Øving 4

1.

$T_n(x)=\cos(n\arccos(x))$

$L=[1,\infty)$

$x\in[-1,1]$

(a)

Vurdere, SA vise at $\frac{1}{2^{n-1}}$ for odtallige stationærverdi med alternerende fortegn og minst i x_i, $i=0,1,\dots,n-1$, samt er de ytterste punktene $x=1$ og en minimum polynom for $x\in[-1,0]$

Ser det $T_n(x)$ alternerer mellom $y=0$ med minimum stationærverdi.

(b) $\frac{d}{dx}T_n(x)=\frac{d}{dx}\cos(n\arccos(x))$

$=-\sin(n\arccos(x))\cdot\frac{-n}{\sqrt{1-x^2}}$

$=n\sin(n\arccos(x))$

Setter $T_n'(x)=0$

$\Rightarrow \sin(n\arccos(x))=0=\sin(n\pi)$

$\Rightarrow \cos(x)=\frac{k\pi}{n}$

$\Rightarrow x=\cos(\frac{k\pi}{n})$

$0\le x\le 1$

$k=0$

(c)

2.

$x\in[0,1]$

$\min_x \int_0^1 x\,y(x)$

$y(0)=y=0$

$y(1)=y=0$

$c\in\mathbb{R}$

A_h LHF

$$A_h:=\begin{pmatrix}-2+c\,h^2 & 1 & & & \\ 1 & -2+c\,h^2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2+c\,h^2 & 1 \\ & & & 1 & -2+c\,h^2\end{pmatrix}$$

$$U:=\begin{pmatrix}U_1\\ U_2\\ \vdots\\ U_N\end{pmatrix}$$

$$F:=\begin{pmatrix}f_1\\ f_2\\ \vdots\\ f_N\end{pmatrix}$$

$h=\frac{1}{N}=(\cos(\pi/N)-1)\pi$

$n^2\dots A$

$\|A_h^{-1}\|_\infty$ tilsvarende som $\|A_h^{-1}f\|\Rightarrow \frac{1}{\pi^2}\|f\|$

$\Rightarrow \frac{1}{\pi^2}\cos(\pi/N)-1)\pi C\le 1$ og $\frac{1}{\pi^2}(\cos(\pi/N)-1)\le C-1$

$\Rightarrow C\ge 1-\frac{1}{\pi^2}(\cos(\pi/N)-1)$

$\lim_{N\to\infty}(\pi/N)=0$

$\Rightarrow C\ge 1 \qquad C\le -1$

3.

(a) $A(t)=2A(t)-2A(t)-B(t)+\cos(t)$

(b) $B'(t)=B'(t)-2B'(t)+A(t)-A'(t)+\sin(t)$

$A(1)=A(t)$

$=2$

$B(1)=1$

$B'(t)=1$

Lager (a) for $\frac{d}{dt}$ og setter inn. : (b)

$B'(t)=\cos(t)-A'(t)+3A(t)+2A(t)$

$-\cos(t)=A''(t)-3A'(t)-3A'(t)-(-\sin(t)-A'(t)-3A(t)+2)-(\cos(t)-A'(t)-2A(t))^2+A(t)-A'(t)+\sin(t)$

$-B''(t)=2A'(t)+3A'(t)+3A'(t)-3A(t)+2\sin(t)+2\cos(t)-2$

$x_1=y \Rightarrow x_1'=x_2$

$x_2=y' \Rightarrow x_2'=y''=x_3$

$x_3=y'' \Rightarrow x_3'=y'''=x_4$

$x_4=y''' \Rightarrow x_4'=y''''=-2\sin(t)-3\cos(t)+2+2x_1^2-4x_1^2-3x_2^2+3x_3$

Sammenlikne med 1. ordens ligninger for (b):

$x_1'=x_2$

$x_2'=x_3$

$x_3'=x_4$

$x_4'=-2\sin(t)-3\cos(t)+2+2x_1-4x_1-3x_2+3x_3$

Initialverdier

$x_1(1)=2$

$x_2(1)=-2$

$x_3(1)=1$

$x_4(1)=$

For (b):

$B'(t)=B'(t)-B'(t)+A'(t)+\sin(t)$

$x_1=y \Rightarrow x_1'=x_2$

$x_2=y' \Rightarrow x_2'=x_1+x_2+\sin(t)+x_1^2$

Sammenlikn med 1. ordens lin. ligninger for (b):

$x_1'=x_2$

$x_2'=x_1+x_2+\sin(t)+x_1-x_1^2$

Initialverdier

$x_1(1)=1$

$x_2(1)=2-x_1^2$

4.

$y'(t)=f(t,y(t))$

$y(t_n)=y_n$

Euler er Taylor utvikle

$y_n=y_n$

$y_{n+1}=y_n+\int_{t_n}^{t_n+h}f(t,y_n+h)$

$=h\,f(t,t_n,y_n)+$

$\int_{t_n}^{t_n+h}(t_n,y_n)=\sum_{i=1}^{\infty}\frac{1}{i!}f^{(i)}(t_n,y_n)$

The local truncation error τ_n is the error done by one step when starting at the exact solution.

$d_n=y(t_n+h)-(y(t_n)+h\,f(t_n,y(t_n)+h))$

5.

0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0
	2	2

Runge

0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0
	0	1

$y'=y\,y=1$

$y(0)=1$

Utbredt på $[0,1]$

Implisitt:

$y_{n+1}=\frac{1}{2}(f(t_n,y_n)+f(t_{n+1},y_{n+1}))$

$y_n=y_n+h$

$y_{n+1}=\frac{1}{2}(y_n+y_n+1)$

$f(t_{n+1},y_{n+1})=f(t_n,y_n+h)$

$=f(t_{n+1},y_n+h(y_n+1+1))$

$=(t_n+1)\cdot(y_n+h(y_n+1+1))$

Da er $y_{n+1}=y_n+\frac{1}{2}(y_n+(t_n+1)\cdot(y_n+h(y_n+1+1)))$

$=y_n+h(y_n+1+1)+\frac{1}{2}(y_n+1+1)$

Midtpunkt:

$y_{n+1}=y_n+h\,f(t+\frac{1}{2}h,y_n+\frac{1}{2}f(t,y_n))$

$f(t_n+\frac{1}{2}h,y_n+\frac{1}{2}f(t_n,y_n))=f(t_n+\frac{1}{2}h,y_n+t_n+1)$

$=t_n+\frac{1}{2}\cdot(y_n+\frac{1}{2}(t_n+y_n+t_n+1))$

$=t_n+\frac{1}{2}\cdot y_n+\frac{1}{2}(t_n+t_n+1)$

Da er $y_{n+1}=y_n+h(t_n+\frac{1}{2}\cdot y_n+\frac{1}{2}(t_n+t_n+1)+1)$

$=y_n+h(t_n+y_n+t_n+1)+\frac{1}{2}(t_n+t_n+1)$

Alltid y_n er lik for begge metoder

6.

7.

7a

$n=3$

$I=[-1,1]$

$2a_0+2a_1=2$

$2a_0+2a_1=8$

$\Rightarrow 2=a_0+a_1+a_2+a_3$

$0=-a_1+3a_2+3a_3+a_4$

$8=a_0+a_1+a_2+a_3$

$0=-a_1+3a_2+3a_3+a_4$

$\Rightarrow a_1=a_3$

$a_2=a_4$

$\Rightarrow a_1+a_2=4$

$a_1+a_2=3$

Skal vise $\|f(x)\|=\|T(x)\|$

$\|f(x)\|=\frac{1}{2^3}(\frac{1}{2}x^3+\frac{1}{2}x^2+T_{-1}\dots)$

$\|T(x)\|=\frac{1}{2^3}(f_0\cdot T_0+f_1\cdot T_1+\dots)$

$f_0=\frac{1}{2}(\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}})$

$\Rightarrow \|T(x)-T(f)\|=\frac{1}{2^3}(f_0\cdot T_0+f_1\cdot T_1+f_2\cdot T_2+\dots)-\frac{1}{2^3}(f_0\cdot T_0+f_1\cdot T_1+f_2\cdot T_2+\dots)$

$=\frac{1}{2^3}(f_0\cdot T_0+f_1\cdot T_1+f_2\cdot T_2+\dots)$

$=\|f(x)\|$

7b

Vet at Chebyshev polynomer $T_n(x)$ danner et ortogonalt system med vektfunksjon $w(x)=(1-x^2)^{-\frac{1}{2}}$ på $[-1,1]$

$\Rightarrow$ Normalisere ved å sette x_j for $n=0$ er null nr $T_{n+1}(x)$

$x_j=\cos(\frac{(2j+1)\pi}{2n+2})$

$j=0,\dots,n$

Vis at det blir bevist for de røtter n

$\frac{d}{dx}\cos(\lambda_j+1)\theta=\frac{\sin(\frac{(2j+1)\pi}{2n+2})\theta}{\sin\theta}$

$n+1$:

$\frac{d}{dx}\cos(\lambda_j+1)\theta=\frac{\sin(\frac{(2j+1)\pi}{2n+2})\theta}{\sin\theta}=\cos(\lambda_j+1)\theta$

$=\frac{\sin(\frac{(2j+1)\pi}{2n+2})\theta\cos(\frac{(2j+1)\pi}{2n+2})\cos\theta}{\sin\theta}$

$=\frac{\sin(\frac{(2j+1)\pi}{2n+2})\theta}{\sin\theta}$ □

Hvis (1) gjøre gr bestemt:

$\frac{d}{dx}\cos(\lambda_j+1)\theta=\frac{d}{dx}(-1)^{\lambda}$

$=(-1)^{\lambda}(\lambda+1)$

$\Rightarrow \frac{d}{dx}\cos(\lambda(2j+1)\pi)/(2n+2))=\begin{cases}0 & j=0,\dots,n\\ (-1)^{\lambda}(\lambda+1) & j=1,\dots,n\end{cases}$

$\Rightarrow \frac{d}{dx}T_n(x)=0$ $k=1,\dots,n$

Hvis T_n er ortogonal med T_m $k\ne 0$ så

$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}T_n(x)dx=0$

$k=1,\dots,n$

og $\frac{d}{dx}T_n(x)=n+1$

og $\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}T_n(x)dx=\pi$

Lagger sterke sammen:

$\frac{1}{2^{n-1}}\frac{d}{dx}T_n(x)=\pi\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}T_n(x)dx$

$\Rightarrow$ Hvis $\lambda_j\ne\frac{1}{2^{n-1}}$

12.2

Trapezoidal error

$T_n\frac{(b-a)}{2n}\frac{(f(a)+f(b))}{2}=\frac{1}{2n}((a_{n,0}f(a_0))+f(a_{n,1}f(a_1)))$

$\int_a^b(x-a_n)(x-a_1)f(x)dx^{\frac{1}{2}}\int_a^b(x-a_n)(x-a_1)(x)\sum_{i=0}^{n}\sum_{j=0}^{n}(L_i(x)x-a_n)y_j(x)dx$

$=\int_a^b(x-a_n)(x-a_1)f(x)dx-\int_a^b((x-a_n)(x-a_1)f(x))\sum_{i=0}^{n}\sum_{j=0}^{n}L_i(x)dx$

$=(a_n-a_n)y_j(a_n)+(a_1-a_n)y_j(a_1)+2f(a_1-a_n)y_j(a_1)$

$\Rightarrow y_j(a_n)-y_j(a_n)+\frac{1}{2}(f(a_n)+f(a_1))-\frac{1}{2}\int_a^b(a_n-x)(x-a_1)f(x)dx$

Bestemt integral, mere ville det

$y(a_n)-y(a_n)+\frac{1}{2}(y(a_n)+y(a_1))-\frac{1}{2}y(\xi_1)\int_a^b(x-a_n)(x-a_1)dx$

$y_n\in(a_1,a_n)$

Vester $x=a_n+ih$

$\int_a^b(a_n-x)(x-a_1)dx=h^2\int_a^b(1-i)dx$

$=h^2\int_a^b(i-i^2)dx$

$=\frac{1}{6}h^3$

$\Rightarrow y(a_n)-y(a_n)+\frac{1}{2}(f(a_n)+f(a_1))-\frac{1}{12}h^3\,y''(\xi_1)$

$\Rightarrow T_n=-\frac{1}{12}h^3\,y''(\xi_1)$

$\Rightarrow |T_n|\le\frac{1}{12}h^3\,M$

$M=\max_{\xi\in[a,b]}|y''(\xi)|$

Finner bestemt for global error av $T=y(a_n)-y_n$

$y(a_n)-y(a_n)+\frac{1}{2}(f(a_n)+f(a_1))-\frac{1}{2}(f(a_n)+f(a_1))+h\,T_n$

$y_n=y+\frac{1}{2}(f(a_n)+f(a_1))-\frac{1}{2}(f(a_n)+f(a_1))$

Vester lifte og vester lipschitz condition på f

$|a_n|\le|a_n|+\frac{1}{2}(|L|a_n|+L|a_1|)+h|T_n|$

$\Rightarrow |a_n|\le|a_1|+\frac{1}{2}(|L|a_n|+L|a_1|)+\frac{1}{12}h^3\,M$

$\Rightarrow |a_n|\le(\frac{1+\frac{1}{2}L}{1-\frac{1}{2}L})a_1+\frac{1}{12}\frac{h^3}{1-\frac{1}{2}L}$

$n\in[1,2,\dots$

$a_0=0$ antatt $0\le k\le 1$

$\Rightarrow |a_n|\le\frac{h^3M}{12}(\frac{1+\frac{1}{2}L}{1-\frac{1}{2}L})^n-1)$

12.7

Taylor serie utvikling

$T_n\frac{(b-a)}{2n}\frac{(f(a)+f(b))}{2}=\frac{1}{2n}((f(a_{n,0}f(a_0))+f(a_{n,1}f(a_1)))$

$\frac{(b-a)}{2n}\frac{(f(a)+f(b))}{2}=\frac{1}{2}(f(a_0)+f(a_{n,0}f(a_0)))$

$=f(a_0)+\frac{1}{2}(a_{n,0}f(a_0))+\frac{1}{2}(a_{n,0}f(a_0))+\frac{1}{2}a_{n,0}f(a_0)+\frac{1}{2}a_{n,0}f(a_0)+\frac{1}{2}(a_{n,0}f(a_0))^2+\frac{1}{2}(a_{n,0}f(a_0))^3+\frac{1}{2}(a_{n,0}f(a_0))^4+O(h^5)$

$\begin{bmatrix}f=f\\ f'=f\\ f''=f\cdot f'=f\cdot f+f\cdot f\\ f'''=f\cdot f''+f'\cdot f'=f\cdot(f\cdot f'+f'\cdot f)+f'\cdot f'=f\cdot f\cdot f'+f\cdot f'\cdot f+f'\cdot f'\cdot f=f\cdot f\cdot f'+3f\cdot f'\cdot f+f'\cdot f'\cdot f\end{bmatrix}$

$=\frac{1}{2}h^2\,(f_0+f_0'\cdot f_0)+\frac{1}{2}(f_0+3f_0\cdot f_0'+f_0\cdot f_0'')\Big|_{a_n}+O(h^5)$

12.8

Runge Kutt:

$y_{n+1}=y_n+\frac{1}{6}h(k_1+2k_2+2k_3+k_4)$

$k_1=f(t_n,y_n)$

$=y_n$

$k_2=f(t_n+\frac{1}{2}h_n,y+\frac{1}{2}hk_1)$

$=f(y_n+\frac{1}{2}hy_n)$

$=y_n+\frac{1}{2}h^2y_n$

$k_3=f(t_n+\frac{1}{2}h_n,y+\frac{1}{2}hk_2)$

$=f(y_n+\frac{1}{2}hy_n+\frac{1}{2}h^2\cdot y_n)$

$=y_n+\frac{1}{2}h^2\cdot y_n+\frac{1}{4}h^3\cdot y_n$

$k_4=f(t_n+h_n,y+hk_3)$

$=f(y_n+h(y_n+\frac{1}{2}h^2\cdot y_n)+\frac{1}{4}h^3\cdot y_n)$

$=y_n+h^2\cdot y_n+\frac{1}{2}h^3\cdot y_n+\frac{1}{4}h^3\cdot y_n$

$=y_n+h^2\cdot y_n+\frac{1}{4}h^3\cdot y_n+\frac{1}{2}h^3\cdot y_n$

$\Rightarrow y_{n+1}=(1+(h)^2+\frac{1}{2}(h)^3+\frac{1}{6}(h)^3+\frac{1}{24}(h)^4)y_n$

For Gauss Seidel

$y(a_n)=a_n$

$=a_n$

$=a_n^{\lambda}y(a_n)$

$\Rightarrow y(a_n)=(1+(h)^2+\frac{1}{2}(h)^3+\frac{1}{6}(h)^3+\frac{1}{24}(h)^4-)y(a_n)$