σ-Algebra

Top-space

(X, τ) ↳ coll. of subsets

$X \in \tau$
$\emptyset \in \tau$
union-closed
intersec. (finite) closed

$X = \{1,2,3\}$
$\tau = \{\emptyset, \{1,2,3\}\}$
$(X, P(X))$

Let S be a non-empty <u>basis set</u> (or "universe"). A family of sets $F \subset P(S)$ is called a <u>σ-Algebra</u> on S, if the following conditions hold:

(I) $S \in F$
(II) From $A \in F$, it follows that $A^c = S \setminus A \in F$
(III) From $A_i \in F$, $i \in \mathbb{N}$, it follows that $\bigcup_{i=1}^{\infty} A_i \in F$

<u>Remark</u>: Any subset $F \subset P(S)$ of the power set $P(S)$ is called a family of sets.

Smallest σ-Algebra: $\{\emptyset, S\}$

σ-Algebra generated by A: If $A \subset S$, then $\sigma(A) = \{\emptyset, A, A^c, S\}$ is called the σ-Algebra generated by A. It is the smallest σ-Algebra which contains A.

generators

Let $\mathcal{E} \subset \mathcal{P}(S)$ be a family of sets. Further, let Σ be the set of <u>all</u> σ-Algebras over S which contain $\mathcal{E}$. Then,

$$\sigma(\mathcal{E}) = \bigcap_{\mathcal{F} \in \Sigma} \mathcal{F}$$

is the σ-Algebra **generated** by $\mathcal{E}$. Also, if for a σ-Algebra A we have

$$\sigma(\mathcal{E}) = A$$

we call $\mathcal{E}$ **generator** of A.

<u>Example</u>: $\mathcal{E} = \{1\}$ and $S = \{1,2,3\}$

$$\sigma(\{\{1\}\}) = \{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}\}$$

Borel σ-Algebra

Of special interest to us is the σ-Algebra of <u>BOREL sets</u> over $\mathbb{R}^n$:

Let S be a topological space and $\mathcal{O}$ the system of <u>open subsets</u> of S. Then,

$$\mathcal{B}(S) = \sigma(\mathcal{O})$$

is called the <u>Borel σ-Algebra</u> over S. Elements $A \in \mathcal{B}(S)$ are called BOREL sets. For $S = \mathbb{R}^n$, we write $\mathcal{B}^n = \mathcal{B}(\mathbb{R}^n)$.

Each of the following families of sets is <u>a generator</u> of the Borel σ-Algebra $\mathcal{B}^n$:

- $\{U \subset \mathbb{R}^n : U \text{ open}\}$
- $\{A \subset \mathbb{R}^n : A \text{ closed}\}$
- $\{]a, b] : a, b \in \mathbb{R}^n \ a \leq b\}$
- $\{]-\infty, c] : c \in \mathbb{R}^n\}$

$$]-\infty, c_1] \times]-\infty, c_2] \times \ldots \times]-\infty, c_n] \subset \mathbb{R}^n$$

$$c = (c_1, \ldots, c_n) \in \mathbb{R}^n$$

Measurable space	if $\mathcal{F}$ is a σ-Alg. over S, we call $$(S, \mathcal{F})$$ a measureable space.
Measurable mapping	given $(S_1, \mathcal{F}_1)$ and $(S_2, \mathcal{F}_2)$ measurable spaces, we call $$f: S_1 \to S_2$$ $(\mathcal{F}_1 - \mathcal{F}_2)$ - measurable (or simply "measureable") if $$\forall E \in \mathcal{F}_2 : \underbrace{f^{-1}(E)}_{\text{pre-image of } E} \in \mathcal{F}_1$$ given $(S_1, \mathcal{F}_1)$ and $(S_2, \mathcal{F}_2)$ with $\mathcal{F}_2 = \sigma(\mathcal{E})$, the mapping $f: S_1 \to S_2$ is measurable if $$\forall E \in \mathcal{E} : f^{-1}(E) \in \mathcal{F}_1$$ So, to check measurability, we can just check the generator.

Example:

$(\mathbb{R}^n, \mathcal{B}^n)$

Generator

Example

given $(S, \mathcal{F})$ a measurable space and

$$f : S \longrightarrow \mathbb{R},$$

then f is measurable if

$$\forall c \in \mathbb{R} : \quad f^{-1}\big(]-\infty, c]\big) = \{ s \in S : f(s) \leq c \} \in \mathcal{F}.$$

(Since $\{]-\infty, c] : c \in \mathbb{R} \}$ is a generator for $\mathcal{B}(\mathbb{R}) = \mathcal{B}$).

Measure

Let $(S, \mathcal{F})$ be a measurable space. A function

$$\mu: \mathcal{F} \to \mathbb{R} \cup \{+\infty, -\infty\} = \overline{\mathbb{R}}$$

is called a measure if

(I) $\mu(\emptyset) = 0$

(II) $\mu(A) \geq 0$ for all $A \in \mathcal{F}$

(III) for every sequence $(A_n)_{n \in \mathbb{N}}$ of disjoint sets from $\mathcal{F}$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

called ("σ-Additivity")

Remark: similar to $\mathcal{B}^n = \mathcal{B}(\mathbb{R}^n)$, we can define $\mathcal{B}(\overline{\mathbb{R}^n})$.

Example

$$\left.\begin{aligned} \mu: \mathcal{F} &\to \overline{\mathbb{R}} \\ A &\mapsto \begin{cases} |A|, & \text{if } A \text{ finite} \\ \infty, & \text{else} \end{cases} \end{aligned}\right\} \text{ called the "counting" measure.}$$

Prob. space

Given that (S, F, P) is a measure space and $P(S) = 1$, we call P a probability measure and (S, F, P) a prob. space.

Random experiments are modeled by prob. spaces: we call

(I) S ... the space of outcomes,

(II) the σ-algebra over S the event set,

(III) and $P: F \to [0,1]$ assigns to each event $A \in F$ a probability $P(A)$.

Given $A, B, A_n \in F$ and $n \in \mathbb{N}$, we have (without proof).

- $P(A^c) = 1 - P(A)$
- if $A \subset B$, then $P(A) \leq P(B)$
- $\underbrace{P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)}_{\text{aka "UNION BOUND"}}$ (A_i's not necessarily disjoint)