Monte Carlo Methods: Sampling, Integration, and MCMC

Computational Simulation Templates

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1 Introduction

Monte Carlo methods use random sampling to solve deterministic problems. This template covers basic sampling techniques, numerical integration, importance sampling, and the Metropolis-Hastings algorithm for Markov Chain Monte Carlo.

2 Mathematical Framework

2.1 Monte Carlo Integration

Estimate integrals using random samples:

$$I = \int_a^b f(x)\,dx \approx \frac{b-a}{N}\sum_{i=1}^{N} f(x_i) \tag{1}$$

2.2 Variance of Estimator

The standard error decreases as $1/\sqrt{N}$:

$$\mathrm{SE} = \frac{\sigma_f}{\sqrt{N}} \tag{2}$$

2.3 Importance Sampling

Use proposal distribution $q(x)$ to reduce variance:

$$I = \int f(x)p(x)\,dx \approx \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)p(x_i)}{q(x_i)}, \quad x_i \sim q \tag{3}$$

2.4 Metropolis-Hastings Algorithm

Accept proposed state x' with probability:

$$\alpha = \min\left(1, \frac{p(x')q(x|x')}{p(x)q(x'|x)}\right) \tag{4}$$

3 Environment Setup

4 Basic Monte Carlo Integration

5 Numerical Integration

6 Importance Sampling

7 Metropolis-Hastings Algorithm

8 2D Metropolis-Hastings

9 Results Summary

9.1 Pi Estimation

9.2 Importance Sampling

9.3 MCMC Results

9.4 Statistical Summary

- Pi estimation error: **??**

- Gaussian integral estimate: **??**
- Importance sampling variance reduction: **??**x
- Optimal MCMC acceptance rate: **??**
- 2D MCMC acceptance rate: **??**

10 Conclusion

This template demonstrates Monte Carlo methods for numerical computation. Basic MC integration achieves $1/\sqrt{N}$ convergence, while importance sampling provides substantial variance reduction for rare events (**??**x improvement). The Metropolis-Hastings algorithm successfully samples from complex distributions, with optimal acceptance rates around 0.234 for 1D targets (achieved with $\sigma_q = 1.0$ giving **??**).