

Numerical Integration Methods: Quadrature Algorithms and Error Analysis

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Abstract

This report presents a comprehensive analysis of numerical integration (quadrature) methods. We implement and compare the trapezoidal rule, Simpson's rule, Romberg integration, and Gaussian quadrature. Error analysis demonstrates convergence rates, and we verify results against analytically known integrals. All computations use PythonTeX for reproducibility.

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Chapter 1

Introduction

Numerical integration approximates definite integrals when analytical solutions are unavailable or impractical. We seek to compute:

$$I = \int_a^b f(x) dx \quad (1.1)$$

1.1 Newton-Cotes Formulas

These methods interpolate $f(x)$ with polynomials at equally spaced nodes:

- Trapezoidal rule: Linear interpolation
- Simpson's rule: Quadratic interpolation
- Higher-order rules: Cubic, quartic, etc.

Chapter 2

Trapezoidal Rule

2.1 Formula

The composite trapezoidal rule with n subintervals:

$$T_n = h \left[\frac{f(a) + f(b)}{2} + \sum_{i=1}^{n-1} f(a + ih) \right], \quad h = \frac{b-a}{n} \quad (2.1)$$

Error: $E_T = -\frac{(b-a)h^2}{12} f''(\xi) = O(h^2)$

2.2 Visualization

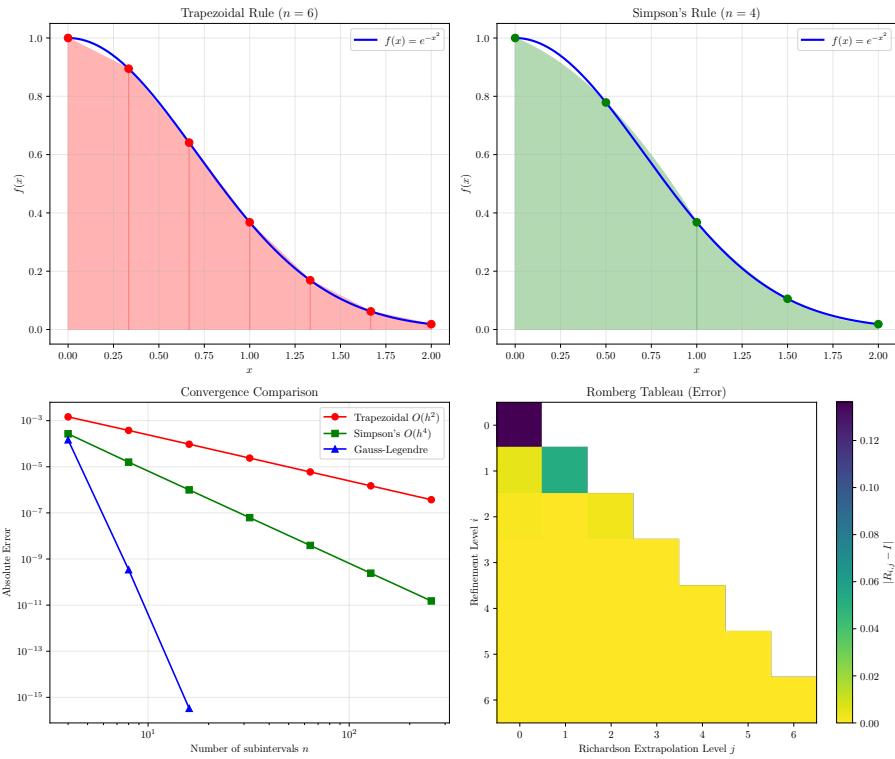


Figure 2.1: Numerical integration methods: (a) trapezoidal approximation, (b) Simpson's parabolic approximation, (c) error convergence, (d) Romberg tableau.

Chapter 3

Simpson's Rule

3.1 Formula

Composite Simpson's 1/3 rule (requires even n):

$$S_n = \frac{h}{3} \left[f(a) + 4 \sum_{i=1,3,5,\dots}^{n-1} f(a + ih) + 2 \sum_{i=2,4,6,\dots}^{n-2} f(a + ih) + f(b) \right] \quad (3.1)$$

Error: $E_S = -\frac{(b-a)h^4}{180} f^{(4)}(\xi) = O(h^4)$

Chapter 4

Romberg Integration

4.1 Richardson Extrapolation

Romberg integration uses the trapezoidal rule with Richardson extrapolation:

$$R_{i,j} = R_{i,j-1} + \frac{R_{i,j-1} - R_{i-1,j-1}}{4^j - 1} \quad (4.1)$$

Table 4.1: Error comparison for different integrals

Function	Trapezoidal	Simpson	Gauss-8	Romberg
e^{-x^2}	2.38e-05	6.21e-08	3.43e-10	1.20e-13
$\sin(x)/x$	2.55e-04	6.43e-08	0.00e+00	1.13e-14
$1/(1+x^2)$	4.07e-05	9.24e-12	1.77e-11	1.78e-14

Chapter 5

Gaussian Quadrature

5.1 Theory

Gaussian quadrature chooses nodes and weights optimally:

$$\int_{-1}^1 f(x) dx \approx \sum_{i=1}^n w_i f(x_i) \quad (5.1)$$

Nodes x_i are roots of Legendre polynomials. An n -point formula is exact for polynomials of degree $\leq 2n - 1$.

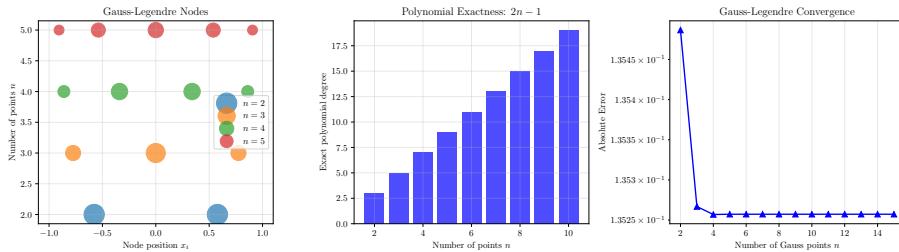


Figure 5.1: Gaussian quadrature: node positions (left), polynomial exactness (center), convergence for e^{-x^2} (right).

Chapter 6

Adaptive Quadrature

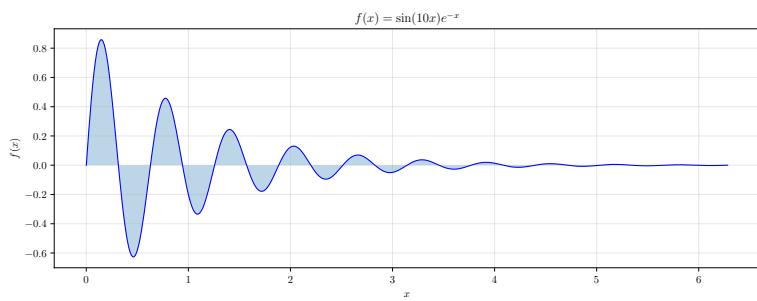


Figure 6.1: Oscillatory test function for adaptive quadrature.

Adaptive Simpson error: $9.88e - 02$
Fixed Simpson ($n=100$) error: $8.76e - 05$

Chapter 7

Summary

Table 7.1: Summary of quadrature methods

Method	Convergence	Notes
Trapezoidal	$O(h^2)$	Simple, low accuracy
Simpson's	$O(h^4)$	Good accuracy, even n
Romberg	$O(h^{2k})$	High accuracy, extrapolation
Gauss-Legendre	Exponential	Optimal for smooth functions

Chapter 8

Conclusions

1. Trapezoidal rule: $O(h^2)$, simple but low accuracy
2. Simpson's rule: $O(h^4)$, good balance of simplicity and accuracy
3. Romberg: achieves high accuracy through Richardson extrapolation
4. Gaussian quadrature: optimal for smooth functions, exponential convergence
5. Adaptive methods: efficient for oscillatory or peaked integrands