AL-KHWARIZMI
INTERNATIONAL MATHEMATICAL AND INFORMATICAL OLYMPIAD

SCIENCE OLYMPIAD CENTER

2023

Problem 1. A parallelogram $ABCD$ with an acute angle A and $2AB = BC$ is given. Point P is a projection of point A on to line CD. Let point M be the midpoint of side BC. Prove that the sum of the angles PMD and BCD equals $90°$.

Proposed by Olimjon Olimov

Problem 2. 2023×2023 board is given. The king is placed in the upper right corner. In even-numbered moves, he repeats the last odd-numbered moves ($2nd$ move repeats the first one, $4th$ move repeats $3rd$ one, and so on). With such moves, can king go around the entire board exactly being in each cell only once?
Comment: The king moves any neighbor that has a common vertex or common side with the square he is on.

Proposed by Tohirbek Tulanov

Problem 3. Given sequence $\{a_n\}$ with $a_1 = 3$, and for any positive integer n the equality $a_{n+1} = 2a_1a_2 \dots a_n + 1$ holds. Prove that for any positive integer n the expression $1 + (a_n - 2)a_{n+1}$ is the cube of some integer.

Proposed by Sardor Gafforov

Problem 4. Positive integers from 1 to 100 are written on the board. Sardor is painting 10 of these numbers in red as the following: for any different a, b painted in red, $|a - b| > 2$. How many different ways can Sardor paint these numbers in red?

Proposed by Sardor Gafforov

Problem 5. Given the positive integers $x_1, x_2, \dots, x_k$ and $m_1, m_2, \dots, m_k$ where $k > 1$. For each positive integer $1 \le i \le k$, $m_i \ge 2023x_i$ holds. If there is a positive integer z satisfying the equality $x_1^{m_1} + x_2^{m_2} + \dots + x_k^{m_k} = z^{m_1 m_2 \dots m_k}$, then prove that $k \ge 2^{2023}$.

Proposed by Sardor Gafforov

Time: 4 hours.
Each problem is worth 10 points.

Part 1: each problem is worth 1.9 points

1. In rebus $UZBEK + IS + TAN$ different letters represent different numbers and same letters represent same numbers. Find the largest possible value of that sum $UZBEK + IS + TAN$?

A) 99631 B) 99190 C) 99387 D) 99423

2. Babur has stones weighing 1 kg, 2 kg, ..., 16 kg and three boxes A, B, C. He put two stones in each box so that the total weight of the stones in each box was M kg. Find the number of all possible values of M.

A) 21 B) 24 C) 19 D) 15

3. Anora wrote 5 integer numbers on the paper. From these numbers, Anora calculated the arithmetic mean of all groups with 4 numbers and got the numbers 37, 44, 25, 46 and 68 as a result. Find the largest number Anora wrote on the paper.

A) 147 B) 120 C) 68 D) 95

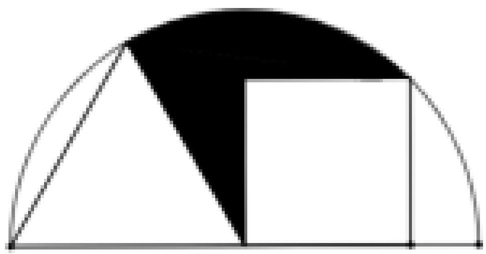
4. $1\cdot2\cdot3\cdot\ldots\cdot29\cdot30$ at least how many of the multipliers can be deleted to form a perfect square?

A) 7 B) 6 C)5 D) 4

5. Numbers from 1 to 2023 are written on the board. At each step, two arbitrary numbers are erased and their difference is written instead. This is done until there is only one number left at the end of the work. Which of the following cannot be the remaining number at the end?

A) 16 B) 1024 C) 2023 D) 2048

6. An equilateral triangle and a square are drawn inside a semicircle with a diameter of 10 cm as shown in the picture. What is the area of the painted part in cm^2?



A) $\frac{25\pi-150}{24}$ B) $\frac{25\pi-150}{12}$ C) $\frac{125\pi-150}{24}$ D) $\frac{45\pi-50}{8}$

7. Find the 1st digit of the smallest number which is divisible by 11 and whose sum of digits is 2023.

A) 9 B) 8 C) 7 D) 6

8. Given 5 consecutive positive integers. If the LCM of the first three numbers is less than twice the LCM of the last two numbers, find the largest value of the sum of these 5 consecutive numbers.

A) 30 B) 40 C) 45 D) 50

9. For the 4-digit number $\overline{abcd}$ function is defined and $f(\overline{abcd}) = a * b * c + d$. For example $f(6542) = 6 * 5 * 4 + 2 = 122$. What is the result of $f(2023) - f(2022) + f(2021) - f(2020) + \cdots + f(1003) - f(1002)$?

A) 509 B) 495 C) 517 D) 511

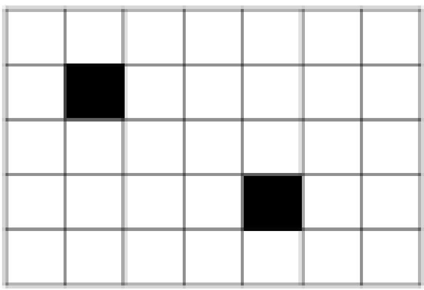
10. Find the smallest positive integer n such that 2023n has exactly 66 positive integer divisors.

A) 2^{33} B) 9216 C) 216 D) 1024

<u>Part 2: each problem is worth 3.1 points</u>

11. There is one rectangular piece of paper on the table. At each step, Olim chooses one of the largest pieces of paper on the table, divides it into two equal pieces and puts them back on the table. After 2023 steps, if the area of the smallest paper on the table is 1 cm^2, what is the area of the initial rectangular paper?

12. How many rectangles are there in the following figure whose sides lie on the lines of this figure and contain at least 1 black square?



13. 1000 straight lines are given and the intersection point of any 2 of them is marked. Find the maximum number of these marked points which can lie on one circle?

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14. Given a parallelogram ABCD. The midpoint of AD is M. The projection of point B intersects the side CM at the point P. If ∠APB=61°, what is the angle ∠PAB?

15. Sardor created one number by writing even numbers from 23 to 2023 in a row. Then he deleted all the odd digits in the decimal notation of this number. How many digits are left?

16. Find the number of integer triples (a, b, c) satisfying the following conditions:

$2 \le a, b, c \le 8$ and $1 \le d \le 12$, $a + b + c + d = 30$.

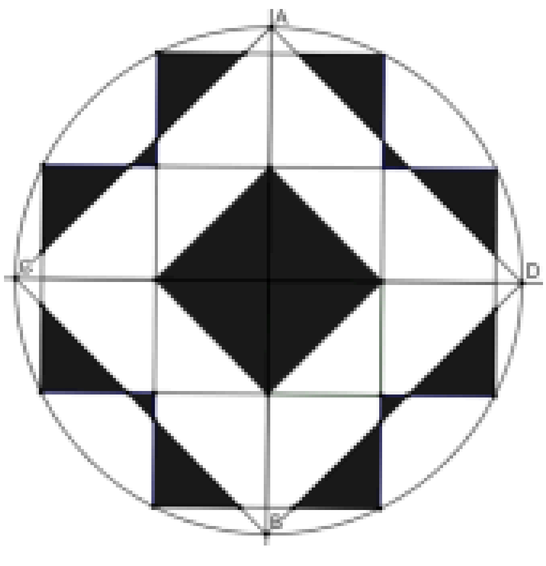
17. The following is appropriate for $\{A_n\}$ sequence of integer numbers:

$$A_{10} < 10 \text{ and } 1 \le A_1 \le 2023$$

$$A_{n+1} = \begin{cases} \frac{A_n}{2}, & \text{if } A_n \text{ is even number} \\ A_n^2 + 1, & \text{if } A_n \text{ is odd number} \end{cases}$$

Find the number of all possible values of A_1.

18. 12 small squares whose sides are parallel to the diameters AB and CD of the circle (AB and CD are perpendicular to each other) are drawn as shown in the picture. AC, AD, BC, BD are the chords of the circle. Find the area of the painted area.



19. Anvar wrote the numbers 1, 2, 4, 5, 6, 9, 10, 11, 13 in these circles and squares. Each number must be written exactly once. In this case, the number in each circle

is equal to the sum of the numbers in its two neighboring squares. If x and y are arranged as follows, find the largest value of x+y.

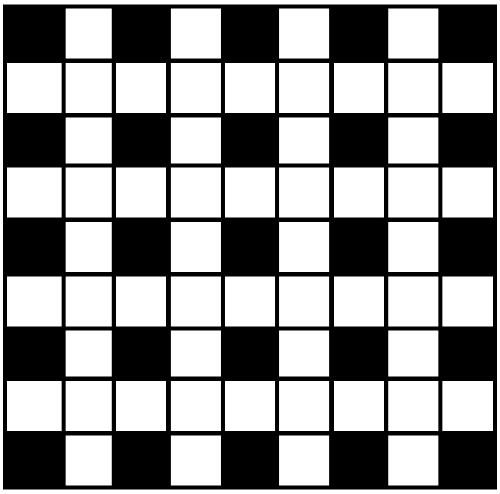
$\boxed{x}$ ○ □ ○ □ ○ □ ○ $\boxed{y}$

20. 10 people came to the ticket office, 5 of them have \$5 and 5 have \$10 each, but the ticket seller has no money. One ticket costs \$5. How many ways can people line up in several different ways so that everyone who needs change (the remaining part of their money) can get it immediately after purchasing the ticket?

Language: English

Answers:

1. $\angle C = \angle A = 2\alpha$. $M \equiv \frac{BC}{2} \Rightarrow AB = BM = MC = CD \Rightarrow \angle DMC = 90^0 - \alpha$ and $\angle ABM = 180^0 - 2\alpha \Rightarrow \angle BMA = \alpha \Rightarrow$ $\angle AMD = 90^0 \Rightarrow \angle AMD + \angle APD = 180^0 \Rightarrow AMDP -$ cyclic $\angle PMD + \angle AMP = 90^0$ and $\angle AMP = \angle ADP, AD \parallel BC \Rightarrow \angle BCP = \angle ADP \Rightarrow \angle BCD + \angle PMD = 90^0$
2. We color the board as follows:



In this coloring, the king begins with a black cell, and after every even move, it will be on a black cell. Also, after every odd moves the king will be on white cell. There are 1012×1012 black cells and $2023 \times 2023 - 1012 \times 1012$ white cells on the board. So if the king finishes on black cell, it will have crossed $1012 \times 1012 - 1$ white cells and if the king finishes on white cells, it will have crossed 1012×1012 white cells. But the number of white are much more than 1012×1012. So the king can not cross all white cells.
3. $a_{n+1} = 2a_1a_2 \dots a_n + 1 = (a_n - 1)a_n + 1$ because $a_n - 1 =$ $= 2a_1a_2 \dots a_{n-1}$. So $a_{n+1} = a_n^2 - a_n + 1 \Rightarrow 1 + (a_n - 2)a_{n+1} = 1 + (a_n - 2)(a_n^2 - a_n + 1) = 1 + a_n^3 - a_n^2 + a_n - 2a_n^2 + 2a_n - 2 = a_n^3 - 3a_n^2 + 3a_n - 1 = (a_n - 1)^3$

4. Let $a_1, a_2, \ldots, a_{10}$ are colored numbers. And $a_1 < a_2 < a_3 < \ldots < a_{10}$. Let us define as follows: $d_i \in \mathbb{N}_0$

$$|a-b| > 2 \Longrightarrow |a-b| \geq 3$$

$$a_2 = a_1 + 3 + d_1$$

$$a_3 = a_2 + 3 + d_2 = a_1 + 6 + d_1 + d_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{10} = a_1 + 27 + d_1 + d_2 + d_3 + \ldots + d_9$$

$a_{10} \leq 100, \Longrightarrow a_1 + d_1 + d_2 + d_3 + \ldots + d_9 \leq 73$

$a_1 \geq 1$, so there exists $d_0 \geq 0$, such that $a_1 = d_0 + 1$

$$d_0 + d_1 + d_2 + \cdots + d_9 \leq 72$$

This means there exists an integer $d_{10} \geq 0$ such that

$$d_0 + d_1 + d_2 + \cdots + d_9 + d_{10} = 72$$

Lemma: The number of non-negative solutions of equation $x_1 + x_2 + x_3 + \ldots + x_m = n$ is C_{m+n-1}^{m-1}.

Proof: $m+n-1$ '1's are written on a row. Let's choose $m-1$ of them and change to '0'. So there are C_{m+n-1}^{m-1} ways.

On the other hand, these $m-1$ '0's divide the '1's into m groups. The number of '1's in groups are $x_1, x_2, \ldots x_m$ And n '1's remain. So This is also the number of all solutions of equation $x_1 + x_2 + \cdots x_n = m$

So, the answer is C_{82}^{10}.

5. $x_1^{m_1} + x_2^{m_2} + \ldots + x_k^{m_k} = z^{m_1 m_2 \ldots m_k}$, $m_i \geq 2023 x_i$

Let $x_i^{m_i}$ is the largest of $\{x_1^{m_1}; x_2^{m_2}; \ldots; x_k^{m_k}\}$

$$z^{m_1 m_2 \ldots m_k} = A^{m_i} \Longrightarrow A \geq x_i + 1 \Longrightarrow k x_i^{m_i} \geq z^{m_1 m_2 \ldots m_k} \geq (x_i + 1)^{m_i}$$

$$k x_i^{m_i} \geq (x_i + 1)^{m_i} \Longrightarrow k \geq \left(1 + \frac{1}{x_i}\right)^{m_i} \geq \left(1 + \frac{1}{x_i}\right)^{2023 x_i}$$

And

$$\left(1 + \frac{1}{x_i}\right)^{x_i} \geq 1 + \binom{x_i}{1} \cdot 1 \cdot \frac{1}{x_i} = 2$$

So $k \geq 2^{2023}$.

Proved.

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Marking scheme (english version)

Problem 1

- If $\angle AMD = 90^{\circ}$ is showed +4 points
- $AMDP$ is cyclic +2 points
- Complete solution 10 points

Problem 2

- No points the right answer, saying no.
- For any coloring (like a chessboard) and eliminating cases(like here is no double move at the diagonal direction ,etc) +4 points
- For a right coloring +6 points
- Full solution 10 points

Problem 3

- Complete solution without proof of $a_{n+1} = a_n^2 - a_n + 1$ 6 points
- If $a_{n+1} = a_n^2 - a_n + 1$ equality is proved 7 points
- Complete solution 10 points

Problem 4

Solution 1:

Let $d_i = a_{i+1} - a_i$

- $a_1 + d_1 + d_2 + \dots + d_9 \leq 100$ +1 point
- Resign and find $x_1 + x_2 + \dots x_{10} \leq 72$ +2 point
- Add elevens term to holds equality x_{11} . $x_1 + x_2 + \dots x_{10} + x_{11} = 72$ +2 point
- Number of solutions of equation $x_1 + x_2 + x_3 + \ldots + x_m = n$ equals C_{m+n-1}^{m-1} is proved +2 point
- C_{82}^{10} is founded as an answer +1 point
- Complete solution 10 point

Solution 2:

Let $d_i = a_{i+1} - a_i$.

- $a_1 + d_1 + d_2 + \dots + d_9 \leq 100$ +1 point
- Resign and find $x_1 + x_2 + \dots x_{10} \leq 72$ +2 point
- Number of solutions of equation $x_1 + x_2 + x_3 + \ldots + x_m = n$ equals C_{m+n-1}^{m-1} is proved +2 point
- If answer remains as $C_{81}^9 + C_{80}^9 + \dots + C_9^9$ +1 point
- $C_p^r + C_{p-1}^r + \dots + C_r^r = C_{p+1}^{r+1}$ is proved +3 point
- C_{82}^{10} is founded as an answer +1 point
- Complete solution 10 point

5-masala.

- For choosing $x_s^{m_s} = \max(x_1^{m_1}, x_2^{m_2}, \dots, x_k^{m_k})$ + 1 point
- For proof of $x_1^{m_1} + x_2^{m_2} + \dots + x_k^{m_k} \geq (x_s + 1)^{m_s}$ +4 points
- For inequality $k \geq \left(1 + \frac{1}{x_s}\right)^{2023x_s}$ +1 point
- Full solution 10 points

For minor mistakes on calculation and etc. upto 3 points can be deducted.