Mathematical Thinking.
Test Flight Assignment Solutions.

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Question 9

Definition 1. $\bigcap\limits_{n=1}^{\infty} A_n := \{x : (\forall n \in \mathbb{N})(x \in A_n)\}$

Lemma 1. *Let* $A_n := \{x : 0 < x < \frac{1}{n}\}$ *for every natural number* n*, then* $A_{n+1} \subset A_n$*.*

Proof. Let n be any natural number and $A_n := \{x : 0 < x < \frac{1}{n}\}$.
Then $A_{n+1} := \{x : 0 < x < \frac{1}{n}\}$.
For every $x \in A_{n+1}$, $0 < x < \frac{1}{n+1}$

$$\begin{aligned}
&\Rightarrow (0 < x) \wedge (x < \frac{1}{n+1}) \\
&\Rightarrow (0 < x) \wedge (x < \frac{1}{n}) && \textit{(Since } \frac{1}{n+1} < \frac{1}{n}\textit{)} \\
&\Rightarrow 0 < x < \frac{1}{n} \\
&\Rightarrow x \in A_n.
\end{aligned}$$

Hence for any natural number n, $A_{n+1} \subset A_n$. □

Lemma 2. *For every* $x \in \mathbb{R}$*, there exists a* $n \in \mathbb{N}$ *such that* $x \notin A_n$*, where*
$A_n := \{y : 0 < y < \frac{1}{n}\}$.

Proof. (By Cases)
Let x be any real number.

Case 1: $(x \leqslant 0)$
Let n be any natural number and $A_n := \{x : 0 < x < \frac{1}{n}\}$.
$\Rightarrow x \notin A_n$ (Since $x \leqslant 0$).

Case 2: $(x > 0)$
Let n be any natural number such that $n \geq \frac{1}{x}$ and $A_n := \{x : 0 < x < \frac{1}{n}\}$.
$\Rightarrow x \geq \frac{1}{n}$
$\Rightarrow x \notin A_n$ (Since $x \geq \frac{1}{n}$).

Hence, for every $x \in \mathbb{R}$, there exists a $n \in \mathbb{N}$ such that $x \notin A_n$, where
$A_n := \{y : 0 < y < \frac{1}{n}\}$. □

Lemma 3. *If* $A_n := \{x : 0 < x < \frac{1}{n}\}$*, for every* $n \in \mathbb{N}$*, then* $\bigcap\limits_{n=1}^{\infty} = \phi$*.*

Proof. Let $A_n := \{x : 0 < x < \frac{1}{n}\}$ for every $n \in \mathbb{N}$.
From Definition **??**, $\bigcap\limits_{n=1}^{\infty} = \phi$ if $\neg(\exists x \in \mathbb{R})(\forall n \in \mathbb{N})(x \in A_n)$
But $\neg(\exists x \in \mathbb{R})(\forall n \in \mathbb{N})(x \in A_n) \Leftrightarrow (\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(x \notin A_n)$.
Hence, $\bigcap\limits_{n=1}^{\infty} = \phi$ if $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(x \notin A_n)$.
By Lemma 2, $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(x \notin A_n)$ is true. Hence, $\bigcap\limits_{n=1}^{\infty} = \phi$ is true. □

Proposition 1. *If $A_n := \{x : 0 < x < \frac{1}{n}\}$ for every $n \in \mathbb{N}$, then:*

1. *$A_{n+1} \subset A_n$ for every $n \in \mathbb{N}$*
2. *$\bigcap\limits_{n=1}^{\infty} = \phi$.*

Proof. Let $A_n := \{x : 0 < x < \frac{1}{n}\}$ for every $n \in \mathbb{N}$. Then property (1) is true by Lemma 1 and property (2) is true by Lemma 3. □