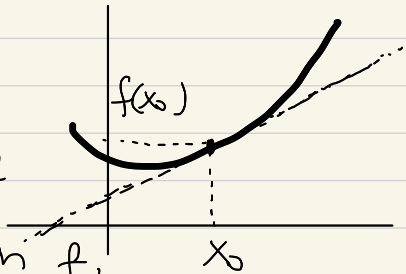
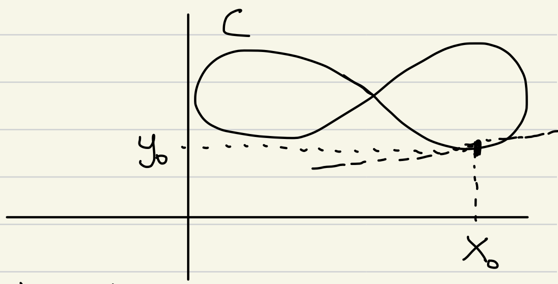
• IMPLICIT DIFFERENTIATION

Suppose we are given a curve $C$ in the plane and we want to find the slope of the curve at some given point $P_0$.




If the equation of the curve is
$$y = f(x)$$
for some differentiable function $f$, we know that the slope of $C$ at $(x_0, f(x_0))$ is $f'(x_0)$, and the tangent line is
$$y = f'(x_0)\cdot(x - x_0) + f(x_0).$$

Very often the equation of the curve has the form
$$F(x, y) = 0$$
for some $F: \mathbb{R}^2 \to \mathbb{R}$, and such curves are not always graphs of functions.

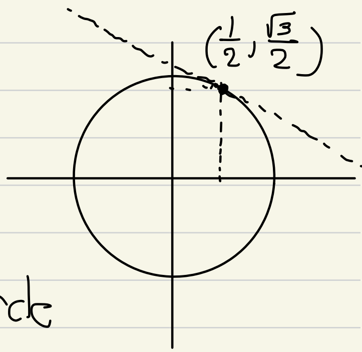



More generally, the relation between $x, y$ might be given implicitly in the form
$$F(x, y) = 0$$
and we might have to find the "rate of change" $\frac{dy}{dx}$.

If we have to find $\frac{dy}{dx}$ when $x=x_0, y=y_0$ then we differentiate the relation

$$F(x,y)=0$$

with respect to $x$.




E.g. Consider the unit circle

$$x^2+y^2-1=0 \quad (1)$$

Find the tangent line of the circle at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

- We differentiate (1) "implicitly" with respect to $x$.

$$x^2+y^2-1=0 \Rightarrow$$

$$2x+2y\cdot y'=0 \Rightarrow$$

$$y'=-\frac{x}{y}.$$

Therefore at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ – i.e. when $x=\frac{1}{2}$, $y=\frac{\sqrt{3}}{2}$ – the slope of the circle is equal to

$$\frac{dy}{dx}=-\frac{1/2}{\sqrt{3}/2}=-\frac{\sqrt{3}}{3}.$$

The equation of the tangent is

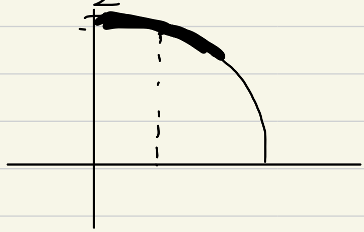
$$y - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{3} \cdot \left(x - \frac{1}{2}\right) \Rightarrow$$

$$y = -\frac{\sqrt{3}}{3} x + \frac{2\sqrt{3}}{3} .$$

* A different argument:

The circle is not the graph of a function, but in some interval around $\frac{1}{2}$ we can "solve wrt $y$" and get

$$y = \sqrt{1 - x^2} .$$



So there is a "part" of the circle that coincides with the graph of

$$f(x) = \sqrt{1 - x^2} .$$

Now we calculate $f'(x) = -\frac{1}{2\sqrt{1-x^2}} \cdot 2x = -\frac{x}{\sqrt{1-x^2}}$

and we find $f'(\frac{1}{2})$, etc...

This is correct, however we cannot always solve for $y$.

E.g. We are given $y \sin x = x^3 + \cos y$ and we need to find $\frac{dy}{dx}$.

(* Here we cannot solve for $y$!)

$$y \sin x = x^3 + \cos y \Rightarrow$$

$$\frac{d}{dx}(y \sin x) = \frac{d}{dx}(x^3 + \cos y) \Rightarrow$$

$$y' \cdot \sin x + y \cdot \cos x = 3x^2 - \sin y \cdot y' \Rightarrow$$

$$y' \cdot (\sin x + \sin y) = 3x^2 - y \cos x \Rightarrow$$

$$y' = \frac{3x^2 - y \cos x}{\sin x + \sin y}.$$

The same procedure can be applied to find derivatives of higher order.

(see Book).

- DERIVATIVE OF THE INVERSE FUNCTION

<u>THEOREM 3.13</u>: Let $f : I \to \mathbb{R}$, $I$ an interval. Assume $f$ is invertible on $I$ and differentiable at $x_0 \in I$ with $f'(x_0) \neq 0$. Then $f^{-1} : f(I) \to \mathbb{R}$ is differentiable at $y_0 = f(x_0)$ with

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} .$$

REMARK: The previous relation is often written as $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$.

This is easy to memorise because it looks like a quotient, but it is not!

To see why Theorem 3.13 is true, recall that

$$f^{-1}(f(x)) = x \qquad \text{for all } x \in I .$$

If we assume that $f^{-1}$ is differentiable on $I$,

$$(f^{-1})'(f(x)) \cdot f'(x) = 1, \quad \forall x \in I \implies$$

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}, \qquad \forall x \in I .$$

Now we can prove that

$$(\arcsin y)' = \frac{1}{\sqrt{1-y^2}}, \quad -1 < y < 1.$$

Let $-1<y<1$.
There exists $-\frac{\pi}{2} < x < \frac{\pi}{2}$ with $\sin x = y$.
Set $f(x) = \sin x$, $f^{-1}(x) = \arcsin x$
so Theorem 3.13 gives

$$(\arcsin y)' = \frac{1}{f'(x)} = \frac{1}{\cos x}$$

$$= \frac{1}{\sqrt{1-\sin^2 x}}$$

$$= \frac{1}{\sqrt{1-y^2}}.$$

* Apply Theorem 3.13 to prove that

$$(\arctan x)' = \frac{1}{1+x^2}.$$

- TAYLOR'S THEOREM

For any $n = 1, 2, \ldots$ we define the factorial of $n$ to be the number

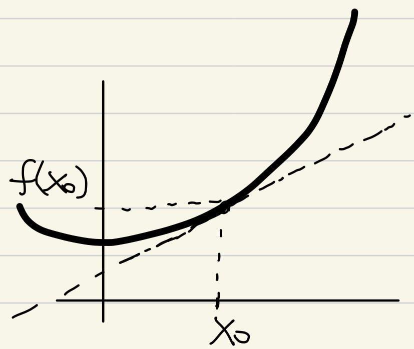
$$n! = 1 \cdot 2 \cdot \ldots \cdot n$$

For $n = 0$, we define $0! = 1$.

Thus
$$2! = 1 \cdot 2 = 2$$
$$3! = 1 \cdot 2 \cdot 3 = 6$$
$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$
$$\vdots$$

Let $f : I \to \mathbb{R}$ be differentiable and $x_0 \in I$. We know that the tangent of $G_f$ at $x_0$ is

$$y = f'(x_0) \cdot (x - x_0) + f(x_0).$$




The function

$$P_1(x) = f(x_0) + f'(x_0) \cdot (x - x_0)$$

is a $1^{st}$-degree polynomial and is called the linearisation of $f$ at $x_0$.

$P_1(x)$ is the best $1^{st}$-degree (linear) approximation to $f$, because

$$P_1(x_0) = f(x_0) \quad \text{and} \quad P_1'(x_0) = f'(x_0).$$

What if we try to approximate $f$ by polynomials of higher degree? Suppose $f$ is twice diff. and we want to find a polynomial $P_2(x)$ of 2nd degree such that

$$\begin{cases} P_2(x_0) = f(x_0) \\ P_2'(x_0) = f'(x_0) \\ P_2''(x_0) = f''(x_0) \end{cases}.$$

Assume $P_2(x) = \underline{a_2} \cdot (x-x_0)^2 + \underline{a_1} \cdot (x-x_0) + \underline{a_0}$.

Then

$$P_2(x_0) = f(x_0) \implies a_0 = f(x_0).$$

$$P_2'(x_0) = f'(x_0) \implies a_1 = f'(x_0)$$

$$P_2''(x_0) = f''(x_0) \implies 2a_2 = f''(x_0)$$

$$\implies a_2 = \frac{f''(x_0)}{2}.$$

$$P_2(x) = f(x_0) + f'(x_0) \cdot (x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2.$$

Even more generally, suppose $f$ is $n$ times diff. and we want to find a polynomial $P_n(x)$ of degree $n$ such that

$$P_n(x_0) = f(x_0), \quad P_n'(x_0) = f'(x_0), \ldots, P_n^{(n)}(x_0) = f^{(n)}(x_0).$$

Set

$$P_n(x) = a_0 + a_1 \cdot (x - x_0) + \ldots + a_n (x - x_0)^n.$$

A direct calculation of the derivatives yields

$$a_0 = f(x_0), \quad a_1 = f'(x_0), \quad a_2 = \frac{1}{2} f''(x_0),$$

$$a_3 = \frac{f^{(3)}(x_0)}{2 \cdot 3} = \frac{f^{(3)}(x_0)}{3!},$$

and generally

$$a_k = \frac{1}{k!} f^{(k)}(x_0), \qquad k = 0, 1, \ldots, n.$$

If $f$ is $n$-times differentiable in $I$ an $x_0 \in I$, then the polynomial

$$P_n(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

is called the <u>n-th Taylor polynomial of $f$ at the point $x_0$.</u>

We have seen that the n-th Taylor pol. has the same derivatives with $f$ at $x_0$, but how good an approximation is it to $f(x)$? (In other words, how small is the difference $f(x) - P_n(x)$?).

THEOREM 3.14 (Taylor's Theorem): Suppose $f: I \to \mathbb{R}$ is $n+1$ times differentiable and $x_0 \in I$. Let

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

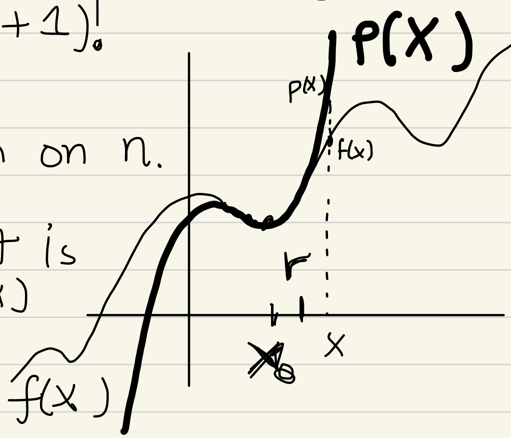
be the n-th Taylor polynomial of $f$ at $x_0$. Then for any $x \in I$ there exists some $r$ between $x_0$ and $x$ such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(r)}{(n+1)!}(x-x_0)^{n+1}.$$

PROOF (NON-EXAMINABLE).

Assume $x > x_0$.

We will use induction on $n$.




- For $n=0$, the statement is that there exists $r \in (x_0, x)$ such that

$$f(x) = f(x_0) + f'(r)\cdot(x-x_0).$$

This is true by the Mean Value Theorem.

- Assume the Theorem is true for some $n$.
- We prove that it is also true for $n+1$.

Set
$$E_n(x) = f(x) - P_n(x)$$

Then $E_n$ is $n$-times diff. and

$$E_n'(x) = f'(x) - P_n'(x)$$
$$= f'(x) - \left(f(x_0) + f'(x_0)(x-x_0) + \dots\right)'.$$

By the Generalised MVT for the functions $E_n(x)$ and $(x-x_0)^{n+1}$, there exists $r \in (x_0, x)$ with

$$\frac{E_n(x) - E_n(x_0)}{(x-x_0)^{n+1}} = \frac{E_n'(r)}{(n+1)\cdot(r-x_0)^n} \Longrightarrow$$

$$\frac{E_n(x)}{(x-x_0)^{n+1}} = \frac{E_n'(r)}{(n+1)(r-x_0)^n}.$$

Now observe that $E_n'(x)$ is the $n$-th degree Taylor polynomial of $f'$, so by the inductive hypothesis it is equal to $\frac{f^{(n+1)}(t)}{n!}(x-x_0)^n$.

E.g. Find the $3^{rd}$ degree Taylor polynomial of $f(x) = e^{x-1}$ at $x_0 = 1$.

- $f(x) = e^{x-1}$
  $f'(x) = e^{x-1}$, $f''(x) = e^{x-1}$, $f'''(x) = e^{x-1}$.

The $3^{rd}$ Taylor pol. at $x_0 = 1$ is

$$P_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= 1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 .$$

In the specific case when $x_0 = 0$, the Taylor polynomial around 0 is called the <u>McLaurin polynomial</u> of $f$.

E.g. Find the McLaurin polynomial of $5^{\text{th}}$ degree of $f(x) = \sin x$.

- $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f^{(3)}(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$f^{(5)}(x) = \cos x$

$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0,$

$f^{(3)}(0) = -1, \quad f^{(4)}(0) = 0, \quad f^{(5)}(0) = 1.$

The McLaurin pol. of $5^{\text{th}}$ degree is

$$P_5(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(5)}(0)}{5!}x^5$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$