- CONTINUITY

Let $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ and $x_0 \in A$.
We say that f is <u>continuous</u> at x_0 if

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0 \text{ such that for all } x \in A,$$
$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

We say that f is <u>continuous</u> on A if it continuous at any $x \in A$.

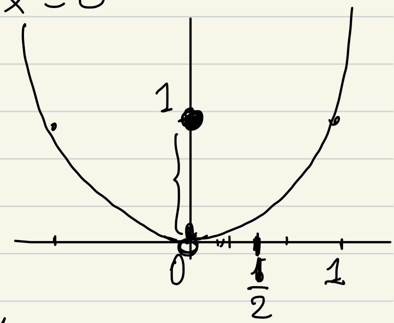
E.g. $f(x) = 2x + 3$ is continuous at $x_0 = 1$.

- Take $\varepsilon > 0$.
  Since $f(x_0) = f(1) = 5$
  and $|f(x) - f(x_0)| = |2x - 2| = 2|x - 1|$
  if we choose $\delta = \dfrac{\varepsilon}{2} > 0$, then

  $|x - 1| < \delta$ implies that

  $$|f(x) - f(1)| = 2|x - 1| < 2\delta = \varepsilon.$$

The function $f(x) = \begin{cases} x^2, & x \neq 0 \\ 1, & x = 0 \end{cases}$



f is not continuous at 0.

Suppose f is continuous at 0.
Then for $\varepsilon = \frac{1}{2} > 0$
there exists $\delta > 0$ such that
$|x - 0| < \delta$ implies $|f(x) - f(0)| = |f(x) - 1| < \frac{1}{2}$.

Choose $x = \frac{1}{2} \min\{\delta, 1\}$.

$x < \frac{1}{2} \Rightarrow f(x) = x^2 < \frac{1}{4}$

and also
$|x| < \delta \Rightarrow |f(x) - 1| < \frac{1}{2}$.

But now
$-\frac{1}{2} < f(x) - 1 < \frac{1}{2} \Rightarrow f(x) > \frac{1}{2}$.

We have shown that
$f(x) < \frac{1}{4}$ AND $f(x) > \frac{1}{2}$;

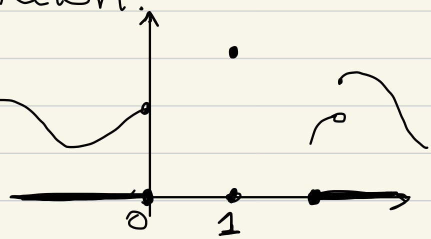
a contradiction.
So f is not continuous at 0.

Suppose $f: I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an open interval, and $x_0 \in I$.

Then it is true that f is continuous at $x_0 \in I$ if and only if
$$\lim_{x \to x_0} f(x) = f(x_0).$$

However, it is not true in general that f is continuous at $x_0 \iff \lim_{x \to x_0} f(x) = f(x_0)$.

Take $A = (-\infty, 0] \cup \{1\} \cup [2, +\infty)$.
Let $f: A \to \mathbb{R}$ be a function.



- The limit $\lim_{x \to 1} f(x)$ is NOT defined.
  (Recall that the domain of f has to contain an interval of the form $(a, 1) \cup (1, b)$.)
- However f is continuous at $x_0 = 1$.
  For any $\varepsilon > 0$, we set $\delta = \frac{1}{2} > 0$.

Then for any $x \in A$,
$$|x - 1| < \delta \Rightarrow |x - 1| < \frac{1}{2} \Rightarrow x = 1$$

hence $|f(x) - f(1)| = 0 < \varepsilon$.

We say that f is continuous on the open interval (a,b) if it is continuous at all $x \in (a,b)$.

We say that f is continuous on the closed interval $[a,b]$ if it continuous on (a,b) and in addition

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

Any sum, product, quotient and composition of continuous functions is continuous on its domain of definition.

E.g. $f(x) = x^2 e^{\sin x}$ is continuous,
$g(x) = \sqrt{x}\sin x + \cos(1 + \sqrt{\ln x})$ is continuous.

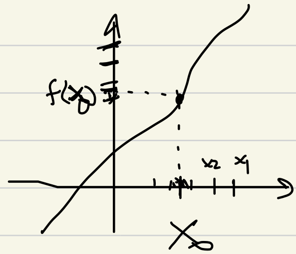
As with limits, we can characterize continuity of functions at some point using sequences (see Prop. 2.9)

PROPOSITION 2.12 : Let $f : A \to \mathbb{R}$ and $x_0 \in A$. The following are equivalent:

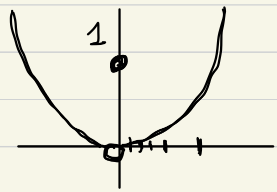
(i) f is continuous at x_0.

(ii) For any sequence $(x_n)_{n=1}^{\infty} \subseteq A$ with $\lim_{n\to\infty} x_n = x_0$, we have

$$\lim_{n\to\infty} f(x_n) = f(x_0).$$



We could have used this Proposition to show that $f(x) = \begin{cases} x^2, & x \neq 0 \\ 1, & x = 0 \end{cases}$ is not continuous at $x_0 = 0$.



- Assume f is continuous at $x_0 = 0$

By Prop. 2.12 for any sequence $(x_n)_{n\geq 1} \subseteq \mathbb{R}$ with $\lim_{n\to\infty} x_n = 0$, we have $\lim_{n\to\infty} f(x_n) = 1$.

Take $x_n = \frac{1}{n}$, $n = 1, 2, \dots$

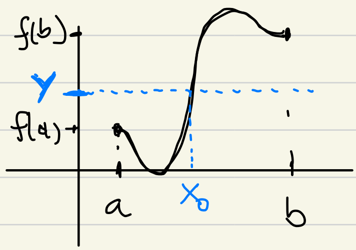
Then $\lim_{n\to\infty} x_n = 0$, so

$$1 = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n^2} = 0;$$

a contradiction.

Continuous functions on closed intervals have some very important properties.

THEOREM 2.13: Let $f:[a,b] \to \mathbb{R}$ be continuous and $f(a) \neq f(b)$.
Then for any y which is between $f(a)$ and $f(b)$, there exists $x_0 \in [a,b]$ such that $f(x_0) = y$.



(This is called the Intermediate Value Theorem or Intersection Theorem)

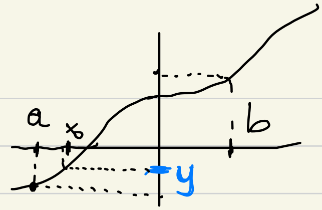
In other words, Theorem 2.13 says that if $f:[a,b] \to \mathbb{R}$ is continuous then the range $f([a,b])$ is an interval.

- Take $f(x) = x^6 + 3x - 1$
  Then $f(0) = -1 < 0$, $f(1) = 3 > 0$
  so by Theorem 2.13 there exists some $p \in (0,1)$ with $f(p) = 0$.

  I.e. the equation $x^6 + 3x - 1 = 0$
  has a root in $(0,1)$
  - even though we cannot find this root.

(Of course f is continuous as a polynomial).

- Let $f(x) = x^3 + x + 1$.
We can prove that $f(\mathbb{R}) = \mathbb{R}$.



Take some $y \in \mathbb{R}$.

- Since $\lim_{x \to -\infty} f(x) = -\infty$,
(by def. of limits) there exists some $a \in \mathbb{R}$ with $f(a) < y$.

- Since $\lim_{x \to +\infty} f(x) = +\infty$,
there exists $b \in \mathbb{R}$, $b > a$ with $f(b) > y$.

So by the Intermediate Value Theorem (since f is continuous), there exists some $x_0 \in (a, b)$ such that $f(x_0) = y$.

We have shown that an arbitrary $y \in \mathbb{R}$ is in the range $f(\mathbb{R})$.
Thus $f(\mathbb{R}) = \mathbb{R}$.

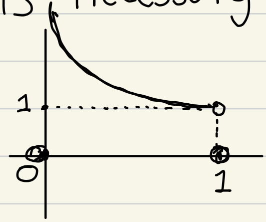
Actually, the same conclusion is true for <u>any</u> polynomial of <u>odd</u> degree.

Extreme Value Theorem / Min-Max Theorem (known as the Heine-Borel Theorem):

<u>THEOREM 2.14</u>: Let $f:[a,b] \to \mathbb{R}$ be continuous. Then f has a maximum and a minimum value on $[a,b]$, that is, there exist $x_1, x_2 \in [a,b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in [a,b].$$

<u>REMARK</u>: (i) The assumption that f is defined on a closed interval is necessary.



Consider e.g.

$$f:(0,1) \to \mathbb{R}, \quad f(x) = \frac{1}{x}$$

This function is not bounded from above, even though it is continuous: $\sup\{f(x) : 0 < x < 1\} = +\infty$.

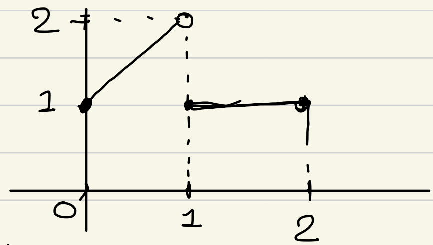
Also f is bounded from below, but there does not exist some $x_0 \in (0,1)$ with $f(x) \geq f(x_0)$ for any $x \in (0,1)$.

In other words, $\inf\{f(x) : x \in (0,1)\} = 1$ but f does not "attain its infimum".

(ii) The assumption of continuity is also necessary. Take

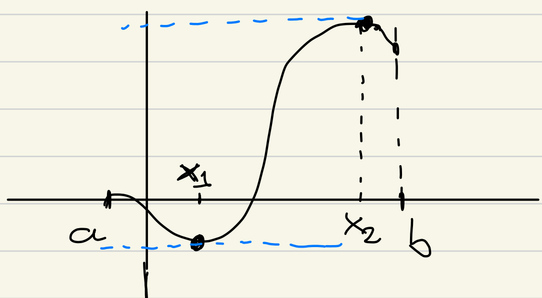
$$g:[0,2]\to\mathbb{R}, \quad g(x)=\begin{cases}1+x, & x\in[0,1)\\ 1, & x\in[1,2].\end{cases}$$

Here even though g is bounded from above, the "supremum is not attained", i.e.



there does not exist $x_0\in[0,2]$ such that $f(x)\le f(x_0)$ for all $x\in[0,2]$.

The Heine-Borel Theorem states that for continuous functions on $[a,b]$ the minimum and maximum values always exist.



$$f(x_1)\le f(x)\le f(x_2)\quad\forall x\in[a,b].$$

$$f(x_1)=\min\{f(x): x\in[a,b]\},\quad f(x_2)=\max\{f(x): x\in[a,b]\}.$$

We now prove Theorem 2.14.

Let $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$.
A <u>subsequence</u> of $(x_n)_{n=1}$ is a sequence of the form $(x_{k_n})_{n=1}^{\infty}$, where $k_1 < k_2 < \dots < k_n < k_{n+1} < \dots$ are positive integers.

E.g.
- if $a_n = n$, $n \geq 1$ then $(a_{n^2})_{n=1}^{\infty} = (n^2)_{n=1}^{\infty}$ is a subsequence.
- given some sequence $(b_n)_{n=1}^{\infty}$, then $(b_{2n+1})_{n=1}^{\infty}$ is a subsequence.
- $(x_n)_{n=1}^{\infty}$ is always a subsequence of itself.

A subsequence of $(x_n)_{n=1}^{\infty}$ is denoted by $(x_{k_n})_{n=1}^{\infty}$ or $(x_{n_i})_{i=1}^{\infty}$ or

$(x_n)_{n \in M}$, where $M \subseteq \mathbb{N}$ infinite.

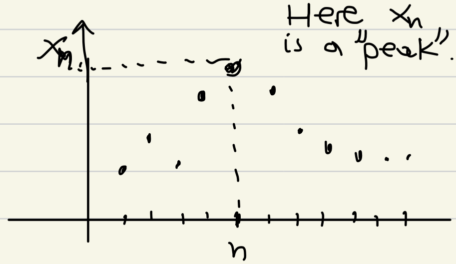
Also we might simply say that $(y_n)_{n=1}^{\infty}$ is a subsequence of $(x_n)_{n=1}^{\infty}$.

LEMMA 2.15: Every sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ has a monotone subsequence (i.e. either an increasing or a decreasing subsequence).

PROOF

We shall say that x_n is a *peak* of $(x_n)_{n=1}^{\infty}$ iff

$$x_n > x_m \text{ for all } m > n.$$



Here x_n is a "peak".

There are two cases:

(I). $(x_n)_{n=1}^{\infty}$ has inf. many peaks.
Call these peaks $x_{n_1}, x_{n_2}, x_{n_3}, \dots$.
For each $i = 1, 2, \dots$
x_{n_i} is a peak, so by definition

$$x_{n_i} > x_{n_{i+1}}.$$

Hence $x_{n_1} > x_{n_2} > \dots > x_{n_i} > x_{n_{i+1}} > \dots$
so $(x_{n_i})_{i=1}^{\infty}$ is a decreasing subsequence.

(II). $(x_n)_{n=1}^{\infty}$ has only finitely many peaks.
Call them $x_{m_1}, x_{m_2}, \dots, x_{m_k}$.
Consider the term x_{m_k+1};
this is not a peak, so there exists
$n_1 \in \mathbb{N}$ with $x_{n_1} \geq x_{m_k+1}$.

Similarly x_{n_1} is not a peak,

So there exists $n_2 > n_1$ with $x_{n_2} \geqslant x_{n_1}$.

In turn x_{n_2} is not a peak, so there exists $n_3 > n_2$ with $x_{n_3} \geqslant x_{n_2}$.

So we can recursively define a subsequence $(x_{n_k})_{k=1}^{\infty}$ which is increasing. $\blacksquare$