$$xu' = \frac{1+u}{2-u} - u = \frac{1+u-2u+u^2}{2-u} \Rightarrow$$

$$xu' = \frac{1-u+u^2}{2-u} \Rightarrow$$

$$\int \frac{2-u}{u^2-u+1}\,du = \int \frac{1}{x}\,dx \Rightarrow$$

$$\int \frac{\frac{3}{2}+\frac{1}{2}-u}{u^2-u+1}\,du = \int \frac{dx}{x} \Rightarrow$$

$$\frac{3}{2}\int \frac{du}{u^2-u+1} - \frac{1}{2}\int \frac{2u-1}{u^2-u+1}\,du = \int \frac{dx}{x} \Rightarrow$$

$$\frac{3}{2}\int \frac{du}{\left(u-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{2}\int \frac{(u^2-u+1)'}{u^2-u+1}\,du = \int \frac{dx}{x} \Rightarrow$$

$$\frac{3}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}} \cdot \arctan\left(\frac{2\sqrt{3}}{3}\left(u-\frac{1}{2}\right)\right) - \frac{1}{2}\ln|u^2-u+1| = \ln\sqrt{|x|^2}+C$$

$$\sqrt{3}\arctan\left(\frac{2\sqrt{3}}{3}\left(u-\frac{1}{2}\right)\right) = \ln\sqrt{(u^2-u+1)x^2} + c \Rightarrow$$

$$\sqrt{3}\arctan\left(\frac{2\sqrt{3}}{3}\cdot\frac{y}{x} - \frac{\sqrt{3}}{3}\right) = \ln\sqrt{y^2-xy+x^2} + c.$$

(We have found the solution in implicit form).

E.g. Solve $y' = \dfrac{x^2 + xy}{xy + y^2} = \dfrac{1 + \frac{y}{x}}{\frac{y}{x} + \left(\frac{y}{x}\right)^2}$ $\quad (*)$

Set $u = \dfrac{y}{x} \Rightarrow y = u \cdot x$

$\Rightarrow y = u' x + u.$

$(*) \Rightarrow u'x + u = \dfrac{1+u}{u+u^2} = \dfrac{1+u}{u(1+u)} = \dfrac{1}{u}$

$\Rightarrow u' x = \dfrac{1}{u} - u = \dfrac{1-u^2}{u}$

$\Rightarrow \displaystyle\int \frac{u\,du}{1-u^2} = \int \frac{dx}{x}$ $\qquad x \cdot \dfrac{du}{dx} = \dfrac{1-u^2}{u}$

$\qquad \displaystyle\int \frac{u}{1-u^2}\,du = \int \frac{dx}{x}$

$\Rightarrow -\dfrac{1}{2} \displaystyle\int \frac{-2u}{1-u^2}\,du = \int \frac{dx}{x}$

$\Rightarrow -\dfrac{1}{2} \ln|1-u^2| = \ln|x| + c, \quad c \in \mathbb{R}$

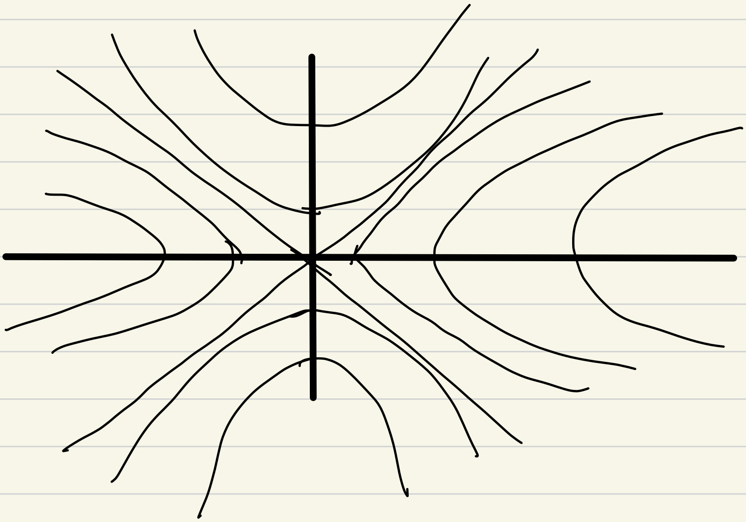
$\Rightarrow \ln|x| + \dfrac{1}{2} \ln|1-u^2| = c, \quad c \in \mathbb{R}$

$\Rightarrow \ln\sqrt{|x^2(1-u^2)|} = c, \quad c \in \mathbb{R}$

$\Rightarrow \sqrt{\left|x^2\left(1 - \dfrac{y^2}{x^2}\right)\right|} = c_1, \quad c_1 > 0$

$$\Rightarrow \quad |x^2 - y^2| = c_2, \qquad c_2 \geq 0$$

$$\Rightarrow \quad x^2 - y^2 = C, \qquad C \in \mathbb{R} \text{ constant.}$$



- Solve the integral equation

$$f(x) = 3 + 2\int_1^x t f(t)\,dt, \qquad x \in \mathbb{R}.$$

- If we differentiate both sides:

$$f'(x) = 2x f(x) \Rightarrow$$

$$f'(x) - 2x f(x) = 0 \Rightarrow$$

$$e^{-x^2} f'(x) - 2x e^{-x^2} f(x) = 0 \Rightarrow$$

$$\left[e^{-x^2} f(x)\right]' = 0 \Rightarrow$$

$$e^{-x^2} f(x) = c, \qquad c \in \mathbb{R} \text{ const.} \Rightarrow$$

$$f(x) = c\, e^{x^2}, \qquad c \in \mathbb{R} \text{ const.}$$

$$f(x) = 3 + 2\int_1^x t f(t)\, dt \Rightarrow f(1) = 3$$

$$\Rightarrow c\, e = 3 \Rightarrow c = 3e^{-1}.$$

Hence

$$f(x) = 3e^{x^2-1}.$$

ASIDE: $f(x) = x + 2\int_0^1 f(t)\, dt$

Set $A = \int_0^1 f(t)\, dt$.

$$f(x) = x + 2A \Rightarrow$$

$$\int_0^1 f(x)\, dx = \frac{1}{2} + 2A \Rightarrow$$

$$\frac{1}{2} + 2A = A \Rightarrow A = -\frac{1}{2}.$$

$$f(x) = x - 1.$$

(OPPGAVE 3, DES. 2017): Solve the initial value problem

$$y' + 2xy = e^{-x^2}, \qquad y(0) = 1.$$

- $y' + 2xy = e^{-x^2} \Rightarrow$

$$e^{x^2} y' + 2xe^{x^2} = 1 \Rightarrow$$

$$(e^{x^2} y)' = (x)' \Rightarrow$$

$$e^{x^2} y = x + c \Rightarrow$$

$$y = (x+c)\, e^{-x^2}, \qquad c \in \mathbb{R} \text{ constant.}$$

$y(0) = 1 \Rightarrow c = 1$

$$y(x) = (x+1)\, e^{-x^2}.$$

(OPPGAVE 2, DES. 2013):

(a) Find

$$\int \frac{2}{u(2+u)}\, du$$

(b) Find all solutions of

$$(2+e^x)\frac{dy}{dx} + 2y = 0.$$

Which solution satisfies $\lim_{x\to+\infty} y(x) = 1$?

ANSWER

(a) We seek constants $A, B \in \mathbb{R}$ s.t.

$$\frac{2}{u(2+u)} = \frac{A}{u} + \frac{B}{2+u} \quad \forall u \in \mathbb{R}\setminus\{0,-2\}$$

$$\Rightarrow \quad A(2+u) + Bu = 2 \quad \forall u \in \mathbb{R}$$

$$\Rightarrow \quad (A+B)u + 2A - 2 = 0 \quad \forall u \in \mathbb{R}$$

Hence $A = 1$ and $B = -1$.

$$\int \frac{2}{u(2+u)}\, du = \int \frac{du}{u} - \int \frac{du}{2+u}$$

$$= \ln|u| - \ln|2+u| + C$$

$$= \ln\left|\frac{u}{2+u}\right| + C.$$

* Alternatively: $\dfrac{2}{u(2+u)} = \dfrac{(u+2)-u}{u(2+u)}$

$$= \frac{1}{u} - \frac{1}{u+2}$$

(b) $(2+e^x)\dfrac{dy}{dx} + 2y = 0$

(Standard Form: $y' + p(x)\cdot y = g(x)$)

$$\frac{dy}{dx} + \frac{2}{2+e^x}y = 0$$

We multiply by $e^{\int \frac{2}{2+e^x}dx}$.

$\displaystyle\int \frac{2}{2+e^x}dx =$

(Set $u = e^x \Rightarrow du = e^x dx$)

$$= \int \frac{2e^x dx}{e^x(2+e^x)} = \int \frac{2\,du}{u(2+u)}$$

$$\overset{(a)}{=} \ln\left|\frac{u}{2+u}\right| + c = \ln\left|\frac{e^x}{2+e^x}\right| + c$$

$$= \ln\left(\frac{e^x}{2+e^x}\right) + c.$$

We multiply by $e^{\ln\left(\frac{e^x}{e^x+2}\right)} = \frac{e^x}{e^x+2}$.

$$y' + \frac{2}{2+e^x} y = 0 \Rightarrow$$

$$\frac{e^x}{e^x+2} y' + \frac{2e^x}{(2+e^x)^2} y = 0 \Rightarrow$$

$$\left(\frac{e^x}{e^x+2} \cdot y\right)' = 0 \Rightarrow$$

$$y = c \cdot e^{-x}(e^x+2) \Rightarrow$$

$$y = c(1 + 2e^{-x}). \quad (c \in \mathbb{R} \text{ const.}).$$

$\lim_{x \to +\infty} \left[c(1+2e^{-x})\right] = c$, hence the solution with $\lim_{x\to+\infty} y(x) = 1$ is

$$y(x) = 1 + 2e^{-x}.$$

(OPPGAVE 8, DES. 2016): Find all solutions of the D.E.

$$y' = y - y^3.$$

- $y(1-y^2) = 0 \iff y = 0$ or $y = 1$ or $y = -1$.

The constant functions $y_0(x) = 0$, $y_1(x) = 1$, $y_2(x) = -1$ are solutions.

$$\frac{y'}{y(1-y^2)} = 1 \Rightarrow \int \frac{dy}{y(1-y^2)} = \int dx$$

We want to find constants $A, B, C \in \mathbb{R}$ s.t.

$$\frac{1}{y(1-y^2)} = \frac{A}{y} + \frac{B}{1+y} + \frac{C}{1-y} \Rightarrow$$

$$A(1-y^2) + By(1-y) + Cy(1+y) = 1, \forall y \in \mathbb{R}$$

$y = 0 \Rightarrow A = 1$
$y = -1 \Rightarrow -2B = 1 \Rightarrow B = -\frac{1}{2}$
$y = 1 \Rightarrow 2C = 1 \Rightarrow C = \frac{1}{2}$

$$\int \frac{dy}{y(1-y^2)} = \int \frac{dy}{y} + \frac{1}{2}\int \left(\frac{1}{1-y} - \frac{1}{1+y}\right) dy$$

$$= \ln|y| - \frac{1}{2}\ln|1-y| - \frac{1}{2}\ln|1+y|$$

$$= \ln|y| - \ln\sqrt{|1-y^2|} + c$$

$$= \ln \frac{|y|}{\sqrt{|1-y^2|}} + c$$

So $\displaystyle\int \frac{dy}{y(1-y^2)} = \int dx \Rightarrow$

$$\ln \frac{|y|}{\sqrt{|1-y^2|}} = x + c \quad , \quad c \in \mathbb{R} \Rightarrow$$

$$\frac{|y|}{\sqrt{|1-y^2|}} = e^{x+c} \quad (c \in \mathbb{R})$$

$$= c_1 \cdot e^x \quad (c_1 > 0 \text{ const.})$$

Thus

$$\frac{y^2}{1-y^2} = c_1^2 e^{2x} \Rightarrow$$

$$y(x) = \pm \sqrt{\frac{c_1^2 e^{2x}}{1 + c_1^2 e^{2x}}}$$