Network Science: Graph Metrics and Community Detection

Computational Science Templates

November 24, 2025

Abstract

Network science provides tools for analyzing complex systems through graph theory. This document demonstrates computational methods for calculating centrality measures, detecting community structure, analyzing random graph models, and characterizing small-world and scale-free properties. Using PythonTeX, we implement algorithms for betweenness centrality, modularity optimization, Erdos-Renyi and Barabasi-Albert models, and compute characteristic path lengths and clustering coefficients.

1 Introduction

Network analysis characterizes complex systems through graph metrics. Networks appear throughout science: social networks, biological networks (protein interactions, neural connections), infrastructure networks (power grids, transportation), and information networks (World Wide Web, citation graphs). This analysis computes centrality measures, identifies community structure, and compares random network models.

2 Mathematical Framework

2.1 Centrality Measures

Degree centrality measures local connectivity:

$$C_D(v) = \frac{\deg(v)}{n-1} \tag{1}$$

Betweenness centrality quantifies importance in shortest paths:

$$C_B(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}} \tag{2}$$

where σ_{st} is the number of shortest paths from s to t, and $\sigma_{st}(v)$ passes through v.

Closeness centrality measures average distance to all nodes:

$$C_C(v) = \frac{n-1}{\sum_{u \neq v} d(v,u)} \tag{3}$$

Eigenvector centrality assigns importance based on neighbor importance:

$$x_v = \frac{1}{\lambda} \sum_{u \in N(v)} x_u \tag{4}$$

2.2 Clustering and Transitivity

Local clustering coefficient:

$$C_i = \frac{2|e_{jk} : v_j, v_k \in N_i, e_{jk} \in E|}{k_i(k_i - 1)} \tag{5}$$

Global clustering coefficient (transitivity):

$$C = \frac{3 \times \text{number of triangles}}{\text{number of connected triples}} \tag{6}$$

2.3 Modularity

Modularity measures community quality:

$$Q = \frac{1}{2m} \sum_{ij} \left[A_{ij} - \frac{k_i k_j}{2m} \right] \delta(c_i, c_j) \tag{7}$$

where A_{ij} is the adjacency matrix, k_i is degree, and $\delta(c_i, c_j) = 1$ if nodes i and j are in the same community.

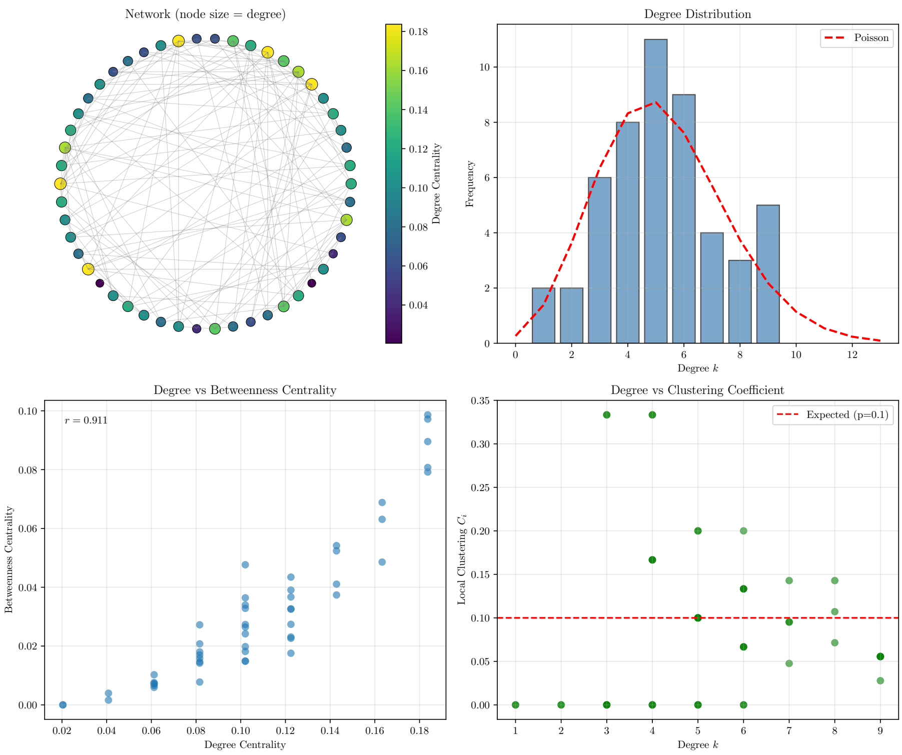

Figure 1: Basic network metrics for Erdos-Renyi random graph

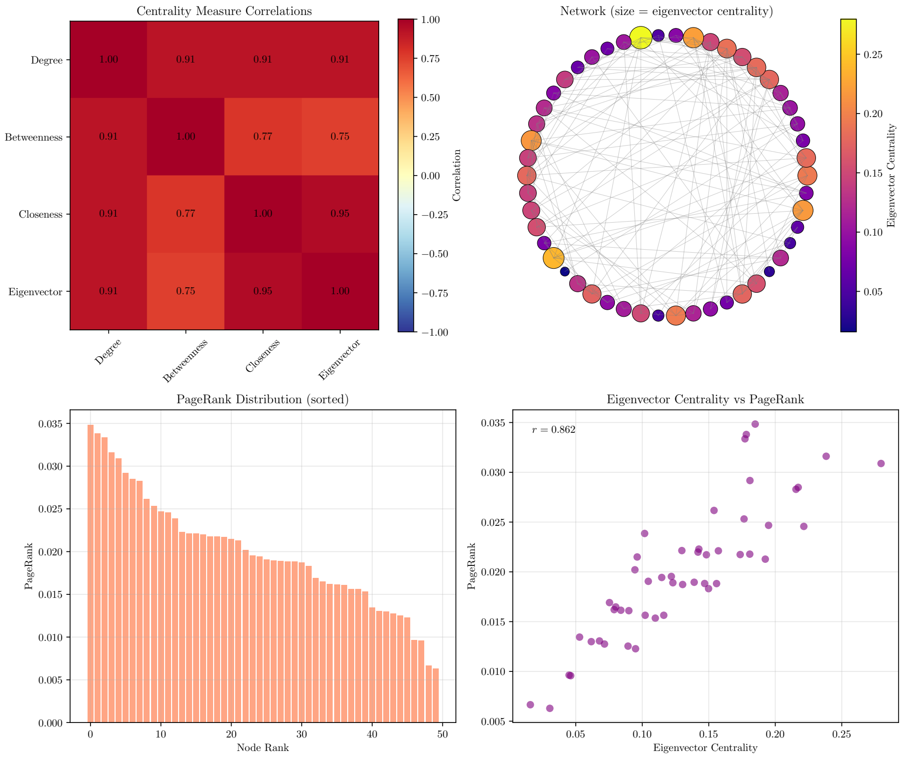

Figure 2: Advanced centrality measures

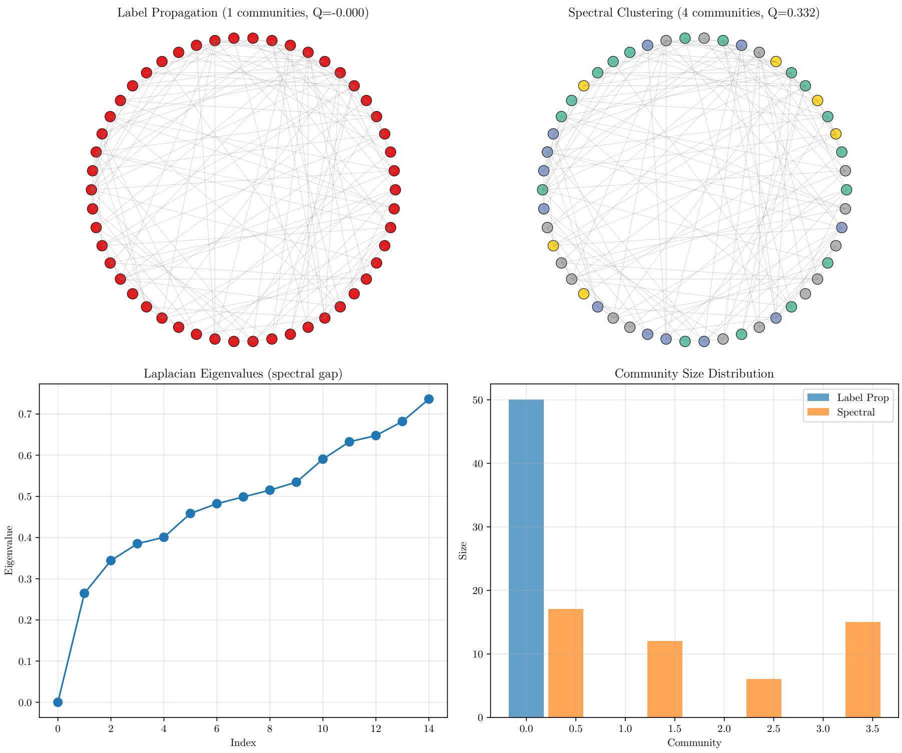

Figure 3: Community detection using label propagation and spectral clustering

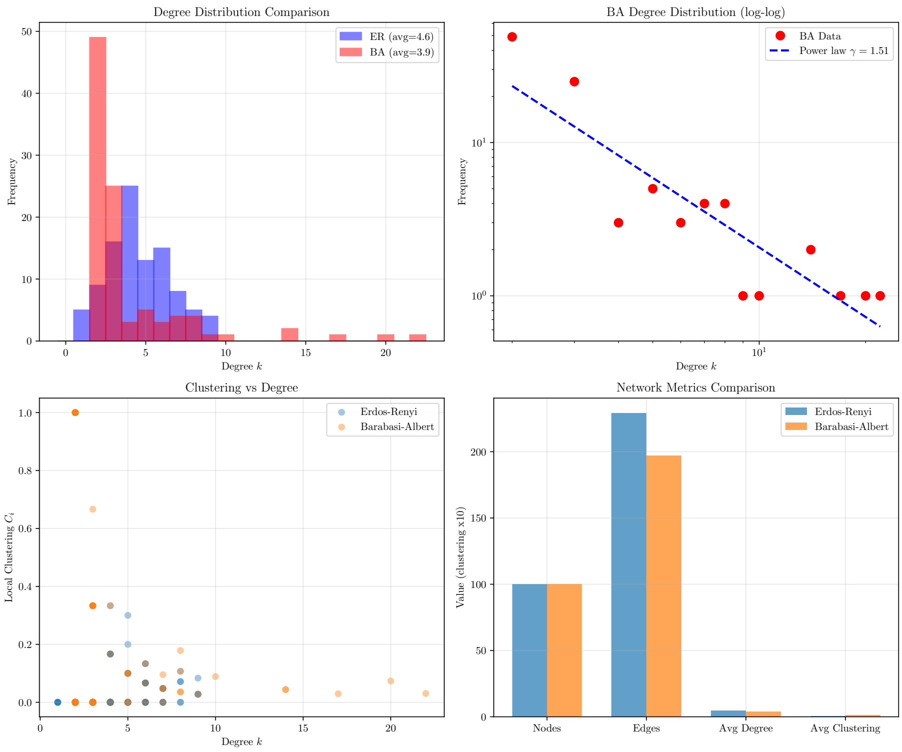

Figure 4: Comparison of Erdos-Renyi and Barabasi-Albert random graph models

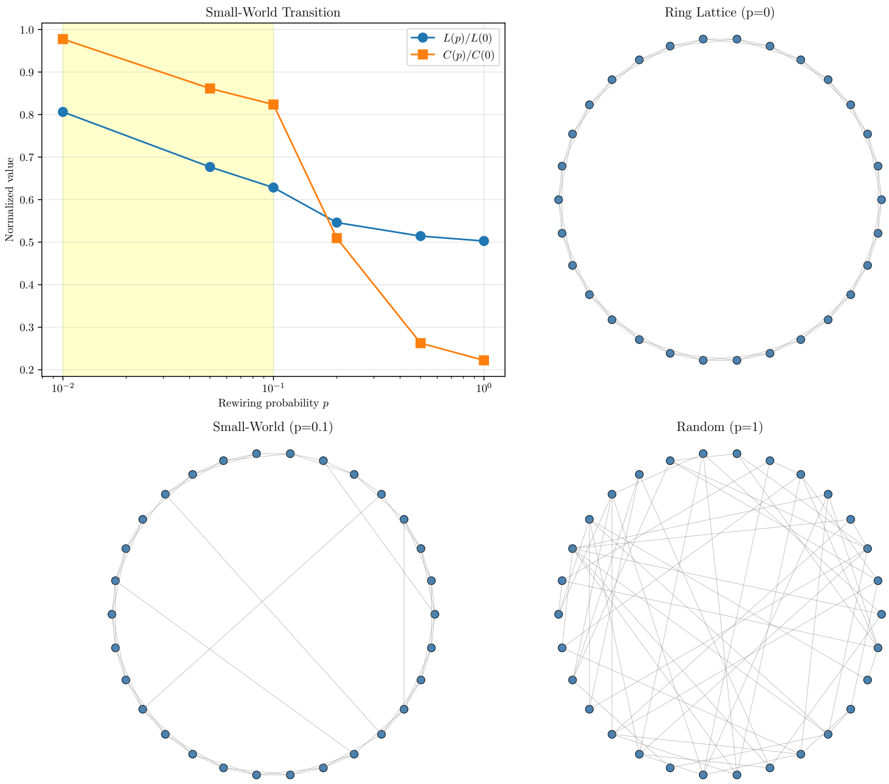

Figure 5: Watts-Strogatz small-world network model

3 Environment Setup

4 Network Generation and Basic Metrics

5 Betweenness Centrality

6 Eigenvector Centrality and PageRank

7 Community Detection

8 Random Graph Models

9 Small-World Networks

10 Network Motifs and Structural Patterns

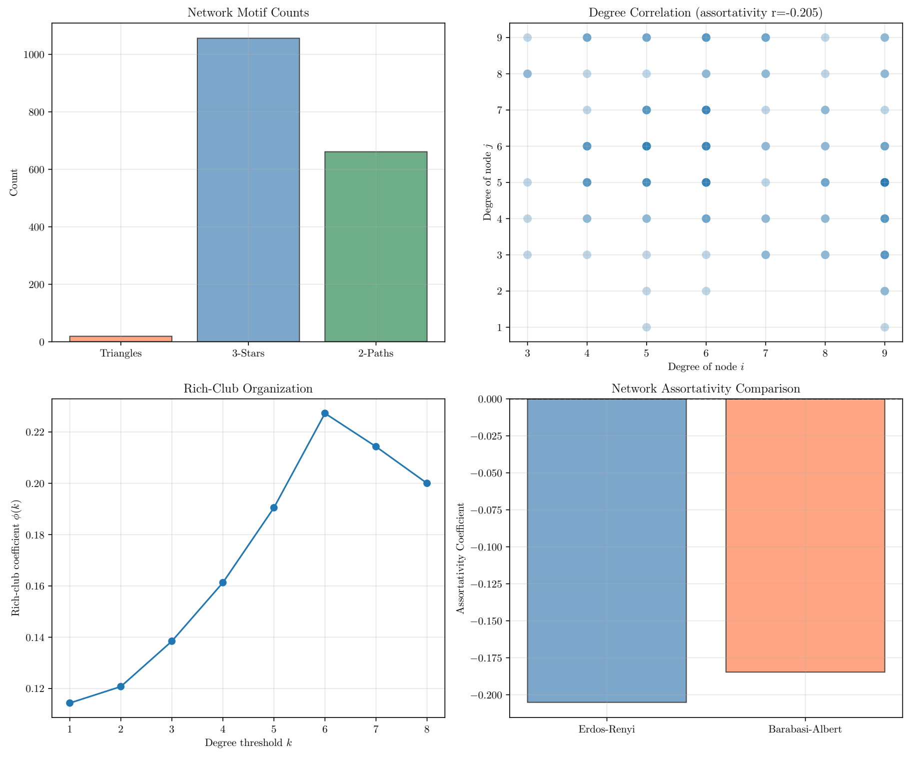

Figure 6: Network motifs, assortativity, and rich-club organization

11 Results Summary

Table 1: Network Analysis Results

Metric	ER Network	BA Network
Nodes	50	100
Edges	131	197
Average Degree	5.24	3.94
Average Clustering	0.0917	0.1355
Characteristic Path Length	2.51	–
Global Clustering	0.0862	–
Assortativity	-0.205	-0.185

Table 2: Top Nodes by Centrality Measures

Rank	Degree	Betweenness	Eigenvector
1	Node 25	Node 25	Node 14
2	Node 30	Node 9	Node 30
3	Node 14	Node 6	Node 11
4	Node 9	Node 14	Node 48
5	Node 6	Node 30	Node 23

Table 3: Community Detection Results

Method	Communities	Modularity
Label Propagation	1	-0.0000
Spectral Clustering	4	0.3325

Table 4: Network Motif Counts

Motif Type	Count
Triangles	19
3-Stars	1056
2-Paths	661

12 Statistical Summary

Network analysis results:

- Nodes: 50, Edges: 131

- Average degree: 5.24
- Average clustering: 0.0917
- Global clustering: 0.0862
- Characteristic path length: 2.51
- Network density: 0.1069
- Number of triangles: 19
- Assortativity coefficient: -0.205
- Communities detected (LP): 1
- Modularity (LP): -0.0000
- Modularity (Spectral): 0.3325

13 Conclusion

Network metrics reveal structural properties of complex systems. This analysis demonstrated:

1. **Centrality measures**: Degree, betweenness, closeness, and eigenvector centrality capture different aspects of node importance. High correlation between measures indicates consistent network structure.
2. **Community detection**: Label propagation and spectral clustering identify community structure with modularity quantifying partition quality.
3. **Random graph models**: Erdos-Renyi produces homogeneous degree distributions, while Barabasi-Albert generates scale-free networks with power-law degree distributions through preferential attachment.
4. **Small-world networks**: Watts-Strogatz model shows that small rewiring probability creates networks with high clustering and short path lengths, characteristic of real-world networks.
5. **Network motifs**: Triangle counting and rich-club analysis reveal hierarchical organization and hub structure.

These tools apply to social networks, biological systems (protein interactions, neural networks), infrastructure (power grids, transportation), and information networks (WWW, citations). Understanding network topology enables prediction of system behavior, identification of critical nodes, and optimization of network performance.