NTNU | Norwegian University of Science and Technology

PLENARY EXERCISES - TMA4145

Week 46, Wednesday 15. November 2023

Problem 1

Let $(X, \|\cdot\|_X)$ be a real Banach space, and assume that $f \in X^* = L(X, \mathbb{R})$.

1. Show that
$$H_f := \{x \in X : f(x) \geq 0\},$$
is a non-empty, closed and convex subset.
2. Show that H_f is not a subspace if $f \neq 0$.

Hint:

1. H_f is closed if for any convergent sequence $\{x_k\}_{k\in\mathbb{N}}$ converges to some $x \in H_f$.
2. H_f is convex if for all $x, y \in H_f$ and any $0 < t < 1$ we have $tx + (1-t)y \in H_f$.

Problem 2

Which of the following statements are true?

1. Let U, V, W be normed spaces and let $S : U \to V$ and $T : V \to W$ be linear maps. If $S : U \to V$ and $T \circ S : U \to W$ are bounded, then $T : V \to W$ is bounded.
2. Let U, V be normed spaces. If $\{T_n\} \subset L(U, V)$ is a sequence such that $T_n u \to 0$ for all $u \in U$, then $T_n \to 0$ in $L(U, V)$.
3. Let U, V be normed spaces, and $S, T \in L(U, V)$. Assume $W \subseteq U$ is a dense subspace such that $Su = Tu$ for all $u \in W$, then $S = T$.

Hint:

1. What do we know about the kernel of linear operators?
2. Consider $U = \ell^1$ and $V = \mathbb{K}$.
3. What does it mean for a subspace to be dense?

Problem 3

Let $a = (a_1, a_2, \ldots) \in \ell^\infty$ and assume there exists $c > 0$ such that $|a_n| > c$ for all $n \in \mathbb{N}$. Define the linear operator $T_a : \ell^2 \to \ell^2$ by

$$T_a x = (a_1 x_1, 0, a_3 x_3, 0, a_5 x_5, \ldots).$$

1. Show that T_a is bounded on ℓ^2.
2. Find the operator norm of T_a.
3. Determine for which sequences $a \in \ell^\infty$ the operator satisfies $T_a = T_a^2$.
4. Show that $\mathrm{Ran}(T_a)$ is closed.
5. Determine the orthogonal complement of $\mathrm{Ker}(T_a)$.

Hint:

1. T_a is bounded if there exists $C > 0$ such that $\|T_a x\|_{\ell^2} \leq C\|x\|_{\ell^2}$ for all $x \in \ell^2$.
2. The operator norm is defined as $\|T\| = \sup_{\|x\|_{\ell^2}=1} \|T_a x\|_{\ell^2}$.
3. $\mathrm{Ran}(T_a)$ is closed if for every convergent sequence $\{y^n\} \subset \mathrm{Ran}(T_a)$ converging to $y \in \ell^2$, then $y \in \mathrm{Ran}(T_a)$.
4. The orthogonal complement is defined as $(\mathrm{Ker}(T_a))^\perp = \{y \in \ell^2 : \langle x, y\rangle = 0, \forall x \in \mathrm{Ker}(T_a)\}$.