A sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ is called a <u>fundamental</u> sequence or a <u>Cauchy</u> sequence, if

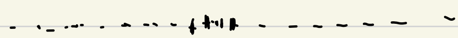
$$\forall \varepsilon > 0 \quad \exists N_0 = N_0(\varepsilon) \geqslant 1 \quad \text{s.t.} \quad m,n \geqslant N_0 : |x_m - x_n| < \varepsilon .$$

(Intuitively, a sequence is Cauchy if its terms tend to become closer and closer to another).

A sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ is Cauchy if and only if it convergent to some $l \in \mathbb{R}$. (I.e. there is no distinction between Cauchy and convergent sequences in $\mathbb{R}$).

The situation is different when we look at Cauchy sequences on other subsets of $\mathbb{R}$. E.g.

- there are Cauchy sequences $(r_n)_{n=1}^{\infty} \subseteq \mathbb{Q}$ which do not converge <u>in $\mathbb{Q}$</u>. (they will, of course, converge in $\mathbb{R}$). — take for example the sequence of decimal approximations to $\sqrt{2}$ (or to π).

$\mathbb{Q}$ 

- the sequence $(x_n)_{n=1}^{\infty} \subseteq (0, \infty)$ with $x_n = 1/n$, $n = 1, 2, \ldots$ is Cauchy but it does not converge in $(0, \infty)$ — it converges to $0 \in \mathbb{R} \setminus (0, \infty)$

PROPOSITION 5.4: Suppose f is uniformly continuous on $A \subseteq \mathbb{R}$. Then whenever $(x_n)_{n=1}^{\infty} \subseteq A$ is a Cauchy sequence, the sequence $(f(x_n))_{n=1}^{\infty} \subseteq \mathbb{R}$ is also Cauchy.

- Show that $f:(0,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$.

→ The sequence $(x_n)_{n=2}^{\infty} \subseteq (0,1)$ with $x_n = 1/n$, $n = 2, 3, \dots$ is a Cauchy sequence.
Suppose for contradiction that f is uniformly continuous.

By Proposition 5.4, $(f(x_n))_{n=2}^{\infty}$ will be a Cauchy sequence; but

$$f(x_n) = \frac{1}{x_n} = n, \quad n = 2, 3, \dots$$

which is NOT a Cauchy sequence.

• RIEMANN INTEGRATION

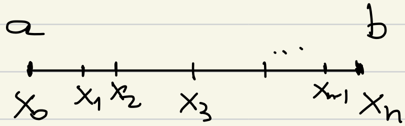
Consider an interval $[a,b]$.

We define a <u>partition</u> of $[a,b]$ to be a set

$$P = \{ x_0, x_1, \dots, x_{n-1}, x_n \}$$

where

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b.$$



Let $f : [a,b] \to \mathbb{R}$ be a function. Given a partition $P = \{ x_0, x_1, \dots, x_n \}$ we set

$$m_j = \inf \{ f(t) : t \in [x_{j-1}, x_j) \},$$

$$M_j = \sup \{ f(t) : t \in [x_{j-1}, x_j) \},$$

for all $j = 1, 2, \dots, n$.

(Note that we have suppressed the dependence of m_j, M_j on the partition P. I.e. we should have written $m_j(P)$ and $M_j(P)$).

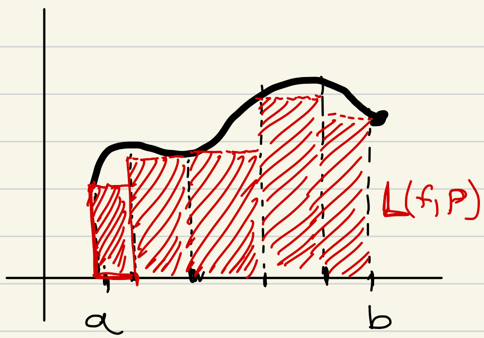
We define:

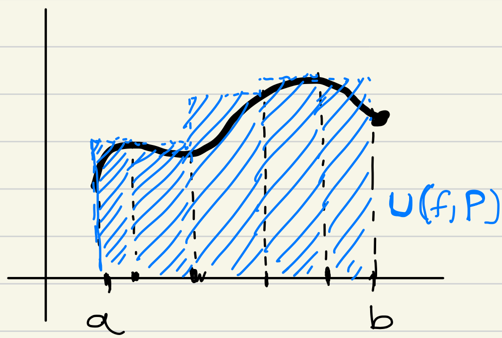
- the <u>upper Darboux sum</u> of f with respect to the partition P to be the number

$$U(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}).$$

- the <u>lower Darboux sum</u> of f with respect to the partition P to be the number

$$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}).$$

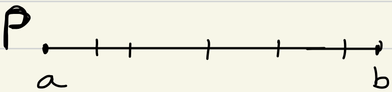


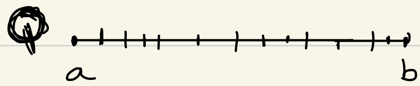


By the definition of m_j, M_j $(j=1,2,\ldots,n)$ we have

$$L(f,P) \leq U(f,P) \quad \text{for any partition } P.$$

A partition $Q = \{ y_0, y_1, \dots, y_m \}$ is called a <u>refinement</u> of the partition $P = \{ x_0, x_1, \dots, x_n \}$ if $P \subseteq Q$.





Whenever Q is a refinement of P, we have

$$L(f, P) \leq L(f, Q) \quad \text{and} \quad U(f, Q) \leq U(f, P).$$

(i.e. when the partition becomes finer, the lower Darboux sums increase and the upper Darboux sums decrease).

Question: Can a lower sum with respect to some partition P_1 be bigger than an upper sum with respect to some partition P_2?

LEMMA 5.5: $\sup_{P} L(f,P) \leq \inf_{P} U(f,P)$

PROOF

Assume the opposite is true.
Then there exist partitions P_1, P_2 of $[a,b]$ such that

$$U(f, P_1) < L(f, P_2).$$

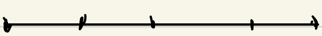
Consider the partition $Q = P_1 \cup P_2$ (the "common refinement" of P_1, P_2).

Then Q is a refinement of both P_1 and P_2, and hence

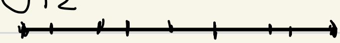
$$U(f,Q) \leq U(f,P_1) < L(f,P_2) \leq L(f,Q),$$

a contradiction, because $L(f,Q) \leq U(f,Q)$. ∎

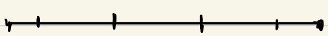
P_1



$P_1 \cup P_2$



P_2



We say that $f:[a,b]\to\mathbb{R}$ is <u>Riemann-integrable</u> if

$$\sup_P L(f,P) = \inf_P U(f,P).$$

In that case, the common value of $\sup\{L(f,P) : P\}$ and $\inf\{U(f,P) : P\}$ is denoted by

$$\int_a^b f(x)\,dx$$

and is called the <u>Riemann integral</u> of f on the interval $[a,b]$.

Equivalent Definition:

The function $f:[a,b]\to\mathbb{R}$ is Riemann-integrable if for any $\varepsilon>0$ there exists a partition $P=P_\varepsilon$ of $[a,b]$ such that

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon.$$

- Any constant function $f:[a,b]\to\mathbb{R}$,
  $f(x) = c \ , \ x\in[a,b]$
  is Riemann integrable with
$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a).$$

  - For any partition $P=\{x_0,x_1,\dots,x_n\}$ we have
$$L(f,P) = \sum_{i=1}^n c(x_i-x_{i-1}) = c(b-a) \quad \text{and}$$
$$U(f,P) = \sum_{i=1}^n c(x_i-x_{i-1}) = c(b-a)$$
  therefore $\sup_P L(f,P) = \inf_P U(f,P) = c(b-a)$

  and $\int_a^b f(x)\,dx = c(b-a)$.

- The function $f:[0,1]\to\mathbb{R}$, $f(x) = \begin{cases} 1, & x\in\mathbb{Q} \\ 0, & x\notin\mathbb{Q}\end{cases}$

  (Dirichlet's function) is not Riemann integrable.
  Indeed, for any partion P of $[0,1]$ we have
$$L(f,P) = \sum_{i=1}^n m_i(x_i-x_{i-1}) = 0$$
$$U(f,P) = \sum_{i=1}^n M_i(x_i-x_{i-1}) = \sum_{i=1}^n (x_i-x_{i-1}) = 1$$
(Each subinterval contains a rational AND an irrational).

THEOREM 5b: If $f:[a,b]\to\mathbb{R}$ is continuous, then it is Riemann integrable.

PROOF

Since $f:[a,b]\to\mathbb{R}$ is continuous it is also uniformly continuous.

Let $\varepsilon>0$. There exists some $\delta=\delta(\varepsilon)>0$ such that

$$|x-y|<\delta \quad \text{implies} \quad |f(x)-f(y)|<\frac{\varepsilon}{b-a}.$$

Let $n\geq 1$ be such that

$$\frac{b-a}{n}<\delta$$

and consider the partition

$$P_n=\left\{a,\ a+\frac{b-a}{n},\ a+\frac{2(b-a)}{n},\ \dots,\ b\right\}.$$