TMA4265 Stochastic Modelling: Exercise 2

Week 35

Problem 1

Consider the Markov chain $\{X_n : n = 0, 1, 2, ...\}$ with state space $\Omega = \{A, B, C\}$ and transition probability matrix given by

$$\mathbf{P} = \begin{array}{c} A \\ B \\ C \end{array} \begin{array}{ccc} A & B & C \\ \begin{pmatrix} 0.1 & 0.7 & 0.2 \\ 0.5 & 0.1 & 0.4 \\ 0.3 & 0.6 & 0.1 \end{pmatrix} \end{array}.$$

The probability distribution of the initial state X_0 is given by $P(X_0 = A) = 0.2$, $P(X_0 = B) = 0.5$ and $P(X_0 = C) = 0.3$. Compute the following probabilities:

a) $P(X_3 = A)$, $P(X_3 = B)$ and $P(X_3 = C)$.

b) $P(X_3 = A | X_1 = B, X_0 = A)$

c) $P(X_3 = A | X_2 = C, X_1 = B, X_0 = A)$

d) $P(X_6 = A | X_3 = C)$

e) $P(X_3 = C | X_6 = A)$

Problem 2

We simplify the real world by 1) classifing each day as either rain or not rain, and 2) imagining that the daily weather is a stochastic process where the probability of rain tomorrow only depends on whether it rains or not today. If it rains today, the probability of rain tomorrow is 0.70, and if it does not rain today, it will rain tomorrow with probability 0.25. Under these assumptions, the daily weather can be modeled as a Markov chain $\{X_n : n = 0, 1, ...\}$, where n denotes day, with two states: "rain" (state 0) and "not rain" (state 1).

a) Explain why the transition probability matrix is given by

$$\mathbf{P} = \begin{bmatrix} 0.70 & 0.30 \\ 0.25 & 0.75 \end{bmatrix}.$$

b) Assume that today is Monday and that it is raining. Compute the probability of rain on Friday.

c) Use `R` to numerically compute the following probabilities:

- $P(X_{100} = 0 | X_0 = 0)$
- $P(X_{100} = 0 | X_0 = 1)$
- $P(X_{100} = 1 | X_0 = 0)$
- $P(X_{100} = 1 | X_0 = 1)$

What do you observe?

Comment: We will discuss the concept of limiting probabilities *and how they can be computed analytically in week 36.*

d) Assume that it rains January 1, and make a script in R where you simulate the weather for one year (365 days) according to the above Markov chain and visualize the results.

Hint: The function `runif()` in R can be useful in your simulations.

e) Run the script N times, where N is a large number, and verify the probabilities you computed in **c)**.

Problem 3

Consider a Markov chain with state space $\Omega = \{0, 1, 2, 3\}$ whose transition probability matrix is given by

$$
\mathbf{P} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0.1 & 0.4 & 0.1 & 0.4 \\
0.2 & 0.1 & 0.6 & 0.1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

Assume that the starting state is state 1.

a) Compute the expected time to absorption.

b) Compute the probability that the Markov chain ends up in state 0.