Exercise 6

Problem 1:

(a) $d: \mathbb{R}\times\mathbb{R}\to\mathbb{R}_{\geq 0}$

$d(x,y)=(x-y)^2$

• Positivity:

$d(x,y)=(x-y)^2\geq 0$ O.K.

• Symetry:

$d(x,y)=(x-y)^2$

$=x^2-2xy+y^2$

$=y^2-2xy+x^2$

$=d(y,x)$ O.K.

• Triangle inequality:

$d(x,z)=(x-z)^2$

$=(x-y+y-z)^2$

$\leq(x-y)^2+(y-z)^2$

$=d(x,y)+d(y,z)$ O.K.

d is a metric.

(b) $d: \mathbb{R}\times\mathbb{R}\to\mathbb{R}_{\geq 0}$

$d(x,y)=\sqrt{|x-y|}$

• Pos.:

$d(x,y)=\sqrt{|x-y|}\geq 0$ because $|x-y|\geq 0$ O.K.

• Symm.:

$d(x,y)=\sqrt{|x-y|}$

$=\sqrt{|y-x|}$

$=d(y,x)$ O.K.

• Tri. ineq.:

$d(x,z)=\sqrt{|x-z|}$

$=\sqrt{|x-y+y-z|}$

$\leq\sqrt{|x-y|+|y-z|}$

$\leq\sqrt{|x-y|}+\sqrt{|y-z|}$

$=d(x,y)+d(y,z)$ O.K

d is a metric.

(c) $d: \mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}_{\geq 0}$

$d(x,y)=\min\{|x_1-y_1|, |x_2-y_2|\}$

• Pos.:

$d(x,y)=\min\{|x-y_1|,|x_2,y_2|\}\geq 0$ O.K.

• Symm.:

$d(x,y)=\min\{|x-y_1|,|x_2,y_2|\}$

$=\min\{|y_1-x_1|,|y_2-x_2|\}$

$=d(y,x)$ O.K.

• Tri. ineq.:

$d(x,z)=\min\{|x_1-z_1|,|x_2-z_2|\}$

$=\min\{|$

(d) $d: \mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}_{\geq 0}$

$d(x,y)=|x_1-y_1|+|x_2-y_2|$

• Pos.:

$d(x,y)=|x_1-y_1|+|x_2-y_2|\geq 0$ O.K.

• Symm.:

$d(x,y)=|x_1-y_1|+|x_2-y_2|$

$=|y_1-x_1|+|y_2-x_2|$

$=d(y,x)$ O.K.

• Tri. ineq.:

$d(x,z)=|x_1-z_1|+|x_2-z_2|$

$=|x_1-y_1+y_1-z_1|+|x_2-y_2+y_2-z_2|$

$\leq|x_1-y_1|+|y_1-z_1|+|x_2-y_2|+|y_2-z_2|$

$=d(x,y)+d(y,z)$ O.K.

d is a metric

Problem 2:

$X=\mathbb{R}$

$d(x,y)=|x-y|$

$E\subset\mathbb{R}$

(a) $E=[0,1)$

Interior:

$E^\circ=(0,1)$

Closure:

$\overline{E}=[0,1]$

Boundary:

$\partial E=\{0,1\}$

(b) $E=\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\}$

Interiour:

$E^\circ=(1,\frac{1}{2})\cup(\frac{1}{2},\frac{1}{3})\cup(\frac{1}{3},\frac{1}{4})\cup\dots$

Closure:

$\overline{E}=[1,\infty)$

Boundary:

$\partial E=E$

(c) $E=\mathbb{Q}$

Interiour:

$E^\circ=\emptyset$

Closure:

$\overline{E}=\mathbb{R}$

Boundary:

$\partial E=\mathbb{R}\setminus\emptyset$

Problem 3:

(X,d) metric space

$x\in X$

$\varepsilon>0$

(a) $B_\varepsilon(x)=\{y\in X \mid d(x,y)<\varepsilon\}$

We pick $y\in B_\varepsilon(x)$, and let $d(x,y)=h<\varepsilon$

$\Rightarrow B_{\varepsilon-h}(y)=\{z\in X \mid d(y,z)<\varepsilon-h\}$

$z\in B_{\varepsilon-h}(y)$

$\Rightarrow d(x,z)\leq \underbrace{d(x,y)}_{=h}+\underbrace{d(y,z)}_{<\varepsilon-h}$

$<h+\varepsilon-h$

$=\varepsilon$

$\Rightarrow d(x,z)<\varepsilon \Rightarrow z\in B_\varepsilon(x)$

$\Rightarrow B_{r-h}(y)\subset B_\varepsilon(x)$

y is in the interior of $B_\varepsilon(x)$, so $B_\varepsilon(x)$ is an open set

(b) $\overline{B}_\varepsilon(x)=\{y\in X \mid d(x,y)\leq\varepsilon\}$

We pick $y\in\overline{B}_\varepsilon(x)$, and let $d(x,y)=h\leq\varepsilon$

$\Rightarrow \overline{B}_{\varepsilon-h}(y)=\{z\in X \mid d(y,z)\leq\varepsilon-h\}$

$z\in\overline{B}_{\varepsilon-h}(y)$

$\Rightarrow d(x,z)\leq \underbrace{d(x,y)}_{=h}+\underbrace{d(y,z)}_{\leq\varepsilon-h}$

$\leq h+\varepsilon-h$

$\leq\varepsilon$

$\Rightarrow d(x,z)\leq\varepsilon \Rightarrow z\in\overline{B}_\varepsilon(x)$

$\Rightarrow \overline{B}_{h-\varepsilon}(y)\subset\overline{B}_\varepsilon(x)$

Problem 4:

Anta $x>y$

$d(x,y)=|\arctan(x)-\arctan(y)|$

$=\int_y^x \frac{1}{x^2+1}dx$

$\arctan(x)$ is bounded in $(-\frac{\pi}{2},\frac{\pi}{2})$, but since it goes to $x\to\infty$, $d(x,y)\to\infty$ if $x\gg y$.

Therefore this is not a complete metric space

Problem 5:

(X,d) metric space.

$\{x_n\}_{x\in\mathbb{N}}$ Cauchy.

Assume there exits $\{x_{n_k}\}_{k\in\mathbb{N}}$ og $x\in X$ such that $\lim_{k\to\infty} x_{n_k}=x$

Since $d(x,x_{n_k})<\varepsilon$, $n_k\geq k>N$ there must exist a N such that $d(x,x_n)<\varepsilon$, $\forall n\geq N$, $\forall\varepsilon>0$