Exercise #4

September 13, 2023

Problem 1.

Assume that U is a finite dimensional vector space and that V is an inner product space with inner product $\langle \cdot, \cdot \rangle_V$. Moreover, let $T: U \to V$ be linear and injective.

We define the mapping $\langle \cdot, \cdot \rangle_U : U \times U \to \mathbb{K}$,

$$\langle u, v \rangle_U := \langle Tu, Tv \rangle_V.$$

Show that the mapping $\langle \cdot, \cdot \rangle_U$ is an inner product on U.

Solution.

We neet to show that $\langle \cdot, \cdot \rangle_U$ satisfies the three conditions of an inner product.

a) Let us start by showing linearity in the first coordinate. Let $u, v, w \in U$ and $\alpha, \beta \in \mathbb{K}$. Then, since T is linear, it follows that

$$\begin{aligned}\langle \alpha u + \beta v, w \rangle_U = \langle T(\alpha u + \beta v), T(w) \rangle_V &= \langle \alpha T(u) + \beta T(v), T(w) \rangle_V \\ &= \alpha \langle T(u), T(w) \rangle_V + \beta \langle T(v), T(w) \rangle_V \\ &= \alpha \langle u, w \rangle_U + \beta \langle v, w \rangle_U.\end{aligned}$$

Here we used the linearity of $\langle \cdot, \cdot \rangle_V$. This shows that $\langle \cdot, \cdot \rangle_U$ is linear in the first coordinate.

b) We need to show conjugate symmetry of $\langle \cdot, \cdot \rangle_U$. Since $\langle \cdot, \cdot \rangle_V$ is an inner product, we have for any $u, v \in U$,

$$\langle u, v \rangle_U = \langle T(u), T(v) \rangle_V = \overline{\langle T(v), T(u) \rangle_V} = \overline{\langle v, u \rangle_U},$$

which proves conjugate symmetry.

c) The last property we need to check is positive definiteness. Let $u \in U$. Then

$$\langle u, u \rangle_U = \langle T(u), T(u) \rangle_V \geq 0.$$

Moreover, $T(u) = 0$ if and only if $u = 0$ as T is injective. Thus, it follows from the positive definiteness of $\langle \cdot, \cdot \rangle_V$ that

$$\langle u, u \rangle_U = \langle T(u), T(u) \rangle_V = 0,$$

if and only if $T(u) = 0$, which is only when $u = 0$.

Combining this three property shows that $\langle \cdot, \cdot \rangle_U$ is an inner product on U.

Problem 2.

Denote by $\mathcal{P}_n$ the space of polynomials of degree $\leq n$ with complex coefficients. Moreover, let $X := \{x_0, x_1, \dots, x_n\} \subset \mathbb{R}$ be a set of $n+1$ distinct points. We define the mapping $T \colon \mathcal{P}_n \to \mathbb{C}^{n+1}$,

$$Tp := \begin{pmatrix} p(x_0) \\ p(x_1) \\ \vdots \\ p(x_n) \end{pmatrix}.$$

a) Show that the mapping T is linear and injective.

   Hint: Recall that a non-zero polynomial of degree $\leq n$ can have at most n zeroes.

b) On $\mathcal{P}_n$ we now define the inner product $\langle p, q \rangle_X := \langle Tp, Tq \rangle_{\mathbb{C}^{n+1}}$, where $\langle \cdot, \cdot \rangle_{\mathbb{C}^{n+1}}$ is the standard (Euclidean) inner product on $\mathbb{C}^{n+1}$. (According to Problem 1 the mapping $\langle \cdot, \cdot \rangle_X$ is indeed an inner product on $\mathcal{P}_n$.)

   We now define the polynomials (the *Lagrange polynomials* for the set X)

$$p_j(x) := \prod_{\substack{k=0,\dots,n \\ k \neq j}} \frac{x - x_k}{x_j - x_k}.$$

   Show that the set $\{p_0, p_1, \dots, p_n\}$ is an orthonormal basis of $\mathcal{P}_n$ for the inner product $\langle \cdot, \cdot \rangle_X$.

Solution.

a) Let us start by showing that the mapping is linear. Let $p, q \in \mathcal{P}_n$ and $\alpha, \beta \in \mathbb{C}$. Then, by the linearity of vector addition,

$$T(\alpha p + \beta q) = \begin{pmatrix} \alpha p(x_0) + \beta q(x_0) \\ \vdots \\ \alpha p(x_n) + \beta q(x_n) \end{pmatrix} = \alpha \begin{pmatrix} p(x_0) \\ \vdots \\ p(x_n) \end{pmatrix} + \beta \begin{pmatrix} q(x_0) \\ \vdots \\ q(x_n) \end{pmatrix} = \alpha Tp + \beta Tq,$$

   which proves linearity. To prove injectivity, let $p, q \in \mathcal{P}_n$ be such that $Tp = Tq$, then $p - q \in \mathcal{P}_n$ and

$$0 = Tp - Tq = T(p - q) = \begin{pmatrix} p(x_0) - q(x_0) \\ \vdots \\ p(x_n) - q(x_n) \end{pmatrix}.$$

   This means that the polynomial $p - q$ has $n + 1$ roots. However, by the fundamental theorem of algebra, $p - q$ can have at most n roots as it is a polynomial of degree less than or equal to n. Hence, $p - q \equiv 0$, which implies that $p = q$ and T is injective.

b) Note the following fact about p_j,

$$p_j(x_l) = \prod_{\substack{k=0,\dots,n \\ k \neq j}} \frac{x_l - x_k}{x_j - x_k} = \begin{cases} 1, & l = j \\ 0, & l \neq j. \end{cases}$$

This implies that $p_j \mapsto Tp_j = e_j$, where e_j is the standard basis element of $\mathbb{C}^n$. In particular,

$$\langle p_i, p_j \rangle_X = \langle Tp_i, Tp_j \rangle_{\mathbb{C}^{n-1}} = \langle e_i, e_j \rangle_{\mathbb{C}^{n-1}} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

This shows that $\{p_j\}_{j=0}^n$ form an orthonormal set of $\mathcal{P}_n$. Since $Tp_j = e_j$, it also follows that T is surjective. By *a*), T injective and therefore a bijection. Since a bijection maps a basis to a basis, we can conclude that $\{p_j\}_{j=0}^n = \{T^{-1}(e_j)\}_{j=1}^n$ is a basis of $\mathcal{P}_n$.

It is also possible to show that $\{p_j\}_{j=0}^n$ form a basis by showing that the set is linear independent. Namely, consider a linear combination

$$0 = \sum_{j=0}^n c_j p_j.$$

Then, by evaluating the polynomial at the different points x_l, we see that

$$0 = \sum_{j=0}^n c_j p_j(x_l) = c_l.$$

Since the same argument holds for all $0 \leq l \leq n$, we see that $c_0 = \ldots = c_n = 0$. This shows that $\{p_j\}_{j=0}^n$ are linear independent and so they form a basis of $\mathcal{P}_n$, as we have $n+1$ linear independent vectors in an $n+1$ dimensional vector space.

Problem 3.

Assume that U is a finite dimensional inner product space and that U_1 and U_2 are sub-spaces of U. Denote moreover by $\pi_{U_i} : U \to U$ the orthogonal projection onto U_i for $i = 1, 2$.

Show that the identity

$$\pi_{U_1} = \pi_{U_1} \circ \pi_{U_2}$$

holds if and only if $U_1 \subset U_2$.

Solution.

Recall that $U = U_1 \oplus U_1^\perp = U_2 \oplus U_2^\perp$.

Assume $U_1 \subset U_2$, then $U_2^\perp \subset U_1^\perp$. In particular $\pi_{U_1}(w) = 0$ for every $w \in U_2^\perp$. For any $x \in U$ we can write it as $x = u_1 + u_1^\perp = u_2 + u_2^\perp$. This gives

$$\pi_{U_1}(x) = u_1 = \pi_{U_1}(u_2) + \pi_{U_1}(u_2^\perp) = \pi_{U_1}(u_2) = \pi_{U_1} \circ \pi_{U_2}(x).$$

Since $x \in U$ is arbitrary, it follows that $\pi_{U_1} = \pi_{U_1} \circ \pi_{U_2}$.

Assume $\pi_{U_1} = \pi_{U_1} \circ \pi_{U_2}$. Let $x \in U_2^\perp$, then

$$\pi_{U_1}(x) = \pi_{U_1} \circ \pi_{U_2}(x) = 0.$$

We can therefore conclude that $U_2^\perp \subset U_1^\perp$. This proves that $U_1 \subset U_2$ as $U_i = (U_i^\perp)^\perp$. We can also show this directly. Fix some $u \in U_1$. Then for any $x \in U_2^\perp$ we have

$$\langle u, x \rangle = 0,$$

as $x \in U_1^\perp$. This means that $U_1 \subset (U_2^\perp)^\perp = U_2$ as u was arbitrarily chosen from U_1.

Problem 4.

Recall that the *trace* of a matrix $C \in \mathbb{K}^{n\times n}$, $C = (c_{ij})_{i,j=1,\dots,n}$ is given as

$$\operatorname{tr} C := \sum_{i=1}^{n} c_{ii}.$$

On the space $\mathrm{Mat}_{m,n}(\mathbb{K}) = \mathbb{K}^{m\times n}$ of $(m \times n)$ matrices over $\mathbb{K}$ we define the mapping $\langle\cdot,\cdot\rangle\colon \mathrm{Mat}_{m,n}(\mathbb{K}) \to \mathbb{K}$,

$$\langle A, B\rangle := \operatorname{tr}\big(B^H A\big),$$

where B^H is the Hermitian conjugate of the matrix B.

a) Show that $\langle\cdot,\cdot\rangle$ is an inner product on $\mathrm{Mat}_{m,n}$.

b) Consider the specific case $m = n$ and $\mathbb{K} = \mathbb{R}$. Let $U := \big\{A \in \mathrm{Mat}_n(\mathbb{R}) : A^H = A\big\}$ be the subspace of Hermitian matrices. Find the orthogonal complement $U^\perp$ of U with respect to this inner product.

Solution.

a) We want to show that $\langle\cdot,\cdot\rangle$ is an inner product on $\mathrm{Mat}_{m,n}$. Let us first make a few remarks of the Hermitian conjugate of a matrix B. The Hermitian conjugate is the $n \times m$ matrix where the components are given

$$B^H_{i,j} = \overline{B}_{j,i},$$

where $B_{i,j}$ are the matrix components of B. As such, the components of the matrix product $B^H A$ is therefore given by

$$(B^H A)_{i,j} = \sum_{l=1}^{m} B^H_{i,l} A_{l,j} = \sum_{l=1}^{m} \overline{B}_{l,i} A_{l,j}.$$

This means that

$$\langle A, B\rangle = \operatorname{tr} B^H A = \sum_{i=1}^{n} (B^H A)_{i,i} = \sum_{i=1}^{n}\sum_{l=1}^{m} \overline{B}_{l,i} A_{l,j}.$$

Moreover, note that

$$\overline{(B^H A)_{i,j}} = \overline{\sum_{l=1}^{m} \overline{B}_{l,i} A_{l,j}} = \sum_{l=1}^{m} \overline{A}_{l,j} B_{l,i} = (A^H B)_{j,i}. \tag{1}$$

For linearity, let $A, B, C \in \mathrm{Mat}_{m,n}$ and $\alpha, \beta \in \mathbb{C}$. Then, by the linearity of the matrix product

$$\begin{aligned}\langle \alpha A + \beta C, B\rangle = \sum_{i=1}^{n} \Big(B^H(\alpha A + \beta C)\Big)_{i,i} = \sum_{i=1}^{n} \Big(\alpha B^H A + \beta B^H C\Big)_{i,i} &= \alpha \sum_{i=1}^{n} (B^H A)_{i,i} + \beta \sum_{i=1}^{n} (B^H C)_{i,i} \\ &= \alpha\langle A, B\rangle + \beta\langle C, B\rangle.\end{aligned}$$

For conjugate symmetry, we have by (1),

$$\langle A, B\rangle = \sum_{i=1}^{n} (B^H A)_{i,i} = \sum_{i=1}^{n} \overline{(A^H B)_{i,i}} = \overline{\sum_{i=1}^{n} (A^H B)_{i,i}} = \overline{\langle B, A\rangle}.$$

For positive definiteness we see that

$$\langle A, A\rangle = \sum_{i=1}^{n} (A^H A)_{i,i} = \sum_{i=1}^{n}\sum_{l=1}^{m} \overline{A}_{l,i} A_{l,i} = \sum_{i=1}^{n}\sum_{l=1}^{m} |A_{l,i}|^2 \geq 0,$$

as we are summing over positive elements. Moreover, if the sum is equal to zero, then $|A_{l,i}| = 0$ for all $1 \leq i \leq n$ and $1 \leq l \leq m$. In particular, all components of the matrix is zero, and thus A is the zero matrix. This proves positive definiteness.

b) Let $U = \{A \in \mathrm{Mat}_n(\mathbb{R}) : A^H = A\}$. We want to find $U^\perp$. Let $X \in U^\perp$, and consider the matrix $E^{i,j}$ given by

$$E^{i,j}_{l,k} = \begin{cases} 1, & i = l, j = k, \\ 0, & \text{else}. \end{cases}$$

Then $E^{i,j} + E^{j,i}$ and $E^{i,i}$ are both Hermitian matrices, and

$$0 = \langle X, E^{i,j} + E^{j,i}\rangle = X_{i,j} + X_{j,i}, \quad 0 = \langle X, E^{i,i}\rangle = X_{i,i}.$$

which shows that $X_{i,j} = -X_{j,i}$ and $X_{i,i} = 0$. This implies that $X^H = -X$, and so X has to be skew-symmetric. We conclude that $U^\perp \subset \mathrm{Skew}_n(\mathbb{R})$.

If $X \in \mathrm{Skew}_n(\mathbb{R})$. Then for any $A \in U$, we have

$$\langle X, A\rangle = \sum_{i=1}^{n}\sum_{l=1}^{n} A_{l,i} X_{l,i} = \sum_{i=1}^{n}\sum_{l=1}^{n} A_{i,l} X_{l,i} = -\sum_{l=1}^{n}\sum_{i=1}^{n} A_{i,l} X_{i,l} = -\langle X, A\rangle,$$

which shows that $\langle X, A\rangle = 0$ and so $\mathrm{Skew}_n(\mathbb{R}) \subset U^\perp$. This shows that $U^\perp = \mathrm{Skew}_n(\mathbb{R})$.