Kandidatnr: 10009. Problem 2

$I(f) := \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx$

$Q_n(f) := \sum_{i=0}^{n} W_i f(x_i)$

$n \geq 0$

$x_i = \cos\left(\frac{2i+1}{2(n+1)} \cdot \pi\right)$

$W_i = \frac{\pi}{n+1}$

(i) When $f(x) = x^4$, $\frac{x^4}{\sqrt{1-x^2}}$ becomes a square function

When we want to approximate a square function we need 3 point

$\Rightarrow n=2$ is the smallest $n = 0,1,2$ satisfying $Q_n(f) = I(f)$

(ii) $f(x) = x^4$

$Q_2(f) = W_0 f(x_0) + W_1 f(x_1) + W_2 f(x_2)$

$= \frac{\pi}{3}\left(\cos^4\left(\frac{\pi}{6}\right) + \cos^4\left(\frac{3}{6}\pi\right) + \cos^4\left(\frac{5}{6}\pi\right)\right)$

$= \frac{3}{8}\pi$

(iii) When $f(x)$ is odd, $\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = 0$

$f(x_i) = \cos\left(\frac{2i+1}{2(n+1)} \pi\right)^p$, $p > 0$ odd

Proof by induction

$n=0$:

$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = W_0 f(x_0)$

$= \pi \cdot \cos\left(\frac{\pi}{2}\right)^p$

$= \pi \cdot 0$

$= 0$

OK

$n=k$:

$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = W_0 f(x_0) + W_1 f(x_1) + \dots + W_k f(x_k)$

$= \frac{\pi}{k+1}\left(\cos^p\left(\frac{\pi}{2(k+1)}\right) + \cos^p\left(\frac{3\pi}{2(k+1)}\right) + \dots + \cos^p\left(\frac{2k+1}{2(k+1)} \cdot \pi\right)\right)$

$= 0$

OK.

$n=k+1$:

$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = W_0 f(x_0) + W_1 f(x_1) + \dots + W_k f(x_k) + W_{k+1} f(x_{k+1})$

$= \frac{\pi}{k+2}\left(\cos^p\left(\frac{\pi}{2(k+2)}\right) + \cos^p\left(\frac{3\pi}{2(k+2)}\right) + \dots + \cos^p\left(\frac{2k+3}{2(k+2)} \cdot \pi\right)\right)$

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