Lecture 3 16/01/23 Week 3.1

Methods Using Mixtures, Multivariate Normals, Rejection Sampling

Example (Student-t Dist'n):

$$X_1 \sim \Gamma\left(\frac{n}{2}, \frac{n}{2}\right), \quad X_2 \mid X_1 \sim N\left(\mu, \frac{\sigma^2}{X_1}\right)$$

Show that $X \sim t_n(\mu, \sigma^2)$

Solution:

$$f_{X_1}(x_1) = \frac{\left(\frac{n}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} x_1^{\frac{n}{2}-1} e^{-\frac{n}{2}x_1}$$

$$f_{X_2|X_1}(x_2|x_1) = \frac{1}{\sqrt{2\pi\sigma^2/x_1}} \exp\left\{-\frac{x_1}{2\sigma^2}(x_2-\mu)^2\right\}$$

$$f_{X_1,X_2}(x_1,x_2) = f_{X_2|X_1}(x_2|x_1)\, f_{X_1}(x_1)$$

$$\propto x_1^{\frac{n}{2}-1} x_1^{\frac{1}{2}} \exp\left\{-\frac{n}{2}x_1 + \frac{x_1}{2\sigma^2}(x_2-\mu)^2\right\}$$

$$f_{X_2}(x_2) \propto \int_0^\infty x_1^{\frac{n}{2}-\frac{1}{2}} \exp\left\{-x_1\left(\frac{n}{2} + \frac{(x_2-\mu)^2}{2\sigma^2}\right)\right\} dx_1$$

$$= \int_0^\infty \left[x_1\left(\frac{n}{2} + \frac{(x_2-\mu)^2}{2\sigma^2}\right)\right]^{\frac{n+1}{2}} \underbrace{\left(\frac{n}{2} + \frac{(x_2-\mu)^2}{2\sigma^2}\right)^{-\frac{n+1}{2}}}_{\text{pull out of integral}} \exp\{\ldots\}\, dx_1$$

$$= \left(\frac{n}{2} + \frac{(x_2-\mu)^2}{2\sigma^2}\right)^{-\frac{n+1}{2}} \int_0^\infty u^{\frac{n+1}{2}} e^{-u}\, du$$

$$\underbrace{}_{=\Gamma\left(\frac{n+1}{2}\right)}$$

$$\propto \left(1 + \frac{1}{n}\left(\frac{x_2 - \mu}{\sigma}\right)^2\right)^{-\frac{n+1}{2}},$$

which is the kernel/core of the $t_n(\mu, \sigma^2)$ dist'n. Hence, $X_2 \sim t_n(\mu, \sigma^2)$. $\square$

Rejection Sampling

$f(x)$: target density

$g(x)$: proposal "

$f(x) \le c\, g(x)$, $\forall x \in \mathbb{R}$ for some $1 \le c < \infty$.

$u \sim Unif(0,1)$ draw

Claim:

If x is drawn from rejection sampling algorithm the $x \sim f$.

Pf: We want to show:

$$p(x \mid c \cdot u\, g(x) \le f(x)) \overset{?}{=} f(x)$$

$$\text{LHS} \propto p(c \cdot u\, g(x) \le f(x) \mid x) \cdot g(x)$$

$$= p\left(u \le \frac{f(x)}{c g(x)} \,\middle|\, x\right) \cdot g(x)$$

$$= \frac{f(x)}{c \cdot g(x)} \cdot g(x)$$

$$\propto f(x).$$

Since $\text{support}(g) \supseteq \text{support}(f)$,

$\text{LHS} = f(x)$. $\square$

$\left(\text{support}(f) \equiv \{x : f(x) > 0\}\right)$

Example (Rejection Sampling of $\mathcal{N}(0,1)$):

Tgt distr'n: $f(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$

proposal distr'n: $g(x) = \frac{\lambda}{2} e^{-\lambda|x|}$, $\lambda > 0$.

$$\frac{f(x)}{g(x)} \propto e^{-\frac{1}{2}x^2 + \lambda|x|}$$

What is $\sup_x e^{-\frac{1}{2}x^2 + \lambda|x|}$?

$$= \sup_x -\frac{1}{2}x^2 + \lambda|x|$$

differentiate:

$$0 = -x + \lambda \operatorname{sgn}(x)$$

$$|x| = \lambda \Rightarrow x = \pm\lambda$$

$$\left.\frac{f(x)}{g(x)}\right|_{x=\pm\lambda} = \sqrt{\frac{2}{\pi}}\, \lambda^{-1} e^{\frac{1}{2}\lambda^2} = c(\lambda)$$

$$\frac{f(x)}{g(x)} \leq \sqrt{\frac{2}{\pi}}\, \lambda^{-1} e^{\frac{1}{2}\lambda^2}$$

Choose λ to minimize $c(\lambda)$:

$$0 = \frac{d}{d\lambda} \sqrt{\frac{2}{\pi}}\, \lambda^{-1} e^{\frac{1}{2}\lambda^2}$$

$$= \sqrt{\frac{2}{\pi}} \left(-\lambda^{-2} e^{\frac{1}{2}\lambda^2} + \cancel{\lambda^{-1}} e^{\frac{1}{2}\lambda^2} \cdot \cancel{\lambda} \right)$$

$$= \sqrt{\frac{2}{\pi}}\, e^{\frac{1}{2}\lambda^2} \left(1 - \lambda^{-2} \right)$$

$$= \sqrt{\frac{2}{\pi}}\, e^{\frac{1}{2}\lambda^2} \left(1 + \lambda^{-1} \right)\left(1 - \lambda^{-1} \right)$$

$$\frac{1}{\lambda} = 1 \Rightarrow \boxed{\lambda = 1}$$