Assignment 1

Question 1:-

Prove the following equality by induction
$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution

I want to prove using the induction method

Base case: Here $n = 1$.
For 1; $\frac{n(n+1)(2n+1)}{6}$

$$\Longrightarrow \frac{1(1+1)(2(1)+1)}{6}$$

$$\Longrightarrow \frac{1(2)(3)}{6} = \frac{1*2*3}{6} = \frac{6}{6} = 1$$

$\Longrightarrow$ For Inductive Hypothesis:
For $n = j$
$\Longrightarrow n = j$ is the statement is True,
$$1^2 + 2^2 + 3^2 + \dots + j^2 = \frac{j(j+1)(2j+1)}{6}$$

For Inductive Step:-
Observe that $n = j+1$
Observe that $1^2+2^2+3^2+\dots+j^2 = \frac{(j+1)(j+2)(2j+3)}{6}$

$$\underbrace{1^2+2^2+3^2+\dots j^2}+(j+1)^2$$

$$\frac{j(j+1)(2j+1)}{6}+(j+1)^2 = \frac{(j+1)(j+2)(2j+3)}{6}$$

$$j(j+1)(2j+1)+6(j+1)^2=(j+1)(j+2)(2j+3)$$

$$\Longrightarrow (j^2+j)(2j+1)+6(j^2+2j+1)=(j+1)(j+2)(2j+3)$$

Expanded both sides will be

$$2j^3+9j^2+13j+6 = 2j^3+9j^2+13j+6$$

$\therefore$ The Left hand side is equal to the right hand side. This proves the inductive step.

Therefore, by induction

$$1^2+2^2+3^2+\dots+n^2=\frac{n(n+1)(2n+1)}{6}$$ is going to

be true for every positive integer n.

Question 2:-

A prime number is a natural number greater than 1 which is not a product of two smaller natural numbers.

Prove or disprove: For every integer q, if $q > 7$, then q can be written as $q = (a + b * c)$ such that the following properties hold

- a and b are prime numbers.
- c is an integer greater than 1.

Solution

Would Prove this by showing direct proof.

I want to show an example that q can be greather than 7.

⇒ Claim 1:-

An integer n is odd if there is an integer m such that $n = 2m + 1$.

∴ So will make q be an odd integer > 7

then if q is an odd integer

$2m + 1 + 2n = 2(m + 1) + 1$ and is an odd

will let j be an odd prime number

$j = 2m + 1$

⇒ Claim 2:-

An integer n is even if there is an integer m such that $n = 2m$.

Will make q be an even integer > 7

⇒ will make j be an even prime number

$j = 2m$

N.B.:- if j is even, $2m + 2n \Rightarrow 2(m+n)$ & is even

Will assume $b \times c$ is an even integer)
& C is greather than 1.
Then $q = 2m + 2n \Longrightarrow 2(m+n)$ and is still even

Further examples to prove if $q > 7$, $q = (a + b * c)$

$q = 8 \Longrightarrow 8 = (a + b \times c)$ Will make a & $b = 2$, $c = 3$

$8 = (2 + 2 \times 3) = 8 = 8$

for $q = 9$

$9 = (3 + 2 \times 3)$ a & $c = 3$, $b = 2$

$9 = 9$

∴ From the above proof from the examples, q increases but b still remain 2 an even number and a switches to 3.

So as q increases, the value can be gotten by multiplying the prime number with a quotient thus.

For $q > 7$, $q = (a + b \times c)$, $c > 1$.

and a and b are prime numbers that holds for every integer value of q

further examples thus

for $q = 10$

$10 = (2 + 2 \times 4)$

$10 = 10$

for $q = 11$

$11 = (3 + 2 \times 4)$

$11 = 11$

for $q = 12$

$12 = (4 + 2 \times 4)$

$12 = 12$ ---- can continuously

additionally :- the difference between 4 and 3 is 1, making any positive number to be reachable/achieveable

Question 3:- True or False?

(a:) True:- Yes, $O(n^2) \Longrightarrow O(n^{1.9})$

(b:) False

(c:) True:- Yes $\Theta(n^4)$ is possible it takes $\Longrightarrow O(n^{3.9})$

(d:) False

(e:) False

Question 4:-

Prove or Disprove: $n^7 = O(7^n)$

$n^7 = O(7^n)$
prove or disprove

So, I would prove $n^7 = O(7^n)$ by using the direct Proof
so, if want to prove, will have to find positive constant C and n_0 such that

$\boxed{n^7 \le C\,n}$ for some $n \ge n_0$

$n^7 \le C \times 7^n$ $n = 7^n$

$\boxed{n_0 = 7 \;\&\; C = 1}$

So, to prove ~~that~~ $n^7 \le 7^n$ that is | $n \ge 7$

Will use log on both sides
$\Longrightarrow 7 \log n \le n \log 7$

$f(n) = 7 \log n \ge 0 - \log 7n$,
Differentiating with respect to $f'(n)$
$f(7) = 0$, $f'(n) > 0$ | for $n \ge 7$

So to prove $\Rightarrow f(n) \ge 0$ | for $n \ge 7$

So every $n \ge 7$ $C^7 \le C(7^n)$ $\overrightarrow{n}K$

$\Longrightarrow n^7 = O(7^n)$

Question 5:-

Prove or Disprove: $n^8 + 3n^{7.9} \log n - n^4 = \Theta(n^8)$

Solution:-

$n^8 + 3n^{7.9} \log n - n^4 = \Theta(n^8)$
prove or disprove

So, if we want to prove, will have to find the positive constant C_n and C_m and n_0 such that

$\Longrightarrow C_m n^8 \leq n^8 + 3n^{7.9} \log n - n^4 \leq C_n \cdot n^8$

$n^8 + 3n^{7.9} \log n - n^4 = \Theta(n^8)$

$\boxed{\begin{array}{l} C_m = 1 \\ C_n = 4 \end{array}}$ $n_0 = 10^{10}$

$\Longrightarrow n^8 \leq n^8 + 3n^{7.9} \log n - n^4$

$n^8 + 3n^{7.9} \log n - n^4 \leq 4n$

$\log n < n^{0.1}$ | when $n \geq 10^{10}$ $3n^{7.9} \log n < 3n^8$

$C_m n^8 \leq n^8 + 3n^{7.9} \log n - n^4 \leq C_n n^8$

| $C_m \to 1$, $C_n \to 4$, $n_0 \to 10^{10}$

$\Longrightarrow n^8 + 3n^{7.9} \log n - n^4 = \Theta(n^8)$

Question 6:

(a.) $f(n) = n^{0.6}$ and $g(n) = (\log n)^2$

$\Longrightarrow$ want to show that $f(x)\ \Omega\ (g(x))$

To show this:

I would assume that $K = 1$ and $n_0 = 1$

So for every $n \geq n_0 \mid n^{0.6} \geq K(\log n)^2$

$\Longrightarrow$ But they are no two (2) positive constants K and n_0 for $n \geq n_0 \mid n^{0.6} \leq K(\log n)^2$

$\Longrightarrow$ $f(n)$ is in $\Omega(g(n))$

(b.) $f(n) = \log^6 n$ and $g(n) = \log^5 n$

$\Longrightarrow$ want to show that $f(n)$ is in $O(g(n))$

Can observe that $Of(n) \leq Og(n)$

$\Longrightarrow$ so to find constant c & n_0:

$f(n) \leq O(g(n)) \mid n \geq n_0$

$\Longrightarrow$ $f(n)$ is in $O(g(n))$

(c.) $f(n) = n$ and $g(n) = \log^{100} n$

$\Longrightarrow$ want to show that $f(n)$ is $\Omega(g(x))$

can observe that $c = 5$,

then n_0 — for every $n \geq n_0 \mid n \geq c \times \log^{100} n$

but they are not positive constants c & n_0

$\Longrightarrow$ $n \geq n_0$, $n \leq c \log^{100} n$

$\Longrightarrow$ $f(n)$ is in $\Omega(g(x))$

Question 6:-

(d.) $f(n) = n \log \log n + n$ and $g(n) = \log n$

$\Longrightarrow$ want to show that $f(n)$ is in $\Omega(g(n))$.

$\Longrightarrow$ I would assume there ~~posi~~ are no positive constant c and n_0

\# $c = 5$, $n_0 = 6$

for every $n \geq n_0$

$\Longrightarrow n \log \log n + n > c \log n$

$\Longrightarrow$ for the constant not been positive

So therefore $| n \geq n - n \log \log n + n \leq c \log n$

$\Longrightarrow f(n)$ is in $\Omega(g(n))$

(e.) $f(n) = 10$ and $g(n) = 10^3$

$\Longrightarrow$ will show that $f(n)$ is in $\Theta(g(n))$

$O f(n) = 10$

$O g(n) = 10^3$

$\Longrightarrow O(g(n) = O(1)$; $O f(n) = O(1)$

$\Longrightarrow O(f(n) = O(g(n))$

$\Longrightarrow O(f(n) = O(g(n))$

$f(n)$ is $O(g(n))$ and $f(n)$ is in $\Omega(g(n))$

$\Longrightarrow$ ~~when f(n)~~ $10 \geq c\, 10^3$ when $n \geq 1$

$g(n) = 10 \geq c\, 10^3$ when $n \geq 1$

$\Longrightarrow f(n) = \Theta(g(n))$

Question 7:-

(a.) $T(n) = 3T(\frac{n}{4}) + 1$

$\Longrightarrow$ the best possible $\Rightarrow \Theta(n \log_4 3)$

$\Longrightarrow$ the $n \log_b^a \Longrightarrow$ is $n \log_4^3$

$\Longrightarrow f(n) \longrightarrow 1$

$\Longrightarrow n \log_b^a$ is $\Theta(n \log_4 3)$

So therefore, 1 is polynomially smaller than $n \log_4^3$

$\Longrightarrow f(n) < n \log_b^a$

(b.) $T(n) = T(n/4) + \sqrt{n}$

$\Longrightarrow$ the best possible $\Longrightarrow O\sqrt{n}$

$\Longrightarrow n \log_b^a$ is 1

and $f(n) = \sqrt{n} \Longrightarrow f(n) = \sqrt{n}$

$\Longrightarrow f(n)$ is larger than $n \log_b^a$

$\Longrightarrow$ Upper bound $\longrightarrow O(\sqrt{n})$

(c.) $T(n) = 2T(n/4) + n^{1.5}$

$\Longrightarrow$ the best possible $\Rightarrow O(n^{1.5})$

$\Longrightarrow n \log_b^a$ is $\log_4^2$

$\Longrightarrow f(n)$ is $n^{1.5}$

$\Longrightarrow f(n)$ is bigger than $n \log_b^a$

$\Longrightarrow f(n) = n^{1.5}$

$\Longrightarrow$ Upper bound $\longrightarrow O(n^{1.5})$

Question 7:-

(d.) $T(n) = 4T(n/4) + n$

$\Longrightarrow$ the best possible $\rightarrow O(n \log n)$

$n^{\log_b a} \rightarrow n^{\log_4 4} = n$

$f(n) \Longrightarrow n$

$f(n)$ is asymptotically equal to $n^{\log_b a}$

$f(n) = n \cdot \log n$

$\Longrightarrow$ the Upper Bound $\rightarrow \underline{O(n \log n)}$

(e.) $T(n) = T(n/5) + 1$

$\Longrightarrow$ the best possible $\rightarrow O(\log n)$

$n^{\log_b a} \rightarrow 1 \quad a \rightarrow 1$

$f(n) \rightarrow 1$

$f(n)$ is asymptotically equal to $n^{\log_b a}$

$\Theta(f(n) \cdot \log(n)) \Longrightarrow 1 \cdot \log n$

$\Theta(\log n)$

$\Longrightarrow$ the Upper Bound $\rightarrow \underline{O(\log n)}$

(f.) $T(n) = T(2n/3) + T(n/5) + 1$

$\Longrightarrow$ the best possible $\rightarrow O(\log n)$

$T(n)$

$T(n/5) \quad T(2n/3)$

$T(n/25) \quad T(2n/15) \quad T(2n/15) \quad T(4n/9)$

$T(2/3)^k n = 1$

$\Longrightarrow k = \frac{\log n}{\log 3/2} \times 1$

$\underline{O(\log n)}$ Upper Bound $\rightarrow \underline{O(\log n)}$

Question 8:-

The best possible asymptotic upper bound is $O\log^3 n$.

$\Longrightarrow$ for $i=1$;
the 1 is constant; $i=$ is constant
and the loop runs $(\log_2 n)$ times

$\Longrightarrow$ while $(j > i)$
$j = j * \frac{1}{3}$

$\Longrightarrow$ the loop runs in $O(\log_3 n)$ times
the upper bound $O(\log_3 n)$

$\Longrightarrow$ while $(K < i)$ {
$K = K * 4$

$\Longrightarrow$ the loop runs in $O(\log_4 n)$ times
the upper bound $O(\log_4 n)$

$\Longrightarrow$ So to get the best possible asymptotic upper bound will multiply all the bounds

$O(\log_3 n) \times O(\log_4 n) \times O(\log_2 n)$

$\Longrightarrow O(\log^3 n)$