

Navier-Stokes Equations: Viscous Flow Analysis and Boundary Layer Theory

Fluid Mechanics Research Laboratory

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Abstract

This technical report presents analytical and computational solutions to the Navier-Stokes equations for canonical viscous flow problems. We analyze Couette flow, Poiseuille flow, and boundary layer development using Python-based numerical methods. Results include velocity profiles, shear stress distributions, and Reynolds number effects on flow characteristics.

1 Introduction

The Navier-Stokes equations govern the motion of viscous fluids and form the foundation of fluid mechanics:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g} \quad (1)$$

Definition 1.1 (Reynolds Number). *The Reynolds number characterizes the ratio of inertial to viscous forces:*

$$Re = \frac{\rho U L}{\mu} = \frac{U L}{\nu} \quad (2)$$

2 Computational Setup

Fluid Properties (Water at 20°C):

- Density: $\rho = ?? \text{ kg/m}^3$
- Dynamic Viscosity: $\mu = ?? \text{ mPa}\cdot\text{s}$
- Kinematic Viscosity: $\nu = ?? \text{ mm}^2/\text{s}$

3 Couette Flow Analysis

Couette flow occurs between two parallel plates where one plate moves relative to the other.

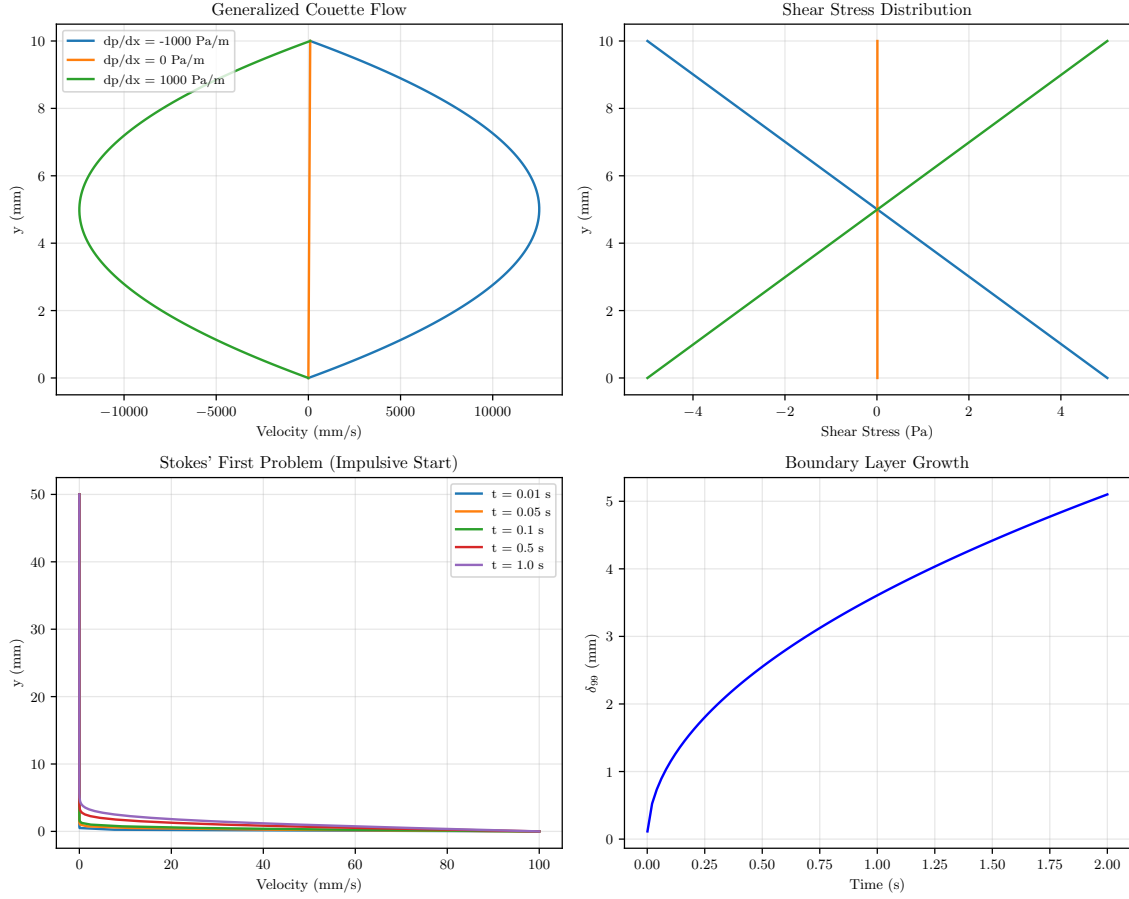


Figure 1: Couette flow analysis: velocity profiles, shear stress, and transient development.

Table 1: Couette Flow Parameters

Parameter	Value	Unit
Wall Velocity	??	mm/s
Channel Height	??	mm
Reynolds Number	??	—
Wall Shear Stress	??	Pa

4 Poiseuille Flow (Pressure-Driven)

Theorem 4.1 (Hagen-Poiseuille Equation). *For laminar flow in a circular pipe, the volumetric flow rate is:*

$$Q = \frac{\pi R^4}{8\mu} \left(-\frac{dp}{dx} \right) \quad (3)$$

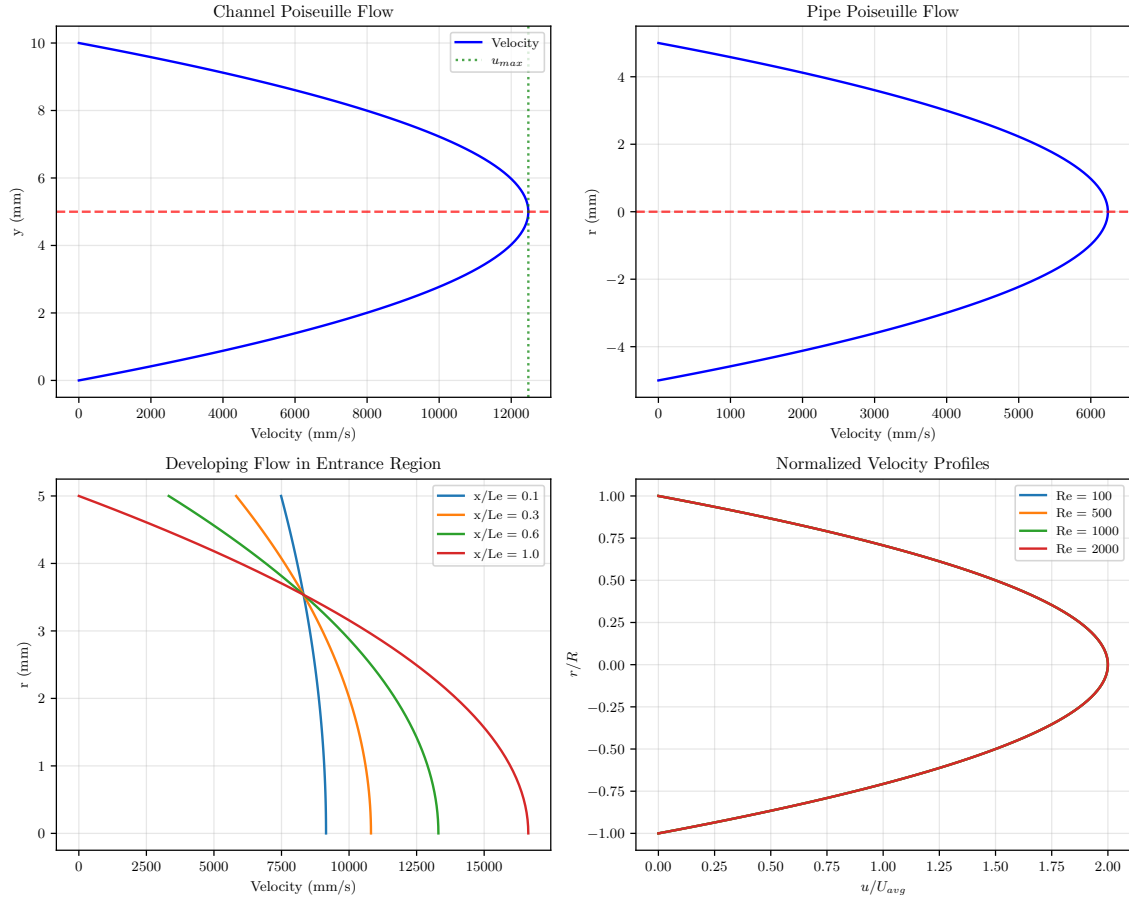


Figure 2: Poiseuille flow analysis: velocity profiles and entrance region development.

Table 2: Poiseuille Flow Results

Parameter	Value	Unit
Maximum Velocity	??	mm/s
Average Velocity	??	mm/s
Flow Rate (channel)	??	mm ² /s
Reynolds Number	??	—
Entrance Length	??	mm

5 Boundary Layer Analysis

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Figure 3: Blasius boundary layer solution: velocity profile, thickness growth, and skin friction.

6 Reynolds Number Effects

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Figure 4: Reynolds number effects on flow characteristics.

7 Vorticity and Stream Function

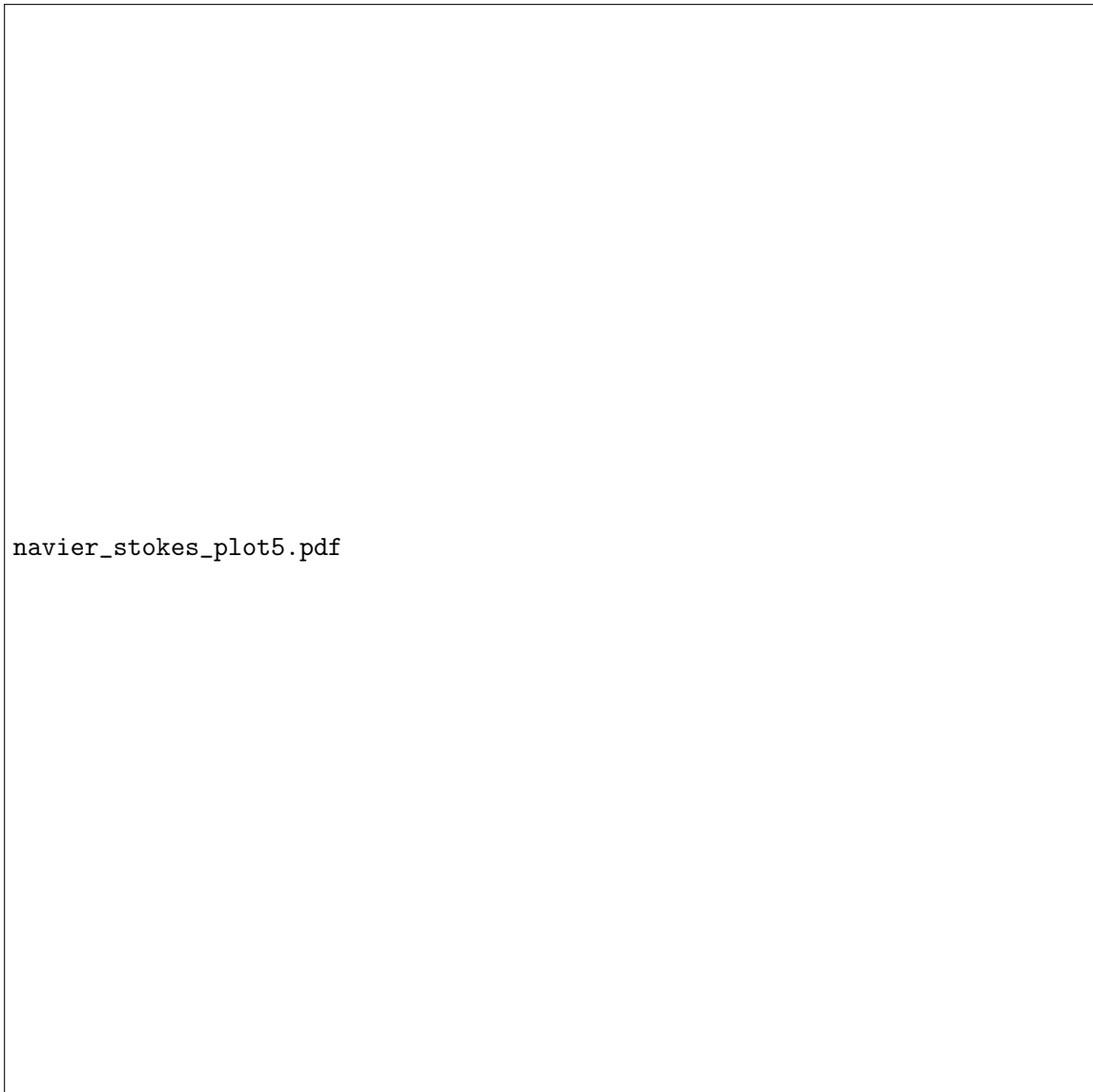


Figure 5: Flow field visualization: stream function, velocity magnitude, vorticity, and vectors.

8 Oscillatory Flow (Stokes' Second Problem)



Figure 6: Oscillatory flow: Stokes' second problem and Stokes layer characteristics.

Table 3: Oscillatory Flow Parameters

Parameter	Value	Unit
Oscillation Frequency	??	Hz
Stokes Layer Thickness	??	mm
Penetration Depth ($3\delta_s$)	??	mm

9 Conclusions

This analysis of the Navier-Stokes equations demonstrated:

1. **Couette Flow:** Linear velocity profile with wall shear stress $\tau_w = ??$ Pa. Generalized solutions with pressure gradients show parabolic modifications.

2. **Poiseuille Flow:** Parabolic velocity profile with maximum velocity $u_{max} = ??$ mm/s. The entrance length scales linearly with Reynolds number for laminar flow.
3. **Boundary Layers:** Blasius solution gives $f''(0) = ??$, with boundary layer thickness $\delta \propto x^{1/2}$.
4. **Reynolds Number Effects:** Critical $Re = 2300$ for pipe flow transition, with friction factor following the Blasius correlation in turbulent regime.
5. **Oscillatory Flows:** Stokes layer thickness $\delta_s = ??$ mm determines penetration depth of oscillations.