Kandidatnr: 10009. Problem 4

$f(x)=x^m$

$m \geq 4$

$x_0 = 0$

$x_1 = 2$

$x_2 = 4$

(i) $p_2(x)=\frac{x-x_1}{x_0-x_1}\cdot\frac{x-x_2}{x_0-x_2}\cdot f(x_0)+\frac{x-x_0}{x_1-x_0}\cdot\frac{x-x_2}{x_1-x_2}\cdot f(x_1)+\frac{x-x_0}{x_2-x_0}\cdot\frac{x-x_1}{x_2-x_1}\cdot f(x_2)$

$=0+\frac{x}{2}\cdot\frac{x-4}{-2}\cdot 2^m+\frac{x}{4}\cdot\frac{x-2}{2}\cdot 4^m$

$=-\frac{x(x-4)}{2^2}\cdot 2^m+\frac{x(x-2)}{2^3}\cdot 2^{2m}$

$\underline{\underline{=-x(x-4)\cdot 2^{m-2}+x(x-2)\cdot 2^{2m-3}}}$

(ii) $|f(x)-p_2(x)| \leq \frac{M_3}{3!}|\pi_3(x)|$

$M_2 = \max_{\xi\in[0,4]}|f^{(3)}(\xi)|$

$\pi_3(x)=(x-x_0)(x-x_1)(x-x_2)$

$f^{(3)}(x)=m(m-1)(m-2)x^{m-3}$

$\max_{\xi\in[0,4]}|f^{(3)}(\xi)|=f^{(3)}(4)$

$=m(m-1)(m-2)4^{m-3}$

$|\pi_3(x)|=|x(x-2)(x-3)|$

$\underline{\underline{\Rightarrow C(m)=\frac{m(m-1)(m-2)4^{m-3}}{6}|x(x-2)(x-3)|}}$

(iii) $g(x):=f(x)-p_2(x)$

$g(x)=x^m-(-x(x-4)\cdot 2^{m-2}+x(x-2)\cdot 2^{2m-3})$

$=x^m+2^{m-2}x(x-4)-2^{2m-3}x(x-2)$

$g'(x)=mx^{m-1}+2^{m-1}x-2^m-2^m-2^{2m-2}x+2^{2m-2}$

$=mx^{m-1}+2^{m-1}x-2^{m+1}-2^{2m-2}x+2^{2m-2}$

$g''(x)=m(m-1)x^{m-2}+2^{m-1}-0-2^{2m-2}+0$

$=m(m-1)x^{m-2}+2^{m-1}-2^{2m-2}$

$g''(0)=2^{m-1}-2^{2m-2}$

$<0,\ m\geq 4$

$g''(4)=m(m-1)\cdot 4^{m-2}+2^{m-1}-2^{2m-2}$

$>0,\ m\geq 4$

<u>So therefore there exist at least one $x_0\in(0,4)$ s.t. $g''(x_0)=0$</u>