$$\min_{\substack{R\in SO(3),\, s\in\mathbb{R}\\ t\in\mathbb{R}^3}} \sum_{i=1}^{n} \|y_i - sRx_i - t\|^2 = E(R,s,t)$$

$$\frac{\partial E}{\partial t} \quad \left(\frac{\partial E}{\partial R}\right) \quad \frac{\partial E}{\partial s}$$

$$- \text{"} - \quad \sum_{i=1}^{n} (y_i - sRx_i - t)^T \cdot (y_i - sRx_i - t)$$

$$(a+b+c)^2 \ldots$$

$$y_i^T y_i + 2y_i^T\cdot(-sRx_i) + (-sRx_i)^T(-sRx_i) + t^T t + 2\cdot(sRx_i)^T t - 2y_i^T t$$

with $\sum$:

$$\sum_{i=1}^{n} y_i^T y_i - 2s\cdot\sum_{i=1}^{n} y_i^T R x_i + s^2 \sum_{i=1}^{n} x_i^T \underbrace{R^T R}_{1} x_i + N t^T t + 2s\left(\sum_{i=1}^{n} x_i^T\right) R^T t$$

$$-2\cdot\left(\sum_{i=1}^{n} y_i^T\right) t = E(R,s,t)$$

$$\boxed{\begin{aligned} &\frac{\partial E}{\partial t} = -\cancel{2}\left(\sum_i y_i^T\right) + \cancel{2}Nt + \cancel{2}s\left(\cdot\sum_i x_i^T\right)R^T = 0 \\ &Nt = \sum_i y_i^T - s\cdot\left(\sum_i x_i^T\right)R^T = \sum_i y_i - sR\cdot\sum_i x_i \end{aligned}}$$

$$Nt = \sum_i y_i - sR \sum_i x_i$$

$$\Rightarrow t^* = \frac{1}{N} \cdot \sum y_i - sR \frac{1}{N} \cdot \sum_i x_i \qquad \bar{x} = \frac{1}{N} \cdot \sum_i x_i, \quad \bar{y} = \frac{1}{N} \cdot \sum y_i$$

$$= \bar{y} - sR\bar{x} \qquad (*)$$

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ & \vdots & \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

avg $\Rightarrow \bar{x}$

Using $(*)$, we have (for the energy term)

$$\sum_{i=1}^{n} \| y_i - sRx_i - \bar{y} + sR\bar{x} \|^2$$

$$= \sum_i \| (y_i - \bar{y}) - sR \cdot (x_i - \bar{x}) \|^2 \qquad \begin{aligned} y_i' &= y_i - \bar{y} \\ x_i' &= x_i - \bar{x} \end{aligned}$$

$$= \sum_1 \| y_i' - sRx_i' \|^2 \qquad (**)$$

Multiplying this (**) out gives

$$y_i'^T y_i' - 2\cdot y_i'^T (sRx_i') + s^2\cdot (Rx_i')^T (Rx_i')$$

with $\sum$: $$\sum_{i=1}^{n} y_i'^T y_i' - 2s\cdot \sum_{i=1}^{n} y_i'^T R x_i' + \underbrace{s^2 \cdot \sum_{i=1}^{n} x_i'^T \underbrace{R^T R}_{1} x_i'}_{s^2\cdot \sum_i x_i'^T x_i'}$$

Now, $$\frac{\partial E}{\partial s} = -\cancel{2}\cdot \sum_{i=1}^{n} y_i'^T R x_i' + \cancel{2}s\cdot \sum_i x_i'^T x_i = 0$$

$$\Rightarrow s^* = \frac{\sum_i y_i'^T R x_i'}{\sum_i x_i'^T x_i} \quad (***)$$

→ dependent on R!

One way to get rid of R in $(***)$ is to "symmetrize"

$$\sum_{i=1}^{n} \|y_i' - sRx_i'\|^2 \longrightarrow \underbrace{\sum_{i=1}^{n} \left\| \frac{1}{\sqrt{s}} y_i' - \sqrt{s} R x_i' \right\|^2}_{(****)}$$

Taking $\frac{\partial E}{\partial s}$ and setting this to zero gives

$$s^* = \frac{\sum_i y_i'^T y_i}{\sum_i x_i'^T x_i}$$

(does <u>not</u> depend on R!)

Multiplying $(****)$ out gives

$$\ldots\ldots - 2 \cdot \underbrace{\sum_i y_i'^T R x_i'}_{T} + \ldots\ldots$$

To minimize energy, we need to <u>maximize</u> T.

This means

$$\underset{R \in SO(3)}{\arg\max} \underbrace{\sum_{i=1}^{n} x_i'^T R y_i'}_{\text{scalar quantity}}$$

cyclic property

$$tr(ABCD) = tr(BCDA) = tr(CDAB)$$

$$\vdots$$

We can write

$$\sum_i x_i'^T R y_i' = \text{trace}\left(\sum_i x_i'^T R y_i'\right)$$

$$= \text{trace}\Big(R \underbrace{\Big(\sum_i x_i' y_i'^T\Big)}_{H}\Big)$$

$$= \text{trace}(R \cdot H)$$

$$y_i' = y_i - \bar{y}$$

$$x_i' = x_i - \bar{x}$$

$$\Rightarrow \underset{R \in SO(3)}{\arg\max} \ \text{trace}(RH)$$

$$R^* = VU^T$$

$$H = U \Sigma V^T$$

(Singular value decomposition)