MA2501 NUMERICAL METHODS
EXAM CONT.
SPRING 2022
NTNU

ENGLISH

Problem 1:	15 points
Problem 2:	15 points
Problem 3:	20 points
Problem 4:	15 points
Problem 5:	15 points
Problem 6:	20 points
Total:	100 points

Problem 1 (15 points)

(i) (3 point) Consider the following interpolation problem: let $f(x) = e^{-x}$. We want to construct a polynomial $p(x)$ of degree at most two satisfying $p(0) = f(0)$, $p'(0) = f'(0)$, and $p(1) = f(1)$.

(ii) (12 point) Let $f(x) = e^{-x}$, and $x_1, ..., x_n$ be distinct n nodes on $[0, 1]$. Prove the following: if there exists a solution $p(x)$ of degree n for the interpolation problem with conditions $p(x_i) = f(x_i)$ for $i = 1, ..., n$, and $p'(x_1) = f'(x_1)$, then the solution is unique.

[Solution] (i) Let $p(x) = a_2x^2 + a_1x + a_0$. We have $p(0) = a_0 = 1$, $p'(0) = a_1 = -1$, and $p(1) = 1/e$. Thus

$$p(x) = \frac{x^2}{e} - x + 1.$$

(ii) Assume that there exist two different polynomials p_n and q_n, which satisfy the given interpolation problem and are of degree at most n. Let $r = p_n - q_n$, then r has n distinct zeros and r is of degree at most n. Due to Rolle's theorem, r' has $n - 1$ distinct zeros, which are different from x_1. Thus r' has n distinct zeros but r' is a polynomial of degree at most $n - 1$. This means r is a constant function. Since r has zeros, r is always zero. This contradicts the assumption that p_n and q_n are different. Therefore the solution is unique.

Problem 2 (15 points)

Let $\mathcal{P}_n$ be the set of all polynomials of degree at most n.

(i) (5 point) Find the polynomial $p_1 \in \mathcal{P}_1$ that minimizes $\|p_1 - x^2 - x - 2\|_\infty$ on $x \in [-1, 1]$.

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(ii) (5 point) Find the polynomial $p_n \in \mathcal{P}_n$ that minimizes $\|p_n - x^{n+1} - x^{n-1} + 1\|_\infty$ on $x \in [-1,1]$.

(iii) (5 point) Let p_n be the solution of (ii). Find all $x \in [-1,1]$ where extrema of $(p_n - x^{n+1} - x^{n-1} + 1)$ are attained.

[Solution]

(i) By Theorem 8.6 from Süli & Mayers, p_1 is such a polynomial that

$$p_1 - x^2 - x - 2 = -2^{-1}T_2(x),$$

where $T_2(x)$ is the Chebyshev polynomial of degree 2. Thus

$$p_1 = -2^{-1}T_2(x) + x^2 + x + 2 \left(= x + \frac{5}{2}\right).$$

(The sign in front of T_2 can also be opposite.)

(ii) As before, using Theorem 8.6 from Süli & Mayers, p_n is such a polynomial that

$$p_n - x^{n+1} - x^{n-1} + 1 = -2^{-n}T_{n+1}(x),$$

where $T_n(x)$ is the Chebyshev polynomial of degree n. Thus

$$p_n = -2^{-n}T_{n+1}(x) + x^{n+1} + x^{n-1} - 1 \left(= -2^{-n}\cos((n+1)\cos^{-1}(x)) + x^{n+1} + x^{n-1} - 1\right).$$

(The sign in front of T_{n+1} can also be opposite.)

(iii) Since $p_n - x^{n+1} - x^{n-1} + 1 = -2^{-n}T_{n+1}(x)$, we only need to find extrema of $T_{n+1}(x)$. We have

$$T_{n+1}(x) = \cos\left((n+1)\cos^{-1}x\right),$$

and $\cos(y)$ gives extrema on $y = k\pi$ for any integer k. This means we want to find x such that $(n+1)\cos^{-1}(x) = k\pi$. Thus the solutions are

$$x = \cos\left(\frac{k\pi}{n+1}\right), \quad k = 0, ..., n+1.$$

PROBLEM 2 (20 points)

(i) (5 points) Consider the following composite rectangle rule on $[0,1]$:

$$Q_n(f) := \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right).$$

Let $f(x) = x^2 - x + \frac{1}{6}$. Compute the exact quadrature error

$$\int_0^1 f(x)\,\mathrm{d}x - Q_n(f),$$

and conclude the order of convergence for this function.

You may use $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

(ii) (15 points) Consider the composite trapezoidal rule of $n+1$ nodes on $[0,1]$. By using the Euler–Maclaurin expansion formula, calculate the exact quadrature error of the composite trapezoidal rule for the integrand function

$$g(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}.$$

Note that first several Bernoulli numbers are $B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42$.

[Solution] (i) $\int_0^1 f(x)\,\mathrm{d}x = 0$. We also have

$$\begin{aligned} Q_n(f) &= \frac{1}{n}\left(\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 - \left(\frac{i}{n}\right) + \frac{1}{6}\right) \\ &= \frac{1}{n}\left(\left(\frac{n(n+1)(2n+1)}{6n^2}\right) - \left(\frac{n(n+1)}{2n}\right) + \frac{n}{6}\right) \\ &= \frac{1}{6n^2}. \end{aligned}$$

Thus the exact error is given by $-\frac{1}{6n^2}$. This gives the second order convergence.

(ii) The Euler–Maclaurin expansion formula states that there exists such $\xi \in [0,1]$

$$\begin{aligned} I(f) - T_n(f) = &-\sum_{r=1}^{k}\frac{B_{2r}}{(2r)!}\left(\frac{1}{n}\right)^{2r}\left(f^{(2r-1)}(1) - f^{(2r-1)}(0)\right) \\ &- \frac{B_{2k+2}}{(2k+2)!}\left(\frac{1}{n}\right)^{2k+2} f^{(2k+2)}(\xi), \end{aligned}$$

where T_n is the composite trapezoidal rule and $f \in C^{2k+2}$. Since our integrand g is infinitely smooth, we can apply the formula for any positive integer k. Notice that $g^{(r)} = 0$ for $r \geq 5$. Thus we have

$$\begin{aligned} I(g) - T_n(g) &= -\sum_{r=1}^{2}\frac{B_{2r}}{(2r)!}\left(\frac{1}{n}\right)^{2r}\left(g^{(2r-1)}(1) - g^{(2r-1)}(0)\right) \\ &= \frac{B_4}{(4)!}\left(\frac{1}{n}\right)^4 24 = \frac{1}{30n^4}. \end{aligned}$$

Problem 4 (15 points)

Exercise: Provide a detailed answer for each of the following problems.

(1) (1 point) We consider a $n \times n$ matrix M with eigenvalues $\alpha_1, \dots, \alpha_n$. Show that the determinant of M equals the product of its eigenvalues (provide details).

(2) (1 point) We consider a $n \times n$ matrix M. Prove that M is singular if and only if zero is an eigenvalue of M.

(3) (2 points) Show that the spectral radius ρ cannot be a matrix norm. (Hint: find a counterexample to the triangle inequality.)

(4) (4 point) We consider a symmetric $n \times n$ matrix M. Show that the 2-norm of M equals its spectral radius.

(5) (5 points) Consider a nonsingular $n \times n$ matrix M and its condition number $\kappa_2(M)$ (defined with respect to the 2-norm). Let α_1 and α_n be the smallest respectively largest eigenvalues of the matrix M^TM. Show that $(\kappa_2(M))^2 = \alpha_n/\alpha_1$.

(2 points) In case of an orthogonal matrix M, what can you deduce for $\kappa_2(M)$?

Solution:

(1) Recall that $\det(M - x\mathrm{id}) = \prod_{i=1}^{n}(\alpha_i - x) = P(M, x)$. Then $P(M, 0) = \prod_{i=1}^{n}\alpha_i = \det(M)$.

(2) Recall that a matrix M is singular if and only if its determinant is zero. Therefore, using (1), at least one of its eigenvalues must be zero.

(3) For instance:

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For these matrices, we have $\rho(M_1) = 1 = \rho(M_2)$, but $\rho(M_1 + M_2) = 3$, which shows that $\rho(M_1 + M_2) > \rho(M_1) + \rho(M_2)$.

(4) Assuming M to be symmetric, we have $M^{\mathrm{T}}M = M^2$. From the eigenvalue equation $Mv = \alpha v$ follows $MMv = \alpha^2 v$. This implies $\rho(M^{\mathrm{T}}M) = \rho(MM) = \rho(M)\rho(M)$. Therefore, $||M||_2 = \rho(M^{\mathrm{T}}M)^{1/2} = \rho(M)$.

(5) Recall that $||M||_2^2 = \rho(M^{\mathrm{T}}M)$, which is the largest eigenvalue of $M^{\mathrm{T}}M$. Let's denote it by α_n. $||M^{-1}||_2^2 = \rho({M^{-1}}^{\mathrm{T}}M^{-1}) = \rho((MM^{\mathrm{T}})^{-1})$. The eigenvalues of MM^{T} equal those of $M^{\mathrm{T}}M$ and the eigenvalue of $(MM^{\mathrm{T}})^{-1}$ is the reciprocal of that of MM^{T}. Hence, $\rho((MM^{\mathrm{T}})^{-1}) = 1/\alpha_1$, where α_1 is the smallest eigenvalue of $M^{\mathrm{T}}M$. Therefore, $\kappa_2^2(M) = ||M||_2^2||M^{-1}||_2^2 = \alpha_n/\alpha_1$.

For an orthogonal matrix M, we know that $M^{\mathrm{T}}M$ equals the identity matrix. All the eigenvalues of the latter are equal to one. Hence, condition number $\kappa_2(M)$ equals one.

Problem 5 (15 points)

Exercise: Provide a detailed answer for the following problem.

Consider a function $f : \mathbb{R}^m \to \mathbb{R}^m$, $f(y) = (f^1(y_1, \dots, y_m), \dots, f^m(y_1, \dots, y_m))^{\mathrm{T}}$. Recall that for a vector $v \in \mathbb{R}^m$, the transposed vector is denoted v^{T}. Show that the identity holds

$$(f_y f)_y f = f^{\mathrm{T}} f_{yy} f - f_y^2 f.$$

Use the upper/lower notation for components respectively partial derivations

$$\frac{\partial f^i}{\partial y_k} =: f^i_k \qquad \frac{\partial^2 f^i}{\partial y^k \partial y^l} =: f^i_{kl}.$$

f_y denotes the Jacobian matrix of $f = f(y)$, i.e., the matrix with entries $(f_y)_{ij} = f^i_j$. f_y^2 is the matrix product and f_{yy} has entries (f^i_{kl}).

Solution: This is an identity among vectors. We show the identity component-wise (1 point).

Using the Leibniz (product) rule, the i-th component on the lefthand side is

$$(f^i_k f^k)_j f^j = f^i_{kj} f^k f^j + f^i_k f^k_j f^j. \qquad \text{(7 points)}$$

The i-th component on the righthand side is

$$f^k f^i_{kn} f^n + f^i_k f^k_j f^j. \qquad \text{(7 points)}$$

Here $f^i_k f^k_j$ is the ij-th component of f_y^2. The two sides coincide.

Problem 5 (20 points)

Exercise: Provide a detailed answer for each of the following problems.

Consider the system of differential equations

$$F^{(3)}(t) - t^2 F(t) F^{(2)}(t) + F(t) G^{(1)}(t) = 0 \tag{1}$$
$$G^{(2)}(t) - t G(t) G^{(1)}(t) - 4F^{(1)}(t) = 0.$$

(1) (5 points) Rewrite the system (1) as a system of first-order differential equations of the form $y^{(1)}(t) = y'(t) = f(t, y)$.

(2) (5 points) Compute the Jacobian matrix (with respect to y-entries) $f_y(t, y)$ for the system displayed in (1) .

(3) (10 points) Determine a Lipschitz constant L for f on $[0, 1] \times \{y \in \mathbb{R}^5 | \ ||y||_1 \leq 1\}$ using the 1-norm $|| \cdot ||_1$.

Solution:

(1) Consider the vector $y^{\mathrm{T}} = (y_1, \ldots, y_5)$ defined by

$$y_1 = F \quad y_2 = F^{(1)} \quad y_3 = F^{(2)} \quad y_4 = G \quad y_5 = G^{(1)}.$$

Then we have the 1st order system:

$$\frac{dy_1}{dt} = y_2 \quad \frac{dy_2}{dt} = y_3 \quad \frac{dy_3}{dt} = t^2 y_1 y_3 - y_1 y_5 \quad \frac{dy_4}{dt} = y_5 \quad \frac{dy_5}{dt} = t y_4 y_5 + 4y_2.$$

(2) The Jacobian is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ t^2 y_3 - y_5 & 0 & t^2 y_1 & 0 & -y_1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & t y_5 & t y_4 \end{pmatrix}$$

(3) Recall the result/method of Ex. 5 in the first exam:

Consider the C^1-function $f(t, y) = (f_1(t, y), \ldots, f_n(t, y))^T : [u, v] \times \mathbb{R}^n \to \mathbb{R}^n$. Let $A_{ij} \in \mathbb{R}_+$ be the constant entries of the $n \times n$-matrix A. Suppose that on $[u, v] \times \mathbb{R}^n$ we have

$$\left| \frac{\partial f_i(t, y)}{\partial y_j} \right| \leq A_{ij}, \qquad i, j = 1, \ldots, n.$$

Determine a Lipschitz constant L of $f(t, y)$ in the 1-norm $|| \cdot ||_1$. Give L in terms of the corresponding matrix norm of A. (Hint: use the mean value theorem for functions of several variables.)

Now, consider

$$\left|\frac{\partial f^i}{\partial y_j}\right| \leq \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ |y_3| + |y_5| & 0 & |y_1| & 0 & |y_1| \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & |y_5| & |y_4| \end{pmatrix} \leq \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 & 1 \end{pmatrix} =: A$$

The last inequality holds on $[0,1] \times \{y \in \mathbb{R}^5 |\ ||y||_1 \leq 1\}$. A valid Lipschitz constant is $L = ||A||_1 = 5$.