NTNU
Norwegian University of
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Department of Mathematical Sciences

Examination paper for
Exercise 8 in TMA4295 Statistical inference

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Examination date: Wednesday in week 40 to Monday in week 42, 2023

Examination time (from–to): Wed 21:00 – Mon 12:00

Permitted examination support material: C

— Tabeller og formler i statistikk, Akademika
– Mathematische Formelsamlung (Matematisk formelsamling) by K. Rottmann
— Stamped yellow A5 sheet with your own handwritten notes
— A specific basic calculator.

Other information:

You may write in English or Norwegian.

All answers must be justified. The answers must include enough details to see how they have been obtained. You must, as always, formulate necessary assumptions as part of the proof of a claim.

All 12 sub-problems carry the same weight for grading.

Language: English

Number of pages: 3

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
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Date Signature

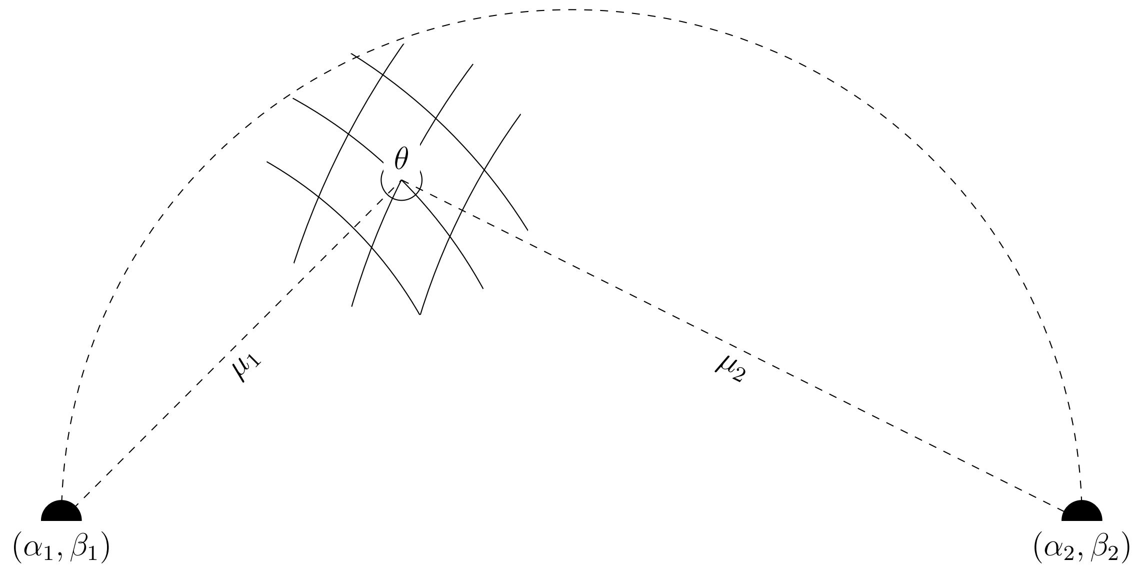

Figure 1: Curved coordinates for an unknown position.

Figure 2: Particles are waves, and waves are particles.

Problem 1 Time and energy Let the data be $x(t) = \mu(t) + \sigma z(t)$ with $\mu(t) = h(t - \tau)$, $t \in [t_0, t_1] \cap (\mathbb{Z} \cdot \delta)$, and z a random sample from $\mathrm{N}(0, 1)$.

a) Let $\overline{\phi} = \sum_t \frac{\phi(t)}{n}$ where n is such that $\overline{1} = 1$. What is n? Give possible interpretations of all symbols above in a real life scenario including SI units.

b) Let the code h and the shift τ be known. Derive the UMVU estimate for the expected noise energy $\epsilon_N = \mathrm{E}\,\overline{\sigma^2 Z^2}$. Find also the scale equivariant UMRU estimate $\hat{\epsilon}_N$ corresponding to the loss $l = \left[10 \log(\hat{\epsilon}_N/\epsilon_0) - 10 \log(\epsilon_N/\epsilon_0)\right]^2$. * Define and find corresponding suitable standard uncertainties, and compare with Cramer-Rao lower bounds. Hint: Consider $x - \mu$ as the data.

c) Assume in the following that the signal arrival time τ is the only unknown. Let $\hat{\tau} = \operatorname{argmin} \overline{(x - \mu)^2}$, $\tilde{\tau} = \operatorname{argmax} \overline{(x - \overline{x})(\mu - \overline{x})}$, and $\overline{\tau} = \operatorname{argmin} \mathrm{E}((\tau - \tau(\Theta))^2 \,|\, x)$ where $\tau(\Theta)$ is the Bayesian signal arrival time by assuming a uniform prior on τ. Give intuitive arguments for the suitability of these estimates. Are the estimates in harmony with the likelihood principle? The sufficiency principle? Are the estimates shift or scale equivariant for actions on τ? Hint: Time shift is defined on data by periodic boundary conditions.

d) Explain how to determine bias and standard deviation for the estimators by computer simulation. How would you calculate a standard uncertainty? Find a formula for the Cramer-Rao lower bound, and explain how this can be used to calculate an optimistic standard uncertainty. Simplify, if possible, when the code h is the triangle function $h(t) = (1 - |t/a|)(|t/a| < 1)h_0$. How does the code design parameters a and h_0 influence the accuracy of the estimates? Is this related to (a suitably defined) energy in the code h?

e) Determine the likelihood test for $H_0 : \tau = \tau_0$ with level $\alpha = 5\%$. Let $[\hat{\tau}_1, \hat{\tau}_2]$ be the smallest interval containing all τ_0 that leads to acceptance of H_0. Prove that $\mathrm{P}(\hat{\tau}_1(X) \leq \tau \leq \hat{\tau}_2(X)) \geq 1 - \alpha$. Let $\breve{\tau}$ be a limit point of the interval as $\alpha \uparrow 1$. Is this estimate related to some of the above estimates?

f) Assume the situation is safe if $H_0 : \tau \geq \tau_0$. Describe a real life situation where this is a good description. Let T be an estimator for τ. Is $B = (T \leq t_\alpha)$ a reasonable test? How would you use the test power function to compare it with the likelihood test for H_0? Choose one of the previous estimators and do the comparison, if possible, when h is the triangle function.

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*The energy dB scale is defined by $10 \log(\epsilon_N/\epsilon_0)$ and a standard reference energy ϵ_0. 'Find' does not necessarily mean an answer by a simple formula, but can mean description of an algorithm.

Problem 2 **Space** Let the data be $x_i = \mu_i + \sigma z_i$ where $z_1, \ldots, z_n$ is a random sample from $\mathrm{N}(0,1)$ where $\mu_i^2 = (\theta_\alpha - \alpha_i)^2 + (\theta_\beta - \beta_i)^2$. Assume that the position $\theta = (\theta_\alpha, \theta_\beta)$ is the unknown model parameter.

a) Figure 1 gives a possible interpretation of the data for the case with $n = 2$ satellites. Explain this.

b) Describe how the MLE and Bayesian estimates of the unknown position θ can be computed.

c) Can you find an explicit formula for an estimator in the $n = 2$ case? Is it unbiased?

d) Calculate the Cramer-Rao lower bound for the $\mathrm{Var}\, T = \mathrm{E}\left[(T-\theta)(T-\theta)^*\right]$ † of an unbiased estimator of θ. Give a geometric interpretation of the result.

e) Explain that $\sqrt{\mathrm{tr}\, \mathrm{Var}\, T}$ is a reasonable measure of the accuracy. Prove that this, together with the Cramer-Rao lower bound, motivates to report the standard uncertainty u by the formula

$$u = \sigma k_D, \quad k_D = \sqrt{\mathrm{tr}\left[(M^* M)^{-1}\right]} \tag{1}$$

where a formula for the matrix M follows from the Cramer-Rao lower bound. The factor k_D is purely geometrical and is the geometrical dilution of precision (GDOP).

f) Let H_0 be a region in the plane. Explain how to implement a likelihood test for $\theta \in H_0$. An alternative test is $B = (T \in R)$ where T is an unbiased estimator of θ. How would you choose the region R? How would you decide on which test is best? Can you think of a real life situation where a test like this is useful?

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†Some authors use the notation $\mathrm{Cov}\, T$ instead of the notation $\mathrm{Var}\, T$ for a random vector T.