NTNU | Norwegian University of Science and Technology

PLENARY EXERCISES - TMA4145

Week 41, Wednesday 11. October 2023

Problem 1

Assume $G : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant L, and that for every $x, y \in \mathbb{R}^n$

$$\langle G(x) - G(y), x - y \rangle \leq 0.$$

1. Show that that G has a unique fixed point x^* and that the iteration

$$x_{n+1} = (1 - \lambda)x_n + \lambda G(x_n),$$

   converges to x^* for all sufficiently small $\lambda > 0$.

Hint:

1. Consider the map $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x) = (1 - \lambda)x + \lambda G(x)$.
2. Show that any fixed point of T is also a fixed point of G.
3. It is benefical to consider $|T(x) - T(y)|^2$ when showing that it is a contraction.

Problem 2

Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be such that

1. Assume there exists $0 < c_1 < 0$ such that

$$\max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{i,j}| \leq c_1 < 1.$$

   Show that A is a contraction for $\|\cdot\|_\infty$.

2. Assume there exists $0 < c_2 < 0$ such that

$$\max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{i,j}| \leq c_2 < 1.$$

   Show that A is a contraction for $\|\cdot\|_1$.

3. Show that neither cases necessarily gives that A is a contraction for $\|\cdot\|_2$.

Hint:

1. For matrix multiplication we have $(Av)_i = \sum_{j=1}^{n} a_{i,j} v_j$.
2. For problem 3, you might consider 2×2 matrices.

Problem 3

Let $g \in C([0,1])$ be such that $\|g\|_\infty \le 1/2$.

1. Show that there exists a unique $f \in C[0,1]$ with $\|f\|_\infty \le 1$ such that

$$f = \frac{1}{4} f * f + g$$

   where

$$f * h(x) = \int_0^x f(y)h(x-y)\,dy.$$

Hint:

1. Show $K = \{f \in C([0,1]) : \|f\|_\infty \le 1\}$ is a closed subset.
2. Show that

$$f * h(x) = \int_0^x f(y)h(x-y)\,dy$$

   is a contraction on K.
3. It might be benefical to add 0 in a clever way for hint 2.

Problem 4

We denote the space of bounded continuous functions on $\mathbb{R}$ by

$$C_b(\mathbb{R}) := \{f \in C(\mathbb{R}) : \|f\|_\infty < \infty\}.$$

1. Show that $(C_b(\mathbb{R}), \|\cdot\|_\infty)$ is a complete metric space.

Hint:

1. $d_\infty(f, g) = \|f - g\|_\infty = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$.
2. You may use that the uniform limit of a sequence of continuous functions is continuous.
3. The strategy for proving completeness: 1) find a suitable candidate for the limit. 2) show that the limit is in C_b. 3) Show that the limit converges to the candidate.