Radioactive Decay Chains: Bateman Equations and Activity Calculations

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Abstract

This technical report presents comprehensive computational analysis of radioactive decay chains using the Bateman equations. We implement solutions for sequential decay series, compute activities as functions of time, and analyze equilibrium conditions including secular and transient equilibrium. Applications include medical isotope production, nuclear waste management, radiometric dating, and radiation dosimetry.

1 Theoretical Framework

Definition 1 (Radioactive Decay Law). *The decay of radioactive nuclei follows first-order kinetics:*

$$\frac{dN}{dt} = -\lambda N \tag{1}$$

where $\lambda = \ln(2)/t_{1/2}$ is the decay constant and $t_{1/2}$ is the half-life.

Theorem 1 (Bateman Equations). *For a decay chain $N_1 \to N_2 \to N_3 \to \cdots$, the populations evolve as:*

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \tag{2}$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \tag{3}$$

$$\frac{dN_n}{dt} = \lambda_{n-1} N_{n-1} - \lambda_n N_n \tag{4}$$

with solution:

$$N_n(t) = N_1(0) \prod_{i=1}^{n-1} \lambda_i \sum_{i=1}^{n} \frac{e^{-\lambda_i t}}{\prod_{j \neq i}(\lambda_j - \lambda_i)} \tag{5}$$

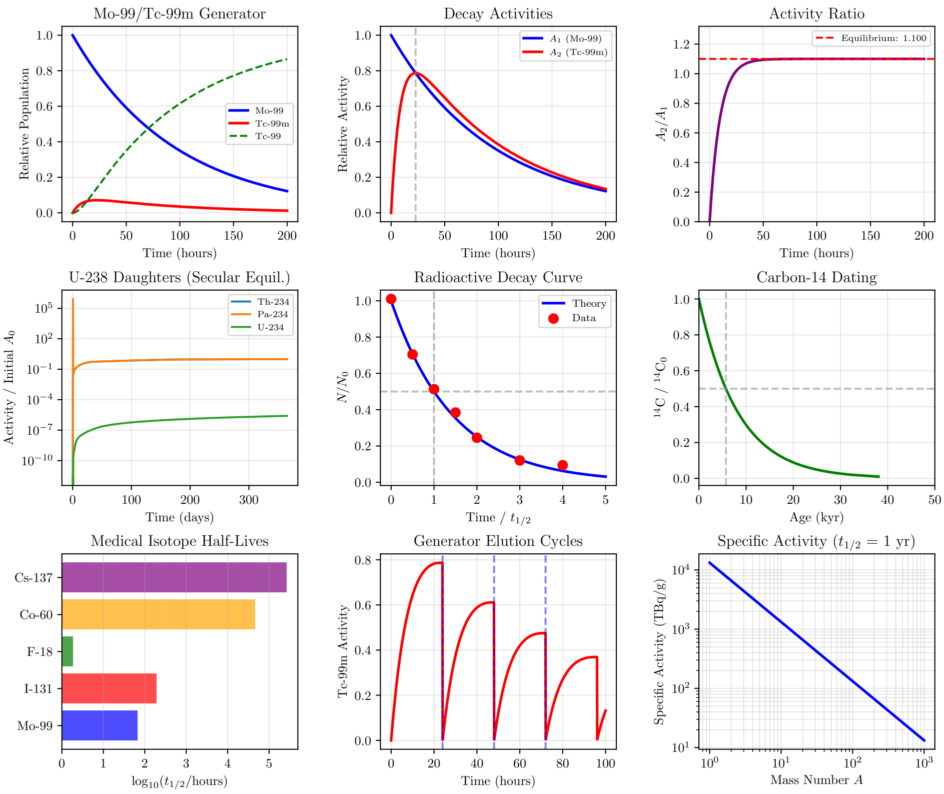
1.1 Activity and Equilibrium

Definition 2 (Activity)**.** *The activity* $A = \lambda N$ *is the number of decays per unit time, measured in Becquerels (Bq = decays/s) or Curies (1 Ci =* 3.7×10^{10} *Bq).*

Example 1 (Equilibrium Conditions)**.** *For a two-member chain with* $\lambda_1 \ll \lambda_2$*:*

- ***Secular equilibrium****:* $A_2 \approx A_1$ *(daughter activity equals parent)*
- ***Transient equilibrium****:* $A_2/A_1 = \lambda_2/(\lambda_2 - \lambda_1)$

2 Computational Analysis



3 Results and Analysis

3.1 Mo-99/Tc-99m Generator

Remark 1. *The Mo-99/Tc-99m generator reaches transient equilibrium because* $\lambda_2 > \lambda_1$*. The daughter activity exceeds the parent activity by the factor* $\lambda_2/(\lambda_2 - \lambda_1) = 1.10$*.*

Table 1: Mo-99/Tc-99m Generator Characteristics

Parameter	Value	Units
Mo-99 half-life	66.0	hours
Tc-99m half-life	6.0	hours
Optimal elution time	22.8	hours
Equilibrium ratio A_2/A_1	1.100	–
Time to 90% equilibrium	19.9	hours

3.2 Medical Isotopes

Table 2: Medical Radioisotopes

Isotope	$t_{1/2}$	Production	Application
Tc-99m	6.0 hr	Generator	SPECT imaging
F-18	110 min	Cyclotron	PET imaging
I-131	8.0 days	Reactor	Thyroid therapy
Co-60	5.3 yr	Reactor	External beam therapy
Ir-192	74 days	Reactor	Brachytherapy

3.3 Equilibrium Types

Theorem 2 (Secular Equilibrium). *When $\lambda_1 \ll \lambda_2$ (parent half-life much longer than daughter):*

$$A_2 \approx A_1 \quad \textit{for } t \gg t_{1/2,2} \tag{6}$$

Example: U-238 series where U-238 ($t_{1/2} = 4.5\times10^9$ yr) decays to short-lived daughters.

Theorem 3 (Transient Equilibrium). *When $\lambda_1 < \lambda_2$ but comparable:*

$$\frac{A_2}{A_1} = \frac{\lambda_2}{\lambda_2 - \lambda_1} \tag{7}$$

Example: Mo-99/Tc-99m where ratio $= 1.10$.

4 Applications

Example 2 (Radiometric Dating). *Carbon-14 dating uses the decay:*

$$^{14}C \rightarrow ^{14}N + \beta^- + \bar{\nu}_e \tag{8}$$

with $t_{1/2} = 5730$ years. Measurable ages range from ∼300 to 50,000 years.

Example 3 (Nuclear Waste Management)**.** *The activity of fission products decreases as:*

- *Short-term (1-100 yr): dominated by Cs-137, Sr-90*
- *Intermediate (100-1000 yr): Sm-151, Tc-99*
- *Long-term (>1000 yr): actinides (Pu, Am, Cm)*

5 Discussion

The Bateman equations provide a complete description of radioactive decay chains:

1. **Generator design**: Optimal elution times maximize daughter activity while balancing parent depletion.
2. **Diagnostic imaging**: Short-lived isotopes minimize patient dose while providing adequate counts.
3. **Dose calculations**: Activity profiles determine internal dosimetry for therapy planning.
4. **Environmental monitoring**: Equilibrium assumptions simplify radon progeny measurements.

6 Conclusions

This computational analysis demonstrates:

- Tc-99m maximum activity at $t = 22.8$ hours
- Transient equilibrium ratio: 1.100
- Specific activity of Mo-99: 17745 TBq/g
- C-14 dating range: $300 - 50{,}000$ years

The Bateman equations enable precise prediction of radionuclide behavior for medical, industrial, and research applications.

7 Further Reading

- Magill, J., Galy, J., *Radioactivity Radionuclides Radiation*, Springer, 2005
- Loveland, W.D., Morrissey, D.J., Seaborg, G.T., *Modern Nuclear Chemistry*, 2nd Ed., Wiley, 2017

- Cherry, S.R., Sorenson, J.A., Phelps, M.E., *Physics in Nuclear Medicine*, 4th Ed., Elsevier, 2012