**Norwegian University of Science and Technology**
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# Numerical Methods (MA2501)

Friday 31 May 2013
Time: 09:15 – 13:00
Grading: Friday 21 June 2013

Permitted Aids:

- Cheney & Kincaid, *Numerical Mathematics and Computing*, 6. or 7. edition.
- Rottmann, *Mathematical Formulae.*
- Note on fixed point iterations.
- Approved calculator.

General:

- All subproblems carry the same weight when grading.
- All answers should include your reasoning.
- All answers should include enough details to make it clear which methods and results have been used.

## Problem 1

**a)** Find the polynomial $p(x)$ of lowest degree that interpolates

$$f(x) = \sqrt[3]{x-2},$$

at the points

$$3, 3.7, 4.$$

Solution: From the uniqueness theorem of interpolating polynomials there exists a unique interpolating polynomial of degree at most 2, i.e. quadratic. There are several ways to go about writing down this polynomial. We here start from the Lagrange form, but divided differences could also have been used. The standard Lagrange form of an interpolating polynomial through 3 points $x_0, x_1$ and $x_2$ is

$$p(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.$$

By inserting the specific values $x_0 = 3$, $x_1 = 3.7$ and $x_2 = 4$, and using our particular function, $f$, we get

$$\begin{aligned} p(x) &= \sqrt[3]{3-2}\frac{(x-3.7)(x-4)}{(3-3.7)(3-4)} + \sqrt[3]{3.7-2}\frac{(x-3)(x-4)}{(3.7-3)(3.7-4)} + \sqrt[3]{4-2}\frac{(x-3)(x-3.7)}{(4-3)(4-3.7)} \\ &= \frac{x^2-7.7x+14.8}{0.7} - \sqrt[3]{1.7}\frac{x^2-7x+12}{0.21} + \sqrt[3]{2}\frac{x^2-6.7x+11.1}{0.3} \\ &\approx -0.054945033x^2 + 0.64453628x - 0.43910355. \end{aligned}$$

**b)** Determine an upper bound for the absolute interpolation error $|f(x)-p(x)|$ on the interval $[3, 4]$.

Solution: The points are not equally spaced. We are therefore unable to employ the Second Interpolation Error Theorem from C & K. Instead we start directly from the First Interpolation Error Theorem. $f(x)$ is smooth on $[3, 4]$ so the theorem applies. With $n = 2$, it follows that for any $x \in [3, 4]$

$$|f(x)-p(x)| = \left|\frac{f^{(3)}(\xi)}{(2+1)!}(x-3)(x-3.7)(x-4)\right| \leq \frac{M}{3!}L, \tag{1}$$

where

$$M = \max_{3\leq x\leq 4}\left|f^{(3)}(x)\right|, \quad \text{and} \quad L = \max_{3\leq x\leq 4}\left|(x-3)(x-3.7)(x-4)\right|.$$

To find $M$ we differentiate $f(x) = \sqrt[3]{x-2}$ three times

$$\begin{aligned} f'(x) &= \frac{1}{3}(x-2)^{-2/3}, \\ f''(x) &= \frac{-2}{9}(x-2)^{-5/3}, \\ f^{(3)}(x) &= \frac{10}{27}(x-2)^{-8/3}. \end{aligned}$$

$f^{(3)}(x)$ is positive and decreasing on the interval $[3,4]$ so

$$M = |f^{(3)}(3)| = f^{(3)}(3) = \frac{10}{27}(3-2)^{-8/3} = \frac{10}{27}.$$

To find $L$ we require the absolute maximum of the cubic polynomial

$$q(x) = (x-3)(x-3.7)(x-4) = x^3 - 10.7x^2 + 37.9x - 44.4,$$

on $[3,4]$. Since $q(x)$ is 0 at both endpoints of this interval, we know from basic calculus that the absolute maximum must occur at some internal point where $q'(x) = 0$. This requirement amounts to solving the quadratic equation

$$q'(x) = 3x^2 - 21.4x + 37.9 = 0.$$

From the standard root finding formula for a quadratic polynomial, we arrieve at the two solutions

$$\begin{aligned} x_1 &= \frac{21.4 + \sqrt{(-21.4)^2 - 4 \cdot 3 \cdot 37.9}}{2 \cdot 3} \approx 3.862939814, \\ x_2 &= \frac{21.4 - \sqrt{(-21.4)^2 - 4 \cdot 3 \cdot 37.9}}{2 \cdot 3} \approx 3.270393519. \end{aligned}$$

Both solutions are contained in our interval. Since $|q(x_1)| \approx 1.927165622 \times 10^{-2}$ and $|q(x_2)| \approx 8.475313770 \times 10^{-2}$, we have the absolute maximum at $x_2$. Therefore

$$L = |q(x_2)| \approx 8.475313770 \times 10^{-2}.$$

Inserting the derived values for $M$ and $L$ into the inequality (1) we find the upper bound

$$|f(x) - p(x)| \le \frac{M}{3!} L \approx \frac{10}{27 \cdot 6} \frac{8.475313770}{100} \approx 5.232 \times 10^{-3},$$

for the absolute interpolation error on the interval $[3,4]$. The actual absolute interpolation error is plotted over the interval $[3,4]$ in Figure 1. We see that it has a maximum of about $1.9 \times 10^{-3}$, around a third of our upper bound.

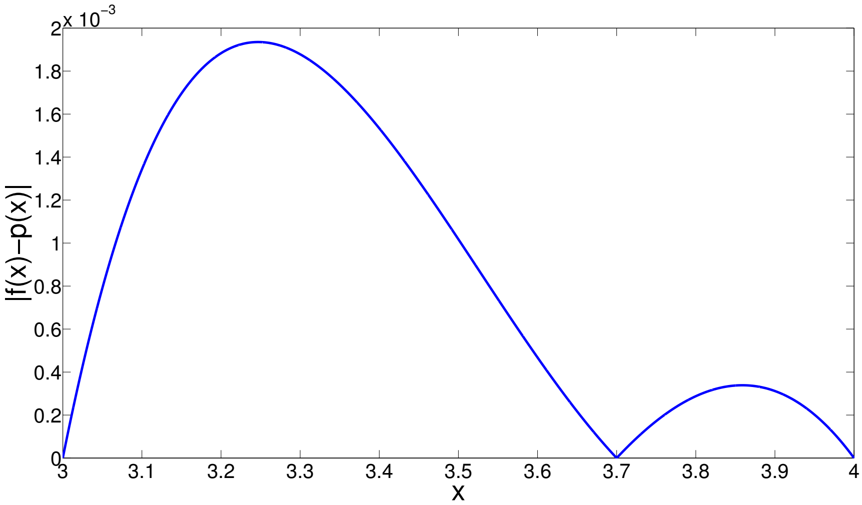


Figure 1: Absolute interpolation error, $|f(x) - p(x)|$, over the interval $[3, 4]$.

**Problem 2** Find real coefficients $a, b, c, d$ such that

$$S(x) = \begin{cases} S_0(x) = 1 + 2x - x^3 & (0 \leq x \leq 1) \\ S_1(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3 & (1 \leq x \leq 2), \end{cases}$$

is a natural cubic spline on $[0, 2]$.

Solution: It is apparent by inspection that, for any real choice of coefficients, the function will be defined on the interval $[0, 2]$, and is at most a cubic polynomial on each of the two subintervals $[0, 1]$ and $[1, 2]$.

Now, from the definition of a cubic spline we must choose our coefficients such that $S$, $S'$ and $S''$ are continous at the internal knot $x = 1$.

Continuity of $S$:

$$\begin{aligned} \lim_{x \to 1^+} S(x) &= \lim_{x \to 1^-} S(x), \\ \lim_{x \to 1^+} S_1(x) &= \lim_{x \to 1^-} S_0(x), \\ \lim_{x \to 1^+} a + b(x-1) + c(x-1)^2 + d(x-1)^3 &= \lim_{x \to 1^-} 1 + 2x - x^3, \\ a &= 2. \end{aligned}$$

Continuity of $S'$:

$$\begin{aligned}
\lim_{x\to 1^+} S'(x) &= \lim_{x\to 1^-} S'(x),\\
\lim_{x\to 1^+} S_1'(x) &= \lim_{x\to 1^-} S_0'(x),\\
\lim_{x\to 1^+} b + 2c(x-1) + 3d(x-1)^2 &= \lim_{x\to 1^-} 2 - 3x^2,\\
b &= -1.
\end{aligned}$$

Continuity of $S''$:

$$\begin{aligned}
\lim_{x\to 1^+} S''(x) &= \lim_{x\to 1^-} S''(x),\\
\lim_{x\to 1^+} S_1''(x) &= \lim_{x\to 1^-} S_0''(x),\\
\lim_{x\to 1^+} 2c + 6d(x-1) &= \lim_{x\to 1^-} -6x,\\
2c &= -6,\\
c &= -3.
\end{aligned}$$

Finally, for the cubic spline to be natural, we must have $S''(2) = S''(0) = 0$. It is a good idea to first verify that $S''(0) = 0$

$$S''(0) = S_0''(0) = -6 \cdot 0 = 0.$$

Finally the requirement $S''(2) = 0$ enable us to determine $d$

$$\begin{aligned}
S''(2) = S_1''(2) = -6 + 6d(2-1) = -6 + 6d = 0,\\
d = 1.
\end{aligned}$$

Thus the choice

$$a = 2, \quad b = -1, \quad c = -3, \quad d = 1,$$

makes $S$ a natural cubic spline on [0,2].

**Problem 3**

**a)** Determine an interval $[a, b]$ containing $x = 3$, such that the sequence generated by the fixed point iteration

$$x_{n+1} = f(x_n) \quad n \geq 0,$$

for the function

$$f(x) = \frac{5}{\sqrt{x}},$$

is guaranteed to converge to a unique fixed point, $x^* \in [a, b]$, for any $x_0 \in [a, b]$. Show that this fixed point is $x^* = \sqrt[3]{25}$. Perform 4 iterations with this method, using $x_0 = 3$.

Solution: Theorem 1 in the note on fixed point iterations gives the requrements for such an interval. We first determine where $|f'(x)| < 1$. Now

$$|f'(x)| = \left| -\frac{5}{2x^{3/2}} \right| = \frac{5}{2x^{3/2}}, \quad \text{for} \quad x > 0,$$

is a decreasing function where it is defined. From this and

$$|f'(x)| = \frac{5}{2x^{3/2}} = 1,$$
$$x = \left(\frac{5}{2}\right)^{2/3} \approx 1.842,$$

it follows that $|f'(x)| \leq k < 1$ on any open interval $(a, b)$ such that $a > (5/2)^{2/3}$. For simplicity we choose $a = 2$.

With our choice of $a$, we must then find an interval $[a, b]$ such that whenever $x \in [a, b]$, $f(x) \in [a, b]$, i.e. the function maps points in the interval into the interval. Since $f(x)$ is also a decreasing function for all $x > 0$, it is sufficient to find $b$ such that $f(a) = f(2) \approx 3.536 \leq b$ and $f(b) \geq a = 2$. After a little bit of trial and error we discover that a possible choice is $b = 6$. For this choice we have $6 \geq f(2) \approx 3.536$ and $f(6) \approx 2.041 \geq 2$. We conclude that the fixed point sequence $\{x_n\}_{n=0}^{\infty}$ is guaranteed to converge to a unique fixed point on the interval $[a, b] = [2, 6]$.

There are of course an unlimited number of other possible choices for such an interval. Though not required here, we can show that for any $x_0 > 0$, the fixed point sequence will eventually enter the interval $[2, 6]$. This implies that the sequence will in fact converge towards the unique fixed point $x^*$ for any $x_0 > 0$.

To show that $x^* = \sqrt[3]{25}$ we simply verify that $f(\sqrt[3]{25}) = \sqrt[3]{25}$

$$f(\sqrt[3]{25}) = \frac{5}{\sqrt{\sqrt[3]{25}}} = \frac{25^{1/2}}{25^{(1/3)(1/2)}} = \frac{25^{3/6}}{25^{1/6}} = 25^{(3/6)-(1/6)} = 25^{1/3} = \sqrt[3]{25}.$$

The results of the 4 fixed point iterations with $x_0 = 3$ is given in the table below with 10 digit accuracy. For comparison $x^* = \sqrt[3]{25} \approx 2.924017738$. We also include the error $|x_n - x^*|$ which we will need in **b)**

| $n$ | $x_n$ | $\lvert x_n - x^*\rvert$ |
|---|---|---|
| 0 | 3 | $7.5982262 \times 10^{-2}$ |
| 1 | 2.886751346 | $3.7266392 \times 10^{-2}$ |
| 2 | 2.942830956 | $1.8813218 \times 10^{-2}$ |
| 3 | 2.914656279 | $9.361459 \times 10^{-3}$ |
| 4 | 2.928709737 | $4.691999 \times 10^{-3}$ |

**b)** Another function with the same unique fixed point (you do not have to show this) is

$$g(x) = \frac{2x^3 + 25}{3x^2}.$$

Perform 4 fixed point iterations using $g(x)$ and the same intitial guess $x_0 = 3$. Based on the computed iterations, estimate the order of convergence for this method and the one used in **a)**. Are the estimated orders as expected?

**Hint:** A method with order of convergence $\alpha$ will behave like

$$|x_{n+1} - x^*| \approx M|x_n - x^*|^\alpha,$$

for some positive constant $M$ when the error becomes sufficiently small.

Solution: We again do 4 fixed point iterations using $g(x)$. The results are given in the following table

| $n$ | $x_n$ | $\lvert x_n - x^*\rvert$ |
|---|---|---|
| 0 | 3 | $7.5982262 \times 10^{-2}$ |
| 1 | 2.925925926 | $1.908188 \times 10^{-3}$ |
| 2 | 2.924018982 | $1.244 \times 10^{-6}$ |
| 3 | 2.924017738 | 0 |
| 4 | 2.924017738 | 0 |

Note that the fact that the computed error is 0 for $n = 3, 4$ does not mean that this error is truly zero. It only tells us that the approximation has converged to within our degree of precision. The initial observation is that the fixed point sequence using $g(x)$ appears to converge much faster towards the fixed point, compared to the sequence using $f(x)$.

Looking at the sequence from **a)** for $f(x)$, the distance to the fixed point appears to roughly half for each iteration. Such behaviour, where the error in the current iteration is close to a fixed percentage of the error in the previous iteration, indicates linear order of convergence, i.e. $\alpha = 1$. To explicitly check this, we follow the hint with $\alpha = 1$ and compute the ratio $|x_{n+1} - x^*|/|x_n - x^*|$. Approximate values are given in the table below

| $n$ | $\frac{\lvert x_{n+1}-x^*\rvert}{\lvert x_n-x^*\rvert}$ |
|---|---|
| 0 | 0.4904 |
| 1 | 0.5048 |
| 2 | 0.4976 |
| 3 | 0.5012 |

Since the resulting ratios are nearly constant, we estimate the convergence of the fixed point iteration in **a)**, using $f(x)$, to be linear.

Now, $f(x)$ is smooth for $x > 0$. We would therefore predict linear convergence if $f'(x^*) \neq 0$, based on Theorem 2 in the note on fixed point iterations. To check this, we calculate

$$f'(x^*) = -\frac{5}{2(x^*)^{3/2}} = -\frac{5}{2\sqrt{25}} = -\frac{1}{2} \neq 0,$$

so our estimated linear convergence is as expected. In fact based on the proof of this theorem, we would predict that the approximate ratio constant $M = |f'(x^*)| = 0.5$. This is precisely what we observe.

Studying the fixed point sequence for $g(x)$, the number of significant digits appear to roughly double for each iteration. This happens until we reach the end of our precision for $x_3$ and beyond. From the discussion of Newtons method, such behaviour leads us to expect quadratic convergence, i.e. $\alpha = 2$. We therefore use the hint with $\alpha = 2$ and compute ratios $|x_{n+1} - x^*|/|x_n - x^*|^2$ for the two cases $n = 0, 1$. Larger values of $n$ cannot be used, because as previously stated, we don't have any real measure of the actual size of the error for $x_3$ and $x_4$.

| $n$ | $\frac{\lvert x_{n+1}-x^*\rvert}{\lvert x_n-x^*\rvert^2}$ |
|---|---|
| 0 | 0.3305 |
| 1 | 0.3416 |

The near constant value of these ratios causes us to estimate the convergence to be quadratic.

$g(x)$ is also readily seen to be smooth for $x > 0$, so from Theorem 2, quadratic convergence is expected when $g'(x^*) = 0$ and $g''(x^*) \neq 0$. We compute

$$\begin{aligned} g'(x) &= \frac{6x^2 \cdot 3x^2 - (2x^3+25)6x}{9x^4} = \frac{6x(3x^3 - 2x^3 - 25)}{9x^4} = \frac{2}{3}\frac{x^3-25}{x^3}, \\ g''(x) &= \frac{2}{3}\frac{3x^5 - (x^3-25)3x^2}{x^6} = 2\frac{25x^2}{x^6} = \frac{50}{x^4}. \end{aligned}$$

From the expressions it is clear that $g'(x^*) = 0$ and $g''(x^*) \neq 0$, and so our estimate of quadratic convergence is also as expected. The proof of Theorem 2 leads us to estimate the error ratios to approximately equal $M = g''(x^*)/2 = 1/\sqrt[3]{25} \approx 0.3420$. This matches our computed values.

An alternate way to explain the observed quadratic convergence is to rewrite $g(x)$ as

$$g(x) = x - \frac{x^3 - 25}{3x^2}.$$

We can then recognize the fixed point iteraton with $g(x)$ as Newton's method for the root of $h(x) = x^3 - 25$, which is precisely $x^* = \sqrt[3]{25}$. Newton's Method Theorem in C & K is readily seen to be valid here. It states that the method converges quadratically towards the root $x^*$, for a sufficiently accurate initial guess.

**Problem 4** Approximate the integral

$$\int_0^1 e^{-x} \sin 2x \, dx$$

by computing $R_{22}$ using Romberg integration.

Solution: Following the Romberg algorithm from C & K, we can compute the Romberg table. From the table we read out $R_{22} = 0.3943560702$.

| | | |
|---|---|---|
| 0.1672559146 | | |
| 0.3388169331 | 0.3960039392 | |
| 0.3805485298 | 0.3944590621 | 0.3943560702 |

Though not required, we can use partial integration to calculate the exact integral

$$I = (2 - e^{-1}(\sin 2 + 2\cos 2))/5 \approx 0.3943343804.$$

Our approximation thus has about 4 digits of accuracy.

**Problem 5** Determine weights $a$ and $b$, and a node $\alpha$, such that the quadrature rule

$$\int_1^4 f(x)\sqrt{x}\, dx \approx af(1) + bf(\alpha),$$

is exact for all polynomials of degree $\leq m$, where the integer $m$ is as large as possible.

Solution: We use the method of undetermined coefficients with the simple polynomial basis $1, x, x^2, \ldots$. Linearity of the quadrature rule and integration guarantees that the method will integrate any polynomial of degree $\leq m$ exactly, provided it integrates exactly a basis for this function space. With 3 free parameters, we expect to be able to at least ensure that the rule be accurate for all polynomials of degree 0, 1 and 2.

We first require that the rule integrate $q_0(x) = 1$ exactly

$$\int_1^4 q_0(x)\sqrt{x}\, dx = \int_1^4 \sqrt{x}\, dx = \frac{2}{3}\left[x^{(3/2)}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3} = a + b.$$

This gives an expression for $a$ in terms of $b$ as

$$a = 14/3 - b. \tag{2}$$

To be exact for all linear polynomials we additionally require it to integrate $q_1(x) = x$ exactly

$$\int_1^4 q_1(x)\sqrt{x}\,dx = \int_1^4 x\sqrt{x}\,dx = \frac{2}{5}\left[x^{(5/2)}\right]_1^4 = \frac{2}{5}(32-1) = \frac{62}{5} = a + b\alpha = \frac{14}{3} + b(\alpha - 1).$$

Here we used (2) in the final equality. From this we can express $b$ in terms of $\alpha$

$$\begin{aligned} \frac{14}{3} + b(\alpha-1) &= \frac{62}{5}, \\ b(\alpha-1) &= \frac{62}{5} - \frac{14}{3} = \frac{116}{15}, \\ b &= \frac{116}{15(\alpha-1)}. \end{aligned} \tag{3}$$

For quadratic polynomials we demand exact integration of the quadratic monomial $q_2(x) = x^2$

$$\begin{aligned} \int_1^4 q_2(x)\sqrt{x}\,dx &= \int_1^4 x^2\sqrt{x}\,dx = \frac{2}{7}\left[x^{(7/2)}\right]_1^4 = \frac{2}{7}(128-1) = \frac{254}{7} = a + b\alpha^2 = \frac{14}{3} + b(\alpha^2-1) \\ &= \frac{14}{3} + b(\alpha-1)(\alpha+1) = \frac{14}{3} + \frac{116}{15(\alpha-1)}(\alpha-1)(\alpha+1) = \frac{14}{3} + \frac{116}{15}(\alpha+1) \\ &= \frac{186}{15} + \frac{116}{15}\alpha \end{aligned}$$

We can solve this last equation for $\alpha$

$$\begin{aligned} \frac{254}{7} &= \frac{186}{15} + \frac{116}{15}\alpha, \\ 1905 &= 651 + 406\alpha, \\ \alpha &= \frac{1254}{406} = \frac{627}{203}. \end{aligned}$$

Now using (3) we can compute $b$ from $\alpha$.

$$b = \frac{116}{15(\alpha-1)} = \frac{116}{15(\frac{627}{203}-1)} = \frac{116}{15\frac{424}{203}} = \frac{5887}{1590},$$

and finally $a$ from $b$ using (2)

$$a = 14/3 - b = 14/3 - \frac{5887}{1590} = \frac{7420-5887}{1590} = \frac{1533}{1590} = \frac{511}{530}.$$

This choice of parameters is the only one guaranteed to integrate any polynomial of degree 2 or lower, and is consequently the best we can achieve for our particular type of quadrature formula.

**Problem 6** A planet is orbiting around a star. With a suitable $(x, y)$-coordinate system centered on the star, the motion will be in the plane. A simplified model for the position $(x(t), y(t))$ (in normalized units) of the center of mass of the planet over time, is given by the ordinary differential equations

$$\ddot{x} = -\frac{x}{(x^2+y^2)^{3/2}}, \quad \ddot{y} = -\frac{y}{(x^2+y^2)^{3/2}}, \quad \text{where } \dot{x} = \frac{dx}{dt}, \ \ddot{x} = \frac{d^2x}{dt^2}, \text{ etc.} \tag{4}$$

The initial conditions are

$$x(0) = 0.4, \ y(0) = 0, \ \dot{x}(0) = 0, \ \dot{y}(0) = 2. \tag{5}$$

**a)** We introduce the new variables $x_1 = x$, $x_2 = y$, $x_3 = \dot{x}$, and $x_4 = \dot{y}$. Rewrite the initial value problem (4) and (5) into a system of first-order differential equations in the variables $\mathbf{X} = [x_1, x_2, x_3, x_4]^T$.

Denote by $\mathbf{X}_i = [x_{1i}, x_{2i}, x_{3i}, x_{4i}]^T$ the approximation from a numerical method to $\mathbf{X}(t_i)$ with $t_i = t_0 + ih$ for $i = 0, 1, 2, \ldots$. Approximate $\mathbf{X}(0.2)$ for this initial value problem, by taking two steps with Euler's method and stepsize $h = 0.1$, i.e. by computing $\mathbf{X}_2$.

Solution: We follow the standard procedure detailed in C & K to replace the old variables with the new ones, and write up the new system of first-order differential equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}) = \begin{bmatrix} x_3 \\ x_4 \\ -\frac{x_1}{(x_1^2+x_2^2)^{3/2}} \\ -\frac{x_2}{(x_1^2+x_2^2)^{3/2}} \end{bmatrix},$$

with

$$\mathbf{X}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \mathbf{S} = \begin{bmatrix} 0.4 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

With the prescribed notation, Euler's method for this system has the component form

$$\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \\ x_{3,i+1} \\ x_{4,i+1} \end{bmatrix} = \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \\ x_{4i} \end{bmatrix} + h \begin{bmatrix} x_{3i} \\ x_{4i} \\ -\frac{x_{1i}}{(x_{1i}^2+x_{2i}^2)^{3/2}} \\ -\frac{x_{2i}}{(x_{1i}^2+x_{2i}^2)^{3/2}} \end{bmatrix}, \quad i \geq 0.$$

We take two steps of size $h = 0.1$ to find $\mathbf{X}_2 \approx \mathbf{X}(0.2)$. Starting with $\mathbf{X}_0 = \mathbf{X}(0)$, we get

$$\mathbf{X}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0 \\ 0 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} 0 \\ 2 \\ -\frac{0.4}{(0.4^2+0^2)^{3/2}} \\ -\frac{0}{(0.4^2+0^2)^{3/2}} \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ -0.625 \\ 2 \end{bmatrix},$$

and then

$$\mathbf{X}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ -0.625 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} -0.625 \\ 2 \\ -\frac{0.4}{(0.4^2+0.2^2)^{3/2}} \\ -\frac{0.2}{(0.4^2+0.2^2)^{3/2}} \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ -0.625 \\ 2 \end{bmatrix} + 0.1 \begin{bmatrix} -0.625 \\ 2 \\ -\frac{2}{\sqrt{0.2}} \\ -\frac{1}{\sqrt{0.2}} \end{bmatrix}$$
$$\approx \begin{bmatrix} 0.3375000000 \\ 0.4000000000 \\ -1.072213595 \\ 1.776393202 \end{bmatrix}.$$

**b)** *Symplectic Euler* is most commonly used for *partitioned systems*, where we can split the unknown vector $\mathbf{X}$ into two parts

$$\mathbf{X} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{P} \end{bmatrix},$$

in such a way that the system of differential equations

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}),$$

can be written in the equivalent form

$$\dot{\mathbf{Q}} = \mathbf{G}(\mathbf{P}), \quad \dot{\mathbf{P}} = \mathbf{H}(\mathbf{Q}). \tag{6}$$

For such a system, Symplectic Euler can be formulated as

$$\mathbf{Q}_{i+1} = \mathbf{Q}_i + h\mathbf{G}(\mathbf{P}_i), \quad \mathbf{P}_{i+1} = \mathbf{P}_i + h\mathbf{H}(\mathbf{Q}_{i+1}).$$

Using $\mathbf{Q} = [x_1, x_2]^T$ and $\mathbf{P} = [x_3, x_4]^T$, write down the Symplectic Euler method for the system of differential equations derived in **a**).

Solution: Using the given partition, $\mathbf{Q} = [x_1, x_2]^T$ and $\mathbf{P} = [x_3, x_4]^T$, Symplectic Euler for our system can be written in component form as

$$\mathbf{X}_{i+1} = \begin{bmatrix} \mathbf{Q}_{i+1} \\ \mathbf{P}_{i+1} \end{bmatrix} = \begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \\ x_{3,i+1} \\ x_{4,i+1} \end{bmatrix} = \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \\ x_{4i} \end{bmatrix} + h \begin{bmatrix} x_{3i} \\ x_{4i} \\ -\frac{x_{1,i+1}}{(x_{1,i+1}^2+x_{2,i+1}^2)^{3/2}} \\ -\frac{x_{2,i+1}}{(x_{1,i+1}^2+x_{2,i+1}^2)^{3/2}} \end{bmatrix}.$$

We can also replace $x_{1,i+1}$ and $x_{2,i+1}$ on the right hand side with their explicit expressions, to highlight the fact that Symplectic Euler is explicit for partitioned systems. By *explicit*, we here mean that the next step can be computed instantly without the need to solve any (generally nonlinear) equations

$$\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \\ x_{3,i+1} \\ x_{4,i+1} \end{bmatrix} = \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \\ x_{4i} \end{bmatrix} + h \begin{bmatrix} x_{3i} \\ x_{4i} \\ -\frac{x_{1i}+hx_{3i}}{((x_{1i}+hx_{3i})^2+(x_{2i}+hx_{4i})^2)^{3/2}} \\ -\frac{x_{2i}+hx_{4i}}{((x_{1i}+hx_{3i})^2+(x_{2i}+hx_{4i})^2)^{3/2}} \end{bmatrix}.$$

This is a main reason why the method is mostly used on partitioned systems on the form (6). In general, Symplectic Euler will be implicit, requiring us to solve equations to perform a time step.

**c)** Complete the MATLAB function, `Symp_Euler_Step`, which should perform one step of Symplectic Euler for the general partitioned system (6).

```
function [Q_new,P_new] = Symp_Euler_Step(Q_old,P_old,G,H,h)
% Performs one step of Symplectic Euler for a partitioned system
% dQ/dt = G(P), dP/dt = H(Q)
%
% INPUT: Q_old, P_old  (Approximation of Q and P at time t_i)
%         G  (Function handle for G(P))
%         H  (Function handle for H(Q))
%         h  (Stepsize)
%
% OUTPUT: Q_new, P_new (Approximation of Q and P at time t_(i+1) = t_i + h)
%

 (WRITE THIS PART)

end
```

**Note:** The functions **G** and **H** are here assumed to already have been implemented, and are given to the function `Symp_Euler_Step` as function handles.

Solution: Since Symplectic Euler is explicit for a system on the form (6), implementation is straightforward. A simple and natural choice for the contents of `Symp_Euler_Step` is

```
Q_new = Q_old + h*G(P_old);
P_new = P_old + h*H(Q_new);
```