Chapter 8: Black and Scholes option pricing - part 1

Exercises - solutions

1. The statement is not correct. The share price is expected to increase during the option's life time, whereas the exercise price remains constant. So if the current share price equals the exercise price, the expected share price at maturity will be larger than the exercise price. The growth rate of the stock depends on the probability measure that is used to model stock prices. Under the equivalent martingale measure, returns are equalized into the risk free interest rate, so that the share grows with r_f. The example in section 8.3.3 of the book illustrates this case. It values an at the money call option on a non-dividend paying share with a current price of €100 and a volatility of 20%. The option's time to maturity is 1 and the risk free interest rate is 10%. The calculations in the book (easily verified with the option price calculator accompanying the book) show that the equivalent martingale probability that the option will be exercised, or $N(d_2)$, is 0.655. Under the real probability measure the share will grow with μ, the expected, continuously compounded return of the share.

2. Under the iid assumption, variance increases with time, so that standard deviation increases with the square root of time. An annual standard deviation of 34% thus gives a daily standard deviation of $\sigma\sqrt{T} = 0.34\sqrt{1/252} = 0.02142$ or 2.14%. In section 3.2.2 (Home-made portfolio optimization) we used the same procedure 'in reverse' to calculate the annual volatility of the stocks in uncle Bob's portfolio from their daily returns.

3. A short position in the shares or a long position in put options on the shares will give a large payoff if a catastrophe happens to the core assets of the company, i.e. its players. According to German media reports, Sergej W. bought 15 000 put options for €78 000 and could have gained up to €3.9m as a result of a large drop in Dortmund's shares. Police were alerted to Sergej W's unusual purchase on the stock exchange by employees of an online broker.
   Selling calls with a high exercise price, so that they will expire out of the money after a disaster, is a correct (albeit less obvious) solution. A long straddle is only half correct, because there is no upside potential to a disaster. A bear spread is only partly correct; it is consistent with the view that a large price decrease is unlikely. A successful bomb attack will have a larger effect than that.

4. Stability over time means that past returns are the best information to assess the distributional properties of the returns. However, they *cannot* be used the predict future returns because they contain no information on the *sequence* of future returns.

5. We use the Black and Scholes option pricing formula:

$$O_{c,0} = S_0 \times N(d_1) - X \times e^{-rT} \times N(d_2)$$

with

$$d_1 = \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2) \times T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

The input data are: $S = 240$, $X = 250$, $\sigma = 25\%$, $r = 6\%$, $T = 1$

$$d_1 = \frac{\ln(240/250) + (0.06 + 0.5 \times 0.25^2) \times 1}{0.25 \times \sqrt{1}} = 0.20171$$

$N(d_1) \rightarrow \mathrm{NormalDist}(0.20171) = 0.579\,93$

$$d_2 = 0.20171 - 0.25 \times \sqrt{1} = -0.04829$$

$N(d_2) \rightarrow \mathrm{NormalDist}(-0.04829) = 0.480\,74$

$$O_{c,0} = 240 \times 0.57993 - 250 \times e^{-0.06 \times 1} \times 0.48074 = 25.997$$

6. The simplest strategy is to buy protective put options with an exercise price of €90 and a time to maturity of 6 months. This allows the investor to profit from her insight that there is a good probability that shares will increase in value over the next six months. It will also lock in a minimum price of €90 when she has to sell in 6 months. Assuming the options are European, the price of such a put is calculated with the Black and Scholes formula:
   $S_0 = 100$, $X = 90$, $r = 0.05$, $\sigma = 0.25$ and $T = 0.5$

$$\begin{aligned} d_1 &= \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ &= \frac{\ln(100/90) + (0.05 + 0.5 \times 0.25^2)0.5}{0.25\sqrt{0.5}} = 0.82582 \end{aligned}$$

   and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.82582 - 0.25\sqrt{0.5} = 0.64904$$

   $\mathrm{NormalDist}(-0.64904) = 0.25816$ and
   $\mathrm{NormalDist}(-0.82582) = 0.20445$ so that the put value is:

$$\begin{aligned} O_{p,0} &= Xe^{-rT}N(-d_2) - S_0N(-d_1) \\ O_{p,0} &= 90 \times e^{-0.05 \times 0.5} \times 0.25816 - 100 \times 0.20445 = 2.22 \end{aligned}$$

   Hence, its costs $1000 \times 2.22 =$ €2220 to set up the position.
   A bull spread and a butterfly spread give no protection against share prices lower than 90, while the investor already has upward potential through the shares. A long straddle is somewhat better because it does provide protection against share price lower than 90.

7. If you exercise the options today, your payoff per option is $\max[0, X - S] = \max[0, 67.5 - 2] =$ € 65.50. At maturity, 9 months from now, this has a value of $65.5 \times e^{0.75 \times 0.08} =$ € 69.55 If you postpone exercising until maturity the payoff cannot be higher than the exercise price, €67.50, because the stock price cannot be lower than 0. Since €69.55 $>$ €67.50 exercising now is the better of the two alternatives. Equivalently, the present value of the maximum payoff at maturity, $67.5 \times e^{-0.75 \times 0.08} =$ € 63.57 is less than the payoff of exercising today: € $63.57 <$ € 65.50.