BASIC COURSE IN ANALYSIS I

1. INTRODUCTION

Sets are collections of elements, e.g.

$$A = \{1, 2, 3, 4\},$$

$$B = \{a, e, i, o, u\},$$

$$P = \{2, 3, 5, 7, \dots\}$$

(the set of prime numbers).

When we are given a set A, the order of writing the elements is not important, ie. $\{a, b\} = \{b, a\}$.

When x is an element of the set A, we write $x \in A$. Otherwise we write $x \notin A$.

There exists the set $\emptyset = \{\ \}$ which does not contain any element. $\emptyset$ is called the <u>empty set</u>.

Given two sets A, B we say $A = B$ if

$$x \in A \iff x \in B$$

(in other words A, B have the same elements).
We say $A \subseteq B$ (A is a subset of B) if

$$x \in A \implies x \in B.$$

Given sets A, B we can define the following operations:

$$A \cup B = \{x : x \in A \text{ or } x \in B\} \quad \text{(union)}$$

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \quad \text{(intersection)}$$

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\} \quad \text{(set-theoretic difference)}$$

$$A \times B = \{(a, b) : a \in A,\ b \in B\} \quad \text{(Cartesian product)}.$$

E.g. when $A = \{1, 2, 5\}$, $B = \{2, 11\}$ then

$$A \cup B = \{1, 2, 5, 11\},$$
$$A \cap B = \{2\}$$
$$A \setminus B = \{1, 5\},$$
$$A \times B = \{(a, b) : a \in A, b \in B\}$$
$$= \{(1,2), (1,11), (2,2), (2,11), (5,2), (5,11)\}$$

Also when we take $A = B = \mathbb{R}$, then the Cartesian product of these two sets is

$$\mathbb{R} \times \mathbb{R} \equiv \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

Two sets X, Y are called <u>disjoint</u> if

$$X \cap Y = \emptyset$$

(i.e. they have no common elements).

Some well-known sets of real numbers:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$
$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$
$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z},\ b \in \mathbb{Z} \setminus \{0\} \right\}$$

$\mathbb{R}$: the set of all real numbers.

$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q}$
and we can also show that $\mathbb{Q} \subsetneq \mathbb{R}$.

<u>LEMMA 1.1</u>: Let $n \in \mathbb{N}$. Then

$$n \text{ is even} \iff n^2 \text{ is even.}$$

<u>PROOF</u>

Suppose n is even. Then

$$n = 2k,\ k \in \mathbb{N} \Rightarrow n^2 = 4k^2,\ k \in \mathbb{N}$$
$$\Rightarrow n^2 \text{ is even.}$$

Suppose n is odd. Then

$$n = 2k+1,\ k \in \mathbb{N} \Rightarrow n^2 = 4k^2 + 4k + 1,\ k \in \mathbb{N}$$
$$\Rightarrow n^2 \text{ is odd.} \quad \blacksquare$$

PROPOSITION 1.2 : $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

PROOF

Let's assume for contradiction that $\sqrt{2} \in \mathbb{Q}$.

Then $\sqrt{2} = \dfrac{p}{q}$, where $p, q \in \mathbb{N}$ and $\gcd(p,q) = 1$.

$$\sqrt{2} = \frac{p}{q} \iff p = \sqrt{2}\, q$$
$$\iff p^2 = 2q^2$$
$$\implies p^2 \text{ is even}$$
$$\implies p \text{ is even} \quad \text{(by Lemma 1.1)}.$$

So $p = 2r$, with $r \in \mathbb{N}$. Now

$$p^2 = 2q^2 \implies 4r^2 = 2q^2$$
$$\implies q^2 = 2r^2$$
$$\implies q^2 \text{ is even}$$
$$\implies q \text{ is even} \quad \text{(by Lemma 1.1)},$$

We have shown that both p and q are even, so $\gcd(p,q) > 1$; a contradiction.

Thus

$$\sqrt{2} \notin \mathbb{Q}.$$

$\blacksquare$

Remark: Whenever a is not a "perfect square" (i.e. $1, 4, 9, 16, 25, \dots$) then $\sqrt{a} \notin \mathbb{Q}$ (the proof is similar).

A subset $I \subseteq \mathbb{R}$ is called an <u>interval</u> if for any $x, y \in I$ with $x < y$, we have

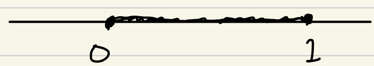
$$x < t < y \Rightarrow t \in I.$$

(If an interval contains two real numbers, it will contain all other numbers in between).

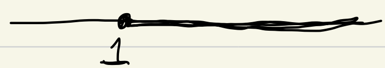
Intervals of $\mathbb{R}$ have one of the following forms:

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \quad \text{(closed interval)}$$
$$(a, b) = \{x \in \mathbb{R} : a < x < b\} \quad \text{(open interval)}$$
$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$
$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

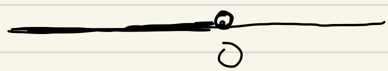
E.g. $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$.



$[1, +\infty) = \{x \in \mathbb{R} : x \geq 1\}$



$(-\infty, 0) = \{x \in \mathbb{R} : x < 0\}$



Note that $\mathbb{R} = (-\infty, +\infty)$ and $\emptyset$ are also intervals.

We want to introduce a way to measure distances between real numbers.

The absolute value $|x|$ of $x \in \mathbb{R}$ is defined by

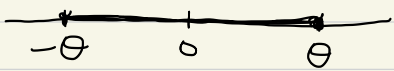
$$|x| = \begin{cases} x, & \text{if } x \geqslant 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

E.g. $|5| = 5$, $\left|\frac{3}{2}\right| = \frac{3}{2}$, $|-1| = 1$,

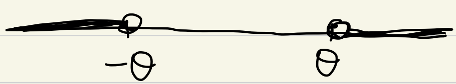
$|-\sqrt{2}| = \sqrt{2}$, $|0| = 0$.

The following hold:

- $|x| = \theta \iff x = \pm\theta$ (where $\theta > 0$).
- $|x| \leq \theta \iff -\theta \leqslant x \leqslant \theta$ $(\theta > 0)$
- $|x| > \theta \iff (x < -\theta$ or $x > \theta)$.

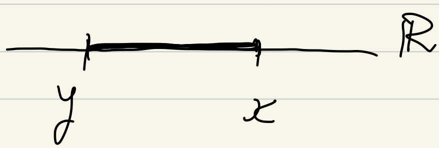


$|x| \leq \theta$



$|x| > \theta$

The distance between $x, y \in \mathbb{R}$ is defined to be $|x-y|$.



THEOREM 1.3 (Triangle Inequality): For any $x, y \in \mathbb{R}$,

$$|x-y| \leqslant |x| + |y|.$$

PROOF

$$|x-y| \leqslant |x| + |y| \iff |x-y|^2 \leqslant (|x|+|y|)^2$$

$$\iff (x-y)^2 \leqslant |x|^2 + |y|^2 + 2|xy|$$

$$\iff \cancel{x^2+y^2} - 2xy \leqslant \cancel{x^2+y^2} + 2|xy|$$

$$\iff -xy \leqslant |xy|$$

$$\iff -xy \leqslant |-xy|$$

True (because $|t| \geqslant t$ for any $t \in \mathbb{R}$), so the equivalent initial inequality will also be correct. ∎

Note that this implies also that

$$|x+y| \leqslant |x| + |y| \text{ for any } x, y \in \mathbb{R}.$$

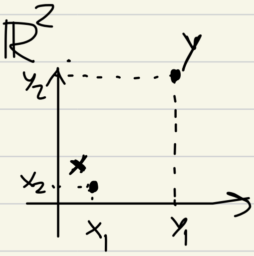
To see this, note that

$$|x+y| = |x-(-y)|$$
$$\leqslant |x| + |-y|$$
$$= |x| + |y|.$$

For any $x \in \mathbb{R}$, $|x| = |-x|$ (by definition).
Also $|x|^2 = x^2$

In two dimensions, we have the set
$$\mathbb{R}^2 = \{ (x,y) : x,y \in \mathbb{R} \}.$$

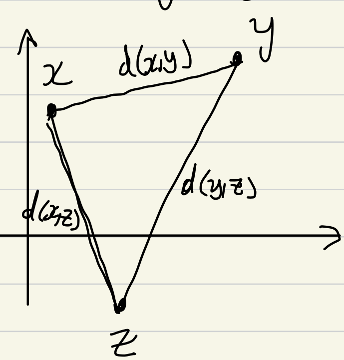
Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$.
The distance between $\mathbf{x}, \mathbf{y}$ is defined as



$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

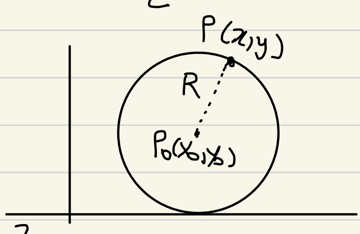
The triangle inequality in two dimensions is the following:

$$d(x,y) \leq d(x,z) + d(z,y) \text{ for any } x,y,z \in \mathbb{R}^2.$$



The "triangle inequality" states that every side of the triangle is less than the sum of the other two sides.

A <u>circle</u> in $\mathbb{R}^2$ with center (x_0, y_0) and radius $R > 0$ is the set of points (x,y) with distance from (x_0, y_0) is equal to R.

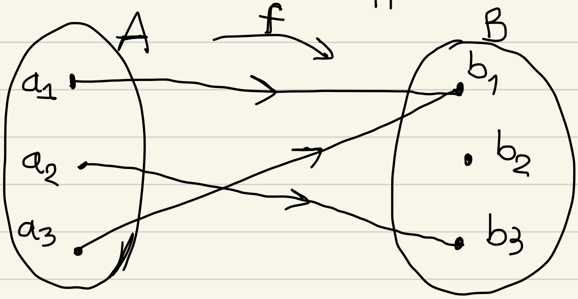


What is the equation <u>of this circle?</u>

$$d(P, P_0) = R \iff \sqrt{(x-x_0)^2 + (y-y_0)^2} = R$$

$$\iff \boxed{(x-x_0)^2 + (y-y_0)^2 = R^2}$$

Given two sets A, B a function $f: A \to B$ is a procedure by which every element of A is mapped to a unique element of B.
For each $a \in A$, the unique element of B to which a is mapped is denoted by $f(a)$.



$f(a_1) = b_1$
$f(a_2) = b_3$
$f(a_3) = b_1$

Formal Definition of a function:

A function $f: A \to B$ is a subset $f \subseteq A \times B$ with the property that for all $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$.
The unique such $b \in B$ is written as $f(a)$.

(Here $A \times B$ is the Cartesian product).

We write $f: A \to B$,

$$x \mapsto f(x).$$

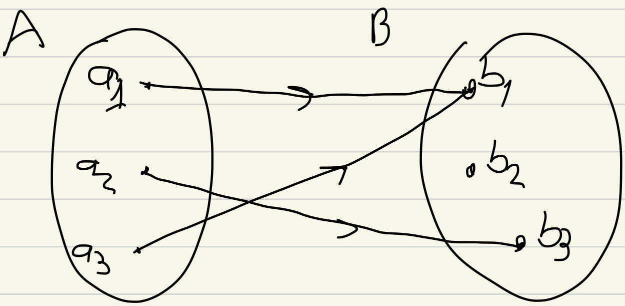
The set A is called the <u>domain</u> of f (or domain of definition of f).

The set B is called the <u>codomain</u> of f. The set

$$f(A) = \{ b \in B : \text{there exists } a \in A \text{ s.t. } b = f(a) \}$$

is called the <u>range</u> or <u>image</u> of f.

In the first example, A is the domain, B is the codomain and the <u>range</u> of f is $f(A) = \{ b_1, b_3 \}$.



$f(a_1) = b_1$

$f(a_2) = b_3$

$f(a_3) = b_1$

$$B = \{ b_1, b_2, b_3 \}.$$

$$f(A) = \{ b_1, b_3 \}$$
$$= \{ f(a) : a \in A \}$$