So there exists $n_2 > n_1$ with $x_{n_2} \geqslant x_{n_1}$.

In turn x_{n_2} is not a peak, so there exists $n_3 > n_2$ with $x_{n_3} \geqslant x_{n_2}$.

So we can recursively define a subsequence $(x_{n_k})_{k=1}^{\infty}$ which is increasing. ∎

THEOREM 2.16 (Bolzano-Weierstrass):
Every sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ which is bounded has a convergent subsequence.

PROOF
By Lemma 2.15, $(x_n)_{n=1}^{\infty}$ has a monotone subsequence, call it $(y_n)_{n=1}^{\infty}$.
Then $(y_n)_{n=1}^{\infty}$ is also bounded - because so is $(x_n)_{n=1}^{\infty}$.
Thus $(y_n)_{n=1}^{\infty}$ is bounded and monotone, so by Theorem 2.7 $(y_n)_{n=1}^{\infty}$ converges. ∎

LEMMA 2.17: Suppose $f:[a,b]\to\mathbb{R}$ is continuous. Then f is bounded.

PROOF

Assume f is not bounded.
Then for each $n=1,2,\dots$ there exists some $x_n \in [a,b]$ such that

$$f(x_n) > n.$$

The sequence $(x_n)_{n=1}^{\infty} \subseteq [a,b]$ is bounded so by Theorem 2.16 it has a convergent subsequence - call it $(x_{k_n})_{n=1}^{\infty}$ - and let

$$\ell = \lim_{n\to\infty} x_{k_n}.$$

Then $a \leqslant x_{k_n} \leqslant b \Rightarrow a \leqslant \ell \leqslant b$.
Since f is continuous, therefore

$$f(\ell) = \lim_{n\to\infty} f(x_{k_n}) \geqslant k_n \quad \text{for all } n=1,2,\dots$$

A contradiction, so f is bounded. ■

Now we can finish the proof of the Heine-Borel Theorem.

If $f:[a,b] \to \mathbb{R}$ is continuous, by Lemma 2.17 it is bounded, so there exist

$$M = \sup\{f(x) : a \leq x \leq b\} \in \mathbb{R}$$

and

$$m = \inf\{f(x) : a \leq x \leq b\} \in \mathbb{R}.$$

We need to prove there exists $x_1 \in [a,b]$ such that $f(x_1) = M$.
Assume this is not true.
Then $f(x) < M$ for all $x \in [a,b]$, so the function

$$g:[a,b] \to \mathbb{R}, \quad g(x) = \frac{1}{M-f(x)}$$

is well-defined and continuous.
By Lemma 2.17 g must be bounded (g is continuous on $[a,b]$), so there exists $A > 0$ with

$$g(x) < A \quad \text{for all } x \in [a,b] \Rightarrow$$

$$M - f(x) > \frac{1}{A} \quad \text{for all } x \in [a,b] \Rightarrow$$

$$f(x) < M - \frac{1}{A} \quad \text{for all } x \in [a,b].$$

A contradiction, because $M = \sup\{f(x) : x \in [a,b]\}$.
So there exists $x_1 \in [a,b]$ s.t. $f(x_1) = M$.
Similarly we show there exists $x_2 \in [a,b]$ such that $f(x_2) = m$. ∎

· TRIGONOMETRIC & HYPERBOLIC FUNCTIONS

$$\sin\theta = y$$

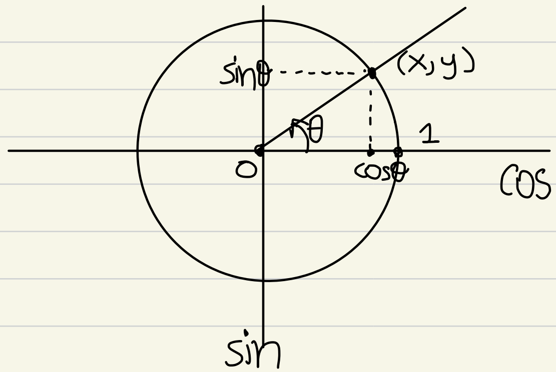
$$\cos\theta = x$$

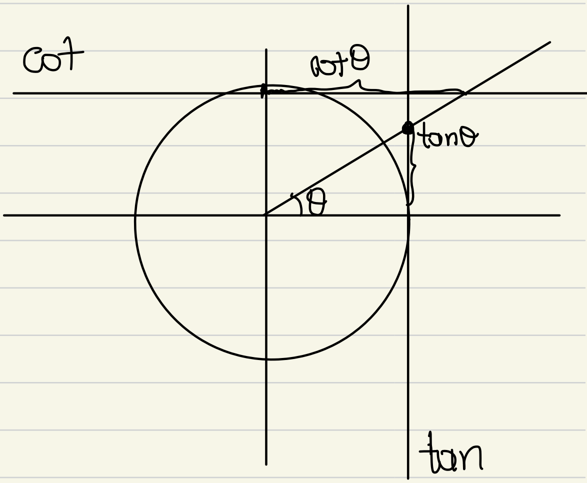
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

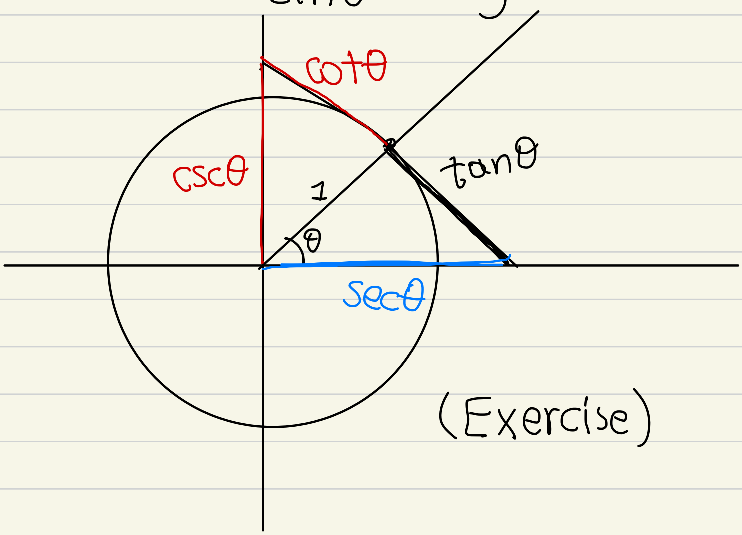
$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}$$

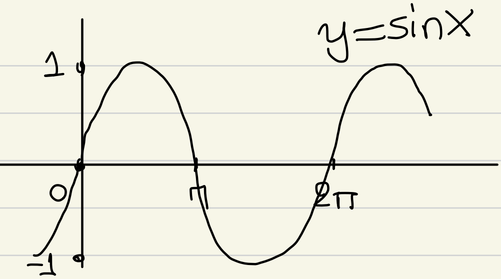
$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{y}$$

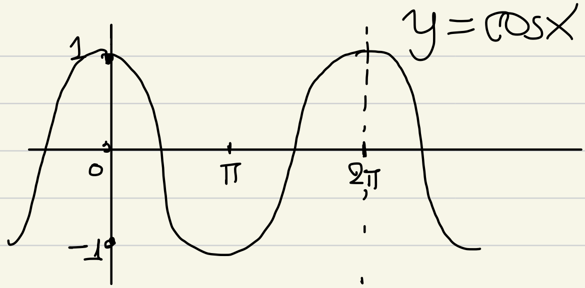


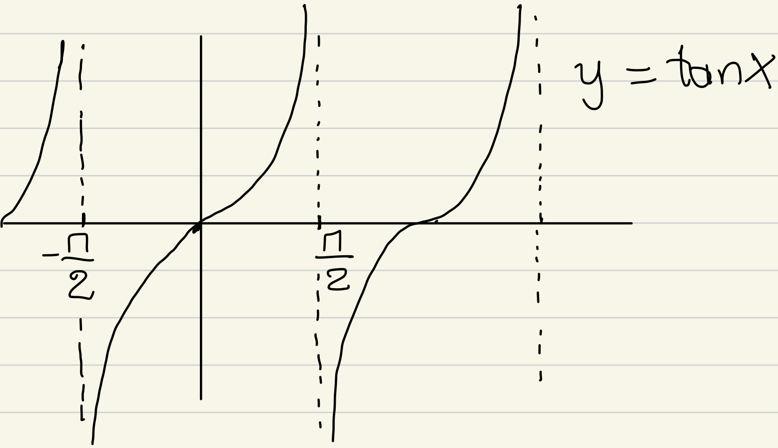


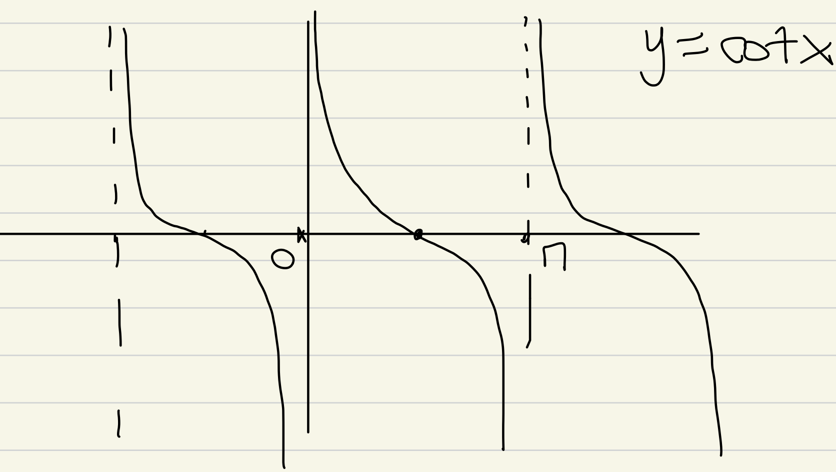


(Exercise)


y=sinx
1
0
-1
π
2π


y=cosx
1
0
-1
π
2π


y = tanx
-π/2
π/2


y=cotx
0
π

Important Identities:

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \frac{1}{\cos^2 x}$$

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$
$$\cot(-x) = -\cot x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$
$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$$
$$\sin(x-y) = \sin x \cdot \cos y - \cos x \cdot \sin y$$
$$\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$$
$$\cos(x-y) = \cos x \cdot \cos y + \sin x \cdot \sin y$$

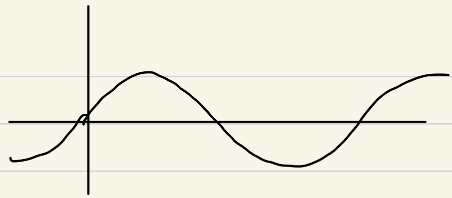
$$\sin 2x = 2 \sin x \cos x$$

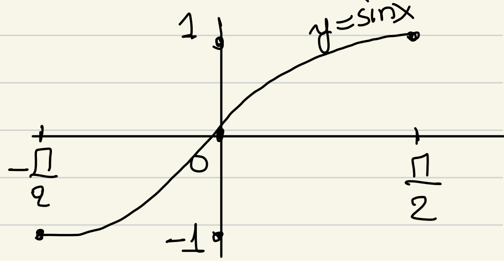
$$\cos 2x = \begin{cases} \cos^2 x - \sin^2 x \\ 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \end{cases}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$ is clearly not invertible – e.g. $f(0) = f(\pi)$.





Consider the function $f: [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$, $f(x) = \sin x$ (i.e. the "restriction" of the sine on $[-\pi/2, \pi/2]$).

This function is 1–1 (and thus invertible) and its range $f([-\pi/2, \pi/2]) = [-1, 1]$ so it has an inverse

$$f^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$

which is called arcsin.

The function $\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ maps every number $x \in [-1, 1]$ to the unique "angle" $\theta \in [-\pi/2, \pi/2]$ such that $\sin\theta = x$.

I.e.

$$\boxed{\sin\theta = x \iff \arcsin x = \theta}$$

E.g. $\arcsin 0 = 0$ because $\sin 0 = 0$.

$$\sin\frac{\pi}{6} = \frac{1}{2} \implies \arcsin\frac{1}{2} = \frac{\pi}{6}$$

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \iff \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}.$$

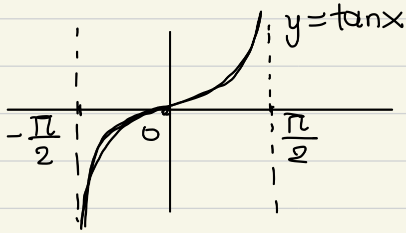
Similarly the function $\cos : [0, \pi] \to [-1, 1]$ is invertible, and its inverse is a function

$$\arccos : [-1, 1] \to [0, \pi]$$

with

$$\boxed{\arccos x = \theta \iff x = \cos\theta}$$

E.g. $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \Rightarrow \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

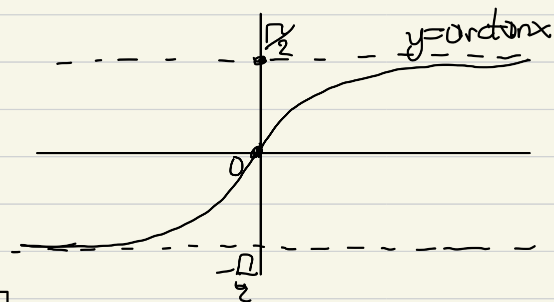


The function

$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is 1-1,

and its inverse function is

$\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



with

$$\boxed{y = \arctan x \iff \tan y = x}$$

$$\boxed{\begin{aligned} &\arctan 0 = 0 \\ &\lim_{x \to +\infty} (\arctan x) = \frac{\pi}{2} \\ &\lim_{x \to -\infty} (\arctan x) = -\frac{\pi}{2} \end{aligned}}$$

Similarly for $\cot : (0,\pi) \to \mathbb{R}$ there exists the inverse function

$$\operatorname{arccot} : \mathbb{R} \to (0,\pi)$$

such that

$$\boxed{y = \operatorname{arccot} x \iff \cot y = x}$$

The inverse trigonometric functions are sometimes denoted as

$$\sin^{-1}, \quad \cos^{-1}, \quad \tan^{-1}, \quad \cot^{-1}.$$

E.g. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

$\tan^{-1}(-1) = \arctan(-1) = -\frac{\pi}{4}$.

(!) Be careful, not to confuse $\sin^{-1} x$ with $(\sin x)^{-1} = \frac{1}{\sin x}$.

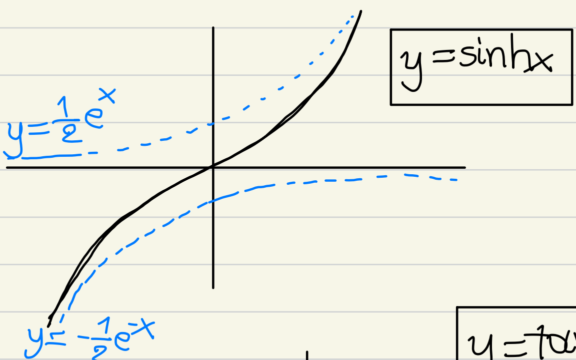
We now define the hyperbolic sinh, cosh, tanh, coth.

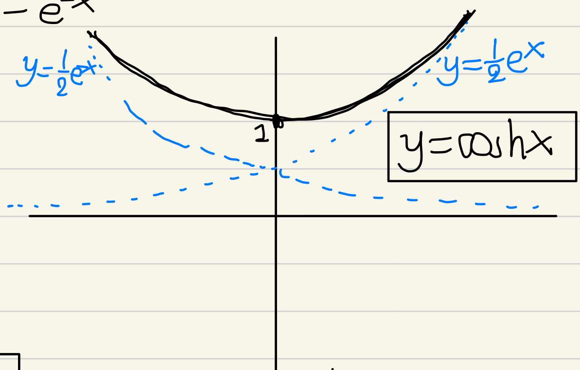
$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad x \in \mathbb{R}$$

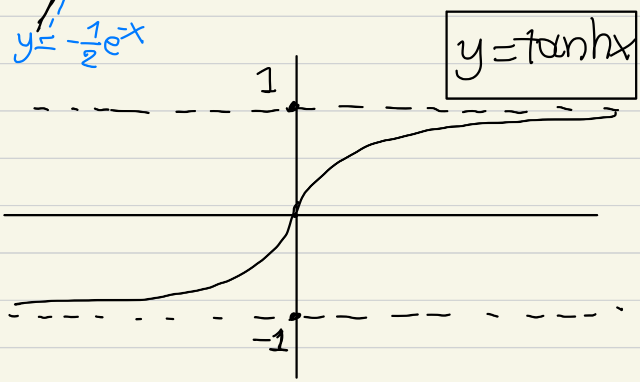
$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad x \in \mathbb{R}$$

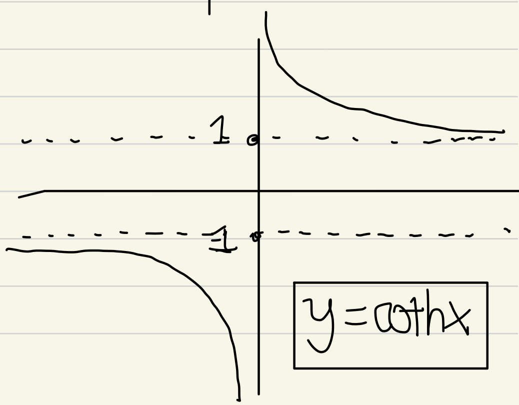
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad x \in \mathbb{R}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \qquad x \neq 0.$$









Important Identities:

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

$$\boxed{\begin{aligned} \sinh(-x) &= -\sinh x \\ \cosh(-x) &= \cosh x \end{aligned}}$$

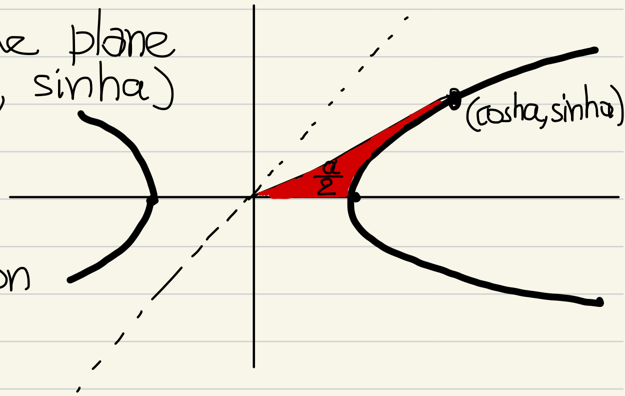
$$\boxed{\begin{aligned} \sinh 2x &= 2\sinh x \cdot \cosh x \\ \cosh 2x &= \begin{cases} \sinh^2 x + \cosh^2 x \\ 2\sinh^2 x + 1 \\ 2\cosh^2 x - 1 \end{cases} \end{aligned}}$$

Suppose a point in the plane has coordinates $(\cosh a, \sinh a)$ where $a \in \mathbb{R}$.

Since

$$\cosh^2 a - \sinh^2 a = 1$$

the point will move on the right branch of the hyperbola $x^2 - y^2 = 1$.



We may also define inverse hyperbolic functions:

$$\sinh^{-1} : \mathbb{R} \to \mathbb{R}, \qquad \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1} : [1, \infty) \to \mathbb{R}, \qquad \cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1} : (-1, 1) \to \mathbb{R}, \qquad \tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).$$

The inverse hyperbolic functions are sometimes written as

arsinh, arcosh, artanh, etc

– Here "ar–" stands for area and not for arc!

3. DERIVATIVES

Suppose $I \subseteq \mathbb{R}$ is an open interval, $f: I \to \mathbb{R}$ is a function and $x_0 \in I$. We say that f is <u>differentiable</u> at x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \qquad \left(= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \right)$$

exists and is a real number.