Fractal Geometry and Self-Similarity: Computational Analysis of Fractal Structures

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Abstract

This report presents a computational exploration of fractal geometry, examining the Mandelbrot set, Julia sets, Sierpinski triangle, and Koch snowflake. We compute fractal dimensions using box-counting methods, analyze escape-time algorithms, and investigate the self-similar structures that characterize these mathematical objects. All visualizations are generated using PythonTeX for complete reproducibility.

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Chapter 1

Introduction

Fractals are geometric objects that exhibit self-similarity at all scales. Unlike classical Euclidean geometry, fractals often have non-integer (fractal) dimensions, reflecting their intricate structure.

1.1 Defining Fractals

A set F is a fractal if:

- It has a fine structure at arbitrarily small scales
- It is too irregular to be described by traditional geometry
- It exhibits self-similarity (exact or statistical)
- Its fractal dimension exceeds its topological dimension

1.2 Fractal Dimension

The box-counting dimension is defined as:

$$D = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)} \tag{1.1}$$

where $N(\epsilon)$ is the number of boxes of size ϵ needed to cover the set.

Chapter 2

The Mandelbrot Set

2.1 Definition

The Mandelbrot set M is defined as the set of complex numbers c for which the iteration:

$$z_{n+1} = z_n^2 + c, \quad z_0 = 0 \tag{2.1}$$

remains bounded as $n \to \infty$.

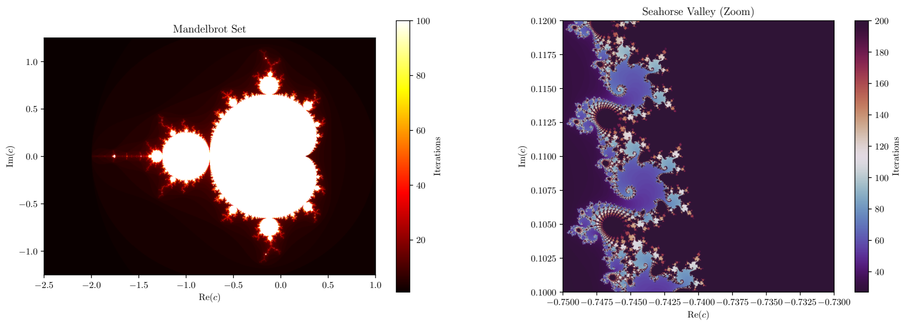

Figure 2.1: The Mandelbrot set: full view (left) and zoomed into the Seahorse Valley (right), demonstrating self-similarity.

2.2 Escape Time Algorithm

Points outside M are colored by escape time—the number of iterations before $|z_n| > 2$:

$$\text{escape time}(c) = \min\{n : |z_n| > 2\} \tag{2.2}$$

Chapter 3

Julia Sets

3.1 Definition

For a fixed c, the filled Julia set K_c consists of initial points z_0 for which $z_{n+1} = z_n^2 + c$ remains bounded.

3.2 Connection to Mandelbrot Set

The Julia set K_c is connected if and only if $c \in M$. Otherwise, K_c is a Cantor set (totally disconnected).

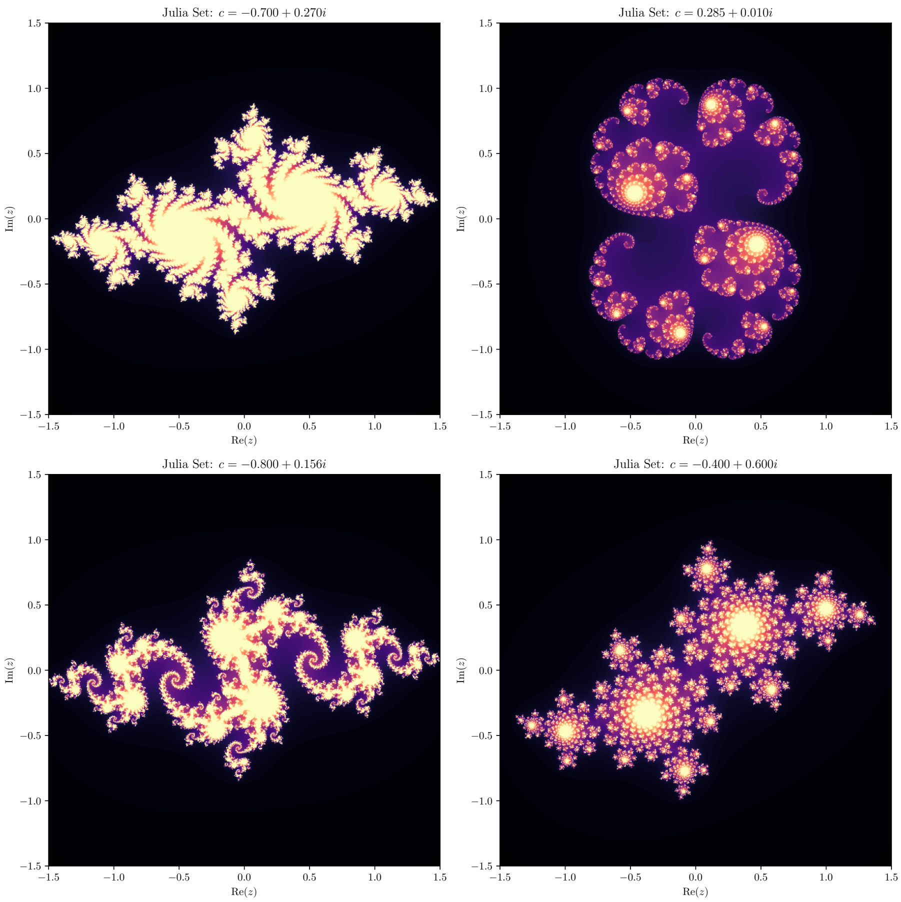

Figure 3.1: Julia sets for different values of c, showing connected ($c \in M$) and disconnected ($c \notin M$) structures.

Chapter 4

Sierpinski Triangle

4.1 Iterated Function System

The Sierpinski triangle can be generated using three affine transformations:

$$T_1(\mathbf{x}) = \frac{1}{2}\mathbf{x} \tag{4.1}$$

$$T_2(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \tag{4.2}$$

$$T_3(\mathbf{x}) = \frac{1}{2}\mathbf{x} + \begin{pmatrix} 1/4 \\ \sqrt{3}/4 \end{pmatrix} \tag{4.3}$$

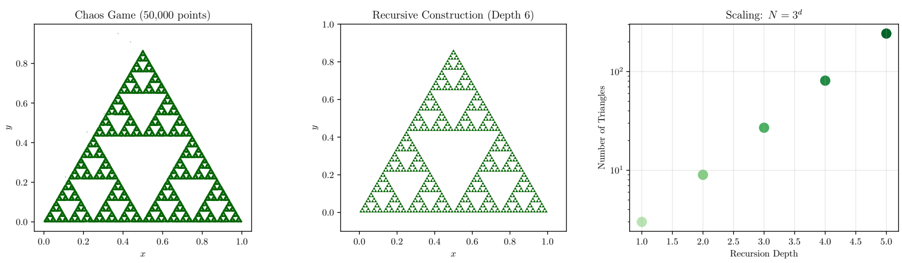

Figure 4.1: Sierpinski triangle: chaos game method (left), recursive IFS (center), scaling behavior (right).

The fractal dimension is $D = \frac{\log 3}{\log 2} \approx 1.5850$.

Chapter 5

Koch Snowflake

5.1 Construction

Each line segment is replaced by four segments of length 1/3:

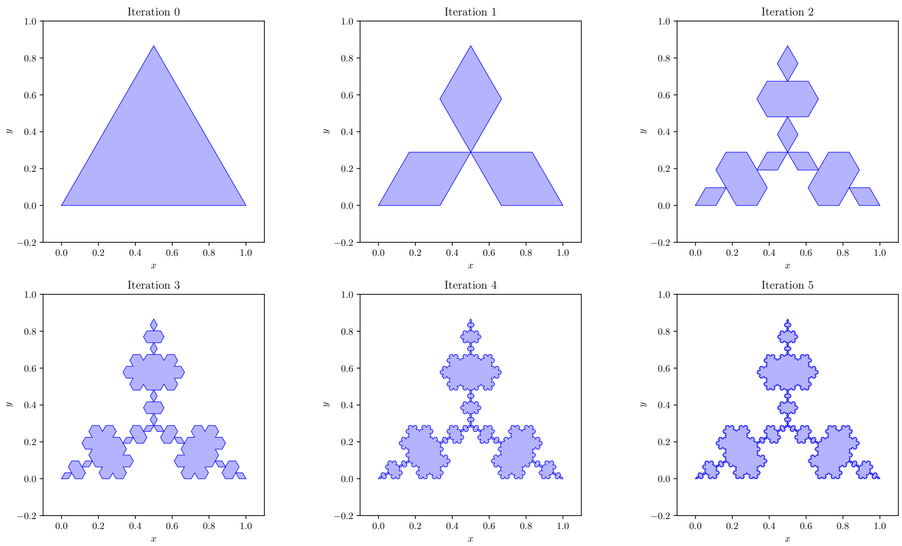

Figure 5.1: Koch snowflake construction: iterations 0-5. The perimeter grows infinitely while the area converges.

5.2 Properties

- Perimeter: $L_n = L_0 \cdot (4/3)^n \to \infty$
- Area: $A_n \to \frac{2\sqrt{3}}{5}s^2$ (finite)
- Fractal dimension: $D = \frac{\log 4}{\log 3} \approx 1.2619$

Chapter 6

Box-Counting Dimension

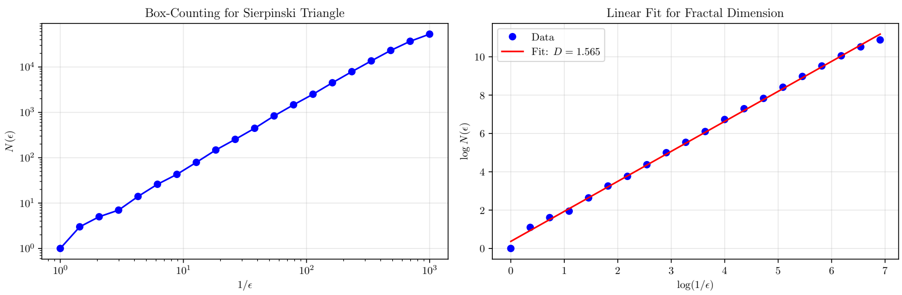

Figure 6.1: Box-counting dimension estimation: log-log plot (left) and linear fit (right).

Computed dimension: $D = 1.5652$ (theoretical: 1.5850)

Chapter 7

Summary of Results

Table 7.1: Fractal dimensions of common fractals

Fractal	Dimension	Property
Mandelbrot boundary	2.0	Coastline
Julia set ($c \in M$)	≈ 2	Connected
Sierpinski triangle	1.5850	Self-similar
Koch snowflake	1.2619	Infinite perimeter
Cantor set	0.6309	Disconnected

Chapter 8

Conclusions

1. The Mandelbrot set contains all c for which Julia sets are connected
2. Julia sets exhibit diverse topologies depending on parameter c
3. IFS methods generate exact self-similar fractals
4. Box-counting provides numerical estimates of fractal dimension
5. Fractals demonstrate that infinite complexity can arise from simple rules