6. DIFFERENTIAL EQUATIONS

A differential equation is an equation of the form

$$F(x, y, y', \dots, y^{(n)}) = 0 \qquad (*)$$

where $y = y(x)$ and $n \geqslant 1$.

The maximum value of $n \geqslant 1$ appearing in $(*)$ is called the order of the D.E.

E.g.
- $y'' + e^x (y')^2 + y = x^2$ has order 2.
- $y^{(3)} = 2y$ has order 3.

The differential equation $(*)$ is called linear if $F(x, y, \dots, y^{(n)})$ is linear on the variables $y, y', \dots, y^{(n)}$.

The general form of a linear D.E. of order n is

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_1(x)\, y' + p_0(x)\, y = g(x).$$

If $g(x) \equiv 0$ the D.E. is called homogeneous. Otherwise it called inhomogenous.

E.g.:
- $y'' + x y' + y = 0$ is a linear, 2^{nd} order homogeneous D.E.
- $y^{(3)} + x^2 y'' + y = e^x$ is a linear, 3^{rd} order inhomogeneous D.E.

An initial value problem

$$\begin{cases} F(x, y, \ldots, y^{(n)}) = 0 \\ y(x_0) = y_0, \ \ldots, \ y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

is a D.E. together with initial conditions.

A D.E. does not in general admit a unique solution. The set of all solutions of a D.E. is called the <u>general solution</u> of the D.E.

* When seeking solutions for a D.E. we find solutions defined on some interval I, without necessarily specifying it.

From now on we only deal with D.E. of order 1.

A typical example:

$$\begin{aligned} y' = y \ &\Rightarrow \ y' - y = 0 \\ &\Rightarrow \ e^{-x} y' - e^{-x} y = 0 \\ &\Rightarrow \ (y \cdot e^{-x})' = 0 \\ &\Rightarrow \ y \cdot e^{-x} = c, \quad c \in \mathbb{R} \text{ const.} \end{aligned}$$

$$\Rightarrow \quad y = ce^{x}, \quad c \in \mathbb{R} \text{ const.}$$

$y = ce^{x}$: the general sol. of the D.E.

- $\boxed{\begin{array}{c}\text{LINEAR D.E. OF ORDER 1} \\ y' + p(x)\,y = q(x)\end{array}}$

$$y' + \frac{1}{x}y = 0 \quad \Rightarrow \quad xy' + y = 0$$

$$\Rightarrow \quad xy' + (x)'y = 0$$

$$\Rightarrow \quad (xy)' = 0$$

$$\Rightarrow \quad xy = c$$

$$\Rightarrow \quad y = \frac{c}{x}, \quad c \in \mathbb{R} \text{ const.}$$

Generally, in order to solve

$$y' + p(x)\, y = q(x)$$

we multiply both sides by $e^{\int_{x_0}^{x} p(t)dt}$.

$$e^{\int_{x_0}^{x} p(t)dt} \cdot y' + p(x)\, e^{\int_{x_0}^{x} p(t)dt} \cdot y = q(x)\, e^{\int_{x_0}^{x} p(t)dt} \Rightarrow$$

$$\left[e^{\int_{x_0}^{x} p(t)dt} \cdot y \right]' = q(x)\, e^{\int_{x_0}^{x} p(t)dt}$$

Now we can integrate and obtain

$$e^{\int_{x_0}^{x} p(t)dt} \cdot y = \int_{x_0}^{x} q(t)\, e^{\int_{x_0}^{t} p(u)du}\, dt + c$$

and therefore we can solve for y.

Examples: (i) $y' + x^2 y = 0$

An antider. of x^2 is $\frac{1}{3}x^3$.

$$e^{x^3/3} y' + x^2 e^{x^3/3} y = 0 \Rightarrow$$

$$(e^{x^3/3} y)' = 0 \Rightarrow$$

$$e^{x^3/3} y = c \Rightarrow$$

$$y = c \cdot e^{-\frac{x^3}{3}}, \quad c \in \mathbb{R} \text{ const.}$$

(ii) $y' + \frac{1}{x} y = 3x$

| multiply with $e^{\int_1^x \frac{1}{t} dt} = e^{\ln x} = x$.

$$x y' + y = 3x^2 \Rightarrow$$

$$(x y)' = (x^3)' \Rightarrow$$

$$x y = x^3 + c \Rightarrow$$

$$y = x^2 + \frac{c}{x}, \quad c \in \mathbb{R} \text{ const.}$$

(iii) $xy' + (2x^2+1)y = x \Rightarrow$

$$y' + \left(2x + \frac{1}{x}\right) y = 1$$

$$\int \left(2x + \frac{1}{x}\right) dx = x^2 + \ln x + c$$

We multiply both sides by

$$e^{x^2 + \ln x} = x e^{x^2}.$$

$$x e^{x^2} y' + (2x^2+1) e^{x^2} y = x e^{x^2} \Rightarrow$$

$$(x e^{x^2} y)' = x e^{x^2} \Rightarrow$$

$$x e^{x^2} y = \frac{1}{2} e^{x^2} + c \Rightarrow$$

$$y = \frac{1}{2x} + \frac{c}{x e^{x^2}}, \quad c \in \mathbb{R} \text{ const.}$$

(iv) Solve the initial value problem

$$\begin{cases} y' - y = \sin x \\ y(0) = 0 \end{cases}.$$

First we solve the D.E. $y' - y = \sin x$; then we find the particular solution which satisfies $y(0)=0$.

$$y' - y = \sin x \Rightarrow$$

$$e^{-x} y' - e^{-x} y = e^{-x} \sin x \Rightarrow$$

$$(y \cdot e^{-x})' = e^{-x} \sin x \Rightarrow$$

$$y \cdot e^{-x} = \int e^{-x} \sin x \, dx$$

$$I = \int e^{-x} \sin x \, dx = \int e^{-x} (-\cos x)' \, dx$$

$$= -e^{-x} \cos x - \int (-e^{-x})(-\cos x) \, dx$$

$$= -e^{-x} \cos x - \int e^{-x} \cos x \, dx$$

$$= -e^{-x} \cos x - \int e^{-x} (\sin x)' \, dx$$

$$= -e^{-x} \cos x - e^{-x} \sin x + \int (-e^{-x}) \sin x \, dx$$

$$= -e^{-x} \cos x - e^{-x} \sin x - I \Rightarrow$$

$$2I = -e^{-x}(\sin x + \cos x) + c \Rightarrow$$

$$I = -\frac{1}{2}e^{-x}(\sin x + \cos x) + c$$

Therefore

$$y e^{-x} = -\frac{1}{2}e^{-x}(\sin x + \cos x) + c \Rightarrow$$

$$y = c e^{x} - \frac{1}{2}(\sin x + \cos x), \quad c \in \mathbb{R}.$$

$$y(0) = 0 \Rightarrow c - \frac{1}{2} = 0 \Rightarrow c = \frac{1}{2}$$

The solution to the initial value problem is

$$y = \frac{1}{2}(e^{x} - \sin x - \cos x).$$

(v) $\begin{cases} y' + \cos x \cdot y = \cos x \\ y(0) = 2 \end{cases}$

$$y' + \cos x \cdot y = \cos x \Rightarrow$$

$$e^{\sin x} \cdot y' + \cos x \, e^{\sin x} y = \cos x \, e^{\sin x} \Rightarrow$$

$$(e^{\sin x} \cdot y)' = (e^{\sin x})' \Rightarrow$$

$$e^{\sin x} y = e^{\sin x} + c \Rightarrow$$

$$y = 1 + c \cdot e^{-\sin x}, \quad c \in \mathbb{R} \text{ const.}$$

$$y(0) = 2 \Rightarrow 1 + c \cdot e^{0} = 2 \Rightarrow c = 1.$$

Hence

$$y = 1 + e^{-\sin x}.$$

- $\boxed{\begin{array}{l}\text{SEPARABLE D.E.,} \\ \text{i.e. D.E. OF THE FORM } \quad y' = f(x)\,g(y)\end{array}}$

$$y' = f(x)\,g(y) \implies \frac{dy}{dx} = f(x)\,g(y)$$

$$\implies \frac{dy}{g(y)} = f(x)\,dx$$

$$\implies \int \frac{dy}{g(y)} = \int f(x)\,dx$$

E.g. (i) $y' - 2x\,y^2 = 0 \implies$

$$y' = 2x\,y^2 \implies$$

$$\int \frac{dy}{y^2} = \int 2x\,dx \implies$$

$$-\frac{1}{y} = x^2 + c \implies$$

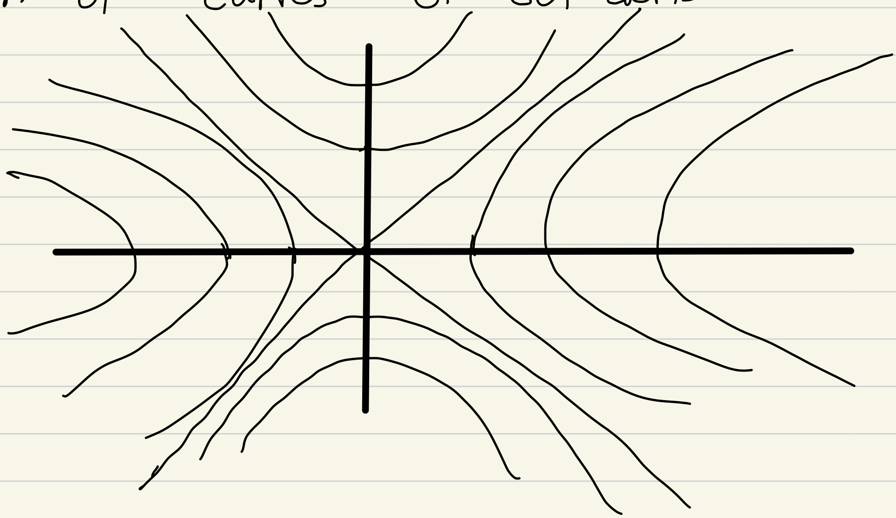
$$y = -\frac{1}{x^2 + c}, \quad c \in \mathbb{R}.$$

(ii) $y' = \frac{x}{y} \Rightarrow \int y \, dy = \int x \, dx$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\Rightarrow y^2 - x^2 = c, \quad c \in \mathbb{R}.$$

Observe that we have found the general solution of the DE in "implicit form", i.e. not as an explicit function $y = y(x)$, but in the form of "curves" of solutions.



(iii) Solve the initial value problem

$$\begin{cases} y' = x^2 y^3 \\ y(1) = 3 \end{cases}$$

$$y' = x^2 y^3 \Rightarrow \int \frac{dy}{y^3} = \int x^2 \, dx$$

$$\Rightarrow -\frac{1}{2y^2} = \frac{x^3}{3} + C$$

$$\Rightarrow \frac{1}{y^2} = -\frac{2x^3}{3} + c,$$

$c \in \mathbb{R}$ const.

$$y(1) = 3 \Rightarrow c - \frac{2}{3} = \frac{1}{9}$$

$$\Rightarrow c = \frac{2}{3} + \frac{1}{9} = \frac{7}{9}$$

Therefore the solution satisfies

$$\frac{1}{y^2} = \frac{7 - 6x^3}{9} \Rightarrow y^2 = \frac{9}{7 - 6x^3}$$

$$\Rightarrow y = \frac{3}{\sqrt{7 - 6x^3}}$$

(Here we chose the positive square root because this is the solution of the D.E. that satisfies $y(1)=3$).

* Some D.E.'s can be viewed as both linear and separable:
e.g. $y' = x^2 y$

We can solve them in any way we like...

• $\boxed{\text{D.E.'s OF THE FORM } \quad y' = f\left(\frac{y}{x}\right)}$

These D.E.'s might be encountered in the form

$$y' = F(x, y)$$

where $F(\lambda x, \lambda y) = F(x, y), \ \forall \lambda > 0$.

We use the substitution

$$u = \frac{y}{x} \qquad \left(\text{i.e. } \ u(x) = \frac{y(x)}{x}\right).$$

Then $y = xu \Rightarrow y' = u + xu'$.

Then $$y' = F\left(\frac{y}{x}\right) \Rightarrow$$
$$u + xu' = F(u) \Rightarrow$$
$$xu' = F(u) - u \Rightarrow$$
$$u' = \frac{F(u)-u}{x}$$

which is separable.

E.g. Solve $y' = \dfrac{x+y}{2x-y}$.

$$y' = \frac{x+y}{2x-y} = \frac{1+\frac{y}{x}}{2-\frac{y}{x}}$$

(alternatively, $f(x,y) = \dfrac{x+y}{2x-y}$ satisfies $f(\lambda x, \lambda y) = f(x,y)$).

Set $u = \dfrac{y}{x} \Rightarrow y = xu$
$\Rightarrow y' = u + xu'$

$$u + xu' = \frac{1+u}{2-u} \Rightarrow$$

$$xu' = \frac{1+u}{2-u} - u = \frac{1+u-2u+u^2}{2-u} \Rightarrow$$

$$xu' = \frac{1-u+u^2}{2-u} \Rightarrow$$

$$\int \frac{2-u}{u^2-u+1} du = \int \frac{1}{x} dx \Rightarrow$$

$$\int \frac{\frac{3}{2}+\frac{1}{2}-u}{u^2-u+1} du = \int \frac{dx}{x} \Rightarrow$$

$$\frac{3}{2} \int \frac{du}{u^2-u+1} - \frac{1}{2} \int \frac{2u-1}{u^2-u+1} du = \int \frac{dx}{x} \Rightarrow$$

$$\frac{3}{2} \int \frac{du}{\left(u-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{2} \int \frac{(u^2-u+1)'}{u^2-u+1} du = \int \frac{dx}{x}$$