STUDENT'S NAME TEJAS SRINIVASAN		TOTAL MARKS OBTAINED
CLASS 4-'A'	SUBJECT LINEAR ALGEBRA	
ROLL NO. 5.	DATE 13/4/2020	

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UE18MA251 - LINEAR ALGEBRA - ASSIGNMENT

1. $y = A + Bx + Cx^2$

Points - $(1,1), (2,-1), (3,1)$

Equations obtained -

$$A+B+C=1$$
$$A+2B+4C=-1$$
$$A+3B+9C=1$$

In $Ax=b$ form -

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Augment matrix - Applying Gaussian elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{array}\right] \quad R_2' = R_2 - R_1, \quad R_3' = R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{array}\right] \quad R_3'' = R_3' - 2R_2'$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{array}\right]$$ (UT Matrix obtained)

$2C=4 \Rightarrow C=2$

$B+3C=-2 \Rightarrow B=-8$

$A+B+C=1 \Rightarrow A=7$

↳ The required equation of the parabola is

$7-8x+2x^2$

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2. LU decomposition for -

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}$$

$R_2' = R_2 - 2R_1$
$R_3' = R_3 + 5R_1$
$R_4' = R_4 - 5R_1$

$$\begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix}$$

$R_3'' = R_3' + 2R_2'$
$R_4'' = R_4' + 2R_2'$

$$\begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix}$$

$R_4''' = R_4'' - 3R_3''$

$$\begin{bmatrix} 2 & 5 & +2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} = U$$

Having obtained the Upper Triangular Matrix, we can find the Lower Triangular Matrix by substitution -

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix}_{4\times4} \quad U = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}_{4\times4}$$

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3. $T: R^3 \rightarrow R^3$

$T(x, y, z) = (x+2y-z,\ y+z,\ x+y-2z)$

i) Matrix T relative to standard basis

$(1,0,0) \rightarrow (1,0,1)$
$(0,1,0) \rightarrow (2,1,1)$
$(0,0,1) \rightarrow (-1,1,-2)$

} for standard basis of R^3 (by expanding each basis in column form)

$$\therefore\ T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

ii) Basis for Four Fundamental Subspaces of T

Augment matrix

$$[A\ b] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 1 & 1 & 2 & b_3 \end{array}\right] \quad R_3' = R_3 - R_1$$

$$[A\ b] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & -1 & -1 & b_3 - b_1 \end{array}\right] \quad R_3'' = R_3' + R_2$$

$$[A\ b] \sim \left[\begin{array}{cccc} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 + b_2 \end{array}\right]$$

$\downarrow$

$-b_1 + b_2 + b_3$

Basis for -

$C(A) = \{(1,0,1),\ (2,1,1)\}$

$C(A^T) = \{(1,-2,-1),\ (0,1,1)\}$

$N(A^T) = \{(-1,1,1)\}$

Caliber

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$$U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1' = R_1 - 2R_2$$

Row reduced form $(R) \rightarrow$

$$R = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\Rightarrow$ free variable

$x - 3z = 0 \Rightarrow x = 3z$

$y + z = 0 \Rightarrow y = -z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} +3z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Basis of

$\therefore N(A) = \{(3,-1,1)\}$

iii) Eigenvalues and Eigenvectors of T

Characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = 0$$

Expanding the determinant -

$$(1-\lambda)\left[(1-\lambda)(-2-\lambda) - 1\right] + 2(1-0) - 1(\lambda-1) = 0$$

$$(1-\lambda)(\lambda^2 + \lambda - 3) + 3 - \lambda = 0$$

$$\Rightarrow \cancel{\lambda^2} + \cancel{\lambda} - \cancel{3} - \lambda^3 - \cancel{\lambda^2} + 3\lambda + \cancel{3} - \cancel{\lambda} = 0$$

$$\lambda^3 - 3\lambda = 0$$

$$\lambda(\lambda^2 - 3) = 0$$

$\therefore$ The eigenvalues are $(0, \sqrt{3}, -\sqrt{3})$

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- Finding eigenvectors for the eigenvalues obtained -

i) $\lambda = 0$

(Basically the Nullspace of T)

$\therefore$ Eigen vector $v_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

ii) $\lambda = \sqrt{3}$

Substituting the value in the determinant - $|A - \sqrt{3} I|$

$$= \begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \frac{x}{2+(1-\sqrt{3})} + \frac{y}{-(1-\sqrt{3})} + \frac{z}{4-2\sqrt{3}}$$

$$v_2 = \begin{bmatrix} (\sqrt{3}+3)/2 \\ (\sqrt{3}+1)/2 \\ 1 \end{bmatrix}$$

iii) $\lambda = -\sqrt{3}$

Substituting the value in the determinant - $|A + \sqrt{3} I|$

$$= \begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_3 = \frac{x}{2+(1+\sqrt{3})} + \frac{y}{-(1+\sqrt{3})} + \frac{z}{(4+2\sqrt{3})}$$

$$v_3 = \begin{bmatrix} (\sqrt{3}+3)/2 \\ (\sqrt{3}+1)/2 \\ 1 \end{bmatrix}$$

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iv) T = QR decomposition

$a = (1,0,1) \quad b = (2,1,1) \quad c = (-1,1,-2)$

Using Gram - Schmidt procedure -

$q_1 = \frac{a}{||a||} \qquad q_1 = (1,0,1)\frac{1}{\sqrt{2}}$

$$q_1^T b = \frac{1}{\sqrt{2}}[1\ 0\ 1]\begin{bmatrix}2\\1\\1\end{bmatrix} = \frac{3}{\sqrt{2}}$$

$q_2 = \frac{B}{||B||}$, where $B = b - (q_1^T b)\, q_1$

$$B = b - (q_1^T b)\, q_1 = (2,1,1) - \frac{3}{\sqrt{2}}(1,0,1)\frac{1}{\sqrt{2}}$$

$$= (2,1,1) - \left(\frac{3}{2}, 0, \frac{3}{2}\right)$$

$$= \frac{1}{2}(1,2,-1) = (1/2, 1, -1/2)$$

$$q_2 = \left(\frac{1/2}{(\sqrt{3}/\sqrt{2})}, \frac{1}{\sqrt{3}/\sqrt{2}}, \frac{-1/2}{\sqrt{3}/\sqrt{2}}\right)$$

$$q_2 = \frac{1}{\sqrt{6}}(1,2,-1)$$

$q_3 = \frac{C}{||C||}$, where $C = c - (q_2^T c)\, q_2 - (q_1^T c)\, q_1$

$$q_1^T c = \frac{1}{\sqrt{2}}[1,0,1]\begin{bmatrix}-1\\1\\-2\end{bmatrix} = \frac{-3}{\sqrt{2}}$$

$$q_2^T c = \frac{1}{\sqrt{6}}[1\ 2\ -1]\begin{bmatrix}-1\\1\\-2\end{bmatrix} = \frac{3}{\sqrt{6}}$$

$$C = (-1,1,2) - \frac{3}{\sqrt{6}} \times \frac{1}{\sqrt{6}}(1,2,-1) + \frac{3}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times (1,0,1)$$

$C = (0,0,0) \rightarrow q_3 = (0,0,0)$

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Factorizing into QR -

$$R = \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}$$

$$(q_1^T a) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \sqrt{2}$$

$$(q_2^T b) = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \sqrt{\frac{3}{2}} = \frac{3}{\sqrt{6}}$$

$$(q_3^T c) = 0$$

$$\therefore R = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{12} & 3\sqrt{3} & -3\sqrt{3} \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow T = QR$$

$$= \frac{1}{6} \begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ \sqrt{3} & -1 & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{3} & 3\sqrt{3} & -3\sqrt{3} \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

4. Best fit line $y = c + dx$ for the given data using least square principles.

x	-4	1	2	3
y	4	6	10	8

Converting this data into matrix form -

$$\begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

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We need to find vector $\hat{x} = (A^T A)^{-1} A^T B$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{116} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix}$$

$$(A^T A)^{-1} A^T = \frac{1}{116} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix}$$

$$= \frac{1}{116} \begin{bmatrix} 38 & 28 & 26 & -24 \\ -18 & 2 & 6 & 10 \end{bmatrix}$$

$$(A^T A)^{-1} A^T . b = \frac{1}{116} \begin{bmatrix} 38 & 28 & 26 & -24 \\ -18 & 2 & 6 & 10 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$= \frac{1}{116} \begin{bmatrix} 772 \\ 80 \end{bmatrix} = \hat{x} \begin{bmatrix} c \\ d \end{bmatrix}$$

$$c = \frac{193}{29} \qquad d = \frac{20}{29}$$

The equation is of the form : $y = c + dx$

$$\therefore y = \frac{193}{29} + \frac{20x}{29}$$

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5 Projection matrices 'P' and 'Q' for plane eqn
$x_1 + x_2 + 3x_3 + 4x_5 = 0$

$Q = A(A^TA)^{-1}A^T$ $\quad A = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix}$

$$Q = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/27 \\ 1/27 \\ 3/27 \\ 0 \\ 4/27 \end{bmatrix}_{5\times1} \begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix}_{1\times5}$$

$$Q = \begin{bmatrix} 1/27 & 1/27 & 3/27 & 0 & 4/27 \\ 1/27 & 1/27 & 3/27 & 0 & 4/27 \\ 3/27 & 3/27 & 1/3 & 0 & 12/27 \\ 0 & 0 & 0 & 0 & 0 \\ 4/27 & 4/27 & 12/27 & 0 & 16/27 \end{bmatrix}_{5\times5}$$

WKT - $P + Q = I$
$P = I - Q$

$$P = \begin{bmatrix} 26/27 & -1/27 & -3/27 & 0 & -4/27 \\ -1/27 & 26/27 & -3/27 & 0 & -4/27 \\ -3/27 & -3/27 & 18/27 & 0 & -12/27 \\ 0 & 0 & 0 & 1 & 0 \\ -4/27 & -4/27 & -12/27 & 0 & 11/27 \end{bmatrix}_{5\times5}$$

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6. Range of number 'a' for which A is positive definite

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$$

All the sub determinants of the matrix should be positive

i) $|a| > 0$ $\therefore a \in (0, \infty)$

ii) $\begin{vmatrix} a & 2 \\ 2 & a \end{vmatrix} > 0$

$a^2 - 4 > 0$ $\therefore a \in (-\infty, -2) \cup (2, \infty)$

iii) $\begin{vmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{vmatrix}$ $(a+4)(a-2)^2 > 0$

$a \in (-4, \infty)$

→ Intersection of all the ranges obtained indicates

$a \in (2, \infty)$

3x3 symmetric matrix B that produces $f = x^T A x$

where $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3)$

$$[x_1 \ x_2 \ x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$= a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + 2a_{12} x_1 x_2 + 2a_{13} x_1 x_3$
$+ 2a_{23} x_2 x_3$

Comparing with the given equation, we get -

$a_{11} = a_{22} = a_{33} = 2$

$a_{12} = -1$ $a_{13} = 0$ $a_{23} = -1$

$$\therefore B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

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7. SVD of A, $U \Sigma V^T$ where

$$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}_{3\times2}$$

The matrix A is a tall matrix

$$\therefore A = U_{3\times3} \Sigma_{3\times2} V^T_{2\times2}$$

$$A^TA = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

Eigen values for A^TA

$$\lambda^2 - 90\lambda = 0$$

$$\lambda(\lambda - 90) = 0$$

$$\lambda = 90, 0$$

Using $\sigma = \sqrt{\lambda}$, $\sigma_1 = \sqrt{90}$ $\sigma_2 = 0$

Finding eigen vectors -

i) $\lambda = 90$

$$\begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -9 & -27 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-9x - 27y = 0$$

$$x = -3y$$

$x_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \rightarrow$ normalising which we get

$$v_1 = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

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ii) $\lambda = 0$

$$\begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 81 & -27 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$81x = 27y$$
$$x = \tfrac{1}{3} y$$

$x_2 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \rightarrow$ normalizing which we get

$V_2 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$

$$\Sigma = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{90}} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$= \frac{1}{\sqrt{90}} \begin{bmatrix} \sqrt{10} \\ -20/\sqrt{10} \\ -20/\sqrt{10} \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

Vectors U_2 and U_3 are orthogonal to u_1

Let $u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\quad u_1^T u = 0$

$\therefore a/3 - 2b/3 - 2c/3 = 0 \rightarrow a = 2b + 2c$

Normalizing y_1 and y_2 -

$U_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$ $\quad U_3 = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$ $\quad \therefore U = \begin{bmatrix} 1/3 & 2/\sqrt{5} & 2/\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 0 \\ -2/3 & 0 & 1/\sqrt{5} \end{bmatrix}$

$A = U \Sigma V^T$

$$= \begin{bmatrix} 1/3 & 2/\sqrt{5} & 2/\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 0 \\ -2/3 & 0 & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$