What we learned untill now:

- Why do we need simulation
- What are pseudo random numbers
- How to simulate from discrete distribution
- How to simulate from (some) continuous distributions
    - Probability integral transform

Review: Sampling from discrete distributions

Let X be a stochastic variable with possible values $\{x_1, \dots, x_k\}$ and $\mathrm{P}(X = x_i) = \mathrm{p}_i$, $\sum_{i=1}^{k} p_i = 1$.

Define: $F_0 = 0, F_1 = p1, F_2 = p_1 + p_2, \dots, F_k = 1$

We can simulate value from F as:

$u \sim U[0, 1]$

for $i = 1, 2, \dots, k$ **do**

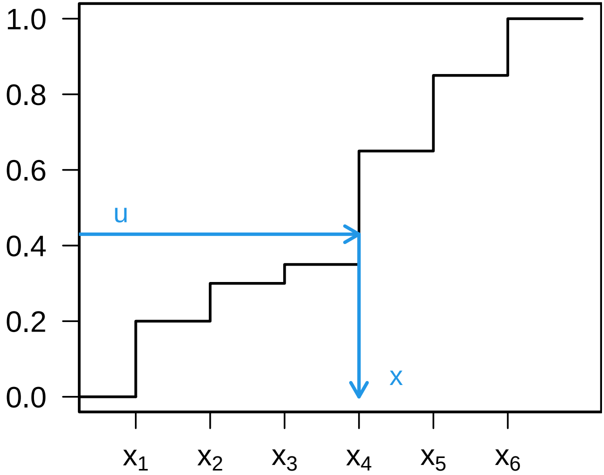
  if $u \in (F_{i-1}, F_i]$ **then**

    $x \leftarrow x_i$

  end if

end for

Review: sampling from discrete distribution (II)



Review: Probability integral transform to sample from continuous distributions

The inversion method (or probability integral transform approach) can be used to generate samples from an arbitrary continuous distribution with density $f(x)$ and CDF $F(x)$:

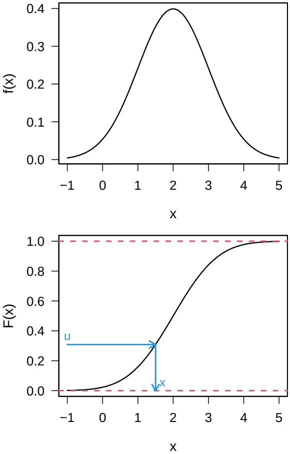
1. Generate random variable U from the standard uniform distribution in the interval $[0, 1]$.
2. Then is

$$X = F^{-1}(U)$$

   a random variable from the target distribution.

Probability integral transform to sample from continuous distributions

Let X have density $f(x)$, $x \in \mathbb{R}$ and CDF $F(x) = \int_{-\infty}^{x} f(z)dz$:



Simulation algorithm:

$u \sim U[0, 1]$

$x = F^{-1}(u)$

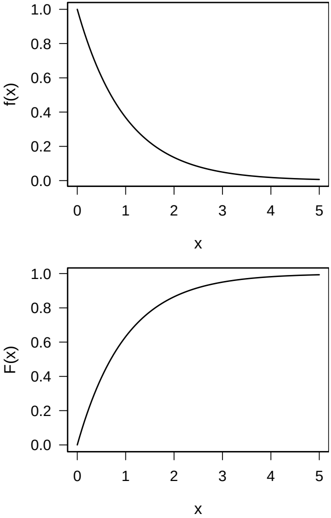
return x

Plan for today

Sampling from continuous distribution

- PIT transform
- Use relationship between random variable
  - Gamma distribution, χ^2 distribution
  - Linear transformation
  - Change of variables
- Bivariate techniques
  - Box-Muller algorithm (Normal distribution)
- Ratio of uniform method

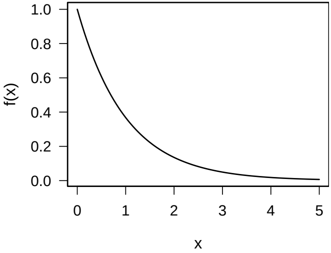
Example - Exponential Distribution

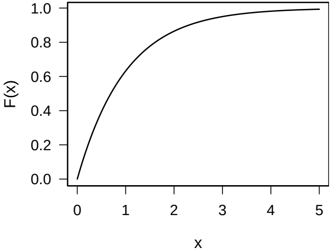


$$f(x) = \lambda \exp(-\lambda x) : \; x > 0$$

$$F(x) = 1 - \exp(-\lambda x)$$

Example - Exponential Distribution





$$f(x) = \lambda \exp(-\lambda x) : \; x > 0$$

$$F(x) = 1 - \exp(-\lambda x)$$

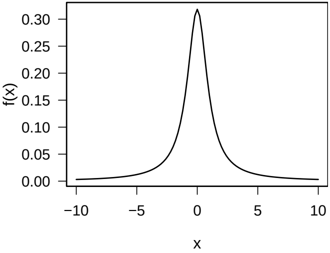
Simulation algorithm:

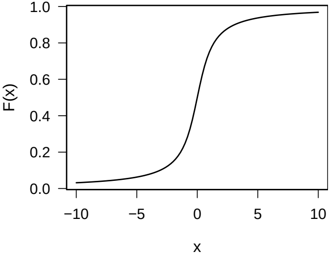
$u \sim U[0, 1]$

$x = -\frac{1}{\lambda} \log(u)$

return x

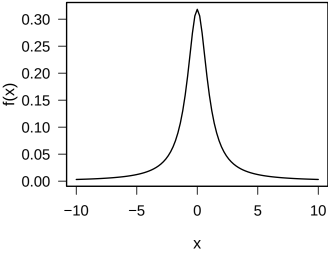
Example - Standard Cauchy distribution

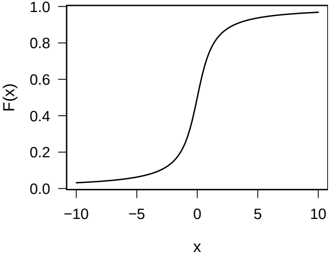




$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1 + x^2}$$
$$F(x) = \frac{1}{2} + \frac{\arctan(x)}{\pi}$$
$$F^{-1}(y) = \tan\left[\pi\left(y - \frac{1}{2}\right)\right]$$

Example - Standard Cauchy distribution





$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$$

$$F(x) = \frac{1}{2} + \frac{\arctan(x)}{\pi}$$

$$F^{-1}(y) = \tan\left[\pi\left(y - \frac{1}{2}\right)\right]$$

Simulation algorithm:

$u \sim U[0,1]$

$x = \tan[\pi(u - \frac{1}{2})]$

return x

Review: inverse transform technique

Let F be a distribution, and let $U \sim \mathcal{U}[0,1]$.

a) Let F be the distribution function of a random variable taking non-negative integer values. The random variable X given by

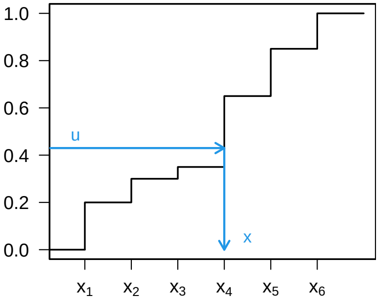
$$X = x_i \quad \text{if and only if} \quad F_{i-1} < u \leq F_i$$

has distribution function F.

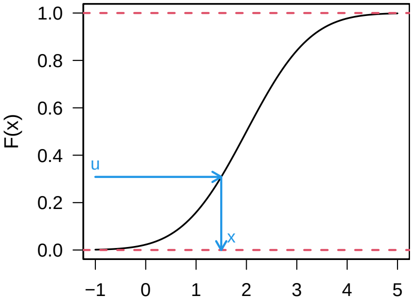
b) If F is a continuous function, the random variable $X = F^{-1}(u)$ has distribution function F.

Review: inverse transform technique (II)

a) Discrete case:



b) Continuous case:



The inverse transform technique is conceptually easy, but

- in the discrete case, a large number of comparisons may be necessary.
- in the continuous case, F^{-1} must be available.

Note

- The inversion method cannot always be used! We must have a formula for $F(x)$ and be able to find $F^{-1}(u)$. This is for example not possible for the normal, χ^2, gamma and t-distributions.
- In some cases we can use known relationships between RV to simulate

Using known relationships - Gamma distribution

Let $X \sim \text{Ga}(\text{shape}{=}\alpha, \text{rate}{=}\beta)$, i.e.

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta \cdot x}, x > 0.$$

If $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Exp}(\beta)$, then $Y = \sum_{i=1}^n X_i \sim \text{Ga}(n, \beta)$.

This gives how to simulate when α is an integer.

$y = 0$

for $i = 1, 2, \ldots, n$ **do**

  generate $u \sim U(0, 1)$

  $x \leftarrow -\frac{1}{\lambda} \log(u)$

  $y \leftarrow y + x$

end for

return y

Using known relationships - χ^2 distribution

Remember: $\chi^2_\nu = \text{Ga}(\frac{\nu}{2}, \frac{1}{2})$,

$X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(0,1) \Rightarrow \sum_{i=1}^n X_i^2 \sim \chi^2_n$.

Thus, we can simulate $X \sim \text{Ga}(\frac{n}{2}, \frac{1}{2})$ by

$x = 0$

for $i = 1, 2, \ldots, n$ **do**

generate $y \sim \mathcal{N}(0,1)$ $\quad\triangleright$ Still have to learn how

$x \leftarrow x + y^2$

end for

return x

Scale and location parameters

In Ga(α, β), β is a rate (inverse scale) parameter

$$X \sim \text{Ga}(\alpha, 1) \qquad \Leftrightarrow \qquad X/\beta \sim \text{Ga}(\alpha, \beta)$$

This gives us a way to sample from a Gamma distribution Ga($\frac{n}{2}, \beta$) where n is an integer

Gamma distribution - simulate $X \sim \text{Ga}(\frac{n}{2}, \beta)$

$x = 0$

for $i = 1, 2, \ldots, n$ **do**

  generate $y \sim \mathcal{N}(0, 1)$ ▷ Still have to learn how

  $x \leftarrow x + y^2$

end for

$x \leftarrow x$ ▷ $\text{Ga}(\frac{n}{2}, \frac{1}{2}), \chi^2_n$

$x \leftarrow \frac{1}{2}x$ ▷ $\text{Ga}(\frac{n}{2}, 1)$

$x \leftarrow \frac{1}{\beta}x$ ▷ $\text{Ga}(\frac{n}{2}, \beta)$

return x

Linear transformations

Many distributions have scale parameters, for example

$$X \sim \mathcal{N}(0,1) \quad \Leftrightarrow \quad \sigma X \sim \mathcal{N}(0,\sigma^2)$$

$$X \sim \mathsf{Exp}(1) \quad \Leftrightarrow \quad \frac{1}{\lambda} X \sim \mathsf{Exp}(\lambda)$$

$$X \sim \mathcal{U}[0,1] \quad \Leftrightarrow \quad \beta X \sim \mathcal{U}[0,\beta]$$

Adding a constant can also help in some situations

$$X \sim \mathcal{N}(0,1) \quad \Leftrightarrow \quad X + \mu \sim \mathcal{N}(\mu,1)$$

and thereby

$$X \sim \mathcal{N}(0,1) \quad \Leftrightarrow \quad \sigma X + \mu \sim \mathcal{N}(\mu,\sigma^2)$$

More general than just linear transformation: Change of variable

let $X \sim f_X(x)$ and $Y = g(X)$ with $g(\cdot)$ being a one-to-one function so that $Y = g^{-1}(X)$, then:

$$f_Y(y) = f_X(g^{-1}(x))|\frac{d\ g^{-1}(x)}{d\ x}|$$

Example: Change of variables

$X \sim \text{Exp}(1)$. We are interested in $Y = \frac{1}{\lambda}X$, i.e. $y = g(x) = \frac{1}{\lambda}x$.

$$g^{-1}(y) = \lambda y \qquad\qquad \frac{dg^{-1}(y)}{dy} = \lambda$$

Application of the change of variables formula leads to:

$$f_Y(y) = \exp(-\lambda y)\lambda$$

It follows: $Y \sim \text{Exp}(\lambda)$.

Exercise: Check other transformations, we mentioned.

Summary

- We can use know relationship between RV to derive samples from a RV we cannt sample directly from.
- If we can simulate from X and we know that $Y = g(X)$ and $g(\cdot)$ is invertible, then we can also get samples from Y
- Location and scale parameter are examples of linear transformation

Bivariate techniques

Remember:

$$\text{If } (x_1, x_2) \sim f_X(x_1, x_2)$$

$$\text{and } (y_1, y_2) = g(x_1, x_2)$$

$$\Updownarrow$$

$$(x_1, x_2) = g^{-1}(y_1, y_2)$$

where g is a one-to-one differentiable transformation. Then

$$f_Y(y_1, y_2) = f_X(g^{-1}(y_1, y_2))|\mathrm{J}|$$

with the determinant of the Jacobian matrix J

$$|\mathrm{J}| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

$\Rightarrow$ Multivariate version of the change-of-variables transformation

Bivariate techniques (II)

If we know how to simulate from $f_X(x_1, x_2)$ we can also simulate from $f_Y(y_1, y_2)$ by

$$(x_1, x_2) \sim f_X(x_1, x_2)$$
$$(y_1, y_2) = g(x_1, x_2)$$

Return (y_1, y_2).

Example: Normal distribution (Box-Muller)

see blackboard

Review: Box-Muller algorithm

Generate

$x_1 \sim U(0, 2\pi)$

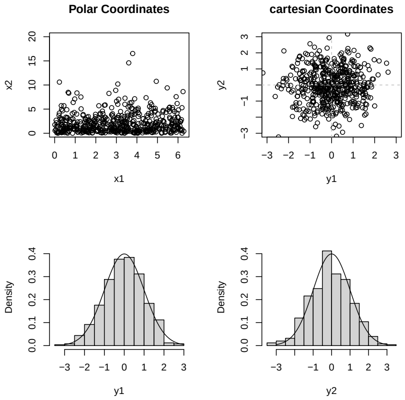
$x_2 \sim \exp(0.5)$

Compute

$y_1 \leftarrow \sqrt(x_2)\cos(x_1)$

$y_2 \leftarrow \sqrt(x_2)\sin(x_1)$

return (y_1, y_2)



Ratio-of-uniforms method

All the techniques seen untill now to sample from $f(x)$ require that we know the normalising constant of $f(x)$.

In many cases this is not the case. Often we only know that:

$$f(x) = \frac{1}{c} f^*(x)$$

where $f^*(x)$ is known while the constant (wrt x) c is unknown and is such that:

$$\int_{\mathcal{R}} f(x)dx = \frac{1}{c} \int_{\mathcal{R}} f^*(x)dx = 1$$

The Ratio of uniform method is a general method for arbitrary densities f known up to a proportionality constant.

Ratio-of-uniforms method

Theorem

Let $f^\star(x)$ be a non-negative function with $\int_{-\infty}^{\infty} f^\star(x)dx < \infty$. Let $C_f = \{(x_1, x_2) \mid 0 \leq x_1 \leq \sqrt{f^\star\left(\frac{x_2}{x_1}\right)}\}$.

a) *Then C_f has a finite area*

Let (x_1, x_2) be uniformly distributed on C_f.

b) *Then $y = \frac{x_2}{x_1}$ has a distribution with density*

$$f(y) = \frac{f^\star(y)}{\int_{-\infty}^{\infty} f^\star(u)du}$$

Example: Standard Cauchy distribution

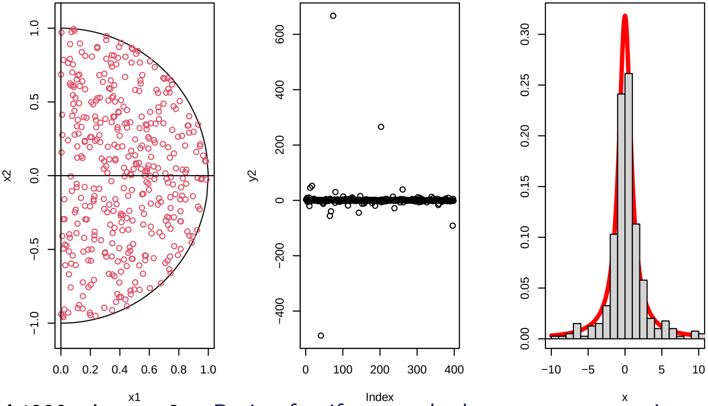
see blackboard

Algorithm to sample form a standard Cauchy

Generate (x_1, x_2) from $\mathcal{U}(C_f)$ ▷ How can we do this?

Compute $y = \frac{x_2}{x_1}$

return y



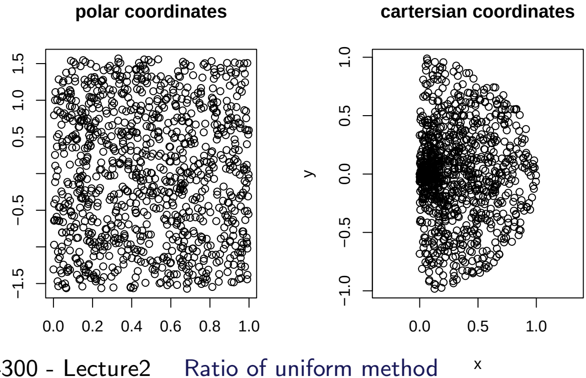
Sampling from the unit half circle

Idea: can we use polar coordinates?

$$x = u * cos(\theta)$$

$$y = u * sin(\theta)$$

can we use $\theta \sim \mathcal{U}(-\pi/2, \pi/2)$ and $u \sim \mathcal{U}(0, 1)$?



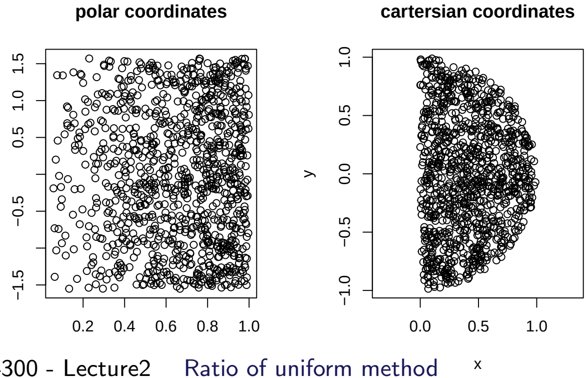
Sampling from the unit half circle

Idea: can we use polar coordinates?

$$x = u * cos(\theta)$$

$$y = u * sin(\theta)$$

Need to have $\theta \sim \mathcal{U}(-\pi/2, \pi/2)$ and $u^2 \sim \mathcal{U}(0, 1)$?



Proof of theorem

see blackboard

Ratio of uniform method

In general it can be hard to sample uniformly from C_f!!

It can be simplified under some contitions:

Theorem

Let $f^\star(x)$ be a non-negative function with $\int_{-\infty}^{\infty} f^\star(x)dx < \infty$. Let $C_f = \{(x_1, x_2) \mid 0 \leq x_1 \leq \sqrt{f^\star\left(\frac{x_2}{x_1}\right)}\}$.

If $f^\star(x)$ and $x^2\ f^\star(x)$ are bounded then $C_f \in [0, a] \times [b_-, b_+]$ with:

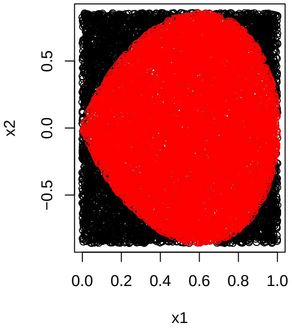
- $a = \sqrt{\sup_x f^\star(x)}$
- $b_- = -\sqrt{\sup_{x\leq 0} x^2\ f^\star(x)}$
- $b_+ = +\sqrt{\sup_{x>0} x^2\ f^\star(x)}$

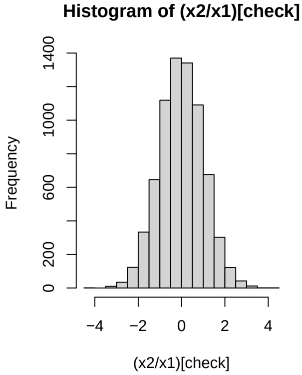
Proof of theorem

see blackboard

Ratio of uniform method: Simplification

- Rather than sampling uniformly from C_f, we can instead sample $(x_2\ x_2)$ uniformly from a rectangle containing C_f
- Reject sample if $(x_1\ x_2) \notin C_f$





Example: Normal distribution

see blackboard

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