1 Quadratic form

1.1 Minimizer and symmetric

Quadratic form

$$f(x) = \frac{1}{2}x^T Ax - b^T x$$

If matrix A is symmetric and positive definite, $f(x)$ is minimized by the solution to $Ax = b$. To minimize $f(x)$ we find the gradient,

$$\frac{\partial f(x)}{\partial x} = \frac{1}{2}\frac{\partial x^T Ax}{\partial x} - \frac{\partial b^T x}{\partial x}$$

$$\frac{\partial x^T Ax}{\partial x} = \frac{\partial x}{\partial x}Ax + \frac{\partial x}{\partial x}A^T x = (A + A^T)x$$

and

$$\frac{\partial b^T x}{\partial x} = b.$$

Altogether

$$\frac{\partial f(x)}{\partial x} = \frac{1}{2}(A + A^T)x - b,$$

meaning it the quadratic form is minimized when x and A is symmetric solves $Ax = b$.

1.2 Positive definite

Assume that A is symmetric and x satisfies $Ax = b$ and minimizes the quadratic form, then let p be some abitrai

$$\begin{aligned} f(x+p) &= \frac{1}{2}(x+p)^T A(x+p) - b^T(x+p) \\ &= \frac{1}{2}(x^T Ax + x^T Ap + p^T Ax + p^T Ap) - (b^T x + b^T p) \end{aligned}$$

By symmetry $p^T Ax = x^T Ap$

$$= \frac{1}{2}x^T Ax - b^T x + \frac{1}{2}p^T Ap + x^T Ap - b^T p$$

By x statisfing $Ax = b$ and of symmetry of A we have $x^T A = b^T$, then the two last terms cancel out

$$= f(x) + \frac{1}{2}p^T Ap$$

Meaning that A has to be positive definite, otherwise we have found a vector $x + p$, where $p \neq -x$, that either is evaluates the function to a smaller value or the same value as $f(x)$.