University of Salzburg | Lecturer: Roland Kwitt

Machine Learning (911.236)

Exercise sheet **D**

Exercise 1. 4 P.

Consider the hypothesis class of *Boolean conjunctions* over n variables, denoted as $\mathcal{H}_{\mathsf{con}_n}$. Take, as an example, $n = 4$: then, *some* valid hypotheses in $\mathcal{H}_{\mathsf{con}_4}$ are

$$x_1 \wedge x_3, \quad x_1 \wedge \neg x_3, \quad \text{or} \quad x_2 \wedge \neg x_3 \wedge x_4$$

The empty hypothesis (no variable is in the conjunction) is taken as the *all positive* hypothesis (i.e., always returns 1), and any hypothesis involving a variable and its negation is considered the *all negative* hypothesis (i.e., always returns 0).

Remark A: by an easy argument (we will do this in the lecture), for finite hypothesis classes, it holds that

$$\mathrm{VC}(\mathcal{H}) \leq \log_2(|\mathcal{H}|) \ .$$

1.1. [2P]. Provide an upper bound on $\mathrm{VC}(\mathcal{H}_{\mathsf{con}_n})$ using *Remark A*.

1.2. [2P]. When we do not allow negations, we have the hypothesis class of *monotone conjunctions*, denoted by $\mathrm{VC}(\mathcal{H}_{\mathsf{mcon}_n})$. Let the empty hypothesis again be the all positive hypothesis (as before) and add an all negative hypothesis to the hypothesis set. Provide an upper bound on $\mathrm{VC}(\mathcal{H}_{\mathsf{mcon}_n})$ using *Remark A*.

Exercise 2. 6 P.

One important result which we will see in the lecture is **Sauer's lemma**.

Lemma 1. *(Sauer-Shelah-Perles) Let $\mathcal{H}$ be a hypothesis class with $VC(\mathcal{H}) \leq d < \infty$. Then, for all m,*

$$\tau_{\mathcal{H}}(m) \leq \sum_{i=0}^{d} \binom{m}{i} \ .$$

In particular, if $m > d + 1$, then

$$\tau_{\mathcal{H}}(m) \leq \left(\frac{em}{d}\right)^d \ .$$

Remark B: Given two hypothesis classes $\mathcal{H}_A$ and $\mathcal{H}_B$, one can show that given $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_B$, we have, for the growth function, $\tau_{\mathcal{H}}(m) \leq \tau_{\mathcal{H}_A}(m) + \tau_{\mathcal{H}_B}(m)$.

Assume w.l.o.g. that $\mathrm{VC}(\mathcal{H}_A) = \mathrm{VC}(\mathcal{H}_B) = d$ and let $k \in \mathbb{N}$ be such that $k \geq 2d + 2$. Show that for $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_B$, we have

$$\tau_{\mathcal{H}}(k) < 2^k \ .$$

Hint: You can, e.g., start with the first result from Sauer's lemma and then use appropriate Binomial coefficient identities, e.g.,

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{for } 0 \leq k \leq n \ .$$

Total #points: 10 P.