# BOISE STATE UNIVERSITY

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**Title: Computing Artifact README**
**Topic: Convergence study of the Snow transport using Garlekin Methods**

### The Einstein summation equation for moisture in the Snow

From the Einstein summation equation represented in one of the literatures reviewed, we will discuss, three cases (process) that occurs in the snow transport systems in the suspension layer using Garlekin method.

Suspension mode is the dominant mode of snow transport, which occurs when falling snow or snow that has already accumulated is picked up and blown by strong winds.

We have three process/cases of study namely:

1. The transport equation
$$\frac{\partial q_L}{\partial t} + \mathbf{u}\frac{\partial q_L}{\partial x} = 0 \tag{1}$$
2. The advection diffusion process
$$\frac{\partial q_L}{\partial t} + \mathbf{u}\frac{\partial q_L}{\partial x} = \nu\frac{\partial^2 q_L}{\partial x^2} \tag{2}$$
3. The advection diffusion process with snow rate term.
$$\frac{\partial q_L}{\partial t} + \mathbf{u}\frac{\partial q_L}{\partial x} = \nu\frac{\partial^2 q_L}{\partial x^2} + C_f q_L \tag{3}$$

### Definition of terms

- $\mathbf{u}$ denote the velocity vector measured in $ms^{-1}$,
- The constant $\nu$ is given by $\frac{\mu_T}{\sigma_T}$, measured in $m^2s^{-1}$,
- $\sigma_T$ the effective Prandtl number of the fully turbulent fluid = 0.9.
- $\mu_T$ is the turbulence kinematic viscosity, measured in $m^2s^{-1}$ and it is given by

$$\mu_T = c_\mu \rho \frac{e^2}{\epsilon},$$

  where;

  - $c_\mu$ is an empirical constant - model coefficient for the turbulent viscosity = 0.09

  - $\rho$ is the density of air $kg/m^3$

- $e$ is the instantaneous turbulence kinetic energy $m^2 s^{-2}$,

$$e = \frac{3}{2}(\mathbf{u}I)^2,$$

- $I$ is the initial turbulence intensity in [%]

$$I = 0.16 Re^{-\frac{1}{8}},$$

  - $Re$ is the turbulent Reynold's number = $10^6$

- $\epsilon$ is the dissipation rate $m^2 s^{-3}$,

$$\epsilon = c_{\mu}^{\frac{3}{4}} e^{\frac{3}{2}} l^{-1},$$

  - $l$ is the turbulence or eddy length scale

- The turbulent length scale in meters can be estimated as

$$l = 0.07L,$$

- $L$ is a characteristic length measured in meters,
- $q_L$ is the humidity (non-vapor part), measured by $kg_{water} kg_{air}^{-1}$,
- $\mathbf{w}_f$ is the free-fall velocity of a particle expressed in $ms^{-1}$,
- $C_f$ is the rate of snow fall, that depends on the absolute velocity of a grain $\mathbf{w}_f$.

$$C_f = \frac{\mathbf{w}_f}{l}$$

  The following conditions hold for every values of $C_f$;

  1. if $C_f > 0$ i.e positive - upward movement of snow from the ground to the atmosphere,
  2. if $C_f < 0$ i.e negative - downward movement of snow to the ground from the atmosphere.

**Continuous Garlekin Method**

The $1D$ advection equation in flux form is

$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = 0, \tag{4}$$

where $f = qu$.

**Transformation in the Garlekin Method Machinery**

We follow these two steps basically;

- multiply the equation by the test function,

- integrate over the domain $\Omega_e$

We present the equation 4 above in the garlekin method stated earlier.

$$\int_{\Omega_e} \psi_i \left( \frac{\partial q_N^{(e)}}{\partial t} + \frac{\partial f_N^{(e)}}{\partial x} \right) d\Omega_e = \mathbf{0} \tag{5}$$

where $\psi$ is the basis function with $i = 0, \cdots, N$, and $\mathbf{0}$ is the zero vector with indices $i = 0, \cdots, N$.

Expanding the variables inside the element and using nodal approximation for N.

$$q_N^{(e)}(x,t) = \sum_{j=0}^{N} q_j^{(e)}(t)\psi_j(x), \quad f_N^{(e)} = f(q_N^{(e)}) = uq_N^{(e)}$$

where $f_N^{(e)}(x,t) = \sum_{j=0}^{N} f_j^{(e)}(t)\psi_j(x)$

Rewriting the equation 5 in terms of the element expansion given above:

$$\int_{\Omega_e} \psi_i \sum_{j=0}^{N} \psi_j \frac{dq_j^{(e)}}{dt} d\Omega_e + \int_{\Omega_e} \psi_i \sum_{j=0}^{N} \frac{d\psi_j}{dx} f_j^{(e)} d\Omega_e = \mathbf{0} \tag{6}$$

The matrix form of equation 6 is written as

$$M_{ij}^{(e)} \frac{dq^{(e)}}{dt} + D_{ij}^{(e)} f_j^{(e)} = \mathbf{0} \tag{7}$$

where the mass matrix is given as

$$M_{ij}^{(e)} = \int_{\Omega_e} \psi_i \psi_j d\Omega_e,$$

and the differentiation matrix

$$D_{ij}^{(e)} = \int_{\Omega_e} \psi_i \frac{d\psi_j}{dx} d\Omega_e$$

**DSS Form**

$$M_{IJ} \frac{dq_J}{dt} + D_{IJ} f_J = \mathbf{0} \tag{8}$$

where $I, J = 1, \cdots, N_P$, $\quad N_P$ is the number of points.

**Initial Conditions**

The initial conditions prescribed for the cases mentioned above are the following:

1. $q(x,0) = e^{-64x^2}$
2. $q(x,0) = \sin 2\pi x e^{-\frac{ux}{2v}} e^{-\left(\frac{u^2}{4v} + 2\pi^2 v\right)t}$
3. $q(x,0) = \sin 2\pi x$

**Simulations**

Here are the steps i followed to get things done.

- I wrote functions in order to construct the element-wise Mass and (Weak form) Differentiation matrices.
- I wrote a DSS function in order to construct the global matrices for Mass and Differentiation matrix.
- I stored the global matrices for D and F.
- I performed convergence rates studies as discussed in the next section.
- I showed results for exact (let $Q = N + 1$ be exact) and inexact integration ($Q = N$).

**Error Norms**

The normalized $L_2$ error norm used is

$$||error||_{L^2} = \sqrt{\frac{\sum_{k=1}^{N_P} \left(q^{numerical} - q^{exact}(x_k)\right)^2}{\sum_{k=1}^{N_P} q^{exact}(x_k)^2}}$$

where:

- $k = 1, ..., N_P$,

- $N_p = N_e N + 1$ global gridpoints,

- $q^{numerical}$ and $q^{exact}$ are the numerical and exact solutions after one full revolution of the wave.

**References**


1. Reynold's number for Turbulence flow; the link is here.
2. Turbulence Kinetic Energy; here and here.
3. Other constants; here.
4. Free fall Velocity; here and here.