Navier-Stokes Equations: Viscous Flow Analysis and Boundary Layer Theory

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Abstract

This technical report presents analytical and computational solutions to the Navier-Stokes equations for canonical viscous flow problems. We analyze Couette flow, Poiseuille flow, and boundary layer development using Python-based numerical methods. Results include velocity profiles, shear stress distributions, and Reynolds number effects on flow characteristics.

1 Introduction

The Navier-Stokes equations govern the motion of viscous fluids and form the foundation of fluid mechanics:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g} \tag{1}$$

Definition 1.1 (Reynolds Number)**.** *The Reynolds number characterizes the ratio of inertial to viscous forces:*

$$Re = \frac{\rho U L}{\mu} = \frac{UL}{\nu} \tag{2}$$

2 Computational Setup

Fluid Properties (Water at 20°C):

- Density: $\rho =$ **??** kg/m^3
- Dynamic Viscosity: $\mu =$ **??** mPa·s
- Kinematic Viscosity: $\nu =$ **??** mm^2/s

3 Couette Flow Analysis

Couette flow occurs between two parallel plates where one plate moves relative to the other.

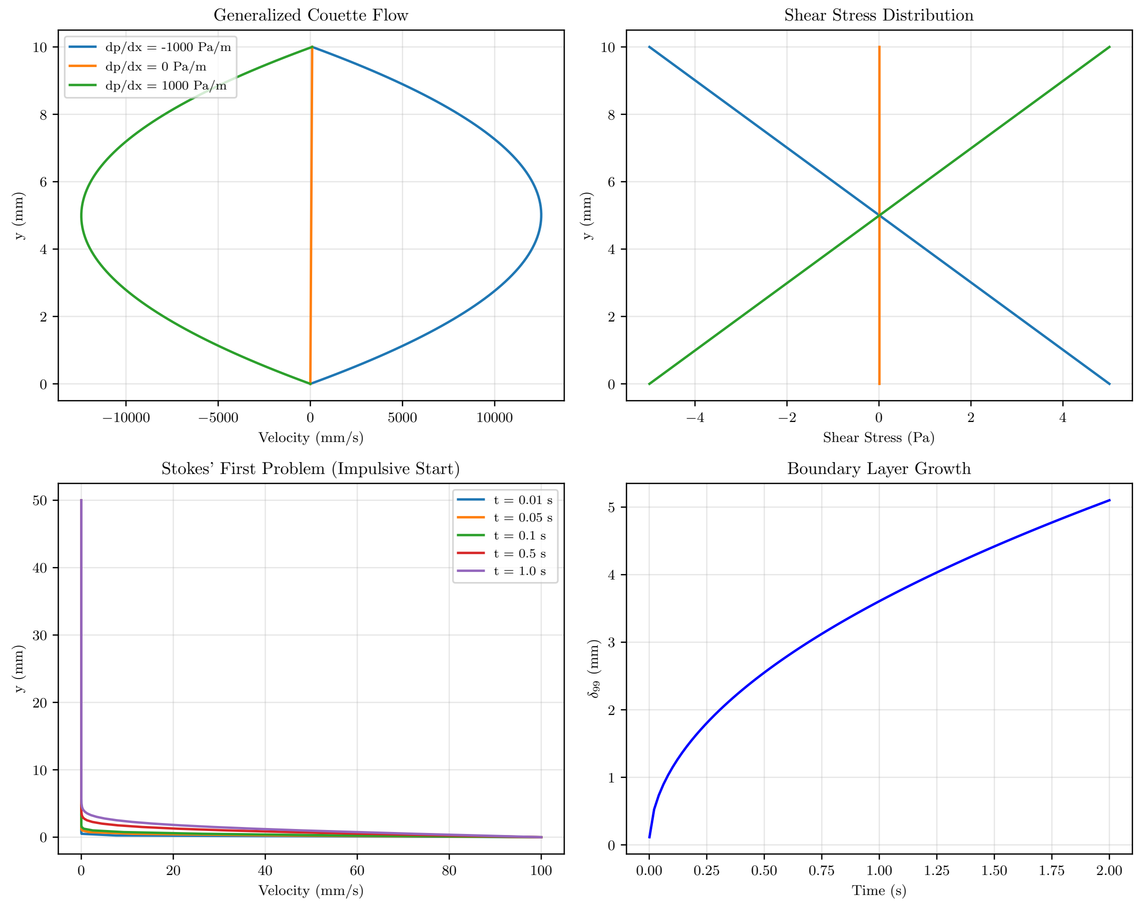

Figure 1: Couette flow analysis: velocity profiles, shear stress, and transient development.

Table 1: Couette Flow Parameters

Parameter	**Value**	**Unit**
Wall Velocity	**??**	mm/s
Channel Height	**??**	mm
Reynolds Number	**??**	–
Wall Shear Stress	**??**	Pa

4 Poiseuille Flow (Pressure-Driven)

Theorem 4.1 (Hagen-Poiseuille Equation). *For laminar flow in a circular pipe, the volumetric flow rate is:*

$$Q = \frac{\pi R^4}{8\mu}\left(-\frac{dp}{dx}\right) \tag{3}$$

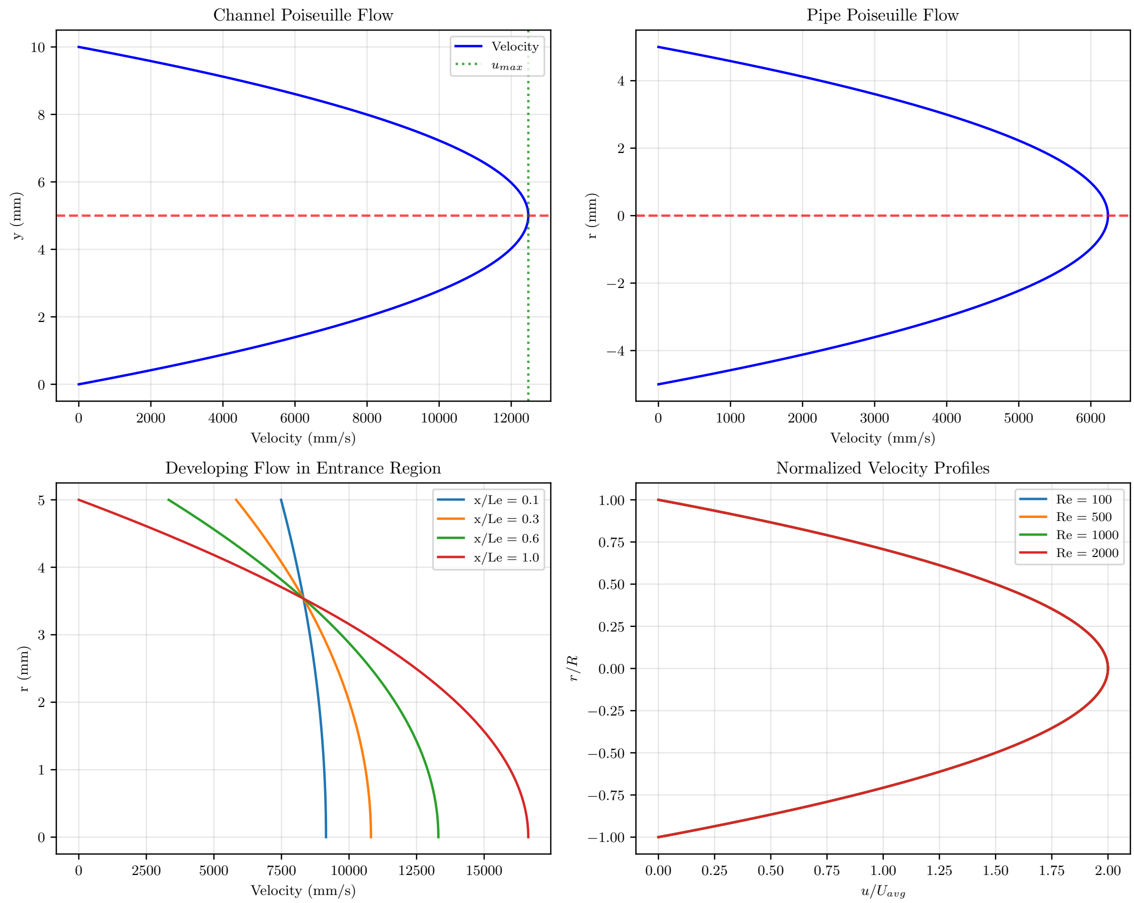

Figure 2: Poiseuille flow analysis: velocity profiles and entrance region development.

Table 2: Poiseuille Flow Results

Parameter	**Value**	**Unit**
Maximum Velocity	??	mm/s
Average Velocity	??	mm/s
Flow Rate (channel)	??	mm^2/s
Reynolds Number	??	–
Entrance Length	??	mm

5 Boundary Layer Analysis

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Figure 3: Blasius boundary layer solution: velocity profile, thickness growth, and skin friction.

6 Reynolds Number Effects

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Figure 4: Reynolds number effects on flow characteristics.

7 Vorticity and Stream Function

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Figure 5: Flow field visualization: stream function, velocity magnitude, vorticity, and vectors.

8 Oscillatory Flow (Stokes' Second Problem)

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Figure 6: Oscillatory flow: Stokes' second problem and Stokes layer characteristics.

Table 3: Oscillatory Flow Parameters

Parameter	**Value**	**Unit**
Oscillation Frequency	**??**	Hz
Stokes Layer Thickness	**??**	mm
Penetration Depth ($3\delta_s$)	**??**	mm

9 Conclusions

This analysis of the Navier-Stokes equations demonstrated:

1. **Couette Flow**: Linear velocity profile with wall shear stress $\tau_w =$ **??** Pa. Generalized solutions with pressure gradients show parabolic modifications.

2. **Poiseuille Flow**: Parabolic velocity profile with maximum velocity u_{max} = **??** mm/s. The entrance length scales linearly with Reynolds number for laminar flow.
3. **Boundary Layers**: Blasius solution gives $f''(0)$ = **??**, with boundary layer thickness $\delta \propto x^{1/2}$.
4. **Reynolds Number Effects**: Critical $Re = 2300$ for pipe flow transition, with friction factor following the Blasius correlation in turbulent regime.
5. **Oscillatory Flows**: Stokes layer thickness δ_s = **??** mm determines penetration depth of oscillations.