Øving 1

Problem 1.

1) $x = 0.d_1 \cdot \ldots \cdot d_k \cdot \ldots \times 10^n$

$fl(x) = 0.d_1 \cdot \ldots \cdot d_k \times 10^n$

Skal vise at 10^{-k-1} er bound:

$\left|\frac{x - fl(x)}{x}\right|$

$|x| \geq 0.d_1 \times 10^n$

$= d_1 \times 10^{n-1}$

$\geq 10^{n-1}$

$\Rightarrow \frac{1}{|x|} \leq \frac{1}{10^{n-1}}$

$|x - fl(x)| = 0.d_{k+1} \cdot \ldots \times 10^n$

$< 10^{n-k}$

$\Rightarrow \left|\frac{x - fl(x)}{x}\right| = |x - fl(x)| \cdot \frac{1}{|x|}$

$< 10^{n-k} \cdot \frac{1}{10^{n-1}}$

$= 10^{-k-1}$

2) $f(t) = t^3 + 2t + s$

$f'(t) = 3t^2 + 2$

$f'(t) > 0 \ \forall t$

$\Rightarrow f(t)$ øker for alle t

$f(t)$ kan være negativ og den kan være positiv

$\Rightarrow f(t)$ krysser t-aksen i akkurat ét punkt

3) $R(t, t_0) = f(t) - p(t)$

$f^{(n+1)}(t)$ kont.

$f(t) = \sum_{n=0} \frac{f^{(n)}(t_0)}{n!}(t - t_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(t - t_0)^{n+1}$

$p(t) = \sum_{n=0} \frac{f^{(n)}(t_0)}{n!}(t - t_0)^n$

$\Rightarrow R(t, t_0) = f(t) - p(t)$

$= \frac{f^{(n+1)}(c)}{(n+1)!}(t - t_0)^{n+1}$

$= \int_{t_0}^{t} \frac{f^{(n+1)}(\tau)}{n!}(t - \tau)^{n+1} d\tau$

4) $f(t) = e^t \cos(t)$

$t_0 = 0$

$f(t_0) = 1$

$f'(t_0) = 1$

$f''(t_0) = 0$

$P_2(t) = f(t_0) + f'(t_0)(t - t_0) + \frac{f''(t_0)}{2!}(t - t_0)^2$

$= 1 + 1t$

$= t + 1$

Remainder:

$R_n(t) = \frac{f^{(n+1)}(c)}{(n+1)!}(t - t_0)^{n+1}$

$n = 2,\ t_0 = 0,\ t = 0.5$:

$f^{(3)}(c) = -2e^c(\cos(c) + \sin(c))$

$R_2 = R_2(0.5)$

$= \frac{(0.5 - 0)^3}{3!} \cdot (-2e^c(\cos(c) + \sin(c)))$

$= -\frac{e^c}{24}(\cos(c) + \sin(c))$

$\cos(c) + \sin(c) = \sqrt{2}\sin(c + \frac{\pi}{4})$

$\Rightarrow R_2 = -\frac{\sqrt{2}}{24} \cdot e^c \sin(c + \frac{\pi}{4})$

Finner upper bound:

$|R_2| = |-\frac{\sqrt{2}}{24} \cdot e^c \sin(c + \frac{\pi}{4})|$

$= \frac{\sqrt{2}}{24} \cdot e^c \cdot |\sin(c + \frac{\pi}{4})|$

$c = 0.5$ gir maks verdi når $c \in [0, 0.5]$

$|R_2| = \frac{\sqrt{2}}{24} \cdot e^{0.5} \cdot |\sin(0.5 + \frac{\pi}{4})|$

≤ 0.0932

< 0.1

Upper bound er 0.1

Problem 2.

$t^2 = a$

$t = F(t)$

$F_1(t) = 0.5(t + at^{-1})$:

$F_1(t) = 0.5(t + t^2 t^{-1})$

$= 0.5(t + t)$

$= 0.5(2t)$

$= t$

$F_2(t) = at^{-1}$:

$F_2(t) = at^{-1}$

$= t^2 t^{-1}$

$= t$

$F_3(t) = 2t - at^{-1}$:

$F_3(t) = 2t - at^{-1}$

$= 2t - t^2 t^{-1}$

$= 2t - t$

$= t$

Konvergens:

$t_{n+1} = F(t_n),\ n \geq 0$

$F_1(t)$:

$t_{n+1} = F_1(t_n) = 0.5(t_n + at_n^{-1})$

$F_1'(t) = 0.5(1 - \frac{a}{t^2})$

$|F_1'(t)| = |0.5(1 - \frac{a}{t^2})| \xrightarrow[t \to a]{} 0.5$

$0.5 < 1 \Rightarrow$ Konvergerer

$F_2(t)$:

$t_{n+1} = F_2(t_n) = at_n^{-1}$

$F_2'(t) = -\frac{a}{t^2}$

$|F_2'(t)| = |-\frac{a}{t^2}| \xrightarrow[t \to a]{} 0$

$|F_2'(t)| = 0 \Rightarrow$ Konvergerer

$F_3(t)$:

$t_{n+1} = F_3(t_n) = 2t - at^{-1}$

$F_3'(t) = 2 - \frac{a}{t^2}$

$|F_3'(t)| = |2 - \frac{a}{t^2}| \xrightarrow[t \to a]{} 2$

$2 > 1 \Rightarrow$ Divergerer

Problem 3.

1.8

$f(c) = 0$

$[X, c],\ X < c$

$f'(x) > 0$

$f''(x) < 0$ kontinuerlig

Fra likning 1.23 i boka følger det at $x_{n+1} < c$ gitt at $x_n \in I = [X, c]$

$f(x)$ er strengt økende og $f(c) = 0$, derfor vil $f(x) < 0$ i I

Så lenge $x_0 \in I$ så vil $\{x_n\}$ være i I, og er også strengt økende

Upper bound er c, som betyr at den konvergerer, og siden c er den eneste rota til $f(x) = 0$ i I så vil $\{x_n\}$ konverger til c $\square$

2.8

(i) $v = (v_1, \dots, v_n)^T \in \mathbb{R}^n$

$\|v\|_\infty = \max_{i=1}^{n} |v_i|$

$\|v\|_2 = \sqrt{\sum_{i=1}^{n} |v_i|^2}$

$\|v\|_1 = \sum_{i=1}^{n} |v_i|$

Skal vise $\|v\|_\infty \leq \|v\|_2$

Hvis $v_j > 0$ og $v_{i\neq j} = 0$

$\Rightarrow \|v\|_\infty = v_j = \|v\|_2$

Hvis $v_j \geq v_{i\neq j}$

$\Rightarrow \|v\|_\infty = v_j < \sqrt{\sum_{i=1}^{n} |v_i|^2} = \sqrt{|v_1|^2 + \ldots + |v_j|^2 + \ldots + |v_n|^2} = \|v\|_2$

$\Rightarrow \|v\|_\infty \leq \|v\|_2$

Eksempel likhet:

$v = (0, 0, \ldots, v_i, 0, 0, \ldots)^T, \ v_i \neq 0$

$\|v\|_\infty = v_i = \|v\|_2$

Skal vise $\|v\|_2^2 \leq \|v\|_1 \, \|v\|_\infty$

$\|v\|_2^2 = \sum_{i=1}^{n} |v_i^2| \leq \left(\sum_{i=1}^{n} |v_i|\right)\left(\max_{i=1}^{n} |v_i|\right)$

Eksempel likhet:

$v = (\pm\alpha, \pm\alpha, \ldots, \pm\alpha)^T$

$\|v\|_2^2 = \alpha^2 + \alpha^2 + \ldots + \alpha^2 = n\alpha^2 = (\alpha + \alpha + \ldots + \alpha) \cdot \alpha$

Siden $\|v\|_\infty \leq \|v\|_2$ og $\|v\|_2^2 \leq \|v\|_1 \, \|v\|_\infty$

$\Rightarrow \|v\|_\infty \leq \|v\|_2 \leq \|v\|_1$

Skal vise $\|v\|_2 \leq \sqrt{n} \|v\|_\infty$:

Vet at $\|v\|_\infty \leq \|v\|_p \leq n^{\frac{1}{p}} \|v\|_\infty$

$p = 2 \Rightarrow \|v\|_2 \leq \sqrt{n} \|v\|_\infty$

ii) $A \in \mathbb{R}^{m \times n}$

$\|A\|_\infty = \max_x \frac{\|Ax\|}{\|x\|}$ (*)

Velger v s.a. (*) holder

Skal vise $\|A\|_\infty \leq \sqrt{n} \|A\|_2$:

$\|A\|_\infty \|v\|_\infty = \|Av\|_\infty \leq \|Av\|_2 = \|A\|_2 \|v\|_2 \leq \|A\|_2 \sqrt{n} \|v\|_\infty$

Deler på $\|v\|_\infty$

$\Rightarrow \|A\|_\infty \leq \sqrt{n} \|A\|_2$

Eksempel likhet:

$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\|A\|_\infty = 2$

$A^T A = 2I$

$\Rightarrow \|A\|_\infty = \sqrt{2}$

$\Rightarrow \|A\|_\infty = 2 = \sqrt{2}\sqrt{2} = \sqrt{2} \|A\|_2$

Skal vise $\|A\|_2 \leq \sqrt{m} \|A\|_\infty$

Velger u s.a. (*) holder

$\|A\|_2 \|u\|_2 = \|Au\|_2 \leq \sqrt{m} \|Au\|_\infty = \sqrt{m} \|A\|_\infty \|u\|_\infty \leq \sqrt{m} \|A\|_\infty \|u\|_2$

Deler på $\|u\|_2$

$\Rightarrow \|A\|_2 \leq \sqrt{m} \|A\|_\infty$

Eksempel likhet:

u har bare ett element ulikt 0

$|a_{ij}| = \alpha \ \forall i,j \geq 1$

u er eigenvektor til $A^T A$

$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\|A\|_\infty = 1$

$A^T A = 2I$

$\|A\|_2 = \sqrt{2}$

$\Rightarrow \|A\|_2 = \sqrt{2} = \sqrt{2} \|A\|_\infty$

$\|A\|_2 = \sqrt{2}$

$\Rightarrow \|A\|_2 = \sqrt{2} = \sqrt{2}\|A\|_\infty$

4.8

$c = \lim_{k\to\infty} x^{(k)}$ i $\mathbb{R}^n$

$x \mapsto f(x)$

$f_1(x_1, x_2) = x_1^2 + x_2^2 - 2$

$f_2(x_1, x_2) = x_1 + x_2 - 2$

Skal vise $f(c) = 0$ når $c = (1,1)^T$

$$f(c) = \begin{pmatrix} x_1^2 + x_2^2 - 2 \\ x_1 + x_2 - 2 \end{pmatrix} = \begin{pmatrix} 1+1-2 \\ 1+1-2 \end{pmatrix} = 0$$

$x_1^{(0)} \neq x_2^{(0)}$

$$J = \begin{pmatrix} 2x_1 & 2x_2 \\ 1 & 1 \end{pmatrix}$$

$$J^{-1} = \frac{1}{2(x_1 - x_2)} \begin{pmatrix} 1 & -2x_2 \\ -1 & 2x_1 \end{pmatrix}$$

$$\begin{pmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \end{pmatrix} = \begin{pmatrix} x_1^{(n)} \\ x_2^{(n)} \end{pmatrix} - \frac{1}{2(x_1^{(n)} - x_2^{(n)})} \begin{pmatrix} 1 & -2x_2^{(n)} \\ -1 & 2x_1^{(n)} \end{pmatrix} \begin{pmatrix} (x_1^{(n)})^2 + (x_2^{(n)})^2 - 2 \\ x_1^{(n)} + x_2^{(n)} - 2 \end{pmatrix}$$

$\Rightarrow x_1^{(n+1)} + x_2^{(n+1)} = 2$

$x_1^{(0)} = 1 + \alpha$

$x_2^{(0)} = 1 - \alpha$

$\alpha \neq 0$

$x_1^{(1)} = 1 + \frac{\alpha}{2}$

$x_2^{(1)} = 1 - \frac{\alpha}{2}$

$x_1^{(0)} \neq x_2^{(0)}$

$x_1^{(1)} + x_2^{(1)} = 2$

$\Rightarrow x_1^{(1)} = 1 + \alpha$

$x_2^{(1)} = 1 - \alpha$

$x_1^{(n)} = 1 + \frac{1}{2^{n-1}}$

$x_2^{(n)} = 1 - \frac{1}{2^{n-1}}$

$\Rightarrow (x^{(n)})$ konvergerer lineært mot $(1,1)^T$ med "rate" $\ln(2)$

Siden Jacobien matrisen av f er singulær i grensepunktet $(1,1)^T$ så er ikke konvergensen kvadratisk