1}\{F(s)\, G(s)\} = L^{-1}\{F(s)\} * L^{-1}\{G(s)\}$

$= L^{-1}\left\{\frac{1}{s-1} \cdot \frac{1}{s-1}\right\} = L^{-1}\left\{\frac{1}{s-1}\right\} * L^{-1}\left\{\frac{1}{s-1}\right\}$

$= e^t * e^t.$

$= \int_0^t e^u e^{t-u}\, du$

$= e^t \int_0^t e^0\, du$

$$= e^t \int_0^t du.$$

$$= t e^t$$

Q Find $L^{-1}\left\{\frac{2}{(s+1)(s^2+1)}\right\}$.

let $\frac{2}{(s+1)(s^2+1)} = \frac{2}{s+1} \cdot \frac{1}{s^2+1}$.

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{2}{s+1}\right\} = 2e^{-t}$.

$L^{-1}\left\{\frac{G(s)}{\cancel{s^2+1}}\right\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$.

let $f(\cancel{t}) = \sin t \quad g(t) = 2e^{-t}$.

$L^{-1}\{F(s)\,G(s)\} = L^{-1}\{F(s)\} * L^{-1}\{G(s)\}$

$L^{-1}\left\{\frac{2}{s+1 \cdot s^2+1}\right\} = 2e^{-t} * \sin t$.

$$= \int_0^t \cancel{2e^{-t}} \sin u \, 2e^{-t+u} du.$$

$$f*g = \int_0^t f(u)\, g(t-u)\, du.$$

$$= 2e^{-t}\int_0^t e^u \sin u\, du.$$

$$= 2e^{-t}\left\{\frac{e^u}{1+1}(1.\sin u - 1.\cos u)\right\}_0^t.$$

$$= 2e^{t}\left\{\frac{e^t}{2}(\sin t - \cos t) - \frac{1}{2}(-1)\right\}.$$

$$= \cancel{2}e^{-t}\{\cancel{e^t}(\sin t - \cos t) + 1).$$

$$= \cancel{e^t}\ \sin t - \cos t + e^{-t}$$

Find $L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\}$ By Using Convolution Theorem.

let $F(s) = \frac{1}{s+3}$ $\quad G(s) = \frac{1}{s-1}$.

$L^{-1}\{F(s)\} = e^{-3t} = f(t)$ $\quad L^{-1}\{G(s)\} = e^{t} = G(t)$ say

If $L^{-1}\{f(s)\} = f(t)$; $L^{-1}\{G(s)\} = G(t)$ then

$$L^{-1}\{f(s) * G(s)\} = f * G = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t e^{-3u} e^{t-u}\, du.$$

$$= e^{t} \int_0^t e^{-3u} \cdot e^{-u}\, du = e^{t} \int_0^t e^{-4u}\, du.$$

$$= e^{t}\left[\frac{e^{-4u}}{-4}\right]_0^t = e^{t}\left[\frac{e^{-4t}}{-4} + \frac{1}{4}\right].$$

$$= \frac{e^{t}}{4}\left[1 - e^{-4t}\right]$$

Find $L^{-1}\left\{\frac{1}{s(s^2-a^2)}\right\}$ by using Convolution theorem.

let $f(s) = \frac{1}{s^2-a^2}$ $\quad L^{-1}\{f(s)\} = \frac{1}{a}\sinh at$.

$g(s) = \frac{1}{s}$ $\quad L^{-1}\{g(s)\} = 1 = g(t)$; $\quad g(t-u) = 1$.

By Convolution theorem $L^{-1}\{f(s)\, g(s)\} = f * g = \int_0^t f(u)\, g(t-u)$.

$$= \int_0^t \frac{1}{a}\sinh au\, du.$$

$$= \frac{1}{a^2}\left[\cosh au\right]_0^t = \frac{1}{a^2}(\cosh at - 1)$$

Solution of Differential Equations By Laplace Transforms.

Laplace Transform can be used to Solve linear differential equations with constant coefficients. The advantage by using Laplace transform is that the particular Solution canbe obtained for given intial condition without obtaining the general Solution.

following formulas are use full.

$$L\{f(t)\} = f(s)$$

$$L\{f'(t)\} = s f(s) - f(0)$$

$$L\{f''(t)\} = s^2 f(s) - s f(0) - f'(0)$$

$$L\{f'''(t)\} = s^3 f(s) - s^2 f(0) - s(f(0)) - f''(0).$$

... likewise

Procedure for finding Solution of Differential Equation by Laplace Transform.

1. Write down the differential and apply Laplace Transform on both sides.
2. Use the given conditions.
3. Re arrange the equations to give transform of the Solution.
4. Take the inverse transform ~~of~~ on both sides to obtain the desired Solution satisfying the given conditions.

Prob: Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 2e^{-x}$.

Given $y=0, y'=-1$ at $x=0$

i.e, $y(0)=0, y'(0)=-1$.

Soln: The given equation

$$y'' + 4y' + 13y = 2e^{-x}.$$

Applying the Laplace transform on both sides

$$L\{y''\} + 4L\{y'\} + 13L\{y\} = 2L\{e^{-x}\}.$$

Apply the formula for derivaties.

$$[s^2 y(s) - s y(0) - y'(0)] + 4[s y(s) - y(0)] + 13 y(s) = \frac{2}{s+1}$$

Apply the conditions

$$[s^2 y(s) + 1) + 4(s(y(s))] + 13 y(s) = \frac{2}{s+1}$$

$$y(s)(s^2+4s+13) + 1 = \frac{2}{s+1}$$

$$y(s)(s^2+4s+13) = \frac{2}{s+1} - 1 = \frac{2-s-1}{s+1} = \frac{1-s}{1+s}.$$

$$L\{y(x)\} = y(s) = \frac{1-s}{(1+s)(s^2+4s+13)}.$$

Apply the Inverse Laplace transform.

$$y(x) = L^{-1}\{y(s)\} = L^{-1}\left\{\frac{1-s}{(1+s)(s^2+4s+13)}\right\}.$$

let $\frac{1-s}{(1+s)(s^2+4s+13)} = \frac{A}{1+s} + \frac{Bs+C}{s^2+4s+13}$.

$$1-s = A(s^2+4s+13) + (Bs+C)(1+s),$$

Put $s=-1$ $\quad A = \frac{1}{5}$

Equate s^2 coeff. $A+B=0$

$B=-A$. $B=-\frac{1}{5}$.

Equate constant terms.

$13A+C=1$. $C=-8/5$.

$$y(x)=L^{-1}\left\{\frac{1}{5}\cdot\frac{1}{s+1}-\frac{1}{5}\cdot\frac{s+8}{s^2+4s+13}\right\}.$$

$$=\frac{1}{5}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{5}L^{-1}\left\{\frac{s+8}{s^2+4s+13}\right\}$$

$$=\frac{1}{5}e^{-x}-\frac{1}{5}\cdot L^{-1}\left\{\frac{s+8}{s^2+4s+4+9}\right\}$$

$$=\frac{1}{5}e^{-x}-\frac{1}{5}L^{-1}\left\{\frac{(s+2)+6}{(s+2)^2+3^2}\right\}$$

$$=\frac{1}{5}e^{-x}-\frac{1}{5}\cdot e^{-2x}L^{-1}\left\{\frac{s+6}{s^2+3^2}\right\}$$

By ILT F.S Thm $L^{-1}\{F(s-a)\}=e^{at}L^{-1}\{F(s)\}$

$$=\frac{1}{5}e^{-x}-\frac{1}{5}e^{-2x}L^{-1}\left\{\frac{s}{s^2+3^2}\right\}+\frac{e^{-2x}}{5}L^{-1}\left(\frac{2\times 3}{s^2+3^2}\right)$$

$$=\frac{1}{5}e^{-x}-\frac{1}{5}e^{-2x}\cdot\cos 3t+\frac{e^{-x}}{5}\cdot 2\sin 3t$$

$$=\frac{1}{5}e^{-x}-\frac{1}{5}e^{-2x}\cos 3t+\frac{2}{5}e^{-x}\sin 3t$$

Prob: Solve $\frac{dy}{dx} + 3y + 2\int_0^t y\,dt = t$. given $y(0)=0$

$$y' + 3y + 2\int_0^t y\,dt = t.$$

Apply L.T.

$$L\{y'\} + 3L\{y\} + 2L\{\int_0^t y\,dt\} = L\{t\}.$$

$$s\,y(s) - y(0) + 3\,y(s) + 2\cdot\frac{1}{s}\,y(s) = \frac{1}{s^2}.$$

$$y(s)\left[s + 3 + \frac{2}{s}\right] = \frac{1}{s^2}.$$

$$y(s)\left[\frac{s^2+3s+2}{s}\right] = \frac{1}{s^2}$$

$$y(s) = \frac{1}{s}\cdot\frac{1}{s^2+3s+2} = \frac{1}{s\cdot(s+1)(s+2)}.$$

Apply Inverse Laplace transform

$$y(x) = L^{-1}\{y(s)\} = L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\}.$$

Now $\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$

$$1 = A(s+1)(s+2) + B(s+2)s + C(s+1)s$$

put $s=0$. $A = \frac{1}{2}$

Put $s=-2$ $C = \frac{1}{2}$

Put $s=-1$ $B=-1$.

Now

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}.$$

$$y(x) = L^{-1}\left\{\frac{1}{2s} - \frac{1}{s+1} - \frac{1}{2(s+2)}\right\}$$

$$= \frac{1}{2} L^{-1}\left\{ 1/s \right\} - L^{-1}\left\{ \frac{1}{s+1} \right\} - \frac{1}{2} L^{-1}\left\{ \frac{1}{s+2} \right\}$$

$$= \frac{1}{2} \cdot 1 - e^{-x} - \frac{1}{2} e^{-2x}.$$

$$y(x) = \frac{1}{2} - e^{-x} - \frac{1}{2} e^{-2x}.$$

L.T of Division by t:

1. $\frac{1-\cos t}{t^2}$, $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt$, $\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt$, $\frac{\cos 4t \sin 2t}{t}$, $\frac{e^{-t}\sin t}{t}$,

$\int_0^\infty \cancel{e^{-t}\sin}$ $\frac{1}{t}\sin 3t$, $\int_0^\infty t e^{-3t}\cos t\,dt = 2/25$, $\int_0^\infty e^{-2t}\sin^3 t\,dt = 6/65$

$\int_0^\infty t^3 e^{-t}\sin t\,dt = 0$, $\frac{\cos at - \cos bt}{t}$

2. $\int_0^t p e^{-3p}\sin 4p\,dp$, $\int_0^t \frac{1-e^u}{u}\,du$, $\int_0^t \frac{1-e^{-2t}}{t}\,dt$, $\int_0^\infty t e^{-3t}\sin t\,dt = 3/50$

Maths

3. $f(t) = \sin \omega t \quad 0<t<\pi/\omega$; $0 \quad \pi/\omega<t<2\pi/\omega$ $\qquad f(t) = t \quad 0<t<a$; $= 2a-t \quad a<t<2a$.

$f(t) = K \quad 0<t<a$; $0 \quad a<t<2a$ $\qquad f(t) = K \quad 0<t<a$; $= -K \quad a<t<2a$

$f(t) = K/p\, t \quad 0<t<p \quad f(t+p) = f(t)$.

4. $\int_0^t \frac{\sin u}{u}\,du = 1/s \cot^{-1} s$.

5. $L^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s^2+1}\right\}$, $\frac{5s-2}{s^2+4s+8}$, $\frac{1}{s^2(s^2+\omega^2)}$, $\frac{3s-14}{s^2-4s+8}$, $\frac{1}{(s+1)^2(s^2+4)}$

$\frac{s+2}{s^2-2s+2}$, $\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$, $\frac{2s}{a+s^4}$

6. ILT S.S.Thm $s e^{-2s}/s^2+3s+2$, $\frac{e^{-3s}}{s^2-2s+5}$, $\frac{s e^{-s/2} + \pi e^{-s}}{s^2+\pi^2}$

ILT of Derivatives.

7. $\cot^{-1}\left(\frac{s+3}{2}\right)$, $\log(1+1/s^2)$, $L^{-1}\{s/(s^2+a^2)^2\}$, $\log\left(\frac{1+s}{s}\right)$, $\tan^{-1} 2/s^2$

8. ILT of Integrals. $\int_s^\infty \ln\frac{u+2}{u+1}\,du$, $\int_s^\infty \tan^{-1}(2/u^2)\,du$

9. Multiplication by s. s^2/s^2+1, $s^2/(s-1)^4$. $\frac{s}{(s^2+a^2)^2}$

10. Division by s. $\frac{1}{s^3(s^2+a^2)}$, $\frac{s+2}{s^2(s+3)}$, $\frac{1}{s^3(s+1)}$, $\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)$

Convolution theorem

11. $\frac{1}{s^2(s+1)^2}$, $\frac{s}{s^2+1\cdot s^2+4}$, $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$, $\frac{1}{(s+2)^2(s-2)}$, $\frac{1}{(s^2+4)(s+1)^2}$

$\frac{s}{\ldots}$, $\frac{s^2}{\ldots}$, $\frac{s^2}{(s^2+4)(s^2+9)}$, $\frac{1}{\ldots}$, $\frac{s^2}{\ldots}$

Applications of DEqn Using L. Transfms.

1. $y'' + 2y + 5y = e^{-t}$ $y(0) = 0, y'(0) = 1$.

2 $y'' + 4y' + 5y = 5$ $y(0) = 0$ $y''(0) = 0$

3. $y'' + 9y = \sin t$ $y(0) = 1, y'(0) = 0$.

4. $(D^2 + 5D - 6)y = x^2 e^{-x}$ $y(0) = a, y'(0) = b$.

5. $(D^2 + 9)y = 18t$ $y(0) = y'(\pi/2) = 0$.

6. $y'' - 2y' + y = e^{2t}$ $y(0) = 0$ $y'(0) = -1$

7. $y'(t) + 3y(t) + 2\int_0^t y(t)\,dt = t$ $y(0) = 1$

8. $y'(t) + 5\int_0^t y(u)\cos 2(t-u)\,du = 10$ if $y(0) = 2$.

9. $x' + x + 3\int_0^t y\,dt = \cos t + 3\sin t$. $x(0) = -3$.

10. $y + \int_0^t y(u)\,du = \sin t$, 11) $y + \int_0^t y(u)\,du = e^{-t}$.

12. $y' + 3y + 2\int_0^t y\,dt = t$ if $y(0) = 0$.

13. $y''(t) + y'(t) = u(t-1)$ & $y(0) = 0, y'(0) = 1$

Ans: $1 - e^{-t} + u(t-1)(-1 + t + e^{-t})$

14 The motion of iron ball attached at the lower end of an elastic string whose upper end is fixed, is given by $y'' + 2y' + 5y = 0$
$y(0) = 2, y'(0) = -4$. Determine free vibration of y.

15 ~~$y'' + n^2 y = n^2 a$, $x = y = x' = y' = 0$~~

16 The current i, charge $q(t)$ capacitor an electric ~~circuit~~ ckt satisfies the equation $L\frac{di}{dt} + ri + q/c = e_0$ $q = \int^t i(t)\,dt$
l, r, c, e_0 constants. if q and i are initially zero s.t

$i = \frac{e_0}{wL} e^{-bt} \sin wt$

EXAMPLE 5

Find $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right]$

Solution :

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s\,(s^2+a^2)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right]$$

$$= \int_0^t L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]dt$$

$$= \int_0^t \frac{t\sin at}{2a}\,dt \qquad \because\ L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}$$

$$= \frac{1}{2a}\int_0^t t\sin at\,dt$$

$$= \frac{1}{2a}\left[t\left(\frac{-\cos at}{a}\right) - 1\left(\frac{-\sin at}{a^2}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{\sin at}{a^2}\right]$$

EXERCISE

Find the inverse L.T of the following

1. $\dfrac{s^2}{(s-2)^2}$ **Ans.** $4e^{2t}\,(1+2t)$

2. $\log\left[\dfrac{1+s}{s^2}\right]$ **Ans.** $\dfrac{2-e^{-t}}{t}$

   (JNTU 2000)

3. $\log\left[\dfrac{1+s}{s}\right]$ **Ans.** $\dfrac{1-e^{-t}}{t}$

4. $\log\left(\dfrac{s+a}{s+b}\right)$ **Ans.** $\dfrac{1}{t}\left(e^{-bt}-e^{-at}\right)$

5. $\log\left(1-\dfrac{a^2}{s^2}\right)$ **Ans.** $\dfrac{2}{t}\left(1-\cosh at\right)$

6. $\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$ **Ans.** $\frac{1}{t}\left(\cos at - \cos bt\right)$

7. $\log\frac{s\,(s+1)}{s^2+1}$ **Ans.** $\frac{2\cos t - e^{-t} - 1}{t}$

8. $\log\left(1-\frac{1}{s^2}\right)$ **Ans.** $\frac{2}{t}\left(1-\cosh t\right)$

9. $\frac{1}{s^3\,(s^2+1)}$ **Ans.** $\frac{1}{2}t^2 + \cos t - 1$

10. $\frac{1}{s^2\,(s+2)}$ **Ans.** $\frac{1}{4}\left(e^{-2t} + 2t - 1\right)$

11. $\frac{1}{s\,(s+2)^3}$ **Ans.** $\frac{1}{8} - \frac{1}{4}\,(t^2+t+2)\,e^{-2t}$

12. $\frac{s}{(s+a)^2}$ **Ans.** $(1-at)\,e^{-at}$

13. $\frac{s^2}{(s+a)^3}$ **Ans.** $\frac{1}{2}\left(a^2t^2 - 4at + 2\right)e^{-at}$

14. $\frac{1}{s\,(s+2)}$ **Ans.** $\frac{1}{2}\,(1-e^{2t})$

15. $\cot^{-1}\left(\frac{s}{a}\right)$ **Ans.** $\frac{\sin at}{t}$

16. $\cot^{-1}(s+1)$ **Ans.** $\frac{e^{-t}\sin t}{t}$

17. $\cot^{-1}\left(\frac{s+3}{2}\right)$ **Ans.** $\frac{2}{t}\,e^{-3t}\sin 2t$

18. $\cot^{-1}(s)$ ***(JNTU 2000)*** **Ans.** $\frac{1}{2}\,t\sin t$

TABLE OF INVERSE LA

1.	$L^{-1}\left[\frac{1}{s}\right]=1$
2.	$L^{-1}\left[\frac{1}{s^2}\right]=t$
3.	$L^{-1}\left[\frac{n!}{s^{n+1}}\right]=t^n$, n is pos
4.	$L^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{\lfloor n+1}$, $n >$
5.	$L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$
6.	$L^{-1}\left[\frac{1}{s-a}\right]=e^{-at}$
7.	$L^{-1}\left[\frac{a}{s^2+a^2}\right]=\sin at$
8.	$L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$
9.	$L^{-1}\left[\frac{a}{s^2-a^2}\right]=\sinh$
10.	$L^{-1}\left[\frac{s}{s^2-a^2}\right]=\cosh$
11.	$L^{-1}\left[\frac{1}{(s-a)^2}\right]=te^{at}$
12.	$L^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$
13.	$L^{-1}\left[\frac{s+a}{(s+a)^2+b^2}\right]$

EXERCISE 5.5

Using Laplace transform method, solve the following differential equation with given conditions.

1. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$

Given at $x = 0,\ y = 0,\ \frac{dy}{dx} = 4$

Ans : $y = e^{x} - e^{-3x}$

2. $(D^2 + 1)\,x = t\cos 2t$, given $x = 0,\ \frac{dx}{dt} = 0$ at $t = 0$

Ans : $x = \frac{4}{9}\sin 2t - \frac{5}{9}\sin t - \frac{1}{3}\,t\cos 2t$ (JNTU 2000)

3. $(D^2 + 4D + 3)\,y = e^{-t}$ Given $y = 1,\ \frac{dy}{dx} = 1$ at $t = 0$ (JNTU 98/S)

Ans : $y(t) = \frac{7}{4}e^{-t} - \frac{7}{8}e^{-3t} + \frac{1}{8}e^{t}$

4. $x''(t) + 2x'(t) + x(t) = 3te^{-t}$ given $x(0) = 4,\ x'(0) = 2$

Ans : $x(t) = \left(4 + 6t + \frac{1}{2}t^2\right)e^{-t}$

5. $(D^2 + 3D + 2)\,y = 2(t^2 + t + 1)$

given $y(0) = 2,\ y'(0) = 0$

Ans : $y = 3 - 2t + t^2 - e^{-2t}$

6. $y''' + 2y'' - y' - 2y = 0$

given $y(0) = y'(0) = 0,\ y''(0) = 6$ (JNTU 05)

Ans : $y = e^{t} - 3e^{-t} + 2e^{-2t}$

Find the Laplace transforms of the following:

1. $t + t^2 + t^3$ **Ans.** $\frac{1}{s^2}+\frac{2}{s^3}+\frac{6}{s^4}$

2. $\sin t \cos t$ **Ans.** $\frac{1}{s^2+4}$

3. $t^{7/2} e^{5t}$ *(M.D.U. Dec. 2009)* **Ans.** $\frac{105\sqrt{\pi}}{16(s-5)^{9/2}}$

4. $\sin^3 2t$ **Ans.** $\frac{48}{\left(s^2+4\right)\left(s^2+36\right)}$

5. $e^{-t}\cos^2 t$ **Ans.** $\frac{1}{2s+2}+\frac{s+1}{2s^2+4s+10}$

6. $\sin 2t \cos 3t$ **Ans.** $\frac{2(s^2-5)}{\left(s^2+1\right)\left(s^2+25\right)}$

7. $\sin 2t \sin 3t$ **Ans.** $\frac{12s}{\left(s^2+1\right)\left(s^2+25\right)}$

8. $\cos at \sinh at$ **Ans.** $\frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2}-\frac{s+a}{(s+a)^2+a^2}\right]$

9. $\sinh^3 t$ **Ans.** $\frac{6}{\left(s^2-1\right)\left(s^2-9\right)}$

10. $\cos t \cos 2t$ **Ans.** $\frac{s\left(s^2+5\right)}{(s^2+1)(s^2+9)}$

11. $\cosh at \sin at$ **Ans.** $\frac{a\left(s^2+2a^2\right)}{s^4+4a^4}$

Find the Laplace transforms of the following :

1. $t_e at$ **Ans.** $\dfrac{1}{(s-a)^2}$

2. $t \cosh at$ **Ans.** $\dfrac{s^2+a^2}{\left(s^2-a^2\right)^2}$

3. $t \cos t$ **Ans.** $\dfrac{s^2-1}{\left(s^2+1\right)^2}$

4. $t \cosh t$ **Ans.** $\dfrac{s^2+1}{\left(s^2-1\right)^2}$

5. $t^2 \sin t$ **Ans.** $\dfrac{2\left(3s^2-1\right)}{\left(s^2+1\right)^3}$

6. $t^3 e^{-3t}$ **Ans.** $\dfrac{6}{(s+3)^4}$

7. $t \sin^2 3t$ **Ans.** $\dfrac{1}{2}\left[\dfrac{1}{s^2}-\dfrac{s^2-36}{\left(s^2+36\right)^2}\right]$

8. $t e^{at} \sin at$ **Ans.** $\dfrac{2a(s-a)}{\left(s^2-2as+2a^2\right)^2}$

9. $te^{-t} \cosh t$ **Ans.** $\dfrac{s^2+2s+2}{\left(s^2+2s\right)^2}$

10. $t^2 e^{-2t} \cos t$ **Ans.** $\dfrac{2(s^3+6s^2+9s+2)}{(s^2+4s+5)^3}$

11. $\int_0^t e^{-2t} t \sin^3 t \, dt$ **Ans.** $\dfrac{3(s+2)}{2s}\left[\dfrac{1}{\left[(s+2)^2+9\right]^2}-\dfrac{1}{\left[(s+2)^2+1\right]^2}\right]$

EXERCISE 42.3

Find Laplace transform of the following:

1. $\frac{1}{t}(1-e^{t})$ **Ans.** $\log\frac{s-1}{s}$

2. $\frac{1}{t}\left(e^{-at}-e^{-bt}\right)$ **Ans.** $\log\frac{s+b}{s+a}$

3. $\frac{1}{t}\left(1-\cos at\right)$ **Ans.** $-\frac{1}{2}\log\frac{s^2}{s^2+a^2}$

4. $\frac{1}{t}\sin^2 t$ **Ans.** $\frac{1}{4}\log\frac{s^2+4}{s^2}$

5. $\frac{1}{t}\sinh t$ **Ans.** $-\frac{1}{2}\log\frac{s-1}{s+1}$

6. $\frac{1}{t}\left(e^{-t}\sin t\right)$ **Ans.** $\cot^{-1}(s+1)$

7. $\frac{1}{t}\left(1-\cos t\right)$ **Ans.** $\frac{1}{2}[\log\left(s^2+1\right)-\log s^{\partial}]$

8. $\int_0^{\infty}\frac{1}{t}e^{-2t}\sin t\,dt$ **Ans.** $\frac{1}{s}\cot^{-1}(s+2)$

9. $\int_0^{\infty}\frac{e^{-t}-e^{-3t}}{t}dt$ **Ans.** $\log 3$

EXERCISE 42.4

Find the Laplace transform of the following:

1. $f(t)=\begin{cases} t-1, & 1<t<2 \\ 0, & \text{otherwise} \end{cases}$ **Ans.** $\dfrac{e^{-s}-e^{-2s}}{s^2}-\dfrac{e^{-2s}}{s}$

2. $e^t u(t-1)$ **Ans.** $\dfrac{e^{-(s-1)}}{s-1}$

3. $\dfrac{1-e^{2t}}{t}+tu(t)+\cosh t.\cos t$ **Ans.** $\log\dfrac{s-2}{s}+\dfrac{1}{s^2}+\dfrac{s^3}{s^4+4}$

4. $t^2u(t-2)$ **Ans.** $\dfrac{e^{-2s}}{s^3}\left(4s^2+4s+2\right)$

5. $\sin t\, u\,(t-4)$ **Ans.** $\dfrac{e^{-4s}}{s^2+1}[\cos 4+s\sin 4]$

6. $f(t)=K(t-2)[u(t-2)-u(t-3)]$ **Ans.** $\dfrac{K}{s^2}\left[e^{-2s}-(s+1)e^{-3s}\right]$

7. $f(t)=K\dfrac{\sin\pi t}{T}[u(t-2T)-u(t-3T)]$ **Ans.** $\dfrac{K\pi T}{s^2T^2+\pi^2}\left(e^{-2sT}-e^{-3sT}\right)$

Express the following in terms of unit step functions and obtain Laplace transforms.

8. $f(t)=\begin{cases} t, & 0<t<2 \\ 0, & 2<t \end{cases}$ **Ans.** $u(t)-u(t-2), \dfrac{1-(2s+1)e^{-2s}}{s^2}$

9. $f(t)=\begin{cases} \sin t, & 0<t<\pi \\ t, & t>\pi \end{cases}$ **Ans.** $\dfrac{1+e^{-\pi s}}{s^2+1}+\dfrac{e^{-\pi s}(\pi s+1)}{s^2}$

1. Find the Laplace transform of the periodic function

$f(t)=e^{t}$ for $0<t<2\pi$ **Ans.** $\dfrac{e^{2(1-s)\pi}-1}{(1-s)\left(1-e^{-2\pi s}\right)}$

2. Obtain Laplace transform of full wave rectified sine wave given by

$f(t)=\sin\omega t, \quad 0<t<\dfrac{\pi}{\omega}$ **Ans.** $\dfrac{\omega}{\left(s^{2}+\omega^{2}\right)}\coth\dfrac{\pi s}{2\omega}$

3. Find the Laplace transform of the staircase function

$f(t)=kn,\ np<t<(n+1)p,\ n=0,1,2,3$ **Ans.** $\dfrac{ke^{ps}}{s\left(1-e^{-ps}\right)}$

Find Laplace transform of the following:

4. $f(t)=t^{2},\ 0<t<2,\ f(t+2)=f(t)$ **Ans.** $\dfrac{2-e^{-2s}-4se^{-2s}-4s^{2}e^{-2s}}{s^{3}\left(1-e^{-2s}\right)}$

5. $f(t)=\begin{cases}1, & 0\le t\le \dfrac{a}{2}\\ -1, & \dfrac{a}{2}\le t<a\end{cases}$ *(U.P. II Semester, 2004)* **Ans.** $\dfrac{1}{s}\tanh\dfrac{as}{4}$

1. Find the Laplace transform of the periodic function

$f(t) = e^t$ for $0 < t < 2\pi$ **Ans.** $\dfrac{e^{2(1-s)\pi} - 1}{(1-s)\left(1-e^{-2\pi s}\right)}$

2. Obtain Laplace transform of full wave rectified sine wave given by

$f(t) = \sin \omega t, \quad 0 < t < \dfrac{\pi}{\omega}$ **Ans.** $\dfrac{\omega}{\left(s^2+\omega^2\right)} \coth \dfrac{\pi s}{2\omega}$

3. Find the Laplace transform of the staircase function

$f(t) = kn, \; np < t < (n+1)p, \; n = 0, 1, 2, 3$ **Ans.** $\dfrac{ke^{ps}}{s\left(1-e^{-ps}\right)}$

Find Laplace transform of the following:

4. $f(t) = t^2, \; 0 < t < 2, \; f(t+2) = f(t)$ **Ans.** $\dfrac{2 - e^{-2s} - 4se^{-2s} - 4s^2e^{-2s}}{s^3\left(1-e^{-2s}\right)}$

5. $f(t) = \begin{cases} 1, & 0 \le t \le \dfrac{a}{2} \\ -1, & \dfrac{a}{2} \le t < a \end{cases}$ *(U.P. II Semester, 2004)* **Ans.** $\dfrac{1}{s} \tanh \dfrac{as}{4}$

EXERCISE 42.8

Evaluate the following by using Laplace Transform:

1. $\int_0^{\infty} t e^{-4t} \sin t \, dt$ **Ans.** $\frac{8}{289}$

2. $\int_0^{\infty} \frac{e^{-2t} \sinh t \sin t}{t} dt$ **Ans.** $\frac{1}{2} \tan^{-1} \frac{1}{2}$

3. $\int_0^{\infty} \frac{\sin^2 t}{t^2} dt$ **Ans.** $i \frac{5}{2}$

4. $\int_0^{\infty} \frac{e^{-t} - e^{-4t}}{t} dt$ **Ans.** $\log 4$