Øving 1: Prove that

$$||a|-|b|| \leq |a-b|.$$

- $||a|-|b|| \leq |a-b| \iff$

$||a|-|b||^2 \leq |a-b|^2 \iff$

$\cancel{|a|^2} + \cancel{|b|^2} - 2|ab| \leq \cancel{a^2} + \cancel{b^2} - 2ab \iff$

$ab \leq |ab|$: True, hence the initial equivalent is also true.

2nd solution:

We know $|x-y| \leq |x| + |y|$.

$|a| = |a-b+b| \leq |a-b| + |b| \Rightarrow$

$|a| - |b| \leq |a-b|$.

Similarly $|b| - |a| \leq |a-b|$.

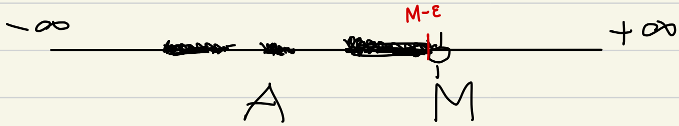
Thus

$$||a|-|b|| = \max\{|a|-|b|, |b|-|a|\} \leq |a-b|.$$

$|x| = \max\{x, -x\}$.

Øving 2

Let $A \subseteq \mathbb{R}$.



We say that $\sup A = M$ $(M \in \mathbb{R})$ if :

- $x \leq M$ for all $x \in A$
- For all $\varepsilon > 0$, there exists $x = x_\varepsilon \in A$ such that

$$M - \varepsilon < x \leq M.$$

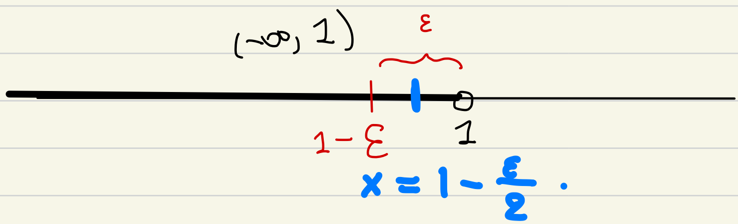
- Show that $\sup(-\infty, 1) = 1$.

$\rightarrow$ For all $x \in (-\infty, 1)$, $x \leq 1$.

Take $\varepsilon > 0$.
We have to prove that there exists some $x \in (-\infty, 1)$ such that

$$1 - \varepsilon < x \leq 1.$$

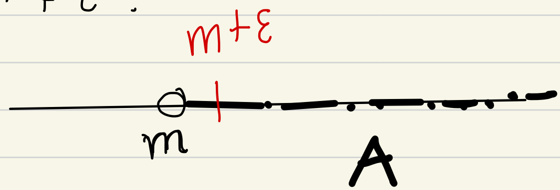
Choose $x = 1 - \frac{\varepsilon}{2}$.



We say $\inf A = m$ $(m \in \mathbb{R})$ if:

- $m \leq x$ for all $x \in A$, and
- For any $\varepsilon > 0$, there exists some $x = x_\varepsilon \in A$ such that

$$m \leq x < m + \varepsilon .$$



3. Let $(x_n)_{n=1}^{\infty}$,

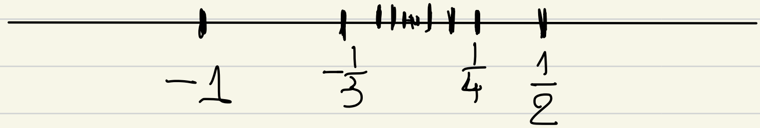
$$x_n = \frac{(-1)^n}{n}, \quad n = 1, 2, \dots$$

and find

- $\sup\{x_n : n \geq 1\}$,
- $\inf\{x_n : n \geq 1\}$,
- $\max\{x_n : n \geq 1\}$,
- $\min\{x_n : n \geq 1\}$.

$$\{x_n : n \geq 1\} = \left\{ \frac{(-1)^n}{n} : n = 1, 2, \ldots \right\}$$

$$= \left\{ -1, \ \frac{1}{2}, \ -\frac{1}{3}, \ \frac{1}{4}, \ -\frac{1}{5}, \ldots \right\}$$



$$\max\{x_n : n \geq 1\} = \frac{1}{2}, \quad \min\{x_n : n \geq 1\} = -1$$

$$\sup\{x_n : n \geq 1\} = \frac{1}{2}, \quad \inf\{x_n : n \geq 1\} = -1.$$

We show that $\max\{x_n : n \geq 1\} = \frac{1}{2}$.

We have to show that

$$x_n \leq \frac{1}{2} \quad \forall n \in \mathbb{N} \quad \text{and} \quad \frac{1}{2} \in \{x_n : n \geq 1\}.$$

The latter is correct because $x_2 = \frac{1}{2}$.

It remains to show that

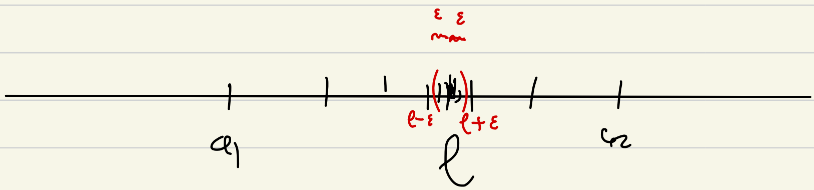
$$x_n \leq \frac{1}{2} \quad \text{for all } n \geq 1.$$

- When n is odd, $x_n < 0 < \frac{1}{2}$
- When n is even, $x_n = x_{2k} = \frac{1}{2k} \leq \frac{1}{2}$.

We say that the sequence $(a_n)_{n=1}^{\infty}$ converges to $\ell \in \mathbb{R}$ if:

For any $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - \ell| < \varepsilon \quad \text{for all } n \geq n_0.$$



We will use this definition to show that

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

Take $\varepsilon > 0$.

We have to find $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\frac{1}{n} = \left|\frac{(-1)^n}{n}\right| = \left|\frac{(-1)^n}{n} - 0\right| < \varepsilon.$$

This is equivalent to $\frac{1}{n} < \varepsilon$.

choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$.

Then for all $n \geq n_0$,

$$\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon \Rightarrow$$

$$\left| \frac{(-1)^n}{n} - 0 \right| < \varepsilon.$$

Hence $\lim_{n \to \infty} \frac{(-1)^n}{n} = 0$.

* We could have specified that n_0 can be chosen to be e.g.

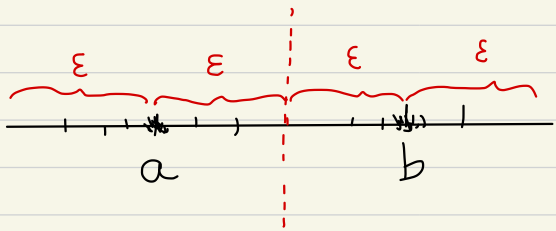
$$n_0 > \frac{2}{\varepsilon} \quad \left(\text{i.e. } n_0 = \left\lfloor \frac{2}{\varepsilon} \right\rfloor + 1\right).$$

– but this is not necessary.

Øving 2, 6: Show that the limit of a sequence $(x_n)_{n=1}^{\infty}$, if it exists, is unique.

i.e., if $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = b$

then $a = b$.



Set $\varepsilon = \frac{1}{2}|a-b| > 0$.

Because $\lim x_n = a$, there exists $n_1 \in \mathbb{N}$ such that

$$|x_n - a| < \varepsilon \quad \text{for all } n \geq n_1.$$

Because $\lim x_n = b$, there exists $n_2 \in \mathbb{N}$ such that

$$|x_n - b| < \varepsilon \quad \text{for all } n \geq n_2.$$

Choose some $N \in \mathbb{N}$ with $N \geq \max\{n_1, n_2\}$.

Then

$$|x_N - a| < \varepsilon \quad \text{and} \quad |x_N - b| < \varepsilon.$$

Thus

$$\begin{aligned} |a-b| &= |(a - x_N) + (x_N - b)| \\ &\leq |a - x_N| + |x_N - b| \\ &< \varepsilon + \varepsilon = 2\varepsilon = |a-b|. \end{aligned}$$

Contradiction.

4. Prove that $A = (0,1)$ has no minimum.

$\ulcorner$ Recall the definition of the minimum element:

The number $m \in \mathbb{R}$ is called the minimum of a set A, if:

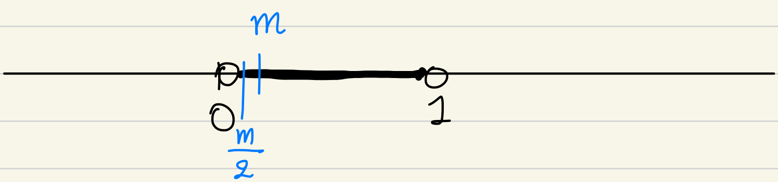
- $m \in A$
- $m \leq a$ for all $a \in A$.

$\llcorner$

Assume, for contradiction, that $A = (0,1)$ has a minimum element m.

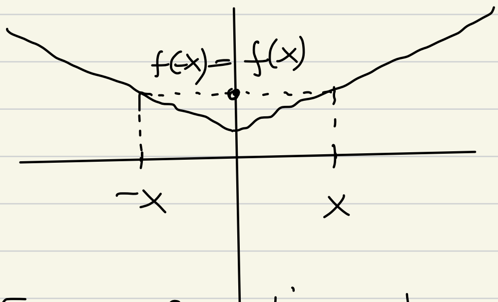
Then $m \in (0,1) \Rightarrow m > 0$.

Take $a = \frac{m}{2}$. Then $0 < a < 1 \Rightarrow a \in (0,1)$ and also $a < m$: contradiction.

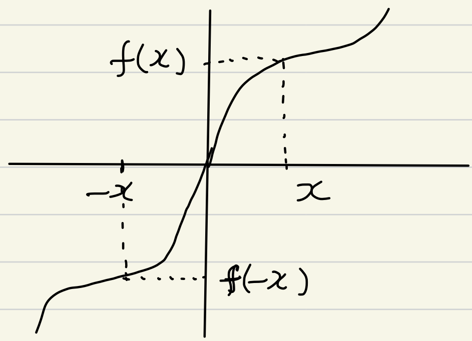


A function $f: \mathbb{R} \to \mathbb{R}$ is called:

- <u>even</u>, if
$$f(-x) = f(x) \quad \text{for all } x \in \mathbb{R}.$$
- <u>odd</u>, if
$$f(-x) = -f(x) \quad \text{for all } x \in \mathbb{R}.$$



Even functions have graphs which are symmetric wrt the y-axis.



Odd functions have graphs that are symmetric wrt the origin.

<u>Exercise:</u> If $f: \mathbb{R} \to \mathbb{R}$ is both odd and even, then $f(x) = 0$ for all $x \in \mathbb{R}$.

- Let $x \in \mathbb{R}$.
  Because f is even: $f(x) = f(-x)$.
  Because f is odd: $f(x) = -f(-x)$
  Thus $2f(x) = f(-x) - f(-x) = 0 \Rightarrow f(x) = 0$.

<u>Exercise:</u> Suppose $A \subseteq \mathbb{R}$ and $\max A = M$. Then also $\sup A = M$.

<u>SOLUTION</u>

- Because $\max A = M$,

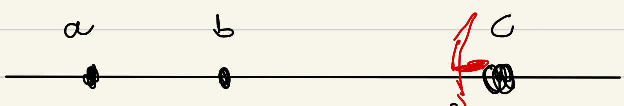
$$a \leq M \quad \text{for all } a \in A.$$

- It remains to prove that for any $\varepsilon > 0$, there exists $x = x_\varepsilon \in A$ such that

$$M - \varepsilon < x \leq M.$$

Take $\varepsilon > 0$. If we set $x = x_\varepsilon = M$, then $x \in A$ and $M - \varepsilon < x \leq M$. $\blacksquare$

(i.e. the maximum is automatically also supremum).



$$A = \{a, b, c\}.$$

$$c = \max A = \sup A.$$

Whenever f, g are differentiable at x_0

and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$

and $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l$ (in $\mathbb{R}$ or $\pm\infty$)

then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = l$.

$$\lim_{t \to 0} \frac{t^2 + 3t}{(t+2)^2 - (t-2)^2} = \lim_{t \to 0} \frac{2t + 3}{2(\cancel{t}+2) - 2(\cancel{t}-2)}$$

$$= \lim_{t \to 0} \frac{2t+3}{8} = \frac{3}{8}$$

• Find $\lim_{x \to +\infty} \frac{e^x + x}{e^x + x^2} = \lim_{x \to +\infty} \frac{e^x + 1}{e^x + 2x}$

$$= \lim_{x \to +\infty} \frac{e^x}{e^x + 2} = \lim_{x \to +\infty} \frac{1}{1 + 2e^{-x}} = 1$$

$2e^{-x} \to 0$

$$\lim_{x\to 0}\left(\frac{1}{\tan x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\tan x}{x\tan x}$$

$$=\lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin x}$$

$$=\lim_{x\to 0}\frac{\cancel{\cos x}-x\sin x-\cancel{\cos x}}{\sin x+x\cos x}=\lim_{x\to 0}\frac{-x\sin x}{\sin x+x\cos x}$$

$$=\lim_{x\to 0}\frac{-\sin x-x\cos x}{\cos x+\cos x-x\sin x}$$

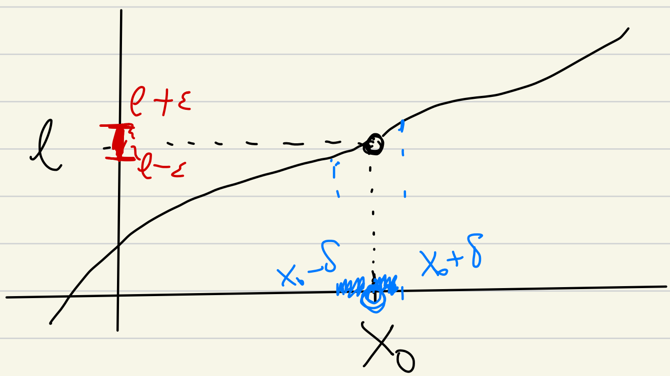
$$=\lim_{x\to 0}-\frac{\sin x+x\cos x}{2\cos x-x\sin x}=0$$

When $x_0 \in \mathbb{R}$ and f is defined on some interval around x_0, we say that

$$\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$$

if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$0 < |x - x_0| < \delta \implies |f(x) - \ell| < \varepsilon.$$



Örnek 3, ε: Show that $\lim_{x \to 2} (5-2x) = 1$.

- We have to show that for any $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$ such that $0 < |x-2| < \delta \Rightarrow |(5-2x) - 1| < \varepsilon$.

Let $\varepsilon > 0$.

$|(5-2x) - 1| = |4-2x| = 2|x-2|$

We need to find $\delta > 0$ such that

$0 < |x-2| < \delta$ implies $2|x-2| < \varepsilon$.

Choose $\delta = \frac{\varepsilon}{2} > 0$.

Then whenever $0 < |x-2| < \delta$ we have

$$2|x-2| < 2\delta = \varepsilon.$$

Øving 3, 2: Show that $\lim_{x \to 1} x^2 = 1$.

- We have to show that, for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $0 < |x-1| < \delta$ implies $|x^2 - 1| < \varepsilon$.

Take $\varepsilon > 0$.

$$|x^2 - 1| = |x+1| \cdot |x-1|$$

When $|x-1| < 1$, then

$$-1 < x - 1 < 1 \Rightarrow$$
$$1 < x + 1 < 3 \Rightarrow$$
$$|x+1| < 3.$$

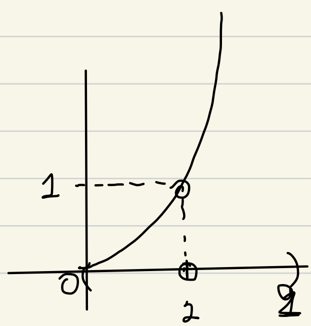
Choose $\delta > 0$ smaller than 1 and $\frac{\varepsilon}{3}$

(e.g. $\delta = \frac{1}{2} \min\{1, \frac{\varepsilon}{3}\} > 0$)

Then $0 < |x-1| < \delta$ implies that

- $|x+1| < 3$
- $|x-1| < \frac{\varepsilon}{3}$

hence $|x^2 - 1| = |x+1|\,|x-1| < \varepsilon$.



Øving 3, 8: (c) $\lim_{x \to +\infty} \left(\sqrt{x^2+2x} - \sqrt{x^2-2x} \right)$

$$\sqrt{x^2+2x} - \sqrt{x^2-2x} = \frac{\left(\sqrt{x^2+2x} - \sqrt{x^2-2x}\right)\left(\sqrt{x^2+2x} + \sqrt{x^2-2x}\right)}{\sqrt{x^2+2x} + \sqrt{x^2-2x}}$$

$$= \frac{(x^2+2x) - (x^2-2x)}{\sqrt{x^2+2x} + \sqrt{x^2-2x}}$$

$$= \frac{4x}{\sqrt{x^2+2x} + \sqrt{x^2-2x}}$$

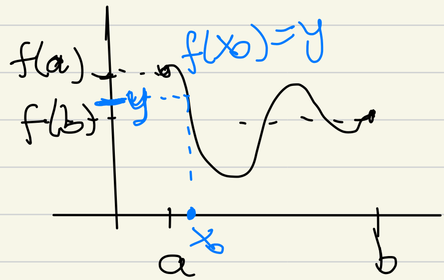
$$= \frac{4\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{2}{x}} + \sqrt{1-\frac{2}{x}}\right)} \quad \text{(for } x>0 \text{ large enough)}$$

$$= \frac{4}{\sqrt{1+\frac{2}{x}} + \sqrt{1-\frac{2}{x}}}$$

So $\lim_{x \to +\infty} \left(\sqrt{x^2+2x} - \sqrt{x^2-2x} \right) = 2.$

THEOREM 2.13 (Intermediate Value Thm): Suppose $f : [a,b] \to \mathbb{R}$ is continuous and y is "between" $f(a)$ and $f(b)$, then there exists $x_0 \in [a,b]$ such that

$$y = f(x_0).$$



Øving 3, Ex. 9: Show that

$$f(x) = x^3 + x - 1$$

has a root in the interval $[0,1]$.

- $f(0) = -1$ and $f(1) = 1$

  f is continuous on $[0,1]$, hence there exists a point $x_0 \in (0,1)$ with

$$f(x_0) = 0.$$

Øving 10, 8: Let

$$f(x) = \begin{cases} 3x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Show f is continuous at $x_0 = 0$

(b) Is f is diff. at 0?

(c) Is f unif. cont. on $[-1, 1]$?

(a) Take $\varepsilon > 0$.

I have to find $\delta = \delta(\varepsilon) > 0$ such that whenever $|x| < \delta$ we have $|f(x)| < \varepsilon$.

$$|f(x)| = \left|3x\sin\left(\frac{1}{x}\right)\right| \leq 3|x|.$$

Choose $\delta = \frac{\varepsilon}{3} > 0$.

Then $|x| < \delta$ implies

$$|f(x)| \leq 3|x| = \varepsilon.$$

f is continuous at $x_0 = 0$.

(b) $\dfrac{f(x)-f(0)}{x-0} \overset{x\neq 0}{=} 3\sin\left(\dfrac{1}{x}\right)$

The limit $\lim\limits_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

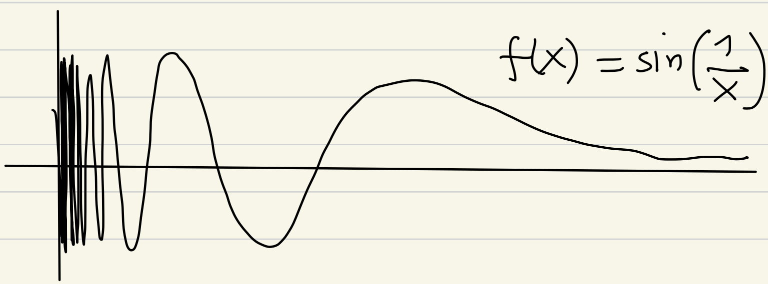
Consider the sequences

$x_n = \dfrac{1}{\pi n}$, $n = 1, 2, \dots$ and

$y_n = \dfrac{1}{2\pi n + \frac{\pi}{2}}$, $n = 1, 2, \dots$

Then $x_n \to 0$, $y_n \to 0$ but

$f(x_n) = \sin(\pi n) = 0 \to 0$,
$f(y_n) = \sin\left(2\pi n + \frac{\pi}{2}\right) = 1 \to 1$.



f is not differentiable at 0.

(c) f is continuous on $[-1, 1]$
therefore it is also uniformly cont. on $[-1, 1]$.

P

The negation of P : $\neg P$

P : $x > 1$
$\neg P$: $x \leq 1$

P : $x \in \mathbb{R}$
$\neg P$: $x \notin \mathbb{R}$

P : all students in the class are shorter than 2m
$\neg P$: there is one st. taller than 2m

P: $\forall s \in C : s < 2$
$\neg P$: $\exists s \in C : s \geq 2$

P: $\exists x \in \mathbb{R} : f(x) = 0$
$\neg P$: $\forall x \in \mathbb{R} : f(x) \neq 0$

P: $\forall x \in \mathbb{R} \; \exists y \in \mathbb{R}$ s.t. $y > x$
$\neg P$: $\exists x \in \mathbb{R} \; \forall y \in \mathbb{R} : y \leq x$

<u>PROPOSITION</u>: The following two statements are equivalent:

(i) $\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$

(ii) For any sequence $(x_n)_{n=1}^{\infty}$ with $x_n \to x_0$ and $x_n \neq x_0$, $n \geq 1$ we have
$$\lim_{n \to \infty} f(x_n) = \ell.$$

<u>PROOF</u>

<u>(i) $\Rightarrow$ (ii)</u>: Take a seq. $(x_n)_{n=1}^{\infty}$ with $x_n \to x_0$ and $x_n \neq x_0$, $n \geq 1$.
Let $\varepsilon > 0$.
There exists $\delta > 0$ such that

$$0 < |x - x_0| < \delta \implies |f(x) - \ell| < \varepsilon.$$

Because $x_n \to x_0$, there exists $n_0 \in \mathbb{N}$ such that

$$|x_n - x_0| < \delta \quad \text{for all } n \geq n_0.$$

Then $|f(x_n) - \ell| < \varepsilon, \quad \forall n \geq n_0.$

$\underline{(ii) \Rightarrow (i)}$: Assume, for contradiction, that it is not true that

$$\lim_{x \to x_0} f(x) = \ell.$$

In other words, it is false that

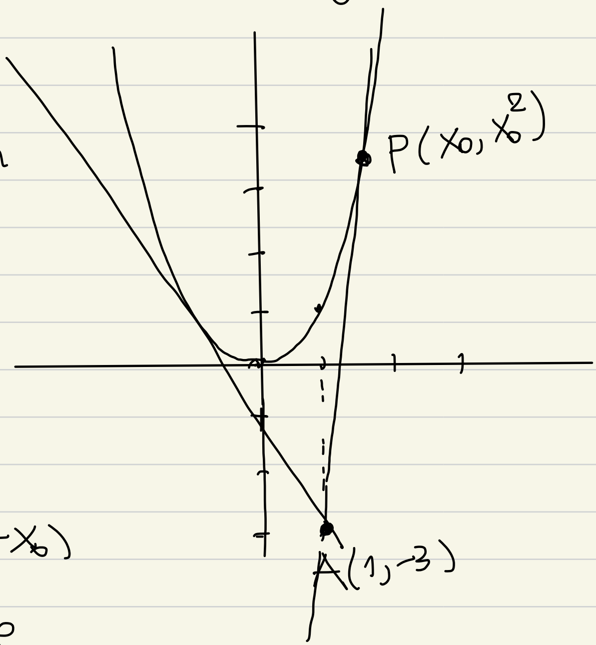
$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{s.t.} \ 0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

Hence there exists some $\varepsilon > 0$ such that $\forall \delta > 0$ there exists x with $0 < |x - x_0| < \delta$ and $|f(x) - \ell| \geq \varepsilon$.

For any $n \geq 1$, I can choose x_n such that $0 < |x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - \ell| \geq \varepsilon$.

Thus I have defined a seq. $(x_n)_{n=1}^{\infty}$ with $x_n \to x_0$ and $x_n \neq x_0$, $n \geq 1$ but $f(x_n) \not\to \ell$; contradiction.

Øving 4, 3: Find the equations of the two lines which pass through the point $(1,-3)$ and are tangents two the curve $y=x^2$.



Suppose the tangent line that goes through A is tangent to the curve $y=x^2$ at the point $P(x_0, x_0^2)$.

The equation of the tangent line is

$$y - f(x_0) = f'(x_0)\cdot(x-x_0)$$

(where $f(x) = x^2$), i.e.

$$y - x_0^2 = 2x_0(x-x_0) \Rightarrow$$

$$y - x_0^2 = 2x_0 x - 2x_0^2 \Rightarrow$$

$$y = 2x_0\cdot x - x_0^2$$

Because the tangent goes through $A(1,-3)$ we have

$$-3 = 2x_0 - x_0^2 \Rightarrow x_0^2 - 2x_0 - 3 = 0$$

$$\Rightarrow (x_0-3)(x_0+1) = 0$$

$$\Rightarrow x_0 = 3 \text{ or } x_0 = -1.$$

For $x_0 = 3$, we have the line

(ℓ_1) : $y = 6x - 9$

which is tangent to $y = x^2$ at the point $(3, 9)$.

For $x_0 = -1$, we have the line

(ℓ_2) : $y = -2x - 1$

which is tangent to $y = x^2$ at the point $(-1, 1)$.

When $f: I \to \mathbb{R}$ is differentiable on the interval I, then f is also <u>continuous</u> on I.

If f is differentiable, is the function f' always continuous on I?

The answer is NO.

<u>Øving 4, 8:</u> Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Prove that f is differentiable, but f' is not continuous on $\mathbb{R}$.

<u>SOLUTION</u>

When $x \neq 0$, f is differentiable at x with

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

For $x = 0$,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0,$$

so f is differentiable at 0 with $f'(0) = 0$.

Therefore

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

We show that f' is not continuous at the point $x_0 = 0$.

Assume for contradiction it is.
Then

$$\lim_{x \to 0} f'(x) = f'(0) = 0 \implies$$

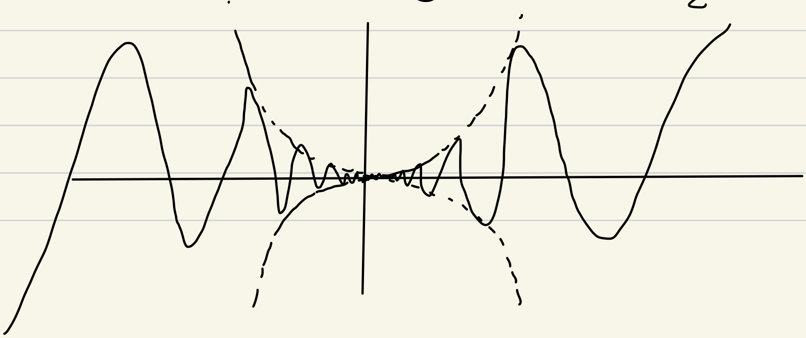
$$\lim_{x \to 0} \left(2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right) = 0. \quad (*).$$

Since $\lim_{x \to 0} 2x\sin\left(\frac{1}{x}\right) = 0$, $(*)$ implies that

$$\lim_{x \to 0} \cos\left(\frac{1}{x}\right) = 0.$$

This is a contradiction, because $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist.
(Take the sequences

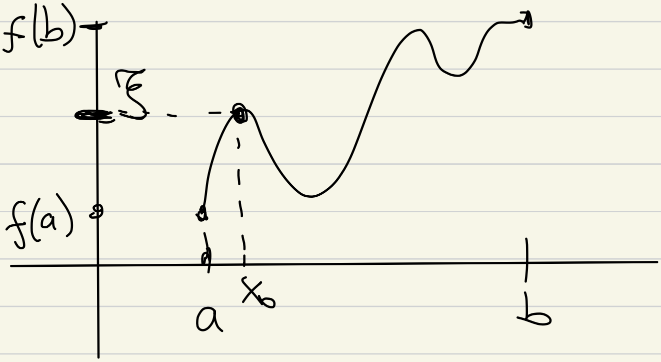
$$x_n = \pi n, \quad y_n = (2n+1)\frac{\pi}{2}, \quad (n \geq 1)).$$



We know from the Intermediate Value Theorem that whenever f is continuous on $[a,b]$ and

ξ is "between" $f(a)$ and $f(b)$

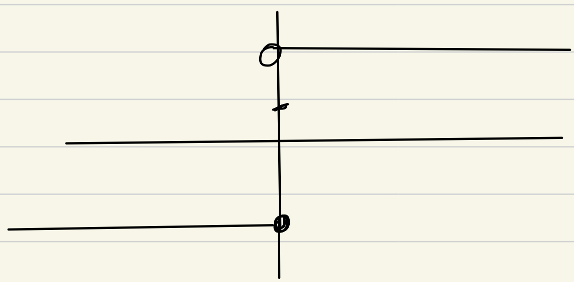
then $\xi = f(x_0)$ for some $x_0 \in [a,b]$.



In other words, any continuous $f : [a,b] \to \mathbb{R}$ must satisfy the "Intermediate Value Property".

Of course if f is not continuous, it does not satisfy necessarily this property.
Take

$$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x \leq 0 \end{cases}$$



As we just saw, even if f is differentiable, the function f' will not always be continuous.

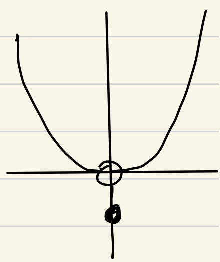
<u>THEOREM (Darboux)</u>: Suppose f is differentiable on the interval I. Then $f' : I \to \mathbb{R}$ has the Intermediate Value Property: i.e. if $a, b \in I$ and ξ is between $f'(a)$ and $f'(b)$, there is $x_0 \in [a, b]$ with

$$\xi = f'(x_0).$$

E.g. the function $f(x) = \begin{cases} 1, & x > 0 \\ -1, & x \leq 0 \end{cases}$

is not the derivative of any function.

The same for $g(x) = \begin{cases} x^2, & x \neq 0 \\ -1, & x = 0 \end{cases}$

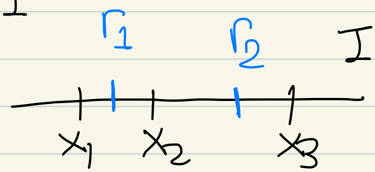


(by Darboux's Theorem).

Øving 5, 9: $f: I \to \mathbb{R}$ is twice differ. and has at least 3 roots. Prove that f'' has at least 1 root in I.

- Assume $x_1 < x_2 < x_3$ are the roots of f in I

$f(x_1) = f(x_2) = f(x_3) = 0$.



By Rolle's theorem, there exists $r_1 \in (x_1, x_2)$ such that

$$f'(r_1) = 0$$

and $r_2 \in (x_2, x_3)$ such that

$$f'(r_2) = 0.$$

Now applying Rolle's theorem to f' on the interval $[r_1, r_2]$, we find some $x_0 \in (r_1, r_2)$ with

$$f''(x_0) = 0.$$

When f is n-times differentiable, the n-th Taylor polynomial of f at $x_0 \in \mathbb{R}$ by

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots$$
$$\dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

In the special case when $x_0 = 0$, the Taylor polynomial of f is called the McLaurin polynomial of f.

Also the first degree Taylor pol. of f at the point x_0 is called the linearisation of f at x_0.

Øving 6, 6: Find the 5^{th} order Taylor polynomial of $f(x) = \sin x$ at the point $x_0 = \pi$.

- $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x$, $f^{(5)}(x) = \cos x$.

$$P(x) = \cancel{f(\pi)} + f'(\pi)(x-\pi) + \cancel{\frac{f''(\pi)}{2}(x-\pi)^2} +$$
$$+ \frac{f'''(\pi)}{3!}(x-\pi)^3 + \cancel{\frac{f^{(4)}(\pi)}{4!}(x-\pi)^4} + \frac{f^{(5)}(\pi)}{5!}(x-\pi)^5$$
$$= -(x-\pi) + \frac{1}{6}(x-\pi)^3 - \frac{1}{120}(x-\pi)^5.$$

Øving 6, 3: Find the linearisation of $f(x) = \tan x$ at the point $x_0 = \frac{\pi}{4}$.

- This is

$$P(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\cdot\left(x - \frac{\pi}{4}\right).$$

Now $f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$,

$$f'(x) = \frac{1}{\cos^2 x} \Rightarrow f'\left(\frac{\pi}{4}\right) = \frac{1}{\cos^2\frac{\pi}{4}} = 2$$

and

$$P(x) = 1 + 2\left(x - \frac{\pi}{4}\right).$$

Inverse Function Theorem: Suppose $f: I \to \mathbb{R}$ is continuous and invertible.
If f is differentiable at the point (x_0, y_0) and $f'(x_0) \neq 0$, then f^{-1} is differentiable at y_0 with

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

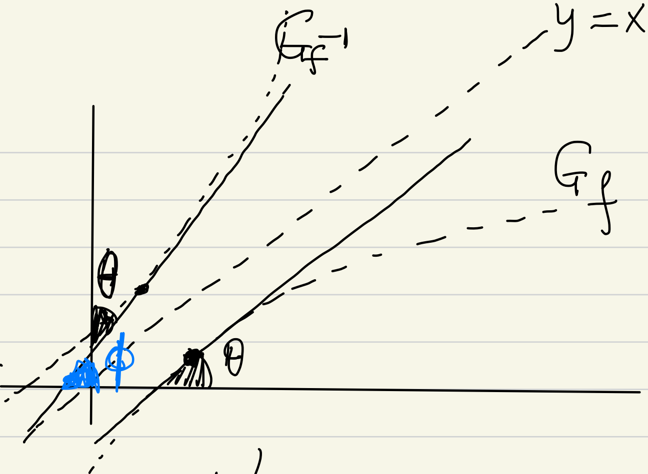
$y_0 = f(x_0)$

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \overset{y=f(x)}{=} \frac{x - x_0}{f(x) - f(x_0)}$$

$$= \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} \to \frac{1}{f'(x_0)}$$

$$f^{-1}(f(x)) = x \Rightarrow (f^{-1})'(f(x)) \cdot f'(x) = 1$$

$$\Rightarrow (f^{-1})'(f(x)) = \frac{1}{f'(x)}$$



$$f'(x_0) \neq 0$$

$$f'(x_0) = \tan\theta$$

$$\phi + \theta = \frac{\pi}{2}$$

$$(f^{-1})'(f(x_0)) = \tan\phi$$

$$= \tan\left(\frac{\pi}{2} - \theta\right)$$

$$= \cot\theta$$

$$= \frac{1}{\tan\theta}$$

$$= \frac{1}{f'(x_0)}$$

Øving 6, 4: Prove that

$$f(x) = \frac{4x^3}{x^2+1}$$

is invertible and find $(f^{-1})'(2)$.

ANSWER

1^{st} answer: $f(x_1) = f(x_2) \Rightarrow \dots \Rightarrow x_1 = x_2$

2^{nd} answer:

$$f'(x) = \frac{12x^2(x^2+1) - 2x \cdot 4x^3}{(x^2+1)^2}$$

$$= \frac{4x^4 + 12x^2}{(x^2+1)^2}$$

Now $f'(x) \geq 0$ for all $x \in \mathbb{R}$ with $f(x)=0 \Leftrightarrow x=0$

hence f is strictly increasing in $\mathbb{R}$.

Therefore f is also invertible in $\mathbb{R}$.

By the inverse function Theorem, if $f'(x_0) \neq 0$,

then at $y_0 = f(x_0)$

we have

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Here $y_0 = 2$, hence $x_0 = f^{-1}(2)$.

We see that $f(1) = \frac{4}{1+1} = 2 \Rightarrow f^{-1}(2) = 1$.

(i.e. $x_0 = 1$),

$$f'(1) = \frac{16}{4} = 4$$

hence $(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{4}$.

Høst 2019 — OPPGAVE 2

(i) $\int_{-1}^{1} x \cosh x \, dx$

(ii) $\int_0^{\pi/2} \frac{\cos t \, dt}{1+\sin^2 t}$ (iii) $\int_0^{\varepsilon} \frac{dx}{\sin x}$ $(0<\varepsilon<1)$.

ANSWER

(i) $\cosh x = \frac{e^x + e^{-x}}{2}$ is even

and $x\cosh x$ is an odd function,
so
$$\int_{-1}^{1} x \cosh x \, dx = 0.$$

(ii) Set $u = \sin t \Rightarrow du = \cos t \, dt$

$t_1 = 0 \Rightarrow u_1 = \sin 0 = 0$
$t_2 = \frac{\pi}{2} \Rightarrow u_2 = \sin\frac{\pi}{2} = 1$

$$\int_0^{\frac{\pi}{2}} \frac{\cos t \, dt}{1+\sin^2 t} = \int_0^1 \frac{du}{1+u^2} = \Big[\arctan u\Big]_0^1$$

$$= \arctan 1 - \arctan 0 = \frac{\pi}{4}.$$

(iii) We know that for $0<x<\varepsilon<1$
$0 < \sin x < x \Rightarrow \frac{1}{\sin x} > \frac{1}{x}$.

We know that

$$\int_0^{\varepsilon} \frac{dx}{x} \quad \text{diverges}$$

hence so does $\displaystyle\int_0^{\varepsilon} \frac{dx}{\sin x}$.

DES. 2019, OPPGAVE 3: Sketch the graph of

$$f(x) = \frac{x^2 - 5x + 6}{x - 1}.$$

Find domain of definition and range of f, zeroes of f, local and global extreme points and potential asymptotes of f.

ANSWER

Domain of def.: $D_f = (-\infty, 1) \cup (1, \infty)$.

$f(x) = 0 \iff x^2 - 5x + 6 = 0 \iff x = 2$ or $x = 3$.

$$f(x) = \frac{x^2 - x - 4x + 6}{x - 1} = \frac{x^2 - x}{x-1} - \frac{4x - 6}{x-1}$$

$$= x - \frac{4x - 4 - 2}{x - 1} = x - 4 + \frac{2}{x-1}, \quad x \neq 1.$$

$$f(x) = x - 4 + \frac{2}{x-1}, \quad x \neq 1$$

$$f'(x) = 1 - \frac{2}{(x-1)^2}, \quad x \neq 1.$$

$$f'(x) > 0 \iff \frac{2}{(x-1)^2} < 1$$

$$\iff (x-1)^2 > 2$$

$$\iff x \in (-\infty, 1-\sqrt{2}) \cup (1+\sqrt{2}, +\infty).$$

$$f'(x) < 0 \iff x \in (1-\sqrt{2}, 1) \cup (1, 1+\sqrt{2}).$$

x	$-\infty$		$1-\sqrt{2}$		1		$1+\sqrt{2}$		$+\infty$
$f'(x)$		$+$	0	$-$	$\Vert$	$-$	0	$+$	
$f(x)$		$\nearrow$	$f(1-\sqrt{2})$	$\searrow$	$\Vert$	$\searrow$		$\nearrow$	

f has a local maximum at $1-\sqrt{2}$, which is the number

$$f(1-\sqrt{2}) = -2\sqrt{2} - 3$$

f has a local minimum at $1+\sqrt{2}$ which is the number

$$f(1+\sqrt{2}) = 2\sqrt{2} - 3.$$

Since $\lim_{x \to +\infty} f(x) = +\infty$,

the local maximum is not a global maximum.
Similarly because $\lim_{x \to -\infty} f(x) = -\infty$

the local minimum is not a global minimum.
(I.e. f has no global extrema).

f is str. incr. on $I_1 = (-\infty, 1-\sqrt{2}]$,
hence

$$f(I_1) = (\lim_{x \to -\infty} f(x), f(1-\sqrt{2})] = (-\infty, -2\sqrt{2}-3].$$

f is str. decreasing on $I_2 = [1-\sqrt{2}, 1)$,
hence

$$f(I_2) = (\lim_{x \to 1} f(x), f(1-\sqrt{2})] = (-\infty, -2\sqrt{2}-3]$$

f is str. decr. on $I_3 = (1, 1+\sqrt{2}]$,
so

$$f(I_3) = [2\sqrt{2}-3, +\infty)$$

f is str. increasing on $I_4 = [1+\sqrt{2}, \infty)$
so

$$f(I_4) = [2\sqrt{2}-3, +\infty).$$

The range of f is

$$f(D_f) = f(I_1) \cup f(I_2) \cup f(I_3) \cup f(I_4)$$
$$= (-\infty, -2\sqrt{2}-3] \cup [2\sqrt{2}-3, +\infty).$$

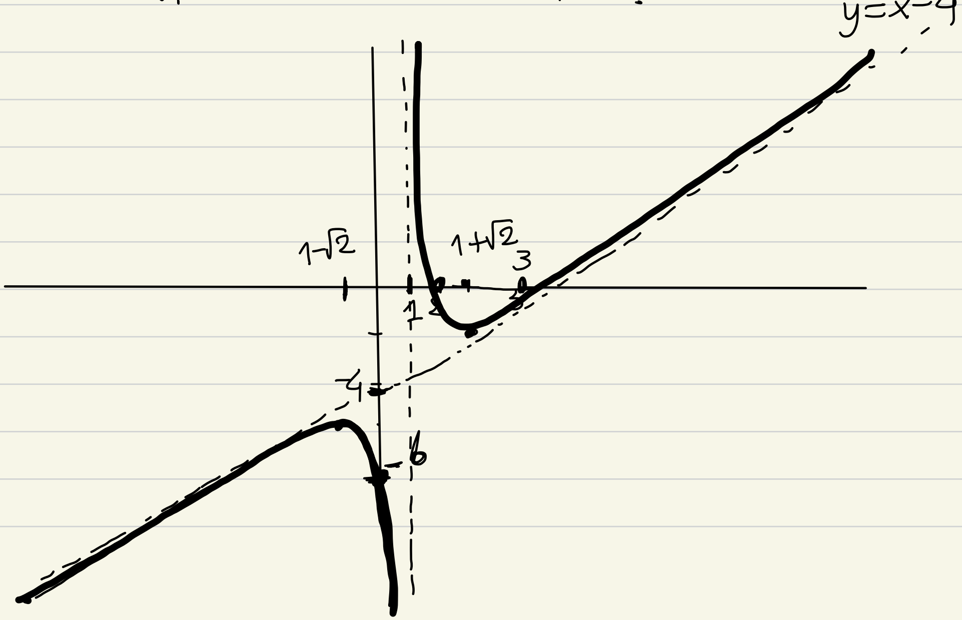
$$f(x) = x-4 + \frac{2}{x-1}, \quad x \neq 1.$$

$$\lim_{x\to 1^-} f(x) = -\infty \quad \text{and} \quad \lim_{x\to 1^+} f(x) = +\infty$$

therefore the line $x=1$ is a vertical asymptote of f.

Recall that $y=ax+b$ is an asymptote of G_f if $\lim_{x\to\pm\infty} (f(x)-ax-b) = 0$.

$\lim_{x\to\pm\infty} (f(x)-(x-4)) = 0$, hence $y=x-4$ is an oblique asymptote at $-\infty$ and $+\infty$.



$$f''(x) = \frac{4}{(x-1)^3}, \quad x \neq 1.$$

f is concave on $(-\infty, 1)$
and f is convex on $(1, \infty)$.

DES. 2019, OPPGAVE 7:

(i) Solve
$$\frac{y'(x)}{2x} - y(x) = 1, \quad y(1) = 2$$

(ii) Show that the solution is uniformly cont. on $[1,2]$ but not on $(1,\infty)$.

SOLUTION

(i)
$$\frac{y'}{2x} - y = 1 \Rightarrow y' - 2xy = 2x$$

$$\Rightarrow e^{-x^2} y' - 2x e^{-x^2} y = 2x e^{-x^2}$$

$$\Rightarrow (e^{-x^2} y)' = (-e^{-x^2})'$$

$$\Rightarrow e^{-x^2} y = -e^{-x^2} + c, \quad c \in \mathbb{R} \text{ const.}$$

$$\Rightarrow y(x) = c e^{x^2} - 1, \quad c \in \mathbb{R} \text{ const.}$$

$y(1) = 2 \Rightarrow ce - 1 = 2 \Rightarrow c = 3e^{-1}$.

Thus $y(x) = 3e^{x^2-1} - 1$.

(ii) $y(x) = 3e^{x^2-1} - 1$.

y is cont. on the closed interval $[1,2]$
therefore y is uniformly cont. on $[1,2]$.

If $1 < x_1 < x_2$,

$|y(x_2) - y(x_1)| = |y'(t)||x_2 - x_1|$ $(x_1 < t < x_2)$

(by the Mean Value Theorem)

$= 6te^{t^2-1}|x_2 - x_1|$

If f is uniformly continuous,
for $\varepsilon = 1$ there will be some $\delta > 0$
such that for all $1 < x_1 < x_2$

$|x_2 - x_1| < \delta \Rightarrow |y(x_2) - y(x_1)| < 1$.

$\Rightarrow 6te^{t^2-1}|x_2 - x_1| < 1$

This is a contradiction.

DES. 2018, OPPGAVE 7: The function

$$\sin : \mathbb{R} \to [-1, 1]$$

is C^∞ and therefore also continuous. Show this is also uniformly continuous. (on its domain of def.).

ANSWER

$f(x) = \sin x$,

We cannot say that continuity automatically implies uniform continuity, because the domain of def. is $\mathbb{R}$ - not of the form $[a,b]$.

$f'(x) = \cos x \Rightarrow |f'(x)| \leq 1$.

f' is bounded, hence f is uniformly continuous on $\mathbb{R}$.

Don't forget: Uniform continuity of a function is defined on a set, not on points (like continuity is).

For example, let $f(x) = x^3$.
Then

- f is uniformly cont. on $[0,1]$
- f is not uniformly cont. on $\mathbb{R}$.

<u>THEOREM</u>: Let $0 \leq f(x) \leq g(x)$, $x \geq a$.

(i) If $\int_a^\infty g(x)\,dx < \infty$, then also $\int_a^\infty f(x)\,dx$.

(ii) If $\int_a^\infty f(x)\,dx = \infty$, then also $\int_a^\infty g(x)\,dx$.

When we cannot calculate an improper integral directly, we have to "compare" with a known integral.

- $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx < \infty$ when $p > 1$

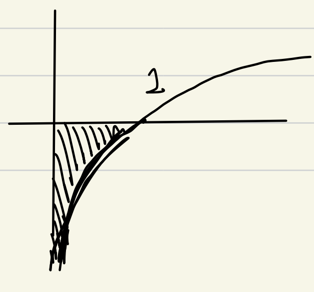
  $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \infty$ when $p \leq 1$

- $\displaystyle\int_0^1 \frac{1}{x^q}\,dx < \infty$ when $q < 1$

  $\displaystyle\int_0^1 \frac{1}{x^q}\,dx = \infty$ when $q \geq 1$.

- $\displaystyle\int_0^\infty e^{-\lambda x}\,dx < \infty$ for any $\lambda > 0$.

- $\displaystyle\int_0^1 \ln x\,dx = -1$ (converges).



ØVING 10, 2: Decide if the improper int. converge or diverge.

(a) $\displaystyle\int_0^{+\infty} \frac{x^2\,dx}{x^5+1}$

For any $x \geq 0$,

$$x^5+1 > x^5 \Rightarrow \frac{1}{x^5+1} < \frac{1}{x^5} \Rightarrow \frac{x^2}{x^5+1} < \frac{1}{x^3}.$$

We know that $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$ converges,

hence so does

$$\int_0^{\infty} \frac{x^2}{x^5+1}\,dx.$$

(b) $\displaystyle\int_0^{\infty} \frac{dx}{1+\sqrt{x}}$ I want to show that this diverges, so I want $\frac{1}{1+\sqrt{x}} \geq$ (sth).

For $x \geq 1$, $1+\sqrt{x} \leq 2\sqrt{x} \Rightarrow$

$$\frac{1}{1+\sqrt{x}} \geq \frac{1}{2\sqrt{x}}$$

and we know that $\displaystyle\int_1^{\infty} \frac{dx}{2\sqrt{x}} = \infty$

therefore $\displaystyle\int_0^{\infty} \frac{dx}{1+\sqrt{x}}$ also diverges.

set $u = \sqrt{x}$
$x = u^2$
$dx = 2u\,du$

Alternatively,

$$\int \frac{dx}{1+\sqrt{x}} = \int \frac{2u\,du}{1+u}$$

$$= \int \frac{2u+2-2}{1+u}\,du$$

$$= \int \left(2 - \frac{2}{1+u}\right) du$$

$$= 2u - 2\ln|1+u| + C$$
$$= 2\sqrt{x} - 2\ln|1+\sqrt{x}| + C$$

hence

$$\int_0^T \frac{dx}{1+\sqrt{x}} = 2\sqrt{T} - 2\ln|1+\sqrt{T}|$$
$$= 2\sqrt{T} - 2\ln(1+\sqrt{T}) \to +\infty.$$

(c) $\int_0^\infty e^{-x^3}\,dx$

When $x \geq 1$, $x^3 \geq x \Rightarrow -x^3 \leq -x$
$\Rightarrow e^{-x^3} \leq e^{-x}$.

We know that $\int_0^\infty e^{-x}\,dx$ converges,

hence so does $\int_0^\infty e^{-x^3}\,dx$.

(d) $\displaystyle\int_{-1}^{1} \frac{e^x}{x+1}\,dx$

We will compare with $\displaystyle\int_{-1}^{1} \frac{dx}{x+1}$.

$$\int_u^1 \frac{dx}{x+1} = \Big[\ln|x+1|\Big]_u^1$$
$$= \ln 2 - \ln|u+1|$$
$$\xrightarrow{u\to -1^+} +\infty$$

hence $\displaystyle\int_{-1}^{1} \frac{dx}{x+1}$ diverges.

For all $-1 \leq x \leq 1$,

$$\left.\begin{array}{l} e^x \geq e^{-1} \\ \dfrac{1}{x+1} > 0 \end{array}\right\} \Rightarrow \frac{e^x}{1+x} \geq \frac{e^{-1}}{1+x}$$

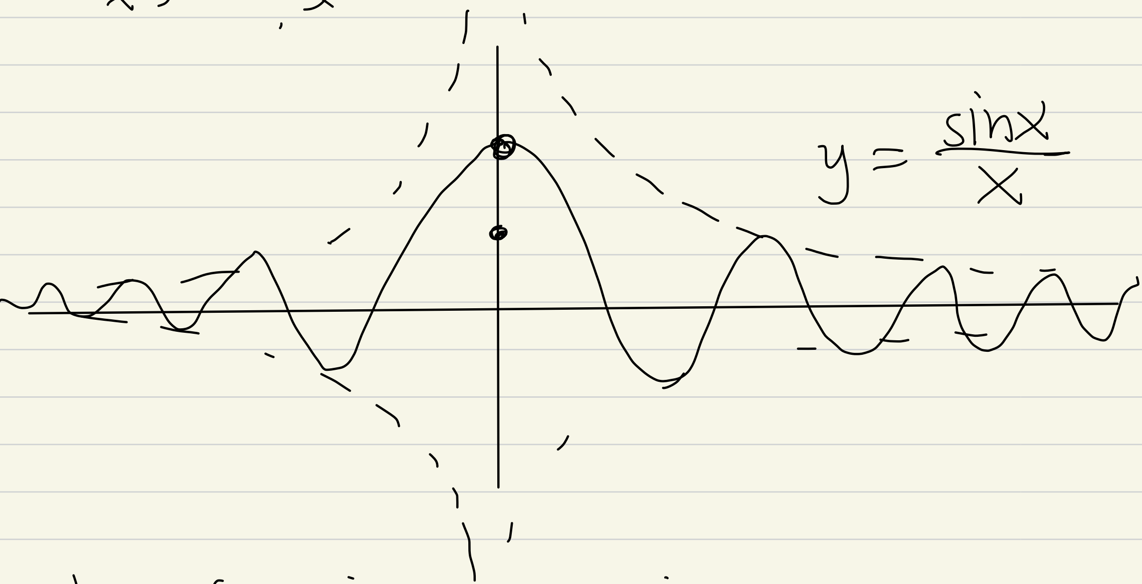
and $\displaystyle\int_{-1}^{1} \frac{e^{-1}\,dx}{1+x} = \infty$,

hence also $\displaystyle\int_{-1}^{1} \frac{e^x}{1+x}\,dx$ diverges.

ØVING 10, 4: Is $\int_0^{\pi} \frac{\sin x}{x}\,dx$ an improper integral or a Riemann integral?

ANSWER

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$



The function $g(x) = \frac{\sin x}{x}$, $x \neq 0$ has a continuous extension

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

So the function $\frac{\sin x}{x}$, $x \neq 0$
has a continuous extension –
it could be considered as
a piecewise continuous function.
Therefore this function is Riemann-integrable
on any interval of the form $[a, b]$.
The integral

$$\int_0^{\pi} \frac{\sin x}{x}\, dx$$

is a Riemann-integral.

When $(a_n)_{n=1}^{\infty}$ is a seq. of real numbers,
consider the series

$$\sum_{n=1}^{\infty} a_n .$$

The sequence of partial sums is

$$S_N = a_1 + a_2 + \dots + a_N \quad , \quad N = 1, 2, \dots$$

We say that $\sum_{n=1}^{\infty} a_n$ converges to
the real number $\ell \in \mathbb{R}$ if $\lim_{N\to\infty} S_N = \ell$.

Otherwise $\sum_{n=1}^{\infty} a_n$ diverges.

Øving 7, 4: (b) $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6}$?

The sequence of partial sums is

$$S_N = \sum_{n=1}^{N} \frac{1}{n^2+5n+6} = \sum_{n=1}^{N} \frac{1}{(n+2)(n+3)}$$

$$= \sum_{n=1}^{N} \frac{(n+3)-(n+2)}{(n+2)(n+3)} = \sum_{n=1}^{N} \left(\frac{1}{n+2} - \frac{1}{n+3} \right)$$

$$= \left(\frac{1}{3} - \cancel{\frac{1}{4}} \right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} \right) + \cancel{\cdots} + \left(\cancel{\frac{1}{N+2}} - \frac{1}{N+3} \right)$$

$$= \frac{1}{3} - \frac{1}{N+3} \longrightarrow \frac{1}{3}.$$

Therefore the series $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6} = \frac{1}{3}$.

Øving 7, 3: Assume $\sum_{n=1}^{\infty} a_n$ converges.

Prove that $\lim_{n\to\infty} (a_n + a_{n+1}) = 0$.

SOLUTION

Since the series $\sum_{n=1}^{\infty} a_n$ converges,

we have $\lim_{n\to\infty} a_n = 0$.

Then also $\lim_{n\to\infty} a_{n+1} = 0$.

Hence $\lim_{n\to\infty} (a_n + a_{n+1}) = 0 + 0 = 0$.

* Whenever the series $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0.$$

Is the converse always true?
NO. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

This **diverges** but $\lim_{n\to\infty} \frac{1}{n} = 0$.

<u>Øving 7, 5:</u> Does $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$ converge or not?

<u>ANSWER</u>

The sequence of partial sums is

$$S_N = \sum_{n=1}^{N} \log\left(\frac{n+1}{n}\right)$$

$$= \sum_{n=1}^{N} \left[\log(n+1) - \log n\right]$$

$$= (\cancel{\log 2} - \log 1) + (\cancel{\log 3} - \cancel{\log 2}) + \dots$$
$$\dots + (\log(N+1) - \cancel{\log N})$$

$$= \log(N+1).$$

Thus $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \log(N+1) = +\infty.$

The series $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$ diverges.

~~Øving 8, 4~~: Find $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sqrt{\frac{k}{n}}$.

⌈Here we need to use Proposition 5.7 (or Corollary 5.8): this states that whenever $f:[a,b]\to\mathbb{R}$ is Riemann-int. then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n}\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right).$$

When $f:[0,1]\to\mathbb{R}$ (ie $a=0$, $b=1$) this is

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

⌊

ANSWER

Set $f(x)=\sqrt{x}$, $x\geqslant 0$.

Then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\sqrt{\frac{k}{n}} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}f\left(\frac{k}{n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

$$= \int_0^1 \sqrt{x}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}.$$

HOST 2017, OPPGAVE 8:

Find the limit
$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}.$$

ANSWER

$$\sum_{k=1}^{n} \frac{n}{n^2+k^2} = \sum_{k=1}^{n} \frac{n}{n^2\left(1+\frac{k^2}{n^2}\right)}$$
$$= \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+\left(\frac{k}{n}\right)^2}$$
$$= \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right), \text{ where } f(x)=\frac{1}{1+x^2}$$

Therefore
$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \int_0^1 \frac{dx}{1+x^2}$$
$$= \Big[\arctan x\Big]_0^1 = \frac{\pi}{4}.$$

Øving 10, 1: Evaluate the following improper integrals, or prove they diverge.

(i) $\int_0^1 \ln x \, dx$

$$\int_u^1 \ln x \, dx = \int_u^1 (x)' \ln x \, dx$$
$$= [x \ln x]_u^1 - \int_u^1 x (\ln x)' dx$$
$$= -u \ln u - \int_u^1 1 \, dx$$
$$= -u \ln u - 1 + u \quad , \text{ for } u > 0.$$

$$\lim_{u \to 0^+} \int_u^1 \ln x \, dx = \lim_{u \to 0^+} \left(-u \ln u - 1 + u \right) = -1.$$

Therefore $\int_0^1 \ln x \, dx = -1$.

(iv) $\int_e^\infty \frac{1}{x \ln x} dx$.

Set $u = \ln x \Rightarrow du = \frac{dx}{x}$
$x_1 = e \Rightarrow u_1 = 1$
$x_2 = T \Rightarrow u_2 = \ln T$

$$\int_e^T \frac{dx}{x \ln x} =$$
$$\int_1^{\ln T} \frac{du}{u} = [\ln|u|]_1^{\ln T} = \ln(\ln T) \to +\infty.$$

(ii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^{1/2} \frac{dx}{\sqrt{x(1-x)}} + \int_{1/2}^1 \frac{dx}{\sqrt{x(1-x)}}$.

We find an antidervatie of $f(x) = \dfrac{1}{\sqrt{x(1-x)}}$.

<u>1^{st} way</u>:

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx = \int \frac{dx}{\sqrt{x}\cdot\sqrt{1-x}}$$

Set $u = \sqrt{x} \Rightarrow du = \dfrac{dx}{2\sqrt{x}}$

$$= \int \frac{2\,du}{\sqrt{1-u^2}}$$

$$= 2\arcsin u + C$$
$$= 2\arcsin\sqrt{x} + C.$$

<u>2^{nd} way</u>: Set $x = \sin^2 u$
$dx = 2\sin u\cos u\,du$

$$\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{2\sin u\cos u\,du}{\sqrt{\sin^2 u\cdot\cos^2 u}} = 2u + C$$

$$= 2\arcsin\sqrt{x} + C.$$

$$\int_0^{1/2} \frac{dx}{\sqrt{x(1-x)}} = \lim_{t\to 0^+} \int_t^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}}$$

$$= \lim_{t\to 0^+} \left[2\arcsin\sqrt{x}\right]_t^{\frac{1}{2}}$$

$$= \lim_{t\to 0^+} \left[2\arcsin\left(\frac{\sqrt{2}}{2}\right) - 2\arcsin\sqrt{t}\right]$$

$$= \frac{\pi}{2} - 2\arcsin 0 = \frac{\pi}{2}.$$

Similarly $\displaystyle\int_{1/2}^{1} \frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{2}.$

Therefore

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \pi.$$

(iii) $\int_0^{\pi/2} \tan x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$x_1 = 0 \Rightarrow u_1 = 1$

$x_2 = T \Rightarrow u_2 = \cos T$

$$\int_0^T \tan x \, dx = \int_0^T \frac{\sin x}{\cos x} \, dx$$

$$= -\int_1^{\cos T} \frac{du}{u}$$

$$= -\left[\ln|u|\right]_1^{\cos T}$$

$$= -\ln|\cos T|$$

so $\lim\limits_{T \to \frac{\pi}{2}^-} \int_0^T \tan x \, dx = \lim\limits_{T \to \frac{\pi}{2}^-} \left[-\ln|\cos T|\right]$

$$= \lim_{u \to 0^+} \left(-\ln|u|\right) = +\infty.$$

Hence $\int_0^{\pi/2} \tan x \, dx$ diverges.

$$\int_1^e \frac{(\ln x)^2 - \ln x}{x}\, dx =$$

$u = \ln x$
$du = \frac{dx}{x}$

$$= \int_0^1 (u^2 - u)\, du$$

$x_1 = 1 \Rightarrow u_1 = 0$
$x_2 = e \Rightarrow u_2 = 1$

$$= \left[\frac{u^3}{3} - \frac{u^2}{2} \right]_0^1 = -\frac{1}{6}.$$

(V) $$\int_{-\infty}^{+\infty} \frac{x\, dx}{1+x^4} = \int_{-\infty}^{0} \frac{x\, dx}{1+x^4} + \int_{0}^{+\infty} \frac{x\, dx}{1+x^4}$$

$$\int_0^T \frac{x}{1+x^4}\, dx =$$

Set $u = x^2$
$du = 2x\, dx$
$x_1 = 0 \Rightarrow u_1 = 0$
$x_2 = T \Rightarrow u_2 = T^2$

$$= \int_0^{T^2} \frac{1}{2} \frac{du}{1+u^2}$$

$$= \frac{1}{2} \int_0^{T^2} \frac{du}{1+u^2} = \frac{1}{2} \arctan T^2 \xrightarrow{T \to +\infty} \frac{\pi}{4}.$$

Hence $$\int_0^{+\infty} \frac{x}{1+x^4}\, dx = \frac{\pi}{4}.$$

Also $$\int_{-\infty}^{0} \frac{x\, dx}{1+x^4} = -\frac{\pi}{4}.$$

$$\int_{-\infty}^{+\infty} \frac{x\,dx}{1+x^4} = \frac{\pi}{4} + \left(-\frac{\pi}{4}\right) = 0.$$

* In general, it is not true that

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{T \to +\infty} \int_{-T}^{T} f(x)\,dx.$$

For example, $f(x) = x$.

Then $\int_{-T}^{T} f(x)\,dx = 0$ for any $T > 0$

but $\int_{-\infty}^{+\infty} f(x)\,dx$ diverges,

** If we show that

- $\int_{-\infty}^{+\infty} f(x)\,dx$ converges
- f is an odd function,

then $\int_{-\infty}^{+\infty} f(x)\,dx = 0.$

Øving 7, 7: Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{3n+1}}$.

Recall the geometric series:

$$\sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda} \quad \text{for any } -1<\lambda<1.$$

$$\sum_{n=1}^{\infty} \frac{1}{2^{3n+1}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2^{3n}} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{8}\right)^n$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n - 1\right]$$

$$= \frac{1}{2}\cdot\frac{1}{1-\frac{1}{8}} - \frac{1}{2}$$

$$= \frac{1}{2}\cdot\frac{8}{7} - \frac{1}{2} = \frac{4}{7} - \frac{1}{2} =$$

$$= \frac{8}{14} - \frac{7}{14} = \frac{1}{14}. \quad \blacksquare$$

Question: Does $\displaystyle\int_1^{+\infty} \frac{x^4-2x^3+x^2-10x+1}{x^5+x^4+3x^2+5x-2}\,dx$

converge or not?

Set $f(x) = \dfrac{x^4-2x^3+x^2-10x+1}{x^5+x^4+3x^2+5x-2}$.

Then $\displaystyle\lim_{x\to+\infty} \frac{f(x)}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{x^5-2x^4+x^3-10x^2+x}{x^5+x^4+3x^3+5x-2}$

$= 1.$

This means that there exists some $a>0$ such that

$\dfrac{f(x)}{\frac{1}{x}} > \dfrac{1}{2}$ for all $x > a \Rightarrow$

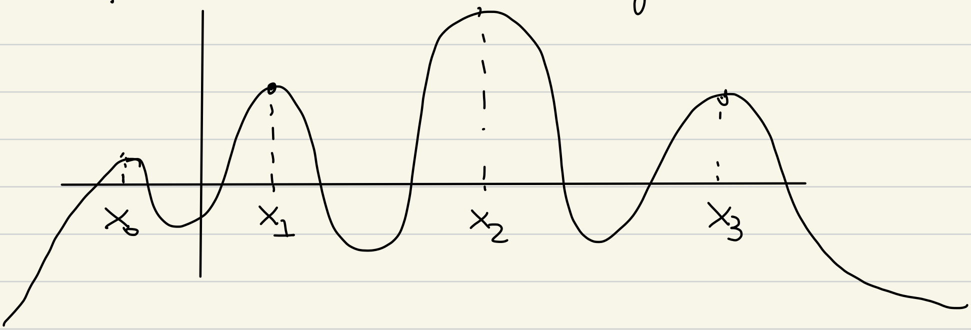
$f(x) > \dfrac{1}{2x}$ for all $x > a$.

We know that $\displaystyle\int_a^{\infty} \frac{1}{2x}\,dx$ diverges,

hence so does $\displaystyle\int_1^{+\infty} f(x)\,dx$.

• How do we determine local and global extreme points? (i.e. how do we tell if a local extremum is also a global one?)

→ Recall that the global extreme points are among the local ones.



The numbers $f(x_0), f(x_1), f(x_2), f(x_3)$ are all local maxima.
Among these, only $f(x_2)$ is also a global maximum.

Generally, in order to decide if a local max./min. is also a global one, we have to find the range of f.

Øving 11, 2: Find the general solution of:

(a) $y' - \frac{2y}{x} = x^2$

An antiderivative of $-\frac{2}{x}$ is

$$-2\ln|x| = \ln|x|^{-2} = \ln\frac{1}{x^2},$$

so we multiply both sides with

$$e^{\ln\left(\frac{1}{x^2}\right)} = \frac{1}{x^2}.$$

$$y' - \frac{2y}{x} = x^2 \Rightarrow \frac{1}{x^2}y' - \frac{2y}{x^3} = 1$$

$$\Rightarrow \left(\frac{1}{x^2}y\right)' = (x)'$$

$$\Rightarrow \frac{1}{x^2}y = x + c, \quad c \in \mathbb{R} \text{ const.}$$

$$\Rightarrow y = x^3 + cx^2, \quad c \in \mathbb{R} \text{ const.}$$

f) $xy' = y + x\cos^2\left(\frac{y}{x}\right)$. $\quad (*)$

Set $u = \frac{y}{x} \Rightarrow y = ux$

$\Rightarrow y' = u'x + u$.

$(*) \Rightarrow y' = \frac{y}{x} + \cos^2\left(\frac{y}{x}\right)$

$\Rightarrow u'x + \cancel{u} = \cancel{u} + \cos^2 u$

$\Rightarrow u'x = \cos^2 u$

$\Rightarrow \frac{du}{\cos^2 u} = \frac{dx}{x}$

$\Rightarrow \int \frac{du}{\cos^2 u} = \int \frac{dx}{x}$

$\Rightarrow \tan u = \ln|x| + c$

$\Rightarrow u = \arctan(\ln|x| + c)$

$\Rightarrow y = x\arctan(\ln|x| + c)$,

$c \in \mathbb{R}$ constant.

g) $y' = y^2(1-y)$

Observe that $y_0(x) = 0$, $y_1(x) = 1$ are solutions of the differential eq.

When $y \neq 0, 1$:

$$\frac{y'}{y^2(1-y)} = 1 \Rightarrow \int \frac{dy}{y^2(1-y)} = \int dx \qquad (*)$$

$$\frac{1}{y^2(1-y)} = \frac{(1-y)+y}{y^2(1-y)}$$

$$= \frac{1}{y^2} + \frac{1}{y(1-y)} = \frac{1}{y^2} + \frac{(1-y)+y}{y(1-y)}$$

$$= \frac{1}{y^2} + \frac{1}{y} + \frac{1}{1-y}.$$

Hence $\displaystyle\int \frac{dy}{y^2(1-y)} = \int \frac{dy}{y^2} + \int \frac{dy}{y} + \int \frac{dy}{1-y}$

$$= -\frac{1}{y} + \ln|y| - \ln|1-y|.$$

$(*) \Rightarrow -\frac{1}{y} + \ln|y| - \ln|1-y| = x + c.$

$\Rightarrow \ln\left|\frac{y}{(1-y)e^{y}}\right| = x + c.$

~~Øving 11, 4~~: Solve the integral equations:

(a) $y(x) = 2 + \int_0^x \frac{t}{y(t)}\,dt \Rightarrow$

$$y'(x) = \frac{x}{y(x)} \Rightarrow$$

$$2y(x)y'(x) = 2x \Rightarrow$$

$$[y^2(x)]' = (x^2)' \Rightarrow$$

$$y^2(x) = x^2 + c \quad (c \in \mathbb{R} \text{ is a constant}).$$

We can evaluate the constant c.
For $x = 0$:

$$y(0) = 2 + \int_0^0 \frac{t\,dt}{y(t)} = 2.$$

Therefore

$$y^2(0) = c \Rightarrow c = 4.$$

Hence $y^2(x) = x^2 + 4 \Rightarrow y(x) = \sqrt{x^2+4}$

(because $y(0) = 2 > 0$).

b) $y(x) = 3 + \int_0^x e^{-y(t)} dt$

$$y'(x) = e^{-y(x)} \Rightarrow$$
$$e^{y(x)} y'(x) = 1 \Rightarrow$$
$$\left[e^{y(x)} \right]' = (x)' \Rightarrow$$
$$e^{y(x)} = x + c \quad , \quad c \in \mathbb{R} \text{ constant.}$$

For $x=0$, $y(0)=3$.
Therefore $e^{y(0)} = 0 + c \Rightarrow c = e^3$, and

$$e^{y(x)} = x + e^3 \Rightarrow$$
$$y(x) = \ln(x + e^3).$$

Øving 11, 5: Decide if the integrals converge or diverge:

(b) $\displaystyle\int_1^{\infty} \frac{x+\sin x}{x^2+\sin x}\,dx$

Set $\displaystyle f(x) = \frac{x+\sin x}{x^2+\sin x}$, $x \geq 1$.

Then $\displaystyle\lim_{x\to+\infty} \frac{f(x)}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{x^2+x\sin x}{x^2+\sin x}$

$$= \lim_{x\to+\infty} \frac{1+\frac{\sin x}{x}}{1+\frac{\sin x}{x^2}} = 1.$$

Thus there exists some $a>1$ such that

$\displaystyle\frac{f(x)}{1/x} \geq \frac{1}{2}$ for all $x>a \Rightarrow$

$\displaystyle f(x) \geq \frac{1}{2x}$ for all $x>a$.

We know that $\displaystyle\int_1^{\infty}\frac{dx}{2x}$ diverges,

hence so does $\displaystyle\int_1^{\infty} f(x)\,dx$.

Alternatively, we could say that

$$x + \sin x \geq x - 1$$

and

$$x^2 + \sin x \leq x^2 + 1$$

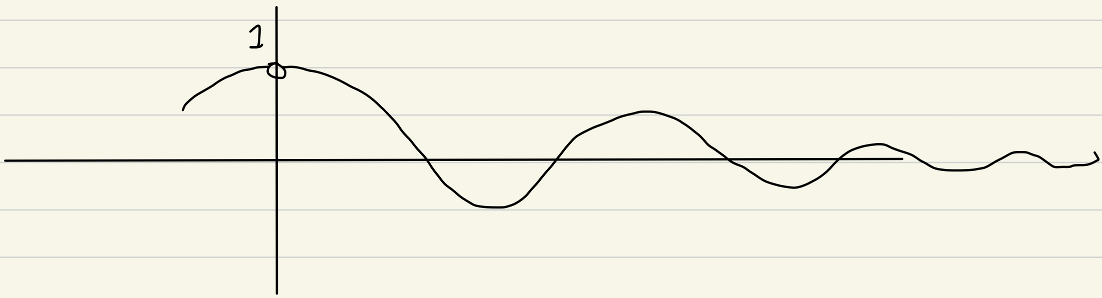
hence

$$\frac{x + \sin x}{x^2 + \sin x} \geq \frac{x-1}{x^2+1}$$

and then show that $\int_1^\infty \frac{x-1}{x^2+1} dx = \infty$.

$$f(x) = \frac{x + \sin x}{x^2 + \sin x} = \frac{1 + \frac{\sin x}{x}}{x + \frac{\sin x}{x}}$$

$$\int_1^\infty f(x) dx = ???$$



$$\frac{x^{\frac{3}{2}}}{x^2} = \frac{1}{x^{7/2}}$$

(d) $\displaystyle\int_2^{+\infty} \frac{x\sqrt{x}}{x^2-1}\,dx$

For $x \geqslant 2$,

$$x^2 - 1 < x^2 \Rightarrow$$

$$\frac{1}{x^2-1} > \frac{1}{x^2} \Rightarrow$$

$$\frac{x\sqrt{x}}{x^2-1} > \frac{1}{\sqrt{x}}$$

and $\displaystyle\int_2^{+\infty} \frac{dx}{\sqrt{x}} = \infty$,

hence also $\displaystyle\int_2^{\infty} \frac{x\sqrt{x}}{x^2-1}\,dx = +\infty.$

(c) $\displaystyle\int_0^{+\infty} \frac{dx}{xe^x} = \int_0^1 \frac{dx}{xe^x} + \int_1^{+\infty} \frac{dx}{xe^x}$.

We look at $\displaystyle\int_1^{\infty} \frac{dx}{xe^x}$.

For $x \geqslant 1$, $\displaystyle\frac{1}{xe^x} \leqslant \frac{1}{e^x} = e^{-x}$

and $\displaystyle\int_1^{\infty} e^{-x}\,dx < \infty$, hence also $\displaystyle\int_1^{\infty} \frac{dx}{xe^x} < \infty$.

We now examine $\displaystyle\int_0^1 \frac{dx}{xe^x}$.

For $0 < x \leqslant 1$, $e^x \leqslant e \Rightarrow$

$$\frac{1}{e^x} \geqslant \frac{1}{e} \Rightarrow \frac{1}{xe^x} \geqslant \frac{1}{ex}$$

and we know that $\displaystyle\int_0^1 \frac{1}{ex}\,dx = +\infty$

hence so does $\displaystyle\int_0^1 \frac{dx}{xe^x}$.

Therefore $\displaystyle\int_0^{+\infty} \frac{dx}{xe^x} = +\infty$.

We know: $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

<u>Øving 7, 1:</u> Show that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

- For all $k = 1, 2, \ldots$

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1 \Rightarrow$$

$$\sum_{k=1}^{n} \left[(k+1)^4 - k^4\right] = \sum_{k=1}^{n} (4k^3 + 6k^2 + 4k + 1) \Rightarrow$$

$$(n+1)^4 - 1 = 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + n \Rightarrow$$

$$4\sum_{k=1}^{n} k^3 + n(n+1)(2n+1) + 2n(n+1) + n = (n+1)^4 - 1$$

$$\Rightarrow \ldots$$

Øving 12, 3: Find $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{2n} (-1)^k k(k+1)$.

SOLUTION

First we find the sum

$$S_n := \sum_{k=1}^{2n} (-1)^k k(k+1)$$

$$= \sum_{\substack{1\le k\le 2n \\ k \text{ even}}} (-1)^k k(k+1) + \sum_{\substack{1\le k\le 2n \\ k \text{ odd}}} (-1)^k k(k+1)$$

$$= \sum_{\substack{1\le k\le 2n \\ k \text{ even}}} k(k+1) - \sum_{\substack{1\le k\le 2n \\ k \text{ odd}}} k(k+1)$$

(Now I observe that the even indices $1\le k\le 2n$ are precisely the numbers $2m$, $m=1,2,\dots,n$;

the odd indices $1\le k\le 2n$ are the numbers $2m-1$, $m=1,2,\dots,n$)

$$= \sum_{m=1}^{n} 2m(2m+1) - \sum_{m=1}^{n} (2m-1)\cdot 2m$$

$$= \sum_{m=1}^{n} 2m[(2m+1)-(2m-1)] = 4\sum_{m=1}^{n} m$$

$$= 4 \cdot \frac{n(n+1)}{2} = 2(n^2+n).$$

Now we find

$$\lim_{n\to\infty} \frac{S_n}{n^2} = \lim_{n\to\infty} \frac{2n^2+2n}{n^2} = 2.$$

<u>Øving 12, 9:</u> Find

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right).$$

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \quad \left(\text{where } f(x) = \frac{1}{1+x}\right)$$

$$= \int_0^1 \frac{1}{1+x}\,dx = \Big[\ln(1+x)\Big]_0^1 = \ln 2.$$

Øving 12, 4:

(a) $$\sum_{n=1}^{\infty} \frac{1}{n^2+7n+12}.$$

The seq. of partial sums is

$$S_N = \sum_{n=1}^{N} \frac{1}{n^2+7n+12} = \sum_{n=1}^{N} \frac{1}{(n+3)(n+4)}$$

$$= \sum_{n=1}^{N} \frac{(n+4)-(n+3)}{(n+3)(n+4)}$$

$$= \sum_{n=1}^{N} \left(\frac{1}{n+3} - \frac{1}{n+4} \right)$$

$$= \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{N+3} - \frac{1}{N+4}\right)$$

$$= \frac{1}{4} - \frac{1}{N+4}$$

so $\lim_{N\to\infty} S_N = \frac{1}{4}$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^2+7n+12} = \frac{1}{4}.$$

(b) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Recall the power series expansion

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}, \quad \text{for all } x \in \mathbb{R}.$$

Set $x=2$: $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$.

Recall :

1. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$$

2. $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R}$$

3. $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad x \in \mathbb{R}$$

4. $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

Example: $\sum_{n=1}^{\infty} \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} - 1 = e - 1.$

Øving 11, 6: Find the values of $p \in \mathbb{R}$ for which

$$\int_0^1 \frac{\sin x}{x^p}\, dx \quad \text{converges.}$$

ANSWER

We know that $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$.

(ie. $\sin x \sim x$, $x \to 0^+$)

therefore $\lim_{x \to 0^+} \frac{\frac{\sin x}{x^p}}{\frac{1}{x^{p-1}}} = 1$,

and we expect $\int_0^1 \frac{\sin x}{x^p}\, dx$ to have the same behaviour with

$$\int_0^1 \frac{1}{x^{p-1}}\, dx.$$

Therefore it will:

- converge, when $p - 1 < 1 \Rightarrow p < 2$
- diverge, when $p - 1 \geqslant 1 \Rightarrow p \geqslant 2$.