Øving 3

1.

(1) $[a,b]=[-1,1]$

$w(x)=1$

$\tau_2=\frac{x}{\sqrt{2}}$

Finner $\tilde{P}_1$:

$\tilde{P}_1(x)=x-c_0\frac{x}{\sqrt{2}}$

$\langle \tilde{P}_1,\tilde{P}_0\rangle=\langle x\tilde{P}_0\rangle-c_0\langle \tilde{P}_0,\tilde{P}_0\rangle$

$=0$

$\Rightarrow c_0=\frac{\langle x\tilde{P}_0,\tilde{P}_0\rangle}{\langle \tilde{P}_0,\tilde{P}_0\rangle}$

$=[\langle \tilde{P}_0,\tilde{P}_0\rangle=1, \langle x,\tilde{P}_0\rangle=0]$

$=0$

$\Rightarrow \tilde{P}_1(x)=x-0\tau_2(x)$

$=x$

$\tilde{P}_1=\frac{\tilde{P}_1}{\|\tilde{P}_1\|_w}$

$=[\|\tilde{P}_1\|_w=(\int_{-1}^{1}w(x)|\tilde{P}_1|^2dx)^{\frac{1}{2}}=\sqrt{\frac{2}{3}}]$

$=\frac{\sqrt{3}x}{\sqrt{2}}$

Finner $\tilde{P}_2$:

$\tilde{P}_2(x)=x^2-(d_1\tilde{P}_1(x)+d_0\tilde{P}_0(x))$

$\langle \tilde{P}_2,\tilde{P}_1\rangle=0$ og $\langle \tilde{P}_2,\tilde{P}_0\rangle=0$

$\begin{cases}\langle x^2,\tilde{P}_1\rangle-d_1\langle \tilde{P}_1,\tilde{P}_1\rangle-d_0\langle \tilde{P}_0,\tilde{P}_1\rangle=0\\ \langle x^2,\tilde{P}_0\rangle-d_1\langle \tilde{P}_1,\tilde{P}_0\rangle-d_0\langle \tilde{P}_0,\tilde{P}_0\rangle=0\end{cases}$

$d_1=\frac{\langle x^2,\tilde{P}_1\rangle}{\langle \tilde{P}_1,\tilde{P}_1\rangle}$

$=[\langle x^2,\tilde{P}_1\rangle=\int_{-1}^{1}x^3\cdot\frac{\sqrt{3}x}{\sqrt{2}}=\frac{\sqrt{3}}{\sqrt{2}}\cdot\int_{-1}^{1}x^4=0]$

$=0$

$d_0=\frac{\langle x^2,\tilde{P}_0\rangle}{\langle \tilde{P}_0,\tilde{P}_0\rangle}$

$=[\langle x^2,\tilde{P}_0\rangle=\int_{-1}^{1}x^2\cdot\frac{1}{\sqrt{2}}dx=\frac{2}{3\sqrt{2}}]$

$=\frac{2}{3}$

$\Rightarrow \tilde{P}_2(x)=x^2-(0\cdot\frac{\sqrt{3}x}{\sqrt{2}}+\frac{1}{3\sqrt{2}})$

$=x^2-\frac{1}{3}$

$\tau_3(x)=\frac{\tilde{P}_2(x)}{\|\tilde{P}_2\|_w}$

$=[\|\tilde{P}_2\|_w=(\int_{-1}^{1}|x^2-\frac{1}{3}|^2dx)^{\frac{1}{2}}=\sqrt{\frac{8}{45}}]$

$=\frac{\sqrt{5}(3x^2-1)}{\sqrt{2}(3x^2-1)}$

$=\frac{\sqrt{45}(3x^2-1)}{\sqrt{8}}$

Finner $\tilde{P}_3$:

$\tilde{P}_3(x)=x^3-(d_2\tilde{P}_2(x)+d_1\tilde{P}_1(x)+d_0\tilde{P}_0(x))$

$\langle \tilde{P}_3,\tilde{P}_2\rangle=0$ $\langle \tilde{P}_3,\tilde{P}_1\rangle=0$ $\langle \tilde{P}_3,\tilde{P}_0\rangle=0$

$\begin{cases}\langle x^3,\tilde{P}_2\rangle-d_2\langle \tilde{P}_2,\tilde{P}_2\rangle-d_1\langle \tilde{P}_1,\tilde{P}_2\rangle-d_0\langle \tilde{P}_0,\tilde{P}_2\rangle=0\\ \langle x^3,\tilde{P}_1\rangle-d_2\langle \tilde{P}_2,\tilde{P}_1\rangle-d_1\langle \tilde{P}_1,\tilde{P}_1\rangle-d_0\langle \tilde{P}_0,\tilde{P}_1\rangle=0\\ \langle x^3,\tilde{P}_0\rangle-d_2\langle \tilde{P}_2,\tilde{P}_0\rangle-d_1\langle \tilde{P}_1,\tilde{P}_0\rangle-d_0\langle \tilde{P}_0,\tilde{P}_0\rangle=0\end{cases}$

$d_2=\frac{\langle x^3,\tilde{P}_2\rangle}{\langle \tilde{P}_2,\tilde{P}_2\rangle}$

$=[\langle x^3,\tilde{P}_2\rangle=\int_{-1}^{1}x^3\cdot\frac{\sqrt{45}(3x^2-1)}{\sqrt{8}}dx=0]$

$=0$

$d_1=\frac{\langle x^3,\tilde{P}_1\rangle}{\langle \tilde{P}_1,\tilde{P}_1\rangle}$

$=[\langle x^3,\tilde{P}_1\rangle=\int_{-1}^{1}x^3\cdot\frac{\sqrt{3}x}{\sqrt{2}}dx=\frac{\sqrt{3}}{\sqrt{2}}\cdot\int_{-1}^{1}x^4]$

$=\frac{3}{5}$

$d_0=\frac{\langle x^3,\tilde{P}_0\rangle}{\langle \tilde{P}_0,\tilde{P}_0\rangle}$

$=[\langle x^3,\tilde{P}_0\rangle=\int_{-1}^{1}x^3\cdot\frac{1}{\sqrt{2}}dx=0]$

$=0$

$\Rightarrow \tilde{P}_3(x)=x^3-(0+\frac{3}{5}x+0)$

$=x^3-\frac{3}{5}x$

$\tau_4(x)=\frac{\tilde{P}_3(x)}{\|\tilde{P}_3\|_w}$

$=[\|\tilde{P}_3\|_w=(\int_{-1}^{1}|x^3-\frac{3}{5}x|^2dx)^{\frac{1}{2}}=\sqrt{\frac{8}{175}}]$

$=\frac{\sqrt{175}(x^3-\frac{3}{5}x)}{\sqrt{8}}$

$=\frac{\sqrt{175}(5x^3-3x)}{\sqrt{8}}$

(2) $I=[-\infty,\infty]$

$w(x)=\exp x^2$, $x\in\mathbb{R}$

$\tau_0=1$

Finner $\tilde{P}_1$:

$\tilde{P}_1=x-c_0\tau_0(x)$

$\langle \tilde{P}_1,\tilde{P}_0\rangle=\langle x\tau_0\rangle-c_0\langle \tau_0,\tau_0\rangle=0$

$c_0=\frac{\langle x\tau_0,\tau_0\rangle}{\langle \tau_0,\tau_0\rangle}$

$=[\langle x\tau_0,\tau_0\rangle]$

$=0$

$\Rightarrow \tilde{P}_1(x)=x$

$\tau_1=\frac{\tilde{P}_1}{\|\tilde{P}_1\|_w}$

$=[\|\tilde{P}_1\|_w=(\int_{-\infty}^{\infty}w(x)|x|^2)^{\frac{1}{2}}=\sqrt{2\int_0^{\infty}}=\sqrt{2}]$

$=x$

Finner $\tilde{P}_2$:

$\tilde{P}_2(x)=x^2-(d_1\tilde{P}_1(x)+d_0\tau_0(x))$

$\langle \tilde{P}_2,\tilde{P}_1\rangle=0$ $\langle \tilde{P}_2,\tau_0\rangle=0$

$\begin{cases}\langle x^2,\tilde{P}_1\rangle-d_1\langle \tilde{P}_1,\tilde{P}_1\rangle-d_0\langle \tau_0,\tilde{P}_1\rangle=0\\ \langle x^2,\tau_0\rangle-d_1\langle \tilde{P}_1,\tau_0\rangle-d_0\langle \tau_0,\tau_0\rangle=0\end{cases}$

$d_1=\frac{\langle x^2,\tilde{P}_1\rangle}{\langle \tilde{P}_1,\tilde{P}_1\rangle}$

$=[\langle x^2,\tilde{P}_1\rangle=\int_{-\infty}^{\infty}x^2\cdot x\cdot w(x)dx=0]$

$=0$

$d_0=\frac{\langle x^2,\tau_0\rangle}{\langle \tau_0,\tau_0\rangle}$

$=[\langle x^2,\tau_0\rangle=\int_{-\infty}^{\infty}x^2\cdot w(x)dx=]$

$=]$

$\Rightarrow \tilde{P}_2=x^2-(0+\tau_0)$

$=x^2-1$

$\tau_2=\frac{\tilde{P}_2}{\|\tilde{P}_2\|_w}$

$=[\|\tilde{P}_2\|_w=(\int_{-\infty}^{\infty}w(x)|\tilde{P}_2|^2dx)^{\frac{1}{2}}=\sqrt{2\int_0^{\infty}}]$

$=\frac{x^2-1}{\sqrt{2}}$

2.

$w(x)=\frac{1}{\sqrt{1-x^2}}$, $x\in(-1,1)$

Gitt τ_n $n=0$ og $n=1$

Lag $\{x\in[-1,1]$ gitt

$\|\tau_0\|_2\leq$

$\|\tau_1\|_2\leq$

Da $\|\tau_n\|_2:=(\int_{-1}^{1}w(x)|\tau_n(x)|^2dx)^{\frac{1}{2}}$

Definerer $L(x)=\frac{\tau_1(x)}{\tau_0(x)}$

$\|\tau_n\|_2:(\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}|\tau_n(x)|^2dx)^{\frac{1}{2}}$

3.

a)

Antar det finnes $\tau_n(\tau_n)$ slik τ_n er en av de ortogonale oppfyller.

$x^n\tau_n=\tau_n$

$\tau(x_j)=0$ $j=0,1,\dots,n$

Ifølge teorem er de slik finnes n punkt x_j, $j=1,\dots,n$

$\tau(x_j)=0$

Ifølge teorem vi er det slik finnes $n-1$ punkt τ_{n-1} $k=1,\dots,n-1$

$\tau'(x_k)=0$

Vil også si $\tau'(x_j)=0$ $j=1,\dots,n$ som vi ser punkt der τ' forsvinner

Nå vi vet ikke om x_j forskjellig for x_j vis det ikke lagde nulp som vi vil si x_k lager indikere $x_k\Rightarrow x_{n-1}$ eg x_k kan være x_n

Antar at $p_n(x)=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4$

$c_0=c_1+c_2+c_3+c_4=1$

$c_0=0$

$c_0+c_1+2c_2+3c_3+4c_4=0$

$2c_2+6c_3$

b)

$L_0(x)$ matt validity

$L_0(x_0)=1$

$L_0(x_1)=L(x_2)$

$=0$

Ikke lineær

$L_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$

$L_2(x)$ matt validity

$L_2(x_2)$

c)

$c_0+c_1+c_2+c_3=1$

$-c_1+c_2+2c_3=\frac{1}{2}$

$c_1+c_2+4c_3=\frac{1}{3}$

$-c_1+c_2+8c_3=\frac{1}{4}$

Løser systemet med determinanten gir:

$c_0=\frac{-1}{24}$

$c_1=\frac{1}{2}$

$c_2=\frac{2}{3}$

$c_3=\frac{-1}{24}$

$E(f)=\int_{-1}^{1}f(x)dx-Q(f)$

$=\int_{-1}^{1}(f(x)-p_n(x))dx$

da $p_n(x)$ er lagrange interpolerer polynomet av f med grad 3 på intervallet $[-1,1]$ med ekvidistanserte $-1,0,1,2$

Ifølge formelen teoremet for lagrange interpolasjon har

$|E(f)|\leq\frac{M_4}{4!}\int_{-1}^{1}|\pi_n(x)|dx$

da $\pi_n(x)=(x+1)(x-0)(x-1)(x-2)$ og $M_4=\max_{[-1,1]}|f^{(4)}(x)|$

$|\pi_n(x)|\leq(1-x^2)(2-x)$ $\forall x\in[0,1]$

Derfor har

$\int_{-1}^{1}|\pi_n(x)|dx\leq\frac{16}{15}$

$\Rightarrow |E(f)|\leq\frac{16}{15}M_4$

7.13

Siden $\cos(nx)$ er oddfunksjon vil $\int_{-\pi}^{\pi}\cos(rx)dx=0$ $\forall r$. Vi har derfor vanskeligheter med frequente tegnskiftene $\cos r$ r nært. Men er nullpunkt som x $r=0$ $\Rightarrow \cos(rx)=1$ OK.

r positiv heltall:

$\int_{-\pi}^{\pi}\cos(rx)dx$ lik

$I_n=4\int_0^{\pi}\cos(-r\pi)+\sum_{j=1}^{r}\cos(-r\pi+j\theta)+\cos(r\pi)),\ \theta=\frac{2\pi}{r}$

$=4\sum_{j=0}^{r}\cos(-r\pi+j\theta)$

$=(-1)^r4\sum_{j=0}^{r}\cos(j\pi\theta)$

$=(-1)^r4(\cos((n+1)\theta)-\cos(3\pi\theta))/\sin(3\pi\theta)$

$=0$

siden $n\theta=2\pi$

Generelt gir frequente tegnskiftene vil 0

Finnerer ikke ikke $\sin(3\pi\theta)=0$, men dog ikke r er nærmest av n

$r+1=n$:

$\cos(-r\pi+j\pi\theta)=\cos(-1\cdot\cos+2j\pi)$

$=\cos(-r\pi+2j\pi)$

$=(-1)^r$

$\Rightarrow I_n=(-1)^r\sin$

$=(-1)^r2\pi$

8?

Minste polynom er slik at $f(x)-p_n(x)$ har tre alternerende ekstremverdier

Siden f er convex, er tre av disse punktene $x=-2$ og $x=1$, og det ville nå ene $x=0$

p_n vil ikke parallell med $(-2,f(-2))$ og $(1,f(1))$

$\Rightarrow p_n(x)=ax-b$

Ekstremverdiene er:

$f(-2)-p_n(-2)=2-(-a+b)$

$=3-a$

$f(0)-p_n(0)=0-b$

$=-b$

$f(1)-p_n(1)=1-(a-b)$

$=3-a$

Dette har samme størrelse

$3-a=-(-b)$

$\Rightarrow p_n(x)=\frac{3}{2}x-\frac{1}{2}$ og $\|f-p_n\|_\infty=\frac{3}{2}$

12)

a) $f(y_n)=2y_n^2$, $x\in[1,\infty)$

$|f(y_n)-f(y_m)|=|2y_n^2|y_n-y_m|$

$(2|y_n-y_m|$

$\forall x\in[1,\infty)$, $\forall y_n\in\mathbb{R}$

Lipschitz kontinuiterlig er slik $L=2$

b) $f(y_n)=e^{-\frac{1}{2}\ln(1/y_n)}$, $x\in[1,\infty)$

$|f(y_n)-f(y_m)|=|\frac{1}{2}\xi_{(y_n)}||y_n-y_m|$ ifølge mean value theorem ξ er y lagger mellom y og z

$\frac{1}{2}\xi_{(y_n)}=e^{-\frac{1}{2}\ln\xi}$

≤ 1

$\forall x\in[1,\infty)$ $\forall y_n\in\mathbb{R}$

Lipschitz kontinuiterlig er slik $L=\frac{1}{2}$

c) $f(y_n)=2y_n(1+y_n^2)^{-1}$ $(1+y_n^2)$ $x\in(-\infty,\infty)$

$|f(y_n)-f(y_m)|=|(1+y_n^2)\frac{1+y_m^2-y_n^2}{1+y_n^2}||y_n-y_m|$

$=|(1+y_n^2)\frac{1+y_m^2-y_n^2}{1+y_n^2}||y_n-y_m|$

$\leq 2\cdot 1\cdot|y_n-y_m|$

$\forall x\in[1,\infty)$ $\forall y_n\in\mathbb{R}$

Lipschitz kontinuiterlig er slik $L=2$

13

$y'=y+x$ $y(0)=1$

Lineærg er

$y(x)=1+\frac{1}{2}(e^x-1)$

Picard iterasjon

$y_0(x)=1$

$y_1(x)=1+\int_0^x(y_0+t)dt$

$=1+y_0x+y_0\frac{x^2}{2}$ $n=0,1,\dots$

y_n er polynom av grad 0

Vår antakelse er y_n er polynom med grad n

$y_0(x)=1$

$y_1(x)=1+(y_0+x)x$

$y_2(x)=1+(y_0+x)x+(y_0+x)\frac{x^2}{2}$

$\vdots$

$y_n(x)=1+\frac{x}{1!}(y_0+\frac{x}{2!}(y_0+\frac{x}{3!}(\dots+\frac{x}{n!}y_0)))$

$=1+\frac{x}{1!}+\frac{x^2}{2!}(y_0+\frac{x}{3!}(\dots-1))$

$\Rightarrow y_n(x)=1+\frac{1}{2}(e^x-1)$

$=y(x)$