1 Bound for conjugate gradient

From CG we know that $x_i \in x_0 + \mathcal{K}^i(A, r_0)$ therefore $x_i = x_0 + Q_{i-1}(A)r_0$, with Q_{i-1} is a polynomial of degree $i-1$ with requirement that $Q_{i-1}(0) = 1$ and that

$$r_0 = b - Ax_0 = Ax - Ax_0 = A(x - x_0),$$

which means that the error, $e_i = x - x_i$, can be written as

$$x - x_i = x - (x_0 + Q_{i-1}(A)r_0) = x - (x_0 + Q_{i-1}(A)A\,(x - x_0)) = x - (x_0 + \tilde{Q}_i(A)\,(x - x_0))$$

$$= I(x - x_0) - \tilde{Q}_i(A)\,(x - x_0) = \left(I - \tilde{Q}_i(A)\right)(x - x_0)$$

This new polynomial $\tilde{Q}_i$ is equal to zero on input zero, $\tilde{Q}_i(0) = 0$. This means the polynomial $Q_i(A) = \left(I - \tilde{Q}_i(A)\right)$ is equal to 1 on input zero. Therefore the i'th error $e_i = P_i(A)e_0$.

As one of the requirements of the matrix A is that it is SPD, meaning that it has n non-zero orthogonal eigenvectors v_j, which are scaled to be unit vectors and corresponding eigenvalues λ_j, therefore we can write e_0 as a linear compination of these vectors

$$e_0 = \sum_{j=1}^{n} \gamma_j v_j$$

which means that

$$e_i = P_i(A) \sum_{j=1}^{n} \gamma_j v_j$$