V2021

08/06/2022 13:09

1.

(a) The fixed-point formulation of a nonlinear (scalar) equation is $f(x)=0$, where f is a nonlinear equation.
$f(x)=0$ can be written as $x-g(x)=0$ giving

$x=g(x)$

This x is then called a fixed point.

The fixed-point iteration is then

$x_{n+1}=g(x_n),\ n=0,1,\dots$

Given functions $a(x)$ and $b(x)$ s.t.

$a(x)=b(x)$

We can reformulate this to

$$\begin{aligned} a(x)-b(x) &= f(x) \\ &= x-g(x) \\ &= 0 \end{aligned}$$

Giving the fixed-point form

$x=g(x)$

(b) $f(x)=x-\tan(\frac{x}{2}),\ |x|<\pi$

$x=x-\alpha f(x)$

$g(x)=x-\alpha f(x)$

From Theorem 1.5 in S-M, we have convergence if $|g'(\xi)|<1$, where ξ is a fixed point

$$|g'(x)|=\left|1-\alpha\left(1-\frac{1}{1+\cos(x)}\right)\right| < 1$$

$$\Rightarrow -2<-\alpha\left(1-\frac{1}{1+\cos(x)}\right)<0$$

$$\Rightarrow 0<\alpha\left(1-\frac{1}{1+\cos(x)}\right)<2$$

Around $\pm r$:

$$0<\alpha\left(1-\frac{1}{1+\cos(\pm r)}\right)<2$$

$0<\alpha\cdot(-2.33)<2$

$\Rightarrow \alpha\in(-0.858, 0)$

Convergence if $\alpha\in(-0.858,0)$ around $\pm r$

2.

(a) Can solve least square problems with

Least squares method

$Ax=b$

$(A^TA)x=A^Tb$

$Bx=A^Tb$

$x=B^{-1}A^Tb$

QR-factorisation

$A=QR$

$QRx=b$

$Rx=Q^Tb$

(b) $A=\begin{pmatrix}\sqrt{2} & 0\\ 0 & 1\\ \sqrt{2} & 0\end{pmatrix}$

$A^T=\begin{pmatrix}\sqrt{2} & 0 & \sqrt{2}\\ 0 & 1 & 0\end{pmatrix}$

$$A^TA=B=\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}$$

$$B^{-1}=\frac{1}{4}\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}=\begin{pmatrix}\frac{1}{4} & 0\\ 0 & 1\end{pmatrix}$$

$$B^{-1}A^T=C=\begin{pmatrix}\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}}\\ 0 & 1 & 0\end{pmatrix}$$

$b=(b_1,b_2,b_3)^T$

$$x=Cb=\begin{pmatrix}\frac{b_1+b_3}{2\sqrt{2}}\\ b_2\end{pmatrix}$$

3.

(a)

(b) Writing

$I_{[a,b]}=I_1$

$I_{[a,c]}=I_2$

$I_{[c,b]}=I_3$

$S_{[a,b]}=S_1$

$S_{[a,c]}=S_2$

$S_{[c,b]}=S_3$

$I_1-S_1=Ch^p+O(h^{p+1})$

$h=b-a$

$c=\frac{a+b}{2}$

$I_1=I_2+I_3$

$I_1=S_1+Ch^p+O(h^{p+1})$

$I_2=S_2+C(\frac{h}{2})^p+O($

$C=\frac{I_1-S_1-O(h^{p+1})}{h^p}$ (1)

$I_1-(S_2+S_3)=2C(\frac{h}{2})^p+O((\frac{h}{2})^p)$

$C=\frac{I_1-S_2-S_3-O((\frac{h}{2})^p)}{2(\frac{h}{2})^p}$ (2)

(1)=(2)

$$\frac{I_1-S_1-O(h^{p+1})}{h^p}=\frac{I_1-S_2-S_3-O((\frac{h}{2})^p)}{2(\frac{h}{2})^p}$$

$$I_1=\frac{-2S_1(\frac{h}{2})^{-p}(\frac{h}{2})^p h^{-p}+S_2(\frac{h}{2})^{-p}+S_3(\frac{h}{2})^{-p}}{(\frac{h}{2})^p-2h^{-p}}$$

$=-2S_1h^{-p}+S_2(\frac{h}{2})^p+$

$I_1-S_1=\dots-S_1$

$I_1-(S_2+S_3)=\dots-(S_2+S_3)$

4.

(a) The local truncation error is

$$T_n=\frac{y(x_{n+1})-y(x_n)}{h}-\Phi(x_n,y(x_n);h)$$

The global error is

$e_n=y(x_n)-y_n$

The order of a RKM is defined by its local truncation error

When the LTE is $O(h^{p+1})$ the RKM is of order p

(b) $A=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$

$b=\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}$

$c=\begin{pmatrix}0\\ 1\end{pmatrix}$

$$\begin{array}{c|cc} 0 & 0 & 0\\ 1 & 1 & 0\\ \hline & \frac{1}{2} & \frac{1}{2}\end{array}$$

$\ddot{y}=-\sin(y)$

$y(0)=y_0$

$\dot{y}(0)=0$

$$X=\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}y\\ \dot{y}\end{pmatrix}$$

$$X'=F(X)=\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}\dot{y}\\ \ddot{y}\end{pmatrix}=\begin{pmatrix}x_2\\ -\sin(x_1)\end{pmatrix}$$

Runge-Kutta:

$y_{n+1}=y_n+\frac{h}{2}(k_1+k_2)$

$k_1=F(t_n,y_n)$

$k_2=F(t_n+h,y_n+hk_1)$

One step:

$X_1=X_0+\frac{h}{2}(k_1+k_2)$

$$k_1=F(X_0)=F\begin{pmatrix}y(0)\\ \dot{y}(0)\end{pmatrix}=F\begin{pmatrix}y_0\\ 0\end{pmatrix}=\begin{pmatrix}0\\ -\sin(y_0)\end{pmatrix}$$

$$k_2=F(X_0+hk_1)=F\begin{pmatrix}y_0+h\cdot 0\\ 0-h\cdot\sin(y_0)\end{pmatrix}=F\begin{pmatrix}y_0\\ -h\sin(y_0)\end{pmatrix}=\begin{pmatrix}-h\sin(y_0)\\ -\sin(y_0)\end{pmatrix}$$

$$X_1=X_0+\frac{h}{2}\begin{pmatrix}0 + -h\sin(y_0)\\ -\sin(y_0) + -\sin(y_0)\end{pmatrix}$$

$$=X_0+\frac{h}{2}\begin{pmatrix}-h\sin(y_0)\\ -2\sin(y_0)\end{pmatrix}$$

$$=\begin{pmatrix}y_0 + -\frac{h^2}{2}\sin(y_0)\\ 0 + -h\sin(y_0)\end{pmatrix}$$

$$=\begin{pmatrix}y_0-\frac{h^2}{2}\sin(y_0)\\ -h\sin(y_0)\end{pmatrix}$$

5.

(a) $G_h\theta=\mathrm{tridiag}(\frac{1}{h^2},-\frac{2}{h^2}+w^2,\frac{1}{h^2})\theta$

$$=\begin{pmatrix}(-\frac{2}{h^2}+w^2)\theta_1+\frac{1}{h^2}\theta_2\\ \frac{1}{h^2}\theta_1+(-\frac{2}{h^2}+w^2)\theta_2+\frac{1}{h^2}\theta_3\\ \vdots\\ \frac{1}{h^2}\theta_{m-2}+(-\frac{2}{h^2}+w^2)\theta_{m-1}+\frac{1}{h^2}\theta_m\\ \frac{1}{h^2}\theta_{m-1}+(-\frac{2}{h^2}+w^2)\theta_m\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{1}{h^2}\theta_0\\ 0\\ \vdots\\ 0\\ -\frac{1}{h^2}\theta_{m+1}\end{pmatrix}$$

$$=-\frac{1}{h^2}\begin{pmatrix}\alpha\\ 0\\ \vdots\\ 0\\ \beta\end{pmatrix}$$

Truncation error. The truncation error is the vector that by definition has components

$$\tau_m := \frac{1}{h^2}(u_{m-1}-2u_m+u_{m+1})-f_m,\quad m=1,\dots,M,\quad \tau_h:=\begin{bmatrix}\tau_1\\ \vdots\\ \tau_M\end{bmatrix}.$$

This is a linear system of equations

$$A_h\mathbf{U}=\mathbf{F}, \tag{2.3}$$

$$\tau_m:=\left(\frac{1}{h^2}(\theta_{m-1}-2\theta_m+\theta_{m+1})+\underbrace{w^2\theta_m}_{f?}\right)-b_m$$

$$T=\begin{cases}\tau_1, & 0\le t<\\ \vdots & \\ \tau_m, & t_m-t_{m-1}<t<t_m\end{cases}$$