NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **MA2501 Numerical Methods**

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Examination time (from–to): 09:00-13:00

Permitted examination support material: Support material code C

- Approved basic calculator.
- The textbook: Cheney & Kincaid, Numerical Mathematics and Computing, 6th or 7th edition, including the list of errata.
- Rottmann, Mathematical formulae.
- Handout: Fixed point iterations.

Other information:
All answers should be justified and include enough details to make it clear which methods and/or results have been used.

Some of the (sub-)problems are worth more points than others. The total value is 100 points

Language: English

Number of pages: 7

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Checked by:

Date Signature

Problem 1

a) Use divided differences and Newton's interpolation formula to find the interpolating polynomial of lowest possible degree for the points in the table:

x	-1	0	1	2
y	3	-4	5	6

(6 points)

Suggested solution: a)

The table of divided differences is ($f[x_i] = y_i$):

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
-1	3			
		-7		
0	-4		8	
		9		-4
1	5		-4	
		1		
2	6			

The polynomial becomes

$$p_3(x) = 3 - 7(x+1) + 8(x+1)x - 4(x+1)x(x-1) = -4x^3 + 8x^2 + 5x - 4.$$

b) Establish the formula

$$f''(x) \approx \frac{2}{h^2}\left[\frac{f(x_0)}{(1+\alpha)} - \frac{f(x_1)}{\alpha} + \frac{f(x_2)}{\alpha(1+\alpha)}\right]$$

using unevenly spaced points $x_0 < x_1 < x_2$, where $x_1 - x_0 = h$ and $x_2 - x_1 = \alpha h$. Notice that this formula for $\alpha = 1$ is reduced to the standard central-difference formula.

(**Hint:** Approximate $f(x)$ by the Newton form of the interpolating polynomial of degree 2.)

(6 points)@

Suggested solution: b)

Given $x_1 - x_0 = h$ and $x_2 - x_1 = \alpha h$.

$$p_2(x) = f(x_0) + f[x_o, x_1](x - x_0) + f[x_o, x_1, x_2](x - x_0)(x - x_1)$$

$$p_2''(x) = 2f[x_o, x_1, x_2] = 2\left(\frac{f[x_1, x_2] - f[x_o, x_1]}{x_2 - x_0}\right) \tag{1}$$

$$\begin{aligned} f[x_o, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{h} \\ f[x_1, x_2] &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{\alpha h} \\ x_2 - x_1 &= (x_2 - x_1) + (x_1 - x_0) = h(\alpha + 1) \end{aligned}$$

Then equation (1) becomes

$$f''(x) \approx p_2''(x) = \frac{2}{h^2}\left[\frac{f(x_0)}{(1+\alpha)} - \frac{f(x_1)}{(\alpha)} + \frac{f(x_2)}{\alpha(1+\alpha)}\right]$$

Problem 2 Use Gaussian elimination with scaled partial pivoting to solve the following linear system.

$$\begin{bmatrix} 3 & 4 & 3 \\ 1 & 5 & -1 \\ 6 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 15 \end{bmatrix}.$$

Suggested solution:
We first compute the scale vector $\mathbf{s} = [4, 5, 7]$. The initial index vector is $l = [1, 2, 3]$.

Comparing the relative pivot sizes $|a_{l_i,1}|/s_{l_i}$ for $i = 1, 2, 3$ we find that $\max\{3/4, 1/5, 6/7\}$ is $6/7 = 0.8571$. Since the maximum happens for $i = 3$, row $l_3 = 3$ is the first pivot row. The new index vector becomes $l = [3, 2, 1]$. After one elimination step the reduced system in augmented form is

$$\begin{bmatrix} 0 & 5/2 & -1/2 & 5/2 \\ 0 & 9/2 & -13/6 & 27/6 \\ 6 & 3 & 7 & 15 \end{bmatrix}$$

Here we have subtracted 1/2 and 1/6 times the third row from the second and first row respectively.

Comparing ratios $|a_{l_i,2}|/s_{l_i}$, $i = 2, 3$ we see that $\max\{9/10, 5/8\}$ is $9/10 = 0.9$. The index corresponding to the maximum is therefore $i = 2$. The final reduced matrix is

$$\begin{bmatrix} 0 & 0 & 19/27 & 0 \\ 0 & 9/2 & -13/6 & 27/6 \\ 6 & 3 & 7 & 15 \end{bmatrix}.$$

Backward substitution is now straightforward.

Solving for x_i, we have

$$\begin{aligned} x_3 &= 0, \\ x_2 &= 1, \\ x_1 &= 2. \end{aligned}$$

Thus $\mathbf{x} = [2, 1, 0]^T$.

Problem 3 Check whether the following function is a natural cubic spline or not.

$$S(x) = \begin{cases} 1 + x - x^3, & 0 \le x \le 1 \\ 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3, & 1 \le x \le 2 \\ 4(x-2) + 9(x-2)^2 - 3(x-2)^3, & 2 \le x \le 3 \end{cases}$$

Justify your answer.

`(10 points)`

Suggested solution:

$$S'(x) = \begin{cases} 1 - 3x^2, & 0 \le x \le 1 \\ -2 - 6(x-1) + 12(x-1)^2, & 1 \le x \le 2 \\ 4 + 18(x-2) - 9(x-2)^2, & 2 \le x \le 3 \end{cases}$$

$$S''(x) = \begin{cases} -6x, & 0 \le x \le 1 \\ -6 + 24(x-1), & 1 \le x \le 2 \\ 18 - 18(x-2), & 2 \le x \le 3 \end{cases}$$

Therefore,

$$\begin{aligned} &S_0(1) = 1 = S_1(1), \quad S_1(2) = 0 = S_2(2) \\ &S_0'(1) = -2 = S_0'(1), \quad S_1'(2) = 4 = S_1'(2) \\ &S_0''(1) = -6 = S_1''(2), \quad S_1''(2) = 18 = S_2''(2) \end{aligned}$$

In addition,

$$S''(0) = 0 = S''(3).$$

Hence, function S is the natural cubic spline.

Problem 4 Find an approximation to the integral

$$\int_0^1 \mathrm{e}^{-(10x)^2} \mathrm{d}x$$

using Romberg integration. Find $R(2,2)$ up to three decimal places.

`(10 points)`

Suggested solution:
Using following formulae:

$$R(0,0) = \frac{1}{2}(b-a)\left[f(a) + f(b)\right]. \tag{2}$$

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\sum_{k=1}^{2^{n-1}} f[a + (2k-1)h], \tag{3}$$

for $n \geq 1$ where $h = \frac{b-a}{2^n}$.

$$R(n,m) = R(n,m-1) + \frac{1}{4^m - 1}\left[R(n,m-1) - R(n-1,m-1)\right] \tag{4}$$

Here we have, $a = 0, b = 1$ and $f(x) = \mathrm{e}^{-(10x)^2}$.

$$\begin{aligned}
R(0,0) &= \frac{1}{2}\left[f(0) + f(1)\right] \approx \frac{1}{2} = 0.5. \\
R(1,0) &= \frac{1}{2}R(0,0) + \frac{1}{2}f(\frac{1}{2}) \approx 0.25 \quad \text{when} \quad (h = \frac{1}{2}) \\
R(2,0) &= \frac{1}{2}R(1,0) + \frac{1}{4}\left[f(\frac{1}{4}) + f(\frac{3}{4})\right] \approx 0.126 \quad \text{when} \quad (h = \frac{1}{4}) \\
R(1,1) &= R(1,0) + \frac{1}{3}\left[R(1,0) - R(0,0)\right] \approx 0.16667 \\
R(2,1) &= R(2,0) + \frac{1}{3}\left[R(2,0) - R(1,0)\right] \approx 0.0847 \\
R(2,2) &= R(2,1) + \frac{1}{15}\left[R(2,1) - R(1,1)\right] \approx 0.079
\end{aligned}$$

Problem 5 Use the method of least squares to find the equation of a parabola of the form $y = ax^2 + b$ that best represents the following data.

x	-1	0	1
y	3.1	0.9	2.9

(10 points)

Suggested solution:
We use the method of least squares. Defining the least squares error

$$\varphi(a,b) = \sum_{k=0}^{2}(ax_k^2 + b - y_k)^2,$$

the normal equations are given by $\partial\varphi/\partial a = 0$ and $\partial\varphi/\partial b = 0$, which when reorganized and written in matrix vector form become

$$\begin{bmatrix} \sum_{k=0}^{2} x_k^4 & \sum_{k=0}^{2} x_k^2 \\ \sum_{k=0}^{2} x_k^2 & \sum_{k=0}^{2} 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{2} y_k x_k^2 \\ \sum_{k=0}^{2} y_k \end{bmatrix}.$$

Inserting values we get

$$\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ 6.9 \end{bmatrix}.$$

The system is now easily solved

$$a = 2.1$$
$$b = 0.9$$

Thus the best polynomial on the desired form is $y = 2.1x^2 + 0.9$

Problem 6 Suppose we have the following initial value problem

$$x' = f(t,x),$$
$$x(1) = 1,$$

with

$$f(t,x) = (tx)^3 - \left(\frac{x}{t}\right).$$

Approximate $x(1.2)$ by taking step size $h = 0.1$ with the following Runge-Kutta method

$$\begin{cases} K_1 = f(t, x), \\ K_2 = f(t+h, x+K_1), \end{cases}$$
$$x(t+h) = x(t) + \frac{h}{2}\left(K_1 + K_2\right).$$

(10 points)

Suggested solution:
Following the description, we compute $x_1 \approx x(1.1)$, by taking a step of size $h = 0.1$ with this Runge-Kutta method for the given IVP, starting at $x_0 = 1, t_0 = 1$

$$\begin{cases} K_1 = f(t_0, x_0) = f(1,1) = 0 \\ K_2 = f(t_0 + h, x_0 + K_1) = f(1.1, 1) = 0.4219, \end{cases}$$
$$x_1 = x_0 + \frac{h}{2}\left(K_1 + K_2\right) \approx 1 + \frac{0.1}{2}\left(0 + 0.4219\right) = 1.0210955.$$

Repeat the same process in order to find $x_2 \approx x(1.2)$ with $t_1 = 1.1, x_1 = 1.0210955$

$$\begin{cases} K_1 = f(t_1, x_1) = f(1.1, 1.0210955) = 0.488755 \\ K_2 = f(t_1 + h, x_1 + K_1) = f(1.2, 1.5098505) = 4.689443, \end{cases}$$
$$x_2 = x_1 + \frac{h}{2}\left(K_1 + K_2\right) \approx 1.0210955 + \frac{0.1}{2}\left(0.488755 + 4.689443\right) = 1.280005$$

$x(1.2) \approx x_2 \approx 1.280005$

Problem 7 Given the initial value problem

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0,$$

where $\mathbf{f} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$. The trapezoidal rule for solving this ODE is given by

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{2}\Big(\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) + \mathbf{f}(t_n, \mathbf{y}_n)\Big),$$

where $h = t_{n+1} - t_n$.
Suppose $\mathbf{f}$ satisfies the L Lipschitz condition

$$\left\|\mathbf{f}(t, \mathbf{y}) - \mathbf{f}(t, \tilde{\mathbf{y}})\right\| \leq L \left\|\mathbf{y} - \tilde{\mathbf{y}}\right\|, \quad \text{for all } t \in \mathbb{R}, \mathbf{y}, \tilde{\mathbf{y}} \in \mathbb{R}^m.$$

The local truncation error for the trapezoidal method

$$\mathbf{d}_{n+1} = \mathbf{y}(t_{n+1}) - \mathbf{y}(t_n) - \frac{h}{2}\left(\mathbf{f}(t_{n+1}, \mathbf{y}(t_{n+1})) + \mathbf{f}(t_n, \mathbf{y}(t_n))\right)$$

satisfies

$$\|\mathbf{d}_{n+1}\| \le \frac{1}{12} h^3 M, \quad M = \max_{\xi \in \mathbb{R}} \|\mathbf{y}'''(\xi)\|.$$

Use this to show that the global error $\mathbf{e}_n = \mathbf{y}(t_n) - \mathbf{y}_n$ satisfies

$$\|\mathbf{e}_{n+1}\| \le \frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL}\|\mathbf{e}_n\| + \frac{\frac{1}{12}Mh^3}{1 - \frac{1}{2}hL}, \quad \text{for } hL < 2.$$

`(10 points)`

Suggested solution:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{2}\Big(\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) + \mathbf{f}(t_n, \mathbf{y}_n)\Big), \tag{5}$$

$$\mathbf{y}(t_{n+1}) = \mathbf{y}(t_n) + \frac{h}{2}\left(\mathbf{f}(t_{n+1}, \mathbf{y}(t_{n+1})) + \mathbf{f}(t_n, \mathbf{y}(t_n))\right) + \mathbf{d}_{n+1} \tag{6}$$

Subtracting (5) from (6), we have

$$\|\mathbf{e}_{n+1}\| \le \|\mathbf{e}_n\| + \frac{hL}{2}(\|\mathbf{e}_{n+1}\| + \|\mathbf{e}_n\|) + \|\mathbf{d}_{n+1}\|$$

$$\left(1 - \frac{hL}{2}\right)\|\mathbf{e}_{n+1}\| \le \left(1 + \frac{hL}{2}\right)\|\mathbf{e}_n\| + \frac{1}{12}h^3 M$$

As long as $\left(1 - \frac{hL}{2}\right) > 0$, dividing by this on both sides gives required result

$$\|\mathbf{e}_{n+1}\| \le \frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL}\|\mathbf{e}_n\| + \frac{\frac{1}{12}Mh^3}{1 - \frac{1}{2}hL}, \quad \text{for } hL < 2.$$