NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **MA2501 Numerical Methods**

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Examination time (from–to): 09:00-13:00

Permitted examination support material: Support material code C

- Approved basic calculator.
- The textbook: Cheney & Kincaid, Numerical Mathematics and Computing, 6th or 7th edition, including the list of errata.
- Rottmann, Mathematical formulae.
- Photo copies of chapter 1 and 4 of the textbook: An Introduction to Numerical Analysis, by Suli and Mayers

Other information:
All answers should be justified and include enough details to make it clear which methods and/or results have been used.

Language: English

Number of pages: 3

Number of pages enclosed: 0

Checked by:

Date Signature

Problem 1

a) Use divided differences and Newton's interpolation formula to find the interpolating polynomial of lowest possible degree for the points in the table:

x	-1	0	1	2
y	3	-4	5	6

`(6 points)`

b) Establish the formula

$$f''(x) \approx \frac{2}{h^2}\left[\frac{f(x_0)}{(1+\alpha)} - \frac{f(x_1)}{\alpha} + \frac{f(x_2)}{\alpha(1+\alpha)}\right]$$

using unevenly spaced points $x_0 < x_1 < x_2$, where $x_1 - x_0 = h$ and $x_2 - x_1 = \alpha h$. Notice that this formula for $\alpha = 1$ is reduced to the standard central-difference formula.

(**Hint:** Approximate $f(x)$ by the Newton form of the interpolating polynomial of degree 2.)

`(6 points)`

Problem 2 Use Gaussian elimination with scaled partial pivoting to solve the following linear system.

$$\begin{bmatrix} 3 & 4 & 3 \\ 1 & 5 & -1 \\ 6 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 15 \end{bmatrix}.$$

`(8 points)`

Problem 3 Check whether the following function is a natural cubic spline or not.

$$S(x) = \begin{cases} 1 + x - x^3, & 0 \le x \le 1 \\ 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3, & 1 \le x \le 2 \\ 4(x-2) + 9(x-2)^2 - 3(x-2)^3, & 2 \le x \le 3 \end{cases}$$

Justify your answer.

`(10 points)`

Problem 4 Find an approximation to the integral

$$\int_0^1 \mathrm{e}^{-(10x)^2} \mathrm{d}x$$

using Romberg integration. Find $R(2,2)$ up to three decimal places.

(10 points)

Problem 5 Use the method of least squares to find the equation of a parabola of the form $y = ax^2 + b$ that best represents the following data.

x	-1	0	1
y	3.1	0.9	2.9

(10 points)

Problem 6 Suppose we have the following initial value problem

$$\begin{aligned} x' &= f(t,x), \\ x(1) &= 1, \end{aligned}$$

with

$$f(t,x) = (tx)^3 - \left(\frac{x}{t}\right).$$

Approximate $x(1.2)$ by taking step size $h = 0.1$ with the following Runge-Kutta method

$$\begin{cases} K_1 = f(t,x), \\ K_2 = f(t+h, x+K_1), \end{cases}$$
$$x(t+h) = x(t) + \frac{h}{2}\left(K_1 + K_2\right).$$

(10 points)

Problem 7 Given the initial value problem

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0,$$

where $\mathbf{f} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$. The trapeziodal rule for solving this ODE is given by

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{2}\Big(\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) + \mathbf{f}(t_n, \mathbf{y}_n)\Big),$$

where $h = t_{n+1} - t_n$.
Suppose $\mathbf{f}$ satisfies the L Lipschitz condition

$$\left\|\mathbf{f}(t, \mathbf{y}) - \mathbf{f}(t, \tilde{\mathbf{y}})\right\| \leq L \left\|\mathbf{y} - \tilde{\mathbf{y}}\right\|, \quad \text{for all}\, t \in \mathbb{R}, \mathbf{y}, \tilde{\mathbf{y}} \in \mathbb{R}^m.$$

The local truncation error for the trapezoidal method

$$\mathbf{d}_{n+1} = \mathbf{y}(t_{n+1}) - \mathbf{y}(t_n) - \frac{h}{2}\left(\mathbf{f}(t_{n+1}, \mathbf{y}(t_{n+1})) + \mathbf{f}(t_n, \mathbf{y}(t_n))\right)$$

satisfies

$$\left\|\mathbf{d}_{n+1}\right\| \leq \frac{1}{12}h^3 M, \quad M = \max_{\xi \in \mathbb{R}} \left\|\mathbf{y}'''(\xi)\right\|.$$

Use this to show that the global error $\mathbf{e}_n = \mathbf{y}(t_n) - \mathbf{y}_n$ satisfies

$$\left\|\mathbf{e}_{n+1}\right\| \leq \frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL}\left\|\mathbf{e}_n\right\| + \frac{\frac{1}{12}Mh^3}{1 - \frac{1}{2}hL}, \quad \text{for } hL < 2.$$

`(10 points)`