Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4295 Statistical inference
Fall 2023

Exercise set 6

Read the questions carefully and make your own assumptions if needed.

1 Casella-Berger

7 46, 47, 49

2 Exponential families

Let $x = (x_1, \ldots, x_n)$ be a random sample from an exponential family with density

$$f(y) = g(y)e^{\theta y - \gamma} \tag{1}$$

with respect to a reference measure $\mu(dy)$.

a) Let the range of models $\mathrm{R}(\Theta) \subset \Omega_\Theta = \mathbb{R}^k$ be the largest possible that allows normalization of f. Is $\mathrm{R}(\Theta)$ convex and open? Hint: Consider first $\mu(dy) = dy$.

b) Show that $\gamma = \gamma(\theta)$ is determined by g and normalization. Prove

$$\mathrm{E}\,Y = \gamma' \tag{2}$$

and that the MLE $\hat{\theta}$ is determined by $y = \gamma'(\hat{\theta})$. Is γ convex? Hint: Show $\gamma'' \geq 0$ by considering second moments of Y.

c) Calculate the Fisher information $\iota = \iota(\theta)$ for Y and for X.

d) What is the Jeffreys prior for θ? What is the Cramer-Rao lower bound for an unbiased estimator of each of the components of θ? When is the lower bound obtained? Hint: Equality in the Cauchy-Schwarz inequality.

e) Find a formula for the Kullbach-Leibler divergence $D(\theta \parallel \theta_0)$ and simplify if possible. Do the same for the metric distance $\mathsf{d}(\theta, \theta_0)$ from the Fisher information metric $g_{ij} = \iota_{ij}$.

f) Calculate explicit expressions, if possible, for all of the above for the cases where the density f is the exponential, the gamma (3 cases), and the normal distribution (3 cases). Hint: Use canonical parameters, and change to conventional parameters only in a final step.

$\boxed{3}$ Posterior for the normal distribution

Let $x_1, \ldots, x_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$.

a) Let the prior be $\mathrm{P}_\Theta(d\mu, d\sigma) = \sigma^{-1}\, d\mu d\sigma$. What is the posterior $\pi_1(\mu, \sigma) = \pi(\mu, \sigma \,|\, x)$ when $n = 1$? Let π_1 be the prior and calculate the posterior when observing x_2 in a single experiment. Compare this posterior with the posterior π_2 from the prior P_Θ and data with $n = 2$. Formulate and prove a generalization of the previous results for a general n. Hint: The normalization constants are more easily calculated using the results in the next task.

b) Prove that the posterior from the right Haar prior P_Θ is given by a gamma distribution for the precision $\lambda = \sigma^{-2}$ and a normal distribution for $\mu \,|\, \lambda$. Hint: Use fiducial posterity, $\overline{x} = \mu + \sigma \overline{z}$ and $s^2 = \sigma^2 s_z^2$, and Basu theorem.

c) Express the normal distribution on standard exponential family form $g(x)\exp(\theta t - \gamma)$. Determine the prior and posterior distribution for θ from the above results. What is the corresponding natural conjugate family of priors? Is the conjugate family an exponential family?

d) Determine the density of $a + b\Theta$. How is this relevant here?

e) Reconsider the above problems for the case when either μ or σ is known.

$\boxed{4}$ Natural conjugate family of priors

Reconsider all relevant sub-problems in $\boxed{3}$ when $\mathrm{N}(\mu, \sigma^2)$ is replaced by the indicated families of distributions.

a) The exponential $\mathrm{Exp}(\beta)$ with prior $\pi(\beta) = 1/\beta$. Hint: Use sufficiency and parameter $\lambda = 1/\beta$.

b) The gamma $\mathrm{G}(\alpha, \beta)$. Hint: Use sufficiency and parameter $\lambda = 1/\beta$. Determine natural initial prior from the limiting case of zero observations starting from the natural conjugate family of priors.

c) The Bernoulli $\mathrm{B}(p)$ with prior $\pi(\beta) = p^{-1}(1-p)^{-1}$. Hint: Use sufficiency and parameter $\eta = \ln(p/(1-p))$.

d) The uniform $\mathrm{U}(a, b) = \mu + \sigma\,\mathrm{U}(1, 2)$ with prior $\pi(\mu, \sigma) = 1/\sigma$. Hint: This is not an exponential family, but still a natural conjugate family of priors can be found.

$\boxed{5}$ Casella-Berger

7 51, 52, 54, 57, 59