Lecture 1 9/1/23 Week 2.1

Introduction & Inversion Sampling

Exercise: Simulating discrete random variables

Assume random variable (RV) X takes values $x_1, x_2, \dots, x_k$ with probabilities $p_1, \dots, p_k$. Let $F_0 = 0$, $F_i = \sum_{j=1}^{i} p_j$ for $i = 1, \dots, k$, and define $I_i := (F_{i-1}, F_i]$ for $i = 1, \dots, k$. For $u \sim \mathrm{Unif}(0,1)$, prove x as drawn from the algorithm on the slides follows the dist'n of X.

Pf:

$$
\begin{aligned}
P(x = x_i) &= P(u \in (F_{i-1}, F_i]) \\
&= \int_{F_{i-1}}^{F_i} 1 \, dx \\
&= F_i - F_{i-1} \\
&= p_i \quad \forall\ i \in \{1, \dots, k\} \\
&= P(X = x_i)
\end{aligned}
$$

$\square$

Exercise: Inversion Sampling

Assume $X \sim F$ (i.e. X has cdf F), with inverse cdf F^{-1}. Assume $U \sim \mathrm{Unif}(0,1)$. Prove

$$F^{-1}(U) \overset{d}{=} X$$

equal in distribution

Pf:

Let $Y = F_X^{-1}(U)$. Want to show $F_Y = F_X$.

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(F_X^{-1}(U) \le y) \\ &= P(U \le F_X(y)) \quad (F_X \text{ monotone increasing}) \\ &= F_U(F_X(y)) \end{aligned}$$

But $F_U(u) = u$ for $u \in [0,1]$, so

$$F_Y(y) = F_X(y)$$

$\square$