Population Dynamics: Predator-Prey Models and Stability Analysis

Mathematical Biology Tutorial

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Abstract

This tutorial presents a comprehensive analysis of predator-prey population dynamics using the Lotka-Volterra equations and their extensions. We examine the classical model, perform stability analysis of equilibrium points, and explore modifications including carrying capacity and functional responses. Computational analysis demonstrates phase portraits, limit cycles, and the ecological implications of parameter variations.

1 Introduction

The Lotka-Volterra predator-prey model is a cornerstone of mathematical ecology, describing the coupled dynamics of two interacting species. Despite its simplicity, the model captures essential features of predator-prey oscillations observed in nature.

Definition 1.1 (Lotka-Volterra Equations) *The classical predator-prey model is:*

$$\frac{dx}{dt} = \alpha x - \beta xy \tag{1}$$

$$\frac{dy}{dt} = \delta xy - \gamma y \tag{2}$$

where x is prey density, y is predator density, α is prey growth rate, β is predation rate, δ is predator growth efficiency, and γ is predator death rate.

2 Theoretical Framework

2.1 Equilibrium Analysis

Theorem 2.1 (Equilibrium Points) *The Lotka-Volterra system has two equilibrium points:*

1. *Trivial equilibrium:* $(x^*, y^*) = (0, 0)$

2. *Coexistence equilibrium:* $(x^*, y^*) = (\gamma/\delta, \alpha/\beta)$

Theorem 2.2 (Stability of Coexistence Equilibrium) *The Jacobian matrix at (x^*, y^*) is:*

$$J = \begin{pmatrix} 0 & -\beta x^* \\ \delta y^* & 0 \end{pmatrix} \tag{3}$$

The eigenvalues are $\lambda = \pm i\sqrt{\alpha\gamma}$, indicating a center (neutral stability). Trajectories form closed orbits around the equilibrium.

2.2 Extensions

Definition 2.1 (Logistic Prey Growth) *Adding carrying capacity to prey growth:*

$$\frac{dx}{dt} = \alpha x \left(1 - \frac{x}{K}\right) - \beta xy \tag{4}$$

$$\frac{dy}{dt} = \delta xy - \gamma y \tag{5}$$

This creates a globally stable equilibrium (spiral sink) for appropriate parameters.

Definition 2.2 (Holling Type II Functional Response) *Predator satiation is modeled by:*

$$\frac{dx}{dt} = \alpha x - \frac{\beta xy}{1 + \beta hx} \tag{6}$$

where h is handling time per prey item.

Remark 2.1 (Functional Responses) *Holling classified functional responses as:*

- ***Type I**: Linear (constant attack rate)*
- ***Type II**: Saturating (handling time limitation)*
- ***Type III**: Sigmoidal (learning or switching behavior)*

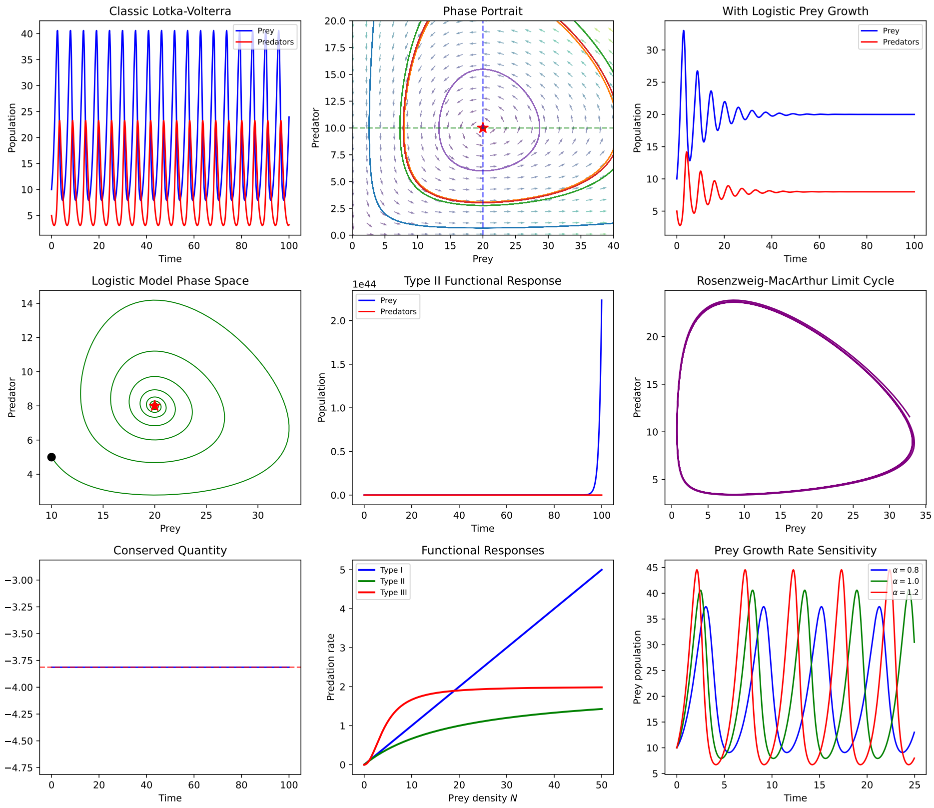

Figure 1: Predator-prey dynamics analysis: (a) Classic Lotka-Volterra time series; (b) Phase portrait with multiple trajectories and isoclines; (c-d) Logistic prey growth model showing damped oscillations; (e) Type II functional response dynamics; (f) Rosenzweig-MacArthur limit cycle; (g) Conserved quantity verification; (h) Comparison of functional responses; (i) Sensitivity to prey growth rate.

Table 1: Lotka-Volterra Model Parameters and Results

Parameter	Symbol	Value
Prey growth rate	α	1.0
Predation rate	β	0.1
Predator efficiency	δ	0.075
Predator death rate	γ	1.5
Equilibrium prey	x^*	20.00
Equilibrium predator	y^*	10.00
Oscillation period	T	5.48

Table 2: Stability Analysis at Coexistence Equilibrium

Property	Value
Eigenvalue 1	$0.0000 + 1.2247j$
Eigenvalue 2	$0.0000 - 1.2247j$
Angular frequency	$\omega = 1.2247$
Theoretical period	$T = 2\pi/\omega = 5.13$
Stability type	Center (neutral)

3 Computational Analysis

4 Results

4.1 Model Parameters

4.2 Stability Analysis

5 Discussion

Example 5.1 (Conservation Law) *The classical Lotka-Volterra system conserves the quantity:*

$$H(x, y) = \delta x - \gamma \ln x + \beta y - \alpha \ln y \tag{7}$$

This integral of motion ensures closed orbits in phase space.

Remark 5.1 (Structural Instability) *The classical model is structurally unstable: any perturbation to the equations (such as adding density dependence) changes the qualitative behavior from neutral cycles to either damped oscillations (stable spiral) or limit cycles.*

Example 5.2 (Paradox of Enrichment) *In the Rosenzweig-MacArthur model, increasing carrying capacity K can destabilize the equilibrium. Beyond a critical K, the system transitions from a stable spiral to a limit cycle via a Hopf bifurcation. This counterintuitive result suggests that enriching an ecosystem can lead to larger oscillations and potential extinction.*

Remark 5.2 (Ecological Implications) *Predator-prey oscillations explain phenomena such as:*

- *Lynx-hare cycles in Canadian fur trade records*
- *Phytoplankton-zooplankton dynamics in lakes*
- *Host-parasitoid population cycles*

- *Microbial predator-prey systems*

The oscillation period depends on the geometric mean of growth and death rates.

6 Conclusions

This analysis demonstrates the rich dynamics of predator-prey systems:

1. Classic Lotka-Volterra produces neutral oscillations with period $T \approx 5.5$
2. Equilibrium at $(x^*, y^*) = (20.0, 10.0)$ with eigenvalues $\lambda = \pm i1.225$
3. Adding logistic prey growth creates damped oscillations (stable spiral)
4. Type II functional response can generate limit cycles
5. The conserved quantity H ensures orbit closure in the classic model

Further Reading

- Murray, J.D. *Mathematical Biology I: An Introduction*, 3rd ed. Springer, 2002.
- Edelstein-Keshet, L. *Mathematical Models in Biology.* SIAM, 2005.
- Kot, M. *Elements of Mathematical Ecology.* Cambridge, 2001.