Exercise 10

Problem 1.

For a matrix $A\in\mathbb{K}^{m\times n}$, we denote by

$\|A\|_\infty := \max_{x\in\mathbb{K}^n\setminus\{0\}} \frac{\|Ax\|_\infty}{\|x\|_\infty}$

its $\infty$-norm.

Show that

$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}|$

for all $A\in\mathbb{K}^{m\times n}$.

Proof:

$$
\begin{aligned}
\|A\|_\infty &:= \max_{x\in\mathbb{K}^n\setminus\{0\}} \frac{\|Ax\|_\infty}{\|x\|_\infty}\\
&= \max_{\substack{x\in\mathbb{K}^n\\ \|x\|_\infty=1}} \|Ax\|_\infty\\
&= \max_{\substack{x\in\mathbb{K}^n\\ \|x\|_\infty=1}} \max_{1\le i\le m} \left|\sum_{j=1}^n a_{ij}x_j\right|\\
&= \max_{1\le i\le m} \max_{\substack{x\in\mathbb{K}^n\\ \|x\|_\infty=1}} \left|\sum_{j=1}^n a_{ij}x_j\right|\\
&= \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}|
\end{aligned}
$$

Problem 2.

For a matrix $A\in\mathbb{K}^{m\times n}$, we denote by

$\|A\|_{\infty\mapsto 1,\mathbb{K}} := \max_{x\in\mathbb{K}^n\setminus\{0\}} \frac{\|Ax\|_1}{\|x\|_\infty}$

its matrix norm with respect to the $\infty$-norm on its domain $\mathbb{K}^n$ and the 1-norm on its codomain $\mathbb{K}^m$.

Specifically, we consider the matrix

$A=\begin{pmatrix} 1 & -1\\ 1 & 1\end{pmatrix}$,

which we can interpret either as an element of $\mathbb{R}^{2\times 2}$ or of $\mathbb{C}^{2\times 2}$.

(a) Compute the matrix norm $\|A\|_{\infty\mapsto 1,\mathbb{R}}$ for the case $\mathbb{K}=\mathbb{R}$.

$$
\begin{aligned}
\|Ax\|_1 &= \sum_{i=1}^2 |(Ax)_i|\\
&= |x_1-x_2|+|x_1+x_2|
\end{aligned}
$$

$\|x\|_\infty=1$

$$
\begin{aligned}
\Rightarrow \|A\|_{\infty\mapsto 1,\mathbb{R}} &= \max_{\substack{x\in\mathbb{R}^n\\ \|x\|_\infty=1}} \left(|x_1-x_2|+|x_1+x_2|\right)\\
&=2
\end{aligned}
$$

(b) Using the vector $x=(1+i,1-i)^T$ in the definition of the matrix norm, show that, for this particular matrix, we have

$\|A\|_{\infty\mapsto 1,\mathbb{C}} > \|A\|_{\infty\mapsto 1,\mathbb{R}}$.

Proof:

$$
\begin{aligned}
\|A\|_1 &= \sum_{i=1}^2 \left|\sum_{j=1}^2 a_{ij}x_j\right|\\
&= |1+i-1+i|+|1+i+1-i|\\
&= |0+2i|+|2+0i|\\
&= 2+2\\
&= 4
\end{aligned}
$$

$\|x\|_\infty = 2$

Problem 3.

Assume that $(U,\langle\cdot,\cdot\rangle)$ is an inner product space with induced norm $\|u\|=(\langle u,u\rangle)^{\frac{1}{2}}$.

Assume moreover that $v\in U$ is some fixed vector and define the mapping

$T: U\to\mathbb{K}$,

$u\mapsto Tu := \langle u,v\rangle$.

Show that $T$ is a bounded linear mapping and compute its norm!

Proof:

Linear:

$$
\begin{aligned}
T(u+w) &= \langle u+w,v\rangle\\
&= \langle u,v\rangle+\langle w,v\rangle \quad \checkmark\\
&= Tu+Tw
\end{aligned}
$$

$$
\begin{aligned}
T(\alpha u) &= \langle \alpha u,v\rangle\\
&= \alpha\langle u,v\rangle\\
&= \alpha Tu \quad \checkmark
\end{aligned}
$$

Bounded:

Induced norm

$$
\begin{aligned}
\|u\| &= (\langle u,u\rangle)^{\frac{1}{2}}\\
&= \left(\sum_{i=1}^n u_i^2\right)^{\frac{1}{2}}
\end{aligned}
$$

$\|Tu\|^2 = \sum_{i=1}^n (Tu_i)^2$

$$
\begin{aligned}
\|Tu\| &= \left(\sum_{i=1}^n (Tu)_i^2\right)^{\frac{1}{2}}\\
&\le \sum_{i=1}^n (Tu)_i\\
&= T\sum_{i=1}^n u_i\\
&= T\|u\|
\end{aligned}
$$

Problem 4.

Define the linear operator $T:\ell^1\to\ell^1$ by

$T(x_1,x_2,x_3,\dots) := (x_1,\frac{x_2}{2},\frac{x_3}{3},\dots)$,

that is, $Tx=\left(\frac{x_k}{k}\right)_{k\in\mathbb{N}}$.

(a) Show that $T$ is bounded with $\|T\|=1$

Proof:

$$
\begin{aligned}
\|Tx\|_{\ell^1} &= \sum_{k=1}^\infty |Tx|_k\\
&= \sum_{k=1}^\infty \frac{x_k}{k}
\end{aligned}
$$