Unit-5 (ordinary Differential eqⁿ)

Definition of differential equation:— An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Examples
$$\frac{dy}{dx} + xy\left(\frac{dy}{dx}\right)^2 = 0 \quad \text{—— (1)}$$
$$\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = u \quad \text{—— (2)}$$
or
$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2xy \quad \text{—— (3)}$$

ordinary Differential eq^n:— A diff^n eq^n involving ordinary derivatives of one or more dependent variable with respect to a single independent variable is called an ordinary differential equation. like eq^n (1) is an O.D.E.

Partial differential equation:— A diff^n eq^n involving partial derivatives of one or more dependent variable with respect to more than one independent variable is called a Partial diff^n eq^n. like eq^n (2) and (3) is an P.D.E.

Order of Differential eqn:— The order of highest order derivative involved in a diff. eqn is called order of the diffn. eqn.

Degree of Diffn eqn:— The degree of diffn eqn is defined as the degree of the highest order derivated involved in the given differential equation after made it free from radical and fractions as per the derivatives are concerned.

Examples (1) $\frac{d^2y}{dx^2} + y = 0$, order-2, degree-01

(2) $y = \sin\left(\frac{dy}{dx}\right)$, order-01, degree-does not exist

(3) $e^{y'''} - y'' + xy = 0$, order-3, degree does not exist

(4) $\frac{d^2y}{dx^2} - \sqrt{\frac{dy}{dx}} = 0$, order-2, degree-2

Linear Differential equation:— A linear ordinary differential eqn of order 'n' in the dependent variable y and independent variable 'x' is an eqn that can be expressed in the form of

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = b(x)$$

Where $a_0 \neq 0$

or

A diff^n eq^n is called linear if dependent variable & every derivative involved, occurs in the first degree only and no product of dependent variables and derivatives occur.

Non-linear differential equation:— A diff^n eq^n, which is not linear is called non-linear diff^n eq^n.

Example ① $\frac{d^2y}{dx^2} + y = 0$ (linear)

(2) $\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^5 + y = x$ (non-linear)

Solution of a Differential equation:— A Solution of a diff. eq^n is a relation b/w the dependent & Independent variable, not containing derivatives, which satisfies the differential eq^n is called a Solution.

Example① $y = ce^{2x}$ is a Solution of $\frac{dy}{dx} = 2y$ —(1)

As by putting $y = e^{2x}$ in eq^{n}①, we get the identity. Hence $y = ce^{2x}$ is the soln of given diffn eqn for any real constant c.

type of Solution D.E :- There are three type of Solution of differential eqns.

(1) General Solution :- Let $f(x, y, y'', --, y^n) = 0$ —(1) be nth order ordinary differential eqn. A soln of (1) containing 'n' arbitrary constants is called a general solution.

Particular Solution :- A soln of (1) obtained from general soln by giving particular values to one or more arbitrary constant is called a particular soln of (1).

Singular Solution :- A solution that can not be obtained from the general soln for any choice of arbitrary constants is called a singular solution.

Example (1) $y^2\left[\left(\frac{dy}{dx}\right)^2 + 1\right] = 9$ has general soln as

$$(x-a)^2 + y^2 = 9,$$ we can also verify that $y(x) = 3$ & $y(x) = -3$ are also solutions of the differential equation, but these both can not be obtained from

general Solution, hence these are singular Solutions.

Exact differential equations:— A differential eqⁿ is said to be exact if it can be derived from its primitive (general Solution) directly by differention, without any subsequent multiplication, elimination etc.

OR

A differential equation of the form $Mdx + Ndy = 0$ is said to be exact differential equation if it satisfies the Condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Where M and N are fⁿ of two variable x and y.

Solution of exact differential equation:— In order to obtain the solution of an exact differential eqⁿ, we have to proceed as follows:

(a) Integrate M with respect to x, keeping y as Constant.

(b) Integrate with respect to y only those terms of N which do not contain x.

(c) Add the two expressions obtained in (a) and (b) above and equate.

In other words, the Solution of an exact differentia equation is

$$\underset{(y \text{ is Constant})}{\int M dx} + \underset{(\text{term not having } x)}{\int N dy} = c$$

Example ① $(hx+by+f)dy + (ax+hy+g)dx = 0$

Solⁿ:- here, $M = ax+hy+g$ and $N = hx+by+f$

Then, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, which show that the given D.E is exact. hence, the required solⁿ is

$$\int M dx + \int N dy = c_1$$

$$\int (ax+hy+g)dx + \int (by+f)dy = c_1$$

$$\frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = c_1$$

$\Rightarrow$ $ax^2 + 2hxy + 2gx + 2fy + c = 0$, where $c = -2c_1$

Example ② Solve $x dx + y dy + \frac{x dy - y dx}{x^2+y^2} = 0$

Solⁿ:- The given D.E can be written as

$$\left(x - \frac{y}{x^2+y^2}\right)dx + \left(y + \frac{x}{x^2+y^2}\right)dy = 0$$

here $M = \frac{x^3+xy^2-y}{x^2+y^2}$, $N = \frac{x^2y+y^3+x}{x^2+y^2}$

Now, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, which show that the given eqn is exact. therefore

$$\int M dx + \int N dy = c_1$$

$$\int \left(x - \frac{y}{(x^2+y^2)}\right) dx + \int y\, dy = c_1$$

$$\frac{x^2}{2} - y\frac{1}{y}\tan^{-1} x/y + y^2/2 = c_1$$

$x^2 - 2\tan^{-1} x/y + y^2 = c$, where $c = 2c_1$

Example ③ Solve $(1+e^{x/y})dx + e^{x/y}(1-x/y)\,dy = 0$

Solnº:- here $M = 1+e^{x/y}$, $N = e^{x/y}(1-x/y)$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, Thus, the given D.E is exact, Now solution is

$$\int M dx + \int N dy = c$$

$$\int (1+e^{x/y})dx + \int e^{x/y}(1-x/y)\,dy = c$$

$\Rightarrow x + y e^{x/y} + 0 = c$

$\Rightarrow x + y e^{x/y} = c$ (which is the required soln)

Integrating factors:- A non-exact differential equation can always be made exact by multiplying it by some functions of x and y. such a f^n is called an integrating factors. Although a differential eq^n of the type $Mdx + Ndy = 0$ always has an integrating factor, there is no general method of finding them. here we shall explain some of Methods for finding the integrating factors.

Method I. In some cases the integrating factor is found by inspection. using the following few exact differentials, it is easy to find the integrating factors:

(a) $d(x/y) = \dfrac{ydx - xdy}{y^2}$ (b) $d(y/x) = \dfrac{xdy - ydx}{x^2}$

(c) $d(xy) = xdy + ydx$ (d) $d(x^2/y) = \dfrac{2yxdx - x^2dy}{y^2}$

(e) $d\left(\dfrac{y^2}{x}\right) = \dfrac{2xydy - y^2dx}{x^2}$ (f) $d\left(\dfrac{x^2}{y^2}\right) = \dfrac{2xy^2dx - 2x^2ydy}{y^4}$

(g) $d(y^2/x^2) = \dfrac{2x^2ydy - 2xy^2dx}{x^4}$ (h) $d\left(\dfrac{1}{xy}\right) = \dfrac{xdy + ydx}{x^2y^2}$

(i) $d(\log y/x) = \dfrac{-ydx+xdy}{xy}$ (j) $d(\log x/y) = \dfrac{ydx-xdy}{xy}$

(k) $d(\tan^{-1} x/y) = \dfrac{ydx-xdy}{x^2+y^2}$ (l) $d(\tan^{-1} y/x) = \dfrac{xdy-ydx}{x^2+y^2}$

(m) $d\left(\dfrac{e^x}{y}\right) = \dfrac{ye^x dx - e^x dy}{y^2}$ (n) $d\left(\dfrac{e^y}{x}\right) = \dfrac{xe^y dy - e^y dx}{x^2}$.

(o) $d\left(-\dfrac{1}{xy}\right) = \dfrac{xdy+ydx}{x^2y^2}$ (p) $d\left[\dfrac{1}{2}\log(x^2+y^2)\right] = \dfrac{xdx+ydy}{x^2+y^2}$.

Example ① solve $(1+xy)ydx + (1-xy)xdy = 0$

Soln:— The given eqn can be written as

$$ydx + xdy + (xy^2dx - x^2ydy) = 0$$

or $\quad d(yx) + xy^2dx - x^2ydy = 0$ from eqn (c)

Dividing by x^2y^2, we get

$$d\left(\frac{1}{xy}\right) + \frac{1}{x}dx - \frac{1}{y}dy = 0$$

Integrating, we get

$$\frac{1}{xy} + \log x - \log y = c$$

$$\frac{1}{xy} + \log x/y = C$$

Which is the required Solution.

Example(2) Solve $(x^3e^x - my^2)dx + mxy\,dy = 0$

Soln:- The given D.E can be written as

$$x^3e^x dx + m(xy\,dy - y^2dx) = 0$$

Dividing by x^3, we get

$$e^x dx + m\frac{xy\,dy - y^2dx}{x^3} = 0$$

$$e^x dx + \frac{m}{2}\frac{x^2 2y\,dy - y^2 2x\,dx}{x^4} = 0$$

$$e^x dx + \frac{1}{2}m\,d(y^2/x^2) = 0$$ from exact differential eqn(g), we get

Integrating, we get the Solution

$$e^x + \frac{1}{2}m\,y^2/x^2 = c$$

$$2x^2e^x + my^2 = 2cx^2$$

which is the required Solution.

Example(3) Solve $x\,dy - y\,dx = a(x^2+y^2)dy$

Soln:- The given D.E eqn can be written as

$$\frac{x\,dy - y\,dx}{x^2+y^2} = a\,dy$$

or

$$d(\tan^{-1}y/x) = a\,dy$$ from exact differential(I)

Integrating, we get the required Soln as $\tan^{-1}y/x = ay + c$

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Method - II :— if the differential equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$, then $1/Mx+Ny$ is integrating factor. Multiplying given D E by this integrating factor to make given D.E is exact D.E.

Example ① solve $x^2y\,dx - (x^3+y^3)dy = 0$

Solⁿ :— here, $Mx + Ny \neq 0 = -y^4$, and the given eqⁿ is homogeneous, Thus, the integrating factor is $-\frac{1}{y^4}$. Multiplying the given eqⁿ by this factor, we get

$$-x^2/y^3\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$$

Now, $M = -x^2/y^3$, $N = \frac{x^3}{y^4} + \frac{1}{y}$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This Shows that the resulting differential eqⁿ is exact.
Thus, it has Solⁿ

$$\int -x^2/y^3\,dx + \int\left(\frac{1}{y}\right)dy = 0$$

$$-\frac{x^3}{3y^3} + \log y = C$$

or $\quad x^3 = 3y^3(\log y - c)$

Example② Solve $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$

Solnº:— The given eqn is homogeneous and $Mx+Ny=x^2y^2$

Then the integrating factor is $1/x^2y^2$. Multiplying the given eqn by this factor, we get

$$\left(\frac{1}{y}dx-\frac{x}{y^2}dy\right)-\frac{2}{x}dx+\frac{3}{y}dy=0$$

OR $$d(x/y)-\frac{2}{x}dx+3/y\,dy=0$$

integrating term by term, we obtain

$$x/y-2\log x+3\log y=c$$

which is the required Soln.

Method-III :— if in the differential equation $Mdx+Ndy=0$, $M=yf_1(x,y)$ and $N=xf_2(x,y)$. Then $\frac{1}{Mx-Ny}$ is an integrating factor.

Example① Solve $(xy+2x^2y^2)ydx+(xy-x^2y^2)xdy=0$

Solnº:— here, $M=yf_1(x,y)$ and $N=xf_2(x,y)$. Thus, the integrating factor $=\frac{1}{Mx-Ny}=\frac{1}{3x^3y^3}$. Multiplying the given eqn by this factor, we get

$$\frac{1}{3}\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\frac{1}{3}\left(\frac{1}{x^2y}-\frac{1}{y}\right)dy=0 \quad \text{——} ①$$

here $M = \frac{1}{3}\left(\frac{1}{x^2y} + \frac{2}{x}\right)$, $N = \frac{1}{3}\left(\frac{1}{xy^2} - \frac{1}{y}\right)$

$$\frac{\partial M}{\partial y} = -\frac{1}{3x^2y^2} = \frac{\partial N}{\partial x}$$

eqⁿ ①, which is an exact differential eqⁿ, whose solⁿ is

$$\frac{1}{3}\left(-\frac{1}{xy} + 2\log x\right) + \frac{1}{3}(-\log y) = C$$

or

$$2\log x - \log y = \frac{1}{xy} + C_1 \quad \text{where } C_1 = 3C.$$

Example ② Solve $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$

Solⁿ:- here, $M = y f_1(x,y)$, $N = x f_2(x,y)$, and thus the integrating factor is $\frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$. Multiplying the given eqⁿ by this factor, we get

$$y\,dx + x\,dy + \left(\frac{1}{x}dx - \frac{1}{y}dy\right) + \frac{1}{x^2y}dx + \frac{1}{xy^2}dy = 0$$

$$d(xy) + \frac{dx}{x} - \frac{dy}{y} + \frac{y\,dx + x\,dy}{x^2y^2} = 0$$

$$\Rightarrow d(xy) + \frac{dx}{x} - \frac{dy}{y} + \frac{d(xy)}{x^2y^2} = 0$$

integrating term by term, we get

$$xy + \log x - \log y + \frac{1}{xy} = C$$

which is the required Solution.

Method IV :- The equation, $Mdx + Ndy = 0$ has $e^{\int f(x)dx}$ as the integrating factor if $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is function of x [say $f(x)$].

Remark if $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = k$ (a Constant) then $I.f = e^{\int k dx}$.

Example① solve $(x^2+y^2)dx - 2xy\,dy = 0$

Solnᵒ:- here, $M = x^2+y^2$, $N = -2xy$

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{2}{x}$ Thus,

$$I.f = e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$$

Multiplying the given eqⁿ by $\frac{1}{x^2}$, we get

$$\left(1+\frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

$$dx + d\left(-\frac{y^2}{x}\right) = 0$$

Integrating term by term, we obtain

$$x - y^2/x = c$$

Example② solve $(y + y^3/3 + x^2/2)dx + \frac{1}{4}(x+xy^2)dy = 0$

Solnᵒ:- here $M = y + y^3/3 + x^2/2$, $N = \frac{1}{4}(x+xy^2)$

and $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=3/x$

Thus, $I.f = e^{\int 3/x\,dx} = e^{3\log x}$ or $= x^3$

Multiply the given eqn by this factor, Then, we get

$$2x^5dx+(x^4dy+4x^3y\,dx)+\frac{1}{3}(x^4 3y^2dy+4x^3y^3dx)=0$$

OR

$$2x^5dx+d(x^4y)+\frac{1}{3}d(x^4y^3)=0$$

Integrating term by term, we have

$$x^6+3x^4y+x^4y^3=3C$$

Which is the required Solution.

Method V if $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is a fn of y [say $f(y)$], then $e^{\int f(y)dy}$ is the integrating factor of $Mdx+Ndy=0$.

Example ① Solve $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$

Solnº— here $M=xy^3+y$, $N=2(x^2y^2+x+y^4)$

Now

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{y}=f(y)$$

Thus, integrating factor $= e^{\int f(y)dy} = e^{\int \frac{1}{y}dy} = y$

Multiplying the given eqn by y, we get

$$(xy^4+y^2)dx+2(x^2y^3+xy+y^5)dy=0$$

Then, $\int (xy^4 + y^2)dx + 2\int y^5 dy = c$

$$\frac{x^2y^4}{2} + y^2x + 2y^5/6 = c$$

$$3x^2y^4 + 6xy^2 + 2y^5 = 6c$$

Which is the required Solution.

Example ② Solve $(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3 + y^2)dy = 0$

Soln:- here, $M = xy^2 - x^2$, $N = 3x^2y^2 + x^2y - 2x^3 + y^2$

Now,
$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 6 \text{ (Constant)}$$

Thus, $I.f = e^{\int 6 dy} = e^{6y}$. Multiplying the given eqn by this factor, we get

$$(xy^2 - x^2)e^{6y}dx + (3x^2y^2 + x^2y - 2x^3 + y^2)e^{6y}dy = 0$$

This can be written as $M_1dx + N_1dy = 0$ —— ①

here $M_1 = xy^2e^{6y} - x^2e^{6y}$ and $N_1 = 3x^2y^2e^{6y} + x^2ye^{6y} - 2x^3e^{6y} + y^2e^{6y}$

Now, $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, therefore eqn ① is exact differential eqn. Therefore, the required Solution is

$$\int M_1 dx + \int N_1 dy = c$$

$$e^{6y}\left(\frac{x^2y^2}{2}-\frac{x^3}{3}\right)+e^{6y}\left(y^2/6-y/18+\frac{1}{108}\right)=c$$

$$\Rightarrow e^{6y}\left(\frac{x^2y^2}{2}-\frac{x^3}{3}+y^2/6-y/18+\frac{1}{108}\right)=c$$

which is the required Solution.

Method VI if the equation $Mdx+Ndy=0$ is of the form

$$x^a y^b(mydx+nxdy)+x^c y^d(pydx+qxdy)=0$$

Where a, b, c, d, m, n, p and q are Constants, then $x^h y^k$ is the integrating factor, where h, k are Constant and Can be obtained by applying the Condition that after multiplication by $x^h y^k$ the given equation is exact

Example ① Solve $(y^2+2x^2y)dx+(2x^3-xy)dy=0$

Solⁿ:- The given eqⁿ Can be written as

$$y(y+2x^2)dx+x(2x^2-y)dy=0$$

Let $x^h y^k$ be the integrating factor. Multiplying the given eqⁿ by this factor, we have

$$(x^h y^{k+2}+2x^{h+2}y^{k+1})dx+(2x^{h+3}y^k-x^{h+1}y^{k+1})dy=0 \quad ①$$

here $M=x^h y^{k+2}+2x^{h+2}y^{k+1}$, $N=2x^{h+3}y^k-x^{h+1}y^{k+1}$

if eqⁿ ① is exact, then $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$$(k+2)x^h y^{k+1} + 2(k+1)x^{h+2}y^k = -(h+1)x^h y^{k+1} + 2(h+3)x^{h+2}y^k$$

equating the coefficients of $x^h y^{k+1}$ and $x^{h+2}y^k$ on sides and solving for k and h, we get

$$h = -5/2, \quad k = -\frac{1}{2}$$

Therefore, the integrating factor is

$$x^h y^k = x^{-5/2} y^{-1/2}$$

Multiplying the given eq^n by this factor, we get

$$\left(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}\right)dx + \left(2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}\right)dy = 0$$

In this equation,

$$M_1 = x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}, \quad N_1 = 2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}$$

and the eq^n is exact. Also

$$\underset{(y = \text{Constant})}{\int M_1 dx} = -2/3\, x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2}$$

and $\int N_1 dy = 0$

(term not having x), hence the required sol^n of the given eq^n is

$$\int M_1 dx + \int N_1 dy = C$$

$-\frac{2}{3}x^{3/2}y^{3/2} + 4x^{1/2}y^{1/2} = c$

Example ② Solve $(2ydx + 3xdy) + 2xy(3ydx + 4xdy) = 0$

Sol^n:- we can write the given eq^n as

$$(2y + 6xy^2)dx + (3x + 8x^2y)dy = 0 \quad \text{---} ①$$

Let $x^h y^k$ be the integrating factor. Multiplying the given eq^n by this factor, we get

$$(2x^h y^{k+1} + 6x^{h+1}y^{k+2})dx + (3x^{h+1}y^k + 8x^{h+2}y^{k+1})dy = 0$$

if this eq^n is exact, we must then have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, Thus

$$2(k+1)y^k x^h + 6(k+2)y^{k+1}x^{k+1} = 3(h+1)x^h y^k + 8(h+2)x^{h+1}y^{k+1}$$

equating the Coefficient of $x^h y^k$ and $x^{h+1}y^{k+1}$ on both sides and solving for h and k, we get $h=1, k=2$, Thus the integrating factor is $x^h y^k = xy^2$. Now, the multiplying the given eq^n ① by xy^2, we have

$$(2xy^3 + 6x^2y^4)dx + (3x^2y^2 + 8x^3y^3)dy = 0$$

here, $M_1 = 2xy^3 + 6x^2y^4$, $N_1 = 3x^2y^2 + 8x^3y^3$ and $\frac{\partial M_1}{\partial y} = \frac{\partial N}{\partial x}$

Therefore, the Sol^n is $\int M_1 dx + \int N_1 dy = c$

$$x^2y^3 + 2x^3y^4 = c$$

first order linear differential equation:—

A differential eqⁿ is said to be linear if the dependent variables y and its differential coefficients occur only in the first degree and are not multiplied together. The linear differential eqⁿ of the first order is of the form

$$\frac{dy}{dx} + Py = Q \quad \text{———} (1)$$

where P and Q are constants or functions of x alone.

To solve such an eqⁿ, multiply both side eqⁿ ① by $e^{\int P dx}$.

Then, we get

$$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = e^{\int P dx} Q$$

OR

$$\frac{d}{dx}\left(y e^{\int P dx}\right) = e^{\int P dx} Q$$

integrating both side, we get

$$y e^{\int P dx} = \int Q e^{\int P dx} + c$$

which is the complete solⁿ of eqⁿ ①.

Remarks ① The factor $e^{\int P dx}$, on multiplying by which the left-hand side of eqⁿ ① becomes the exact differential coefficient of some f^n x and y, is called the integrating factor of the given differential eqn.

Sometimes a given differential eq^n becomes linear if we take x as the dependent variable and y as the independent variable, i.e it can be written in the form

$$\frac{dx}{dy} + P_1 x = Q_1$$

Where P_1 and Q_1 are constant or f^n of y alone, In this case, the solution is

$$x e^{\int P_1 dy} = \int Q_1 e^{\int P_1 dy} dy + c.$$

Example ① solve $(1+x)\frac{dy}{dx} - xy = 1-x$

Sol^n :— The given D.E can be written as

$$dy/dx - \frac{x}{(1+x)} y = \frac{1-x}{(1+x)}$$

Compaire with linear differential eq^n $dy/dx + Py = Q$

here, $P = -\frac{x}{(1+x)} = \frac{1}{(1+x)} - 1$, $Q = \frac{1-x}{(1+x)}$

The integrating factor (I.F) is given by

$$I.f = e^{\int P dx} = e^{\int \frac{1}{(1+x)} - 1 \, dx} = (1+x) \cdot e^{-x}$$

Therefore, The Solution of the given differential is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$y(1+x)e^{-x} = \int \frac{1-x}{(1+x)} (1+x)e^{-x} + c$$

$$y(1+x)e^{-x} = \int e^{-x} dx + \int x e^{-x} + c$$

$$y(1+x)e^{-x} = \cancel{-e^{-x}} - \cancel{-e^{-x}} + x e^{-x} + c$$

$$y(1+x) = x + c e^{x}.$$

Example 2 Solve $y dx - x dy + \log x \, dx = 0$

Solution The given D.E Can be written as

$$\frac{dy}{dx} - \frac{1}{x} y = \frac{1}{x} \log x$$ Comparing with $dy/dx + Py = Q$

here, $P = -1/x$, $Q = \frac{1}{x} \log x$

Therefore $I.F = e^{\int -\frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$

hence, the Complete Sol^n is

$$y/x = \int \frac{1}{x} \log x \cdot \frac{1}{x} dx + c$$

putting, $\log x = t$, and integrating by Parts the integral on righthand side, we get

$$y = cx - (1+\log x)$$

Example (3) Solve $(1-x^2)\frac{dy}{dx} + 2xy = x(1-x^2)^{1/2}$

Solution The given eq^n can be written as

$$\frac{dy}{dx} + \frac{2x}{(1-x^2)}y = \frac{x(1-x^2)^{1/2}}{(1-x^2)}$$

Where $P = \frac{2x}{1-x^2}$, $Q = \frac{x(1-x^2)^{1/2}}{(1-x^2)}$

$$I.F = e^{\int \frac{2x}{1-x^2}dx} = e^{-\log(1-x^2)} = \frac{1}{(1-x^2)}$$

Therefore, the Sol^n of the given differential eq^n is

$$y\, e^{\int P dx} = \int Q\, e^{\int P dx} dx + C$$

$$\frac{y}{(1-x^2)} = \int \frac{x(1-x^2)^{1/2}}{(1-x^2)} \times \frac{1}{(1-x^2)} dx + C$$

putting $t = 1-x^2$ and integrating the integral on the right hand side, we get

$$\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + C$$

$$\Rightarrow \quad y = \sqrt{1-x^2} + C(1-x^2)$$

Example ④ Solve $\sin 2x \, dy/dx - y = \tan x$

Soln:- The given D.E eqn can be written as

$$\frac{dy}{dx} - \operatorname{cosec} 2x \, y = \frac{\tan x}{\sin 2x} = \frac{1}{2}\sec^2 x$$

here, $P = -\operatorname{cosec} 2x$, $Q = \frac{1}{2}\sec^2 x$

$$I.F = e^{\int P dx} = e^{-\int \operatorname{cosec} 2x} = e^{(\log(\tan x)^{-1/2}} = \frac{1}{\sqrt{\tan x}}$$

Therefore, the solution of the given D.E is

$$\frac{y}{\sqrt{\tan x}} = \int \frac{1}{2}\sec^2 x \frac{1}{\sqrt{\tan x}} dx + c$$

substituting $t = \tan x$ in the integral on the right-hand side and simplifying, we get

$$y = c\sqrt{\tan x} + \tan x$$

Example ⑤ Solve $(1+y^2)dx = (\tan^{-1} y - x)dy$

Soln:- The given eqn can be written as

$$\frac{dx}{dy} + \frac{1}{(1+y^2)} x = \frac{\tan^{-1} y}{1+y^2}$$

here, $P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1} y}{(1+y^2)}$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

The Sol^n of the given differential eq^n is

$$x e^{\int P dy} = \int Q e^{\int P dy} dy + C$$

$$x e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{(1+y^2)} e^{\tan^{-1} y} dy + C$$

putting $t = \tan^{-1} y$ in the integral on the righthand side, and integrate, we get

$$x e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + C$$

Example ⑥ Solve $(x+2y^3) dy/dx = y$.

Sol^n:— Write the given eq^n as

$$\frac{dx}{dy} - \frac{1}{y} x = 2y^2$$

here $P = -1/y$, $Q = 2y^2$ then $I.F = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$

The Sol^n of given D.E is

$$\frac{x}{y} = \int \frac{2y^{\cancel{2}}}{\cancel{y}} dy + C$$

$$x/y = y^2 + C$$

$$\Rightarrow \quad x = y^3 + Cy$$

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Bernoulli's differential equation:- An eq^n of the form

$$\frac{dy}{dx} + Py = Qy^n \text{ --- } (2)$$

Where P and Q are Constants or f^n of x alone and n is Constant except 0 and 1, is Called a Bernoulli's eq^n.

The Sol^n of (2) is obtained as follow:- Divides eq^n (2) by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + P\frac{y}{y^n} = Q, \text{ put } y^{-n+1} = u, \text{ So that}$$

$(1-n)\, y^{-n}\, dy/dx = du/dx$ and eq^n (2), Thus reduces to

$$\frac{1}{(1-n)}\frac{du}{dx} + Pu = Q$$

OR

$$\frac{du}{dx} + P(1-n)u = Q(1-n)$$

Which is a linear differential eq^n in u and can be solved by previous Method.

Example (1) Solve $(1-x^2)\frac{dy}{dx} + xy = xy^2$

Sol^n:- The given eq^n can be written as

$$y^{-2}\frac{dy}{dx} + \frac{x}{(1-x^2)}y^{-1} = \frac{x}{(1-x^2)}$$

put $y^{-1} = u \Rightarrow -y^{-2}\, dy/dx = du/dx$ and the given eq^n reduces to

the $$\frac{du}{dx} - \frac{x}{(1-x^2)}u = -\frac{x}{(1-x^2)}$$

which is linear D.E in u. its integrating factor is

$$\Rightarrow I.f = e^{\int P dx} = e^{-\int x/(1-x^2) dx} = e^{\log(1-x^2)^{1/2}} = (1-x^2)^{1/2}$$

The required soln of the given D.E is

$$u e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$u(1-x^2)^{1/2} = \int -\frac{x}{(1-x^2)}(1-x^2)^{\frac{1}{2}} dx + c$$

In the integral on the right hand side, put $t = 1-x^2$ and integrate, we get

$$u(1-x^2)^{1/2} = (1-x^2)^{1/2} + c$$

$$\frac{(1-x^2)^{1/2}}{y} = (1-x^2)^{1/2} + c \quad \text{Since } u = y^{-1}$$

$$\Rightarrow (1-y)\sqrt{1-x^2} = cy$$

Example (2) Solve $x\frac{dy}{dx} + y = y^2 \log x$

Soln:- The given eqn can be written as

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y} = \frac{1}{x}\log x$$

putting $-y' = u \Rightarrow \frac{1}{y^2} dy/dx = \frac{du}{dx}$ in the above eqn, we get

$$\frac{du}{dx} - \frac{1}{x}u = \frac{1}{x}\log x$$

Which is linear eqn in u, and the I.F is

$$I.F = e^{\int P dx} = e^{-\int \frac{1}{x} dx} = \frac{1}{x}$$

Therefore, The required soln is

$$u/x = \int \frac{\log x}{x^2} dx + c$$

put $t = \log x$ in the integral on the right hand side and integrate, we get

$$u/x = c - (1+\log x) e^{-\log x}$$

OR

$$-\frac{1}{xy} = c - (1+\log x)\frac{1}{x}$$

$$\Rightarrow \quad 1 = (1+\log x)y - cxy$$

Example ③ Solve $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

Soln:- The given eqn can be written as

$$\sec^2 y \, dy/dx + 2x\tan y = x^3 \quad \text{③}$$

put $\tan y = u \Rightarrow \sec^2 y \, dy/dx = \frac{du}{dx}$, and eqn ③ reduces to

Eq3,

$$\frac{du}{dx} + 2xu = x^3$$

here, $P = 2x$, $Q = x^3$

$I.F = e^{\int P dx} = e^{x^2}$, Then the required solⁿ is given by

$$u e^{x^2} = \int x^3 e^{x^2} dx$$, put $x^2 = t \Rightarrow 2x dx = dt$

$$= \frac{1}{2}\int t e^t dt = \frac{1}{2} e^t(t-1) + c$$

$$\tan y \, e^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + c$$

$$\Rightarrow \tan y = \frac{1}{2}(x^2-1) + c e^{-x^2}.$$

Which is the required solⁿ.

Example ④ solve $y(2xy + e^x)dx - e^x dy = 0$

Solⁿ:- The given eqⁿ can be written as

$$y^{-2} dy/dx - y^{-1} = 2x e^{-x}$$

Now, put $y^{-1} = u \Rightarrow -\frac{1}{y^2} dy/dx = \frac{du}{dx}$ and the above eqⁿ reduces to

$$\frac{du}{dx} + u = -2x e^{-x}$$

Which is a linear D.E eqⁿ in u and hence the solⁿ

$$u e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$u e^{\int dx} = \int -2x e^{-x} e^{x} dx$

$u e^{x} = -\int 2x e^{\alpha} dx$

$u e^{x} = -x^2 + c$

$e^{x}/y = -x^2 + c$ which is the required soln.

Orthogonal Trajectories :— A curve which has cuts every members of a given family of curves at a Constant angle are called isogonal trajectories.
A curve which cuts every members of a given family of curve at a right angle is called an orthogonal trajectory of the given family of curves.

Application of orthogonal trajecteries :—

Examples ① In the electrical field the path along which the current flows. are the orthogonal trajecteries of the equipotential lines.

② In fluid dynamic the stream lies and equipotential lines are orthogonal trajecteries.

(3) In thermo dynamics the lines of heat flow are OT.

Method to find orthogonal trajectories :-

Let $f(x,y,c)=0$ — (1) represents the equation of a given family of curves with single parameter c.

Differentiating (1) w.r.t. to x and eliminating c. we get the differential equation of the form

$$\phi(x, y, dy/dx) = 0$$

To find the family of curve which touches the given family of curves (1) replace $\frac{dy}{dx} = -\frac{dx}{dy}$

i.e $\phi(x, y, -\frac{dx}{dy}) = 0$ or $\phi(x, y, -1/y') = 0$ — (2)

Solving the differential eqn (2), we get the OT of (1).

Self orthogonal :- A given family of curves is said to be self orthogonal if the family of orthogonal trajectory is the same as given family of curves.

Example (1) find the OT of family of line $y = mx$.

Soln :- given $y = mx$, diff. w.r.t. to x, we get

$$dy/dx = m, \quad \text{put } m = y/x$$

$$\Rightarrow \quad x\,dy/dx - y = 0$$

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Replace $dy/dx = -\frac{dx}{dy}$, we have

$$x\left(-\frac{dx}{dy}\right) - y = 0$$

$$-x\,dx - y\,dy = 0$$

$$-\frac{x^2}{2} - \frac{y^2}{2} = c$$

$$x^2 + y^2 = c_1, \quad c_1 = -2c$$

family of circles is OT of family of straight lines.

Example ② find the orthogonal trajectories of family of semi-cubical Parabola $ay^2 = x^3$ — ① a is a parameter

Solⁿ:— given $ay^2 = x^3$, diff. ① w.r.t. x, we get

$$2ay\,dy/dx = 3x^2 \Rightarrow a = \frac{3x^2}{2y\,dy/dx}$$

Since $a = x^3/y^2$

$$\frac{\cancel{x^3}}{\cancel{y^2}} = \frac{3\cancel{x^2}}{2\cancel{y}\,dy/dx}$$

$$\Rightarrow \quad 3y = 2x\,dy/dx$$

or $\quad 2x\,dy/dx - 3y = 0$ — (2)

Replace $dy/dx = -dx/dy$, $\quad -2x\frac{dx}{dy} - 3y = 0$

$$\Rightarrow -2x\,dx - 3y\,dy = 0$$

$$-\cancel{2}x^2/\cancel{2} - 3y^2/2 = c$$

$$2x^2 + 3y^2 = 2c_1 \qquad c_1 = -2c$$

Example ③ find the O.T of $xy = c$ — ①

Soln:- $xy = c$, diff. w.r.to x, we get

$x\,dy/dx + y = 0$

Replace $dy/dx = -\frac{dx}{dy}$

$-x\frac{dx}{dy} + y = 0$

$-x\,dx + y\,dy = 0$

integrate, we get

$-\frac{x^2}{2} + \frac{y^2}{2} = c$

$x^2 - y^2 = c_1$, $c_1 = -2c$

Example ④ find OT of $x^2 - y^2 = c_1$.

Soln:- given $x^2 - y^2 = c_1$, diff. w.r.to x, we get

$2x - 2y\,dy/dx = 0 \Rightarrow x - y\,dy/dx = 0$

Replace dy/dx by $-dx/dy$

$x + y\,dx/dy = 0$

$y\,dx/dy = -x$, seperable the variables

$\frac{1}{x}dx + \frac{1}{y}dy = 0 \Rightarrow \log x + \log y = \log c$

$\Rightarrow xy = c$

Example ⑤ Show that $y^2 = 4a(x+a)$ is self orthogonal.

Soln:- given family of curves

$$y^2 = 4a(x+a) \quad \text{--- (1)}$$

diff. w.r.to x, we get

$$2y\, dy/dx = 4a \Rightarrow a = \frac{y}{2}\frac{dy}{dx}$$

sub. in ①, we get

$$y^2 = \left(2y\frac{dy}{dx}\right)x + \cancel{4}\cdot\frac{y^2}{\cancel{4}}\left(\frac{dy}{dx}\right)^2$$

$$y^2 = 2xy\, dy/dx + y^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow y = 2x\, dy/dx + y(dy/dx)^2 \quad \text{--- (2)}$$

which is the D.E of ①.

Replace $dy/dx = -\frac{dx}{dy}$ in ②, we get

The differential equation of O.T is given by

$$y = 2x\left(-\frac{dx}{dy}\right) + y\left(-\frac{dx}{dy}\right)^2$$

$$y = -2x\frac{dx}{dy} + y\left(\frac{dx}{dy}\right)^2 \text{ or}$$

$$y(dy/dx)^2 + 2x(dy/dx) = y \quad \text{--- (3)}$$

which is the same as ②.

The differential eqn of O.T of (3) is same as the D.E of given family of curve (1).

hence, given family of curves self orthogonal.

Orthogonal trajectories (in Polar Coordinate):—

Let $f(r, \theta, a) = 0$ — (1) ('a' being a arbitrary Constant

eqn (1) be the equation of given system of curves.

diff. eqn (1) w.r.to θ and eliminate (a) gives the D.E of (1)

$$\phi\left(r, \theta, \frac{dr}{d\theta}\right) = 0$$

Replace, $\dfrac{dr}{d\theta} = -r^2 \dfrac{d\theta}{dr}$ then $\phi\left(r, \theta, -r^2\dfrac{d\theta}{dr}\right) = 0$ is O.T differential eqn. Solving this eqn, we get system of O.T.

Example (1) find O.T of the curve $r = a(1-\cos\theta)$ — (1)

Soln:— Taking log both sides

$$\log r = \log a + \log(1-\cos\theta)$$

diff. w.r.to θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta},$$ which is D.E of given family of curve.

Replacing by $\dfrac{dr}{d\theta} = -r^2\dfrac{d\theta}{dr}$, we have

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\sin\theta}{1-\cos\theta}$$

$$-r\frac{d\theta}{dr} = \frac{\sin\theta}{1-\cos\theta}$$

OR $$\frac{1-\cos\theta}{\sin\theta}d\theta + \frac{1}{r}dr = 0$$

$$\frac{\cancel{1}-\cancel{1}+\cancel{2}\sin^2\theta/2}{\cancel{2}\sin\theta/2\cos\theta/2}d\theta + \frac{1}{r}dr = 0$$

$$\tan\theta/2\,d\theta + \frac{1}{r}dr = 0$$

integrating both sides, we get

$$\log r + 2\log\sec\theta/2 = \log C$$

$$r\sec^2\theta/2 = k$$

$$r = \frac{k2\cos^2\theta/2}{2}$$

$$r = \frac{k}{2}(1+\cos\theta)$$

Which is the required O.T of given curve.

Example ② find O.T of $r^n\sin n\theta = a^n$.

Solnº:- given $r^n\sin n\theta = a^n$

taking log both side, we get

$$n\log r + \log\sin n\theta = n\log a$$

diff. w.r.to θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} + \frac{n\cos n\theta}{\sin n\theta} = 0$$

$$\Rightarrow \frac{dr}{d\theta} + r\cot n\theta = 0 \quad \text{--- (2)}$$

Replace $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

then (2) becomes, $-r^2\frac{d\theta}{dr} + r\cot n\theta = 0$

$$\cancel{r^2}\frac{d\theta}{dr} = \cancel{r}\cot n\theta$$

$\tan n\theta \, d\theta = \frac{1}{r}dr$, integrate both sides we get

$$\log r - \frac{1}{n}\log\cos n\theta = \log c$$

$$n\log r - \log\cos n\theta = \log c$$

$$\log r^n - \log\cos n\theta = \log c$$

$$\log\frac{r^n}{\cos n\theta} = \log c$$

$$r^n \sec n\theta = c$$

Initial value problem:— A first order differential eqn is an equation

$$\frac{dy}{dx} = f(x,y) \quad \text{--- (1)}$$

Where $f(x,y)$ is a fn of two variable defined on a region in the XY-plane.

A Solution of such eqⁿ ① is a differentiable fⁿ $y = y(x)$ defined on an interval I of x-values such that $\frac{d}{dx} y(x) = f(x, y(x))$. on the interval. that is when $y(x)$ and derivative $y'(x)$ are substitute into the eqⁿ ① the resulting eqⁿ is true for all x over the interval I. The general Solution to first order D.E is a Solution that Contain all possible Solution.
The general Solution always Contain an arbitrary Constant, but having a property does not mean a Solution is general Solution, that is a Solution may Contain an arbitrary Constant without being the general Solution.
Establishing that a Solution is the general Solution may deeper results from the theory of differential eqⁿ and is best Studied in more advanced case.

Example ① Show that a fⁿ $y = (x+1) - \frac{1}{3} e^x$ is a Solution to the first order IVP

$$\frac{dy}{dx} = y - x, \quad y(0) = 2/3$$

Solⁿ :— The eqⁿ $dy/dx = y - x = f(x, y)$ is a first order diff. eqⁿ with $f(x, y) = y - x$

on the left side,

$$\frac{dy}{dx} = \frac{d}{dx}\left((x+1) - \frac{e^x}{3}\right) = 1 - \frac{1}{3}e^x$$

on the right hand side, $y - x = (x+1) - \frac{e^x}{3} - x = 1 - \frac{e^x}{3}$

and also f^n satisfies initial condition

$y(0) = 0 + 1 - \frac{1}{3} = 2/3$. hence $y = (x+1) - \frac{1}{3}e^x$

is a solⁿ of $dy/dx = y - x$.

Picards Theorem:— suppose that $f(x,y)$ and its partial derivative $\partial f/\partial y$ are continuous on the interior of a rectangle R. and the pt (x_0, y_0) is an interior pt of R then the IVP

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

has a unique solⁿ $y = y(x)$.

Picards approximation:— Consider IVP

$$\frac{dy}{dx} = f(x,y) \quad \text{——(1)}$$

$$y(x_0) = y_0$$

integral eqⁿ ① w.r.to x, with limits x_0 to x, we get

$$\int_{x_0}^{x} y' \, dx = \int_{x_0}^{x} f(x,y) \, dx$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\,dx$$

$$y(x) = y(x_0) + \int_{x_0}^{x} f(x,y)\,dx \quad -(2)$$

This is Solution of IVP of (1)

Conversely if $y(x)$ Satisfies (2) then Satisfies the IVP (1)

The interval eqn (2) is Solved iteratively.

assuming first approximation to $y(x)$ as $y_0(x) = y(x_0) = y_0$

Now, next iterative is called y_1

$$y_1(x) = y(x_0) + \int_{x_0}^{x} f(x, y_0(x))\,dx = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

second approximation

$$y_2(x) = y(x_0) + \int_{x_0}^{x} f(x, y_1(x))\,dx$$

sequence of approximation $y_0, y_1, -- y_{n+1}$ is given by

$$\boxed{y_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f(x, y_n(x))\,dx}$$

This is formula for Picards iteration.

Converges to $y(x)$ for n sufficiently Close to x_0. The Soln $y(x)$ Thus obtained, is the unique Soln of IVP.

Example (1) find the Solution of IVP $y' = 2y - x$ with $y(0) = 1$, using Picards Iteration Method. Compare with exact Solution.

Sol^{n}:- given $y' = 2y - x = f(x, y)$, and $y_0(0) = 1 = y_0$

$y_0(x_0) = 1 = y_0$

using Picards iteration formula to found Sol^{n}

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(x, y_n)\, dx$$

for $n = 0, 1, 2, 3$, therefore, first iteration

$$y_1(x) = 1 + \int_{0}^{x} (2y_0 - x)\, dx = 1 + \int_{0}^{x} (2 - x)\, dx$$

$$y_1(x) = 1 + 2x - x^2/2$$

second $y_2(x) = y_0 + \int_{0}^{x} f(x, y_1)\, dx$

$$= 1 + \int_{0}^{x} 2(1 + 2x - x^2/2 - x)\, dx$$

$$= 1 + 2x + 3/2\, x^2 - x^3/3$$

$$y_3(x) = y_0 + \int_{0}^{x} f(x, y_2)\, dx$$

$$= 1 + \int_{0}^{x} \left[\left[2(1 + 2x + 3/2\, x^2 - x^3/3 \right] - x \right] dx$$

$$y_3(x) = 1 + \int_0^x \left(2 + 4x + 3x^2 - \frac{2}{3}x^3 - x\right)dx$$

$$= 1 + \left(2x + 2x^2 + x^3 - \frac{2}{3}\frac{x^4}{4} - \frac{x^2}{2}\right.$$

$$y_3(x) = 1 + 2x + \frac{3x^2}{2} + x^3 - \frac{x^4}{6}$$

Compare with exact Solution :— $dy/dx = 2y - x$, $y(0) = 1$

given D.E is first order linear diff. eqn

$I.F = e^{-2x}$, Soln is

$$y e^{-2x} = -\int x e^{-2x} dx$$

$$y e^{-2x} = \frac{1}{4}(2x+1)e^{-2x} + c$$

appling Condition $y(0) = 1$

$$1 = \frac{1}{4}(1) + c \Rightarrow c = 3/4$$

then $y = \frac{1}{4}\left[(2x+1)e^{-2x} + \frac{3}{4}\right]$

$$y = \frac{1}{4}\left[(2x+1) + 3e^{2x}\right]$$

$$y = \frac{1}{4}\left[(2x+1) + 3\left(1 + 2x + \frac{4x^2}{2!} + \cdots\right)\right]$$

$$y = 1 + 2x + 3/2\, x^2 + x^3 + \frac{1}{2}x^4 + \frac{1}{5}x^5.$$

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