Øving 10

1. $\int_C F \cdot dr$

$F(x,y,z) = (x,y,z,y)$

$C: (x-1)^2 + (y-1)^2 = 1$

2. $\int_C \sqrt{y}\, ds$

$ds = |\dot{\gamma}(x)|\, dx$

$\gamma(t) = (\cos(t), \cos^2(t)),\ t \in [0, \frac{\pi}{2}]$

$\dot{\gamma}(t) = (-\sin(t), -2\cos(t)\sin(t))$

$= (-\sin(t), -\sin(2t))$

3. $F(x,y,z) = (zy^2e^{xz}, 2ye^{xz}, xy^2e^{xz})$

(i) $F(x,y,z) = F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k$

$= \nabla\phi(x,y,z)$

$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zy^2e^{xz} & 2ye^{xz} & xy^2e^{xz} \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)k$$

$$= (2xye^{xz} - 2xye^{xz})i + (y^2e^{xz} + xzy^2e^{xz} - y^2e^{xz} - xzy^2e^{xz})j + (2yze^{xz} - 2yze^{xz})k$$

$= 0 \cdot i + 0 \cdot j + 0 \cdot k$

$\Rightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$ ■

$\int zy^2e^{xz}\, dx = y^2e^{xz} + g_1(y,z)$

$\int 2ye^{xz}\, dy = y^2e^{xz} + g_2(x,z)$

$\int xy^2e^{xz}\, dz = y^2e^{xz} + g_3(x,y)$

$\underline{f(x,y,z) = y^2e^{xz} + C}$, C konstant

(ii) $\int F \cdot dr$

$r(t) = (\cos(2t), \sin(t), t(\pi - 2t)),\ t \in [0, \frac{\pi}{2}]$

F konservativt $\Rightarrow \int_0^{\frac{\pi}{2}} F \cdot dr = f(r(\frac{\pi}{2})) - f(r(0))$

$f(r(t)) = \sin^2(t)\, e^{\cos(2t)t(\pi-2t)}$

$f(r(\frac{\pi}{2})) = 1 \cdot e^0 = 1$

$f(r(0)) = 0$

$\underline{\int_0^{\frac{\pi}{2}} F \cdot dr = 1}$

(iii) $\int F \cdot dr$

$r(t) = (\cos(2t), \sin(t), t(\pi - 2t)),\ t \in [0, \frac{\pi}{2}]$

$r'(t) = (-2\sin(t), \cos(t), \pi - 2t - 2t)$

$r(t) = (\cos(2t), \sin(t), t(\pi - 2t)),\ t \in [0, \frac{\pi}{2}]$

$r'(t) = (-2\sin(t), \cos(t), \pi - 2t - 2t)$

$= (-2\sin(t), \cos(t), \pi - 4t)$

$F \cdot r'(t) =$

4. $F(x,y,z) = (yz, xz, xy)$

$r(t) = (\cos(t), \sin(t), t),\ t \in [0, \frac{\pi}{4}]$

$r'(t) = (-\sin(t), \cos(t), 1)$

$F(r(t)) = (t\sin(t), t\cos(t), \cos(t)\sin(t))$

$F(r(t)) \cdot r'(t) = -t\sin^2(t) + t\cos^2(t) + \cos(t)\sin(t)$

5. $F = (F_1, F_2, F_3)$

$G = (G_1, G_2, G_3)$

$G_1 = \frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}$

$G_2 = \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}$

$G_3 = \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}$

(a) F konservativ

$\Rightarrow \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$

$\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$

$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$

$\Rightarrow G_1 = 0$

$G_2 = 0$

$G_3 = 0$

$\Rightarrow G = 0$ $\boxed{\text{M}}$

(b) $(x,y,z) \mapsto (x, y, \pi x)$

$\frac{\partial F_3}{\partial y} = 0 = \frac{\partial F_2}{\partial z}$

$\frac{\partial F_1}{\partial z} = 0 \neq \frac{\partial F_3}{\partial x}$

Ikke konservativt

(c) $F(x,y,z) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$

$\frac{\partial F_3}{\partial y} = 0 = \frac{\partial F_2}{\partial z}$

$\frac{\partial F_1}{\partial z} = 0 = \frac{\partial F_3}{\partial x}$

$\frac{\partial F_2}{\partial x} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2}$

$= \frac{-x^2 + y^2}{(x^2+y^2)^2}$

$\frac{\partial F_1}{\partial y} = -\left(\frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}\right)$

$= -(x^2 - y^2$

6. $\int_C x\,ds$

7.(a) $F(x,y)=(x,y)$

$\frac{dx}{x}=\frac{dy}{y}$

$\int\frac{dx}{x}=\int\frac{dy}{y}$

$\ln|x|=\ln|y|+c_1$

$x=y\cdot e^{c_1}=x=y\cdot C$

$y=\frac{x}{C}$, C konstant

(b) $F(x,y)=(x,x^2)$

$\frac{dx}{x}=\frac{dy}{x^2}$

$dy=x\,dx$

$y=\frac{x^2}{2}+C$, C konstant

(c) $F(x,y)=(x\ln(x),y)$

$\frac{dx}{x\ln(x)}=\frac{dy}{y}$

$\int\frac{dx}{x\ln(x)}=\int\frac{dy}{y}$

$\int\frac{du}{u}=\int\frac{dy}{y}$

$\ln|u|=\ln|y|+c_1$

$\ln|\ln(x)|=\ln|y|+c_1$

$\ln(x)=y\cdot e^{c_1}=y\cdot C$

$y=\frac{\ln(x)}{C}$, C konstant

8. $\int_C\frac{-y\,dx+x\,dy}{x^2+y^2}$

$P=-\frac{y}{x^2+y^2}$

$Q=\frac{x}{x^2+y^2}$

(a) $\int_{-a}^{a}\int_{-a}^{a}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$

$\frac{\partial Q}{\partial x}=\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}=\frac{-x^2+y^2}{(x^2+y^2)^2}$

$\frac{\partial P}{\partial y}=-\left(\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right)=\frac{-x^2+y^2}{(x^2+y^2)^2}$

$\int_{-a}^{a}\int_{-a}^{a}$