Innlevering 1

1.

$X \sim N(\mu, \Sigma)$

$\mu = E[X]$

$= \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\Sigma = Cov[X]$

$= \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} X$

(a) $E[Y] = E\left[\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}\right]$

$= E\left[\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} X\right]$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} E[X]$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$

$\Sigma_Y = Var[Y]$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} Var[X] \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$

Y is a linear combination of $X \sim N(\mu_X, \Sigma_{XX})$, wich means that $Y \sim N(\mu_Y, \Sigma_{YY})$

Because $Cov[Y_1, Y_2] = 0$ and $Cov[Y_2, Y_1] = 0 \Rightarrow Y_1$ and Y_2 are independent

(b) $f(x) = a,\ a > 0$

$(x-\mu)^T \Sigma^{-1} (x-\mu) = b,\ b > 0$

2.

$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

$C = I - \frac{1}{n}(1 1)^T$

$= \begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & 1-\frac{1}{n} \end{pmatrix}$

$X = (X_1, X_2, \ldots, X_n)^T$

(a) Show that $\bar{X} = \frac{1}{n} 1^T X$

Then I have to show that $\sum_{i=1}^{n} X_i = 1^T X$

$\sum_{i=1}^{n} X_i = X_1 + X_2 + \ldots + X_n$

$1^T X = (1, 1, \ldots, 1) \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$

$= 1 \cdot X_1 + 1 \cdot X_2 + \ldots + 1 \cdot X_n$

$\Rightarrow \bar{X} = \frac{1}{n} 1^T X$

Show that $S^2 = \frac{1}{n-1} X^T C X$

Then I have to show that $\sum_{i=1}^{n} (X_i - \bar{X})^2 = X^T C X$

$CX = \begin{pmatrix} X_1 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix}$

$= X - \bar{X}$

$X^T C X = X^T C C X$

$= (X_1 - \bar{X}, \ldots, X_n - \bar{X}) \begin{pmatrix} X_1 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix}$

$= \sum_{i=1}^{n} (X_i - \bar{X})^2$

$\Rightarrow S = \frac{1}{n-1} X^T C X$

(b) Show that $\frac{1}{n} 1^T C = 0^T$:

$\frac{1}{n} 1^T C = (\frac{1}{n}, \ldots, \frac{1}{n}) \begin{pmatrix} 1-\frac{1}{n} & & -\frac{1}{n} \\ & \ddots & \\ -\frac{1}{n} & & 1-\frac{1}{n} \end{pmatrix}$

$= (\frac{n-n}{n^2}, \ldots, \frac{n-n}{n^2})$

$= (\frac{0}{n^2}, \ldots, \frac{0}{n^2})$

$= (0, \ldots, 0)$

$= 0^T$

O.K.

$\Rightarrow$ If $X \sim N(\mu, I)$, then $\frac{1}{n} 1^T X$ and CX are independent

We then use $\bar{X}$ and S^2 from (a) and we get that $\bar{X}$ and S^2 are independent

(c) Derive distribution of $\frac{(n-1)S^2}{\sigma^2}$:

$\bar{X}$ and S^2 are independent

$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 + \left(\frac{\bar{X} - \mu}{\sigma / \sqrt{n}}\right)^2$

Denote:

$u = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$

$v = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2$

$w = \left(\frac{\bar{X} - \mu}{\sigma / \sqrt{n}}\right)^2$

$\Rightarrow u \sim \chi^2_n$

$v \sim \frac{(n-1)S^2}{\sigma^2}$

$w \sim \chi^2_1$

Since $\bar{X}$ and S^2 are independent, so are v and w

We get,

$M_u(t) = M_v(t) + M_w(t)$

Where $M_X(t)$ is the MGF

$\frac{1}{(1-2t)^{\frac{n}{2}}} = M_v(t) \frac{1}{(1-2t)^{\frac{1}{2}}}$

$\Rightarrow M_v(t) = \frac{1}{(1-2t)^{(n-1)/2}}$

$\Rightarrow v = S^2 \sim \chi^2_{n-1}$