We have seen that every continuous $f:[a,b]\to\mathbb{R}$ has the intermediate value property; i.e. if $f(a)<t<f(b)$ then $t=f(x_0)$ for some $x_0\in(a,b)$.

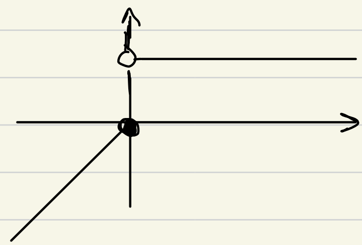
When f is diff. on $[a,b]$, then f' is not always continuous. However it always has the intermediate value property!

<u>THEOREM 3.11</u> (Darboux): Let $f:I\to\mathbb{R}$ be differentiable on the interval I. If $a,b\in I$ and $f'(a)<y_0<f'(b)$ then there exists $x_0\in I$ with

$$f'(x_0)=y_0.$$

PROOF: Exercise (see book).

E.g. Let $g(x)=\begin{cases}1, & x>0\\ x, & x\leq 0\end{cases}$



Does there exist a function $f:\mathbb{R}\to\mathbb{R}$ such that

$$f'(x)=g(x) \text{ for all } x\in\mathbb{R}\,?$$

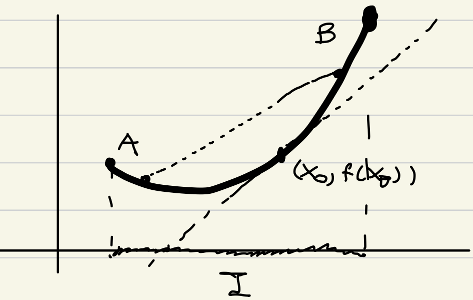
The answer is NO.
If $g=f'$ for some $f:\mathbb{R}\to\mathbb{R}$, since $f'(0)=0$ and $f'(1)=1$, there should exist $t\in(0,1)$ with $f'(t)=\frac{1}{2}$.

· CONVEXITY

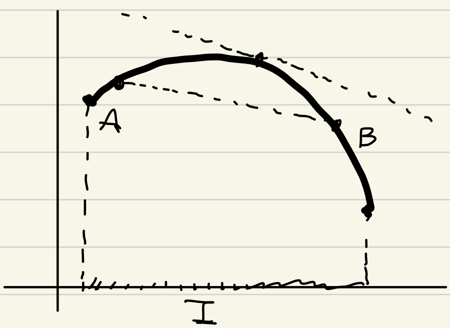
Let $f: I \to \mathbb{R}$ be differentiable on the interval I. We say that f is:

(i) <u>convex</u> (or concave up) in I if f' is increasing in I.

(ii) <u>concave</u> (or concave down) in I if f' is decreasing in I.

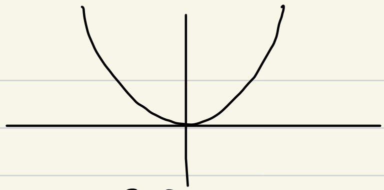


When f is convex, the graph of f is:
- above the tangent at any point $(x_0, f(x_0))$
- below any line segment joining two points of the graph.



When f is concave on I the graph of f is:
- below the tangent at any point
- above any line segment joining two points of the graph.

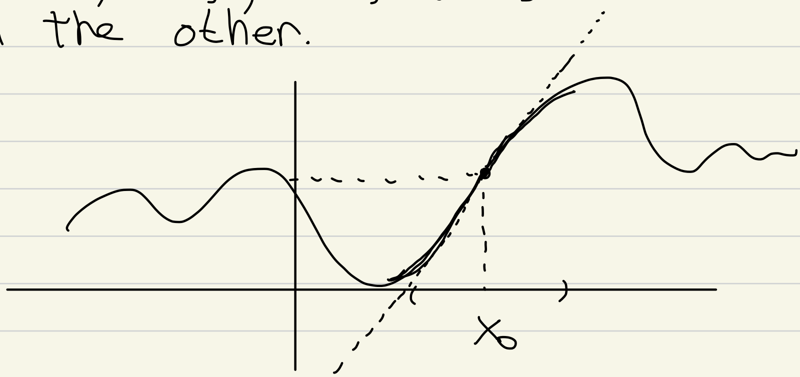
E.g. $f(x) = x^2$.



$f'(x) = 2x$, so f' is increasing in $\mathbb{R}$, so f is convex.

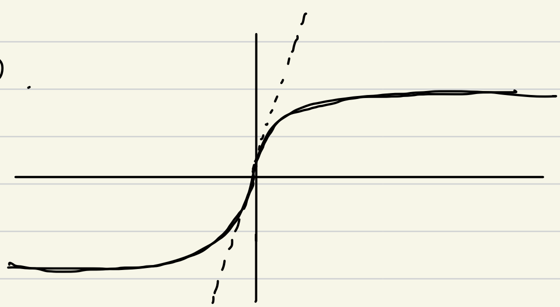
We say that the graph of $f : I \to \mathbb{R}$ has an <u>inflection point</u> at $x_0 \in I$ if:

- G_f has a tangent line at $(x_0, f(x_0))$
- there is some $\delta > 0$ such that f is convex on one of the intervals $(x_0 - \delta, x_0)$, $(x_0, x_0 + \delta)$ and concave on the other.



E.g. $f(x) = \arctan x$, $x_0 = 0$.

Then $f'(x) = \dfrac{1}{1+x^2}$



f' is str. incr. in $(-\infty, 0]$ and str. decr. in $[0, \infty)$ and G_f has a tangent at $(0,0)$. So $(0,0)$ is an inflection point of f.

THEOREM 3.12: Let $f: I \to \mathbb{R}$ be twice differentiable.

(a) If $f''(x) > 0$ for all $x \in I$, then f is convex in I.

(b) If $f''(x) < 0$ for all $x \in I$, then f is concave in I.

(c) If f has an inflection point at x_0 then $f''(x_0) = 0$.

The proof follows directly from the definition of concavity and theorem 3.

The converse to Theorem 3.12 is not true. Take for example
$f(x) = x^4$, $x \in \mathbb{R}$.

$f'(x) = 4x^3$, $x \in \mathbb{R}$

f' increasing $\Rightarrow f$ convex

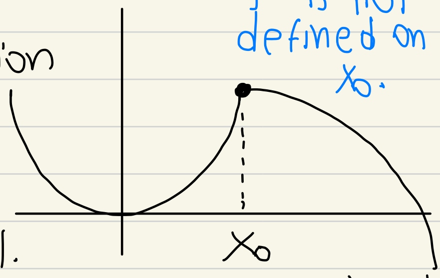
BUT: $f''(0) = 0$
(because $f''(x) = 12x^2$).

i.e. if f is convex, this does not imply $f''(x) > 0 \ \forall x \in I$)

Also $f''(0) = 0$ but f does not have an inflection point at 0.

- The requirement that f has a tangent line at $(x_0, f(x_0))$ is necessary in the definition of inflection points.

f' is not defined on x_0.

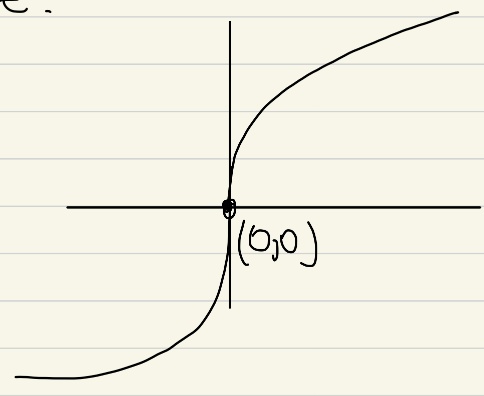


The graph of f in the figure does not have an infl. at $(x_0, f(x_0))$ because the tangent is not defined there.

- Also if f has an inflection point at $(x_0, f(x_0))$ it is not always differentiable there.

E.g.

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x < 0 \end{cases}$$



• ASYMPTOTES

We say that :

(a) the line $x = x_0$ is a <u>vertical asymptote</u> of the graph of f if

$$\lim_{x \to x_0^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to x_0^+} f(x) = \pm\infty$$

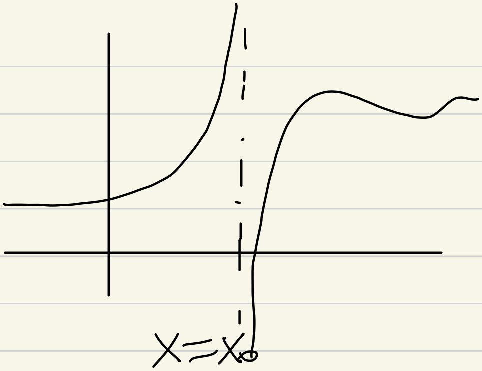
and also x_0 is not in the domain of f.

(b) the line $y = y_0$ is a <u>horizontal asymptote</u> of the graph of f <u>at</u> $+\infty$ (or $-\infty$) if

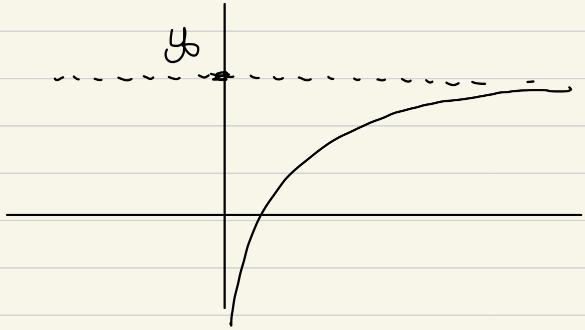
$$\lim_{x \to +\infty} f(x) = y_0 \qquad \left(\text{resp. } \lim_{x \to -\infty} f(x) = y_0\right).$$

(c) the line $y = ax + b$ $(a \neq 0)$ is an <u>oblique asymptote</u> of the graph of f at $+\infty$ (or $-\infty$) if

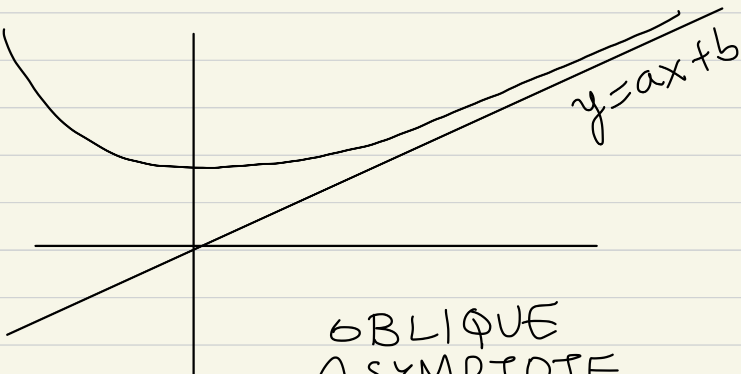
$$\lim_{x \to +\infty} \left(f(x) - ax - b\right) = 0 \qquad \left(\text{resp. } \lim_{x \to -\infty} \left(f(x) - ax - b\right) = 0\right).$$



VERTICAL ASYMPTOTE

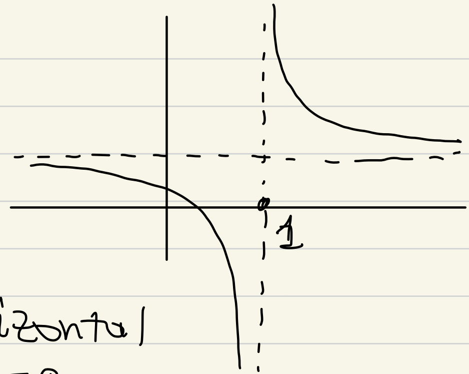


HORIZONTAL ASYMPTOTE



OBLIQUE ASYMPTOTE

E.g. • $f(x) = \frac{1}{x-1} + 1$



The line $x=1$ is a vertical asymptote.

The line $y=1$ is a horizontal asymptote at $+\infty$ and $-\infty$.

• $g(x) = \frac{x^2+1}{2x+1}$, $x \in (-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, +\infty)$

The line $x = -\frac{1}{2}$ a vertical asymptote.

$\lim_{x \to -\frac{1}{2}^-} g(x) = -\infty$ and $\lim_{x \to -\frac{1}{2}^+} g(x) = +\infty$.

We want to look for oblique asymptotes.

$$\lim_{x \to +\infty} \frac{g(x)}{x} = \lim_{x \to +\infty} \frac{x^2+1}{2x^2+x} = \frac{1}{2}$$

and

$$\lim_{x \to +\infty} \left(g(x) - \frac{x}{2}\right) = \lim_{x \to +\infty} \left(\frac{x^2+1}{2x+1} - \frac{x(x+\frac{1}{2})}{2x+1}\right)$$

$$= \lim_{x \to +\infty} \frac{1 - \frac{x}{2}}{2x+1} = -\frac{1}{4}.$$

Therefore $y = \frac{1}{2}x - \frac{1}{4}$ is an oblique asymptote of G_f at $+\infty$.

* If $y = ax + b$ is an oblique asymptote, then $\lim\limits_{x \to +\infty} \frac{f(x)}{x} = a$,
so in order to find the coeff. a, we first calculate $\lim\limits_{x \to +\infty} \frac{f(x)}{x}$.

The line $y = \frac{1}{2}x - \frac{1}{4}$ is also an oblique asymptote of G_f at $-\infty$

(we simply repeat the same procedure but we take limits at $-\infty$ instead of $+\infty$).

· SKETCHING THE GRAPH OF A FUNCTION

In order to draw the graph of a given function f, we find:

1. the domain of f
2. points of intersection with the axes
3. asymptotes
4. monoticity of f & extreme points (maxima and minima of f)
5. convexity & inflection points.

E.g. Sketch the graph of $f(x) = x e^{-x^2}$.

- Domain: $D_f = \mathbb{R}$

$f(0) = 0$, and f intersects with the axes at $(0,0)$.

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x}{e^{x^2}} = 0$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x}{e^{x^2}} = 0$$

So $y = 0$ (the horiz. axis) is a horizontal asymptote of f at $+\infty$ and $-\infty$.

$$f'(x) = (1-2x^2)\,e^{-x^2}$$

$$f'(x) > 0 \iff -\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}$$

$$f'(x) < 0 \iff \left(x < -\frac{\sqrt{2}}{2} \quad \text{OR} \quad x > \frac{\sqrt{2}}{2}\right).$$

x	$-\infty$		$-\sqrt{2}/2$		$\sqrt{2}/2$		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	
$f(x)$	0	$\searrow$	$-\frac{1}{\sqrt{2e}}$	$\nearrow$	$\frac{1}{\sqrt{2e}}$	$\searrow$	0

$f \searrow (-\infty, -\sqrt{2}/2]$, $\quad f \nearrow [-\sqrt{2}/2, \sqrt{2}/2]$, $\quad f \searrow [\sqrt{2}/2, +\infty)$

$$f\left(-\frac{\sqrt{2}}{2}\right) = -\frac{1}{\sqrt{2e}}, \quad f\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{\sqrt{2e}}$$

f has a local minimum at $-\sqrt{2}/2$, the number $f(-\sqrt{2}/2) = -\frac{1}{\sqrt{2e}}$.

This is also a global minimum.

f has a local maximum at $\sqrt{2}/2$ the number $f(\sqrt{2}/2) = \frac{1}{\sqrt{2e}}$.

This is also a global maximum.

$$f''(x) = (4x^3 - 6x)\, e^{-x^2}$$
$$= 4x\left(x - \sqrt{\tfrac{3}{2}}\right)\left(x + \sqrt{\tfrac{3}{2}}\right) e^{-x^2}$$

x	$-\infty$	$-\sqrt{3/2}$		0		$\sqrt{3/2}$	$+\infty$
$f''(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\frown$		$\smile$		$\frown$		$\smile$

Inflection Points: $\left(-\sqrt{\frac{3}{2}}, f\left(-\sqrt{\frac{3}{2}}\right)\right)$, $(0, f(0))$, $\left(\sqrt{\frac{3}{2}}, f\left(\sqrt{\frac{3}{2}}\right)\right)$.