MID-TERM EXAM : 5 OCT – 18:30

3. DERIVATIVES

Suppose $I \subseteq \mathbb{R}$ is an open interval, $f: I \to \mathbb{R}$ is a function and $x_0 \in I$. We say that f is <u>differentiable</u> at x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \qquad \left(= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \right)$$

exists and is a real number.

In that case we write

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

and the number $f'(x_0)$ is called the (first) derivative of f at x_0.

The first derivative of f at x_0 is also denoted by $\left.\frac{df}{dx}\right|_{x = x_0}$.

Let $I_1 \subseteq I$ be the set of points in I on which f is differentiable. The function

$$f': I_1 \to \mathbb{R}, \quad x \mapsto f'(x)$$

is called the (first) derivative of f.

If for some $n \in \mathbb{N}$ the n-th derivative $f^{(n)}$ has been defined, let $I_{n+1} \subseteq I$ be the set of points in I on which $f^{(n)}$ is differentiable.

Then we define the $(n+1)$-st derivative of f to be the function

$$f^{(n+1)} : I_{n+1} \to \mathbb{R}, \qquad f^{(n+1)}(x) = \left(f^{(n)}\right)'(x).$$

E.g. • the function $f(x) = x^2$ is differentiable on any $x \in \mathbb{R}$. Indeed, if $x \in \mathbb{R}$

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = 2x$$

therefore f is diff. on $x \in \mathbb{R}$ with $f'(x) = 2x$.

• the function $g(x) = \sqrt{x}$ is not diff. on $x_0 = 0$.

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{1}{\sqrt{x}} = +\infty$$

However g is diff. at any $x > 0$ with $g'(x) = \frac{1}{2\sqrt{x}}$, $x > 0$.

Assume f, g are differentiable at some point x in their domain. Then

$$f+g, \quad \lambda \cdot f, \quad f \cdot g, \quad \frac{f}{g}$$

are also differentiable at x and

$$(f+g)'(x) = f'(x) + g'(x),$$

$$(\lambda f)'(x) = \lambda f'(x),$$

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x),$$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

(provided $g(x) \neq 0$).

THEOREM 3.1 (Chain Rule) : Suppose f, g are such that $f \circ g$ is well-defined. If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x with

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

The previous relation is also written as

$$\frac{du}{dt} = \frac{du}{dv} \cdot \frac{dv}{dt}$$

(so here u is $f \circ g$, v is g and t is the variable.). Be careful, this is not a multiplication of fractions but only a way to remember the chain rule.

E.g. if $g(x) = f(x^2)$ and f is differentiable, then

$$g'(x) = f'(x^2) \cdot (x^2)' = 2x f'(x^2).$$

Derivatives of Basic Functions.

$$(x^n)' = n x^{n-1}, \quad n \geqslant 1, \quad x \in \mathbb{R}$$

$$(x^k)' = k x^{k-1}, \quad k \in \mathbb{Z} \quad x>0 \text{ or } x<0.$$

$$(x^a)' = a x^{a-1}, \quad a \in \mathbb{R} \quad x>0$$

$$(e^x)' = e^x$$

$$(a^x)' = a^x \ln a \quad (a>0)$$

$$(\ln|x|)' = \frac{1}{x}$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \frac{1}{\cos^2 x}$$

$$(\cot x)' = -\frac{1}{\sin^2 x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

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$$(\cot x)' = -\frac{1}{\sin^2 x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\sinh x)' = \cosh x$$

$$(\cosh x)' = \sinh x$$

$$(\tanh x)' = \frac{1}{\cosh^2 x}$$

$$\cos\left(\frac{\pi}{2}-x\right) = \sin x \Rightarrow \arccos x = \frac{\pi}{2} - \arcsin x \Rightarrow (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

THEOREM 3.2 : Suppose f is differentiable at $x_0 \in \mathbb{R}$. Then f is continuous at x_0.

PROOF

By the hypothesis, $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$.

Therefore

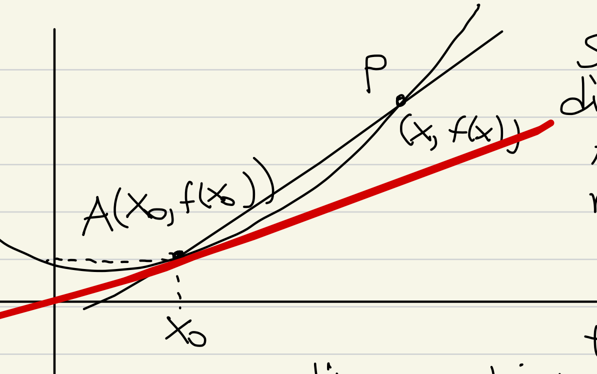
$$\lim_{x \to x_0} \left(f(x)-f(x_0)\right) = \lim_{x \to x_0} \left[\frac{f(x)-f(x_0)}{x-x_0} \cdot (x-x_0)\right]$$

$$= f'(x_0) \cdot 0$$

$$= 0$$

and $\lim_{x \to x_0} f(x) = f(x_0)$.

So f is continuous at x_0. ■



Suppose f is differentiable at x_0. As the point $(x, f(x))$ moves arbitrarily close to $A(x_0, f(x_0))$ the line segment AP tends to become a line which is called the <u>tangent</u> of G_f at the point $(x_0, f(x_0))$.

The <u>slope</u> of G_f at $(x_0, f(x_0))$ is the slope of the tangent line at this point, which is

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

The equation of the tangent line at $(x_0, f(x_0))$ is

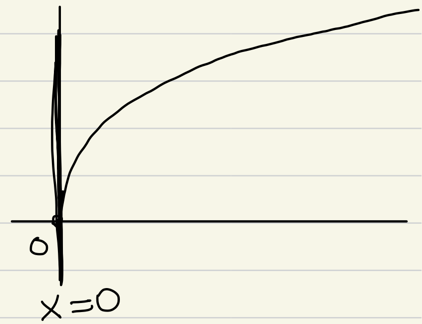
$$\boxed{y = f'(x_0) \cdot (x - x_0) + f(x_0)}$$

If f is not differentiable at x_0 but

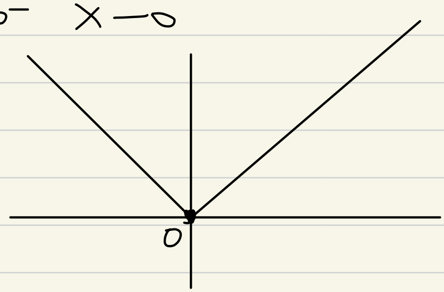
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = +\infty \text{ or } -\infty$$

then we say that the line $x = x_0$ is a <u>vertical tangent</u> of G_f at the point $(x_0, f(x_0))$.

E.g. • if $f(x)=\sqrt{x}$ then
$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=+\infty$$
so the line $x=0$ is a vertical tangent of G_f.

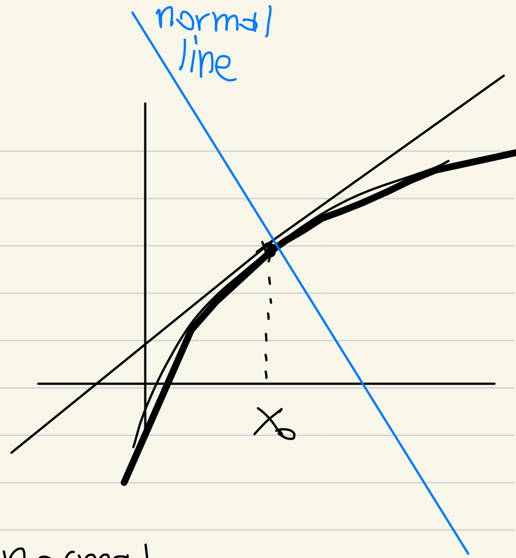


• if $g(x)=|x|$ then
$$\lim_{x\to 0^+}\frac{g(x)-g(0)}{x-0}=1\ ,\quad \lim_{x\to 0^-}\frac{g(x)-g(0)}{x-0}=-1$$
and the graph of g does not have a tangent line at the point $(0,0)$.



If f is diff. at x_0 then the function
$$x\mapsto f'(x_0)(x-x_0)+f(x_0)$$
is called the "best linear approximation" to f near x_0, in the sense that its values are sufficiently close to the values of f in some small interval around x_0.

If the graph of f has a tangent line at $P(x_0, f(x_0))$, then we define the <u>normal</u> line of G_f at $P(x_0, f(x_0))$ to be the line through P which is vertical to the tangent.



- If $f'(x_0) \neq 0$, the normal line at P has slope

$$\lambda = -\frac{1}{f'(x_0)}$$

- If $f'(x_0) = 0$ the tangent line is horizontal and the normal line is vertical.

- If the tangent line is vertical then the normal line is horizontal.

Eg $f(x) = x^2$

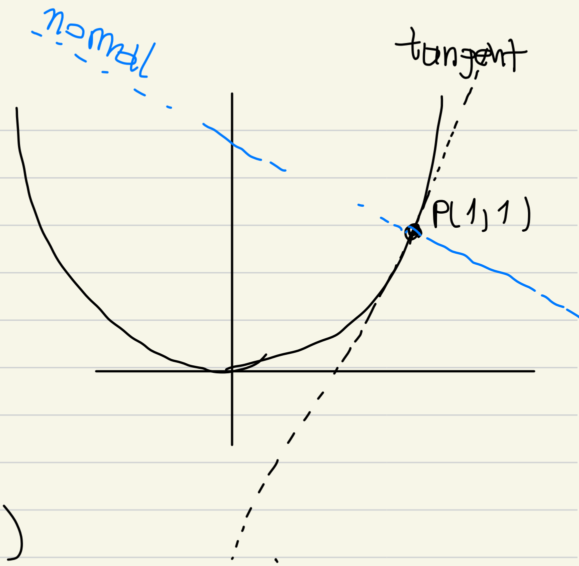
$f'(x) = 2x$

So $f'(1) = 2$ and the tangent at $(1,1)$ is

$y - f(1) = f'(1)\cdot(x-1) \Rightarrow$

$y - 1 = 2(x-1) \Rightarrow$

$$\boxed{y = 2x - 1}$$



The normal at $(1,1)$ is vertical to the tangent so it has slope

$$\lambda = -\frac{1}{2}.$$

Since it goes through $(1,1)$ its equation is

$$y - 1 = -\frac{1}{2}(x-1) \Rightarrow$$

$$\boxed{y = -\frac{1}{2}x + \frac{3}{2}}$$

Let $f : I \to \mathbb{R}$ where I is an interval and $x_0 \in I$.

We say that f has:

(i) a <u>local minimum</u> at x_0 if $f(x) \geq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap I$ for some $\delta > 0$.

(ii) a <u>local maximum</u> at x_0 if $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap I$ for some $\delta > 0$.

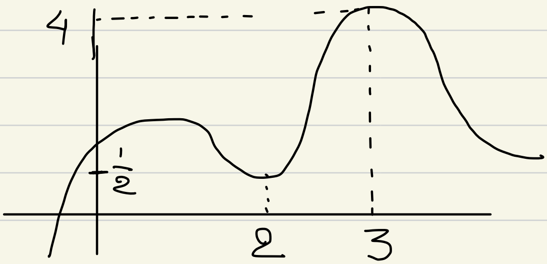
(iii) a <u>(global) minimum</u> at x_0 if $f(x) \geq f(x_0)$ for all $x \in I$.

(iv) a <u>(global) maximum</u> at x_0 if $f(x) \leq f(x_0)$ for all $x \in I$.

Local or global minima / maxima are called <u>extrema</u> of f.

In the previous definitions, x_0 is a position of an extremum of f and the extremum of f is the number $f(x_0)$.

E.g. • f has a local minimum at 2, which is $f(2) = \frac{1}{2}$.



• f has a local maximum at 3, which is $f(3) = 4$. This is also a global maximum.

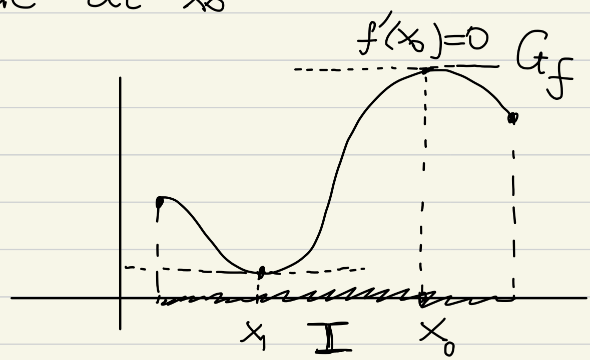
<u>REMARK</u>: Global extrema are also (trivially) local.

THEOREM 3.3 (Fermat) : Let $f: I \to \mathbb{R}$, I an interval and $x_0 \in I$. If

- x_0 is an internal point of I
- f has a local max. or min. at x_0
- f is differentiable at x_0

then

$$f'(x_0) = 0.$$



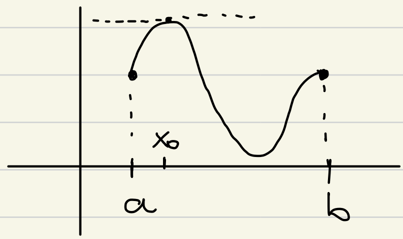
According to Fermat's Theorem the potential positions of local minima and maxima of are:

1. points $x \in I$ where $f'(x) = 0$ (these are called <u>critical points</u> of f)
2. points $x \in I$ where f is not differentiable
3. the endpoints of I (if they belong to the interval I).

THEOREM 3.4 (Rolle): Let $f:[a,b] \to \mathbb{R}$. If

- f is continuous on $[a,b]$.
- f is differentiable on (a,b)
- $f(a) = f(b)$

then there exists $x_0 \in (a,b)$ such that $f'(x_0) = 0$.



PROOF

Since $f:[a,b] \to \mathbb{R}$ is continuous, by the Heine-Borel Theorem there exist $x_1, x_2 \in [a,b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in [a,b].$$

Consider two cases:

I. If f is constant on $[a,b]$.
The conclusion is trivially true.

II. If f is not constant on $[a,b]$.
Then one of x_1, x_2 (let's say x_1) must be an internal point of $[a,b]$.
Then by Fermat's theorem, $f'(x_1) = 0$. ∎