Ordinary Differential Equations: Phase Portraits, Stability Analysis, and Limit Cycles

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Abstract

This report provides a comprehensive computational analysis of ordinary differential equations (ODEs). We examine first and second-order ODEs, construct phase portraits for autonomous systems, perform stability analysis of equilibrium points, and investigate limit cycles in nonlinear oscillators. The van der Pol oscillator is analyzed in detail to demonstrate relaxation oscillations and Hopf bifurcations.

Contents

Chapter 1

Introduction

Ordinary differential equations describe the evolution of systems in terms of derivatives with respect to a single variable. This report focuses on qualitative analysis through phase portraits and stability theory.

1.1 Classification of ODEs

- **First-order**: $\frac{dy}{dt} = f(t, y)$
- **Second-order**: $\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$
- **Linear**: Coefficients are functions of t only
- **Autonomous**: No explicit time dependence

Chapter 2

First-Order ODEs

2.1 Analytical Solutions

Consider the first-order linear ODE:

$$\frac{dy}{dt} + p(t)y = q(t) \tag{2.1}$$

Solution via integrating factor $\mu(t) = e^{\int p(t)dt}$:

$$y(t) = \frac{1}{\mu(t)} \left[\int \mu(t)q(t)dt + C \right] \tag{2.2}$$

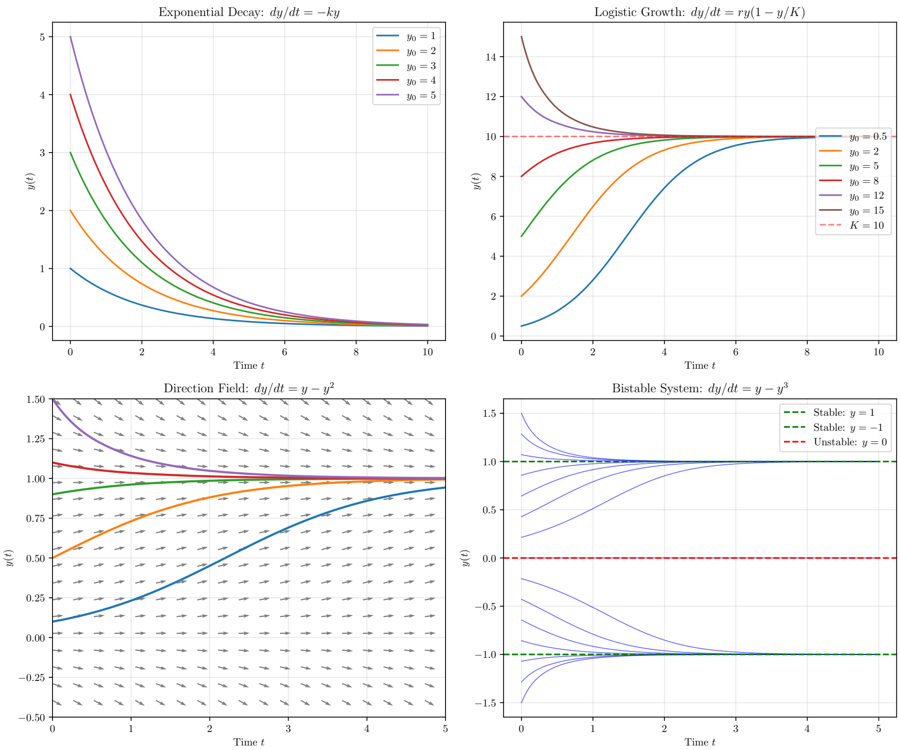

Figure 2.1: First-order ODE solutions: (a) exponential decay, (b) logistic growth, (c) direction field, (d) bistable system.

Chapter 3

Second-Order Linear ODEs

3.1 Harmonic Oscillator

The damped harmonic oscillator:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{3.1}$$

Characteristic equation: $mr^2 + cr + k = 0$

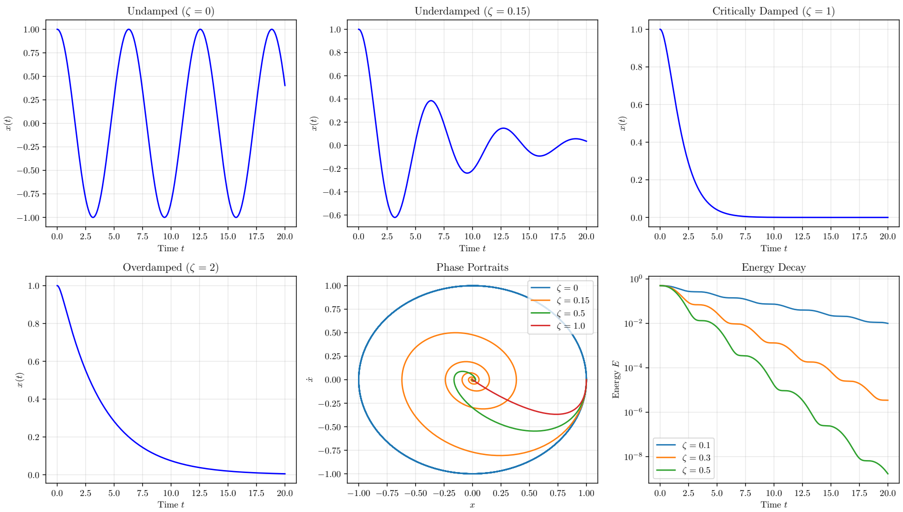

Figure 3.1: Second-order ODE: damped harmonic oscillator with various damping ratios.

Chapter 4

Phase Portrait Analysis

4.1 2D Autonomous Systems

Consider the system:

$$\frac{dx}{dt} = f(x, y) \tag{4.1}$$

$$\frac{dy}{dt} = g(x, y) \tag{4.2}$$

Equilibrium points satisfy $f(x^*, y^*) = g(x^*, y^*) = 0$.

4.2 Linear Stability Analysis

Near an equilibrium, linearize using the Jacobian:

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \tag{4.3}$$

Eigenvalues λ determine stability.

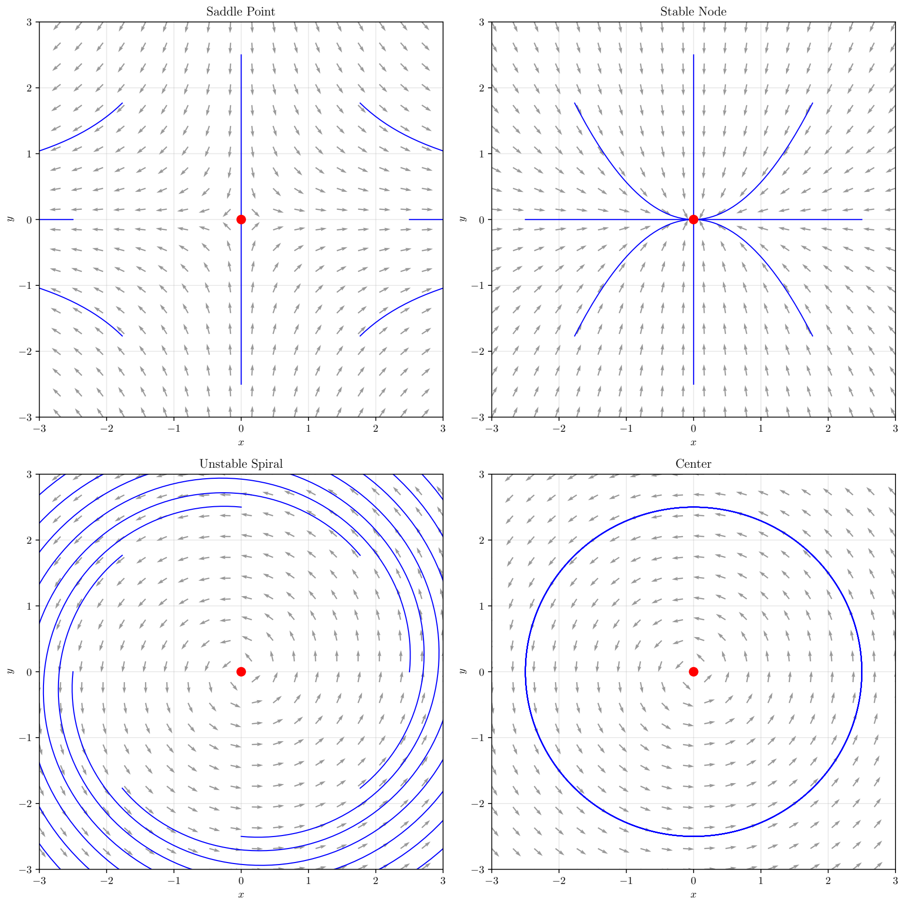

Figure 4.1: Phase portraits showing different equilibrium types based on eigenvalue classification.

Chapter 5

Limit Cycles

5.1 Van der Pol Oscillator

The van der Pol oscillator exhibits self-sustained oscillations:

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0 \tag{5.1}$$

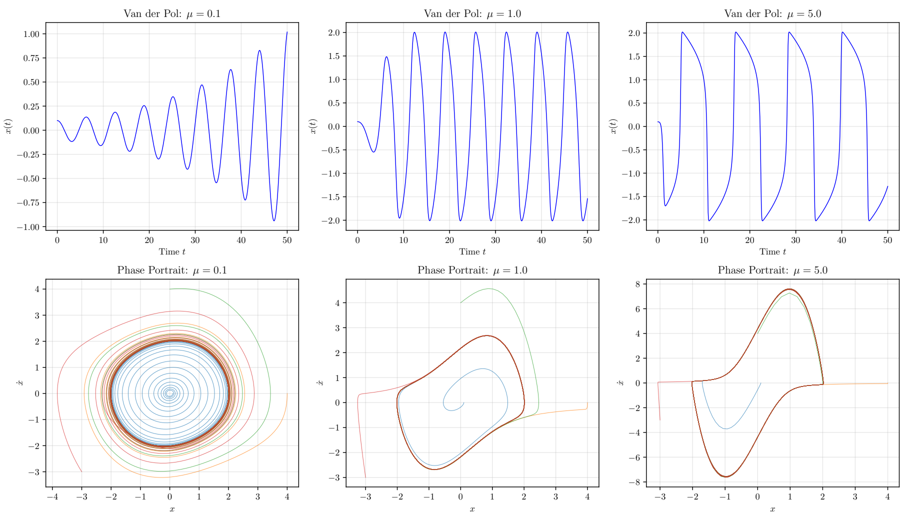

Figure 5.1: Van der Pol oscillator: time series (top) and phase portraits (bottom) showing limit cycles.

The limit cycle amplitude for $\mu = 1$ is approximately 2.014.

Chapter 6

Predator-Prey Dynamics

6.1 Lotka-Volterra Equations

$$\frac{dx}{dt} = \alpha x - \beta xy \tag{6.1}$$

$$\frac{dy}{dt} = \delta xy - \gamma y \tag{6.2}$$

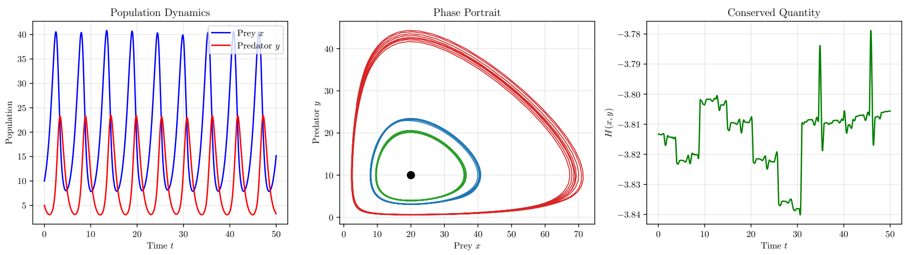

Figure 6.1: Lotka-Volterra predator-prey model: populations oscillate with period $T \approx 5.47$.

Chapter 7

Numerical Results

Table 7.1: Equilibrium classification by eigenvalues

Eigenvalues	Type	Stability
Real, opposite signs	Saddle	Unstable
Real, both negative	Stable node	Stable
Real, both positive	Unstable node	Unstable
Complex, Re < 0	Stable spiral	Stable
Complex, Re > 0	Unstable spiral	Unstable
Pure imaginary	Center	Neutral

Chapter 8

Conclusions

1. First-order ODEs: direction fields reveal solution behavior
2. Second-order ODEs: damping ratio determines oscillation character
3. Phase portraits: eigenvalues classify equilibrium types
4. Limit cycles: nonlinear systems can have isolated periodic orbits
5. Predator-prey: conservative systems show closed orbits