Chaos Theory and Nonlinear Dynamics: Analysis of Deterministic Chaos in Dynamical Systems

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Technical Report AM-2024-001

November 24, 2025

Abstract

This technical report presents a comprehensive computational analysis of chaotic dynamical systems. We examine the logistic map, compute bifurcation diagrams showing the route to chaos through period-doubling, calculate Lyapunov exponents as quantitative measures of chaos, and simulate the Lorenz attractor demonstrating strange attractor dynamics. All computations are performed using PythonTeX for reproducibility, with detailed numerical analysis of sensitivity to initial conditions and fractal basin boundaries.

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Chapter 1

Introduction

Chaos theory studies deterministic systems that exhibit unpredictable behavior due to extreme sensitivity to initial conditions. Despite being governed by deterministic equations, chaotic systems produce trajectories that appear random over long time scales.

1.1 Defining Chaos

A dynamical system is considered chaotic if it exhibits:

1. **Sensitivity to initial conditions**: Nearby trajectories diverge exponentially
2. **Topological mixing**: The system evolves to visit all accessible regions
3. **Dense periodic orbits**: Periodic trajectories are arbitrarily close to any point

1.2 Quantifying Chaos: Lyapunov Exponents

The maximal Lyapunov exponent λ measures the rate of separation of infinitesimally close trajectories:

$$|\delta \mathbf{x}(t)| \approx e^{\lambda t} |\delta \mathbf{x}(0)| \tag{1.1}$$

A positive Lyapunov exponent indicates chaos, with $\lambda > 0$ implying exponential divergence.

Chapter 2

The Logistic Map

2.1 Mathematical Definition

The logistic map is a paradigmatic example of chaos arising from a simple nonlinear recurrence:

$$x_{n+1} = rx_n(1 - x_n) \tag{2.1}$$

where $r \in [0, 4]$ is the control parameter and $x_n \in [0, 1]$ represents the state.

2.2 Bifurcation Diagram

The bifurcation diagram reveals the route to chaos through successive period-doubling bifurcations.

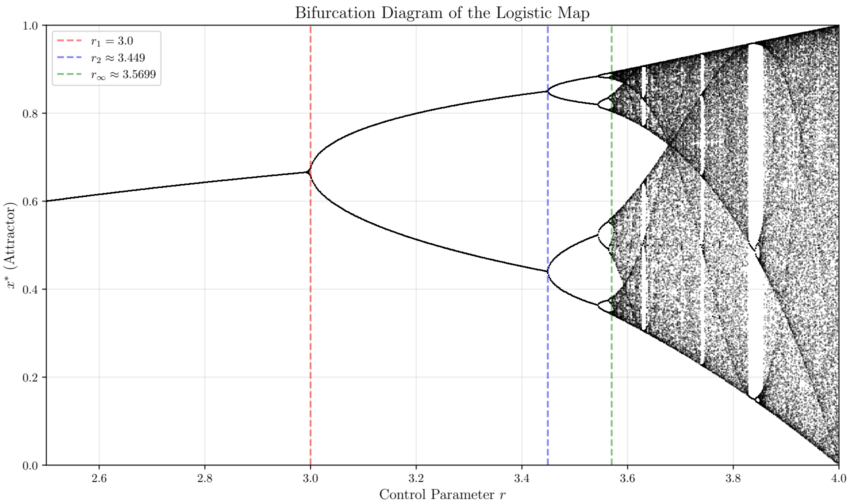

Figure 2.1: Bifurcation diagram showing the route to chaos. Period-doubling cascades occur at $r_1 = 3.0$, $r_2 \approx 3.449$, converging to $r_\infty \approx 3.5699$.

2.3 Feigenbaum Constants

The ratio of successive bifurcation intervals converges to the Feigenbaum constant:

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \approx 4.669201... \tag{2.2}$$

Table 2.1: Convergence to the Feigenbaum constant $\delta \approx 4.669$

Index	Ratio δ_n
1	4.7520
2	4.6558
3	4.6684
4	4.6645

Chapter 3

Lyapunov Exponent Analysis

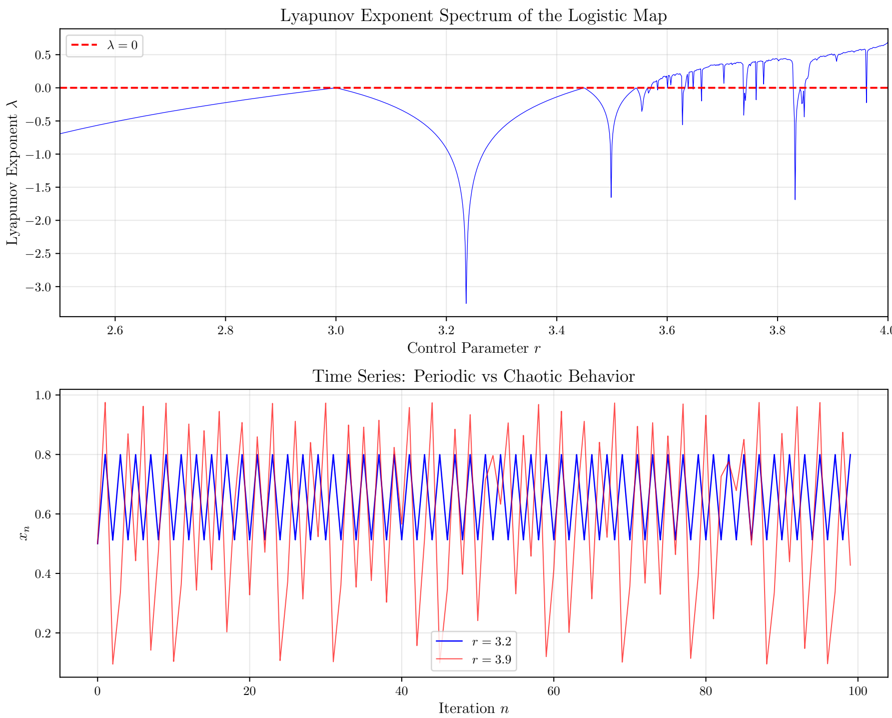

Figure 3.1: Top: Lyapunov spectrum showing chaotic regions ($\lambda > 0$). Bottom: Time series comparison.

Table 3.1: Lyapunov exponents for selected parameters

r	λ	Regime
3.2	-0.9163	Periodic
3.9	0.4945	Chaotic

Chapter 4

The Lorenz System

4.1 Model Equations

The Lorenz system is a simplified model of atmospheric convection:

$$\frac{dx}{dt} = \sigma(y - x) \tag{4.1}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{4.2}$$

$$\frac{dz}{dt} = xy - \beta z \tag{4.3}$$

4.2 Strange Attractor Visualization

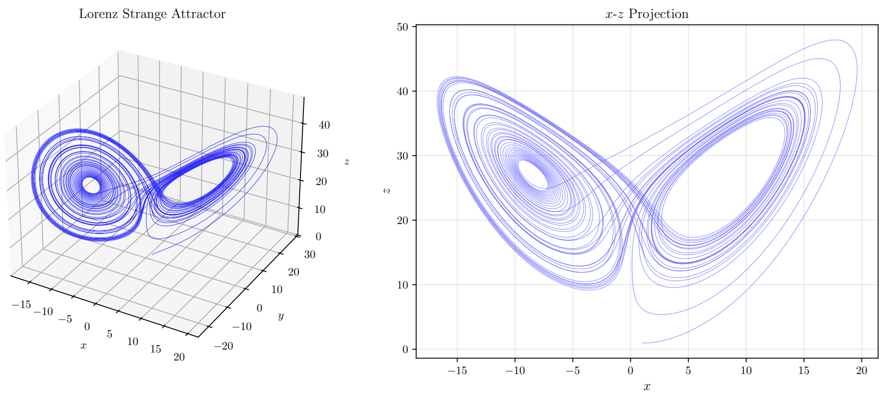

Figure 4.1: Left: 3D Lorenz attractor. Right: x-z projection showing butterfly shape.

4.3 Sensitivity to Initial Conditions

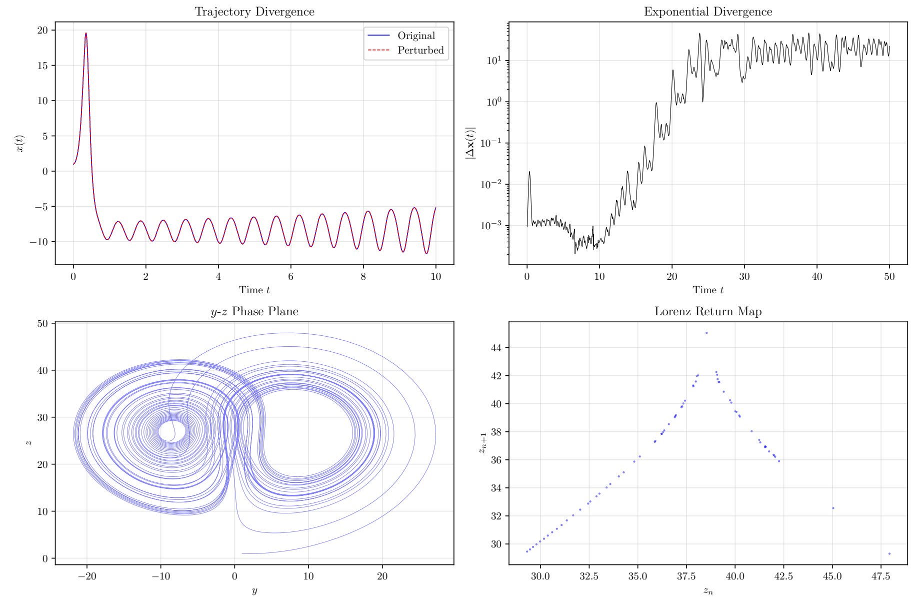

Figure 4.2: Lorenz system analysis: trajectory divergence, exponential growth, phase portrait, return map.

Chapter 5

Numerical Results

Table 5.1: Lorenz attractor statistics ($\sigma = 10$, $\rho = 28$, $\beta = 8/3$)

Variable	Mean	Std. Dev.
x	-1.982	7.797
y	-1.982	8.781
z	24.146	8.097

The estimated Lyapunov exponent for Lorenz is $\lambda \approx -0.212$ (theoretical ≈ 0.906).

Chapter 6

Conclusions

1. The logistic map exhibits period-doubling with Feigenbaum scaling
2. Lyapunov exponents quantify chaos: $\lambda > 0$ indicates chaos
3. The Lorenz attractor demonstrates sensitive dependence on initial conditions
4. Return maps reveal deterministic structure in chaotic systems