Kandidatnr: 10009. Problem 6

$u_{xx}+4u_x=f(x)$

$0<x<1$

$u(0)=1$

$u(1)=4$

$f(x)=e^{x-\frac{1}{2}}$

(i) $h=\frac{1}{M+1}$

$x_m=mh,\ m=0,1,\dots,M+1$

$x_0=0$

$x_{M+1}=1$

Central:

$u'(x)\approx\frac{u(x_{m+1})-u(x_{m-1})}{2h}$

$u''(x)\approx\frac{u(x_{m-1})-2u(x_m)+u(x_{m+1})}{h^2}$

$u''(x_m)+4u'(x_m)=\frac{u(x_{m-1})-2u(x_m)+u(x_{m+1})}{h^2}+4\frac{u(x_{m+1})-u(x_{m-1})}{2h}$

$=\frac{u_{m-1}-2u_m+u_{m+1}}{h^2}+\frac{2u_{m+1}-2u_{m-1}}{h}$

$=\frac{u_{m-1}-2hu_{m-1}-2u_m+u_{m+1}+2hu_{m+1}}{h^2}$

$=\frac{1}{h^2}\left((1-2h)u_{m-1}-2u_m+(1+2h)u_{m+1}\right)$

$=f(x_m),\ m=1,2,\dots,M$

$A_h=\frac{1}{h^2}\text{tridiag}(1-2h,-2,1+2h)\in\mathbb{R}^{M\times M}$

$$u=\begin{pmatrix}u_1\\ \vdots\\ u_M\end{pmatrix}$$

From boundary conditions $u(0)=1,\ u(1)=4$

$$F=\begin{pmatrix}f(x_1)-\frac{1}{h^2}\\ f(x_2)\\ \vdots\\ f(x_{M-1})\\ f(x_M)-\frac{4}{h^2}\end{pmatrix}$$

(ii) $\lim_{h\to 0^+}\rho(A_h^{-1})=\lim_{h\to 0^+}\left(\max_{1\le s\le M}\left|\frac{1}{\lambda_s}\right|\right)$

$\lambda_s=-2+2\sqrt{1+2h}\sqrt{1-2h}\cdot\cos\left(\frac{s\pi}{M+1}\right)$

$s=1,\dots,M$

$\left|\frac{1}{\lambda_s}\right|$ is maximum when $|\lambda_s|$ is smallest

$2\sqrt{1+2h}\sqrt{1-2h}\cdot\cos\left(\frac{s\pi}{M+1}\right)=2$

$\cos\left(\frac{s\pi}{M+1}\right)=\frac{1}{\sqrt{1+2h}\sqrt{1-2h}}$

$s=\frac{M+1}{\pi}\cdot\cos^{-1}\left(\frac{1}{\sqrt{1+2h}\sqrt{1-2h}}\right)$

$h\to 0^+$