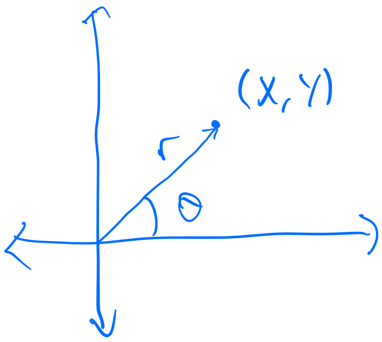
Lecture 2 10/1/23 Week 2.2

Bivariate sampling, transformations, ratio of uniforms

Box-Müller

Goal: Sample $X, Y \overset{iid}{\sim} N(0,1)$

Idea: Represent (X, Y) in polar coordinates



$r^2 = X^2 + Y^2$

$X = r\cos\theta$

$Y = r\sin\theta$

$$\rightarrow \quad r^2 = \underbrace{\underbrace{X^2}_{\chi^2_1} + \underbrace{Y^2}_{\chi^2_1}}_{\chi^2_2}$$

$$r^2 \sim \chi^2_2$$
$$\sim \text{Gamma}(1, 1/2)$$
$$\sim \text{Exp}(1/2)$$

rates

We can draw from this distr'n!

$\left(r^2 \overset{d}{=} -2\log(U_1),\ \text{for } U_1 \sim \text{Unif}(0,1)\right)$

Now, by symmetry, Θ is distributed uniformly on $[0, 2\pi)$, so $\Theta \overset{d}{=} 2\pi U_2$ for $U_2 \sim \text{Unif}(0,1)$.

Note that $\theta \mid r \overset{d}{=} 2\pi U_2$ also, so $\theta \perp r$, & we can draw θ & r independently!

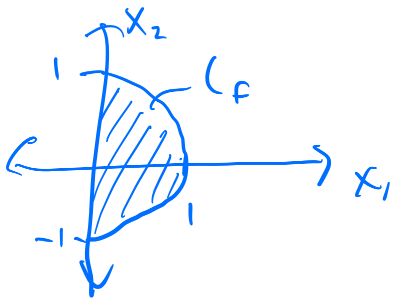
<u>Exercise</u>: Use the Jacobian method for a more rigorous proof.

<u>Example: Cauchy dist'n, ratio of uniforms</u>

$$f^*(x) = \frac{1}{1+x^2} \propto f(x)$$

$$\begin{aligned} C_f &= \left\{ (x_1, x_2) : 0 \le x_1 \le \sqrt{f^*\left(\tfrac{x_2}{x_1}\right)} \right\} \\ &= \left\{ (x_1, x_2) : 0 \le x_1 \le \sqrt{\frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}} \right\} \\ &= \left\{ (x_1, x_2) : 0 \le x_1^2 \le \frac{1}{1+\frac{x_2^2}{x_1^2}},\ x_1 \ge 0 \right\} \\ &= \left\{ (x_1, x_2) : 0 \le x_1^2 + x_1^2 \le 1,\ x_1 \ge 0 \right\} \end{aligned}$$

$\Rightarrow C_f$ is a semi-circle!



Pf (Ratio of Uniforms):

Assume $(X_1, X_2) \sim \text{Unif}(C_f)$ with:

$$C_f = \left\{ (x_1, x_2) : 0 \le x_1 \le \sqrt{f^*\left(\tfrac{x_2}{x_1}\right)} \right\}.$$

Put transformation $g(x_1, x_2) = (y, z)$ for $y = \frac{x_2}{x_1}$, $z = x_1$, so that $g^{-1}(y, z) = (yz, z)$. Then:

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_2}{y} \\ \frac{\partial x_1}{\partial z} & \frac{\partial x_2}{\partial z} \end{vmatrix} = \begin{vmatrix} z & 0 \\ y & 1 \end{vmatrix} = z.$$

So:

$$f_{YZ}(y, z) = f_{X_1 X_2}(yz, z) \cdot z, \quad \text{for } 0 \le z \le \sqrt{f^*(y)}$$

$$= kz,$$

where $k = \frac{1}{\text{Area}(C_f)} = \frac{1}{|C_f|}$. Hence,

$$f_Y(y) = \int_0^{\sqrt{f^*(y)}} kz \, dz = \frac{k}{2} z^2 \Big|_0^{\sqrt{f^*(y)}}$$

$$= \frac{k}{2} f^*(y)$$

$$\Rightarrow \quad f_Y(y) \propto f(y)$$

target density

$\square$

Pf (Ratio of Uniforms Simplification):

$(x_1, x_2) \in C_f \Rightarrow$ if $x_2 \geq 0$, then:

$$x_1 \leq \sqrt{f^*\left(\frac{x_2}{x_1}\right)} \Rightarrow x_2 \leq \frac{x_2}{x_1}\sqrt{f^*\left(\frac{x_2}{x_1}\right)} = \sqrt{\left(\frac{x_2}{x_1}\right)^2 f^*\left(\frac{x_2}{x_1}\right)}$$

$$\leq \sqrt{\sup_{x \geq 0} x^2 f^*\left(\frac{x_2}{x_1}\right)}.$$

$$\equiv b_+$$

If $x_2 \leq 0$, then:

$$x_1 \leq \sqrt{f^*\left(\frac{x_2}{x_1}\right)} \Rightarrow x_2 \geq \frac{x_2}{x_1}\sqrt{f^*\left(\frac{x_2}{x_1}\right)} = -\sqrt{\left(\frac{x_2}{x_1}\right)^2 f^*\left(\frac{x_2}{x_1}\right)}$$

$$\geq -\sqrt{\sup_{x \leq 0} x^2 f^*\left(\frac{x_2}{x_1}\right)}.$$

$$\equiv b_-$$

Also, $0 \leq x_1 \leq \sqrt{f^*\left(\frac{x_2}{x_1}\right)} \leq \sup_x \sqrt{f^*(x)} = a.$

$\Rightarrow (x_1, x_2) \in [0, a] \times [b_-, b_+]$

$\Rightarrow C_f \in [0, a] \times [b_-, b_+].$ $\square$

Example (Normal distribution):

Sufficient to draw $X \sim N(0,1)$ w/ density

$f(x) \propto \exp\{-\frac{1}{2}x^2\} = f^*(x)$. Then

$$a = \sqrt{\sup_x f^*(x)} = f^*(0) = 1.$$

Now consider $\sup_x x^2 f^*(x)$ (solve via differentiation).

$$0 = \frac{d}{dx} x^2 f^*(x) = 2x f^*(x) + x^2(-x) f^*(x)$$

$$2x f^*(x) = x^3 f^*(x)$$

$$(f^*(x) > 0) \Rightarrow \quad 2x = x^3$$

$$0 = x^3 - 2x$$

$$= x(x-\sqrt{2})(x+\sqrt{2})$$

maxima @ $x = \pm\sqrt{2}$

$$b_+ = \sqrt{\sup_{x \geq 0} x^2 f^*(x)} = \sqrt{2e^{-1}}$$

$$b_- = \sqrt{\sup_{x \leq 0} \text{''}} = -\sqrt{2e^{-1}}$$