Assume a non-stochastic option pricing model in which the price of a stock option depends on time $t$ and stock price $S$:

$$O = F(t, S)$$

What is the change in O as a result of marginal changes in the determinants $t$ and $S$?

- we calculate that with the total differential:

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When $O$ and $S$ are modelled as stochastic processes, i.e.

- dynamic processes
- with a random (Brownian motion) term in them

this is no longer correct.

Brownian motion gives fluctuations, like deviation $\varepsilon$ in interest calculations
The equation becomes:

$$dO = \underbrace{\frac{\partial O}{\partial t}dt + \frac{\partial O}{\partial S}dS}_{\substack{\textit{normal}\\ \textit{calculus}}} + \underbrace{\frac{1}{2}\frac{\partial^2 O}{\partial S^2}(dS)^2}_{\substack{\textit{stochastic}\\ \textit{calculus}}}$$

Look at where the additional term comes from

- start with model of stock price dynamics
- then introduce Itô's lemma

Black and Scholes model of stock price dynamics:

$$dS_t = \mu S_t dt + \sigma S_t d\widetilde{W}_t$$
$$S_0 > 0$$

where:
$d =$ next instant's incremental change
$S_t =$ stock price at time $t$
$\mu =$ drift coefficient, representing the stock's return
$\sigma =$ diffusion coefficient, representing the stock's volatility
$\widetilde{W} =$ standard Brownian motion
$S_0 =$ initial condition (a process has to start somewhere)

Look at the ingredients

Standard Brownian motion:

- continuous time analogue of random walk:
- series of very small (infinitesimal) steps
- each step drawn randomly from standard normal distribution

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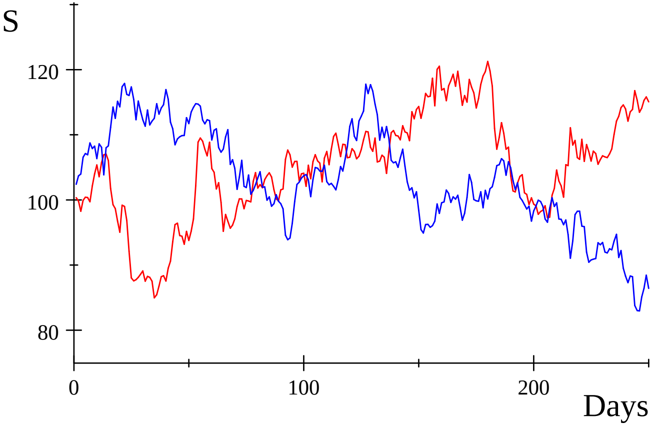
Formally, a process $\widetilde{W}$ is standard Brownian motion if:

- $\widetilde{W}_t$ is continuous and $\widetilde{W}_0 = 0$,
- it has independent increments
- increments $\widetilde{W}_{s+t} - \widetilde{W}_s$ are $\backsim N(0, \sqrt{t})$
- means that increments are stationary, i.e. only function of time interval $t$ not of location $s$.

Recall that Brownian motion has remarkable properties:

- increments drawn from normal distribution
  - it has no upper or lower bounds
  - behaves 'wildly', will eventually hit any barrier, no matter how high/low
- It is continuous everywhere, but differentiable nowhere
  - a truly remarkable combination
  - never smooths out
  - if motion's scale is compressed/stretched, picture remains jagged (as in next slide)
- Is a fractal.




Sample paths of geometric Brownian motion with $\mu = 0.15$, $\sigma = 0.3$ and T=250

Re-write a bit to clarify:

$$\frac{S_{t+dt} - S_t}{S_t} = \mu dt + \sigma d\widetilde{W}_t$$

- $(S_{t+dt} - S_t)/S_t$ stock return over time interval $dt$
- $\mu dt$ continuously compounded expected return over time interval $dt$ (comparable with (1+r)$^T$ in discrete time)
- $d\widetilde{W}_t$ is random element ('surprise') in stock return
- $\sigma$ stock's volatility (s.d.) to scale the random element (some stocks are more volatile than others)

The Brownian motion term makes Itô calculus necessary

Itô's lemma allows manipulation stochastic differential equations
It states that:

- if a process $Z$ has a stochastic differential of the form

$$dZ = a(Z, t)dt + b(Z, t)dW$$

- where $dW$ is a standard Brownian motion
- and both drift coefficient $a$ and diffusion coefficient $b$ are functions of $Z$ and $t$
- then a function $H$ of $Z$ and $t$ (i.e. $H_t = f(t, Z_t)$) follows the process:

$$dH = \left( \frac{\partial H}{\partial Z} a(Z, t) + \frac{\partial H}{\partial t} + \frac{1}{2} \frac{\partial^2 H}{\partial Z^2} b^2(Z, t) \right) dt + \frac{\partial H}{\partial Z} b(Z, t) dW$$

Requires $H$ to be twice differentiable w.r.t. $Z$ and once w.r.t. $t$
Formal proof of Itô's lemma is complicated

Result can be understood as a second order Taylor expansion:

$$dH = \frac{\partial H}{\partial t}dt + \frac{\partial H}{\partial Z}dZ + \frac{1}{2}\frac{\partial^2 H}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 H}{\partial Z^2}(dZ)^2 + \frac{\partial^2 H}{\partial Z \partial t}dtdZ$$

with the following multiplication rules:

$$\begin{aligned} (dt)^2 &= 0 \\ dt \times dW &= 0 \\ (dW)^2 &= dt \end{aligned}$$

The last multiplication rule was very difficult to prove
means not all quadratic terms can be ignored

Equation can be simplified:

- first multiplication rule means $3^{rd}$ term RHS is 0
- Our original function was

$$dZ = a(Z,t)dt + b(Z,t)dW$$

  so that

$$dtdZ = a(Z,t)\,(dt)^2 + b(Z,t)dWdt = 0$$

  rules mean that $5^{th}$ term RHS is 0

Result simplifies to:

$$dH = \frac{\partial H}{\partial t}dt + \frac{\partial H}{\partial Z}dZ + \frac{1}{2}\frac{\partial^2 H}{\partial Z^2}(dZ)^2$$

That is where the third term comes from!

How do we get Itô's lemma from this result?

1. substitute $dZ = a(Z,t)dt + b(Z,t)dW$
2. calculate $(dZ)^2$ and substitute
3. simplify result

$$\begin{aligned}(dZ)^2 &= a^2(Z,t)(dt)^2 + b^2(Z,t)(dW)^2 + 2a(Z,t)b(Z,t)(dt)(dW)\\ &= 0 + b^2(Z,t)dt + 0\end{aligned}$$

substituting all this:

$$dH = \frac{\partial H}{\partial t}dt + \frac{\partial H}{\partial Z}\left(a(Z,t)dt + b(Z,t)dW\right) + \frac{1}{2}\frac{\partial^2 H}{\partial Z^2}b^2(Z,t)dt$$

$$dH = \left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial Z}a(Z,t) + \frac{1}{2}\frac{\partial^2 H}{\partial Z^2}b^2(Z,t)\right)dt + \frac{\partial H}{\partial Z}b(Z,t)dW$$

$$
\begin{array}{ccccc}
\text{Binomial model} & & & & \text{Calculus of variations} \\
\downarrow & & & & \downarrow \\
\text{Risk free portfolio} & & & & \text{stochastic calculus} \\
& \searrow & & \swarrow & \\
& & \text{Black \& Scholes formula} & &
\end{array}
$$

# How to make risk disappear

## Black and Scholes' Nobel prize winning breakthrough

Combining the elements we have:
Price process of a stock (under objective probability measure):

$$dS_t = \mu S_t dt + \sigma S_t d\widetilde{W}_t$$

A derivative is written on the stock such that

- derivative price is function of:
  - the price of the stock
  - and time

$\Rightarrow$ can use Itô's lemma to find derivative's stochastic process

Call the derivative $H$, its price at time $t$ is $H_t$
its stochastic price process is by Itô's lemma:

$$dH_t = \left( \frac{\partial H_t}{\partial S_t} \mu S_t + \frac{\partial H_t}{\partial t} + \frac{1}{2} \frac{\partial^2 H_t}{\partial S_t^2} \sigma^2 S_t^2 \right) dt + \frac{\partial H_t}{\partial S_t} \sigma S_t d\widetilde{W}_t$$

Two elements are worth noticing about this equation:

1. Nature of the derivative (put, call, Eur., Am.) not specified, so it is a very general relation
2. Brownian motions in $dS_t$ and $dH_t$ are the same

Means it is possible to make portfolio of $S$ and $H$ from which Brownian motion disappears

Call that portfolio $V$, what is its composition?
can guess (or recall from analyses in discrete time):

- a short position in the derivative
- long position in a fraction $\Delta$ of the stock:

$$V = -H + \Delta S$$

portfolio dynamics are:

$$dV_t = -dH_t + \Delta dS_t$$

We have expressions for $dS_t$ and $dH_t$; substituting them gives a long equation

$$dV_t = -\left(\frac{\partial H_t}{\partial S_t}\mu S_t + \frac{\partial H_t}{\partial t} + \frac{1}{2}\frac{\partial^2 H_t}{\partial S_t^2}\sigma^2 S_t^2\right) dt - \frac{\partial H_t}{\partial S_t}\sigma S_t d\widetilde{W}_t + ..$$
$$.. + \Delta\mu S_t dt + \Delta\sigma S_t d\widetilde{W}_t$$

$$dV_t = -\left(\frac{\partial H_t}{\partial S_t}\mu S_t + \frac{\partial H_t}{\partial t} + \frac{1}{2}\frac{\partial^2 H_t}{\partial S_t^2}\sigma^2 S_t^2\right) dt - \frac{\partial H_t}{\partial S_t}\sigma S_t d\widetilde{W}_t + ..$$
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How do we eliminate risk? What is risk? What variable do we give a value? (Black and Scholes' Aha-erlebnis)

$$dV_t = -\left(\frac{\partial H_t}{\partial S_t}\mu S_t + \frac{\partial H_t}{\partial t} + \frac{1}{2}\frac{\partial^2 H_t}{\partial S_t^2}\sigma^2 S_t^2\right) dt - \frac{\partial H_t}{\partial S_t}\sigma S_t d\widetilde{W}_t + ..$$
$$.. + \Delta\mu S_t dt + \Delta\sigma S_t d\widetilde{W}_t$$

How do we eliminate risk? What is risk? What variable do we give a value? (Black and Scholes' Aha-erlebnis)

- Risk is the Brownian motion $d\widetilde{W}_t$
- we choose the fraction of the stock $\Delta$
- Brownian motion terms cancel out if $\Delta = \partial H_t / \partial S_t$ (which is our old friend the option delta)
- with $\Delta = \partial H_t / \partial S_t$ the $\mu S_t dt$ terms also cancel out

Portfolio dynamics reduce to:

$$dV_t = -\left(\frac{\partial H_t}{\partial t} + \frac{1}{2}\frac{\partial^2 H_t}{\partial S_t^2}\sigma^2 S_t^2\right) dt$$

- process contains only a drift term, no diffusion
- $\Rightarrow$ portfolio is riskless
- in arbitrage free market, return is risk free interest rate $r$ :

$$dV_t = rV_t dt$$

- Substitute for $dV_t$ in left hand side
- portfolio definition $V_t = -H_t + (\partial H_t/\partial S_t)S_t$ into the right hand

$$
-\left(\frac{\partial H_t}{\partial t} + \frac{1}{2}\frac{\partial^2 H_t}{\partial S_t^2}\sigma^2 S_t^2\right) dt = r\left(-H_t + \frac{\partial H_t}{\partial S_t} S_t\right) dt
$$

Dropping $dt$ and rearranging terms gives:

$$
\frac{\partial H_t}{\partial t} + \frac{1}{2}\frac{\partial^2 H_t}{\partial S_t^2}\sigma^2 S_t^2 + \frac{\partial H_t}{\partial S_t} S_t r - rH_t = 0
$$

This is the famous Black and Scholes partial differential equation

- or in short: *Black and Scholes equation*
- not be confused with *Black and Scholes formula*

Notice: return $\mu$ not included, only volatility $\sigma$

All derivatives described by function $H_t$

- must satisfy the Black and Scholes equation
- if they are to be arbitrage free

Solutions to Black and Scholes equation are functions whose partial derivatives satisfy the equation

- there are many such functions
- mathematical way of saying it is valid for different derivatives

To find solution for a particular derivative we:

- use derivative's characteristics
- formulate them as *boundary conditions*

For example:

- when stock price is 0 it will stay 0 (multiplicative model)
  - value of a European call is then also 0
- mathematically this gives a boundary condition:
  - when $S_t = 0$ then $H_t = 0$

Another example:

- we know call's payoff at maturity:
  - maximum of (stock price - exercise price) or zero
- gives boundary condition:
  - $H_T = \max[0, S_T - X]$
  - used by Black and Scholes

The combination of

- a partial differential equation
- boundary condition(s)

is called *boundary value problem*

This problem is well posed, i.e. a unique solution exists
solved by change of variable transformation, e.g.:

- using temporary variables for complex expressions
- taking logarithms to transforms variable coefficient equation into constant coefficient equation

Needs several pages of mathematics, no economics involved

- We follow Black and Scholes good example
- refer to literature for solution to boundary value problem

Solution is the celebrated Black and Scholes formula for the value of a European call:

$$O_{c,0} = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

with the corresponding value of a European put:

$$O_{p,0} = Xe^{-rT} N(-d_2) - S_0 N(-d_1)$$

where:

$$d_1 = \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{\ln(S_0/X) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$