第 1 讲: 测试

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评分：　　　　　　评阅：

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这是适用于问题求解作业的 Typst 模板,
同时也可用于写作实验报告等。
但该模板仍在进行测试中,
请谨慎使用。

1 作业 (必做部分)

Problem 1 (AC 1.2-3)

Solution:

a) when $\min\left(\|\boldsymbol{x}\|_2\right)$, $\boldsymbol{x} = \boldsymbol{x}^*$ is the solution to the problem, which is $x^* = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

b) We have a matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$, the projection operator is

$$\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^T A)^{-1}\boldsymbol{A}^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

hence,

$$\boldsymbol{x}^* = \boldsymbol{P}\boldsymbol{v} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

c) We have a matrix $\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 2 & 2 \end{pmatrix}$, the projection operator is

$$\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^T A)^{-1}\boldsymbol{A}^T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

hence,

$$\boldsymbol{x}^* = \boldsymbol{P}\boldsymbol{v} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}.$$

Problem 2

Solution:

1. we know that:

$$\operatorname{prox}_{\varphi}(z) = \arg\min_{x\in\mathbb{R}}\left\{\frac{1}{2}\|x - z\|^2 + \phi(x - c)\right\}.$$

let $x' = x - c$

$$\operatorname{prox}_{\varphi}(z) = \arg\min_{x\in\mathbb{R}}\left\{\frac{1}{2}\|x' - (z - c)\|^2 + \phi(x' + c - c)\right\} + c = \operatorname{prox}_{\phi}(z - c) + c.$$

2. if we want to $f(x) = \frac{1}{2}\|x-z\|^2 + \phi(x)$ to be minimized, we need to find the x that makes the derivative of the function equal to zero.

we know

$$\partial f(x) = \begin{cases} x - z + \lambda \text{ when } x > 0 \\ [x - z - \lambda, x - z + \lambda] \text{ when } x = 0 \\ x - z - \lambda \text{ when } x < 0 \end{cases}$$

. Hence, let

$$\partial f(x) = 0$$

, we have

$$\text{prox}_{\phi(z)} = x^* = \begin{cases} z - \lambda \text{ when } z > \lambda \\ [z - \lambda, z + \lambda] \text{ when } z \in [-\lambda, \lambda]. \\ z + \lambda \text{ when } z < -\lambda \end{cases}$$

3. if $\varphi(x) = \lambda|x-c|$, where $c \in \mathbb{R}$ and $\lambda > 0$. Use the result from part a.

$$\text{prox}_{\varphi(z)} = \text{prox}_{\phi(z-c)} + c = \begin{cases} z - \lambda \text{ when } z > \lambda + c \\ [z - \lambda, z + \lambda] \text{ when } z \in [-\lambda + c, \lambda + c] \\ z + \lambda \text{ when } z < -\lambda + c \end{cases}$$

Problem 3

Solution:

1. If we take the derivative of $\frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}^{t-1}\|^2 + \gamma g(\boldsymbol{x})$, we have

$$\boldsymbol{x^t} = \text{prox}_{\gamma g}(\boldsymbol{x}^{t-1}) = \boldsymbol{x}^{t-1} - \gamma \nabla g(\boldsymbol{x^t})$$

2. By the convexity of g, we know that $g(\boldsymbol{x}^t) + \nabla g(\boldsymbol{x^t})^T(\boldsymbol{x}^{t-1} - \boldsymbol{x^t}) \le g(\boldsymbol{x^{t-1}})$. Hence, we have

$$g(\boldsymbol{x}^t) \le g(\boldsymbol{x^{t-1}}) - \nabla g(\boldsymbol{x^t})^T(\boldsymbol{x}^{t-1} - \boldsymbol{x^t}) = g(\boldsymbol{x^{t-1}}) - \gamma \nabla \|g(\boldsymbol{x^t})\|_2^2$$

3. because $\boldsymbol{x^t} = \boldsymbol{x}^{t-1} - \gamma \nabla g(\boldsymbol{x^t})$ which is a gradient descent method, so

$$-\infty < g(\boldsymbol{x}^t) \le g(\boldsymbol{x}^{t-1})$$

and we have

$$g(\boldsymbol{x}^t) \le g(\boldsymbol{x^{t-1}}) - \gamma \nabla \|g(\boldsymbol{x^t})\|_2^2$$

hence

$$0 \le \gamma \nabla \|g(\boldsymbol{x^t})\|_2^2 \le 0$$

if

$$t \to +\infty$$

Problem 4

Solution:

1. because

$$\partial f(\boldsymbol{x}) = \{\boldsymbol{v} \in \mathbb{R}^n : f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n\}$$

if $g(\boldsymbol{x}) = \theta f(\boldsymbol{x})$,

$$\partial g(\boldsymbol{x}) = \{\boldsymbol{v} \in \mathbb{R}^n : g(\boldsymbol{y}) \geq g(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n\}$$

$$\partial g(\boldsymbol{x}) = \{\boldsymbol{v} \in \mathbb{R}^n : \theta f(\boldsymbol{y}) \geq \theta f(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n\}$$

$$\partial g(\boldsymbol{x}) = \left\{\boldsymbol{v} \in \mathbb{R}^n : f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \frac{\boldsymbol{v}^T}{\theta}(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n\right\}$$

$$\partial g(\boldsymbol{x}) = \theta\{\boldsymbol{v} \in \mathbb{R}^n : f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n\} = \theta \partial f(\boldsymbol{x})$$

2.

$$\partial h(\boldsymbol{x}) = \{\boldsymbol{v} \in \mathbb{R}^n : f(\boldsymbol{y}) + g(\boldsymbol{y}) \geq f(\boldsymbol{x}) + g(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n\}$$

all of the elements that satisfy

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n$$

and

$$g(\boldsymbol{y}) \geq g(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n$$

are in the set

$$\partial h(\boldsymbol{x})$$

hence

$$\partial f(\boldsymbol{x}) + \partial g(\boldsymbol{x}) \subseteq \partial h(\boldsymbol{x})$$

3. we know that

$$\partial \|x\|_1 = \begin{cases} 1 \text{ when } x > 0 \\ [-1, 1] \text{ when } x = 0 \\ -1 \text{ when } x < 0 \end{cases}$$

.

hence $\mathrm{sgn}(x) \in \partial \|x\|_1$.

Problem 5

Solution:

```
This a test for code blocks.
```

For rust:

```rust
pub fn main() {
    println!("Hello, world!");
}
```

For haskell:

```haskell
zipWith' :: (a -> b -> c) -> [a] -> [b] -> [c]
zipWith' _ [] _ = []
zipWith' _ _ [] = []
zipWith' f (x:xs) (y:ys) = f x y : zipWith' f xs ys
```

Select only a range of lines to show:

```python
def fibonaci(n):
  if n <= 1:
    return n
  else:
    return(fibonaci(n-1) + fibonaci(n-2))
```

Disable line numbers:

```cpp
int main() {
  cout << "Hello, World!"; // 你好，世界
  return 0;
}
```

Then pseudocode.

Algorithm 1: The Euclidean algorithm

input: integers a and b
output: greatest common divisor of a and b

1 **while** $a \neq b$ **do**
2   **if** $a > b$ **then**
3         $a \leftarrow a - b$
4   **else**
5         $b \leftarrow b - a$ ▷ comment test
6   **end** ▷ another comment test
7 **end**
8 **return** a

In Line 1, we have a while loop.

The algorithm figure’s breakable.

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2 作业 (选做部分)

Problem 1 (EoSD 9961)

How to pass 「レッドマジック」?

Solution:

Practice more.
