5. INTEGRATION

· INDEFINITE INTEGRALS

Let $f: I \to \mathbb{R}$ I an interval.
We say that F is an <u>antiderivative</u> of the f in I if

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

<u>THEOREM 5.1</u>: Let $f: I \to \mathbb{R}$ and suppose F, G are two antiderivatives of f in I.
Then

$$F(x) = G(x) + c \,, \quad x \in I$$

where $c \in \mathbb{R}$ is a constant.

<u>PROOF</u>
Since F, G are antiderivatives of f in I, we have

$$F'(x) = G'(x) = f(x) \quad \forall x \in I$$

thus by Corollary 3.8 we have

$$F(x) = G(x) + c \quad \text{for all } x \in I.$$

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E.g. $f(x) = x^2$, $x \in \mathbb{R}$
Then $F(x) = \frac{1}{3} x^3$ is an antider. of f

and also $G(x) = \frac{1}{3} x^3 + 1$,

and so is $H(x) = \frac{1}{3} x^3 - 5$, etc.

Clearly, if F is an antiderivative of f then any function of the form

$$G(x) = F(x) + c \quad , \quad x \in I$$

will also be an antiderivative.

Together with Theorem 5.1, this implies that whenever we have an antider. F of f every other antiderivative of f will have the form $F(x) + c$.

We write

$$\int f(x)\,dx = F(x) + c .$$

E.g., for $f(x) = x^2$, we write

$$\int x^2\,dx = \frac{1}{3}x^3 + c .$$

<u>REMARK</u>: Whenever we write

$$\int f(x)\,dx = F(x) + c$$

we mean that $F(x)$ is an antiderivative of f in some <u>interval</u> where f is defined and not necessarily in the whole domain of f.

The rules of differentiation imply that

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int \lambda f(x)\,dx = \lambda \int f(x)\,dx \quad (\text{when } \lambda \neq 0).$$

Indefinite Integrals of Certain Functions

$$\int 1\,dx = x + c$$

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + c, \quad a \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int e^x\,dx = e^x + c$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c, \quad 0 < a \neq 1$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + c$$

$$\int \frac{1}{\sin^2 x}\,dx = -\cot x + c$$

$$\int \sinh x \, dx = \cosh x + c$$

$$\int \cosh x \, dx = \sinh x + c$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$$

$$\int \frac{1}{1+x^2} dx = \arctan x + c$$

Recall the chain rule in differentiation:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x) .$$

This implies that

$$\int f'(g(x)) \, g'(x) \, dx = f(g(x)) + c .$$

That is, for the integral

$$\int f'(g(x)) \, g'(x) \, dx$$

we set $u = g(x)$.

then $\dfrac{du}{dx} = g'(x) \Rightarrow du = g'(x)dx$

(dx, du are called the "differentials").

$$\int f'(g(x)) g'(x) dx = \int f'(u) \, du = f(u) + c$$
$$= f(g(x)) + C .$$

E.g. $\int x e^{x^2} dx =$ Set $u = x^2 \Rightarrow du = 2x\,dx$

$$= \int \frac{1}{2} e^u du = \frac{1}{2} \int e^u du$$

$$= \frac{1}{2} e^u + c = \frac{1}{2} e^{x^2} + c.$$

$$\int \frac{\cos\sqrt{x+1}}{\sqrt{x+1}} dx =$$ Set $u = \sqrt{x+1}$

$du = \frac{1}{2\sqrt{x+1}} dx$

$$= \int 2\cos u \, du$$

$$= 2\sin u + c$$

$$= 2\sin\sqrt{x+1} + c$$

Integration by Parts / Factorial Integration

The product rule $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$ implies that

$$\boxed{\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx}$$

Sometimes this is also written as

$$\int u\,dv = uv - \int v\,du.$$

- $\int x e^x dx = \int x (e^x)' dx$

$= x e^x - \int (x)' e^x dx$

$= x e^x - \int e^x dx$

$= x e^x - e^x + c.$

- $\int x^2 e^x dx = \int x^2 (e^x)' dx$

$= x^2 e^x - \int (x^2)' e^x dx$

$= x^2 e^x - 2\int x e^x dx$

$= x^2 e^x - 2(x-1)e^x + c.$

- $\int x^2 \cos x \, dx = \int x^2 (\sin x)' dx$

$= x^2 \sin x - \int 2x \sin x \, dx$

$= x^2 \sin x + 2\int x (\cos x)' dx$

$= x^2 \sin x + 2x \cos x - 2\int \cos x \, dx$

$= x^2 \sin x + 2x \cos x - 2\sin x + c.$

The method of integration by parts can be used to calculate all integrals of the form

$$\int P(x)e^{ax}\,dx\ ,\quad \int P(x)\sin(ax)\,dx\ ,\quad \int P(x)\cos(ax)\,dx$$

where $P(x)$ is a polynomial.

- Integrals of the form $\int F(x)dx$ when we know $F'(x)$.

$$\int F(x)dx = \int (x)'F(x)dx = xF(x) - \int xF'(x)dx$$

- $\int \ln x\, dx = \int (x)' \ln x\, dx$

$$= x\ln x - \int x(\ln x)'dx$$

$$= x\ln x - \int x \cdot \frac{1}{x}\,dx$$

$$= x\ln x - \int 1\,dx$$

$$= x\ln x - x + c$$

- $\int \arctan x \, dx = \int (x)' \arctan x \, dx$

$$= x \arctan x - \int x (\arctan x)' \, dx$$

$$= x \arctan x - \int \frac{x}{1+x^2} dx$$

Set $u = 1 + x^2 \Rightarrow du = 2x \, dx$

$$= x\arctan x - \int \frac{du}{2u}$$

$$= x \arctan x - \frac{1}{2} \ln|u| + C$$

$$= x \arctan x - \frac{1}{2} \ln(1+x^2) + C.$$

▸ Integrals of Rational Functions $\int \frac{P(x)}{Q(x)} dx$.

$$\int \frac{1}{x^2+1} dx = \arctan x + c$$

$$\int \frac{1}{x^2+2} dx = \int \frac{1}{2u^2+2} \sqrt{2} \, du$$

Set $x = \sqrt{2}\, u \Rightarrow dx = \sqrt{2}\, du$

$$= \frac{\sqrt{2}}{2} \int \frac{du}{u^2+1} = \frac{\sqrt{2}}{2} \arctan u + c$$

$$= \frac{\sqrt{2}}{2} \arctan\left(\frac{x}{\sqrt{2}}\right) + c$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int \frac{1}{x+a}\,dx = \ln|x+a| + c$$

Generally, to find $\int \frac{dx}{ax^2+bx+c}$

we factorise ax^2+bx+c.

I. If it has two real roots, then e.g.

$$\int \frac{1}{x^2-3x+2}\,dx = \int \frac{dx}{(x-1)(x-2)}$$

I look for coefficients $A, B \in \mathbb{R}$ such that

$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \quad \forall x \neq 1,2 \Rightarrow$$

$$A(x-2) + B(x-1) = 1 \quad \forall x \neq 1,2 \Rightarrow$$

$$\begin{cases} A + B = 0 \\ 2A + B = -1 \end{cases} \quad \begin{matrix} A = -1 \\ B = 1 \end{matrix}$$

$$\int \frac{dx}{x^2-3x+2} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1}$$

$$= \ln|x-2| - \ln|x-1| + c$$

$$= \ln\left|\frac{x-2}{x-1}\right| + c.$$

II. If it has a double real root, then e.g.

$$\int \frac{dx}{(2x-1)^2} = -\frac{1}{2}\int \left(\frac{1}{2x-1}\right)' dx$$

$$= -\frac{1}{2}\cdot\frac{1}{2x-1} + C$$

$$= \frac{1}{2-4x} + C.$$

III. If it has no real roots, e.g.

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$

$$= \int \frac{dx}{(x+1)^2+1}$$

$$= \arctan(x+1) + C$$

Generally for $\int \frac{P(x)}{Q(x)}dx$

we apply the partial fraction decomposition.

E.g. $\int \frac{dx}{(x+1)(x^2+x+1)} = \int \left(\frac{1}{x+1} - \frac{x}{x^2+x+1}\right)dx$

see Chapter 4

$$= \int \frac{dx}{x+1} - \int \frac{x}{x^2+x+1}\,dx$$

$$= \ln(x+1) - \frac{1}{2}\int \frac{2x+1-1}{x^2+x+1}\,dx$$

$$= \ln(x+1) - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{1}{2}\int \frac{dx}{x^2+x+1}$$

$$= \ln|x+1) - \frac{1}{2}\ln(x^2+x+1) + \frac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

Set $x+\frac{1}{2} = \frac{\sqrt{3}}{2}u \Rightarrow$

$dx = \frac{\sqrt{3}}{2}du$

$$= \ln|x+1| - \frac{1}{2}\ln(x^2+x+1) + \frac{1}{2}\,\frac{\sqrt{3}}{2}\int \frac{du}{\frac{3}{4}u^2+\frac{3}{4}}$$

$$= \ln|x+1| - \frac{1}{2}\ln(x^2+x+1) + \frac{\sqrt{3}}{\cancel{4}\cdot\frac{3}{\cancel{4}}}\int \frac{du}{u^2+1}$$

$$= \ln|x+1| - \frac{1}{2}\ln(x^2+x+1) + \frac{\sqrt{3}}{3}\arctan u + c$$

$$= \ln\frac{|x+1|}{\sqrt{x^2+x+1}} + \frac{\sqrt{3}}{3}\arctan\left(\frac{2\sqrt{3}x}{3}+\frac{\sqrt{3}}{3}\right) + c.$$

For $\int \frac{P(x)}{Q(x)} dx$ where $\deg P(x) \geqslant \deg Q(x)$

we first perform the Euclidean division algorithm.

E.g. $\int \frac{x^3 + 3x^2}{x^2+1} dx$

$$\begin{aligned} x^3 + 3x^2 &= x^2(x+3) \\ &= (x^2+1-1)(x+3) \\ &= (x^2+1)(x+3) - (x+3) \end{aligned}$$

so $$\frac{x^3+3x^2}{x^2+1} = x+3 - \frac{x+3}{x^2+1}$$

and

$$\begin{aligned} \int \frac{x^3+3x^2}{x^2+1} dx &= \int (x+3)\, dx - \int \frac{x+3}{x^2+1} dx \\ &= \frac{x^2}{2} + 3x - \frac{1}{2}\int \frac{2x}{x^2+1} dx - 3\int \frac{dx}{x^2+1} \\ &= \frac{x^2}{2} + 3x - \frac{1}{2}\ln(x^2+1) - 3\arctan x + c. \end{aligned}$$

▸ Integrals involving $\sqrt{a^2-x^2}$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + c$$

Generally we may set $x = a\sin\theta$.

$$\int (5-x^2)^{-\frac{3}{2}}\,dx = \int \frac{dx}{\sqrt{5-x^2}^{\,3}}$$

Set $x = \sqrt{5}\sin\theta$

$dx = \sqrt{5}\cos\theta\,d\theta$

$5-x^2 = 5-5\sin^2\theta = 5\cos^2\theta$

$$= \int \frac{\sqrt{5}\cos\theta}{(\sqrt{5}\cos\theta)^3}\,d\theta = \frac{1}{5}\int \frac{d\theta}{\cos^2\theta}$$

$$= \frac{1}{5}\tan\theta + c$$

$$= \frac{1}{5}\,\frac{x}{\sqrt{5-x^2}} + c.$$

$\sin\theta = \frac{1}{\sqrt{5}}x$

$\cos\theta = \sqrt{1-\sin^2 x}$

$= \frac{1}{\sqrt{5}}\sqrt{5-x^2}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$