Cosmological Expansion: From Hubble's Law to Dark Energy

A Comprehensive Analysis of the ΛCDM Model

Cosmology Division
Computational Science Templates

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Abstract

This comprehensive analysis explores the expansion history of the universe from observational foundations to theoretical frameworks. We examine Hubble's law and its modern calibrations, derive the Friedmann equations governing cosmic evolution, and analyze different cosmological models including matter-dominated, radiation-dominated, and dark energy-dominated universes. The ΛCDM concordance model is developed in detail, with computational analysis of the scale factor evolution, distance-redshift relations, and the cosmic age problem. We explore observational evidence from Type Ia supernovae, baryon acoustic oscillations, and cosmic microwave background measurements that constrain cosmological parameters.

1 Introduction

The discovery that the universe is expanding stands as one of the most profound achievements of twentieth-century science. Edwin Hubble's 1929 observation of a linear relationship between galaxy distances and their recession velocities transformed our understanding of cosmic structure and evolution.

Definition 1 (Hubble's Law) *The recession velocity v of a galaxy is proportional to its distance d:*

$$v = H_0 d \tag{1}$$

where H_0 is the Hubble constant, currently measured at approximately $70\,\mathrm{km\,s^{-1}}$.

2 Theoretical Framework

2.1 The Friedmann Equations

General relativity applied to a homogeneous, isotropic universe yields the Friedmann equations:

Theorem 1 (First Friedmann Equation) *The expansion rate of the universe is determined by its energy content:*

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \tag{2}$$

where $a(t)$ is the scale factor, ρ is the total energy density, k is the spatial curvature, and Λ is the cosmological constant.

Theorem 2 (Second Friedmann Equation) *The acceleration of expansion:*

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{3}$$

where p is the pressure.

2.2 Density Parameters

We define dimensionless density parameters relative to the critical density $\rho_c = 3H_0^2/(8\pi G)$:

$$\Omega_m = \frac{\rho_m}{\rho_c} \quad \text{(matter)} \tag{4}$$

$$\Omega_r = \frac{\rho_r}{\rho_c} \quad \text{(radiation)} \tag{5}$$

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} \quad \text{(dark energy)} \tag{6}$$

$$\Omega_k = -\frac{kc^2}{H_0^2 a_0^2} \quad \text{(curvature)} \tag{7}$$

Remark 1 (Closure Relation) *For a universe with these components:*

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1 \tag{8}$$

2.3 Redshift and Scale Factor

The cosmological redshift z relates to the scale factor:

$$1 + z = \frac{a_0}{a(t)} = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} \tag{9}$$

The Hubble parameter at redshift z:

$$H(z) = H_0\sqrt{\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda} \tag{10}$$

3 Computational Analysis

4 Results and Discussion

4.1 Key Cosmological Parameters

Example 1 (ΛCDM Universe) *For the concordance cosmological model with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$:*

- *Age of the universe: ?? Gyr*
- *Current deceleration parameter: $q_0 =$??*
- *Matter-Λ equality redshift: $z_{eq} =$??*
- *Hubble time: $1/H_0 =$?? Gyr*

4.2 Distance Measures

4.3 Observational Evidence for Dark Energy

The evidence for cosmic acceleration comes from multiple independent probes:

1. **Type Ia Supernovae**: Standardizable candles show that distant SNe are fainter than expected in a decelerating universe
2. **Baryon Acoustic Oscillations**: Standard ruler measurements in galaxy surveys constrain $D_A(z)/r_s$
3. **Cosmic Microwave Background**: Angular power spectrum constrains $\Omega_m h^2$ and Ω_Λ
4. **Cluster Counts**: Evolution of galaxy cluster abundance sensitive to σ_8 and Ω_m

Remark 2 (Cosmic Coincidence Problem) *We appear to live at a special epoch when $\Omega_m \approx \Omega_\Lambda$. Given that these densities scale differently with redshift ($\rho_m \propto a^{-3}$ vs. $\rho_\Lambda = const$), this equality is a remarkable coincidence.*

5 Dark Energy Models

5.1 Cosmological Constant

The simplest dark energy model is Einstein's cosmological constant Λ:

- Equation of state: $w = p/\rho = -1$ (constant)
- Energy density: constant in time
- Interpretation: vacuum energy

5.2 Quintessence

Dynamical dark energy from a scalar field ϕ:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{11}$$

This gives $-1 < w < 1$ with possible time evolution.

6 The Cosmic Age Problem

7 Limitations and Extensions

7.1 Model Limitations

1. **Perfect homogeneity**: Real universe has structure
2. **Constant** w: Dark energy EoS may evolve
3. **Flat geometry**: Ω_k constrained but not zero
4. **No perturbations**: Linear theory not included

7.2 Possible Extensions

- Time-varying dark energy: $w(z) = w_0 + w_a \cdot z/(1+z)$
- Modified gravity: $f(R)$ theories
- Interacting dark sector
- Backreaction from inhomogeneities

8 Conclusion

This analysis demonstrates the fundamental framework of modern cosmology:

- The ΛCDM model provides an excellent fit to observations
- Dark energy dominates the current epoch ($\Omega_\Lambda \approx 0.7$)
- The universe is accelerating ($q_0 =$ **??** < 0)
- Cosmic age of **??** Gyr is consistent with stellar ages
- Future observations will constrain dark energy properties

Further Reading

- Peebles, P. J. E. (1993). *Principles of Physical Cosmology*. Princeton University Press.
- Weinberg, S. (2008). *Cosmology*. Oxford University Press.
- Planck Collaboration (2020). Planck 2018 results. VI. Cosmological parameters.