We define $\lim_{n\to\infty} a_n = \ell \in \mathbb{R}$ iff

$\forall \varepsilon > 0 \quad \exists n_0 = n_0(\varepsilon) \in \mathbb{N}$ s.t. for all $n \geq n_0$: $|a_n - \ell| < \varepsilon$.

We can show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Take some $\varepsilon > 0$.
We need to find some $n_0 \in \mathbb{N}$
with the property that for all $n \geq n_0$,

$$\left|\frac{1}{n} - 0\right| < \varepsilon \iff$$

$$\frac{1}{n} < \varepsilon \iff$$

$$n > \frac{1}{\varepsilon}.$$

Choose $n_0 > \frac{1}{\varepsilon}$ (we can find such an n_0 by the Archimedean Property).
Then for all $n \geq n_0$,

$$n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon.$$

If $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ are such that

$$\lim_{n\to\infty} a_n = l_1 \quad \text{and} \quad \lim_{n\to\infty} b_n = l_2$$

(with $l_1, l_2 \in \mathbb{R}$) then:

- $\lim_{n\to\infty} (a_n + b_n) = l_1 + l_2$
- $\lim_{n\to\infty} (a_n - b_n) = l_1 - l_2$
- $\lim_{n\to\infty} (\lambda a_n + \mu b_n) = \lambda l_1 + \mu l_2 \quad (\forall \lambda, \mu \in \mathbb{R})$
- $\lim_{n\to\infty} (a_n b_n) = l_1 l_2$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{l_1}{l_2}$ provided that $l_2 \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$.

PROPOSITION 2.4: If $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ are such that $\lim_{n\to\infty} a_n = l_1$ and $\lim_{n\to\infty} b_n = l_2$ with $l_1, l_2 \in \mathbb{R}$ then $\lim_{n\to\infty} (a_n + b_n) = l_1 + l_2$.

PROOF

Let $\varepsilon > 0$.

(We want to find $n_0 \in \mathbb{N}$ s.t.

$$|(a_n + b_n) - (l_1 + l_2)| < \varepsilon \quad \forall n \geq n_0.)$$

Because $\lim_{n\to\infty} a_n = l_1$, there exists $n_1 \in \mathbb{N}$ such that

$$|a_n - l_1| < \frac{\varepsilon}{2} \quad \text{for all } n \geq n_1.$$

Since $\lim_{n\to\infty} b_n = l_2$, there exists $n_2 \in \mathbb{N}$ such that

$$|b_n - l_2| < \frac{\varepsilon}{2} \quad \text{for all } n \geq n_2.$$

Set $n_0 = \max\{n_1, n_2\}$. Then for all $n \geq n_0$,

$$\begin{aligned}
|(a_n + b_n) - (l_1 + l_2)| &= |(a_n - l_1) + (b_n - l_2)| \\
&\leq |a_n - l_1| + |b_n - l_2| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

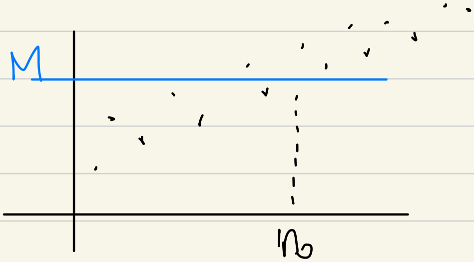
So $\lim_{n\to\infty} (a_n + b_n) = l_1 + l_2$. ■

We say that the sequence $(a_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ <u>diverges</u> if it does not converge to any real number (i.e. it does not have a limit in $\mathbb{R}$).

A special case of divergent sequences are those which diverge to $+\infty$.

We say that $(a_n)_{n=1}^{\infty}$ diverges to $+\infty$ and we write $\lim_{n\to\infty} a_n = +\infty$, if

$\forall M>0 \ \exists n_0 = n_0(M) \in \mathbb{N}$ such that $a_n > M$ for all $n \geq n_0$.



For example, $\lim_{n\to\infty} (2n+1) = +\infty$.

REMARK: If a sequence does not converge, this does not necessarily mean that it will diverge to $+\infty$.
E.g. take

$$a_n = (-1)^n, \quad n = 1, 2, \dots$$

We say that $(a_n)_{n=1}^{\infty}$ diverges to $-\infty$ and we write $\lim_{n\to\infty} a_n = -\infty$ if

$\forall M>0 \ \exists n_0 = n_0(M) \in \mathbb{N}$ s.t. $\forall n \geq n_0 : a_n < -M$.
(equivalently : $\lim_{n\to\infty} (-a_n) = +\infty$).

THEOREM 2.5 (Interpolation Theorem): Suppose $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ are sequences such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

for some $L \in \mathbb{R}$. If $(c_n)_{n=1}^{\infty}$ is such that

$$a_n \leq c_n \leq b_n \quad \text{for all } n = 1, 2, \ldots$$

then

$$\lim_{n\to\infty} c_n = L.$$

PROOF

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$,

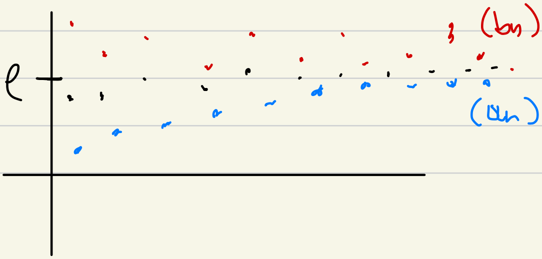
$$|a_n - L| < \varepsilon \implies L - \varepsilon < a_n < L + \varepsilon.$$

Since $\lim_{n\to\infty} b_n = L$, there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$,

$$|b_n - L| < \varepsilon \implies L - \varepsilon < b_n < L + \varepsilon.$$

Set $n_0 = \max\{n_1, n_2\} \in \mathbb{N}$. Then for all $n \geq n_0$,

$$L - \varepsilon < a_n \leq c_n \leq b_n < L + \varepsilon \implies$$
$$L - \varepsilon < c_n < L + \varepsilon \implies$$
$$|c_n - L| < \varepsilon. \quad \blacksquare$$



Examples

(a)

$$\lim_{n\to\infty} \frac{n^2-n+1}{2n^2+2n-2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(1-\frac{1}{n}+\frac{1}{n^2}\right)}{2\cancel{n^2}\left(1+\frac{1}{n}-\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty}\left(\frac{1}{2}\cdot\frac{\overset{\to 1}{1}-\overset{\to 0}{\frac{1}{n}}+\overset{\to 0}{\frac{1}{n^2}}}{\underset{\to 1}{1}+\underset{\to 0}{\frac{1}{n}}-\underset{\to 0}{\frac{1}{n^2}}}\right) = \frac{1}{2}.$$

(b) $\lim_{n\to\infty} \frac{\cos n}{n} = ?$

Here we recall that $|\cos x| \leq 1$ for all $x \in \mathbb{R}$.
Thus

$$\left|\frac{\cos n}{n}\right| = \frac{|\cos n|}{n} \leq \frac{1}{n} \Rightarrow$$

$$-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}.$$

Since $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n} = 0$

by the interpolation Theorem,

$$\lim_{n\to\infty}\frac{\cos n}{n} = 0.$$

(c) $\lim\limits_{n\to\infty} \left(\overset{+\infty}{\sqrt{n^2+1}} - \overset{+\infty}{n}\right) =$

$$\lim_{n\to\infty} \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n} =$$

$$\lim_{n\to\infty} \frac{(n^2+1)-n^2}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} = 0.$$

(The last limit is 0 because

$$\sqrt{n^2+1} + n > \sqrt{n^2} + n = 2n \Rightarrow$$

$$0 < \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{2n},$$

and now we can use the Interpolation Thm.).

(So we cannot say that "$(+\infty)-(+\infty)=0$").

* Show that if $\lim\limits_{n\to\infty} a_n = +\infty$

then $\lim\limits_{n\to\infty} \frac{1}{a_n} = 0$.

(using the definitions only.)

Recall that $(a_n)_{n=1}^{\infty}$ is bounded iff there exists $M>0$ such that

$$|a_n| < M \quad \text{for all } n=1,2,\dots$$

<u>THEOREM 2.6</u>: Every convergent sequence is bounded.

<u>PROOF</u>

Let $(a_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ be a sequence which converges to $L \in \mathbb{R}$. Let $\varepsilon = 1$. There exists $n_0 \in \mathbb{N}$ such that

$$|a_n - L| < 1 \quad \text{for all } n \geq n_0.$$

Thus

$$\begin{aligned} |a_n| &= |a_n - L + L| \\ &\leq |a_n - L| + |L| \\ &< 1 + |L| \quad \text{for all } n \geq n_0. \end{aligned}$$

Set

$$M = \max\{|a_1|, |a_2|, \dots, |a_{n_0-1}|, 1+|L|\}.$$

Then for any $n \in \mathbb{N}$, we have

$$|a_n| \leq M \quad \text{if } n < n_0$$

$$|a_n| < 1+|L| \leq M, \quad \text{if } n \geq n_0,$$

Hence $|a_n| \leq M$ for all $n \geq 1$ and $(a_n)_{n=1}^{\infty}$ is bounded. ■

E.g. the sequence $\left(\frac{n-1}{n}\right)_{n=1}^{\infty}$ converges to 1

so it is bounded.

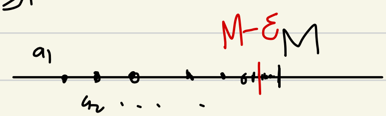
The converse of Theorem 2.6 is NOT true: the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is bounded but it does not converge.

However if we add the assumption of monotonicity, then we have convergence.

<u>THEOREM 2.7</u>: (i) If $(a_n)_{n=1}^{\infty}$ is bounded and increasing, then it converges to $M = \sup_{n\geq 1} a_n$.

(ii) If $(a_n)_{n=1}^{\infty}$ is bounded and decreasing, then it converges to $m = \inf_{n\geq 1} a_n$.

<u>PROOF</u>



(i) Let $\varepsilon > 0$.

Since $M = \sup_{n\geq 1} a_n$, there exist $n_0 \in \mathbb{N}$ such that

$$M - \varepsilon < a_{n_0} \leq M.$$

Since $(a_n)_{n=1}^{\infty}$ is increasing, for all $n \geq n_0$,

$$M - \varepsilon < a_{n_0} \leq a_n \leq M$$

therefore

$$|a_n - M| < \varepsilon.$$

We have proved that for any $\varepsilon > 0$, $\exists\, n_0 = n_0(\varepsilon) \in \mathbb{N}$ s.t. $|a_n - M| < \varepsilon$ $\forall n \geq n_0$. So $\lim_{n\to\infty} a_n = M$.

(ii) Similar. ∎

Example: Consider the sequence $(a_n)_{n=1}^{\infty}$ defined by

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \sqrt{6+a_n}, \quad n = 1, 2, \dots \end{cases}$$

Does this sequence converge?
If so, to which limit?

- First we show $(a_n)_{n\geq 1}$ is increasing.
  We need to show that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.
  We use induction on $n \in \mathbb{N}$:
  - For $n = 1$: $a_1 = 1$, $a_2 = \sqrt{6+1} = \sqrt{7} > a_1$.
  - Assume $a_{n+1} > a_n$ for some $n \in \mathbb{N}$.
  - Then $a_{n+2} = \sqrt{6 + a_{n+1}} > \sqrt{6+a_n} = a_{n+1}$.

  So $a_{n+1} > a_n$ for all $n \geq 1$
  and $(a_n)_{n=1}^{\infty}$ is increasing.

- Now, we show $(a_n)_{n=1}^{\infty}$ is bounded above by 3.
  Again we use induction on $n \geq 1$.
  - For $n = 1$: $a_1 = 1 < 3$.
  - Assume $a_n < 3$ for some $n \in \mathbb{N}$.
  - Then $a_{n+1} = \sqrt{6+a_n} < \sqrt{6+3} = \sqrt{9} = 3$.

  So indeed $a_n < 3$ for all $n \in \mathbb{N}$.

Thus $(a_n)_{n=1}^{\infty}$ is bounded and increasing
so by Theorem 2.7 it converges to some $\ell \in \mathbb{R}$.

$$a_{n+1} = \sqrt{6 + a_n} \Rightarrow$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6+a_n} \Rightarrow$$

$$\ell = \sqrt{6+\ell} \Rightarrow$$
$$\ell^2 = 6 + \ell \Rightarrow$$
$$\ell^2 - \ell - 6 = 0 \Rightarrow$$
$$(\ell - 3)(\ell + 2) = 0 \Rightarrow$$
$$\ell = 3 \quad \text{or} \quad \ell = -2.$$

We have $a_n > 0$ for all $n \geq 1$, hence

$$\lim_{n\to\infty} a_n = 3.$$

Some more limits of sequences (without proof):

<u>THEOREM 2.8</u>: The following hold:

(i) $\lim_{n\to\infty} a^n = 0$ whenever $|a| < 1$.

(ii) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(iii) $\lim_{n\to\infty} \sqrt[n]{a} = 1$ when $a > 0$.

E.g.: $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$ (because $\frac{2}{3} < 1$).

- $$\lim_{n\to\infty} \frac{3^n+4^n}{3^n+5^n} = \lim_{n\to\infty} \frac{4^n\left(1+\left(\frac{3}{4}\right)^n\right)}{5^n\left(1+\left(\frac{3}{5}\right)^n\right)}$$

$$= \lim_{n\to\infty}\left[\underbrace{\left(\frac{4}{5}\right)^n}_{\to 0} \cdot \frac{\overbrace{1+\left(\frac{3}{4}\right)^n}^{\to 1}}{\underbrace{1+\left(\frac{3}{5}\right)^n}_{\to 1}}\right] = 0 \cdot 1 = 0.$$

- $$\lim_{n\to\infty} \sqrt[n]{2n+3} = ?$$

We know that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$,

$$\lim_{n\to\infty} \sqrt[n]{2n+3} = \lim_{n\to\infty} \sqrt[n]{2n\left(1+\frac{3}{2n}\right)}$$

$$= \lim_{n\to\infty} \sqrt[n]{2}\cdot\sqrt[n]{n}\cdot\sqrt[n]{1+\frac{3}{2n}} = 1.$$

Also the number $e \cong 2,71,\ldots$ is defined as the limit of the sequence

$$a_n = \left(1+\frac{1}{n}\right)^n, \quad n\in\mathbb{N}.$$

So

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

In general this is not true that if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = +\infty$$

then $\lim_{n\to\infty} (a_n - b_n) = 0$.

Take $a_n = 2n^2 + 2$,

$b_n = 2n^2$, $n = 1, 2, \ldots$

Then $\lim_{n\to\infty} a_n = +\infty$, $\lim_{n\to\infty} b_n = +\infty$

but $\lim_{n\to\infty} (a_n - b_n) = 2$.