TIØ4146 Finance for Science and Technology Students

Chapter 2
Fundamental concepts and techniques

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Time Value of Money

Two basic rules in finance

1. €1 now is worth more than €1 later
   - time value of money
2. A safe €1 is more worth than a risky €1
   - market price of risk

Both are combined in risk adjusted discount / return rates

The notion of "Time value of money"

- means: €1 now worth more than €1 later
- is expressed in *risk free interest rate*
- price for postponing/advancing consumption
- does not include risk premium

Two reasons why money now has higher value than money later:

1. Time preference or 'human impatience'
   - Income not necessary synchronous with needs
   - borrow money to buy house, save for retirement
2. Productive investment opportunities
   - increase consumption later by giving up consumption now
   - sow grain (instead of eating it) to eat more tomorrow

Consequence of time value of money:

- Amounts on different points in time cannot be directly compared
  - cannot say that €100 now is worth less (or more) than €108 next year
- amounts have to be moved through time to same point, adjusting for time value, called:
  - *compounding* if moved forward in time
  - *discounting* if moved backward in time

Interest is *compounded* when it

- is added to principal sum
- starts earning interest (interest on interest)

Simple example: yearly interest rate 10%, compounded yearly

- deposit €100 in a bank
- after 1 year, 10% is added to your account $\Rightarrow$ €110
- second year, interest over €110 is €11 $\Rightarrow$ €121, etc.

Formula for future value, FV, after T years is

$$FV_T = PV(1 + r)^T$$

PV is present value, r is interest rate.

Same principle applies to *discounting*, moving money back in time

- Future value of €100 at time T
- has value of 100/1.1=€90.90 at T-1
- which has value of 90.90/1.1=€82.60 at T-2, etc.

In formula, simply move interest rate factor to other side:

$$PV = \frac{FV_T}{(1+r)^T}$$

Can also re-write formula for the interest rate:

$$r = \sqrt[T]{\frac{FV_T}{PV}} - 1$$

is geometric average rate, $<$ than arithmetic if r fluctuates

Compounding periods not necessarily same as interest periods

- e.g. corporate bonds often pay interest $2\times$ per year
- even though interest is annual rate
- 10% bond pays 5% every half year
- bondholders earn interest on interest in second half year
- *effective annual rate* is $1.05^2 = 1.1025$ or 10.25%
- if compounded quarterly $1.025^4 = 1.1038$ or 10.38%

Future value formula with variable compounding frequency, n, is:

$$FV_T = PV\left(1 + \frac{r}{n}\right)^{Tn}$$

If compounding frequency $n \to \infty$

- compounding periods become infinitesimal
- compounding becomes continuous

Future value formula found by multiplying Tn by r/r and splitting in n/r and rT :

$$FV_T = PV \left[\left(1 + \frac{r}{n} \right)^{n/r} \right]^{rT}$$

Defining $c = n/r$

$$FV_T = PV \left[\left(1 + \frac{1}{c} \right)^{c} \right]^{rT}$$

As $c \to \infty$, $(1+\frac{1}{c})^c \to e = 2.7183..$, base of natural logarithms

$$\lim_{c\to\infty}\left(1+\frac{1}{c}\right)^c = e = 2.71828\ldots$$

Formulae then become:

$$FV_T = PVe^{rT} \quad \text{and} \quad PV = FV_T e^{-rT}$$

re-writing for the interest rate gives $FV_T/PV = e^{rT}$
Taking logarithms:

$$\ln\frac{FV_T}{PV} = \ln e^{rT} = rT$$

These logarithmic rates of return are frequently used in continuous time finance (option pricing)

Advantages of continuously compounded log-returns:

- easily calculated from e.g. daily stock prices $S_0, S_1, S_2, etc.$
- additive over time:

$$\ln\left(\frac{S_1}{S_0}\times\frac{S_2}{S_1}\right) = \ln\frac{S_1}{S_0} + \ln\frac{S_2}{S_1} = \ln e^{r_1} + \ln e^{r_2} = r_1 + r_2$$

- week-return sum of day-returns
- But *not* additive across investments:
  - logarithmic transformation not linear
  - log of a sum $\neq$ sum of logs

Discretely compounded returns $\frac{S_1 - S_0}{S_0}$, $\frac{S_2 - S_1}{S_1}$:

- easily aggregated across investments
  - weighted returns are additive
  - for example, two stocks A and B
  - return A, r_A = 10%, return B, r_B = 20%
  - equally weighted portfolio of A and B gives

$$\frac{1}{2} \times 10 + \frac{1}{2} \times 20 = 15$$

- But: not additive over time:
  - 5% over 10 years is

$$1.05^{10} = 1.629$$

    or 62.9%, not 50%

Annuities and perpetuities

- Cash flows (payments and receipts) often come in series
- called annuity (yearly) and perpetuity (for ever)
- use mathematical series properties to calculate value
- e.g. series of n payments of amount A:

$$PV = \frac{A}{1+r} + \frac{A}{(1+r)^2} + \dots + \frac{A}{(1+r)^n}$$

look to the book to see different anuities

One exception: *Gordon growth model*

- present value of perpetuity
- perpetuity = annuity with infinite number of payments

Formula easily derived (see book):

$$PV = \frac{A}{r}$$

Formula for perpetuity with growth rate g is:

$$PV = \frac{A}{r - g}$$

assumes $r > g$

Gordon growth model:

- often used for its simplicity
- also in exam questions (easy for students)
- usually applied such that number for A is given

Example: Stock price as discounted dividends

A stock is expected to pay €10 in dividends 1 year from now

- Dividends are expected to continue forever and to grow with the inflation rate of 2%
- Discount rate of 10%

Value of the stock is:

$$\frac{10}{.1-.02} = €125$$