NTNU | Norwegian University of Science and Technology

PLENARY EXERCISES - TMA4145

Week 37, Wednesday 13. September 2023

Problem 1

Let V be a vector space, and consider $P : V \to V$ such that $P^2 = P$. Let $m = \dim(\ker P)$ and $n = \dim(\mathrm{range}(P))$. Find the characteristic polynomial of P. Find, and verify, the minimal polynomial of P.

Hint:

1. What are the eigenvalues of P?
2. What happens if $n = 0$?

Problem 2

Let V be a vector space. Assume there exists $V_1, \ldots, V_n$ T-invariant subspaces such that

$$V = V_1 \oplus \ldots \oplus V_n.$$

Let $T : V \to V$ be a linear operator, and T_j is the restriction of T to V_j. Show that the characteristic polynomial χ_T can be written as

$$\chi_T = \chi_{T_1} \cdots \chi_{T_n}.$$

Hint:

1. The restriction of T_j is defined such that $T(v_j) = T_j(v_j)$ for all $v_j \in V_j$.
2. Let W be a vector space, and $S : W \to W$. Then $W = \bigoplus_l G(\mu_l, S)$.
3. How does the matrix representation of T look like?

Problem 3

Let $A, B \in \mathrm{Mat}_n(\mathbb{C})$. Show that AB and BA have the same eigenvalues with the same multiplicities. Show that the dimension of the respective eigenspaces corresponding to $\lambda \neq 0$ are the same.

Hint:

1. Find a mapping from one eigenspace to the other.
2. Consider $\lambda \neq 0$, and $\lambda = 0$ seperately.
3. How many eigenvalues are there?