

Machine Learning (911.236)

Exercise sheet D

Exercise 1.

4 P.

Consider the hypothesis class of *Boolean conjunctions* over n variables, denoted as $\mathcal{H}_{\text{con}_n}$. Take, as an example, $n = 4$: then, *some* valid hypotheses in $\mathcal{H}_{\text{con}_4}$ are

$$x_1 \wedge x_3, \quad x_1 \wedge \neg x_3, \quad \text{or} \quad x_2 \wedge \neg x_3 \wedge x_4$$

The empty hypothesis (no variable is in the conjunction) is taken as the *all positive* hypothesis (i.e., always returns 1), and any hypothesis involving a variable and its negation is considered the *all negative* hypothesis (i.e., always returns 0).

Remark A: by an easy argument (we will do this in the lecture), for finite hypothesis classes, it holds that

$$\text{VC}(\mathcal{H}) \leq \log_2(|\mathcal{H}|) .$$

1.1. [2P]. Provide an upper bound on $\text{VC}(\mathcal{H}_{\text{con}_n})$ using *Remark A*.

1.2. [2P]. When we do not allow negations, we have the hypothesis class of *monotone conjunctions*, denoted by $\mathcal{H}_{\text{mcon}_n}$. Let the empty hypothesis again be the all positive hypothesis (as before) and add an all negative hypothesis to the hypothesis set. Provide an upper bound on $\text{VC}(\mathcal{H}_{\text{mcon}_n})$ using *Remark A*.

Exercise 2.

6 P.

Remark B: Given two hypothesis classes \mathcal{H}_A and \mathcal{H}_B , one can show that given $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_B$, we have, for the growth function, $\tau_{\mathcal{H}}(m) \leq \tau_{\mathcal{H}_A}(m) + \tau_{\mathcal{H}_B}(m)$.

Assume w.l.o.g. that $\text{VC}(\mathcal{H}_A) = \text{VC}(\mathcal{H}_B) = d$ and let $k \in \mathbb{N}$ be such that $k \geq 2d + 2$. Show that for $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_B$, we have

$$\tau_{\mathcal{H}}(k) < 2^k$$

Hint: Binomial coefficient identities will be helpful, e.g.,

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{for } 0 \leq k \leq n$$