TIØ4146 Finance for Science and Technology Students

Revisions

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The normal distribution

The most famous continuous distribution is the *normal distribution* (introduced by Abraham de Moivre, 1667-1754). The normal probability density function is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

The cumulative distribution function does not have a close form solution:

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

When a random variable X follows a normal distribution with parameters μ and σ^2 we write $X \sim N\left(\mu, \sigma^2\right)$.

Properties:

1. Moment generating function $M_X(t) = e^{\left(\mu t + 0.5\sigma^2 t^2\right)}$
2. $E(X) = \mu$.
3. $Var(X) = \sigma^2$

There is no closed form solution to the CDF of a normal distribution, which means that one has to use an adequate software to compute the probabilities. Alternatively, one may use the tables with probabilities for the normal distribution with mean equal to 0 and variance equal to 1. To use this strategy one has to notice that

$$X \sim N(\mu, \sigma^2) \Rightarrow \frac{X - \mu}{\sigma} \sim N(0, 1)$$

z=	0	0.01	0.02	0.03	0.04	0.05	0.06	0.09
0	0.500	0.504	0.508	0.512	0.516	0.520	0.524	0.536
0.1	0.540	0.544	0.548	0.552	0.556	0.560	0.564	0.575
0.2	0.579	0.583	0.587	0.591	0.595	0.599	0.603	0.614
0.3	0.618	0.622	0.626	0.629	0.633	0.637	0.641	0.652
0.4	0.655	0.659	0.663	0.666	0.670	0.674	0.677	0.688
0.5	0.691	0.695	0.698	0.702	0.705	0.709	0.712	0.722
0.6	0.726	0.729	0.732	0.736	0.739	0.742	0.745	0.755
0.7	0.758	0.761	0.764	0.767	0.770	0.773	0.776	0.785
0.8	0.788	0.791	0.794	0.797	0.800	0.802	0.805	0.813
0.9	0.816	0.819	0.821	0.824	0.826	0.829	0.831	0.839
1	0.841	0.844	0.846	0.848	0.851	0.853	0.855	0.862
2.5	0.994	0.994	0.994	0.994	0.994	0.995	0.995	0.995

When $\mu = 0$ and $\sigma^2 = 1$, the distribution is denoted as standard normal distribution.

The probability density function of the standard normal distribution is denoted $\phi(z)$ and it is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

The standard normal cumulative distribution function is denoted as

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \phi(t)\, dt.$$

Properties of the standard normal cumulative distribution function:

- $P(Z > z) = 1 - \Phi(z)$.
- $P(Z < -z) = P(Z > z)$.
- $P(|Z| > z) = 2[1 - \Phi(z)]$, for $z > 0$.

Examples: Assume that the weight of a certain population is modeled by a normal distribution with a mean 60 Kg and a standard deviation 5kg.

Question: What is the probability that someone weighs more than 65kg?

Solution: Let X be the random variable that represents the weight of a certain person in the given population. The required probability is

$$
\begin{aligned}
P(X > 65) = P\left(\frac{X-\mu}{\sigma} > \frac{65-\mu}{\sigma}\right) = P(Z > 1) = 1 - \Phi(1) \\
= 1 - 0.841 = 0.159,
\end{aligned}
$$

where

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1).$$

Theorem: (Linear combinations of Normal random variables): Let X and Y be two independent random variables such that $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$. Let $V = aX + bY + c$, then

$$V \sim N(\mu_V, \sigma_V^2)$$

where

$$\begin{aligned} \mu_V &= a\mu_X + b\mu_Y + c \\ \sigma_V^2 &= a^2\sigma_X^2 + b^2\sigma_Y^2. \end{aligned}$$

Remarks:

- A special case is obtained when $b = 0$, if $V = aX + c$, then $V \sim N(\mu_V, \sigma_V^2)$ where $\mu_V = a\mu_X + c$, $\sigma_V^2 = a^2\sigma_X^2$.
- if $X \sim N\left(\mu, \sigma^2\right)$, $Z = \frac{X-\mu}{\sigma} \sim N\left(0, 1\right)$.

Example: Let X and Y be two independent random variables such that

$$X \sim N(\mu = 10, \sigma^2 = 4) \quad \text{and} \quad Y \sim N(\mu = 12, \sigma^2 = 5).$$

Question: Compute the following probability

$$P(X + Y > 19).$$

Solution: Firstly, we notice that

$$X + Y \sim N(22, 9).$$

Therefore,

$$P\left(\frac{X + Y - 22}{3} > \frac{19 - 22}{3}\right) = P(Z > -1) = \Phi(1) = 0.8413,$$

where

$$Z = \frac{X + Y - 22}{3} \sim N(0, 1).$$