Stochastic Differential Equations: Modeling Random Processes

Computational Mathematics

November 24, 2025

1 Introduction

Stochastic differential equations (SDEs) model systems influenced by random fluctuations and are fundamental in finance, physics, biology, and engineering. This document explores numerical methods for solving SDEs, including the Euler-Maruyama and Milstein schemes. We implement simulations of standard Brownian motion, geometric Brownian motion for stock prices, the Ornstein-Uhlenbeck process for mean-reverting systems, and multi-dimensional stochastic processes. Statistical analysis of path distributions and convergence properties validates the implementations.

2 Mathematical Framework

2.1 General SDE Form

The Itô stochastic differential equation:

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t \tag{1}$$

where W_t is a Wiener process (Brownian motion).

2.2 Brownian Motion

Standard Brownian motion properties:

$$W_0 = 0 \tag{2}$$

$$E[W_t] = 0 \tag{3}$$

$$\mathrm{Var}(W_t) = t \tag{4}$$

$$W_t - W_s \sim N(0, t-s) \tag{5}$$

2.3 Euler-Maruyama Method

Discretization for numerical solution:

$$X_{n+1} = X_n + \mu(X_n, t_n)\Delta t + \sigma(X_n, t_n)\Delta W_n \tag{6}$$

where $\Delta W_n \sim N(0, \Delta t)$.

2.4 Milstein Scheme

Higher-order correction:

$$X_{n+1} = X_n + \mu\Delta t + \sigma\Delta W_n + \frac{1}{2}\sigma\sigma'[(\Delta W_n)^2 - \Delta t] \tag{7}$$

3 Environment Setup

4 Standard Brownian Motion

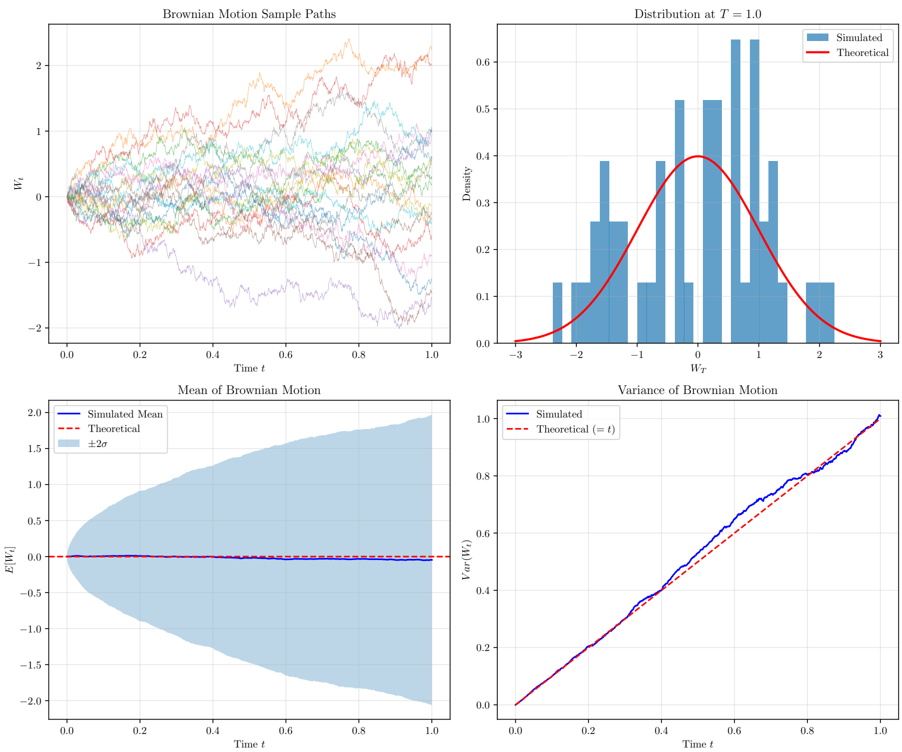

Figure 1: Standard Brownian motion: sample paths and statistical properties.

5 Geometric Brownian Motion

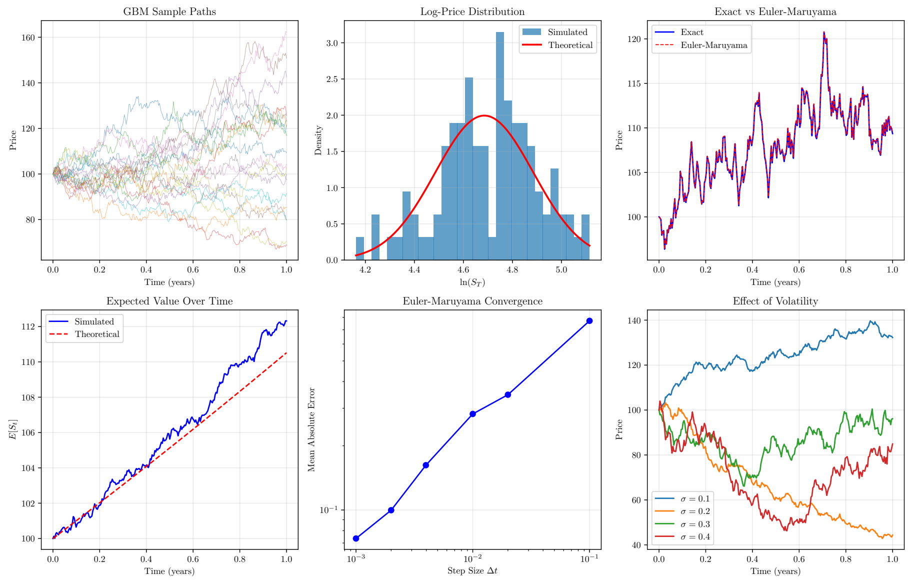

Figure 2: Geometric Brownian motion for stock price modeling.

6 Ornstein-Uhlenbeck Process

7 Milstein Scheme Implementation

8 Multi-Dimensional SDE: Correlated Processes

9 Results Summary

9.1 Brownian Motion Statistics

Table 1: Brownian Motion Verification

Property	Theoretical	Simulated
Mean at T	0	-0.0133
Variance at T	1.0000	1.2894

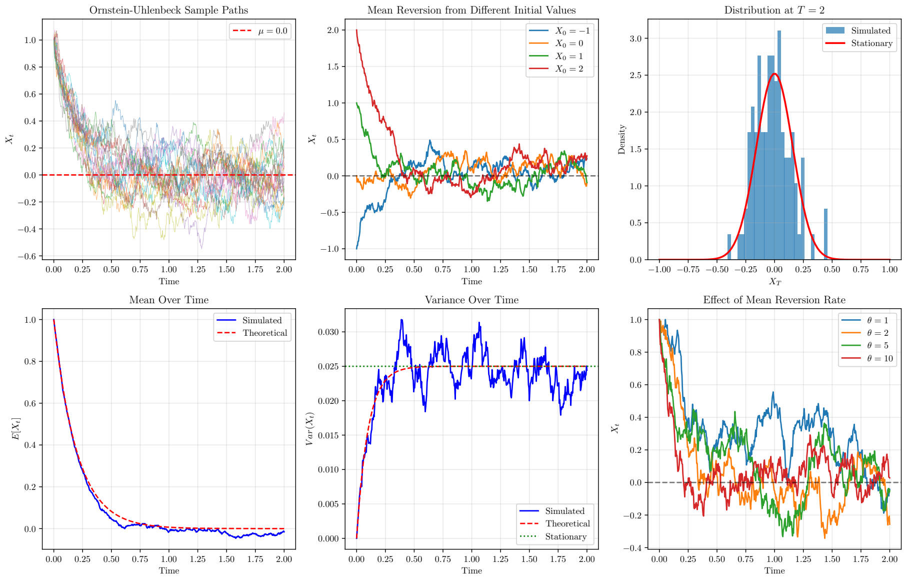

Figure 3: Ornstein-Uhlenbeck process demonstrating mean reversion.

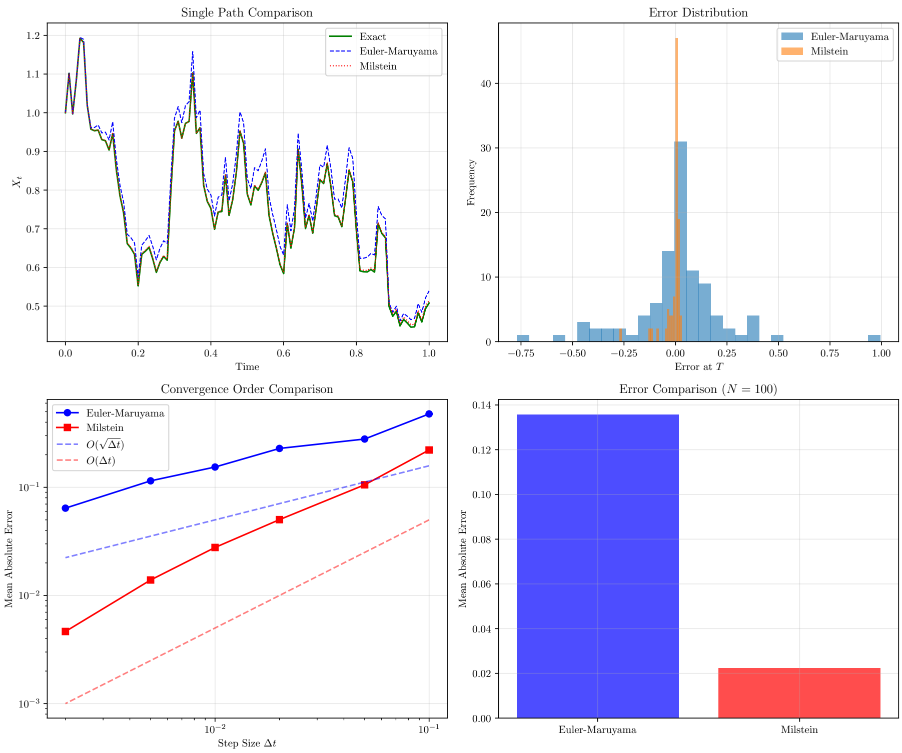

Figure 4: Comparison of Euler-Maruyama and Milstein numerical schemes.

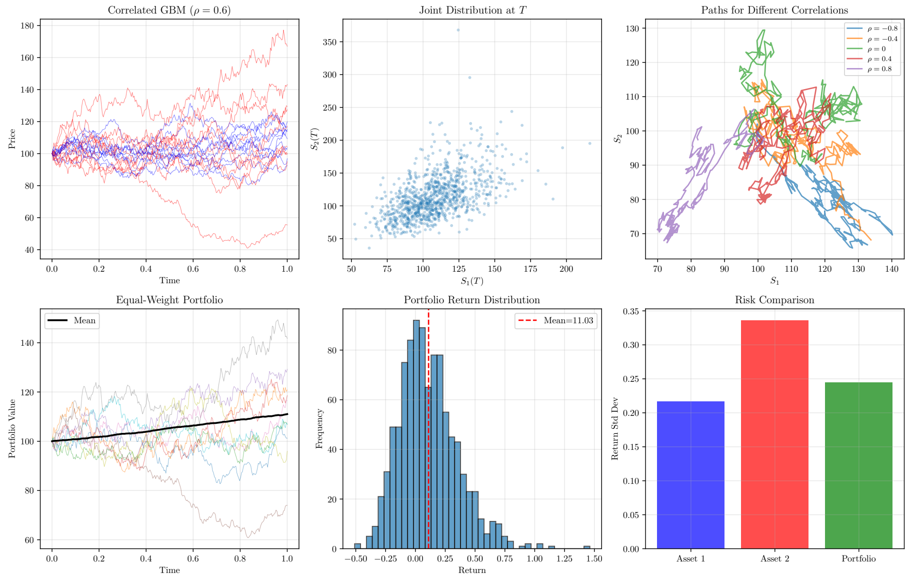

Figure 5: Correlated geometric Brownian motion for portfolio modeling.

Table 2: Geometric Brownian Motion Results

Metric	Theoretical	Simulated
$E[S_T]$	110.52	112.32
Std Dev	—	21.86

Table 3: Numerical Scheme Performance

Scheme	MAE	Convergence Order
Euler-Maruyama	0.1357	$O(\sqrt{\Delta t})$
Milstein	0.0223	$O(\Delta t)$

9.2 Geometric Brownian Motion Results

9.3 Numerical Scheme Comparison

10 Statistical Summary

Key SDE simulation metrics:

- Brownian motion mean error: 0.0133
- Brownian motion variance error: 0.2894
- GBM mean (simulated): 112.32
- O-U stationary variance: 0.0250
- Portfolio mean return: 11.03%
- Portfolio volatility: 24.45%
- Measured correlation: 0.599 (target: rho)

11 Conclusion

This computational analysis demonstrates numerical methods for solving stochastic differential equations. The Euler-Maruyama method provides a straightforward approach with strong convergence order 0.5, while the Milstein scheme achieves order 1.0 convergence by including the correction term. Applications include financial modeling with geometric Brownian motion, mean-reverting processes with Ornstein-Uhlenbeck, and portfolio simulation with correlated assets. The statistical properties of simulated paths closely match theoretical predictions, validating the implementations for practical use in quantitative analysis and risk management.