Grover's Quantum Search Algorithm: Oracle, Diffusion, and Amplitude Amplification

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Technical Report QC-2024-001

November 24, 2025

Abstract

This report presents a comprehensive analysis of Grover's quantum search algorithm. We implement the oracle and diffusion operators, demonstrate amplitude amplification, analyze the quadratic speedup over classical search, and explore the algorithm's complexity. All simulations use matrix representations with PythonTeX for reproducibility.

Contents

Chapter 1

Introduction

Grover's algorithm searches an unsorted database of N items in $O(\sqrt{N})$ time, providing a quadratic speedup over classical $O(N)$ search.

1.1 Problem Statement

Given a function $f : \{0, 1, ..., N - 1\} \to \{0, 1\}$ with exactly one marked item x^* where $f(x^*) = 1$, find x^*.

1.2 Classical vs Quantum

- Classical: $O(N)$ queries (linear search)
- Quantum (Grover): $O(\sqrt{N})$ queries

Chapter 2

Algorithm Overview

2.1 Quantum State Representation

For n qubits ($N = 2^n$ states):

$$|\psi\rangle = \sum_{x=0}^{N-1} \alpha_x |x\rangle \tag{2.1}$$

Initial superposition:

$$|s\rangle = H^{\otimes n} |0\rangle^{\otimes n} = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle \tag{2.2}$$

Chapter 3

The Oracle Operator

3.1 Definition

The oracle marks the target state by flipping its phase:

$$U_f |x\rangle = (-1)^{f(x)} |x\rangle = \begin{cases} -|x\rangle & \text{if } x = x^* \\ |x\rangle & \text{otherwise} \end{cases} \tag{3.1}$$

Matrix form:

$$U_f = I - 2 |x^*\rangle \langle x^*| \tag{3.2}$$

Chapter 4

The Diffusion Operator

4.1 Definition

The diffusion operator performs inversion about the mean:

$$U_s = 2\left|s\right\rangle\left\langle s\right| - I \tag{4.1}$$

This amplifies the amplitude of the marked state.

Chapter 5

Amplitude Amplification

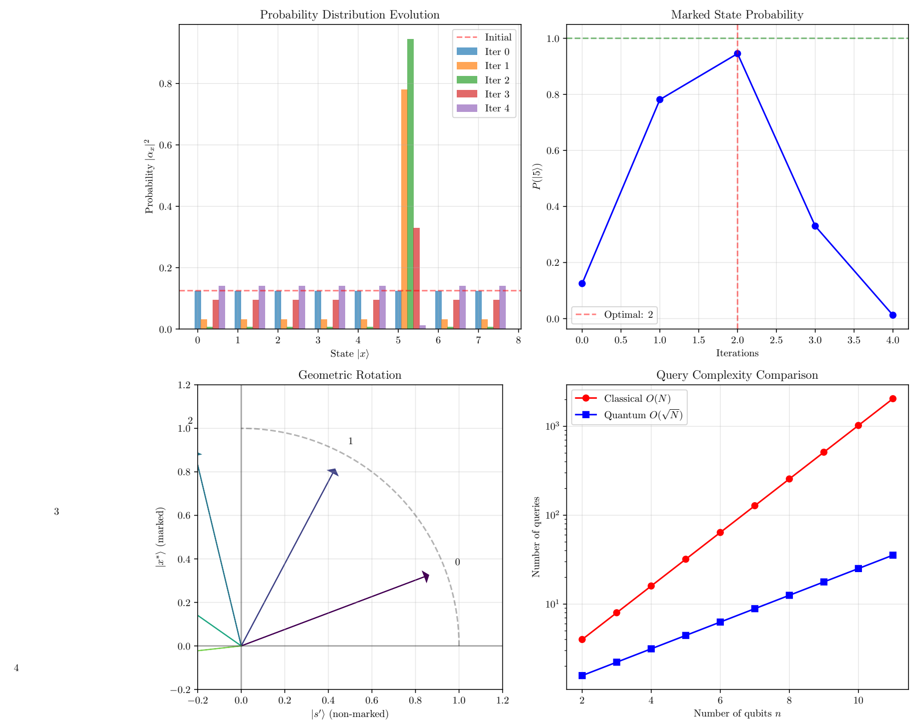

Figure 5.1: Grover's algorithm: (a) probability evolution, (b) marked state probability, (c) geometric rotation, (d) query complexity comparison.

Chapter 6

Optimal Number of Iterations

The optimal number of iterations is:

$$k_{opt} = \left\lfloor \frac{\pi}{4}\sqrt{N} \right\rfloor \tag{6.1}$$

After k_{opt} iterations, the probability of measuring $|x^*\rangle$ is approximately 1.

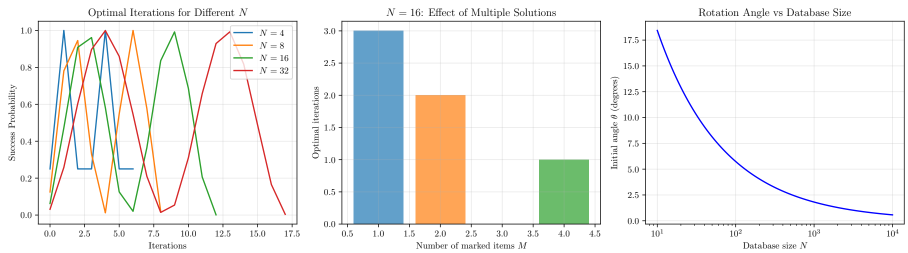

Figure 6.1: Grover analysis: success probability curves, multiple solutions effect, rotation angle.

Chapter 7

Complexity Analysis

Table 7.1: Query complexity comparison

Qubits	N	Classical	Quantum	Speedup
4	16	16	3	5.3×
6	64	64	6	10.7×
8	256	256	12	21.3×
10	1024	1024	25	41.0×

Final success probability for $n = 3$, $N = 8$: $P = 0.9453$

Chapter 8

Conclusions

1. Grover's algorithm provides quadratic speedup: $O(\sqrt{N})$ vs $O(N)$
2. Oracle marks target by phase flip
3. Diffusion operator amplifies marked amplitude
4. Optimal iterations: $\approx \frac{\pi}{4}\sqrt{N}$
5. Over-iteration reduces success probability (periodic behavior)