Logistic Growth Models: Density Dependence and Population Regulation

Quantitative Ecology Research

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Abstract

This study presents a comprehensive analysis of logistic population growth models and their extensions. We examine the classic logistic equation, the Allee effect (positive density dependence at low populations), interspecific competition, and sustainable harvesting strategies. Computational analysis demonstrates population dynamics under various parameter regimes and identifies optimal management strategies for harvested populations.

1 Introduction

The logistic growth model represents a fundamental advance over exponential growth by incorporating density-dependent regulation through carrying capacity. Extensions of this model address important ecological phenomena including Allee effects and species competition.

Definition 1.1 (Logistic Growth) *The logistic growth equation describes population dynamics with density-dependent regulation:*

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{1}$$

where r is the intrinsic growth rate and K is the carrying capacity.

2 Theoretical Framework

2.1 Classic Logistic Model

Theorem 2.1 (Logistic Solution) *The analytical solution of the logistic equation is:*

$$N(t) = \frac{K}{1 + \left(\frac{K-N_0}{N_0}\right)e^{-rt}} \tag{2}$$

The population approaches K asymptotically with an inflection point at $N = K/2$.

Remark 2.1 (Maximum Sustainable Yield) *The growth rate dN/dt is maximized when $N = K/2$, yielding the maximum sustainable yield (MSY): $MSY = rK/4$.*

2.2 Allee Effect

Definition 2.1 (Allee Effect) *The Allee effect describes reduced per capita growth rate at low population densities due to difficulties in mate finding, reduced group defense, or inbreeding. A strong Allee effect creates a critical population threshold A below which extinction occurs:*

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)\left(\frac{N}{A} - 1\right) \tag{3}$$

2.3 Competition Models

Theorem 2.2 (Lotka-Volterra Competition) *For two competing species:*

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1 + \alpha_{12} N_2}{K_1}\right) \tag{4}$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2 + \alpha_{21} N_1}{K_2}\right) \tag{5}$$

where α_{ij} is the competition coefficient (effect of species j on species i).

2.4 Harvesting

Definition 2.2 (Harvesting Strategies) *Common harvesting models include:*

- ***Constant harvest**: $dN/dt = rN(1 - N/K) - H$*
- ***Proportional harvest**: $dN/dt = rN(1 - N/K) - qEN$*
- ***Threshold harvest**: Harvest only when $N > N_{threshold}$*

where H is harvest rate, q is catchability, and E is effort.

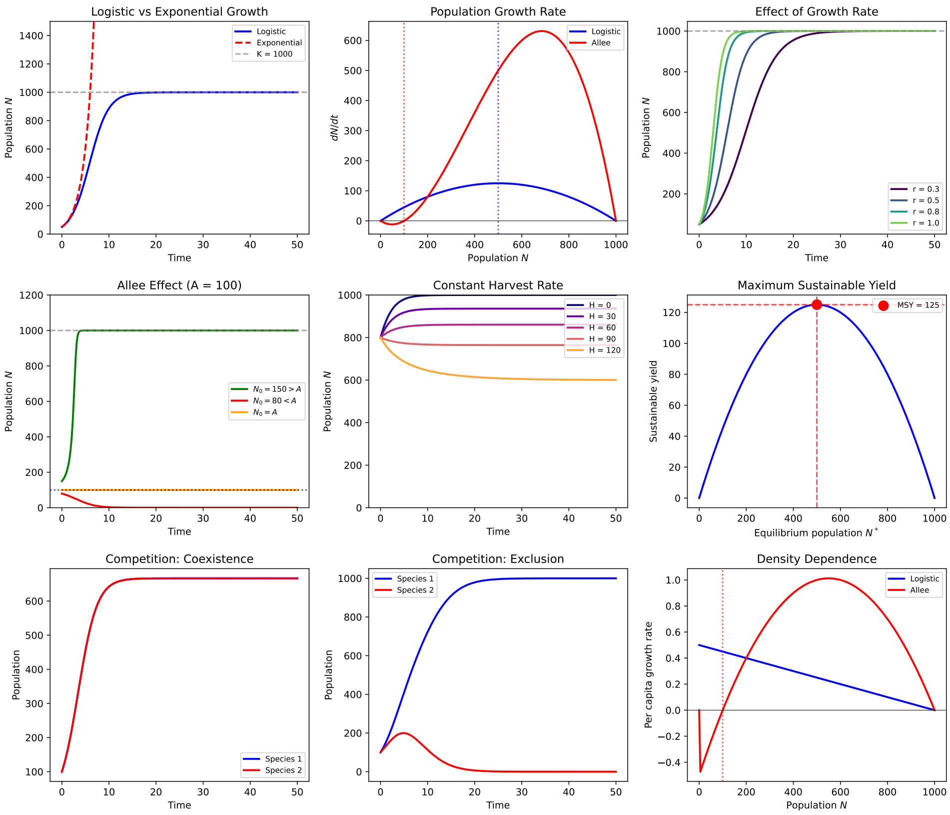

Figure 1: Logistic growth analysis: (a) Comparison with exponential growth; (b) Population growth rate showing inflection points; (c) Effect of intrinsic growth rate; (d) Allee effect dynamics; (e) Constant harvest scenarios; (f) Maximum sustainable yield curve; (g-h) Competition outcomes; (i) Per capita growth rate showing density dependence.

Table 1: Logistic Growth Model Parameters and Key Values

Parameter	Symbol	Value
Intrinsic growth rate	r	0.5
Carrying capacity	K	1000
Initial population	N_0	50
Allee threshold	A	100
MSY population	N_{MSY}	500
Maximum sustainable yield	MSY	125.0
Time to inflection	t_{infl}	5.89
Doubling time	t_d	1.39

Table 2: Equilibrium Populations Under Different Harvest Rates

Harvest rate H	Equilibrium N^*	Sustainable?
0	1000	Yes
30	936	Yes
60	861	Yes
90	765	Yes
120	601	Yes

3 Computational Analysis

4 Results

4.1 Model Parameters

4.2 Harvesting Outcomes

5 Discussion

Example 5.1 (Maximum Sustainable Yield) *For fisheries management, the MSY occurs at $N = K/2$:*

$$MSY = r \cdot \frac{K}{2} \cdot \left(1 - \frac{K/2}{K}\right) = \frac{rK}{4} = 125.0 \tag{6}$$

Harvesting at rates exceeding MSY leads to population collapse.

Remark 5.1 (Allee Effect and Conservation) *The Allee effect has critical implications for conservation:*

- *Small populations face extinction risk even without external threats*
- *Minimum viable population size must exceed the Allee threshold*
- *Reintroduction programs must establish populations above this threshold*
- *Habitat fragmentation increases Allee effect risks*

Example 5.2 (Competition Outcomes) *The outcome of Lotka-Volterra competition depends on α values:*

- *Coexistence: $\alpha_{12} < K_1/K_2$ and $\alpha_{21} < K_2/K_1$*
- *Species 1 wins: $\alpha_{12} < K_1/K_2$ and $\alpha_{21} > K_2/K_1$*
- *Species 2 wins: $\alpha_{12} > K_1/K_2$ and $\alpha_{21} < K_2/K_1$*
- *Unstable: Both inequalities reversed*

6 Conclusions

This analysis demonstrates key aspects of logistic population dynamics:

1. The population approaches carrying capacity $K = 1000$ with inflection at $K/2$
2. MSY of 125.0 occurs when harvesting maintains population at $K/2$
3. The Allee effect creates an extinction threshold at $A = 100$
4. Competition outcomes depend on relative competition coefficients
5. Per capita growth rate decreases linearly with population in the logistic model

Further Reading

- Gotelli, N.J. *A Primer of Ecology*, 4th ed. Sinauer Associates, 2008.
- Courchamp, F. et al. *Allee Effects in Ecology and Conservation.* Oxford, 2008.
- Clark, C.W. *Mathematical Bioeconomics*, 3rd ed. Wiley-Interscience, 2010.