1. $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x\sin(x) + e^x$

(a) $f'(x) = x' \cdot \sin(x) + x \cdot (\sin(x))' + e^x = \sin(x) + x\cos(x) + e^x$

$f''(x) = \cos(x) + x' \cdot \cos(x) + x \cdot (\cos(x))' + e^x = \cos(x) + \cos(x) - x\sin(x) + e^x = 2\cos(x) - x\sin(x) + e^x$

$f^{(3)}(x) = -2\sin(x) - \sin(x) - x\cos(x) + e^x = -3\sin(x) - x\cos(x) + e^x$

$x_0 = 0$

$f(0) = 1$

$f'(0) = 1$

$f''(0) = 2 + 1 = 3$

$f^{(3)}(0) = 1$

$\underline{\underline{P_3(x) = 1 + x + \frac{3}{2}x^2 + \frac{1}{6}x^3}}$

(b) $P = (\pi, f(\pi))$

$f'(x) = \sin(x) + x\cos(x) + e^x$

$f'(\pi) = 0 - \pi + e^\pi = e^\pi - \pi$

$f(\pi) = 0 + e^\pi = e^\pi$

$y - f(\pi) = f'(\pi)(x - \pi) \Rightarrow$

$\underline{\underline{y = (e^\pi - \pi)x - (e^\pi\pi - \pi^2) + e^\pi = (e^\pi - \pi)x - e^\pi\pi + \pi^2 + e^\pi}}$

2(a) $\int_0^1 \arcsin^2(x)\,dx = \int_0^1 x' \arcsin^2(x)\,dx)$

$= [x\arcsin^2(x)]_0^1 - \int_0^1 x \cdot 2\arcsin(x) \cdot \frac{1}{\sqrt{1-x^2}}dx$

$= \frac{\pi^2}{4} - 2\int_0^1 \frac{x\arcsin(x)}{\sqrt{1-x^2}}\,dx$

$= \frac{\pi^2}{4} - 2\int_0^1 (-\sqrt{1-x^2})' \cdot \arcsin(x)\,dx$

$= \frac{\pi^2}{4} - 2([-\sqrt{1-x^2} \cdot \arcsin(x)]_0^1 - \int_0^1 -1\,dx)$

$= \frac{\pi^2}{4} - 2(0 - [-x]_0^1)$

$= \frac{\pi^2}{4} - 2$

(b) $\int_0^2 \frac{x^2+2}{x+1}dx = \int_0^2 \frac{x^2}{x+1}dx + \int_0^2 \frac{2}{x+1}dx$

$u = x+1 \Rightarrow du = dx \Rightarrow$

$\int_0^2 \frac{x^2+2}{x+1}dx = \int_1^3 \frac{(u-1)^2}{u}du + 2[\ln|x+1|]_0^2$

$= \int_1^3 \frac{u^2-2u+1}{u}du + 2\ln(3)$

$= \int_1^3 u\,du - \int_1^3 2\,du + \int_1^3 \frac{du}{u} + 2\ln(3)$

$= [\frac{1}{2}u^2]_1^3 - 2[u]_1^3 + [\ln|u|]_1^3 + 2\ln(3)$

$= 4 - 4 + \ln(3) + 2\ln(3)$

$= 3\ln(3)$

(c) $\int_{-\pi}^{\pi} x^4 \sin(x)\,dx$

$x^4 \sin(x)$ er odde $\Rightarrow$

$\int_{-\pi}^{\pi} x^4 \sin(x)\,dx = 0$

3. $g:(0,+\infty)\to\mathbb{R}$

$g(x)=\frac{\ln(x)}{x}$

Horisontal asymptote:

$x=0$

$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}\frac{\ln(x)}{x}$

$=\lim_{x\to 0^+}\left(\ln(x)\cdot\frac{1}{x}\right)$

$=-\infty\cdot\infty$

$=-\infty$

Vertibal asymptote

$\lim_{x\to+\infty} g(x)=\lim_{x\to+\infty}\frac{\ln(x)}{x}$

$=\lim_{x\to+\infty}\frac{(\ln(x))'}{x'}$

$=\lim_{x\to+\infty}\frac{1}{x}$

$=0$

$y=0$

$g(x)=0 \Rightarrow$

$\frac{\ln(x)}{x}=0 \Rightarrow$

$\ln(x)=0 \Rightarrow$

$x=1$

$$g'(x)=\frac{\frac{1}{x}\cdot x-\ln(x)}{x^2}=\frac{1-\ln(x)}{x^2}$$

$g'(x)=0 \Rightarrow$

$\frac{1-\ln(x)}{x^2}=0 \Rightarrow$

$1-\ln(x)=0 \Rightarrow$

$\ln(x)=1 \Rightarrow$

$x=e$

$g(e)=\frac{1}{e}$

$$g''(x)=\frac{-\frac{1}{x}\cdot x^2-(1-\ln(x))\cdot 2x}{x^4}=\frac{-x-2x+2x\ln(x)}{x^4}=\frac{-3+2\ln(x)}{x^3}$$

$g''(x)=0 \Rightarrow$

$\frac{-3+2\ln(x)}{x^3}=0 \Rightarrow$

$-3+2\ln(x)=0 \Rightarrow$

$2\ln(x)=3 \Rightarrow$

$\ln(x)=\frac{3}{2} \Rightarrow$

$x=e^{\frac{3}{2}}$

$g(e^{\frac{3}{2}})=\frac{3}{2e^{\frac{3}{2}}}$

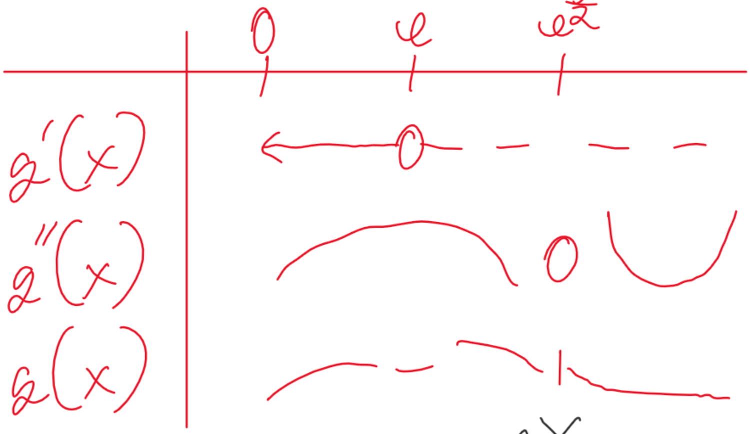
$g''(x)>0 \Rightarrow$

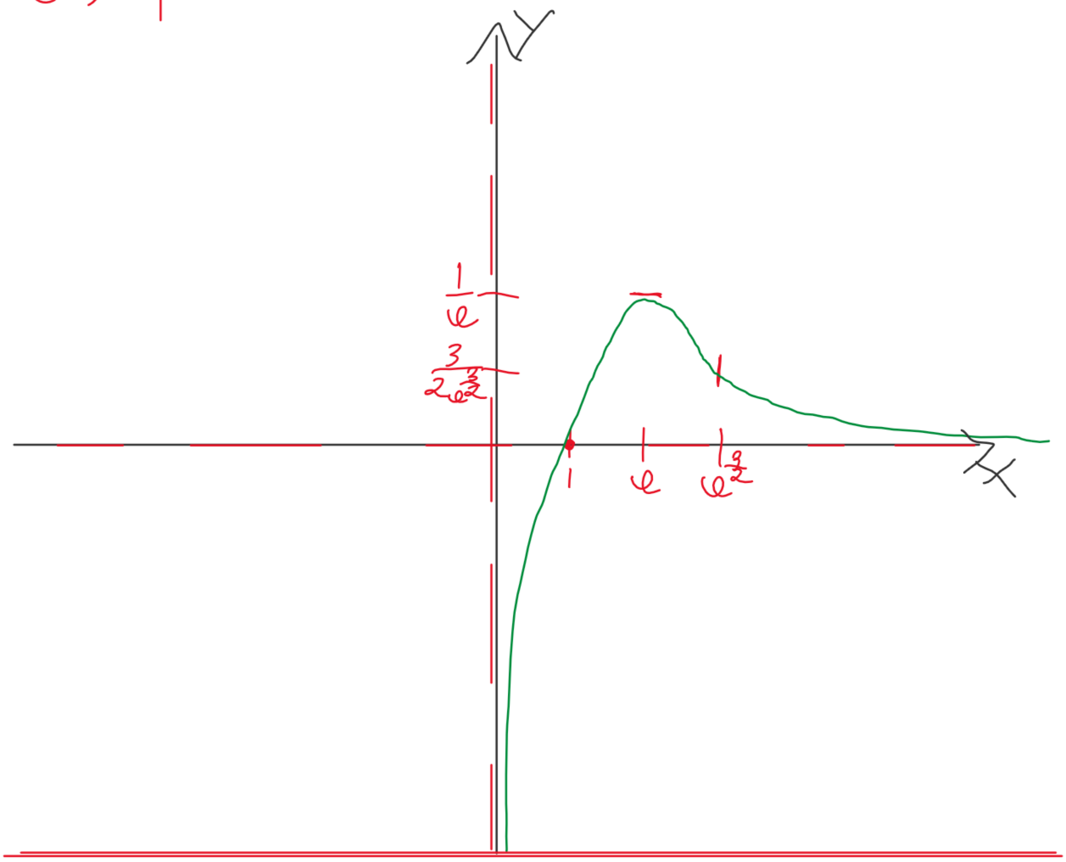
$x>e^{\frac{3}{2}}$

$g''(x)<0 \Rightarrow$

$x<e^{\frac{3}{2}}$

$x<e^{\frac{3}{2}}$





4 $(1+x^2)y'+xy=0 \Rightarrow$

$y'+\frac{x}{1+x^2}y=0$

$\int \frac{x}{1+x^2}dx$

$u=1+x^2 \Rightarrow du=2xdx \Rightarrow$

$\int \frac{x}{1+x^2}dx=\int \frac{1}{2}\cdot\frac{1}{u}du$

$=\frac{1}{2}\ln|u|$

$=\frac{1}{2}\ln|1+x^2| \Rightarrow$

$e^{\frac{1}{2}\ln|1+x^2|}y'+e^{\frac{1}{2}\ln|1+x^2|}\frac{x}{1+x^2}y=0 \Rightarrow$

$\left(e^{\frac{1}{2}\ln|1+x^2|}y\right)'=\int 0dx \Rightarrow$

$e^{\frac{1}{2}\ln|1+x^2|}y=c,\ c\in\mathbb{R} \Rightarrow$

$e^{\ln|1+x^2|^{\frac{1}{2}}}y=c,\ c\in\mathbb{R} \Rightarrow$

$\sqrt{1+x^2}\,y=c,\ c\in\mathbb{R} \Rightarrow$

$\underline{\underline{y=\frac{c}{\sqrt{1+x^2}},\ c\in\mathbb{R}}}$

<u>Differensiallikningen er av 1. orden, lineær og homogent</u>

5(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$

$n^2+3n = n(n+3)$

$\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} \Rightarrow$

$A(n+3) + Bn = 1$

$n=0 \Rightarrow A = \frac{1}{3}$

$n=-3 \Rightarrow B = -\frac{1}{3} \Rightarrow$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n} - \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n+3}$$

$$= \frac{1}{3}\left(\sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=1}^{\infty}\frac{1}{n+3}\right)$$

$$= \frac{1}{3}\left(\left(1-\cancel{\frac{1}{4}}\right) + \left(\frac{1}{2}-\cancel{\frac{1}{5}}\right) + \left(\frac{1}{3}-\cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{7}}\right) + \cancel{(\ldots)}\right)$$

$$= \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right)$$

$$= \frac{11}{18}$$

(b) Vet at $\sum_{n=1}^{\infty} a_n < \infty \Rightarrow \lim_{n\to\infty} a_n = 0$, men $\lim_{n\to\infty} a_n = 0$ impliserer ikke at $\sum_{n=1}^{\infty} a_n < \infty$. Usant

6. $\int_0^{\infty} \frac{x\cos^2(x)}{x^3+1}\,dx$

$x\cos^2(x) \leq x$

$x^3+1 > x^3 \Rightarrow$

$\frac{x\cos^2(x)}{x^3+1} < \frac{x}{x^3} = \frac{1}{x^2}$

$\int_0^{\infty} \frac{1}{x^2}\,dx < \infty \Rightarrow$

$\int_1^{\infty} \frac{x\cos^2(x)}{x^3+1}\,dx < \infty$ fordi $\int_0^{\infty} \frac{1}{x^2}\,dx < \infty$

7.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2n+k}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{2+\frac{k}{n}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2+\frac{k}{n}} \Rightarrow$$

$$f(x)=\frac{1}{2+x} \Rightarrow$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2n+k}=\int_0^1\frac{1}{2+x}\,dx$$

$$=\Big[\ln|2+x|\Big]_0^1$$

$$=\ln(3)-\ln(2)$$

$$=\ln\left(\frac{3}{2}\right)$$

8 $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x, & x<0 \\ 1+x^2, & x \geq 0 \end{cases}$$

(a) $(\forall \varepsilon > 0 \; \exists \delta = \delta(\varepsilon) > 0$ S.A. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon)$

$x_0 = 1$

For $x \geq 0$

$|x-1| < \delta \Rightarrow |f(x) - f(1)| < \varepsilon$

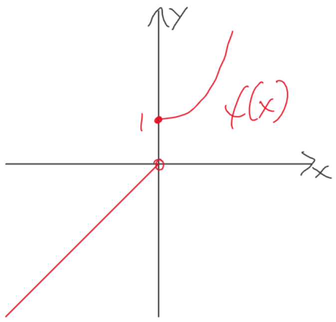
$|1 + x^2 - 2| = |x^2 - 1| = |x+1| \cdot |x-1| < \varepsilon$

Velger $\delta > |x-1| \Rightarrow$

$|f(x) - f(1)| < \delta |x+1| < \varepsilon$ ∎

(b) Darboux teorem

$a, b \in \mathbb{R}$, $F(a) < y_0 < F(b) \Rightarrow \exists x_0 \in \mathbb{R}$ der $f'(x_0) = y_0$



Hvis $F'(x) = f(x)$ for $F: \mathbb{R} \to \mathbb{R}$

Siden $F(0) = 0$ og $F(1) = 1$ skal det finnes $t \in (0,1)$ der $f(t) = \frac{1}{2}$

Men $f(t)$ er aldri $\frac{1}{2}$

Så svaret er nei

9.(a) $\varepsilon = 1,\ L = 0$

$\exists n_0 \in \mathbb{N}$ S.A. $|a_n| < 1 \quad \forall n \geq n_0$

$|a_n| = |a_n - L + L|$

$= |a_n - L| + |L|$

$< 1 + |L| \quad \forall n \geq n_0$

$M = \max\{|a_1|, |a_2|, \dots, |a_{n_0 - 1}|, 1 + |L|\}$

$\forall n \in \mathbb{N}$: $|a_n| \leq M$ når $n < n_0$

$|a_n| < 1 + |L| \leq M$ når $n \geq n_0$

$\Rightarrow$ $|a_n| \leq M$ for alle $n \geq 1$ så a_n er begrenset

(b) Følgen $\{(-1)^n\}_{n=1}^{\infty}$ er begrenset men konvergerer ikke

10 $g(x) = \begin{cases} \frac{\sin^2(\frac{\pi x}{2})}{x^2}, & x \neq 0 \\ \frac{\pi^2}{4}, & x = 0 \end{cases}$

$F(x) = \int_1^x g(u)\,du$

(a) Når $x < 0$

$F(x) = -\int_x^1 \frac{\sin^2(\frac{\pi u}{2})}{u^2}\,du$

Når $x = 0$

$F(x) = -\int_x^1 \frac{\pi^2}{4}\,du = -\left[\frac{\pi^2 u}{4}\right]_x^1 = -\left(\frac{\pi^2}{4} - \frac{\pi^2 x}{4}\right) = \frac{\pi^2 x}{4} - \frac{\pi^2}{4} = \frac{\pi^4}{4}(x-1)$

$\int_1^x \frac{\sin^2(\frac{\pi u}{2})}{u^2}\,du$