Chapter 3: Modern Portfolio Theory - part 1

Exercises - solutions

1. We can use the formula for the variance of a 2-stock portfolio and deduce that in such a portfolio the weight of one stock is 1 minus the weight of the other: $x_A = 1 - x_B$. Given the stocks' variances and correlation, the portfolio variance becomes a function of one variable (weight x) and we can easily find its minimum.

   $$\sigma_p^2 = x^2.15^2 + (1-x)^2.2^2 + 2x(1-x) \times -1 \times 0.15 \times 0.2$$

   which simplifies to:

   $$\sigma_p^2 = 0.1225x^2 - 0.14x + 0.04$$

   Taking the derivative with respect to x gives:

   $$\frac{\partial}{\partial x}\left(0.1225x^2 - 0.14x + 0.04\right) = 0.245\,x - 0.14$$

   This is zero when $x = 0.571$. Taking the second derivative confirms that we have found a minimum:

   $$\frac{\partial}{\partial x}(0.245\,x - 0.14) = 0.245 > 0$$

   So the minimum variance portfolio consists of 0.571 in A and 0.429 in B. It variance is $\sigma_p^2 = 0.1225 \times 0.571^2 - 0.14 \times 0.571 + 0.04 = 0$.
   Alternatively, we know that a zero variance portfolio is possible if two stocks are perfectly negatively correlated. If $\rho_{A,B} = -1$ portfolio variance is

   $$\sigma_p^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 - 2x_Ax_B\sigma_A\sigma_B = (\sigma_A x_A - \sigma_B x_B)^2$$

   This is only zero if $\sigma_A x_A - \sigma_B x_B = 0$. We also know that $x_A + x_B = 1$. Solving the two equations for the two unknown weights $x_{A,B}$ we get

   $$x_A = \frac{\sigma_B}{\sigma_A + \sigma_B} \quad and \quad x_B = \frac{\sigma_A}{\sigma_A + \sigma_B}$$

   so $x_A = 0.2/(0.15 + 0.2) = 0.571$ and $x_B = 0.15/(0.15 + 0.2) = 0.429$.

2. (a) Yes, if the index fund is a good proxy for the market as a whole, then combinations of this fund and risk free investing lie on the CML.

   (b) $.5 \times .05 + .5 \times .15 \ = 0.1$ or 10%

   (c) $\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\rho_{1,2}\sigma_1\sigma_2$
   $\sigma_p^2 = .5^2 \times 0 + .5^2 \times .2^2 + 2 \times .5 \times .5 \times 0 \times 0 \times .2 = 0.01 \Rightarrow$
   $\sigma_p = \sqrt{.01} = 0.1$ or 10%

   (d) $.5 \times 0 + .5 \times 1 = 0.5$

3. (a) By borrowing an amount equal to 50% of his money and investing this together with his own money.

(b) Yes, if the index fund is a good proxy for the market as a whole, then combinations of this fund and risk free investing lie on the CML.

(c) $-.5 \times .05 + 1.5 \times .15 = 0.2$

(d) $\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\rho_{1,2}\sigma_1\sigma_2$
$\sigma_p^2 = -.5^2 \times 0 + 1.5^2 \times .2^2 + 2 \times -.5 \times 1.5 \times 0 \times 0 \times .2 = 0.09 \Rightarrow$
$\sigma_p = \sqrt{.09} = 0.3$ or 30%

(e) $-.5 \times 0 + 1.5 \times 1 = 1.5$

4. (a) No, two stocks are not enough to capture the better part of the diversification effect, that takes 20-30 stocks. So the portfolio is exposed to unique (or unsystematic, or diversifiable) risk, for which the market return contains no compensation.

(b) In times of financial crises practically all investments drop in value, so their correlation coefficients are high (approach 1).

(c) $.5 \times .2 + .5 \times .125 = 0.1625$

(d) $\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\rho_{1,2}\sigma_1\sigma_2$
$\sigma_p^2 = .5^2 \times .4^2 + .5^2 \times .25^2 + 2 \times .5 \times .5 \times 0 \times .4 \times .2 = .055625 \Rightarrow$
$\sigma_p = \sqrt{.055625} = 0.235\,85$ or 23.5%

5. (a) Portfolio variance is calculated as:

$$\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\rho_{1,2}\sigma_1\sigma_2$$

Filling in the numbers gives:

$$0.6^2 \times 0.363^2 + 0.4^2 \times 0.34^2 + 2 \times 0.6 \times 0.4 \times 0.34 \times 0.363 \times 0.34 = 0.086075$$

So the portfolio standard deviation is:

$$\sqrt{0.086075} = 0.29339 \text{ or } 29.3\%$$

(b) The relative contribution of each stock to this portfolio's variance is the sum of the stock's row (or column) entries in the variance-covariance matrix, divided by the portfolio's variance. Cisco's contribution is:

$$0.6^2 \times 0.363^2 + 0.6 \times 0.4 \times 0.34 \times 0.363 \times 0.34 = 0.0575$$

so its relative contribution is:

$$0.0575/0.086075 = 0.668$$

Similarly, for Amazon:

$$0.4^2 \times 0.34^2 + 0.6 \times 0.4 \times 0.34 \times 0.363 \times 0.34 = 0.028567$$

So its relative contribution is:

$$0.028567/0.086075 = 0.332$$

(c) We know that $\beta_i = cov_{i,p}/\sigma_p^2$ and that the contribution of a stock to portfolio variance can be written as: $contr_i = weight_i \times cov_{i,p}$. The relative contribution is $contr_i$ divided by portfolio variance: $rel.contr_i = (weight_i \times cov_{i,p})/\sigma_p^2 = weight_i \times (cov_{i,p}/\sigma_p^2) = weight_i \times \beta_i$. So $weight_i \times \beta_i = contr_i/\sigma_p^2$. We calculated the relative contributions of both stocks to portfolio variance under (b) so:

$$0.6 \times \beta_1 = 0.668 \;\Rightarrow\; \beta_1 = 1.113$$

and

$$0.4 \times \beta_2 = 0.332 \;\Rightarrow\; \beta_2 = 0.830$$

We can check this by calculating the portfolio β as the weighted average of the two stock βs, which should be 1: $0.6 \times 1.113 + 0.4 \times 0.830 = 1$

(d) We vary portfolio composition in steps of 10 percentage points, starting with 100% in Cisco. For example, with 90% in Cisco and 10% in Amazon, portfolio return is

$$r_p = 0.9 \times 0.075 + 0.1 \times 0.125 = .08$$

and portfolio variance and standard deviation are:

$$\begin{aligned}
\sigma_p^2 &= 0.9^2 \times 0.363^2 + 0.1^2 \times 0.34^2 + 2 \times 0.9 \times 0.1 \times 0.34 \times 0.363 \times 0.34 \\
\sigma_p^2 &= 0.115\,44 \\
\sigma_p &= \sqrt{0.115\,44} = 0.33976
\end{aligned}$$

Repeating these calculations for different portfolio weights produces the results in Table 1.

Table 1: Portfolio return and risk

Cisco	Amazon	r_p	σ_p
1.0	0.0	0.075	0.363
0.9	0.1	0.080	0.340
0.8	0.2	0.085	0.320
0.7	0.3	0.090	0.304
0.6	0.4	0.095	0.293
0.5	0.5	0.100	0.288
0.4	0.6	0.105	0.288
0.3	0.7	0.110	0.293
0.2	0.8	0.115	0.304
0.1	0.9	0.120	0.320
0.0	1.0	0.125	0.340

Portfolio return is plotted against portfolio risk in Figure 1.

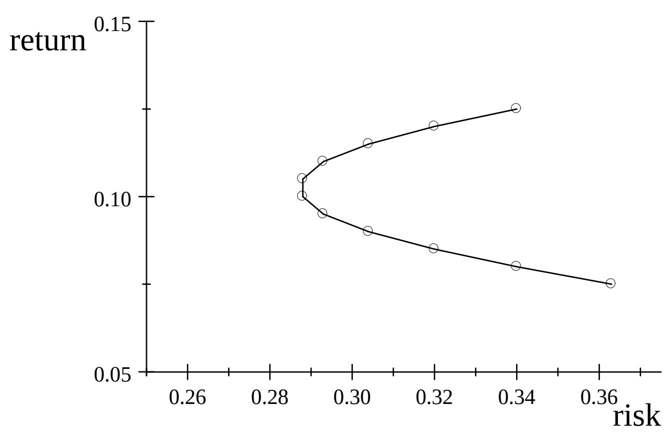

Figure 1: Portfolio risk and return

As could be expected for two stocks that are moderately positively correlated ($\rho = 0.34$), Figure 1 has the shape of a Markowitz bullet.

(e) Yes, we just saw that in a 2-stock portfolio the weight of one stock is 1 minus the weight of the other, so that the portfolio variance becomes a function of one variable (weight) and we can easily find its minimum. Writing the variance of a 2-stock portfolio as:

$$\sigma_p^2 = x^2\sigma_1^2 + (1-x)^2\sigma_2^2 + 2x(1-x)\rho_{1,2}\sigma_1\sigma_2$$

and filling in the numbers for Cisco and Amazon we get:

$$\sigma_p^2 = x^2 \times 0.363^2 + (1-x)^2 0.34^2 + 2x(1-x) \times 0.34 \times 0.363 \times 0.34$$

which simplifies to:

$$\sigma_p^2 = 0.16344x^2 - 0.14727x + 0.1156$$

Taking the derivative with respect to x gives:

$$\frac{\partial}{\partial x}\left(0.16344x^2 - 0.14727x + 0.1156\right) = 0.32688x - 0.14727$$

This is zero when $x = 0.45$. Taking the second derivative confirms that we have found a minimum:

$$\frac{\partial}{\partial x}(0.32688x - 0.14727) = 0.32688 > 0$$

So the minimum variance portfolio consists of 0.45 in Cisco and 0.55 in Amazon. It variance is $\sigma_p^2 = 0.16344 \times 0.45^2 - 0.14727 \times 0.45 + 0.1156 = 0.082425$ so its standard deviation is $\sqrt{0.082425} = 0.287$. Its return is $0.45 \times 0.075 + 0.55 \times 0.125 = 0.1025$.

(f) If the two stocks would be perfectly positively correlated there would be no diversification effect and the portfolios would plot on a straight line. Figure 2 shows the graph.

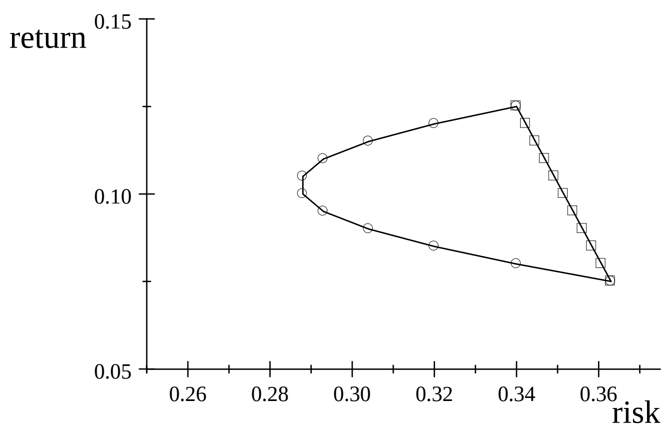

Figure 2: Portfolio risk and return with $\rho = 0.34$ and $\rho = 1$

6. (a) No, in the presence of a money market, i.e. the risk free investment opportunity, two fund separation obtains and all efficient portfolios lie on the capital market line.

(b) If two fund separation obtains all efficient portfolios are a combination of risk free borrowing or lending and the market portfolio. Using the index as the market portfolio we can write the variance of efficient portfolios as:

$$\sigma_p^2 = x^2\sigma_m^2 + (1-x)^2\sigma_{rf}^2 + 2x(1-x)\rho_{m,rf}\sigma_m\sigma_{rf}$$

where the subscript m denotes the the market and rf the risk free investment. Since the variance of and the correlation with the risk free investment are zero, this reduces to:

$$\sigma_p^2 = x^2\sigma_m^2$$

Uncle Bob's minimum variance portfolio has a standard deviation of 0.232 and the market portfolio has a standard deviation of 0.24. Substituting these numbers into the formula for the variance of efficient portfolios and solving for the weight x we get:

$$0.232^2 = x^2 \times 0.24^2 \Rightarrow x = 0.967$$

A portfolio with 0.967 in the market portfolio and 0.033 risk free has a return of:

$$r_p = 0.967 \times 0.12 + 0.033 \times 0.025 = 0.117$$

So aunt Agatha can get $0.117 - 0.092 = 0.025$ or 2.5% more return for the same risk.

(c) First we calculate the portfolio composition for the given return:

$$r_p = x \times 0.12 + (1-x) \times 0.025 = 0.092$$

Solving for x gives $x = 0.705$. The variance of a portfolio with 70.5% in the market portfolio and 29.5% risk free is:

$$\sigma_p^2 = x^2\sigma_m^2 = 0.705^2 \times 0.24^2 = 0.028629$$

so its standard deviation is $\sqrt{0.028629} = 0.169$ or 16.9%, which is $0.232 - 0.169 = 0.063$ or 6.3% lower than uncle Bob's minimum variance portfolio. The situation is illustrated in Figure 3, which replicates Figure 3.7 in the book with rescaled axes and the Capital Market Line and market portfolio M drawn in. Notice that uncle Bob's portfolios are inefficient; 5 stocks are not enough to be fully diversified.

7. (a) To maximize the expected return of his portfolio uncle Bob should invest the other 50% in the stock with the highest return, Amazon. The expected return of this portfolio is $0.5 \times 0.06 + 0.5 \times 0.125 = 0.0925$ or $9\frac{1}{4}$%.

(b) The variance of an equally weighted 2-stock portfolio is

$$\sigma_p^2 = 0.5^2\sigma_1^2 + 0.5^2\sigma_2^2 + 2 \times 0.5 \times 0.5 \times \rho_{1,2}\sigma_1\sigma_2$$

Filling in the numbers Logitech and Google we get:

$$\begin{aligned}
\sigma_p^2 &= 0.5^2 \times 0.462^2 + 0.5^2 \times 0.287^2 + 2 \times 0.5 \times 0.5 \times 0.34 \times 0.462 \times 0.287 \\
\sigma_p^2 &= 0.09649 \text{ so } \sigma_p = \sqrt{0.09649} = 0.311
\end{aligned}$$

for Logitech and Cisco:

$$\begin{aligned}
\sigma_p^2 &= 0.5^2 \times 0.462^2 + 0.5^2 \times 0.363^2 + 2 \times 0.5 \times 0.5 \times 0.28 \times 0.462 \times 0.363 \\
\sigma_p^2 &= 0.10978 \text{ so } \sigma_p = \sqrt{0.10978} = 0.331
\end{aligned}$$

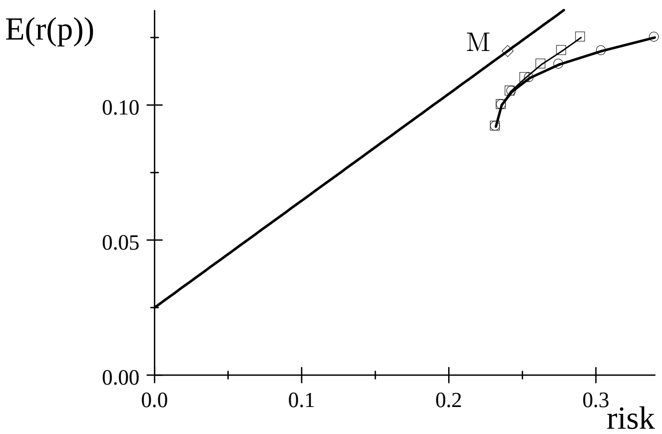

Figure 3: Capital market line, market porfolio M and stock portfolios with (lower) and without (upper) short selling restriction

for Logitech and Amazon:

$$\begin{aligned}\sigma_p^2 &= 0.5^2 \times 0.462^2 + 0.5^2 \times 0.340^2 + 2 \times 0.5 \times 0.5 \times 0.35 \times 0.462 \times 0.340 \\ \sigma_p^2 &= 0.10975 \text{ so } \sigma_p = \sqrt{0.10975} = 0.331\end{aligned}$$

and, finally, for Logitech and Apple:

$$\begin{aligned}\sigma_p^2 &= 0.5^2 \times 0.462^2 + 0.5^2 \times 0.250^2 + 2 \times 0.5 \times 0.5 \times 0.39 \times 0.462 \times 0.250 \\ \sigma_p^2 &= 0.09151 \text{ so } \sigma_p = \sqrt{0.09151} = 0.303\end{aligned}$$

We see that the portfolio of Logitech and Apple has the lowest standard deviation.