NTNU | Norwegian University of Science and Technology

PLENARY EXERCISES - TMA4145

Week 36, Wednesday 6. September 2023

Problem 1

Let U and W be subspaces of a vector space V. Show that

$$U + W = U \oplus W,$$

if and only if

$$\dim(U + W) = \dim(U) + \dim(W).$$

Hint:

1. Can use $\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$.
2. What can we say about $U \cap W$?

Problem 2

Let $T : X \to X$ be a linear operator.

a) Show that the image of T, $\mathrm{im}(T)$, is T-invariant.

Let U and V be T-invariant subspaces of X.

b) Show that $U \cap V$ is T-invariant.

c) Show that $U + V$ is T-invariant.

Hint:

1. A subset A of X is called T-invariant if $T(A) \subseteq A$.
2. $x \in U \cap V$ if and only if $x \in U$ and $x \in V$.
3. $x \in U + V$ if and only if $x = u + v$ for $u \in U$ and $v \in V$.

Problem 3

Let $T : V \to V$ be a linear operator. Let v be an eigenvector of T with eigenvalue λ. Show that v is an eigenvector of $p(T)$ for any polynomial $p \in \mathcal{P}$, and find the corresponding eigenvalue.

Hint:

1. For a polynomial $p(x) = \sum_{i=0}^{n} c_i x^i$, the operator $p(T) = \sum_{i=0}^{n} c_i T^i$.
2. The operator $T^0 = I$ is the identity operator.
3. What happens for $p(x) = x^2$?