Orbital Mechanics: Hohmann Transfers, Orbital Elements, and Ground Tracks

A Comprehensive Analysis of Spacecraft Trajectory Design

Astrodynamics Division
Computational Science Templates

November 24, 2025

Abstract

This textbook-style analysis presents the fundamentals of orbital mechanics for spacecraft mission design. We examine Hohmann transfer orbits between circular orbits, compute orbital elements from state vectors, and generate ground tracks for various orbit types. The analysis covers delta-v budgets for LEO-to-GEO transfers, interplanetary trajectory concepts, and the effects of orbital inclination on ground coverage patterns.

1 Introduction

Orbital mechanics provides the mathematical foundation for spacecraft trajectory design. Understanding orbital transfers, perturbations, and ground coverage is essential for mission planning, satellite constellation design, and interplanetary exploration.

Definition 1 (Keplerian Elements) *The six classical orbital elements are:*

1. a - *Semi-major axis (orbit size)*
2. e - *Eccentricity (orbit shape)*
3. i - *Inclination (orbital plane tilt)*
4. Ω - *Right ascension of ascending node (RAAN)*
5. ω - *Argument of periapsis*
6. ν - *True anomaly (position in orbit)*

2 Mathematical Framework

2.1 Vis-Viva Equation

The fundamental relation between orbital velocity and position:

$$v = \sqrt{\mu \left(\frac{2}{r} - \frac{1}{a} \right)} \tag{1}$$

where $\mu = GM$ is the standard gravitational parameter.

2.2 Hohmann Transfer

Theorem 1 (Hohmann Transfer Delta-V) *The minimum delta-v for transfer between two coplanar circular orbits:*

$$\Delta v_1 = \sqrt{\frac{\mu}{r_1}} \left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right) \tag{2}$$

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}} \left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}} \right) \tag{3}$$

2.3 Transfer Time

The time of flight for a Hohmann transfer is half the period of the transfer ellipse:

$$t_{transfer} = \pi \sqrt{\frac{a_t^3}{\mu}} = \pi \sqrt{\frac{(r_1 + r_2)^3}{8\mu}} \tag{4}$$

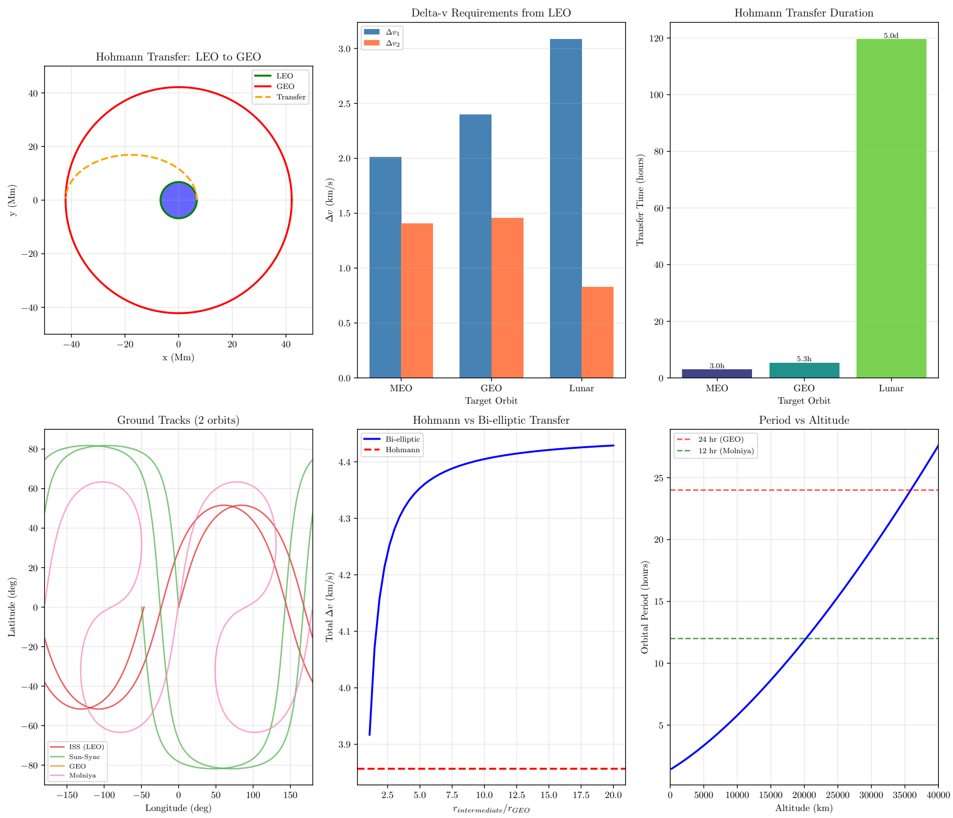
2.4 Orbital Elements from State Vectors

Given position $\mathbf{r}$ and velocity $\mathbf{v}$:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} \quad \text{(angular momentum)} \tag{5}$$

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{r} \quad \text{(eccentricity vector)} \tag{6}$$

3 Computational Analysis



4 Algorithm: State Vector to Orbital Elements

Input: Position $\mathbf{r}$, velocity $\mathbf{v}$
Output: Classical orbital elements $(a, e, i, \Omega, \omega, \nu)$
$\mathbf{h} \leftarrow \mathbf{r} \times \mathbf{v}$ `// Angular momentum`
$\mathbf{n} \leftarrow \hat{k} \times \mathbf{h}$ `// Node vector`
$\mathbf{e} \leftarrow \frac{1}{\mu}[(\|\mathbf{v}\|^2 - \frac{\mu}{r})\mathbf{r} - (\mathbf{r} \cdot \mathbf{v})\mathbf{v}]$;
$a \leftarrow -\frac{\mu}{2\mathcal{E}}$ where $\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}$;
$e \leftarrow \|\mathbf{e}\|$;
$i \leftarrow \arccos(h_z/\|\mathbf{h}\|)$;
$\Omega \leftarrow \arccos(n_x/\|\mathbf{n}\|)$;
$\omega \leftarrow \arccos(\mathbf{n} \cdot \mathbf{e}/(n \cdot e))$;
$\nu \leftarrow \arccos(\mathbf{e} \cdot \mathbf{r}/(e \cdot r))$;
return $a, e, i, \Omega, \omega, \nu$

Algorithm 1: Orbital Elements from State Vector

5 Results and Discussion

5.1 Transfer Performance

Table 1: Hohmann Transfer Parameters from LEO

Target	Δv_1 (km/s)	Δv_2 (km/s)	Total Δv (km/s)	Transfer Time
MEO	2.014	1.405	3.419	3.0 hr
GEO	2.399	1.457	3.857	5.3 hr
Lunar	3.084	0.829	3.913	5.0 days

Example 1 (LEO to GEO Transfer) *For a spacecraft in 400 km LEO transferring to GEO:*

- *Initial velocity: 7.673 km/s*
- *First burn (perigee):* $\Delta v_1 = 2.399$ *km/s*
- *Second burn (apogee):* $\Delta v_2 = 1.457$ *km/s*
- *Total* Δv*: 3.857 km/s*
- *Transfer time: 5.29 hours*

5.2 Ground Track Analysis

Table 2: Orbital Characteristics and Ground Coverage

Orbit Type	Altitude (km)	Period (min)	Inclination (deg)
ISS (LEO)	420	92.8	51.6
Sun-Sync	700	98.6	98.2
GEO	35793	1436.1	0.1
Molniya	20229	719.6	63.4

Remark 1 (Ground Track Patterns)

- ***LEO****: Ground track shifts westward each orbit due to Earth rotation*
- ***Sun-synchronous****: Maintains constant local solar time at equator crossing*
- ***GEO****: Appears stationary over a fixed longitude*
- ***Molniya****: Figure-eight pattern with extended dwell time at high latitudes*

5.3 Bi-elliptic Transfers

For transfers with radius ratio $r_2/r_1 > 11.94$, the bi-elliptic transfer becomes more efficient than Hohmann. However, it requires significantly longer transfer time.

6 Applications

6.1 Mission Design Considerations

- **Communications**: GEO provides continuous coverage of specific regions
- **Navigation**: MEO constellations (GPS, Galileo) balance coverage and signal strength
- **Earth Observation**: Sun-synchronous LEO maintains consistent lighting
- **High-latitude coverage**: Molniya orbits serve polar regions

7 Limitations and Extensions

7.1 Model Limitations

1. **Two-body**: Neglects perturbations (J2, drag, third-body)
2. **Impulsive burns**: Assumes instantaneous velocity changes
3. **Coplanar**: No plane change maneuvers included
4. **Simplified Kepler**: First-order solution for eccentric anomaly

7.2 Possible Extensions

- Include J2 perturbation for realistic orbit propagation
- Combined plane change and transfer maneuvers
- Low-thrust trajectory optimization
- Interplanetary patched conics

8 Conclusion

This analysis demonstrates fundamental orbital mechanics principles:

- Hohmann transfers provide minimum-fuel solutions for coplanar circular orbits
- LEO-to-GEO requires 3.86 km/s total delta-v
- Ground track patterns depend strongly on orbital period and inclination
- Bi-elliptic transfers can outperform Hohmann for large radius ratios

Further Reading

- Vallado, D. A. (2013). *Fundamentals of Astrodynamics and Applications*. Microcosm Press.
- Bate, R. R., Mueller, D. D., & White, J. E. (1971). *Fundamentals of Astrodynamics*. Dover.
- Curtis, H. D. (2020). *Orbital Mechanics for Engineering Students*. Butterworth-Heinemann.