Øving 5

12.1

14.

(a) $u=v(x)+w(y)$ $\quad u_{xy}=0$ (1)

$u=v(x)w(y)$ $\quad uu_{xy}=u_x u_y$ (2)

$u=v(x+2t)+w(x-2t)$ $\quad u_{tt}=4u_{xx}$ (3)

(1):

$$u_{xy}=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}u\right)=\frac{\partial}{\partial y}\left(v'(x)\right)=0$$

O.K.

(2):

$$uu_{xy}=u\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}u\right)=u\frac{\partial}{\partial y}\left(v'(x)w(y)\right)=uv'(x)w'(y)=v(x)v'(x)w(y)w'(y)$$

$$u_x u_y=v'(x)w(y)\cdot v(x)w'(y)=uu_{xy}$$

O.K

(3):

$$u_{tt}=\frac{\partial^2}{\partial t^2}\left(v(x+2t)+w(x-2t)\right)=\frac{\partial}{\partial t}\left(2v'(x+2t)-2w'(x-2t)\right)=4v''(x+2t)+4w''(x-2t)$$

$$u_{xx}=\frac{\partial^2}{\partial x^2}\left(v(x+2t)+w(x-2t)\right)=\frac{\partial}{\partial x}\left(v'(x+2t)+w'(x-2t)\right)=v''(x+2t)+w''(x-2t)$$

$\Rightarrow u_{tt}=4u_{xx}$

O.K

15.

$u(x,y)=a\cdot\ln(x^2+y^2)+b$

Laplace:

$u_{xx}+u_{yy}=0$

Sjekk:

$$u_{xx}+u_{yy}=\frac{\partial}{\partial x}\left(\frac{2ax}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{2ay}{x^2+y^2}\right)$$

$$=2a\left(\frac{1\cdot(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}+\frac{1\cdot(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2}\right)$$

$$=2a\left(\frac{\cancel{x^2}+\cancel{y^2}-\cancel{2x^2}+\cancel{x^2}+\cancel{y^2}-\cancel{2y^2}}{(x^2+y^2)^2}\right)$$

$$=2a\left(\frac{0}{(x^2+y^2)^2}\right)=0$$

OK.

$x^2+y^2=1,\ u=110 \Rightarrow a\cdot\ln(1)+b=110$ (1)

$x^2+y^2=100,\ u=110 \Rightarrow a\cdot\ln(100)+b=110$ (2)

Fra (1)

$a\cdot 0+b=110 \Rightarrow b=110$

$$\Rightarrow a=\frac{100-110}{\ln(100)}=-\frac{5}{\ln(10)}$$

12.3

5.

$c^2=1,\ L=1$

$u(0,t)=0=u(L,t),\ t\geq 0$

$u(x,0)=f(x)=k\sin(3\pi x)$

$u_x(x,0)=g(x)=0$

$$u(x,0)=\begin{cases}0, & x\leq 0\\ k\sin(3\pi x), & 0<x<\frac{1}{3}\\ 0, & x\geq\frac{1}{3}\end{cases}$$

$$u(x,t)=\sum_{n=1}^{\infty}\left(B_n\cos(\lambda_n t)+B_n^*\sin(\lambda_n t)\right)\sin(n\pi x),\ \lambda_n=n\pi$$

$$=\sum_{n=1}^{\infty}\left(B_n\cos(n\pi t)+B_n^*\sin(n\pi t)\right)\sin(n\pi x)$$

$$B_n=\frac{2}{L}\int_0^L f(x)\sin(n\pi x)\,dx$$

$$=2\int_0^{\frac{1}{3}}k\sin(3\pi x)\sin(n\pi x)\,dx$$

$$=2k\int_0^{\frac{1}{3}}\frac{-\cos(3\pi x+n\pi x)+\cos(3\pi x-n\pi x)}{2}\,dx$$

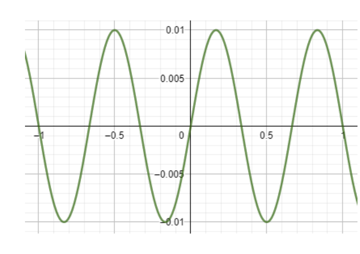
$$=-k\left[\frac{\sin(3\pi x+n\pi x)}{3\pi+n\pi}\right]_0^{\frac{1}{3}}+k\left[\frac{\sin(3\pi x-n\pi x)}{3\pi-n\pi}\right]_0^{\frac{1}{3}}$$

$$=-k\cdot\frac{\sin(\pi+n\pi/3)}{3\pi+n\pi}+k\cdot\frac{\sin(\pi-n\pi/3)}{3\pi+n\pi}$$

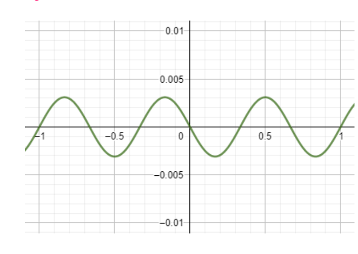
$$=\begin{cases}k, & n=3\\ 0, & n\neq 3\end{cases}$$

$\Rightarrow u(x,t)=k\cos(3\pi t)\sin(3\pi x)$

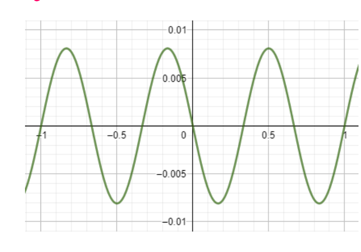
$t=0$:



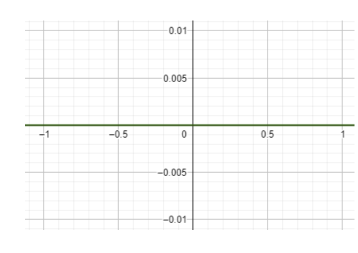
$t=\frac{1}{5}$:



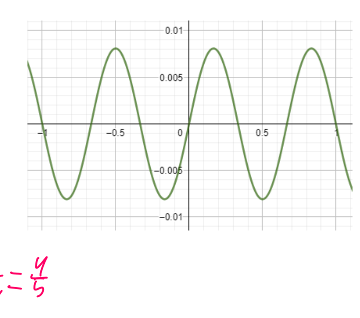
$t=\frac{2}{5}$:



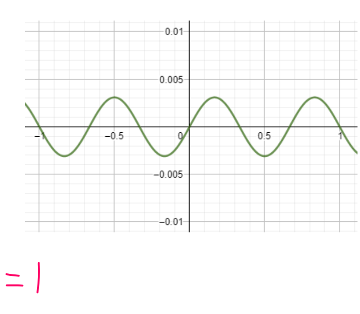
$t=\frac{1}{2}$:



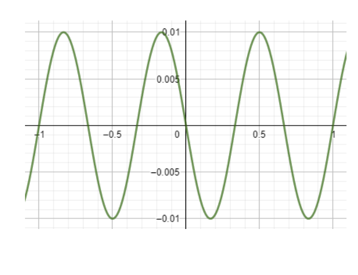
$t=\frac{3}{5}$:



$t=\frac{4}{5}$



$t=1$



7.

$f(x)=kx(1-x)$

$$B_n=2\int_0^1 kx(1-x)\sin(n\pi x)\,dx$$

$$=2k\left(\int_0^1 x\sin(n\pi x)\,dx-\int_0^1 x^2\sin(n\pi x)\,dx\right)$$

= [Tidl. øvinger:]

$$=2k\left(\frac{-2(-1)^n+2}{\pi^3 n^3}\right)$$

$$=\begin{cases}\frac{8k}{\pi^3 n^3}, & n\ \text{odde}\\ 0, & n\ \text{lik}\end{cases}$$

$$\Rightarrow u(x,t)=\sum_{\substack{n=1\\ n\ \text{odde}}}^{\infty}\frac{8k}{\pi^3 n^3}\cos(n\pi t)\sin(n\pi x)$$

$$=\frac{8k}{\pi^3}\sum_{n=1}^{\infty}\frac{\cos((2n-1)\pi t)\sin((2n-1)\pi x)}{(2n-1)^3}$$

14.

$L=\pi$

$c^2=1$

$g(x)=u_t(x,0)$

$$=\begin{cases}0.01x, & 0\leq x\leq\frac{\pi}{2}\\ 0.01(\pi-x), & \frac{\pi}{2}\leq x\leq\pi\end{cases}$$

$f(x)=0 \Rightarrow B_n=0$

$$B_n^*=\frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{2}{n\pi}\int_0^{\pi}g(x)\sin(nx)\,dx$$

$$=\frac{2}{n\pi}\left(\int_0^{\pi/2}0.01x\cdot\sin(nx)\,dx+\int_{\pi/2}^{\pi}0.01(\pi-x)\sin(nx)\,dx\right)$$

$$=\frac{0.02}{n\pi}\left(\left[-\frac{x\cos(nx)}{n}\right]_0^{\pi/2}+\frac{1}{n}\int_0^{\pi/2}\cos(nx)\,dx+\left[-\frac{\pi\cos(nx)}{n}\right]_{\frac{\pi}{2}}^{\pi}-\int_{\frac{\pi}{2}}^{\pi}x\sin(nx)\,dx\right)$$

$$=\frac{0.02}{n\pi}\left(-\frac{2\pi}{n}+\frac{1}{n}\left[\frac{\sin(nx)}{n}\right]_0^{\frac{\pi}{2}}+\left(-\frac{\pi(-1)^n}{n}\right)-(\right.$$

15.

$$\frac{F^{(4)}}{F}=-\frac{\ddot{G}}{c^2G}=\beta^4=\text{const.}$$

$F(x)=A\cos(\beta x)+B\sin(\beta x)+C\cosh(\beta x)+D\sinh(\beta x)$

$G(t)=a\cos(c\beta^2 t)+b\sin(c\beta^2 t)$

(21) $\frac{\partial^2 u}{\partial t^2}=-c^2\frac{\partial^4 u}{\partial x^4}$

$\frac{\partial^2 u}{\partial t^2}=F(x)\ddot{G}(t)$

$\frac{\partial^4 u}{\partial x^4}=F^{(4)}(x)G(t)$

$F\ddot{G}=-c^2F^{(4)}G \Rightarrow \frac{F^{(4)}}{F}=-\frac{\ddot{G}}{c^2G}=k$

$F^{(4)}-kF=0$:

$$s^4-k=0 \Rightarrow s=\pm\sqrt[4]{k}=\pm\sqrt[4]{\beta^4}=\pm\beta$$

$\ddot{G}-kc^2G=0$:

$$s^2-kc^2=0 \Rightarrow s=\pm\sqrt{kc^2}=\pm\sqrt{\beta^4c^2}=\pm c\beta^2$$

$\Rightarrow F(x)=A\cos(\beta x)+B\sin(\beta x)+C\cosh(\beta x)+D\sinh(\beta x)$

$G(t)=a\cos(c\beta^2 t)+b\sin(c\beta^2 t)$

12.4

19.

$u_{tt}=c^2u_{xx},\ c^2=\frac{E}{\rho}$

$u(0,t)=0$

$u_x(L,t)=0$

$u(x,0)=f(x)$

$u(x,t)=\frac{1}{2}\left(f(x+ct)+f(x-ct)\right)$

Vet at

$$B=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$u(x,t)=\sum_{n=1}^{\infty}B_n\cos(\lambda_n t)\sin\left(\frac{n\pi}{L}\right)x$$

$$\lambda_n=\frac{cn\pi}{L}$$

Skriver $p_n=\frac{n\pi}{L}$

12.rev

18.

$u_{xx}+u_x=0$

$u(0,y)=f(y)$

$u_x(0,y)=g(y)$

$$u(x,t)=\frac{1}{2}\left(f(x+ct)+f(x-ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)\,ds$$