▶ INTEGRALS INVOLVING $\sqrt{a^2+x^2}$

Recall the identity $\cosh^2 x - \sinh^2 x = 1$.
Here the substitution

$$x = a\sinh y$$

might work.

E.g. $\displaystyle\int \frac{dx}{\sqrt{1+x^2}} =$ Set $x = \sinh y$, $dx = \cosh y\, dy$

$$= \int \frac{\cosh y\, dy}{\sqrt{1+\sinh^2 y}} = \int \frac{\cosh y\, dy}{\sqrt{\cosh^2 y}}$$

$$= y + c$$

$$= \operatorname{arsinh} x + c$$
$$= \ln(x + \sqrt{x^2+1}) + c.$$

Also in view of the identity

$$1 + \tan^2 x = \frac{1}{\cos^2 x}$$

the substitution $x = a\tan\theta$ might work.

$\displaystyle\int \frac{dx}{\sqrt{1+x^2}} =$ Set $x = \tan\theta$, $dx = \dfrac{d\theta}{\cos^2\theta}$

$$= \int \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \frac{1}{\cos^2\theta}\, d\theta = \int \frac{1}{\sqrt{\frac{1}{\cos^2\theta}}} \cdot \frac{1}{\cos^2\theta}\, d\theta$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$= \int \frac{d\theta}{\cos\theta} = \int \frac{\cos\theta\,d\theta}{\cos^2\theta} = \int \frac{\cos\theta\,d\theta}{1-\sin^2\theta}$$

Set $u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \int \frac{du}{1-u^2} = \int \frac{du}{(1-u)(1+u)}$$

(we need to use partial fraction decomp.)

$$= \frac{1}{2}\int \frac{du}{1+u} + \frac{1}{2}\int \frac{du}{1-u}$$

$$= \frac{1}{2}\ln|1+u| - \frac{1}{2}\ln|1-u| + C$$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\frac{1}{\cos\theta}+\tan\theta}{\frac{1}{\cos\theta}-\tan\theta}\right| + C$$

$$= \frac{1}{2}\ln\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right) + C$$

$$= \ln\left(x+\sqrt{1+x^2}\right) + C.$$

$x = \tan\theta$

$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta \Rightarrow$

$\frac{1}{\cos^2\theta} = 1 + x^2 \Rightarrow$

$\frac{1}{\cos\theta} = \sqrt{1+x^2}$

▸ INTEGRALS INVOLVING $\sqrt{x^2-a^2}$

In view of the identity

$$\cosh^2 x - \sinh^2 x = 1$$

the substitution $x = a\cosh y$ might be useful.

$$\int \sqrt{x^2-1}\, dx \qquad \text{Set } x = \cosh u, \quad dx = \sinh u\, du$$

$$= \int \sqrt{\cosh^2 u - 1} \cdot \sinh u\, du$$

$$= \int \sinh^2 u\, du = \frac{1}{2}\int (\cosh 2u - 1)\, du$$

$$= \frac{1}{2}\int \cosh 2u\, du - \frac{1}{2}\int 1\, du$$

$$= \frac{1}{4}\sinh 2u - \frac{u}{2} + C$$

$$= \frac{1}{2}\sinh u \cdot \cosh u - \frac{u}{2} + C$$

$$= \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\operatorname{arcosh} x + C$$

$$= \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln\left(x + \sqrt{x^2-1}\right) + C.$$

➤ INTEGRALS OF THE FORM $\int \varphi(\sin\theta, \cos\theta)\, d\theta$ WHERE $\varphi(x, y)$ IS A RATIONAL FUNCTION.

We have already seen

$$\int \frac{d\theta}{\cos\theta} = \int \frac{\cos\theta\, d\theta}{1-\sin^2\theta} = \cdots = \frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right) + C.$$

Generally for $\int \varphi(\sin\theta, \cos\theta)\, d\theta$ we apply the substitution

$$x = \tan\frac{\theta}{2}.$$

Then $\theta = 2\arctan x \Rightarrow d\theta = \dfrac{2dx}{1+x^2}.$

Also

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{1+\tan^2\frac{\theta}{2}} - 1 \Rightarrow$$

$$\cos\theta = \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \frac{2x}{1+x^2}.$$

E.g. $$\int \frac{d\theta}{2+\sin\theta} = \int \frac{1}{2+\frac{2x}{1+x^2}} \cdot \frac{2}{1+x^2} \cdot dx =$$

(we set $x = \tan\left(\frac{\theta}{2}\right)$)

$$D = 1^2 - 4 \cdot 1 = -3.$$

$$= \int \frac{dx}{x^2 + x + 1} = \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

Set $x + \frac{1}{2} = \frac{\sqrt{3}}{2} u \Rightarrow$

$$dx = \frac{\sqrt{3}}{2} du$$

$$= \int \frac{\sqrt{3}}{2} \frac{du}{\left(\frac{\sqrt{3}}{2}\right)^2 \cdot (u^2 + 1)}$$

$$= \frac{2\sqrt{3}}{3} \arctan u + C$$

$$= \frac{2\sqrt{3}}{3} \arctan\left(\frac{2\sqrt{3}\,x}{3} + \frac{\sqrt{3}}{3}\right) + C.$$

$$= \frac{2\sqrt{3}}{3} \arctan\left(\frac{2\sqrt{3}\tan\frac{\theta}{2}}{3} + \frac{\sqrt{3}}{3}\right) + C.$$

▶ INTEGRALS OF THE FORM

$$\int e^{ax}\sin bx\,dx \quad , \quad \int e^{ax}\cos bx\,dx$$

E.g. Find $\int e^{-x}\cos x\,dx$.

$$I = \int e^{-x}\cos x\,dx$$

$$= \int e^{-x}(\sin x)'\,dx$$

$$= e^{-x}\cdot\sin x - \int (e^{-x})'\cdot\sin x\,dx$$

$$= e^{-x}\sin x + \int e^{-x}\sin x\,dx$$

$$= e^{-x}\sin x + \int e^{-x}(-\cos x)'\,dx$$

$$= e^{-x}\sin x - e^{-x}\cos x - \int (e^{-x})'(-\cos x)\,dx$$

$$= e^{-x}(\sin x - \cos x) - \int e^{-x}\cos x\,dx$$

$$= e^{-x}(\sin x - \cos x) - I \implies$$

$$2I = e^{-x}(\sin x - \cos x) + c \implies$$

$$I = \frac{1}{2}e^{-x}(\sin x - \cos x) + c.$$

- INTEGRALS OF THE FORM

$$\int \sin^k x \cdot \cos^m x \, dx \qquad (k, m \geq 1)$$

- When one of k, m is odd we use it to create a new differential.

$$\int \sin^8 x \cdot \cos^7 x \, dx = \int \sin^8 x \cdot \cos^6 x \cdot \cos x \, dx$$

$$= \int \sin^8 x \cdot (1 - \sin^2 x)^3 \cos x \, dx$$

Set $u = \sin x$
$du = \cos x \, dx$

$$= \int u^8 (1 - u^2)^3 \, du$$

$$= \frac{1}{9} u^9 - \frac{3}{11} u^{11} + \frac{3}{13} u^{13} - \frac{1}{15} u^{15} + C$$

$$= \frac{1}{9} \sin^9 x - \frac{3}{11} \sin^{11} x + \frac{3}{13} \sin^{13} x - \frac{1}{15} \sin^{15} x + c.$$

- When both k, m are even, we may use the "desquaring formulas":

$$\cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin^2 x = \frac{1 - \cos 2x}{2}.$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\int \sin^2 x \cos^2 x \, dx = \frac{1}{4}\int \sin^2 2x \, dx$$

$$= \frac{1}{4}\int \frac{1-\cos 4x}{2} \, dx$$

$$= \frac{1}{8}\int dx - \frac{1}{8}\int \cos 4x \, dx$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + c.$$

• UNIFORM CONTINUITY

Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuous. Let $\varepsilon>0$.

- Since f is cont. at $x=1$,
  $\exists\, \delta_1>0$ s.t. $|x-1|<\delta_1 \Rightarrow |f(x)-f(1)|<\varepsilon$.

- Since f is cont. at $x=\frac{3}{2}$,
  $\exists\, \delta_{\frac{3}{2}}>0$ s.t. $|x-\frac{3}{2}|<\delta_{3/2} \Rightarrow |f(x)-f(\frac{3}{2})|<\varepsilon$.

- Since f is cont. at $x=-\sqrt{2}$
  $\exists\, \delta_{-\sqrt{2}}>0$ s.t. $|x+\sqrt{2}|<\delta_{-\sqrt{2}} \Rightarrow |f(x)-f(-\sqrt{2})|<\varepsilon$.

⋮

In general, the values of $\delta_1, \delta_{3/2}, \delta_{\sqrt{2}}, \ldots$ etc are not the same. If they can be chosen to have the same value, f is called uniformly continuous.

We say f is <u>uniformly continuous</u> on a set $A \subseteq D_f$ if:
For all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, y \in A$:

$$|x-y| < \delta \implies |f(x)-f(y)| < \varepsilon .$$

<u>REMARKS</u>: (i) Continuity of f is defined on a given point x_0 while uniform continuity is defined on a set A.

(ii) If f is uniformly continuous on $A \subseteq \mathbb{R}$ and $B \subseteq A$, then f is also uniformly continuous on B.

<u>THEOREM 5.2</u>: Let $f : [a,b] \to \mathbb{R}$. If f is continuous on $[a,b]$, then it is also uniformly continuous on $[a,b]$.

This is the second instance of a special property of continuous functions on closed intervals $[a,b]$. The first was that they always have maximum and minimum.

PROPOSITION 5.3: Suppose $f: I \to \mathbb{R}$ (I is an interval) is differentiable and there exists $M>0$ such that
$$|f'(x)| \leq M \qquad \text{for all } x \in I.$$

Then f is uniformly continuous on I.

PROOF

By the Mean Value Theorem, for all $x, y \in I$ with $x \neq y$ there exists $\xi = \xi(x,y) \in I$ such that
$$f(x) - f(y) = f'(\xi) \cdot (x-y).$$

Let $\varepsilon > 0$. If we set $\delta = \frac{\varepsilon}{M} > 0$, then for all $x, y \in I$,
$$|x-y| < \delta \quad \text{implies} \quad |f(x)-f(y)| = |f'(\xi)| \cdot |x-y| < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

So f is unif. cont. on I. ∎

Exercise: Show that $f:(0,1)\to\mathbb{R}$, $f(x)=\frac{1}{x}$ is not unif. continuous on $(0,1)$.

(Here the idea is that f' is not bounded).

Assume for contradiction f is uniformly continuous on $(0,1)$. Take $\varepsilon=1$. Because f is unif. cont., there exists $\delta>0$ such that

$$|x-y|<\delta \Rightarrow \left|\frac{1}{x}-\frac{1}{y}\right|<1.$$

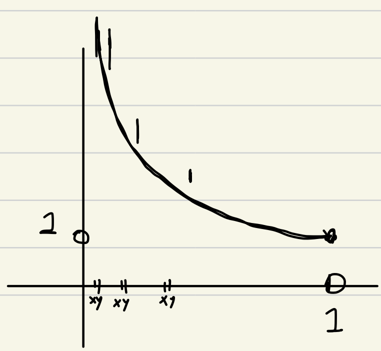
Let $N>1$ be such $\frac{1}{N}<\delta$.

Then for any $0<x<y<1$ such that $\frac{1}{2N}<y-x<\frac{1}{N}$ we have $\frac{1}{x}-\frac{1}{y}<1$. Then

$$1>\frac{1}{x}-\frac{1}{y}=|f'(\xi)|\cdot|y-x|$$

$$=\frac{1}{\xi^2}(y-x)$$

$$>\frac{1}{y^2}\cdot\frac{1}{2N}.$$



Now if we choose $y=\frac{1}{N}$, we get

$$1>\frac{N}{2};\ \text{contradiction.}$$