Unit-4

Higher order linear differential equation with Constant Coefficient :—

Definition :— A linear differential equation of order n is an equation of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = Q(x) \quad \text{——} (1)$$

Where $P_0, P_1, P_2, \cdots, P_n$ are Constant and $Q(x)$ are Continuous real f^n on a Common interval I. The right-hand side of eq^n ① is Called the nonhomogeneous term.

if $Q(x)$ is identically zero, then eq^n ① reduces to

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = 0 \text{ —} (2)$$

then eq^n ② is Called a homogeneous linear differential equation of order n.

Solution of homogeneous linear differential equations of order n with Constant Coefficients :—

Suppose that a possible solution of eq^n ② is

$$y = e^{mx}$$

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Since $\frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2e^{mx}$ ... $\frac{d^ny}{dx^n} = m^n e^{mx}$

putting these value in eqn (2) then it takes the form

$$P_0 m^n e^{mx} + P_1 m^{n-1} e^{mx} + \cdots + P_{n-1} m e^{mx} + P_n e^{mx} = 0 \quad \text{---(3)}$$

As $e^{mx} \neq 0$ for all m and x, we can divide eqn (3) by e^{mx} it to get

$$P_0 m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0 \quad \text{---(4)}$$

Now, for each value of m for which eqn (4) hold will make $y = e^{mx}$ a solution of eqn (2). But eqn (4) is an algebraic eqn in m of degree n and, therefore, by the fundamental theorem of algebra, it has at least one and not more than n-distinct roots. we denote these roots by $m_1, m_2, \ldots, m_n$, where m's need not all be distinct, then each f^n

$y_1 = e^{m_1x}$, $y_2 = e^{m_2x}$, ... $y_n = e^{m_nx}$ is a solution of eqn (2).

Equation (4) is called the auxiliary equation (AE) or the characteristic eqn (C.E) of eqn (2) and can easily be obtained from this eqn by simply replacing y' with m, y'' with m^2 and so on, and y^n with m^n.

While solving the auxiliary equation, the following three

Cases may occur.

1. All the roots are distinct and real.
2. All the roots are real but some are repeated.
3. All the roots are imaginary.

We shall discuss all these three cases seperately.

Case–I if the n roots $m_1, m_2, \ldots, m_n$ of A.E eqn ④ are distinct then n solution of eqn ② are $y_1 = e^{m_1 x}$; $y_2 = e^{m_2 x}$ $\ldots$, $y_n = e^{m_n x}$, But these n solution are different and linearly independent and thus the general solution of eqn ② is

$$y_c = y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

here, y_c is known as the complementary function.

Examples related to Case–I.

Example ① solve $(y''' + 6y'' + 11)y' + 6y = 0$

Soln :— here. A.E is

$$m^3 + 6m^2 + 11m + 6 = 0$$

$$\text{or } (m+1)(m+2)(m+3) = 0$$

Which gives $m = -1, -2, -3$

Therefore, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$$

Example ② solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ with $y=0$ when $x=0$ and dy/dx

Solⁿ :- The A.E is $m^2 - 3m + 2 = 0$

$\Rightarrow m = 1, 2$ then solution is

$$y = c_1 e^x + c_2 e^{2x} \quad -① $$

given that $y=0$ when $x=0$ then eqⁿ ① becomes

$$0 = c_1 + c_2 \quad -②$$

Also, $dy/dx = 0$, when $x=0$

$$dy/dx = c_1 e^x + 2c_2 e^{2x}$$

$$\Rightarrow \quad 0 = c_1 + 2c_2 \quad -③$$

solving ② and eqⁿ ③, we get $c_1 = c_2 = 0$ and the general solution of given eqⁿ is $y=0$.

Example ③ solve $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 0$

Solⁿ :- The A.E is $m^3 - m^2 - 6m = 0$

$\Rightarrow m(m^2 - m - 6) = 0$

OR $m = 0, -2, 3$ Thus, The general solⁿ is

$$y = c_1 + c_2 e^{-2x} + c_3 e^{3x}$$

Case-II if the A.E ④ has a root $m=a$, which repeats n times, then the general solⁿ of eqⁿ ② in this case is

$$y = (c_1 + x c_2 + x^2 c_3 + \cdots + c_n x^{n-1}) e^{ax}$$

if A.E eqⁿ ④ has K roots each equal m_1 and remaining $(n-k)$ roots are all different, then the Solⁿ of eqⁿ ② is

$$y = (C_1 + C_2x + C_3x^2 + \cdots + C_kx^{k-1})e^{m_1x} + C_{k+1}e^{m_{k+1}x} + \cdots C_ne^{m_nx}$$

Examples related to this above Case.

Example ① Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Solution:- The A.E is $m^3 - 3m + 2 = 0$

Which gives $m = 1, 1, -2$ and the Solution is

$$y = (C_1 + C_2x)e^{x} + C_3e^{-2x}$$

Example ② Solve $(16D^2 + 24D + 9)y = 0$

Solⁿ:- The A.E is $16m^2 + 24m + 9 = 0$

$\Rightarrow (4m+3)^2 = 0$

$\Rightarrow m = -3/4, -3/4$

The general Solⁿ is $y = (C_1 + C_2x)e^{-3/4x}$

Example ③ Solve $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2y = 0$

Solⁿ:- here, A.E is $m^2 - 2am + a^2 = 0$

$\Rightarrow (m-a)^2 = 0$

and has double root $m = a$, then the general Solⁿ is

$$y = (C_1 + xC_2)e^{ax}$$

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Case-III if the A.E ④ has imaginary roots i.e $\alpha - i\beta$ and $\alpha + i\beta$ then general solⁿ of eqⁿ ② is of the forms $y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

if $\alpha + i\beta$ and $\alpha - i\beta$ each occurs twice as a root, then the general solⁿ of this case is of the form

$$y = e^{\alpha x}\left[(c_1 + x c_2)\cos\beta x + (c_3 + x c_4)\sin\beta x\right]$$

Examples related to this above case.

Example ① solve $(D^2+4)y=0$

Solⁿ:- The A.E is $m^2+4=0$ has a pair of roots $m = \pm 2i$, Then the general solⁿ is of the form

$$y = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$y = c_1 \cos 2x + c_2 \sin 2x$$

Example ② solve $(D^4+8D^2+16)y=0$

Solⁿ:- The A.E is $m^4+8m^2+16=0$ has a double roots $m = \pm 2i$, The solⁿ is

$$y = (c_1 + x c_2)\cos 2x + (c_3 + x c_4)\sin 2x.$$

Example ③ solve $(D^3+1)y=0$

Solⁿ:- The A.E is $m^3+1=0$ or $(m+1)(m^2-m+1)=0$

Which gives $m = -1, \frac{1 \pm \sqrt{3}i}{2}$ therefore, The general soln is

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right).$$

Example ④ Solve $\left(\frac{dy}{dx} - y\right)^2 \left(\frac{d^2y}{dx^2} + y\right)^2 = 0$

Soln :— The A.E is $(m-1)^2(m^2+1)^2 = 0$

Which gives $m = 1, 1, \pm i, \pm i$, Therefore, The general Soln is

$$y = (c_1 + x c_2)e^{x} + (c_3 + x c_4)\cos x + (c_5 + x c_6)\sin x.$$

Solution of Nonhomogeneous linear differential eqn with Constant Coefficients by Means of Polynomial operators :— The general Solution of a non-homogeneous linear differential eqn

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots P_{n-1}\frac{dy}{dx} + P_n y = Q(x) \quad \text{—③}$$

, where $P_0 \neq 0$, $Q(x) \neq 0$ is

$$y = y_c + y_p$$

here, y_c and y_p are, respectively, known as the Complementary f^n (C.f) and Particular integral (P.I) or Particular Solution. it was seen in the earlier section that when $Q(x) = 0$ then the general soln is $y = y_c$.

and we had also discussed the method of finding the Complementary function. it only remains to find the particular integral. to start the Method of finding Particular integral, we have discuss the following definition of inverse operator.

inverse operator:— The nonhomogeneous linear D.E eqn (3) can be written as in the form of operator D i.e

$$(P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + P_{n-1} D + P_n) y = Q(x)$$

OR $f(D) y = Q(x)$

Let $f(D) = P_0 D^n + P_1 D^{n-1} + \cdots + P_{n-1} D + P_n$ be the operator polynomial for the differential eqn $f(D)y = Q$, then $\frac{1}{f(D)} Q(x)$ is defined to be that fn of x, not Containing arbitrary Constants, which when operated upon by $f(D)$ gives $Q(x)$. then $\frac{1}{f(D)}$ is Called inverse operator.

Example $\frac{1}{(D^2+D)}(2x+3) = \frac{1}{D(1+D)}(2x+3) = \frac{1}{D}(1+D)^{-1} 2x+3$

$$\frac{1}{(D^2+D)}(2x+3) = \frac{1}{D}(1-D+D^2+\cdots)(2x+3) = \frac{1}{D}(2x+1)$$

$$\frac{1}{(D^2+D)}(2x+3) = \frac{1}{D}(2x+1) = \int (2x+1)dx = x^2+x$$

here D means differential operator and $\frac{1}{D}$ is integral operator

Therefore, $\frac{1}{D} Q(x) = \int Q(x)\,dx$ and $D Q(x) = \frac{d}{dx} Q(x)$.

Definition of Particular integral :— Let $f(D)y = Q(x)$ be a linear differential equation with constant coefficients. Then the particular integral or particular soln of this differential equation is that fn of x, which when operated by $f(D)$ gives $Q(x)$. it follows from this defn and the meaning of operator $\frac{1}{f(D)}$ given in earlier section that $\frac{1}{f(D)} Q(x)$ is the particular integral of $f(D)y = Q(x)$ hence, $P.I = \frac{1}{f(D)} Q$.

General Method of finding Particular integral :—

In this section, we will discuss various methods of finding particular integrals of linear D.E with constant coefficients.

first general Method :— Let the D.E $f(D)y = Q$, where $f(D) = P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_{n-1} D + P_n$ is operator polynomial. Suppose $f(D)$ can be broken up into linear factors. which may be taken in any order.

Let $f(D) = (D-m_1)(D-m_2) \cdots (D-m_n)$, then

$$P.I = \frac{1}{f(D)} Q(x)$$

$$\Rightarrow P.I = \frac{1}{(D-m_1)(D-m_2) \cdots (D-m_n)} Q(x)$$

$$\Rightarrow P.I = \frac{1}{(D-m_1)} \times \frac{1}{(D-m_2)} \times \cdots \frac{1}{(D-m_n)} Q(x)$$

$$P.I = \frac{1}{(D-m_1)} \times \frac{1}{(D-m_2)} \times \cdots \times \frac{1}{(D-m_{n-1})} \times e^{m_n x} \int e^{-m_n x} Q(x)\,dx$$

$$P.I = \frac{1}{(D-m_1)} \times \frac{1}{(D-m_2)} \times \cdots \times \frac{1}{(D-m_{n-1})} Q_1(x), \text{ Where } Q_1(x) = e^{m_n x} \int e^{-m_n x} Q(x)\,dx$$

$$\Rightarrow P.I = \frac{1}{(D-m_1)} \times \frac{1}{(D-m_2)} \times \cdots \times \frac{1}{(D-m_{n-2})} e^{m_{n-1} x} \int e^{-m_{n-1} x} Q_1(x)\,dx$$

$$\text{OR } P.I = \frac{1}{(D-m_1)} \times \frac{1}{(D-m_2)} \times \cdots \times \frac{1}{(D-m_{n-2})} Q_2(x), \text{ Where } Q_2(x) = e^{m_{n-1} x} \int e^{-m_{n-1} x} Q_1(x)\,dx$$

This process is Continued untill all the n-operational factors have been utilised. The final expression obtained is the P.I of the differential equation $f(D)y = Q$.

Second general Method:— In this Method the inverse operator $\frac{1}{f(D)} = \frac{1}{(D-m_1)(D-m_2) \cdots (D-m_n)}$ is resolved into partial fractions. Let the partial fraction of

$$\frac{1}{(D-m_1)(D-m_2)\cdots(D-m_n)} = \frac{A_1}{(D-m_1)} + \frac{A_2}{(D-m_2)} + \cdots + \frac{A_n}{(D-m_n)}$$

where $A_1, A_2, \ldots, A_n$ are constants. i.e

$$P.I = \frac{1}{f(D)}Q = \frac{1}{(D-m_1)(D-m_2)\cdots(D-m_n)}Q = \left\{\frac{A_1}{(D-m_1)} + \frac{A_2}{(D-m_2)} + \cdots + \frac{A_n}{(D-m_n)}\right\}Q$$

$$\Rightarrow P.I = A_1\left\{\frac{1}{(D-m_1)}\right\}Q + A_2\left\{\frac{1}{(D-m_2)}\right\}Q + \cdots + A_n\left\{\frac{1}{(D-m_n)}\right\}Q$$

$$\Rightarrow P.I = A_1 e^{m_1x}\int e^{-m_1x} Q(x)dx + A_2 e^{m_2x}\int e^{-m_2x} Q(x)dx + \cdots + A_n e^{m_nx}\int e^{-m_nx} Q(x)dx$$

Remark In short the particular integral of the D.E of the form $(D-\alpha)y = Q(x)$ is $P.I = \frac{Q}{(D-\alpha)} = e^{\alpha x}\int e^{-\alpha x} Q(x)dx$

Example ① Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x}$.

Solⁿ:— The given diff. eqⁿ in operator notation can be written as $(D^2-5D+6)y = e^{3x}$, where $d/dx = D$

The general solution of this equation will consist of two parts (i) Complementary function (C.f) y_c (ii) Particular integral (P.I) y_p.

The A.E is $m^2-5m+6=0$

$\Rightarrow (m-2)(m-3)$

$\Rightarrow m = 2, 3$ then $C.f = y_c = c_1e^{2x} + c_2e^{3x}$

Now, finding Particular integral by general Method

$$P.I = \frac{1}{(D^2-5D+6)} e^{3x} = \frac{1}{(D-2)(D-3)} e^{3x} = \frac{1}{(D-2)}\left\{\frac{1}{(D-3)} e^{3x}\right\}$$

$$= P.I = \frac{1}{(D-2)}\left[e^{3x}\int e^{-3x} e^{3x} dx\right] = \frac{1}{(D-2)} x e^{3x} = e^{2x}\int e^{-2x} e^{3x} x\, dx$$

$$= P.I = e^{2x}\int x e^{x} dx = e^{2x}(xe^{x} - e^{x}) = e^{3x}(x-1).$$

by Second Method

$$P.I = \frac{1}{(D-2)(D-3)} e^{3x} = \left\{\frac{1}{(D-3)} - \frac{1}{(D-2)}\right\} e^{3x} = \frac{1}{(D-3)} e^{3x} - \frac{1}{(D-2)} e^{3x}$$

$$P.I = e^{3x}\int e^{-3x} e^{3x} dx - e^{2x}\int e^{-2x} e^{3x} dx$$

$$\Rightarrow P.I = e^{3x} x - e^{3x} = e^{3x}(x-1)$$

hence, the general Solution of the given differential equation is given by $y = C.f + P.I$

$$y = c_1 e^{2x} + c_2 e^{3x} + e^{3x}(x-1) = c_1 e^{2x} + (c_2 - 1)e^{3x} + xe^{3x}$$

$$y = c_1 e^{2x} + c_3 e^{3x} + x e^{3x}, \text{ where } c_3 = c_2 - 1 \text{ is a new constant}$$

Example ② solve the differential eqn:

$$\frac{d^2y}{dx^2} + n^2 y = \sec nx$$

Soln:— The A.E is $m^2 + n^2 = 0$

$$\Rightarrow m = \pm in$$

$$y_c = C.F = C_1 \cos nx + C_2 \sin nx$$

Now,

$$P.I = \frac{1}{(D^2+n^2)} \sec nx = \frac{1}{(D+in)(D-in)} \sec nx = \frac{1}{2in}\left\{\frac{1}{(D-in)} - \frac{1}{(D+in)}\right\} \sec nx$$

$$P.I = \frac{1}{2in}\left[\frac{\sec nx}{(D-in)} - \frac{1}{(D+in)} \sec nx\right] = \frac{1}{2in}\left[e^{inx}\int e^{-inx} \sec nx \, dx - e^{-inx}\int e^{inx} \sec nx \, dx\right]$$

$$P.I = \frac{1}{2in}\left\{e^{inx}\int (\cos nx - i \sin nx) \sec nx \, dx - e^{-inx}\int (\cos nx + i \sin nx) \sec nx \, dx\right\}$$

$$P.I = \frac{1}{2in}\left\{e^{inx}\int (1 - i \tan nx) dx - e^{-inx}\int (1 + i \tan nx) dx\right\}$$

$$P.I = \frac{1}{2in}\left\{e^{inx}\left(x + \frac{i}{n} \log \cos nx\right) - e^{-inx}\left(x - \frac{i}{n} \log \cos nx\right)\right\}$$

$$P.I = \frac{1}{2in}\left\{x\left(e^{inx} - e^{-inx}\right) + \frac{i}{n} \log \cos nx \left(e^{inx} + e^{-inx}\right)\right\}$$

$$P.I = \left\{\frac{x}{n}\left(\frac{e^{inx} - e^{-inx}}{2i}\right) + \frac{1}{n^2} \log \cos nx \left(\frac{e^{inx} + e^{-inx}}{2}\right)\right\}$$

$$P.I = x/n \sin nx + \frac{1}{n^2}(\log \cos nx) \cos nx$$

hence, the complete solⁿ of the given D.E is given by

$$y = C.f + P.I = C_1 \cos nx + C_2 \sin nx + \frac{x}{n} \sin nx + \frac{1}{n^2} \log \cos nx \cos nx$$

$$\Rightarrow \quad y = C_1 \cos nx + C_2 \sin nx + \frac{x}{n} \sin nx + \frac{1}{n^2}(\log \cos nx) \cos nx$$

Example ③ Solve $(D^2+4)y = \tan 2x$

Soln :- The A.E is $m^2+4=0$

$\Rightarrow m = \pm 2i$

$\Rightarrow y_c = C.F = C_1 \cos 2x + C_2 \sin 2x$

$= C_1 \cos 2x + C_2 \sin 2x$

Now,

$P.I = \frac{1}{(D^2+4)} \tan 2x = \frac{1}{(D+2i)(D-2i)} \tan 2x = \frac{1}{4i}\left\{\frac{1}{(D-2i)} - \frac{1}{(D+2i)}\right\} \tan 2x$

$P.I = \frac{1}{4i}\left\{\frac{1}{(D-2i)} \tan 2x - \frac{1}{(D+2i)} \tan 2x\right\} = \frac{1}{4i}\left\{e^{2ix}\int e^{-2ix} \tan 2x\,dx - e^{-2ix}\int e^{2ix} \tan 2x\,dx\right\}$

Now, $\int e^{-2ix} \tan 2x\,dx = \int (\cos 2x - i\sin 2x)\tan 2x\,dx$ —— ①

$= \int \sin 2x - \frac{i\sin^2 2x}{\cos 2x}\,dx$

$= \int \sin 2x - i\left(\frac{1-\cos^2 2x}{\cos 2x}\right)dx$

$= \int \sin 2x - i\sec 2x + i\cos 2x\,dx$

$= \int \sin 2x\,dx - i\int \sec 2x\,dx + i\int \cos 2x\,dx$

$= -\frac{1}{2}\cos 2x - \frac{i}{2}\log(\sec 2x + \tan 2x) + \frac{i}{2}\sin 2x$

and $\int e^{2ix} \tan 2x\,dx = \int (\cos 2x + i\sin 2x)\tan 2x\,dx$ —— (2)

$= \int \sin 2x + i\frac{\sin^2 2x}{\cos 2x}\,dx$

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$$\int e^{2ix}\tan 2x\,dx = \int \sin 2x + i\left(\frac{1-\cos^2 2x}{\cos 2x}\right)dx$$

$$= \int \sin 2x + i\sec 2x - i\cos 2x\,dx$$

$$= \int \sin 2x\,dx + i\int \sec 2x\,dx - i\int \cos 2x\,dx$$

$$= -\frac{1}{2}\cos 2x + \frac{i}{2}\log(\sec 2x + \tan 2x) - \frac{i}{2}\sin 2x \quad \text{--- (3)}$$

from eq^n (1) (2) and (3), we have

$$P.I = \frac{1}{4i}\left[e^{i2x}\left\{-\frac{1}{2}\cos 2x - \frac{i}{2}\log(\sec 2x + \tan 2x) + \frac{i}{2}\sin 2x\right\} - e^{-i2x}\left\{-\frac{1}{2}\cos 2x + \frac{i}{2}\log(\sec 2x + \tan 2x) - \frac{i}{2}\sin 2x\right\}\right]$$

$$P.I = \frac{1}{4}\left[-\cos 2x\left(\frac{e^{2ix} - e^{-2ix}}{2i}\right) - \left(\frac{e^{2ix} + e^{-2ix}}{2}\right)\log(\sec 2x + \tan 2x) + \left(\frac{e^{2ix} + e^{-2ix}}{2}\right)\sin 2x\right]$$

$$P.I = \frac{1}{4}\left\{-\cos 2x \sin 2x - \cos 2x \log(\sec 2x + \tan 2x) + \cos 2x \sin 2x\right\}$$

$$P.I = -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Hence, The general Sol^n of the given D.E is $y = C.F + P.I$

$$y = C_1\cos 2x + C_2\sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Example(4) Solve $\frac{d^2y}{dx^2}+\frac{dy}{dx}=(1+e^x)^{-1}$

Sol^n:- The A.E is $m^2+m=0$

$m(m+1)=0$

$\Rightarrow m=0,-1$

$C.F = C_1+C_2e^{-x}$

Now, $P.I = \frac{1}{(D^2+D)}(1+e^x)^{-1} = \frac{1}{D(D+1)}\frac{1}{1+e^x}$

$P.I = \left(\frac{1}{D}-\frac{1}{D+1}\right)\frac{1}{1+e^x} = \frac{1}{D}\frac{1}{1+e^x}-\frac{1}{(D+1)}\frac{1}{1+e^x}$

$P.I = \int\frac{1}{1+e^x}dx - e^{-x}\int\frac{e^x}{1+e^x}dx$

$P.I = \int\frac{e^{-x}}{1+e^{-x}}dx - e^{-x}\int\frac{e^x}{1+e^x}dx$

$P.I = -\log(1+e^{-x}) - e^{-x}\log(1+e^x)$

hence, the general solution is given by

$y=C_1+C_2e^{-x}-\log(1+e^{-x})+e^{-x}\log(1+e^x)$

Example(5) Solve $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{e^x}$

Sol^n:- The A.E is $m^2+3m+2=0$

$\Rightarrow (m+1)(m+2)=0$

$\Rightarrow m=-1,-2$

$C.F = C_1e^{-x}+C_2e^{-2x}$

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Now, $P.I = \frac{1}{(D^2+3D+2)} e^{e^x} = \frac{1}{(D+1)(D+2)} e^{e^x}$

$P.I = \left(\frac{1}{(D+1)} - \frac{1}{(D+2)}\right) e^{e^x} = \frac{1}{(D+1)} e^{e^x} - \frac{1}{(D+2)} e^{e^x}$

$P.I = e^{-x}\int e^{x} e^{e^x} dx - e^{-2x}\int e^{2x} e^{e^x} dx$

$= e^{-x}\int e^{u} du - e^{-2x}\int u e^{u} du$, where $e^x = u$

$P.I = e^{-x} e^{u} - e^{-2x}\{(u-1)e^{u}\} = e^{-x} e^{e^x} - e^{-2x}(e^{x}-1) e^{e^x}$

$P.I = (e^{-x} - \cancel{e^{-x}} + e^{-2x}) e^{e^x} = e^{-2x} e^{e^x}$

hence, the general solution of the given D.E is

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$$

Shortcut Method of finding Particular integral:—

In the previous section, we have learnt about general Method of finding particular integral of the D.E of the type $f(D)y = Q$. The process of Computing particular integral becomes quiet easy, if the f^n $Q(x)$ assumes any of the following forms: (i) e^{ax} (ii) x^n, $n \in N$ (iii) $\sin ax$ or $\cos ax$ (iv) $e^{ax} V(x)$ (V) $x V(x)$, where V is any f^n of x.

Case-I Particular integral when $Q(x)$ is of the form e^{ax}, when $f(a) \neq 0$

$$P.I = \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)}, \text{ provided the } f(a) \neq 0$$

Thus if $Q = e^{ax}$, then particular integral can be easily obtained by the substition of a in place of D, a^2 in place of D^2 and so on in $f(D)$, provided that $f(a) \neq 0$

Example ① solve: $\frac{d^2y}{dx^2} + 31\frac{dy}{dx} + 240y = 272e^{-x}$

Solⁿ:— The A.E is $m^2 + 31m + 240 = 0$

$\Rightarrow (m+16)(m+15) = 0$

$\Rightarrow m = -15, -16$

$C.f = c_1e^{-15x} + c_2e^{-16x}$

Now, $P.I = \frac{Q}{f(D)} = \frac{272e^{-x}}{(D^2+31D+240)}$

$= 272 \frac{e^{-x}}{(-1)^2 + 31\times -1 + 240} = \frac{272e^{-x}}{210} = \frac{136e^{-x}}{105}$

hence, The general solⁿ of the differential eqⁿ is

$$y = c_1e^{-15x} + c_2e^{-16x} + \frac{136e^{-x}}{105}$$

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Example ② Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^x \cosh 2x$

Sol^n:- We have $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^x \cosh 2x$

or $(D^2+D-6)y = e^x \cosh 2x$

The A.E is $m^2+m-6=0$

$\Rightarrow (m+3)(m-2)=0$

$\Rightarrow m=-3,2$

$\therefore C.F = C_1 e^{-3x} + C_2 e^{2x}$

Now, $P.I = \frac{Q}{f(D)} = \frac{1}{(D^2+D-6)} e^x \cosh 2x$

put $\cosh 2x = \frac{e^{2x}+e^{-2x}}{2}$

$P.I = \frac{1}{(D^2+D-6)} e^x \left(\frac{e^{2x}+e^{-2x}}{2}\right) = \frac{1}{(D^2+D-6)} \frac{e^{3x}+e^{-x}}{2}$

$P.I = \frac{1}{2} \times \frac{1}{(D^2+D-6)} e^{3x} + \frac{1}{2} \times \frac{1}{(D^2+D-6)} e^{-x}$

$P.I = \frac{1}{2(9+3-6)} e^{3x} + \frac{1}{2(1-1-6)} e^{-x}$

$P.I = \frac{1}{12} e^{3x} - \frac{1}{12} e^{-x}$, hence the general sol^n of the given differential is given by

$y = C_1 e^{-3x} + C_2 e^{2x} + \frac{1}{12}(e^{3x} - e^{-x})$.

Case-II :- Particular integral when Q is of the form e^{ax} and $f(a)=0$

$P.I = \frac{e^{ax}}{f(D)}$, when $f(a)=0$, then we proceed as

$P.I = \frac{x e^{ax}}{f'(D)} = \frac{x e^{ax}}{f'(a)}$, provided $f'(a) \neq 0$

again, if $f'(a)=0$, then we proceed as

$P.I = \frac{x^2 e^{ax}}{f''(a)}$, provided $f''(a) \neq 0$, and continue this process. and we use this formula

$$P.I = \frac{e^{ax}}{f(D)} = \frac{1}{(D-a)^r \phi(D)} e^{ax} = \frac{x^r e^{ax}}{r!\, \phi(a)}$$

Example ① Solve $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = e^{4x}$

Soln :- The A.E is $m^2 - 8m + 16 = 0$

$\Rightarrow (m-4)^2 = 0$

$\Rightarrow m = 4, 4$

$\therefore C.f = (C_1 + xC_2)e^{4x}$

$$P.I = \frac{e^{4x}}{(D^2 - 8D + 16)} = \frac{e^{4x}}{(D-4)^2} = \frac{x^2 e^{4x}}{2!}$$

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Thus, the general Soln of D.E is given by

$$y = C.F + P.I = (c_1 + xc_2)e^{4x} + \frac{x^2}{2}e^{4x}$$

Example ② Solve $(D-1)^2(D^2+1)^2 y = e^x$

Solnº:- The A.E is $(m-1)^2(m^2+1)^2 = 0$

$$\Rightarrow m = 1 \text{(twice)}, \pm i \text{(twice)}$$

$$\therefore C.f = (c_1 + xc_2)e^x + (c_3 + xc_4)\cos x + (c_5 + xc_6)\sin x$$

$$P.I = \frac{e^x}{(D-1)^2(D^2+1)^2} = \frac{e^x}{(D-1)^2 4} = \frac{1}{4}\frac{e^x}{(D-1)^2} = \frac{x^2 e^x}{4 \times 2!} = \frac{x^2 e^x}{8}$$

Hence, the general Solution is given by

$$y = C.f + P.I = (c_1 + xc_2)e^x + (c_3 + xc_4)\cos x + (c_5 + xc_6)\sin x + \frac{x^2 e^x}{8}.$$

Example ③ Solve $(D^2 - a^2)y = \cosh ax$

Solnº:- AE is $m^2 - a^2 = 0$

$$m = \pm a$$

$$\therefore C.f = c_1 e^{ax} + c_2 e^{-ax}$$

Now, $P.I = \dfrac{\cosh ax}{(D^2 - a^2)} = \dfrac{1}{(D^2-a^2)}\left(\dfrac{e^{ax} + e^{-ax}}{2}\right)$

$$= \frac{1}{2}\left\{\frac{e^{ax}}{(D^2-a^2)} + \frac{e^{-ax}}{(D^2-a^2)}\right\}$$

$$P.I = \frac{1}{2}\left\{\frac{1}{(D+a)(D-a)}e^{ax} + \frac{1}{(D+a)(D-a)}e^{-ax}\right\}$$

$$P.I = \frac{1}{2}\left\{\frac{1}{2a(D-a)}e^{ax} + \frac{1}{-2a(D+a)}e^{-ax}\right\}$$

$$P.I = \frac{1}{2}\left\{\frac{1}{2a}\frac{xe^{ax}}{1!} - \frac{1}{2a}\frac{xe^{-ax}}{1!}\right\} = \frac{x}{2a}\left(\frac{e^{ax}-e^{-ax}}{2}\right)$$

$$P.I = \frac{x}{2a}\sinh ax$$

Hence, the general soln is given by $y = C.f + P.I$

$$y = c_1 e^{ax} + c_2 e^{-ax} + \frac{x}{2a}\sinh ax.$$

Example(4) solve $\frac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$

Soln:— Given D.E can be written as $(D^2-4)y = \cosh(2x-1)+3^x$

The A.E is $m^2-4=0 \Rightarrow m = \pm 2$

$\therefore$ $C.f = c_1 e^{-2x} + c_2 e^{2x}$

Now,

$$P.I = \frac{1}{(D^2-4)}\{\cosh(2x-1)+3^x\} = \frac{1}{(D^2-4)}\left\{\frac{e^{2x-1}+e^{-2x+1}}{2}\right\} + \frac{1}{(D^2-4)}e^{\log 3^x}$$

$$P.I = \frac{1}{(D^2-4)}\left\{\frac{1}{2e}e^{2x} - e^{-2x}\cdot e\right\} + \frac{1}{(D^2-4)}e^{x\log 3}$$

$$P.I = \frac{1}{2e}\frac{1}{(D^2-4)}e^{2x} - \frac{e}{2}\frac{e^{-2x}}{(D^2-4)} + \frac{1}{(\log 3)^2-4}e^{x\log 3}$$

$$P.I = \frac{1}{2e}\frac{xe^{2x}}{2D} - \frac{e}{2}\frac{xe^{-2x}}{2D} + \frac{3^x}{(\log 3)^2-4}$$

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$$= \frac{x}{4}\left(\frac{e^{2x-1} - e^{-(2x-1)}}{2}\right) + \frac{3^x}{(\log 3)^2 - 4} = \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2-4}$$

Hence, The general solⁿ is given by

$$y = C_1 e^{2x} + C_2 e^{-2x} + \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}$$

Case-III Particular integral when Q is of the form x^n, $n \in N$

$$P.I = \frac{x^n}{f(D)} = \frac{1}{(P_0 D^n + P_1 D^{n-1} + \cdots + P_{n-1}D + P_n)} x^n$$

$$P.I = \frac{1}{P_n\left(1 + \frac{P_{n-1}}{P_n}D + \cdots + \frac{P_1}{P_n}D^{n-1} + \frac{P_0}{P_n}D^n\right)} x^n$$

$$P.I = \frac{1}{P_n}\left(1 + \frac{P_{n-1}}{P_n}D + \cdots + \frac{P_0}{P_n}D^n\right)^{-1} x^n$$

Where $\left(1 + \frac{P_{n-1}}{P_n}D + \cdots + \frac{P_0}{P_n}D^n\right)^{-1}$ the term expand by using Binomial Expansion. The following expansions will be helpful in expanding $\{f(D)\}^{-1}$ in ascending power of D.

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

(ii) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

(iii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$

(iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

(v) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$

(vi) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

Example ① solve $\frac{d^2y}{dx^2} - y = 2 + 3x$

Sol^n:– The A.E is

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$\therefore$ C.f $= c_1 e^{x} + c_2 e^{-x}$

Now, $P.I = \frac{2+3x}{(D^2-1)} = \frac{2+3x}{-(1-D^2)}$

$$P.I = -(1-D^2)^{-1}(2+3x)$$

$$= -(1+D^2+D^4)(2+3x)$$

$$= -(2+3x) + 0 + 0 = -(2x+3)$$

hence, The Complete Sol^n is

$$y = c_1 e^{x} + c_2 e^{-x} - (2x+3).$$

Example ② solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = x^2 + 2e^{2x}$

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Sol^n:- The given D.E, can be written as symbolic form, is $(D^2-6D+9)y = x^2+2e^{2x}$

The A.E is $(m-3)^2=0 \Rightarrow m=3,3$

$\therefore$ C.f $= (c_1+xc_2)e^{3x}$

Now, P.I $= \dfrac{1}{(D^2-6D+9)}(x^2+2e^{2x}) = \dfrac{1}{(D-3)^2}x^2 + \dfrac{2}{(D-3)^2}e^{2x}$

$= \dfrac{1}{9(1-D/3)^2}x^2 + 2\cdot\dfrac{e^{2x}}{(2-3)^2}$

P.I $= \dfrac{1}{9}(1-D/3)^{-2}x^2 + 2e^{2x}$

P.I $= \dfrac{1}{9}\left(1+2D/3+\dfrac{3D^2}{9}+\cdots\right)x^2+2e^{2x}$

P.I $= \dfrac{1}{9}\left(x^2+\dfrac{4x}{3}+2/3\right)+2e^{2x}$

hence, the general sol^n is given by $y = C.f + P.I$

$$y = (c_1+xc_2)e^{3x} + \frac{1}{27}(3x^2+4x+2)+2e^{2x}$$

Example ② solve $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 2x^2$, given that $y(0)=0$, $y'(0)=0$.

Sol^n:- The A.E is $m^2-m-2=0$

$\Rightarrow (m-2)(m+1)=0 \Rightarrow m=2,-1$

$\therefore$ C.f $= c_1e^{2x}+c_2e^{-x}$

Now. $P.I = \frac{1}{(D^2-D-2)} 2x^2 = 2 \times \frac{1}{-2\left\{1+\frac{(D-D^2)}{2}\right\}} x^2$

$$P.I = -\left(1+\frac{(D-D^2)}{2}\right)^{-1} x^2$$

$$P.I = -\left(1-\frac{D}{2}+\frac{D^2}{2}+\frac{(D-D^2)^2}{4}+\cdots\right)x^2$$

$$P.I = -\left(1-\frac{D}{2}+\frac{D^2}{2}+\frac{(D^2+D^4-2D^3}{4}+\cdots\right)x^2$$

$$P.I = -\left(1-\frac{D}{2}+\frac{3D^2}{4}+\cdots\right)x^2 = -(x^2-x+3/2)$$

Therefore, the general Sol^n of given D.E is

$$y = c_1e^{2x}+c_2e^{-x}-x^2+x-3/2 \quad -(1)$$

diff. (1). w.r. to x, we get

$$dy/dx = 2c_1e^{2x}-c_2e^{-x}-2x+1 \quad -(2)$$

it is given that $y=0$ and $dy/dx=0$ at $x=0$ putting $x=0, y=0$ and $dy/dx=0$ in eq^n (1) and (2), we get

$$c_1+c_2-3/2=0 \quad \text{and} \quad 2c_1-c_2+1=0$$

on solving these two eq^n, we get $c_1=1/6$ and $c_2=4/3$

Substituting these value of c_1 and c_2 in eq^n (1), we obtain

$y = \frac{1}{6}e^{2x}+4/3\,e^{-x}-x^2+x-3/2$ is the required Sol^n.

Case-IV :- Particular Integral when Q is of the form $\sin ax$ or $\cos ax$

$$P.I = \frac{1}{P(D^2)}\sin ax = \frac{1}{P(-a^2)}\sin ax$$

$$P.I = \frac{\cos ax}{P(D^2)} = \frac{1}{P(-a^2)}\cos ax$$ Simply replace D^2 by $-a^2$.

The above Method fail when $P(-a^2)=0$ and in such a case, we proceed as follows:

$$\frac{1}{(D^2+a^2)}\cos ax = \frac{x\sin ax}{2a}$$

$$\frac{1}{(D^2+a^2)}\sin ax = -\frac{x\cos ax}{2a}$$

Example ① solve $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\cos 2x$

Soln :- The given differential eqn, in symbolic form, is

$$(D^2+D+1)y=\cos 2x$$

The A.E is $m^2+m+1=0$

$$m=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

$$\therefore C.f = e^{-x/2}\left(C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right)$$

Now,

$$P.I = \frac{1}{(D^2+D+1)}\cos 2x,$$ replace $D^2=-4=-(2)^2$

$$P.I = \frac{1}{-4+D+1}\cos 2x = \frac{1}{(D-3)}\cos 2x = \frac{(D+3)\cos 2x}{D^2-9} = -\frac{(D+3)\cos 2x}{13}$$

$P.I = -\frac{1}{13}(-2\sin 2x + 3\cos 2x$

hence, the general Sol^n is given by

$$y = e^{-x/2}\left\{c_1\cos\frac{\sqrt{3}x}{2} + c_2\sin\frac{\sqrt{3}}{2}x\right\} - \frac{1}{13}(-2\sin 2x + 3\cos 2x)$$

Example ② Solve $(D^2+D-2)y = x + \sin x$

Sol^n:— The A.E is $m^2+m-2=0$

$$\Rightarrow (m+2)(m-1)=0$$

$$\therefore C.f = c_1e^{-2x} + c_2e^{x}$$

Now,

$$P.I = \frac{1}{(D^2+D-2)}(x+\sin x) = \frac{1}{(D^2+D-2)}x + \frac{1}{(D^2+D-2)}\sin x$$

$$P.I = \frac{1}{-2\left\{1-\frac{(D+D^2)}{2}\right\}}x + \frac{1}{-1+D-2}\sin x$$

$$P.I = -\frac{1}{2}\left\{1-\frac{(D+D^2)}{2}\right\}^{-1}x + \frac{1}{(D-3)}\sin x$$

$$P.I = -\frac{1}{2}\left(1+\frac{D+D^2}{2}+\cdots\right\}x + \frac{(D+3)}{D^2-9}\sin x$$

$$P.I = -\frac{1}{2}\left\{1+\frac{D}{2}+\frac{D^2}{2}+\cdots\right\}x + \frac{1}{-10}(\cos x + 3\sin x$$

$$P.I = -\frac{1}{2}\left\{x+\frac{1}{2}\right\} - \frac{1}{10}(\cos x + 3\sin x) = -\frac{1}{4}(2x+1) - \frac{1}{10}(\cos x + 3\sin x)$$

hence, The general Sol^n is given by

$$y = C.f + P.I = c_1e^{-2x} + c_2e^{x} - \frac{1}{4}(2x+1) - \frac{1}{10}(\cos x + 3\sin x)$$

Example(3) Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} + \cos^2 x$

Solnº:– The A.E is $m^2+2m+1=0$

$\Rightarrow (m+1)^2 = 0$

$\Rightarrow m = -1, -1$

$C.f = (c_1 + x c_2)e^{-x}$

Now,

$P.I = \frac{1}{(D+1)^2}(e^{2x} + \cos^2 x) = \frac{1}{(D+1)^2} e^{2x} + \frac{1}{(D+1)^2} \cos^2 x$

$P.I = \frac{1}{9} e^{2x} + \frac{1}{(D+1)^2}\left(\frac{1+\cos 2x}{2}\right) = \frac{1}{9} e^{2x} + \frac{1}{2} \times \frac{1}{(D+1)^2}(1+\cos 2x)$

$P.I = \frac{1}{9} e^{2x} + \frac{1}{2}\frac{1}{(D+1)} \cdot 1 + \frac{1}{2(D+1)^2} \cos 2x$

$P.I = \frac{1}{9} e^{2x} + \frac{1}{2}\frac{1}{(D+1)^2} e^{0x} + \frac{1}{2(D^2+2D+1)} \cos 2x$

$P.I = \frac{1}{9} e^{2x} + \frac{1}{2(0+1)^2} e^{0x} + \frac{1}{2(-4+2D+1)} \cos 2x$

$P.I = \frac{1}{9} e^{2x} + \frac{1}{2} + \frac{1}{2(2D-3)} \cos 2x$

$P.I = \frac{1}{9} e^{2x} + \frac{1}{2} + \frac{1}{2}\frac{(2D+3)\cos 2x}{4D^2-9} = \frac{1}{9} e^{2x} + \frac{1}{2}\frac{(2D+3)\cos 2x}{-25}$

$P.I = \frac{1}{9} e^{2x} + \frac{1}{2} - \frac{1}{50}(-4\sin 2x + 3\cos 2x)$

Hence, the general soln is given by

$y = (c_1 + x c_2)e^{-x} + \frac{e^{2x}}{9} + \frac{1}{2} - \frac{1}{50}(-4\sin 2x + 3\cos 2x)$

Example(4) Solve $\frac{d^2y}{dx^2}+2y=\sin 2x+3+5x$

Soln:- The A.E is $m^2+2=0$

$\Rightarrow m=\pm\sqrt{2}i$

$\therefore C.F = C_1\cos\sqrt{2}x+C_2\sin\sqrt{2}x$

Now,

$P.I=\frac{1}{(D^2+2)}(\sin 2x+3+5x) = \frac{1}{(D^2+2)}\sin 2x+\frac{3}{(D^2+2)}e^{0x}+\frac{5}{(D^2+2)}\cdot x$

$P.I=\frac{1}{-4+2}\sin 2x+\frac{3\cdot e^{0x}}{0+2}+5\frac{1}{2(1+\frac{D^2}{2})}x$

$P.I=-\frac{1}{2}\sin 2x+3/2+\frac{5}{2}(1+\frac{D^2}{2})^{-1}x$

$P.I=-\frac{1}{2}\sin 2x+3/2+\frac{5}{2}(1-\frac{D^2}{2}+\dots)x$

$P.I=-\frac{1}{2}\sin 2x+3/2+\frac{5}{2}x$

Hence, the general soln is given by

$y=C_1\cos\sqrt{2}x+C_2\sin\sqrt{2}x-\frac{1}{2}\sin 2x+3/2+\frac{5}{2}x.$

Example(5) Solve $\frac{d^2y}{dx^2}+y=\sin 3x\cos 2x$

Soln:- The A.E is $m^2+1=0$

$\Rightarrow m=\pm i$

$\therefore C.F = C_1\cos x+C_2\sin x$

Now,

$P.I=\frac{1}{(D^2+1)}\sin 3x\cos 2x=\frac{1}{2}\times\frac{1}{(D^2+1)}2\sin 3x\cos 2x$

$$P.I = \frac{1}{2(D^2+1)} \sin 5x + \sin x = \frac{1}{2} \times \left\{ \frac{1}{(D^2+1)} \sin 5x + \frac{1}{(D^2+1)} \sin x \right\}$$

$$P.I = \frac{1}{2} \times \left\{ \frac{1}{-25+1} \sin 5x + \frac{x \cos x}{2} \right\} \quad \because \frac{1}{(D^2+a^2)} \cos ax = \frac{-x \sin ax}{2a}$$

$$P.I = -\frac{1}{48} \sin 5x - \frac{1}{4} x \cos x \qquad \frac{1}{(D^2+a^2)} \sin ax = \frac{-x \cos ax}{2a}$$

Example ⑤ solve $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$

Solnᵒ:– The A.E is $m^2 + 4 = 0$

$$\Rightarrow m^2 = 4i^2$$

$$\Rightarrow m = \pm 2i$$

$$\therefore C.F = C_1 \cos 2x + C_2 \sin 2x$$

Now,

$$P.I = \frac{1}{(D^2+4)} e^x + \frac{1}{(D^2+4)} \sin 2x$$

$$P.I = \frac{1 e^x}{1+4} + \frac{x \cos 2x}{4} \qquad \because \frac{1}{(D^2+a^2)} \sin ax = -\frac{x \cos ax}{2a}$$

$$P.I = \frac{1 e^x}{5} - \frac{x}{4} \cos 2x$$

Hence, The Complete Solⁿ is given by

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{e^x}{5} - \frac{x}{4} \cos 2x$$

Case–II Particular integral when Q is of the form $e^{ax}V(x)$, where V is any function of x, then P.I is

$$P.I = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V(x)$$

Example ① Solve $(D^2+2)y = x^2 e^{3x}$

Soln:– The A.E is $m^2+2=0$

$\Rightarrow m = \pm i\sqrt{2}$

$\therefore C.f = C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x$

Now

$$P.I = \frac{1}{(D^2+2)} x^2 e^{3x} = e^{3x} \frac{1}{(D+3)^2+2} x^2 = e^{3x} \frac{1}{(D^2+6D+11)} x^2$$

$$P.I = \frac{e^{3x}}{11} \frac{1}{\left(1+\frac{(6D+D^2)}{11}\right)} x^2 = \frac{e^{3x}}{11}\left(1+\frac{(6D+D^2)}{11}\right)^{-1} x^2$$

$$P.I = \frac{e^{3x}}{11}\left(1-\frac{(6D+D^2)}{11}+\left(\frac{6D+D^2}{11}\right)^2+\cdots\right)x^2$$

$$P.I = \frac{e^{3x}}{11}\left\{1-\frac{6D}{11}+\frac{25}{121}D^2+\cdots\right\}x^2$$

$$P.I = \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$$

Hence, The general solution of given D.E is

$$y = C_1\cos\sqrt{2}x + C_2\sin\sqrt{2}x + \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$$

Example (2) Solve $\frac{d^2y}{dx^2}+2y = x^2e^{3x}+e^x\cos x$

Soln:- The A.E is $m^2+2=0$

$\Rightarrow m = \pm\sqrt{2}i$

$\therefore C.F = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x$

Now,

$P.I = \frac{1}{(D^2+2)}(x^2e^{3x}+e^x\cos x) = \frac{1}{(D^2+2)}x^2e^{3x} + \frac{1}{(D^2+2)}e^x\cos x$

$P.I = \frac{e^{3x}}{11}\left(x^2-\frac{12}{11}x+\frac{50}{121}\right) + e^x\frac{1}{(D+1)^2+2}\cos x$

$P.I = \frac{e^{3x}}{11}\left(x^2-\frac{12}{11}x+\frac{50}{121}\right) + e^x\frac{1}{D^2+1+2D+2}\cos x$

$P.I = \frac{e^{3x}}{11}\left(x^2-\frac{12}{11}x+\frac{50}{121}\right) + e^x\frac{1}{D^2+2D+3}\cos x$

$P.I = \frac{e^{3x}}{11}\left(x^2-\frac{12}{11}x+\frac{50}{121}\right) + e^x\frac{1}{(-1+2D+3)}\cos x$

$P.I = \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right) + \frac{e^x}{2}\frac{(D-1)}{(D+1)(D-1)}\cos x$

$P.I = \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right. + \frac{e^x}{2}\frac{(D-1)\cos x}{D^2-1}$

$P.I = \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right) + \frac{e^x}{2}\times -\frac{1}{2}(\sin x-\cos x)$

$P.I = \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{21}\right) + \frac{e^x}{4}(\sin x+\cos x)$

Hence, the general soln is given by

$y = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right) + \frac{e^x}{4}(\sin x+\cos x)$

Case-VI Particular integral when Q is of the form xV, where V is any function of x, then P.I is

$$P.I = \frac{xV}{f(D)} = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V$$

Example ① solve $(D^2-2D+1)y = x\sin x$

Soln:- The A.E is $m^2-2m+1=0$

$\Rightarrow m=1,1$

$\therefore C.f = (C_1+xC_2)e^x$

Now, $P.I = \frac{1}{(D^2-2D+1)}x\sin x = x\frac{1}{(D^2-2D+1)}\sin x - \frac{(2D-2)}{(D^2-2D+1)^2}\sin x$

$P.I = x\frac{1}{-2D}\sin x - \frac{2(D-1)\sin x}{4_2 D^2}$

$P.I = \frac{x}{2}\cos x + \frac{(\sin x-\cos x)}{D^2} = \frac{x}{2}\cos x + \frac{1}{2}\frac{(-\cos x-\sin x)}{D}$

$P.I = \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$

Hence, the general soln is given by

$$y = (C_1+xC_2)e^x + \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$$

Example ② solve $(D^2-2D+1)y = xe^x\sin x$

Soln:- The A.E is $(m-1)^2=0 \Rightarrow m=1,1$

$\therefore C.f = (C_1+xC_2)e^x$

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Now,

$$P.I = \frac{1}{(D^2-2D+1)} xe^x \sin x = \frac{1}{(D-1)^2} xe^x \sin x = e^x \frac{1}{(D+1-1)^2} x\sin x$$

$$P.I = e^x \frac{1}{D^2} x\sin x = e^x \frac{1}{D}\left(\int x\sin x\,dx\right) = e^x \frac{1}{D}(-x\cos x + \sin x)$$

$$P.I = e^x \int(-x\cos x + \sin x)dx = e^x(-x\sin x - 2\cos x)$$

Hence, The general solⁿ is

$$y = (c_1 + xc_2)e^x - e^x(x\sin x + 2\cos x)$$

Example ③ Solve $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$

Solⁿ:- The A.E is $m^2-4m+4=0$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m = 2, 2$$

$$\therefore C.F = (c_1 + xc_2)e^{2x}$$

Now, $P.I = \frac{1}{(D-2)^2} 8x^2e^{2x}\sin 2x = 8e^{2x}\frac{1}{(D+2-2)^2} x^2\sin 2x$

$$P.I = 8e^{2x}\frac{1}{D^2}x^2\sin 2x = 8e^{2x}\frac{1}{D}\left\{\int x^2\sin 2x\,dx\right\}$$

$$P.I = 8e^{2x}\frac{1}{D}\left\{-\frac{x^2\cos 2x}{2} + \int x\cos 2x\,dx\right\}$$

$$P.I = 8e^{2x}\frac{1}{D}\left\{-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right\}$$

$$P.I = 8e^{2x}\int\left(-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right)dx$$

$$P.I = 8e^{2x}\left\{-\frac{x^2}{4}\sin 2x - \frac{x}{4}\cos 2x + \frac{1}{8}\sin 2x - \frac{x}{4}\cos 2x + \frac{1}{8}\sin 2x + \frac{1}{8}\sin 2x\right\}$$

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$$P.I = 8e^{2x}\left\{-\frac{x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{3}{8}\sin 2x\right\}$$

Hence, The general Sol^n is

$$y = (c_1 + xc_2)e^{2x} + 8e^{2x}\left\{-\frac{x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{3}{8}\sin 2x\right\}.$$

Example(4) Solve $(D^4 + 2D^2 + 1)y = x^2\cos x$

Sol^n:- The A.E is $m^4 + 2m^2 + 1 = 0$

$$\Rightarrow m^4 + m^2 + m^2 + 1 = 0$$

$$(m^2+1)^2 = 0$$

$$\Rightarrow m = \pm i \text{(twice)}$$

$$\therefore C.f = (c_1 + xc_2)\cos x + (c_3 + xc_4)\sin x$$

Now,

$$P.I = \frac{1}{(D^2+1)^2}x^2\cos x = \text{Real Part of } \frac{x^2e^{ix}}{(D^2+1)^2}$$

$$P.I = e^{ix}\frac{1}{((D+i)^2+1)^2}x^2 = e^{ix}\frac{1}{(D^2+2iD)^2}x^2$$

$$P.I = e^{ix}\frac{1}{(2iD)^2}\left(1+\frac{D}{2i}\right)^{-2}x^2$$

$$P.I = -\frac{1}{4}e^{ix}\frac{1}{D^2}\left\{1 - \frac{D}{i} + \frac{3}{4}\frac{D^2}{i^2} - \dots\right\}x^2$$

$$P.I = -\frac{1}{4}e^{ix}\frac{1}{D^2}\left\{x^2 + 2ix - 3/2\right\} = -\frac{1}{4}e^{ix}\left\{\frac{x^4}{12} - \frac{3}{4}x^2 + \frac{x^3}{3}i\right\}$$

$$P.I = -\frac{1}{4}(\cos x + i\sin x)\left\{\frac{x^4}{12} - \frac{3}{4}x^2 + \frac{x^3}{3}i\right\}$$

$$P.I = \text{Real Parts of } \frac{1}{(D^2+1)^2}x^2e^{ix} = -\frac{1}{4}\left\{\left(\frac{x^4 - 3x^2}{12}\right)\cos x - \frac{x^3}{3}\sin x\right\}$$

$$\Rightarrow P.I = -\frac{1}{4}\left\{\left(\frac{x^4-9x^2}{12}\right)\cos x - \frac{x^3}{3}\sin x\right\}$$

Hence, the general sol^n is given by

$$y = (c_1 + x c_2)\cos x + (c_3 + x c_4)\sin x - \frac{1}{4}\left\{\left(\frac{x^4-9x^2}{12}\right)\cos x - \frac{x^3}{3}\sin x\right\}.$$

Cauchy-Euler equation:— An equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1 x\frac{dy}{dx} + a_0 y = Q(x) \quad \text{——— (1)}$$

Where $a_0, a_1, \dots, a_n$ are Constant, is Called a Cauchy-Euler equation of order n. To know the Solution of such an equation, we make some suitable substitution so that eq^n (1) may reduce to an equation for which the Methods of sol^n are known. such a substitution is $x = e^t$. This substitution reduces eq^n (1) to a linear eq^n with Constant Coefficients. To see how this Can be done.

take $x = e^t$ Then $\log x = t$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dt} \Rightarrow xDy = D_1 y \quad \text{Where } D = d/dx,\ D_1 = \frac{d}{dt}$$

Similarly $$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$\Rightarrow x^2D^2y = D_1(D_1-1)$$

and replace

$$x^3D^3y = D_1(D_1-1)(D_1-2)$$

$$- \; - \; - \; - \; -$$

$$x^nD^ny = D_1(D_1-1)(D_1-2)\ldots(D_1-(n-1))y$$

after substituting these value in eq^n ① then eq^n ① reduce to a linear differential with constant coefficient. And can be solved by previous method.

Example ① Solve $(x^2D^2 + xD - 4)y = x^2$

Sol^n :- putting $x = e^t$ or $\log x = t$ and $D_1 = d/dt$ the given D.E can be written as

$$[D_1(D_1-1) + D_1 - 4]y = e^{2t}$$

$$\Rightarrow (D_1^2 - 4) = e^{2t}$$

here, A.E is $m^2 - 4 = 0$

$$\Rightarrow m = \pm 2$$

$$\therefore C.F = c_1e^{2t} + c_2e^{-2t} = c_1x^2 + c_2x^{-2}$$

Now,

$$P.I = \frac{1}{(D_1^2-4)}e^{2t} = \frac{e^{2t}}{(D_1+2)(D_1-2)} = \frac{e^{2t}}{4(D_1-2)} = \frac{te^{2t}}{4}$$

$$P.I = x^2\log x/4$$

Hence, The required Sol^n is

$$y = y_c + y_p = c_1x^2 + c_2x^2 + \frac{1}{4}x^2\log x$$

Example ② Solve $(x^2D^2 - xD - 3)y = x^2\log x$

Sol^n:- Putting $x = e^t$ or $\log x = t$ and $D_1 = d/dt$
the given eq^n becomes

$$(D_1^2 - 2D_1 - 3)y = te^{2t}$$

The A.E is $m^2 - 2m - 3 = 0$

$$m^2 + 3m + m - 3 = 0$$

$$\Rightarrow m = 3, -1$$

$$C.f = y_c = c_1e^{-t} + c_2e^{3t} = c_1x^{-1} + c_2x^3$$

Now,

$$P.I = \frac{1}{(D_1^2 - 2D_1 - 3)} te^{2t} = e^{2t}\frac{t}{(D_1+2)^2 - 2(D_1+2) - 3}$$

$$P.I = \frac{e^{2t}}{-3}\left(1 - \frac{2}{3}D_1 - \frac{1}{3}D_1^2\right)^{-1} t = -\frac{1}{3}e^{2t}\left(1 + \frac{2}{3}D_1 + \cdots\right)t$$

$$P.I = -\frac{1}{3}e^{2t}(t + 2/3) = -\frac{1}{3}x^2(\log x + 2/3)$$

hence, The required Sol^n is

$$y = y_c + y_p = c_1x^{-1} + c_2x^3 - \frac{x^2}{3}(\log x + 2/3)$$

Legendre's linear equation:— An equation of the form

$$k_n(ax+b)^n\frac{d^ny}{dx^n}+k_{n-1}(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+k_0y=Q(x) \quad —(1)$$

Where $k_0, k_1, \ldots, k_n$ are constants and $Q(x)$ is a f^n of x, is called Legendre's linear equation.

Such eq^n can be reduced to linear eq^n with constant coefficients by the substitution

$(ax+b)=e^t$, $t=\log(ax+b)$ and

$(ax+b)Dy = aD_1y$, where $D=d/dx$ & $D_1=d/dt$

$(ax+b)^2D^2y=a^2D_1(D_1-1)y$

Similarly $(ax+b)^3D^3y = a^3D_1(D_1-1)(D_1-2)y$

and so on.

by making these replacements in eq^n (1), we get a linear D.E with constant coefficient.

Example (1) solve $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=2\sin[\log(1+x)]$

Soln:— put $1+x=e^t$, $t=\log(1+x)$ and $D_1=d/dt$ Then given D.E becomes $(D_1^2+1)y=2\sin t$

which is a linear D.E eqn with Constant Coefficient,

The A.E is $m^2+1=0$

$\Rightarrow m = \pm i$

$\therefore$ C.F $= c_1\cos t + c_2\sin t = c_1\cos\log(1+x) + c_2\sin\log(1+x)$

Now,

$$P.I = \frac{2\sin t}{(D_1^2+1)} = -\frac{2t\cos t}{2} = -\log(1+x)\cos\log(1+x)$$

Therefore, The Soln of the given eqn is

$$y = c_1\cos\log(1+x) + c_2\sin\log(1+x) - \log(1+x)\cos\log(1+x).$$

Simultaneous differential equation:-

Solution of a system of linear equations with Constant Coefficient:-

Definition:- The system of equation

$$\left.\begin{aligned}
&P_{11}(D)y_1 + P_{12}(D)y_2 + \cdots + P_{1n}(D)y_n = Q_1(t)\\
&P_{21}(D)y_1 + P_{22}(D)y_2 + \cdots + P_{2n}(D)y_n = Q_2(t)\\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
&P_{n1}(D)y_1 + P_{n2}(D)y_2 + \cdots P_{nn}(D)y_n = Q_n(t)
\end{aligned}\right\} \quad \text{--- } (1)$$

Where $D = d/dt$ and the Coefficient of $y_1, y_2, \dots, y_n$ are polynomial operator in D, is called a system of n-linear differential eq^n. A Solution of eq^n ① is a set of f^n $y_1(t), y_2(t), \dots, y_n(t)$, each defined on a Common interval I, Satisfying all equations of eq^n ① identically. The Solution is general if the set of f^n $y_1(t), y_2(t), \dots, y_n(t)$ Contain the Correct number of Constants by the following theorem.

Theorem 1:— Consider the pair of equations

$$\left.\begin{array}{l} P_1(D)x + P_2(D)y = q_1(t) \\ P_3(D)x + P_4(D)y = q_2(t) \end{array}\right\} \quad \text{—— ②}$$

The number of arbitrary Constants in the general Solution $x(t), y(t)$ of eq^n ② is equal to the order of

$$\begin{vmatrix} P_1(D) & P_2(D) \\ P_3(D) & P_4(D) \end{vmatrix} = P_1(D)P_4(D) - P_2(D)P_3(D)$$

provided $P_1(D)P_4(D) - P_2(D)P_3(D) \neq 0$

Example ① Solve $2\frac{dx}{dt} - x + \frac{dy}{dt} + 4y = 1$,

$$\frac{dx}{dt} - \frac{dy}{dt} = t - 1$$

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Solution:- given system of above eq^n can be written as $(2D-1)x + (D+4)y = 1$ — ①

$Dx - Dy = t-1$, where $d/dt = D$ — ②

Multiplying the first eq^n by D, the second by $(D+4)$, and adding them, we get

$$(3D^2+3D)x = 4t-3$$

OR $(D^2+D)x = 4/3\,t - 1$

The general sol^n of above eq^n is

$$x(t) = c_1 + c_2 e^{-t} + 2/3\,t^2 - 7/3\,t \quad -(3)$$

substituting the value of eq^n(3) in eq^{n}②, we get

$$Dy = -c_2 e^{-t} + t/3 - 4/3$$

then $y(t) = \dfrac{-c_2 e^{-t} + t/3 - 4/3}{D}$

$$y(t) = c_2 e^{-t} + \frac{t^2}{6} - 4/3\,t + c_3 \quad -(4)$$

here, $\begin{vmatrix} 2D-1 & D+4 \\ D & -D \end{vmatrix} = -3D^2 - 3D$, which is of order 2

hence by theorem①, the number of constant appering in the general solutions must be two. but from eq^n

eqⁿ③ and (4), we have three Constant. To find the relation b/w these three Constants, we use the fact that the Solution of a system of equation is a set of functions, which Satisfies each eqⁿ of the system identically. Now substituting, the value of $x(t)$ and $y(t)$ in eqⁿ ①, we get

$$(2D-1)\left(c_1+c_2e^{-t}+\tfrac{2}{3}t^2-\tfrac{7}{3}t\right)+(D+4)\left(c_2e^{-t}+t^2/6-\tfrac{4}{3}t+c_3\right) = 1 \quad \text{—⑤}$$

performing the indicated operation in eqⁿ ⑤, we obtain

$$c_3 = \frac{c_1+7}{4}$$

putting above value in ④, we get

$$y(t) = c_2e^{-t}+\frac{t^2}{6}-\tfrac{4}{3}t+\frac{c_1+7}{4} \quad \text{—⑥}$$

The Pair of fⁿ $x(t)$, $y(t)$ defined by eqⁿ③ and eqⁿ⑥, Contain only two arbitrary Constants and is the general solⁿ of eqⁿ ① and ②.

Example ② Solve

$$\left.\begin{aligned}(D-1)x+(D+1)y&=0\\ (2D+2)x+(2D-2)y&=t\end{aligned}\right\} \quad \text{—①}$$

Solⁿ:- Multiplying the first eqⁿ in ① by $(2D+2)$ and second

y $-(D-1)$ and then adding the both eqⁿ, we get

$$Dy = t/8 - 1/8$$

Which has a solution of the form

$$y(t) = \frac{t^2}{16} - t/8 + c_1 \quad \text{—— (A)}$$

Substituting the value of $y(t)$ in the first eqⁿ of (1), we get

$$(D-1)x = -\frac{t^2}{16} + 1/8 - c_1 \quad \text{—— (2)}$$

A.E is

$$m-1=0$$

$$\Rightarrow m=1$$

$$C.f = c_2 e^t$$

$$P.I = \frac{1}{(D-1)}\left(1/8 - t^2/16 - c_1\right)$$

$$= -(1-D)^{-1}\left(1/8 - t^2/16 - c_1\right)$$

$$= -(1+D+D^2+\cdots)\left(1/8 - t^2/16 - c_1\right)$$

$$= -\left[1/8 - t^2/16 - c_1 - t/8 - 1/8\right]$$

$$= t^2/16 + c_1 + t/8$$

Then the general solⁿ of (2) is

$$x(t) = t^2/16 + c_1 + t/8 + c_2 e^t \quad \text{—— (B)}$$

The determinant of eqⁿ (1) is $-8D$, which is of order one.

hence, pair of function eqⁿ (A) and (B)

Should have only one Constant, but it has two Constant. To obtain, a relationship b/w the two, substitute the value of (A) and (B) in the second eqⁿ of (1), we get

$$2(D+1)\left(\frac{t^2}{16}+t/8+C_1+C_2e^{t}\right)+2(D-1)\left(\frac{t^2}{16}-t/8+C_1\right)=t$$

Which Simplifies to

$$t/2+2C_2e^{t}=t/2$$

$$\Rightarrow C_2=0,\ e^{t}\neq 0$$

Putting this value in eqⁿ(B), we get

$$x(t)=\frac{t^2}{16}+t/8+C_1.$$

Example ③
H.W

$$(D+3)x+(D+1)y=e^{t}$$

$$(D+1)x+(D-1)y=t$$

Example ④
H.W

$$(3D^2+3D)x=4t-3$$

$$(D-1)x-D^2y=t^2$$

Method of Variation of Parameter :- ① This Method is used to find the Particular integral of second order non-homogeneous L.D.E with Constant Coefficient, whose C.F is known.

② This Method is applicable to eqⁿ of the form

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=Q(x) \quad \text{—— (1)}$$

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where p, q and Q(x) are f^n of x. Then we use the following formula to find Particular by this Method

$$P.I = -y_1\int \frac{y_2 Q(x)}{W}dx + y_2\int \frac{y_1 Q(x)}{W}dx$$

where y_1 and y_2 are the two L.I Solution of eq^n ①.

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called Wronskian of y_1, y_2.

Example ① Solve by M.V.P $\frac{d^2y}{dx^2} + a^2y = \operatorname{Cosec} ax$

Sol^n:- The A.E is $m^2 + a^2 = 0 \Rightarrow m = \pm ai$

$\therefore$ C.F $= C_1 \cos ax + C_2 \sin ax$

and $W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a \neq 0$

then $P.I = -y_1\int \frac{y_2 Q(x)}{W}dx + y_2\int \frac{y_1 Q(x)}{W}dx$

$$P.I = -\cos ax\int \frac{\sin ax \operatorname{Cosec} ax}{a}dx + \sin ax\int \frac{\cos ax \operatorname{Cosec} ax}{a}dx$$

$$P.I = -\frac{x}{a}\cos ax + \frac{\sin ax}{a^2}\log(\sin ax)$$

hence, the general sol^n of given D.E is given by

$$y = C_1\cos ax + C_2 \sin ax - \frac{x}{a}\cos ax + \frac{\sin ax}{a^2}\log \sin ax$$

Example ② Solve $(D^2 - 3D + 2)y = \sin e^{-x}$

Solⁿ:- The A.E is $m^2-3m+2=0$

$\Rightarrow (m-1)(m-2)=0$

$\Rightarrow m=1,2$

$y_c = c_1e^x + c_2e^{2x}$ here $y_1 = e^x$, $y_2 = e^{2x}$

and $|W| = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$ then P.I is

$$P.I = -y_1\int \frac{y_2Q(x)}{|W|}dx + y_2\int \frac{y_1Q(x)}{|W|}dx$$

$$= -e^x\int \frac{e^{2x}\sin e^{-x}}{e^{3x}}dx + e^{2x}\int \frac{e^x \sin e^{-x}}{e^{3x}}dx$$

$$= -e^x\int e^{-x}\sin e^{-x}dx + e^{2x}\int e^{-2x}\sin e^{-x}dx$$

put $e^{-x} = t \Rightarrow -e^{-x}dx = dt$

$$= e^x\int \sin t\,dt - e^{2x}\int t\sin t\,dt$$

$$= -e^x\cos e^{-x} - e^{2x}[-e^{-x}\cos e^{-x} + \sin e^{-x}]$$

$$P.I = -\cancel{e^x\cos e^{-x}} + \cancel{e^x\cos e^{-x}} - e^{2x}\sin e^{-x} = -e^{2x}\sin e^{-x}$$

Hence, The general Solⁿ is $y = y_c + y_p$

$$y = c_1e^x + c_2e^{2x} - e^{2x}\sin e^{-x}$$