Øving 11

15.3

5.

$$\sum_{n=2}^{\infty} \underbrace{\frac{n(n-1)}{4^n}}_{a_n}(z-2i)^n$$

(a)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)\cancel{n}}{4^{n+1}} \Big/ \frac{\cancel{n}(n-1)}{4^n}\right|$$
$$= \frac{1}{4}\left|\frac{n+1}{n-1}\right| \xrightarrow{n\to\infty} \frac{1}{4}$$

Kan bruke Cauchy-Hadamard:

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$
$$= 4$$

(b) $\sum_{n=2}^{\infty} \frac{n(n-1)}{4^n}\cdot n\cdot(z-2i)^{n-1} = \sum_{n=2}^{\infty} \frac{n^2(n-1)}{4^n}(z-2i)^{n-1}$

$$a_n = \frac{n^2(n-1)}{4^n}$$
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2 n}{4^{n+1}} \Big/ \frac{n^2(n-1)}{4^n}\right|$$
$$= \frac{1}{4}\left|\frac{(n+1)^2}{n(n-1)}\right| \xrightarrow{n\to\infty} \frac{1}{4}$$
$$\Rightarrow R = 4$$

8.

$$\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)} z^n$$

(a) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}}{\cancel{(n+1)}(n+2)} \Big/ \frac{3^n}{n\cancel{(n+1)}}\right|$

$$= \left|\frac{3n}{n+2}\right|$$
$$= 3\left|\frac{n}{n+2}\right| \xrightarrow{n\to\infty} 3$$
$$\Rightarrow R = \frac{1}{3}$$

(b) $\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)}\cdot n z^{n-1} = \sum_{n=1}^{\infty} \frac{3^n}{n+1} z^{n-1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}}{n+2} \Big/ \frac{3^n}{n+1}\right|$$
$$= 3\left|\frac{n+1}{n+2}\right| \xrightarrow{n\to\infty} 3$$
$$\Rightarrow R = \frac{1}{3}$$

10.

$$\sum_{n=k}^{\infty} \binom{n}{k}\left(\frac{z}{2}\right)^n = \sum_{n=k}^{\infty} \binom{n}{k}\frac{1}{2^n}\cdot z^n$$

(a)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\binom{n+1}{k}\frac{1}{2^{n+1}} \Big/ \binom{n}{k}\frac{1}{2^n}\right|$$
$$= \left|\frac{(n+1)!}{\cancel{k!}(n+1-k)!}\cdot\frac{1}{2} \Big/ \frac{n!}{\cancel{k!}(n-k)!}\right|$$
$$= \frac{1}{2}\left|\frac{n+1}{n+1-k}\right| \xrightarrow{n\to\infty} \frac{1}{2}$$
$$\Rightarrow R = 2$$

(b) $\sum_{n=k}^{\infty} \binom{n}{k}\cdot\frac{n}{2}\left(\frac{z}{2}\right)^{n+1} = \sum_{n=k}^{\infty} \binom{n}{k}\cdot\frac{n}{2^{n+2}}\cdot z^{n+1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\binom{n+1}{k}\frac{n+1}{2^{n+3}} \Big/ \binom{n}{k}\frac{n}{2^{n+2}}\right|$$
$$= \frac{1}{2}\left|\frac{(n+1)(n+1)}{(n+1-k)\cdot n}\right| \xrightarrow{n\to\infty} \frac{1}{2}$$
$$\Rightarrow R = 2$$

15.4

3.

$$f(z) = \sin\left(\frac{z^2}{2}\right)$$

$u = \frac{z^2}{2}$:

$$\sin(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{2^{2n+1}(2n+1)!}$$
$$a_n = (-1)^n \frac{1}{2^{2n+1}(2n+1)!}$$
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^{n+1}\frac{1}{2^{2n+3}(2n+3)!} \Big/ (-1)^n \frac{1}{2^{2n+1}(2n+1)!}\right|$$
$$= \left|-\frac{1}{2^2(2n+3)(2n+2)}\right|$$
$$= \frac{1}{4}\left|\frac{1}{(2n+3)(2n+2)}\right| \xrightarrow{n\to\infty} 0$$
$$\Rightarrow R = \infty$$

4.

$$f(z) = \frac{z+2}{1-z^2}$$
$$= \frac{z+1+1}{(1+z)(1-z)}$$
$$= \frac{z+1}{(1+z)(1-z)} + \frac{1}{(1+z)(1-z)}$$
$$= \frac{1}{1-z} + \left(\frac{A}{1+z} + \frac{B}{1-z}\right)$$
$$= \begin{bmatrix} 1 = A(1-z) + B(1+z) \\ 0 = -Az + Bz \\ A = B \\ 1 = A + B \\ A = \frac{1}{2} = B \end{bmatrix}$$
$$= \frac{1}{1-z} + \frac{1}{2}\cdot\frac{1}{1-z} + \frac{1}{2}\cdot\frac{1}{1+z}$$
$$= \frac{3}{2}\cdot\frac{1}{1-z} + \frac{1}{2}\cdot\frac{1}{1+z}$$
$$= \frac{3}{2}\cdot\sum_{n=0}^{\infty} z^n + \frac{1}{2}\cdot\sum_{n=0}^{\infty}(-1)^n z^n$$
$$= \sum_{n=0}^{\infty}\left(\frac{3}{2}z^n + \frac{1}{2}(-1)^n z^n\right)$$
$$= \sum_{n=0}^{\infty} \frac{3+(-1)^n}{2} z^n$$
$$\left|\frac{3+(-1)^{n+1}}{2} \Big/ \frac{3+(-1)^n}{2}\right| \xrightarrow{n\to\infty} 1$$
$$R = 1$$

8.

$$f(z) = \sin^2(z)$$
$$=$$

23.

$\frac{1}{(z-i)^2}$, $z_0 = -i$

$$\frac{1}{(z-i)^2} = \frac{1}{((z+i)-2i)^2}$$
$$= \frac{1}{(-2i)^2}\cdot\frac{1}{(1+(z+i)/(-2i))^2}$$
$$= \frac{1}{(-2i)^2}\cdot\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{z+i}{-2i}\right)^n$$
$$= \sum_{n=0}^{\infty}\binom{-2}{n}\frac{1}{(-2i)^{n+2}}(z+i)^n$$
$$= \sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{i}{2}\right)^{n+2}(z+i)^n$$
$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{i}{2}\right)^{n+2}(z+i)^n$$
$$a_n = (-1)^n\left(\frac{i}{2}\right)^n$$
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^{n+1}\left(\frac{i}{2}\right)^{n+1} \Big/ (-1)^n\left(\frac{i}{2}\right)^n\right|$$
$$= \left|\frac{i}{2}\right|$$
$$= \frac{1}{2}|i|$$
$$= \frac{1}{2}$$
$$\Rightarrow R = 2$$

24.

$e^{z(z-2)}$, $z_0 = 1$

$$e^{z(z-2)} = e^{z^2-2z}$$
$$= \sum_{n=0}^{\infty}\frac{(z^2-2z)^n}{n!}$$

16.1

2.

$$f(z) = \frac{e^{(-1/z^2)}}{z^2}$$
$$e^{-\frac{1}{z^2}} = \sum_{n=0}^{\infty}\frac{(-1/z^2)^n}{n!}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^{-2n}$$
$$\Rightarrow f(z) = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^{-2n}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^{-2n-2}$$

6.

$$f(z) = \frac{1}{z^2(z-i)}$$
$$= \frac{1}{z^2}\cdot\frac{1}{z(1-i/z)}$$
$$= \frac{1}{z^3}\cdot\sum_{n=0}^{\infty}\frac{1}{(z/i)^{n+1}}, \quad \left|\frac{z}{i}\right| > 1$$
$$f(z) = \frac{1}{z^2}\cdot\frac{-1}{i(1-z/i)}$$
$$= \frac{-1}{iz^2}\sum_{n=0}^{\infty}(iz)^n, \quad \left|\frac{z}{i}\right| < 1$$
$$= -\sum_{n=0}^{\infty} i^{n-1}z^{n-2}$$

$z_0 = i$

$$\Rightarrow f(z) = -\sum_{n=0}^{\infty} i^{n-1}(z-i)^{n-2}$$
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{i^n}{i^{n-1}}\right|$$
$$= |i|$$
$$= 1$$

$R = 1$

$0 < |z-i| < 1$

13.

$f(z) = \frac{z^8}{1-z^4}$, $z_0 = 0$

$$= \begin{cases} z^8\sum_{n=0}^{\infty} z^{4n} = \sum_{n=0}^{\infty} z^{4n+8}, & |z^4| < 1 \\ z^8\cdot\frac{-1}{z^4(1-z^{-4})} = -z^4\sum_{n=0}^{\infty}\frac{1}{z^{4n+4}} = -\sum_{n=0}^{\infty}\frac{1}{z^{4n}}, & |z^4| > 1 \end{cases}$$