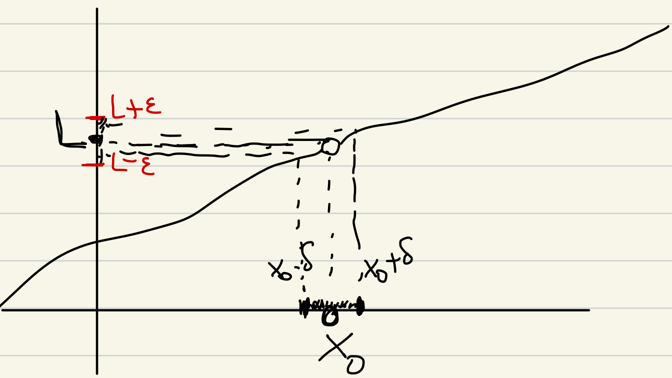
• LIMITS OF REAL FUNCTIONS

Let $f: A \to \mathbb{R}$, where A contains a set of the form $(a, x_0) \cup (x_0, b)$. We say that

$$\lim_{x \to x_0} f(x) = L \in \mathbb{R}$$

if

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0 \quad \text{such that}$$
$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon .$$



- x_0 is not necessarily in the domain of definition of f.
- $f(x_0)$ – if it exists – has nothing to do with $\lim\limits_{x \to x_0} f(x)$.

- Prove that $\lim_{x \to 1} (5x-1) = 4$.

Take $\varepsilon > 0$.
I have to find some $\delta > 0$ with the property that whenever
$$0 < |x-1| < \delta$$
we have
$$|(5x-1) - 4| < \varepsilon.$$

But $|(5x-1) - 4| = |5x-5| = 5 \cdot |x-1|$.
Choose $\delta = \frac{\varepsilon}{5} > 0$.
Then $0 < |x-1| < \delta$ implies
$$\begin{aligned} |(5x-1)-4| &= 5 \cdot |x-1| \\ &< 5\delta \\ &= 5 \cdot \frac{\varepsilon}{5} \\ &= \varepsilon. \end{aligned}$$

So $\lim_{x \to 1} (5x-1) = 4$.

- Prove that $\lim_{x \to 2} (x^2 - 4) = 0$.

Let $\varepsilon > 0$.

I have to find $\delta > 0$ such that $0 < |x-2| < \delta$ implies $|(x^2-4) - 0| < \varepsilon$.

$$|(x^2-4)-0| = |x^2-4| = |x+2| \cdot |x-2|$$

Observe that whenever $|x-2| < 1$ then

$$1 < x < 3 \Rightarrow$$
$$3 < x+2 < 5 \Rightarrow$$
$$|x+2| < 5.$$

Choose $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\} > 0$.

Then $0 < |x-2| < \delta$ implies that

$|x-2| < 1 \Rightarrow |x+2| < 5$

and also $|x-2| < \frac{\varepsilon}{5}$

so

$$|(x^2-4)-0| = |x+2| \cdot |x-2| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon.$$

For any $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that $0 < |x-2| < \delta$ we have $|x^2 - 4| < \varepsilon$.

So $\lim_{x \to 2} (x^2-4) = 0$.

BOOK: 1.2, 1.3

When $f: A \to \mathbb{R}$ and A contains an interval of the form $(a, +\infty)$, we can define the limit $\lim_{x \to +\infty} f(x)$.

We say that $\lim_{x \to +\infty} f(x) = L \in \mathbb{R}$ if
$\forall \varepsilon > 0 \quad \exists M = M(\varepsilon) > 0$ such that
$$x > M \implies |f(x) - L| < \varepsilon.$$

E.g. $\lim_{x \to +\infty} \left(2 + \frac{1}{x} - \frac{1}{x^2}\right) = 2.$

Let $\varepsilon > 0$.
There exists $M > 0$ such that
$$x > M \implies \frac{1}{x} < \frac{\varepsilon}{2}.$$
So for $x > M$,
$$\left|\frac{1}{x}\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|\frac{1}{x^2}\right| < \left|\frac{1}{x}\right| < \frac{\varepsilon}{2}.$$
Hence
$$\left|\left(2 + \frac{1}{x} - \frac{1}{x^2}\right) - 2\right| \leq \left|\frac{1}{x}\right| + \left|\frac{1}{x^2}\right|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

A similar definition for when $\lim_{x \to -\infty} f(x) = L$.

The following rules hold:

If $\lim_{x \to x_0} f(x) = L_1$, $\lim_{x \to x_0} g(x) = L_2$ (with $L_1, L_2 \in \mathbb{R}$ and x_0 can be real or $\pm\infty$), then

- $\lim_{x \to x_0} \left(af(x) + bg(x) \right) = aL_1 + bL_2$, for any $a, b \in \mathbb{R}$

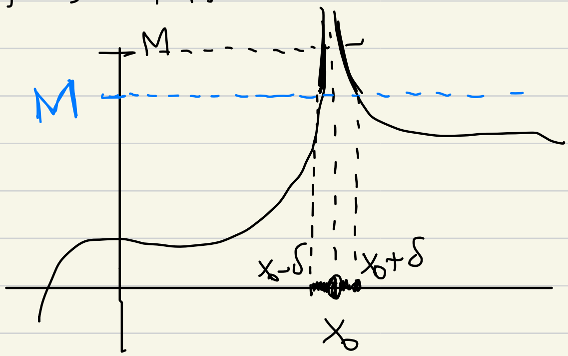
- $\lim_{x \to x_0} \left(f(x) g(x) \right) = L_1 L_2$

- $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$, provided $L_2 \neq 0$ and $g(x) \neq 0$ near x_0.

Book: 1.2 & 1.3.

We say that $\lim_{x \to x_0} f(x) = +\infty$ if

$\forall M > 0 \quad \exists \delta = \delta(M) > 0$ such that
$0 < |x - x_0| < \delta \implies f(x) > M.$



E.g. Show that $\lim_{x \to 1} \frac{1}{|x-1|} = +\infty.$

Let $M > 0$. Set $\delta = \frac{1}{M} > 0$

then $0 < |x-1| < \delta$ implies that

$$\frac{1}{|x-1|} > \frac{1}{\delta} = \frac{1}{\frac{1}{M}} = M.$$

So $\lim_{x \to 1} \frac{1}{|x-1|} = +\infty.$

We say that $\lim_{x \to x_0} f(x) = -\infty$

iff $\lim_{x \to x_0} (-f(x)) = +\infty.$

We have defined when: $\lim_{x \to x_0} f(x) = \ell$,

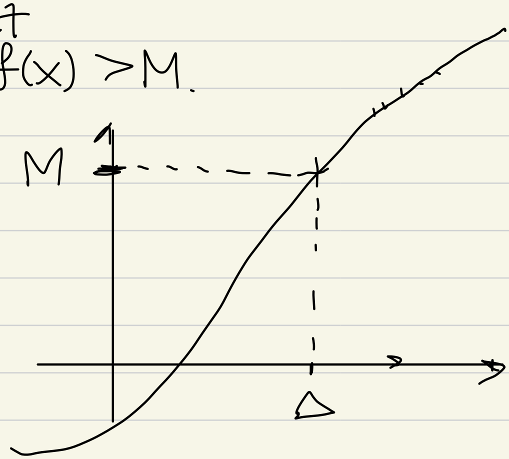
$\lim_{x \to +\infty} f(x) = \ell$,

$\lim_{x \to x_0} f(x) = \pm\infty$.

Now we define when $\lim_{x \to +\infty} f(x) = +\infty$.

We say that $\lim_{x \to +\infty} f(x) = +\infty$ if for any $M > 0$ there exists some $\Delta = \Delta(M) > 0$ such that

$x > \Delta$ implies $f(x) > M$.

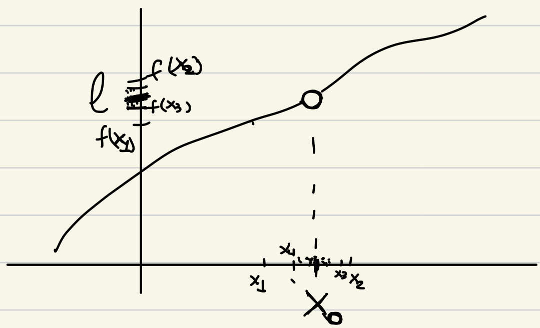


There exists the "sequential characterization" of limits.

PROPOSITION 2.9 : Let x_0 be a real number or $\pm\infty$, and f is defined on an interval around x_0. The following are equivalent:

(i) $\lim_{x \to x_0} f(x) = l$

(ii) For any sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ such that $x_n \neq x_0$ for all $n = 1, 2, \ldots$ and $\lim_{n \to \infty} x_n = x_0$, we have $\lim_{n \to \infty} f(x_n) = l$.



This Proposition states that in order to find $\lim_{x \to x_0} f(x)$ it suffices to let $x \to x_0$ "along sequences".

Proposition 2.9 is useful for showing that a limit does NOT exist.

PROPOSITION 2.10: The limits
$$\lim_{x\to\pm\infty} \sin x \,, \quad \lim_{x\to\pm\infty} \cos x$$
do not exist.

PROOF

Suppose $\lim_{x\to+\infty} \sin x = \ell$.

Then (by Prop. 2.9) for any sequence $(x_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = +\infty$ we have
$$\lim_{n\to\infty} \sin x_n = \ell .$$

Take $a_n = \pi n$, $n = 1, 2, \dots$

Then $\lim a_n = +\infty$, so
$$\ell = \lim_{n\to\infty} \sin a_n = \lim_{n\to\infty} \sin(\pi n) = 0 .$$

Take $b_n = 2\pi n + \frac{\pi}{2}$, $n = 1, 2, \dots$

Then
$$\lim_{n\to\infty} b_n = +\infty$$
so
$$\ell = \lim_{n\to\infty} \sin b_n = \lim_{n\to\infty} \sin\left(2\pi n + \frac{\pi}{2}\right) = 1 .$$

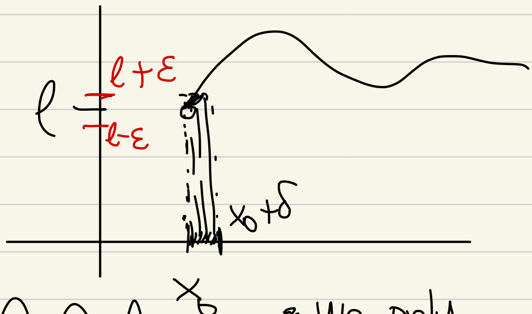
We have shown that $\ell = 0$ and $\ell = 1$; contradiction. ∎

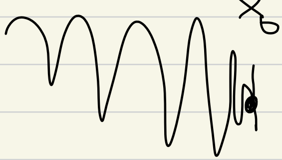
However
$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1 .}$$

We can also define the side-limits $\lim_{x\to x_0^+} f(x)$ and $\lim_{x\to x_0^-} f(x)$.

We say that $\lim_{x\to x_0^+} f(x) = \ell \in \mathbb{R}$ if

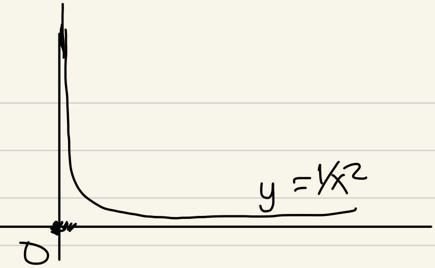
$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0$ such that
$$0 < x - x_0 < \delta \implies |f(x) - \ell| < \varepsilon.$$
$$x_0 < x < x_0 + \delta$$



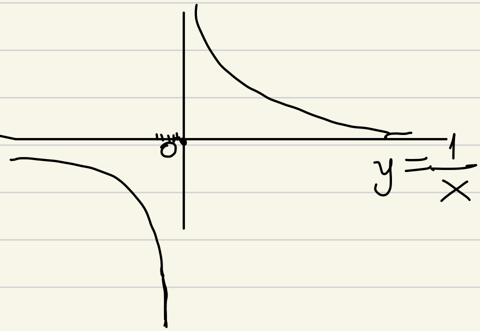


* We only care about values $x_0 < x < x_0 + \delta$.

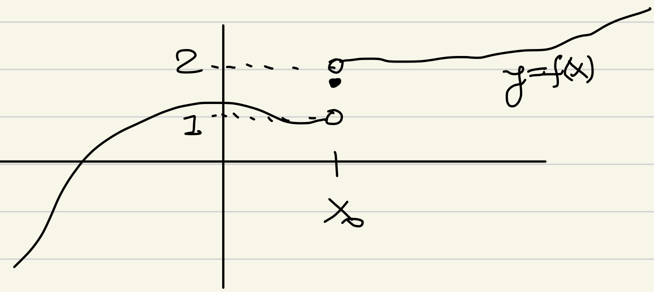
A similar definition holds for the left side-limit of f at x_0 (whether it is finite or infinite).



$$\lim_{x \to 0^+} \frac{1}{x^2} = +\infty$$



$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$



$$\lim_{x \to x_0^-} f(x) = 1$$

$$\lim_{x \to x_0^+} f(x) = 2.$$

- If f is defined on $(a, x_0) \cup (x_0, b)$ then $\lim\limits_{x \to x_0} f(x) = l$ if and only if

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = l.$$

E.g. $\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$ & $\lim\limits_{x \to 0^+} \frac{1}{x} = +\infty$

so the limit $\lim\limits_{x \to 0} \frac{1}{x}$ does not exist.

<u>Rational Functions</u> are those of the form $\frac{P(x)}{\varphi(x)}$, where $P(x), \varphi(x)$ are polynomials.

Limits of Rational Functions at $\pm\infty$:

- $$\lim_{x\to-\infty} \frac{x^5 - x^2 + 1}{2x^2 + x - 5} = \lim_{x\to-\infty} \frac{x^5\left(1 - \frac{1}{x^3} + \frac{1}{x^5}\right)}{2x^2\left(1 + \frac{1}{2x} - \frac{5}{2x^2}\right)}$$

$$= \lim_{x\to-\infty} \left(\underbrace{\frac{x^3}{2}}_{-\infty} \cdot \frac{1 - \overset{\to 0}{\frac{1}{x^3}} + \overset{\to 0}{\frac{1}{x^5}}}{\underbrace{1 + \underset{\to 0}{\frac{1}{2x}} - \underset{\to 0}{\frac{5}{2x^2}}}_{1}} \right) = -\infty.$$

- $$\lim_{x\to+\infty} \frac{2x^3 + 1}{3x^3 - x^2} = \lim_{x\to+\infty} \frac{2\cancel{x^3}\left(1 + \frac{1}{2x^3}\right)}{3\cancel{x^3}\left(1 - \frac{1}{3x}\right)} = \frac{2}{3}$$

In order to find $\lim_{x\to x_0} \frac{P(x)}{\varphi(x)}$ when $x_0 \in \mathbb{R}$ we first have to check if $\varphi(x_0) = 0$ or not.

$$\lim_{x\to 1} \frac{x^3 - 1}{x^2 + 1} = \frac{1^3 - 1}{1^2 + 1} = 0$$

- $$\lim_{x\to 2} \frac{2x-4}{x^2-4} = \lim_{x\to 2} \frac{2\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$

$$= \lim_{x\to 2} \frac{2}{x+2} \quad (\text{because } x-2 \neq 0)$$

$$= \frac{1}{2}.$$

- $$\lim_{x\to 1} \frac{x^4+x+1}{x^2-1} = \lim_{x\to 1} \frac{x^4+x+1}{(x+1)(x-1)}$$

Now $\displaystyle\lim_{x\to 1^+} \frac{x^4+x+1}{x+1} = \frac{3}{2}$

and $\displaystyle\lim_{x\to 1^+} \frac{1}{x-1} = +\infty$

so $\displaystyle\lim_{x\to 1^+} \frac{x^4+x+1}{(x+1)(x-1)} = +\infty.$

Similarly $\displaystyle\lim_{x\to 1^-} \frac{x^4+x+1}{x+1} = \frac{3}{2}$ and $\displaystyle\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$

So $\displaystyle\lim_{x\to 1^-} \frac{x^4+x+1}{(x+1)(x-1)} = -\infty.$

Hence $\displaystyle\lim_{x\to 1} \frac{x^4+x+1}{x^2-1}$ does not exist.

- $$\lim_{x\to+\infty} \frac{x+1}{\sqrt{x^2+1}} = \lim_{x\to+\infty} \frac{x\left(1+\frac{1}{x}\right)}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}$$

$$= \lim_{x\to+\infty} \frac{x\left(1+\frac{1}{x}\right)}{\sqrt{x^2}\cdot\sqrt{1+\frac{1}{x^2}}}$$

$$= \lim_{x\to+\infty} \frac{x\left(1+\frac{1}{x}\right)}{|x|\cdot\sqrt{1+\frac{1}{x^2}}}$$

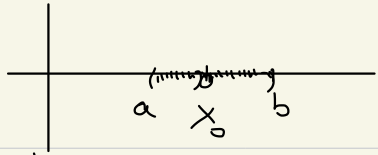
$$= \lim_{x\to+\infty} \frac{\cancel{x}\cdot\left(1+\frac{1}{x}\right)}{\cancel{x}\cdot\sqrt{1+\frac{1}{x^2}}} = 1.$$

- $$\lim_{x\to 0} \frac{x}{|x|} = ?$$

$$\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} \frac{x}{x} = 1, \quad \text{and}$$

$$\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{\cancel{x}}{-\cancel{x}} = -1.$$

So $\lim_{x\to 0} \frac{x}{|x|}$ does not exist.



THEOREM 2.11: Suppose f, g, h are defined on a set of the form $(a, x_0) \cup (x_0, b)$.
If:

(i) $f(x) \leq h(x) \leq g(x)$ for all $x \in (a, x_0) \cup (x_0, b)$

(ii) $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = L$

then also $\lim_{x \to x_0} h(x) = L$.

(* The same conclusion is true also when dealing with side limits).

E.g. find $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

$$\left| x \sin\left(\frac{1}{x}\right) \right| = |x| \cdot \left| \sin\left(\frac{1}{x}\right) \right| \leq |x| \implies$$

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|, \text{ for all } x \neq 0.$$

So by Theorem 2.11 we have

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

(* In this example, regarding the domains of definition, we could take $a = -\infty$, $b = +\infty$).