TMA4265 Stochastic Modelling: Exercise 6

Week 41

Problem 1 – Exercises 6.1.1 and 6.1.2 in the book

A pure birth process starting from $X(0) = 0$ has birth parameters $\lambda_0 = 1$, $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = 5$. Let W_1, W_2 and W_3 be the stochastic times it takes the process to reach states 1, 2 and 3, respectively.

a) Determine the transition probability functions $P_n(t) = \Pr\{X(t) = n | X(0) = 0\}$ for $n = 0, 1, 2, 3$. *Hint: Read Section 6.1.2 in the book, and remember what we did in the lectures.*

b) Write W_3 as a sum of sojourn times and thereby deduce that the mean time is $\mathrm{E}[W_3] = \frac{11}{6}$.

c) Determine the mean of $W_1 + W_2 + W_3$.

d) Determine the variance of W_3.

Problem 2: Parking garage

A parking garage opens at 08:00 and closes at 16:00. We model the arrival of cars to the parking garage during opening hours as a Poisson process with rate $\lambda = 0.5$ cars/min. In part **a)** and **b)**, we assume that the garage has room for infinitely many cars.

a)

- Calculate the probability that no car has arrived at 08:05.
- Compute the expected number of cars that will arrive during the first 15 minutes of opening hours.
- Given that two cars will arrive during the first 10 minutes, compute the probability that no cars arrive during the first 5 minutes.

b) Customers spend on average 100 kr for parking, independently of each other, with standard deviation of 10 kr.

- Calculate the expected value of the total income at the garage during one day.
- Calculate the variance of the total income at the garage during one day.

Assume additionally that cars, independently of each other and their arrival times, spend a stochastic time in the parking garage, which follows an exponential distribution with expectation $1/\mu = 30$ minutes. After the time has passed, the car immediately exists the parking garage. Further, the parking garage has a maximum capacity of $N_{\max} = 20$ cars, and cars that arrive when the garage is full will drive past the garage without forming a queue. Under these assumptions, the number of cars $X(t)$ in the garage at time t is a birth-and-death process.

c)

- Determine the birth and death rates of the process.
- Draw a transition diagram for the process.

- Two cars are in the parking garage at 16:00, and no new cars may enter. The two cars will leave according to the rates described above. Let T be the time until the parking garage is empty, and determine the probability density, $f_T(t)$.

d) Ignore the opening hours and assume that the parking garage is always open. There are currently 16 cars in the parking garage.

- Determine the distribution of the time until the number of cars in the garage changes.
- Calculate the probability that the next time the number of cars in the garage changes, it will be a car that leaves.