Oppg. 3

3. $f_T(t) = \lambda e^{-\lambda t}$

$t > 0$

$\lambda > 0$

$\lambda = 0.8$

(a) $E[T] = \frac{1}{\lambda}$

$$= \frac{1}{0.8}$$

$f_T(t) = \frac{1}{2}$

$0.8 e^{-0.8t} = \frac{1}{2}$

$e^{-0.8t} = \frac{5}{8}$

$-0.8t = \ln\left(\frac{5}{8}\right)$

$t = -\frac{\ln\left(\frac{5}{8}\right)}{0.8}$

$$\approx 0.5875$$

$P(T > 1.2) = 1 - P(T \leq 1.2)$

$= 1 - \left(1 - e^{-0.8 \cdot 1.2}\right)$

$= e^{-0.96}$

$$\approx 0.3829$$

(b) $M_T(t) = \int_0^\infty e^{ty} \cdot 0.8 e^{-0.8y} dy$

$= \int_0^\infty 0.8 e^{-(0.8-t)y} \, dy$

$= \begin{bmatrix} u = (0.8 - t)y \\ dy = \frac{du}{0.8 - t} \end{bmatrix}$

$$\left.dy = \frac{du}{0.8-t}\right]$$

$$= \int_0^\infty 0.8 e^{-u} \frac{du}{0.8-t}$$

$$= \frac{0.8}{0.8-t} \int_0^\infty e^{-u}\, du$$

$$= \frac{0.8}{0.8-t} \left[-e^{-u} \right]_0^\infty$$

$$\lim_{u \to \infty} \left(-e^{-u} \right) = 0$$

$$\Rightarrow M_T(t) = \frac{0.8}{0.8-t} (0+1)$$

$$= \frac{0.8}{0.8-t}$$

$$Var[T] = E[T^2] - (E[T])^2$$

$$E[T^2] = M_T''(0)$$

$$M_T''(t) = \left(\frac{0.8}{0.8-t} \right)''$$

$$= \left(\frac{0.8}{(0.8-t)^2} \right)'$$

$$= \frac{2 \cdot 0.8}{(0.8-t)^3}$$

$$M_T''(0) = \frac{2 \cdot 0.8}{0.8^3}$$

$$= \frac{2}{0.8^2}$$

$$\Rightarrow Var[T] = \frac{2}{0.8^2} - \left(\frac{1}{0.8} \right)^2$$

$$= \frac{1}{0.8^2}$$

(c) $T_1, T_2 \sim Exp(0.8)$

$$Y = T_1 + T_2$$

T_i uavhengig $\Rightarrow Y \sim \text{gamma}(0.8, 2)$

$$P(Y \geq 2.4) = \int_{2.4}^\infty \frac{0.8^2}{\Gamma(2)} t e^{-0.8t}\, dt$$

$$= 0.8^2 \int_{2.4}^\infty t e^{-0.8t}\, dt$$

$$= \left[u = t \quad\quad v' = \frac{1}{\ldots} e^{-0.8t} \right]$$

$$= \begin{bmatrix} u=t & v=-\frac{1}{0.8}e^{-0.8t} \\ du=dt & dv=e^{-0.8t}dt \end{bmatrix}$$

$$=0.8^2\left(\left[t\cdot\left(-\frac{1}{0.8}e^{-0.8t}\right)\right]_{2.4}^{\infty}-\int_{2.4}^{\infty}-\frac{1}{0.8}e^{-0.8t}dt\right)$$

$$=0.8\left(\left[-te^{-0.8t}\right]_{2.4}^{\infty}+\int_{2.4}^{\infty}e^{-0.8t}dt\right)$$

$$=0.8\left(\left[-te^{-0.8t}\right]_{2.4}^{\infty}+\left[-\frac{e^{-0.8t}}{0.8}\right]_{2.4}^{\infty}\right)$$

$$\lim_{t\to\infty}\left(-te^{-0.8t}\right)=\lim_{t\to\infty}\left(-\frac{t}{e^{0.8t}}\right)$$

$$=\lim_{t\to\infty}\left(-\frac{1}{0.8e^{0.8t}}\right)$$

$$=0$$

$$\Rightarrow P(Y\geq 2.4)=0.8(0+0.3519+0.1833)$$

$$\simeq 0.4282$$

(d) $P(\min(T_1,T_2)<0.4)=1-\prod_{i=1}^{2}(1-F_{T_i}(0.4))$

$$F_{T_i}(t)=\int_0^t 0.8^2 ye^{-0.8y}dy$$

$$=0.8\int_0^t ye^{-0.8y}dy$$

$$=0.8\left(\left[y\cdot\left(-\frac{1}{0.8}e^{-0.8y}\right)\right]_0^t+\left[-\frac{1}{0.8}e^{-0.8y}\right]_0^t\right)$$

$$=\left[ye^{-0.8y}\right]_0^t+\left[-e^{-0.8y}\right]_0^t$$

$$=te^{-0.8t}-e^{-0.8t}+1$$

$$F_{T_i}(0.4)\simeq 0.5643$$

$$\Rightarrow P(\min(T_1,T_2)<0.4)=1-(1-0.5643)^2$$

$$\simeq 0.8102$$

$P(T_2<\frac{T_1}{2})$