• ASYMPTOTIC ESTIMATES

Let x_0 be a real number or $\pm\infty$.

We write
$$f(x) = O(g(x)), \quad x \to x_0$$
if there exists some $M > 0$ such that
$$|f(x)| \leq M|g(x)|$$
for all x in an interval around x_0.
(we read f is big Oh of g).

We write
$$f(x) = o(g(x)), \quad x \to x_0$$
if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$ (we read f is little oh of g)

The idea behind this definition:

- $f(x) = O(g(x)), \quad x \to x_0$
  when f has order of magnitude at most $g(x)$ near x_0.

- $f(x) = o(g(x)), \quad x \to x_0$
  when f has order of magnitude smaller than $g(x)$ near x_0.

E.g. we can write

$$\cdot\ \frac{\cos x}{x} = O\left(\frac{1}{x}\right), \qquad x \to +\infty$$

$$\cdot\ \sin x = O(1), \qquad x \to +\infty$$

$$\cdot\ \sin x = O(x), \qquad x \to 0.$$

The first of these relations follows since

$$\left|\frac{\cos x}{x}\right| \leq \frac{1}{x} \quad \text{for all } x > 0.$$

The second follows since

$$|\sin x| \leq 1, \quad x \in \mathbb{R}.$$

The O- and o- notation is mostly used to describe error terms in some approximation relation.

E.g. We have

$$1 + \frac{1}{2} + \dots + \frac{1}{n} = \ln n + \gamma + O\left(\frac{1}{n}\right), \quad n \to \infty.$$

This means that the difference

$$\left|\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) - (\ln n + \gamma)\right| \leq \frac{C}{n}$$

(for some constant $C > 0$ which is not specified).

Example: If $f(x) = \dfrac{x^3 + x + 1}{x - 2}$,

then

$$f(x) = O(x^2), \quad x \to +\infty$$

but

$$f(x) = o(x^2 \log x), \quad x \to +\infty.$$

Back to Taylor polynomials, if

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

then Taylor's theorem states that

$$f(x) = P_n(x) + O((x - x_0)^{n+1}), \quad x \to x_0.$$

4. INTRODUCTION TO SERIES

- FINITE SUMS

Let $x_1, x_2, \dots, x_n$ be real numbers, we write

$$\sum_{k=1}^{n} x_k = x_1 + x_2 + \dots + x_n .$$

The index k does not appear in the right hand side, and

$$\sum_{k=1}^{n} x_k = \sum_{i=1}^{n} x_i = \sum_{j=1}^{n} x_j = \sum_{\ell=1}^{n} x_\ell \quad \dots$$

The summation symbol can be also used in different manners, e.g.

- if $A \subseteq \mathbb{N}$ we may write $\sum_{\substack{n \in A \\ n \leq X}} a_n$ for the sum of numbers a_n such that $n \in A$ and $n \leq X$.
- if $f : \mathbb{R} \to \mathbb{R}$ and $I = \{i_1, i_2, \dots, i_m\}$ then
$$\sum_{i \in I} f(i) = f(i_1) + f(i_2) + \dots + f(i_m)$$
(and the i_k's do not have to be integers).

Of course

$$\sum_{k=1}^{n}(a x_k + b y_k) = a\sum_{k=1}^{n} x_k + b\sum_{k=1}^{n} y_k$$

for any $a, b \in \mathbb{R}$.

Using this notation, the Taylor polynomial can be written as

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x - x_0)^k .$$

<u>THEOREM 4.1</u> : The following hold.

(i) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

(ii) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

(iii) $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

<u>PROOF</u>

(i) Set $S = \displaystyle\sum_{k=1}^{n} k$. Then

$$S = 1 + 2 + \dots + (n-1) + n$$
$$S = n + (n-1) + \dots + 2 + 1 .$$

Thus $2S = (n+1) + (n+1) + \dots + (n+1) = n(n+1)$,

and $S = \dfrac{n(n+1)}{2}$.

(ii) Set $S = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \dots + n^2$.

For any $k = 1, 2, \dots, n$ we have

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1.$$

For $k=1$: $\cancel{2^3} = 1^3 + 3\cdot 1^2 + 3\cdot 1 + 1$

For $k=2$: $\cancel{3^3} = \cancel{2^3} + 3\cdot 2^2 + 3\cdot 2 + 1$

For $k=3$: $\cancel{4^3} = \cancel{3^3} + 3\cdot 3^2 + 3\cdot 3 + 1$

$\vdots$

For $k=n$: $(n+1)^3 = \cancel{n^3} + 3n^2 + 3n + 1$

Adding by parts,

$$(n+1)^3 = 1 + 3S + 3\cdot(1+2+\dots+n) + n \Rightarrow$$

$$3S + \frac{3n(n+1)}{2} + n + \cancel{1} = n^3 + 3n^2 + 3n + \cancel{1} \Rightarrow$$

$$3S = n^3 + 3n^2 + 2n - \frac{3n^2}{2} - \frac{3n}{2}$$

$$= n^3 + \frac{3n^2}{2} + \frac{n}{2}$$

$$= \frac{2n^3 + 3n^2 + n}{2}$$

$$= \frac{n(2n^2 + 3n + 1)}{2}$$

$$= \frac{n(n+1)(2n+1)}{2}.$$

$\blacksquare$

(iii) Exercise.

Alternatively we could have also used induction, but the previous proof also shows how to derive the formulae.

In the language of the asymptotic notation of the previous chapter, we have

$$1 + 2 + \dots + n = O(n^2), \quad n \to \infty$$
$$1^2 + 2^2 + \dots + n^2 = O(n^3), \quad n \to \infty$$
$$1^3 + 2^3 + \dots + n^3 = O(n^4), \quad n \to \infty.$$

A sequence $(a_n)_{n=0}^{\infty}$ is called a geometric progression if there exists some $\lambda \neq 1$ such that

$$\frac{a_{n+1}}{a_n} = \lambda, \quad n = 0, 1, 2, \dots$$

In that case the terms of the sequence are $a_0, \lambda a_0, \lambda^2 a_0, \lambda^3 a_0, \dots$

The sum of the terms of a geometric progression is called a <u>geometric sum</u>.

THEOREM 4.2: For $\lambda \neq 1$,

$$\sum_{k=0}^{n} \lambda^k = \frac{1-\lambda^{n+1}}{1-\lambda} .$$

PROOF

Set $S = \sum_{k=0}^{n} \lambda^k$. Then

$$S = 1 + \cancel{\lambda} + \cancel{\lambda^2} + \dots + \cancel{\lambda^n}$$
$$\lambda S = \cancel{\lambda} + \cancel{\lambda^2} + \dots + \cancel{\lambda^n} + \lambda^{n+1}$$

and subtracting by parts

$$(1-\lambda) S = 1 - \lambda^{n+1} \Longrightarrow$$

$$S = \frac{1-\lambda^{n+1}}{1-\lambda} . \qquad \blacksquare$$

Alternatie Proof: For all $n \geq 1$, $x, y \in \mathbb{R}$

$$x^{n+1} - y^{n+1} = (x-y) \cdot (x^n + x^{n-1} y + x^{n-2} y^2 + \dots + y^n).$$

Set $x = \lambda$, $y = 1$.

$$\lambda^{n+1} - 1 = (\lambda - 1)(\lambda^n + \lambda^{n-1} + \dots + 1) \Longrightarrow$$
$$1 + \lambda + \dots + \lambda^n = \frac{\lambda^{n+1} - 1}{\lambda - 1} = \frac{1-\lambda^{n+1}}{1-\lambda} .$$

- INFINITE SERIES

Let $(a_k)_{k=1}^{\infty}$ be a sequence.
The operation "+" is binary, i.e. we can only add 2 real numbers. Sums of more than 2 numbers are defined recursively:

$$a_1 + a_2 + a_3 = (a_1 + a_2) + a_3 ,$$
$$a_1 + a_2 + a_3 + a_4 = (a_1 + a_2 + a_3) + a_4 .$$

Algebra only allows sums of finitely many reals, and there do not exist infinite sums of real numbers!

The infinite series associated with $(a_k)_{k=1}^{\infty}$ is the formal symbol

$$\sum_{k=1}^{\infty} a_k ,$$

The sequence of <u>partial sums</u> of $(a_k)_{k=1}^{\infty}$ is the sequence $(S_n)_{n=1}^{\infty}$ defined by

$$S_n = a_1 + a_2 + \dots + a_n , \quad n = 1, 2, \dots$$

i.e. $S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$, etc.

We say that the series $\sum_{k=1}^{\infty} a_k$ converges to the real number $s \in \mathbb{R}$ if

$$\lim_{n\to\infty} S_n = s.$$

In that case, we write $\sum_{k=1}^{\infty} a_k = s$.

We say that the series $\sum_{k=1}^{\infty} a_k$ converges if it converges to some real number. In that case, we write

$$\sum_{k=1}^{\infty} a_k < \infty.$$

Otherwise, we say that the series $\sum_{k=1}^{\infty} a_k$ diverges.

THEOREM 4.3: If $\sum_{k=1}^{\infty} a_k$ converges, then

$$\lim_{n\to\infty} a_n = 0.$$

PROOF

Let $(S_n)_{n=1}^{\infty}$ be the sequence of partial sums of $(a_n)_{n=1}^{\infty}$. There exists some $s \in \mathbb{R}$ so that $\lim_{n\to\infty} S_n = s$.

But

$$\begin{aligned} a_n &= (a_1 + a_2 + \dots + a_n) - (a_1 + \dots + a_{n-1}) \\ &= S_n - S_{n-1} \longrightarrow s - s = 0. \end{aligned}$$

$\blacksquare$

COROLLARY 4.4: If $(a_n)_{n=1}^{\infty}$ is a sequence such that $a_n \not\to 0$, then

$$\sum_{k=1}^{\infty} a_n \quad \text{diverges.}$$

* A remark regarding divergent series:
If $(S_n)_{n=1}^{\infty}$ is the seq. of partial sums of $(a_n)_{n=1}^{\infty}$, then whenever

$$\lim_{n\to\infty} S_n = \infty$$

we say that the series $\sum_{k=1}^{\infty} a_k$ diverges to ∞ and we write

$$\sum_{k=1}^{\infty} a_k = \infty .$$

Divergent series do not always diverge to ∞. Take for example

$$\sum_{k=1}^{\infty} (-1)^k .$$

The n-th partial sum is

$$S_n = (-1) + (+1) + \dots + (-1)^n = \begin{cases} 0, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

which does not converge. Thus the series $\sum_{k=1}^{\infty} (-1)^k$ diverges, but not to ∞.

For any $\lambda \neq 1$, the series $\sum_{k=0}^{\infty} \lambda^k$ is called a <u>geometric series</u>.

<u>THEOREM 4.5</u>: When $|\lambda| < 1$, the geometric series

$$\sum_{k=0}^{\infty} \lambda^k = \frac{1}{1-\lambda}.$$

When $|\lambda| \geq 1$, the geom. series $\sum_{k=0}^{\infty} \lambda^k$ diverges.

<u>PROOF</u>

Let $(S_n)_{n=1}^{\infty}$ be the seq. of partial sums. Then

$$S_n = 1 + \lambda + \lambda^2 + \dots + \lambda^n = \frac{1-\lambda^{n+1}}{1-\lambda}.$$

When $|\lambda| < 1$, then $\lim_{n\to\infty} \lambda^{n+1} = 0$ and thus

$$\lim_{n\to\infty} S_n = \frac{1}{1-\lambda}.$$

When $|\lambda| > 1$, then $\lim_{n\to\infty} \lambda^{2n} = \infty$ therefore S_n does not converge.

When $\lambda = 1$, $S_n = n \to \infty$

When $\lambda = -1$, $S_n = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$ $\blacksquare$

E.g. $$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$$

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} \frac{1}{3^n} - 1 = \frac{1}{1-\frac{1}{3}} - 1 = \frac{1}{2}.$$

THEOREM 4.6: Assume the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ converge to $a, b \in \mathbb{R}$, respectively.
For any $\lambda_1, \lambda_2 \in \mathbb{R}$ the series $\sum_{n=1}^{\infty}(\lambda_1 a_n + \lambda_2 b_n) = \lambda_1 a + \lambda_2 b$.

PROOF.

Let A_n, B_n, S_n, $(n=1,2,\dots)$ the partial sums of the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty}(\lambda_1 a_n + \lambda_2 b_n)$ resp.

By the hypothesis, $\lim_{n\to\infty} A_n = a$ and $\lim_{n\to\infty} B_n = b$.

Then $$S_n = \sum_{k=1}^{n} (\lambda_1 a_k + \lambda_2 b_k)$$
$$= \lambda_1 A_n + \lambda_2 B_n \to \lambda_1 a + \lambda_2 b. \blacksquare$$