Partial Differential Equations: Finite Difference Methods and Stability Analysis

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Abstract

This report presents finite difference methods for solving partial differential equations (PDEs). We implement explicit and implicit schemes for the heat equation and wave equation, analyze stability using von Neumann analysis, and demonstrate the CFL condition. Numerical examples illustrate the importance of stability constraints in time-stepping methods.

Contents

Chapter 1

Introduction

Partial differential equations describe physical phenomena involving multiple independent variables. We focus on two canonical PDEs:

- **Heat equation** (parabolic): $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$
- **Wave equation** (hyperbolic): $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

1.1 Finite Difference Approximations

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t} \tag{1.1}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{1.2}$$

Chapter 2

Heat Equation

2.1 Explicit (FTCS) Scheme

Forward Time, Centered Space:

$$u_i^{n+1} = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n), \quad r = \frac{\alpha \Delta t}{(\Delta x)^2} \tag{2.1}$$

Stability requires: $r \leq \frac{1}{2}$

2.2 Numerical Results

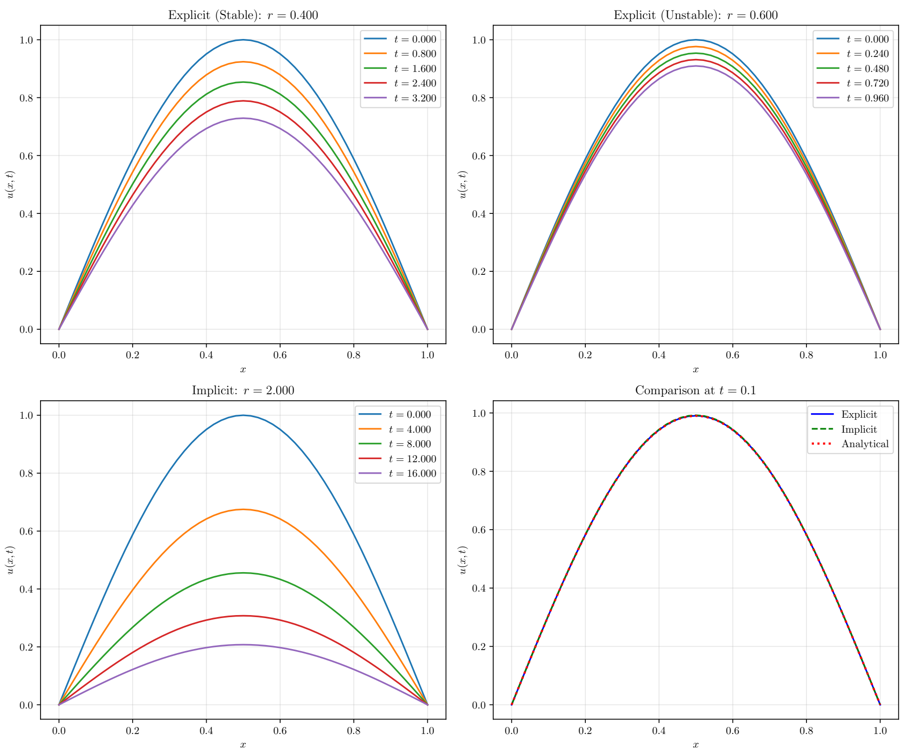

Figure 2.1: Heat equation solutions: (a) stable explicit, (b) unstable explicit showing oscillations, (c) implicit scheme, (d) comparison with analytical solution.

Chapter 3

Wave Equation

3.1 Explicit Scheme

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + s^2(u_{i+1}^n - 2u_i^n + u_{i-1}^n), \quad s = \frac{c\Delta t}{\Delta x} \tag{3.1}$$

CFL condition: $s = \frac{c\Delta t}{\Delta x} \leq 1$

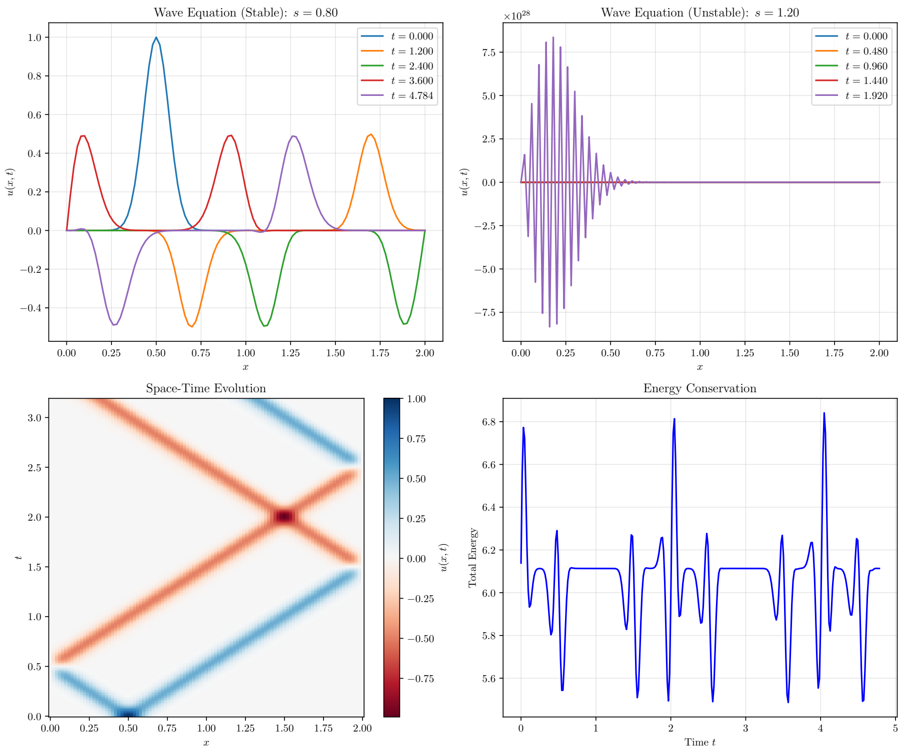

Figure 3.1: Wave equation: (a) stable propagation, (b) unstable CFL violation, (c) space-time diagram, (d) energy conservation.

Chapter 4

Stability Analysis

4.1 Von Neumann Analysis

Assume solution of form $u_j^n = G^n e^{ikj\Delta x}$

Heat equation (explicit):

$$G = 1 - 4r\sin^2\left(\frac{k\Delta x}{2}\right) \tag{4.1}$$

Stability: $|G| \leq 1 \Rightarrow r \leq \frac{1}{2}$

Wave equation:

$$G = 1 - 2s^2\sin^2\left(\frac{k\Delta x}{2}\right) \pm is\sin(k\Delta x)\sqrt{1 - s^2\sin^2\left(\frac{k\Delta x}{2}\right)} \tag{4.2}$$

Stability: $|G| = 1$ when $s \leq 1$

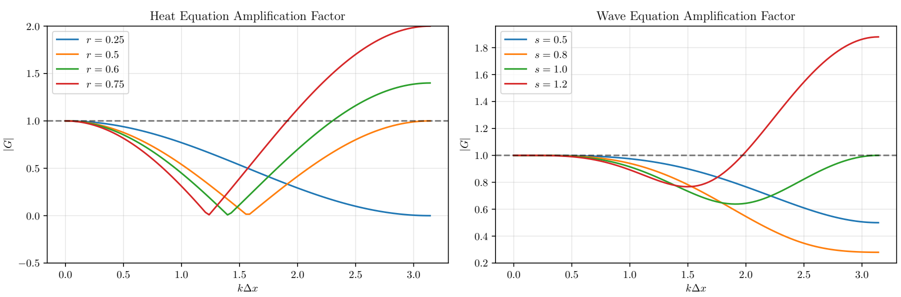

Figure 4.1: Von Neumann stability analysis: amplification factors for heat (left) and wave (right) equations.

Chapter 5

Numerical Results

Table 5.1: Stability conditions for finite difference schemes

Scheme	Stability Condition	Type
Heat Explicit	$r \leq 0.5$	Conditional
Heat Implicit	None	Unconditional
Heat Crank-Nicolson	None	Unconditional
Wave Explicit	$s \leq 1$	Conditional (CFL)

Chapter 6

Conclusions

1. Explicit schemes are simple but require stability constraints
2. Implicit schemes are unconditionally stable but require solving linear systems
3. CFL condition is essential for hyperbolic PDEs
4. Von Neumann analysis provides stability criteria
5. Choice of method depends on problem type and accuracy requirements