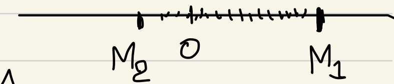
PROPOSITION 2.1: The set $A \subseteq \mathbb{R}$ is bounded if and only if there exists some $M > 0$ such that

$$|a| \leq M \quad \text{for all } a \in A.$$

PROOF

$\Rightarrow$: Assume A is bounded.
There exist $M_1, M_2 \in \mathbb{R}$ such that



$$M_2 \leq a \leq M_1, \quad \text{for all } a \in A.$$

Set $M = \max\{|M_1|, |M_2|\}$, then:

$$\left.\begin{array}{l} a \leq M_1 \leq |M_1| \leq M \\ a \geq M_2 \geq -|M_2| \geq -M \end{array}\right\} \Rightarrow -M \leq a \leq M \quad \forall a \in A$$

so we have shown that $|a| \leq M \quad \forall a \in A$.

$\Leftarrow$ If $|a| \leq M \quad \forall a \in A$, then
$-M \leq a \leq M$ for all $a \in A$,
so A is both bounded from above and below, so A is bounded. $\blacksquare$

Examples: The following sets are bounded:

- $[0,2)$, an upper bound is 2 - and also 3 will do.
- $[-1,0] \cup \{2,3\}$, an upper bound is 3, a lower is -2.
- $\{-2,-1,0,1,2,3\}$, 5 is an upper bound, -4 is a lower bound.

The following sets are not bounded:
$[0,\infty)$, $[-1,0) \cup (1,\infty)$, $(-\infty,5)$.

Let $A \subseteq \mathbb{R}$. We say that M is the <u>maximum</u> of A and we write $M = \max A$, if:

(i) $M \in A$

(ii) $a \leq M$ for all $a \in A$.

We say that m is the <u>minimum</u> of A and we write $m = \min A$ if:

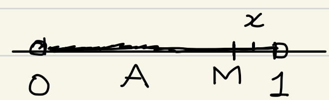
(i) $m \in A$

(ii) $m \leq a$ for all $a \in A$.

E.g. if $A = \{1, 2, 4\}$, $\max A = 4$ and $\min A = 1$
if $B = [0, 1]$, $\max B = 1$ and $\min B = 0$.

Not all sets have maximum and minimum, for example $A = (1, \infty)$, $B = (-\infty, 0]$.
Even bounded sets might not have maximum and minimum, take for example the set $A = (0, 1)$.

Suppose A has a maximum, $M = \max A$.
Then $M \in A \Rightarrow M \in (0, 1) \Rightarrow M < 1$.
Set $x = \frac{1+M}{2}$. Then



$$0 < x = \frac{1+M}{2} < \frac{1+1}{2} = 1$$

so $x \in A$. Also $x = \frac{1+M}{2} > \frac{M+M}{2} = M$;
a contradiction.

$\forall$: for all $\exists$: there exists

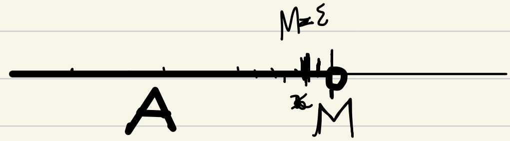
Let $A \subseteq \mathbb{R}$. We say $M \in \mathbb{R}$ is the supremum (or least upper bound) of A and we write $M = \sup A$, if

- M is an upper bound of A, and
- for any upper bound M' of A we have $M \leq M'$.

This definition has an equivalent formulation.
$M = \sup A$ if and only if :

(i) $x \leq M$ for all $x \in A$, and

(ii) $\forall \varepsilon > 0$ $\exists\, x = x_\varepsilon \in A$ such that $M - \varepsilon < x$.



Example: $\sup(-\infty, 1) = 1$. Indeed:

(i) $x < 1$ for all $x \in (-\infty, 1)$.

(ii) Take an arbitrary $\varepsilon > 0$.
Set $x = 1 - \frac{\varepsilon}{2}$.

Then $1 - \varepsilon < x < 1$.
We have shown that for any $\varepsilon > 0$ there exists $x = x_\varepsilon \in (-\infty, 1)$ such that $1 - \varepsilon < x$.

Thus $\sup(-\infty, 1) = 1$.

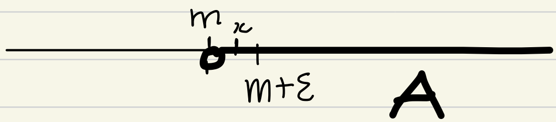
We say $m \in \mathbb{R}$ is the <u>infimum</u> of $A \subseteq \mathbb{R}$ (or greatest lower bound) and we write $m = \inf A$ if

- m is a lower bound of A, and
- for any lower bound m' of A, we have $m' \leq m$.

Equivalently: $m = \inf A$ if and only if:

(i) $m \leq x$ for all $x \in A$

(ii) $\forall \varepsilon > 0$ $\exists x = x_\varepsilon \in A$ such that $x < m + \varepsilon$.



For example, $\inf(0,\infty) = 0$. Indeed:

(i) $0 < x$ for all $x \in (0,\infty)$.

(ii) For any $\varepsilon > 0$, the element $x = \frac{\varepsilon}{2}$ is in $(0,\infty)$ and
$$0 < x < 0 + \varepsilon.$$

* If a set is not bounded from above, we sometimes write $\sup A = \infty$.

We have seen that there exist subsets of $\mathbb{R}$ that do not have max and min (even some bounded sets). But what happens with the supremum and the infimum?

The answer to this follows from the definition / construction of the set $\mathbb{R}$ from the rationals.

COMPLETENESS AXIOM: Every bounded set $A \subseteq \mathbb{R}$ has a supremum (and an infimum) in $\mathbb{R}$.

Using the completeness Axiom we can prove properties which otherwise seem profound.

THEOREM 2.2 (Archimedean Property):
The set $\mathbb{N} \subseteq \mathbb{R}$ is not bounded.

PROOF
Assume $\mathbb{N} \subseteq \mathbb{R}$ is bounded.
By the Completeness Axiom, there exists

$$M = \sup \mathbb{N} \in \mathbb{R}.$$

Now $M-1$ is not an upper bound of $\mathbb{N}$, so there exists $n \in \mathbb{N}$ such that

$$M-1 < n \implies n+1 > M.$$

But $n+1 \in \mathbb{N}$; a contradiction. ∎

COROLLARY 2.3 (Archimedean Property):
For any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

PROOF
Consider the number $\frac{1}{\varepsilon} > 0$.
Since $\mathbb{N}$ is not bounded, there exists $n \in \mathbb{N}$ such that

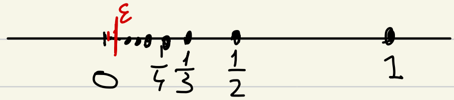
$$n > \frac{1}{\varepsilon} \iff \frac{1}{n} < \varepsilon.$$

∎

Consider the set

$$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}.$$

Now we can show that $\inf A = 0$.



- $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$.

- Take an arbitrary $\varepsilon > 0$.
  By the Archimedean Property (Corollary 2.3) there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$, i.e. $0 < \frac{1}{n} < 0 + \varepsilon$.

So $\inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0$.

(Here, of course, $\sup\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 1$).

- SEQUENCES AND LIMITS OF SEQUENCES.

A <u>sequence</u> is a function with domain of definition equal to $\mathbb{N}$.

Instead of writing $a : \mathbb{N} \to \mathbb{R}$, $n \mapsto a(n)$ for a sequence of real numbers, we denote by a_n for the n-th term of the sequence (rather than $a(n)$), and the sequence itself is written as

$$(a_n)_{n=1}^{\infty}\,, \quad (a_n)_{n\geq 1}\,, \quad (a_n)_{n\in\mathbb{N}} \quad \text{or} \quad \{a_n\}_{n\in\mathbb{N}}.$$

The sequence itself is a function, while the set $\underline{\{a_n : n\in\mathbb{N}\}}$ of all terms of the sequence is a set of real numbers. However we often call both objects "the sequence $(a_n)_{n=1}^{\infty}$".

E.g. When $a_n = \frac{1}{n}$, $n=1,2,\ldots$

we have the sequence

$$\left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{3}, \cdots\right)$$

When $b_n = (-1)^n = \cos(\pi n)$, $n=1,2,\ldots$ then the sequence $(b_n)_{n=1}^{\infty}$ is the sequence

$$((-1)^n)_{n=1}^{\infty} = (-1, +1, -1, +1, \ldots).$$

Although we often have an explicit formula for a sequence, e.g.

$$a_n = 2^n + 1 \quad \forall n\in\mathbb{N}$$

sometimes a sequence $(a_n)_{n=1}^{\infty}$ might be defined through a <u>recursive relation</u>, for example:

$$\begin{cases} a_1 = 1, \\ a_{n+1} = 2a_n + 1, \quad n=1,2,3,\ldots \end{cases}$$

Here we can find: $a_2 = 2a_1 + 1 = 3$, $a_3 = 2a_2 + 1 = 7$, ...

The most common example of a recursively defined sequence is the sequence $(F_n)_{n=1}^{\infty}$ of fibonacci numbers:

$$\begin{cases} F_1 = F_2 = 1 \\ F_{n+2} = F_{n+1} + F_n, \quad n = 1, 2, \dots \end{cases}$$

We can calculate $F_3 = 2$, $F_4 = 3$, etc.

We say that the sequence $(a_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ is:

(i) <u>bounded above</u> by M, if

$$a_n \leq M \quad \text{for all } n \geq 1.$$

(ii) <u>bounded below</u> by m, if

$$a_n \geq m \quad \text{for all } n \geq 1.$$

(iii) <u>bounded</u>, if it is both bounded from above and below by some $M, m \in \mathbb{R}$ respectively.

(iv) <u>increasing</u> if $a_{n+1} \geq a_n$ for all $n \geq 1$, and <u>strictly increasing</u> if $a_{n+1} > a_n$ for all $n \geq 1$.

(v) <u>decreasing</u> if $a_{n+1} \leq a_n$ for all $n \geq 1$, and <u>strictly decreasing</u> if $a_{n+1} < a_n$ for all $n \geq 1$.

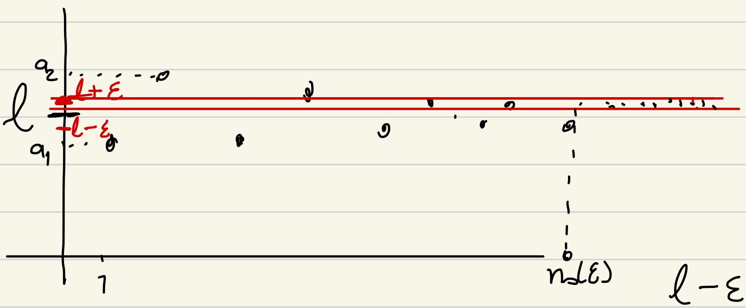
A sequence $(a_n)_{n=1}^{\infty}$ is called <u>constant</u> if $a_{n+1} = a_n$ for all $n \in \mathbb{N}$ (e.g. $a_n = 0 \ \forall n \in \mathbb{N}$).

We know that the limit of a sequence $(a_n)_{n=1}^{\infty}$ is equal to the number ℓ if "finally all terms of the sequence become arbitrarily close to ℓ".

We want to give a formal definition for this.

We say that the sequence $(a_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ <u>converges</u> to the real number ℓ and we write $\lim_{n\to\infty} a_n = \ell$, if:

$$\forall \varepsilon > 0 \quad \exists\, n_0 = n_0(\varepsilon) \in \mathbb{N} \text{ such that: for all } n \geq n_0,\ |a_n - \ell| < \varepsilon.$$



However small I choose $\varepsilon > 0$, there will exist some $n_0 \in \mathbb{N}$ such that $\ell - \varepsilon < a_n < \ell + \varepsilon \quad \forall n \geq n_0$.

E.g. we can show $\lim_{n\to\infty} \frac{1}{n} = 0$.