Innlevering 4

1.

Not every matrix is a covariance matrix of a random vector because not every matrix is square, symmetric or positive semi-definite

Not every matrix is a covariance matrix of two random vectors either for the same reason as above.

2.

$X=(X_1,X_2,X_3)^T$ trivariate random vector

$X_i \sim N(\cdot,\cdot)$

$i=1,2,3$

(a) No. X is not always a trivariate normal random vector.

X is a normal random vector iff (if and only if) any linear combination $\sum_{i=1}^{3} a_i X_i,\ a_i \in \mathbb{R}$ is a normal random variable

Since the X_i-s are not independent, this is not always the case

(b) If we now suppose that the X_i-s are independent, the condition holds, and X is therefore a normal random vector

(c) Denote

$Y_1 = X_1$

$Y_2 = (X_2, X_3)$

If Y_2 is normally distributed, then the linear combination $\sum_{i=1}^{2} a_i Y_i$ is also normally distributed, which implies that $X=(Y_1,Y_2)$ is normally distributed

But for Y_2 to be normally distributed, X_2 and X_3 needs to be independent, and we dont know that they are

So this condition does not simply imply that X is normally distributed

3.

$X=(X_1,\dots,X_n) \sim N(\beta_0, 1)$

$Y=(Y_1,\dots,Y_m)$

$Y_i = \beta_0 + \beta_1 y_i + \varepsilon_i$

$i=1,\dots,m$

y_i-s known

$\varepsilon_i \sim N(0,1)$

(a) Finds $\hat{\beta}_0$:

Uses X_i-s

$\sum_{i=1}^{n}(X_i-\bar{X})^2 = \sum_{i=1}^{n}(X_i-\beta_0)^2$

$\frac{\partial}{\partial \beta_0}\left(\sum_{i=1}^{n}(X_i-\beta_0)^2\right) = \frac{\partial}{\partial \beta_0}\left(\sum_{i=1}^{n} X_i^2 - 2\sum_{i=1}^{n} X_i\beta_0 + \sum_{i=1}^{n}\beta_0^2\right)$

$= \frac{\partial}{\partial \beta_0}\left(\sum_{i=1}^{n} X_i^2 - 2\beta_0\sum_{i=1}^{n} X_i + n\beta_0^2\right)$

$= -2\sum_{i=1}^{n} X_i + 2n\beta_0$

Equal to zero:

$-2\sum_{i=1}^{n} X_i + 2n\beta_0 = 0$

$2n\beta_0 = 2\sum_{i=1}^{n} X_i$

$\hat{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

Finds out if it is unbiased

For $\hat{\beta}_0$ to be unbiased, the following condition must hold:

$E[\hat{\beta}_0] = \beta_0$

$E[\hat{\beta}_0] = E[\frac{1}{n}\sum_{i=1}^{n} X_i]$

$= \frac{1}{n}E[\sum_{i=1}^{n} X_i]$

$= \frac{1}{n}\cdot n\beta_0$

$= \beta_0$

Unbiased estimator for β_0 is thus $\hat{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

Finds $\hat{\beta}_1$: Uses Y_i-s

$\sum_{i=1}^{m}(Y_i-\bar{Y})^2 = \sum_{i=1}^{m}(Y_i-(\hat{\beta}_0+\hat{\beta}_1 y_i))^2$

$\frac{\partial}{\partial \beta_1}\left(\sum_{i=1}^{m}(Y_i-(\hat{\beta}_0+\hat{\beta}_1 y_i))^2\right) = \sum_{i=1}^{m}\frac{\partial}{\partial \beta_1}\left((Y_i-(\hat{\beta}_0+\hat{\beta}_1 y_i))^2\right)$

$= \sum_{i=1}^{m} 2(Y_i-(\hat{\beta}_0+\hat{\beta}_1 y_i))(-y_i)$

$= -2\sum_{i=1}^{m} y_i(Y_i-(\hat{\beta}_0+\hat{\beta}_1 y_i))$

Set it equal to zero

$-2\sum_{i=1}^{m} y_i(Y_i-(\hat{\beta}_0+\hat{\beta}_1 y_i))=0$

$\sum_{i=1}^{m} y_i\left(Y_i-\left(\frac{\sum_{j=1}^{n} X_j}{n}+\hat{\beta}_1 y_i\right)\right)=0$

$\sum_{i=1}^{m} y_i Y_i - \sum_{i=1}^{m} y_i \frac{\sum_{j=1}^{n} X_j}{n} - \sum_{i=1}^{m}\hat{\beta}_1 y_i^2 = 0$

$\hat{\beta}_1 = \left(-\sum_{i=1}^{m} y_i Y_i + \sum_{i=1}^{m} y_i \frac{\sum_{j=1}^{n} X_j}{n}\right) / \left(-\sum_{i=1}^{m} y_i^2\right)$

$= \left(\sum_{i=1}^{m} y_i \left(Y_i - \frac{\sum_{j=1}^{n} X_j}{n}\right)\right) / \left(\sum_{i=1}^{m} y_i^2\right)$

Finds out if its unbiased

$E[\hat{\beta}_1] =$

4.

$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$

ε-s are independent and $\varepsilon_i \sim N(0,\sigma^2),\ i=1,\dots,30$

$x_{i1} = \begin{cases} 1, & 11 \le i \le 20 \\ 0, & \text{otherwise} \end{cases}$

$x_{i2} = \begin{cases} 1, & 21 \le i \le 30 \\ 0, & \text{otherwise} \end{cases}$

$R^2 = 0.4246$

$SST = 42.6397$

(a) $H_0:\ \beta_1 = \beta_2 + 1$

$H_1:\ \beta_1 \neq \beta_2 + 1$

$\alpha = 0.05$

F-test:

Let $A = (0,1,-1)$

$\Rightarrow H_0:\ A\beta = 1$

$H_1:\ A\beta \neq 1$

Test statistic:

$F = \frac{(A\hat{\beta}-1)^T (A(X^TX)^{-1}A^T)^{-1}(A\hat{\beta}-d)}{SSE/(n-p)}$

$n=30,\ p=3,\ r=1$

$\Rightarrow F \overset{\text{Under } H_0}{\sim} f_{1,27}$

Finds SSE

$R^2 = 1 - \frac{SSE}{SST}$

$\Rightarrow SSE = SST(1-R^2)$

$= 42.6397(1-0.4246)$

$= 24.53488338$

≈ 24.5349

$X = \begin{pmatrix} 1_{10} & 0_{10} & 0_{10} \\ 1_{10} & 1_{10} & 0_{10} \\ 1_{10} & 0_{10} & 1_{10} \end{pmatrix}$

0_{10} (10×1) vector of zeros

1_{10} (10×1) vector of ones

$X^TX = \begin{pmatrix} 30 & 10 & 10 \\ 10 & 10 & 0 \\ 10 & 0 & 10 \end{pmatrix}$

$(X^TX)^{-1} = \begin{pmatrix} \frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{1}{5} & \frac{1}{10} \\ -\frac{1}{10} & \frac{1}{10} & \frac{1}{5} \end{pmatrix}$

Finds $(A(X^TX)^{-1}A)^{-1}$

$A(X^TX)^{-1}A^T = (0,1,-1)\begin{pmatrix} \frac{1}{10} & -\frac{1}{10} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{1}{5} & \frac{1}{10} \\ -\frac{1}{10} & \frac{1}{10} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$= \frac{1}{5}$

$(A(X^TX)^{-1}A^T)^{-1} = 5$

$\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)^T$

$Y_1 = \hat{\beta}_0 + \varepsilon_1$

$\vdots$

$Y_{10} = \hat{\beta}_0 + \varepsilon_1$

$Y_{11} = \hat{\beta}_0 + \hat{\beta}_1 + \varepsilon_{11}$

$\vdots$

$Y_{20} = \hat{\beta}_0 + \hat{\beta}_1 + \varepsilon_{20}$

$Y_{21} = \hat{\beta}_0 + \hat{\beta}_2 + \varepsilon_{21}$

$\vdots$

$Y_{30} = \hat{\beta}_0 + \hat{\beta}_2 + \varepsilon_{30}$

Finner $\hat{\beta}_0$, $\hat{\beta}_1$ og $\hat{\beta}_2$:

$\hat{\beta}_0 = Y_2 - \varepsilon_2$

$= 1.4 + 1.1$

$= 2.5$

$\hat{\beta}_1 = Y_{12} - \hat{\beta}_0 - \varepsilon_{12}$

$= 2.5 - 2.5 + 0.8$

$= 0.8$

$\hat{\beta}_2 = Y_{25} - \hat{\beta}_0 - \varepsilon_{25}$

$= 5.7 - 2.5 - 1.4$

$= 1.8$

$\Rightarrow \hat{\beta} = (2.5, 0.8, 1.8)^T$

Finds $(A\hat{\beta}-1)^T$ and $(A\hat{\beta}-1)$

$A\hat{\beta} - 1 = (0,1,-1)\cdot(2.5,0.8,1.8)^T - 1$

$= -2$

$\Rightarrow F = \frac{(-2)\cdot 5\cdot(-2)}{24.5349/27}$

$= 22.01$

$f_{0.05,1,27} = 4.21$

$F > f_{0.05,1,27}$ so H_0 is rejected