O základných vlastnostiach lineárnych systémov

1 Stabilita

1.1 Lyapunovova teória stability

Definícia 1.1 (Stabilita podľa Lyapunova, priama metóda)**.** $V : \mathbb{R}^n \to \mathbb{R}$

1.
$$V(\mathbf{0}) = 0 \tag{1a}$$

2.
$$V(\mathbf{x}) > 0, \quad \mathbf{x} \neq \mathbf{0} \tag{1b}$$

3.
$$\dot{V}(\mathbf{x}) = \sum_i \frac{\partial V}{\partial x^i} f^i(\mathbf{x}) = \nabla V^\top \mathbf{f}(\mathbf{x}) \leq 0 \tag{1c}$$

$$\dot{V}(\mathbf{x}) < 0, \quad \dot{V}(\mathbf{0}) = 0 \tag{1d}$$

1.2 Diskrétny systém

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n \tag{2}$$

$$V_n = \mathbf{x}_n^\top \mathbf{P}\mathbf{x}_n \tag{3}$$

$$V_{n+1} - V_n < 0 \tag{4}$$

$$\mathbf{x}_{n+1}^\top \mathbf{P}\mathbf{x}_{n+1} - \mathbf{x}_n^\top \mathbf{P}\mathbf{x}_n < 0 \tag{5}$$

$$\mathbf{x}_n^\top \left(\mathbf{A}^\top \mathbf{P}\mathbf{A} - \mathbf{P}\right) \mathbf{x}_n < 0 \tag{6}$$

$$\mathbf{A}^\top \mathbf{P}\mathbf{A} - \mathbf{P} < 0 \tag{7}$$

$$\mathbf{A}^\top \mathbf{P}\mathbf{A} - \mathbf{P} = -\mathbf{Q} \tag{8}$$

1.3 Spojitý systém

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{9}$$

$$V = \mathbf{x}^\top \mathbf{P}\mathbf{x} \tag{10}$$

$$\dot{V} < 0 \tag{11}$$

$$\dot{\mathbf{x}}^\top \mathbf{P}\mathbf{x} + \mathbf{x}^\top \mathbf{P}\dot{\mathbf{x}} < 0 \tag{12}$$

$$\mathbf{x}^\top \left(\mathbf{A}^\top \mathbf{P} + \mathbf{P}\mathbf{A}\right) \mathbf{x} < 0 \tag{13}$$

$$\mathbf{A}^\top \mathbf{P} + \mathbf{P}\mathbf{A} < 0 \tag{14}$$

$$\mathbf{A}^\top \mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \tag{15}$$

2 Pozorovateľnosť

2.1 Diskrétny systém

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n \tag{16}$$

$$\mathbf{y}_n = \mathbf{C}\mathbf{x}_n \tag{17}$$

$$\begin{aligned} \mathbf{y}_0 &= \mathbf{C}\mathbf{x}_0 && \text{(18a)} \\ \mathbf{y}_1 &= \mathbf{C}\mathbf{A}\mathbf{x}_0 && \text{(18b)} \\ &\vdots && \text{(18c)} \\ \mathbf{y}_{n-1} &= \mathbf{C}\mathbf{A}^{n-1}\mathbf{x}_0 && \text{(18d)} \end{aligned}$$

$$\begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_{n-1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{x}_0 \tag{19}$$

Veta 2.1. Diskrétny lineárny systém (16) je pozorovateľný pokiaľ matica pozorovateľnosti $\mathcal{O}$ má plnú hodnosť:

$$\operatorname{rank}\{\mathcal{O}\} = \operatorname{rank}\left\{\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}\right\} = n \tag{20}$$

2.2 Spojitý systém

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{21}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{22}$$

$$\begin{aligned}
\dot{\mathbf{y}}(t_i) &= \mathbf{C}\mathbf{x}_0 && \text{(23a)}\\
\ddot{\mathbf{y}}(t_i) &= \mathbf{C}\mathbf{A}\mathbf{x}_0 && \text{(23b)}\\
&\vdots && \text{(23c)}\\
\mathbf{y}^{(n-1)}(t_i) &= \mathbf{C}\mathbf{A}^{n-1}\mathbf{x}_0 && \text{(23d)}
\end{aligned}$$

$$\begin{bmatrix} \dot{\mathbf{y}}(t_i) \\ \ddot{\mathbf{y}}(t_i) \\ \vdots \\ \mathbf{y}^{(n-1)}(t_i) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{x}(t_i) \tag{24}$$

Veta 2.2. Spojitý lineárny systém (21) je pozorovateľný pokiaľ matica pozorovateľnosti $\mathcal{O}$ má plnú hodnosť:

$$\operatorname{rank}\{\mathcal{O}\} = \operatorname{rank}\left\{\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}\right\} = n \tag{25}$$

3 Riaditeľnosť

3.1 Diskrétny systém

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n + \mathbf{B}\mathbf{u}_n \tag{26}$$

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{A}\mathbf{x}_0 + \mathbf{B}\mathbf{u}_0 && \text{(27a)}\\
\mathbf{x}_2 &= \mathbf{A}^2\mathbf{x}_0 + \mathbf{A}\mathbf{B}\mathbf{u}_0 + \mathbf{B}\mathbf{u}_1 && \text{(27b)}\\
&\vdots \\
\mathbf{x}_n &= \mathbf{A}^n\mathbf{x}_0 + \mathbf{A}^{n-1}\mathbf{B}\mathbf{u}_0 + \cdots + \mathbf{A}\mathbf{B}\mathbf{u}_{n-2} + \mathbf{B}\mathbf{u}_{n-1} && \text{(27c)}
\end{aligned}$$

$$\mathbf{x}_n - \mathbf{A}^n\mathbf{x}_0 = \underbrace{\begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} \mathbf{u}_{n-1} \\ \mathbf{u}_{n-2} \\ \vdots \\ \mathbf{u}_0 \end{bmatrix} \tag{28}$$

kde $\mathcal{C}$ označuje maticu riaditeľnosti. Matica $\mathcal{C}$ má rozmery $n \times nr$, kde r je počet vstupov (rozmer vektora $\mathbf{u}$).

Veta 3.1. Diskrétny lineárny systém (26) je riaditeľný pokiaľ matica riaditeľnosti má plnú hodnosť:

$$\operatorname{rank}\{\mathcal{C}\} = \operatorname{rank}\left\{\begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}\right\} = n \tag{29}$$

3.2 Spojitý systém

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{30}$$

$$\mathbf{x}(t) = \exp\left(\mathbf{A}(t - t_i)\right) + \int_{t_i}^{t} \exp(\mathbf{A}(t-\tau))\mathbf{B}\mathbf{u}(\tau)d\tau \tag{31}$$

$$\exp(-\mathbf{A}t)\mathbf{x}(t) - \exp(-\mathbf{A}t_i)\mathbf{x}(t_i) = \int_{t_i}^{t} \exp(-\mathbf{A}\tau)\mathbf{B}\mathbf{u}(\tau)d\tau \tag{32}$$

$$\exp(\mathbf{A}\tau) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n \tau^n \tag{33}$$

$$\det\left\{\mathbf{A} - \lambda I\right\} = 0 \tag{34}$$

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0 = 0 \tag{35}$$

$$\mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \cdots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I} = 0 \tag{36}$$

$$\mathbf{A}^n = \sum_{i=0}^{n-1} \alpha_i \mathbf{A}^i \tag{37}$$

$$\exp(\mathbf{A}\tau) = \sum_{n=0}^{n-1} \alpha_i(\tau)\mathbf{A}^i \tag{38}$$

$$\exp(-\mathbf{A}t)\mathbf{x}(t) - \exp(-\mathbf{A}t_i)\mathbf{x}(t_i) = \sum_{n=0}^{n-1}\left(\mathbf{A}^i\mathbf{B}\int_{t_i}^{t} \alpha_i(\tau)\mathbf{u}(\tau)d\tau\right) \tag{39}$$

$$\begin{aligned}&\exp(-\mathbf{A}t)\mathbf{x}(t) - \exp(-\mathbf{A}t_i)\mathbf{x}(t_i)\\ &\qquad = \underbrace{\begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} \int_{t_i}^{t} \alpha_0(\tau)\mathbf{u}(\tau)d\tau \\ \int_{t_i}^{t} \alpha_1(\tau)\mathbf{u}(\tau)d\tau \\ \vdots \\ \int_{t_i}^{t} \alpha_{n-1}(\tau)\mathbf{u}(\tau)d\tau \end{bmatrix}\end{aligned} \tag{40}$$

Veta 3.2. Spojitý lineárny systém (30) je riaditeľný pokiaľ matica riaditeľnosti $\mathcal{C}$ má plnú hodnosť:

$$\operatorname{rank}\left\{\mathcal{C}\right\} = \operatorname{rank}\left\{\begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}\right\} = n \tag{41}$$