Eksamen

1.(a) $y''+2y'+y=\sin(2x)$

$y=y_h+y_p$

Finner $y_h$:

$y''+2y'+y=0$

$r^2+2r+1=0$

$r=\frac{-2\pm\sqrt{4-4}}{2}$

$=-1$

Én rot $\Rightarrow$ $y_h=(A+Bt)e^{-t}$, $A,B\in\mathbb{R}$

Finner $y_p$:

$y_p=C\sin(2t)+D\cos(2t)$, $C,D\in\mathbb{R}$

$y_p''+2y'+y=\sin(2t)$

$y_p''=-4C\sin(2t)-4D\cos(2t)$

$2y_p'=2(2C\cos(2t)-2D\sin(2t))$

$=4C\cos(2t)-4D\sin(2t)$

$\Rightarrow$ $y_p''+2y'+y=(-4C\sin(2t)-4D\cos(2t))+(4C\cos(2t)-4D\sin(2t))+(C\sin(2t)+D\cos(2t))$

$=(-3C-4D)\sin(2t)+(-3D+4C)\cos(2t)$

$(-3C-4D)\sin(2t)+(-3D+4D)\cos(2t)=\sin(2t)$ når $(-3C-4D)=1$ og $(-3D+4C)=0$

$\Rightarrow$ $C=-\frac{3}{25}$

$D=-\frac{4}{25}$

$\Rightarrow$ $y_p=-\frac{3}{25}\sin(2t)-\frac{4}{25}\cos(2t)$

$\underline{\underline{y=(A+Bt)e^{-t}-\frac{3}{25}\sin(2t)-\frac{4}{25}\cos(2t)}}$

(b) $y'''+8y=0$

$r^3+2^3=0$

$r=\sqrt[3]{-2^3}$

$r=-2$

$(r+2)(r^2-2r+2^2)=0$

$r^2-2r+2^2=0$

$r=\frac{2\pm\sqrt{4-16}}{2}$

$=\frac{2\pm2\sqrt{3}i}{2}$

$=1\pm\sqrt{3}i$

$r=-2 \lor r=1+\sqrt{3}i \lor r=1-\sqrt{3}i$

$\underline{\underline{y=Ae^{-2t}+e^{t}(B\cos(\sqrt{3}t)+C\sin(\sqrt{3}t))}}$

2. $x=\cos^4(t)$

$y=\sin^4(t)$

$0\leq t\leq\frac{\pi}{2}$

$2(x+y)=1+(x-y)^2$

$2x+2y-x^2+2xy-y^2-1=0$

$-x^2+2xy-y^2+2x+2y-1=0$

$x=u\cos(\theta)-v\sin(\theta)$

$y=u\sin(\theta)+v\cos(\theta)$

$A'u^2+B'uv+C'v^2+D'u+E'v+F=0$

$A'=\frac{1}{2}\left(-(1+\cos(2\theta))+2\sin(2\theta)-(1-\cos(2\theta))\right)$

$=\frac{1}{2}(-2+2\sin(2\theta))$

$=-1+\sin(2\theta)$

$B'=(-1-1)\sin(2\theta)+2\cos(2\theta)$

$=-2\sin(2\theta)+2\cos(2\theta)$

$C'=\frac{1}{2}\left(-(1-\cos(2\theta))-2\sin(2\theta)-(1+\cos(2\theta))\right)$

$=-1-\sin(2\theta)$

$D'=2\cos(\theta)+2\sin(\theta)$

$E'=-2\sin(\theta)+2\cos(\theta)$

$F=-1$

Velger $\theta=\frac{\pi}{4}$ siden $A=C$, $B\neq 0$

$\Rightarrow$ $A'=0$

$B'=0$

$C'=-2$

$D'=2\sqrt{2}$

$E'=0$

$\Rightarrow$ $-2v^2+2\sqrt{2}u-1=0$

$2\sqrt{2}u-2v^2=1$

$\frac{u}{\frac{1}{2\sqrt{2}}}-\left(\frac{v}{\frac{1}{\sqrt{2}}}\right)^2=1$

$\left(\frac{u}{\sqrt{\frac{1}{2\sqrt{2}}}}\right)^2-\left(\frac{v}{\frac{1}{\sqrt{2}}}\right)^2=1$

Ligningen beskriver en hyperbel

3. Punktvis konvergens betyr at

$\lim_{n\to\infty}f_n(x)=f(x)\quad\forall x\in I$

der $f_n(x)=nx^{\alpha}e^{-nx}$ og $f(x):I\to\mathbb{R}$

$I=[0,\infty)$

der $f_n(x) = n x^{\alpha} e^{-nx}$ og $f(x) : I \Rightarrow \mathbb{R}$

$I = [0, \infty)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(n x^{\alpha} e^{-nx}\right)$$

$$= x^{\alpha} \lim_{n\to\infty} \left(n e^{-nx}\right)$$

$$= x^{\alpha} \lim_{n\to\infty} \left(\frac{n}{e^{nx}}\right)$$

$$= x^{\alpha} \lim_{n\to\infty} \left(\frac{n}{e^{nx}}\right)$$

$$\overset{\text{L.Hop.}}{=} x^{\alpha} \lim_{n\to\infty} \left(\frac{1}{e^{nx}\, x}\right)$$

$$= x^{\alpha-1} \lim_{n\to\infty} \left(\frac{1}{e^{nx}}\right)$$

$$= 0$$

$\Rightarrow \lim_{n\to\infty} f_n = 0 \quad \forall x \quad \square$

$f(x) = 0$

Uniform konvergens betyr at

$$\lim_{n\to\infty} \left(\sup_{x\in I} |f_n(x) - f(x)|\right) = 0$$

$$\sup_{x\in I} |n x^{\alpha} e^{-nx}| = \sup_{x\in I} \left|\frac{n x^{\alpha}}{e^{nx}}\right|$$

$$= 0$$

Konvergensen er uniform for alle $\alpha$

4. (a) $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{\ln(n)}$

Bruker forholdstest:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{\ln(n+1)}}{\frac{x^n}{\ln(n)}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x \ln(n)}{\ln(n+1)}\right|$$

$$\overset{\text{L.Hop.}}{=} |x| \cdot \lim_{n\to\infty} \left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right|$$

$$= |x| \cdot 1$$

$$= |x|$$

$|x| < 1$ fordi da konvergerer rekken

$\Rightarrow$ Konvergeringsintervallet til rekken er $(-1, 1)$

(b) $\lim_{n\to\infty}$

5. $f$ kont på $[0, 2]$

$f(0) = -1$

$f(1) = 1$

$f', f''$ kont på $(0, 2)$

$f'(x) > 0, \ 0 < x < 2$

$f''(x) > 0, \ 0 < x < 2$

$f''(x)>0,\ 0<x<2$

ⓐ $f(x)=0$ har nøyaktig én løsning for $0<x<1$

fordi $f$ er kont. og kan dermed ikke gå fra $f(x)=-1$ til $f(x)=1$ uten å gå igjennom x-aksen, og $f'(x)>1$ for $0<x<2$ så ligningen vokser på intervallet.

$f(x)>f(y)+f'(y)(x-y),\ x\neq y,\ x,y\in(0,1]$

$f(y)+f'(y)(x-y)$ er tangentlinjen som går igjennom $(y,f(y))$

$f''(x)>0$ betyr at $f(x)$ er "Concave up", så tangentlinjen vil alltid ligge under $f(x)$

d.v.s. $f(x)>f(y)+f'(y)(x-y)$

ⓑ $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

$x_0=1$

Induksjon

Steg 1:

$n=1$

$x_1=1-\frac{f(1)}{f'(1)}$

$=1-\frac{1}{f'(1)}<x_0$

Steg 2:

$n=k$

$x_k=x_{k-1}-\frac{f(x_{k-1})}{f'(x_{k-1})}<x_{k-1}$

Steg 3:

$n=k+1$

$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}$

$<x_{k-1}-\frac{f(x_{k-1})}{f'(x_{k-1})}=x_k$

6.ⓐ $a_0=a_1=1$

$a_n+(n+1)(n+2)a_{n+1}+(n+1)(n+2)a_{n+2}=0$

Steg 1:

$n=0$

$\left|\frac{a_1}{a_0}\right|=1>\frac{1}{2}$ ✓

Steg 2:

$n=k$

$\left|\frac{a_{k+1}}{a_k}\right|\geq\frac{1}{2}$

Steg 3:

$n=k+1$

$\left|\frac{a_{k+2}}{a_{k+1}}\right|$

Er

$\left|\frac{a_{k+2}}{a_{k+1}}\right|$

Finner $a_{k+2}$

$$a_k + (k+1)(k+2)a_{k+1} + (k+1)(k+2)a_{k+2} = 0$$

$$a_{k+2} = -\frac{a_k}{(k+1)(k+2)} - a_{k+1}$$

$$|a_{k+2}| = \left|\frac{a_k}{(k+1)(k+2)} + a_{k+1}\right|$$

$$\left|\frac{a_{k+2}}{a_{k+1}}\right| = \left|-\frac{a_k}{(k+1)(k+2)} - a_{k+1}\right| \cdot \frac{1}{|a_{k+1}|}$$

$$= \left|-\frac{1}{(k+1)(k+2} \cdot \frac{a_k}{a_{k+1}} - 1\right|$$

$$= \left|\frac{1}{(k+1)(k+2)} \cdot \frac{a_k}{a_{k+1}} + 1\right|$$

$$=$$

(b) $y = f(x)$

$$(x+1)y'' + 2y' + y = 0$$

$$y(0) = y'(0) = 1$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Setter inn:

$$\underbrace{\sum_{n=2}^{\infty} a_n n(n-1) x^{n-1}}_{\substack{k=n-1 \\ n=k+1}} - \underbrace{\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}}_{\substack{k=n-2 \\ n=k+2}} + \underbrace{2\sum_{n=1}^{\infty} a_n n x^{n-1}}_{\substack{k=n-1 \\ n=k+1}} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{\substack{k=n \\ n=k}} = 0$$

$$\sum_{k=1}^{\infty} a_{k+1}(k+1)k x^k - \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + 2\sum_{k=0}^{\infty} a_{k+1}(k+1)x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=1}^{\infty} a_{k+1}(k+1)k x^k - 2a_2 - \sum_{k=1}^{\infty} a_{k+2}(k+2)(k+1)x^k + 2a_1 + 2\sum_{k=1}^{\infty} a_{k+1}(k+1)x^k + a_0 + \sum_{k=1}^{\infty} a_k x^k = 0$$

$$\sum_{k=1}^{\infty}\left[a_{k+1}(k+1)k x^k - a_{k+2}(k+2)(k+1)x^k + 2a_{k+1}(k+1)x^k + a_k x^k\right] - 2a_2 + 2a_1 + a_0 = 0$$

$$\sum_{k=1}^{\infty}\left[k^2(a_{k+1} - a_{k+2}) + 3k(a_{k+1} - a_{k+2}) + 2(a_{k+1} - a_{k+2}) + a_k\right] - 2a_2 + 2a_1 + a_0 = 0$$