Mathematical Thinking.
Test Flight Assignment Solutions.

Okon Samuel

Question 10

Lemma 1. *If n is any natural number and $A_n := \{x : 0 \leqslant x < \frac{1}{n}\}$, then $A_{n+1} \subset A_n$.*

Proof. Let n be any natural number and
$A_n := \{x : 0 \leqslant x < \frac{1}{n}\}$, then $A_{n+1} := \{x : 0 \leqslant x < \frac{1}{n+1}\}$.
For every $x \in A_{n+1}$, $0 \leqslant x < \frac{1}{n+1}$

$$\begin{aligned}
&\Rightarrow (0 \leq x) \wedge (x < \frac{1}{n+1}) \\
&\Rightarrow (0 \leq x) \wedge (x < \frac{1}{n}) && \textit{(Since } \frac{1}{n+1} < \frac{1}{n}\textit{)} \\
&\Rightarrow 0 \leqslant x < \frac{1}{n} \\
&\Rightarrow x \in A_n.
\end{aligned}$$

Hence for any natural number n, $A_{n+1} \subset A_n$. □

Lemma 2. *For every $x \in (\mathbb{R}^+ \cup \mathbb{R}^-)$, there exists a $n \in \mathbb{N}$ such that $x \notin A_n$ where $A_n := \{y : 0 \leqslant y < \frac{1}{n}\}$.*

Proof. (By Cases)
Let x be any non-zero real number (i.e $x \in (\mathbb{R}^+ \cup \mathbb{R}^-)$).

Case 1: $(x < 0)$
Let n be any natural number and $A_n := \{x : 0 \leqslant x < \frac{1}{n}\}$.
$\Rightarrow x \notin A_n$ (Since $x < 0$).

Case 2: $(x > 0)$
Let n be any natural number such that $n \geqslant \frac{1}{x}$ and $A_n := \{x : 0 \leqslant x < \frac{1}{n}\}$.
$\Rightarrow x \leqslant \frac{1}{n}$
$\Rightarrow x \notin A_n$.

Hence for every non-zero real number, there exists a $n \in \mathbb{N}$ such that $x \notin A_n$, where $A_n := \{x : 0 \leqslant x < \frac{1}{n}\}$. □

Lemma 3. *If $A_n := \{x : 0 \leqslant x < \frac{1}{n}\}$ for every $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} = \{0\}$.*

Proof. From Definition 1,
$\bigcap_{n=1}^{\infty} = \{0\}$ if $(\forall x \in \mathbb{R})((\forall n \in \mathbb{N})(x \in A_n) \Rightarrow x = 0)$
But $(\forall x \in \mathbb{R})((\forall n \in \mathbb{N})(x \in A_n) \Rightarrow x = 0) \Leftrightarrow (\forall x \in \mathbb{R})(x \neq 0 \Rightarrow (\exists n \in \mathbb{N})(x \notin A_n))$.
Hence

$\bigcap_{n=1}^{\infty} = \{0\}$ if $(\forall x \in \mathbb{R})(x \neq 0 \Rightarrow (\exists n \in \mathbb{N})(x \notin A_n))$.

By Lemma 2, $(\forall x \in \mathbb{R})(x \neq 0 \Rightarrow (\exists n \in \mathbb{N})(x \notin A_n))$ is true. Hence $\bigcap_{n=1}^{\infty} = \{0\}$ is true. $\square$

Proposition 1. *If* $A_n := \{x : 0 \leqslant x < \frac{1}{n}\}$ *for every* $n \in \mathbb{N}$*, then:*

1. $A_{n+1} \subset A_n$ *for every* $n \in \mathbb{N}$
2. $\bigcap_{n=1}^{\infty} = \{0\}$.

Proof. Let $A_n := \{x : 0 \leqslant x < \frac{1}{n}\}$ for every $n \in \mathbb{N}$. Then property (1) is true by Lemma 1 and property (2) is true by Lemma 3. $\square$