NTNU | Norwegian University of Science and Technology

PLENARY EXERCISES - TMA4145

Week 39, Wednesday 27. September 2023

Problem 1 (Riesz representation Theorem)

Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space, and let $T : V \to \mathbb{K}$ be linear. Show that there exists $u \in V$ such that

$$T(v) = \langle v, u\rangle, \qquad \text{for all } v \in V.$$

Hint: There are several ways to solve the problem, here are a few options:

1. What is the singular value decomposition of T?
2. Recall that any $x \in V$ can be written on the form $x = \sum_{i=1}^{n} \langle x, e_i\rangle e_i$, where $\{e_i\}_{i=1}^{n}$ is a orthonormal basis of V.
3. Choose a basis of V. What is the corresponding matrix representation of V?

Problem 2

Let $(U, \langle\cdot,\cdot\rangle_U)$ and $(V, \langle\cdot,\cdot\rangle_V)$ be two inner product spaces, and consider $T : U \to V$ a linear transformation.

1. Show that $\mathrm{Ker}(T) = (\mathrm{Ran}(T^*))^\perp$.

Hint:

1. $x \in W^\perp$ if $\langle x, w\rangle = 0$ for all $w \in W$.
2. $\langle Tu, v\rangle_V = \langle u, T^* v\rangle_U$ for all $u \in U$ and $v \in V$.

Problem 3

Let $(U, \langle \cdot, \cdot \rangle_U)$ and $(V, \langle \cdot, \cdot \rangle_V)$ be two inner product spaces, and consider $T : U \to V$ a linear transformation.

1. Show that $T : U \to V$ is injective if and only if $T^* : V \to U$ is surjective.

Hint:

1. Recall $\mathrm{Ker}(T) = (\mathrm{Ran}(T^*))^\perp$.

Problem 4

Let $(U, \langle\cdot,\cdot\rangle)$ be an inner product spaces. Let $P : U \to U$ be a unitary operator, and $T : U \to U$ a self-adjoint operator. Define $S : U \to U$ as $S = PTP^*$.

1. Show that S is self-adjoint.
2. Show that S is positive semi-definite if and only if T is positive semi-definite.

Hint:

1. $P : U \to V$ is called unitary if $P^* P = \mathrm{Id}_U$.
2. A transformation T is called positive semi-definite if $\langle Tu, u\rangle \geq 0$ for all $u \in U$.