Root Finding Algorithms: Convergence Analysis and Comparison

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Abstract

This report presents a comprehensive analysis of root-finding algorithms for nonlinear equations. We implement and compare bisection, Newton-Raphson, secant, and Brent's methods. Convergence rates are analyzed theoretically and verified computationally. The importance of initial guesses and potential pitfalls of each method are demonstrated through examples.

Contents

Chapter 1

Introduction

Root finding seeks solutions to $f(x) = 0$. We classify methods by their convergence rate:

- **Linear**: $|e_{n+1}| \leq C|e_n|$, where $C < 1$
- **Superlinear**: $|e_{n+1}| \leq C|e_n|^p$, $1 < p < 2$
- **Quadratic**: $|e_{n+1}| \leq C|e_n|^2$

Chapter 2

Bisection Method

2.1 Algorithm

Given $f(a) \cdot f(b) < 0$:

1. Compute midpoint: $c = \frac{a+b}{2}$
2. If $f(a) \cdot f(c) < 0$, set $b = c$; else set $a = c$
3. Repeat until $|b - a| < \epsilon$

Convergence: Linear, $|e_{n+1}| = \frac{1}{2}|e_n|$

2.2 Comparison of Methods

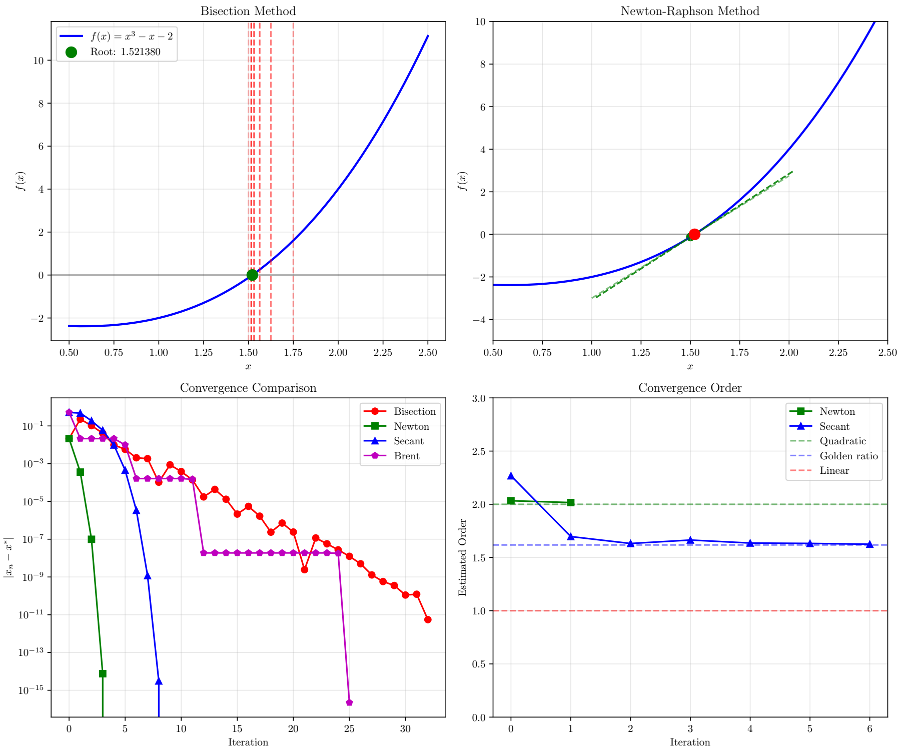

Figure 2.1: Root finding methods: (a) bisection visualization, (b) Newton tangent lines, (c) error convergence, (d) convergence order estimation.

Chapter 3

Newton-Raphson Method

3.1 Algorithm

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3.1}$$

Convergence: Quadratic near root (when $f'(x^*) \neq 0$)

3.2 Pitfalls

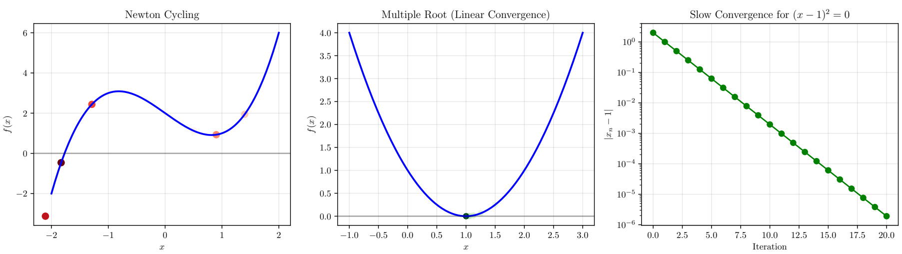

Figure 3.1: Newton method pitfalls: cycling, slow convergence for multiple roots.

Chapter 4

Secant Method

4.1 Algorithm

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \tag{4.1}$$

Convergence: Superlinear, $p = \frac{1+\sqrt{5}}{2} \approx 1.618$ (golden ratio)
Advantage: No derivative required.

Chapter 5

Brent's Method

Combines bisection, secant, and inverse quadratic interpolation. Guaranteed convergence with superlinear speed for smooth functions.

Chapter 6

Numerical Results

Table 6.1: Comparison for $f(x) = x^3 - x - 2$

Method	Iterations	Order	Notes
Bisection	33	Linear (1)	Guaranteed
Newton	5	Quadratic (2)	Requires f'
Secant	10	Superlinear (1.618)	No derivative
Brent	26	Superlinear	Robust

Chapter 7

Multiple Test Functions

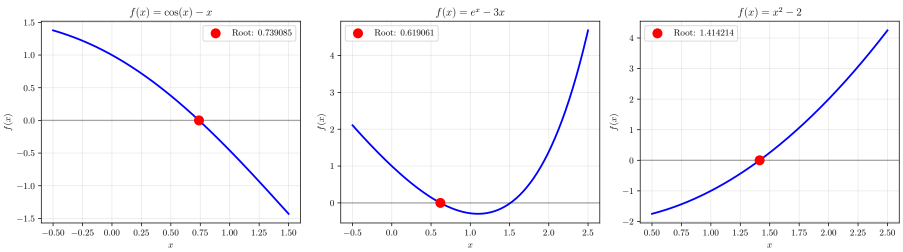

Figure 7.1: Root finding for various test functions.

Chapter 8

Conclusions

1. Bisection: Always converges but slow (linear)
2. Newton: Fast (quadratic) but requires derivative and good initial guess
3. Secant: Superlinear without derivatives
4. Brent: Best general-purpose method (robust + fast)
5. Multiple roots reduce Newton to linear convergence
6. Choice depends on: derivative availability, robustness needs, speed requirements