Exercise 5

Problem 1.

U, V, W finite dimensional inner product spaces

$T: U \to V$ linear

$S: V \to W$ linear

(a) Show $(T^*)^* = T$:

T^* is defined as

$\langle Tu, v\rangle_V = \langle u, T^* v\rangle_U, \ \forall u \in U, \forall v \in V$

where $T: U \to V$.

Proof:

$$\langle Tu, v\rangle_V = \langle u, T^* v\rangle_U$$
$$= \langle (T^*)^* u, v\rangle_V$$
$$\Rightarrow (T^*)^* = T$$

(b) Show $(S \circ T)^* = T^* \circ S^*$

$(S \circ T): U \to W$

$(S \circ T)^*: W \to U$

Proof

$$\langle u, (S\circ T)^* v\rangle_U = \langle (S \circ T)u, w\rangle_W$$
$$= \langle S(Tu), w\rangle_W$$
$$= \langle Tu, S^* w\rangle_V$$
$$= \langle u, T^* S^* w\rangle_U$$
$$= \langle u, (T^* \circ S^*) w\rangle_U$$
$$\Rightarrow (S \circ T)^* = T^* \circ S^*$$

Problem 2.

$A := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$AA^H = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

Eigenvalues of AA^H are 0 and 2

The singular value of A is therefore $\sigma = \sqrt{2}$

The singular vector of A is therefore $q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Rightarrow AA^H = Q\Sigma^2 Q^H$$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sqrt{2}^2 \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$A = Q\Sigma P^H \Rightarrow P = A^H Q \Sigma^{-1}$$
$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow A^+ = P\Sigma^{-1}Q^H$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$(A^+)^2 = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}^2$$
$$= \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$(A^2)^+ = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^+$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

Problem 3.

$A = \begin{pmatrix} 10 & 10 \\ -1 & 7 \\ 5 & 5 \\ -2 & 14 \end{pmatrix}$

SVD of A:

$$A^H A = \begin{pmatrix} 10 & -1 & 5 & -2 \\ 10 & 7 & 5 & 14 \end{pmatrix}\begin{pmatrix} 10 & 10 \\ -1 & 7 \\ 5 & 5 \\ -2 & 14 \end{pmatrix}$$
$$= \begin{pmatrix} 130 & 90 \\ 90 & 370 \end{pmatrix}$$

Finding σ's:

$A^H A v = \lambda v$

$$\det\left(A^H A - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 130-\lambda & 90 \\ 90 & 370-\lambda \end{pmatrix}$$
$$= (130-\lambda)(370-\lambda) - 90^2$$
$$= 48100 - 130\lambda - 370\lambda + \lambda^2 - 8100$$
$$= \lambda^2 - 500\lambda + 4000$$

$$\det\left(A^H A - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0 \Rightarrow \lambda_1 = 100$$
$$\lambda_2 = 400$$

$\Rightarrow \underline{\sigma_2 = 10}$

$\underline{\sigma_1 = 20}$

$\Rightarrow \Sigma = \begin{pmatrix} 20 & 0 \\ 0 & 10 \end{pmatrix}$

Finding P:

$$\left(A^H A - 100\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)v = 0$$
$$\Rightarrow \begin{pmatrix} 30 & 90 \\ 90 & 270 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$
$$\Rightarrow 30x + 90y = 0$$
$$90x + 270y = 0$$
$$x = -3y \Rightarrow -270y + 270y = 0$$
$$y = 1 \Rightarrow x = -3$$
$$\underline{q_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}}$$

$$\left(A^H A - 400\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)v = 0$$
$$\Rightarrow \begin{pmatrix} -270 & 90 \\ 90 & -30 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$
$$\Rightarrow -270x + 90y = 0$$
$$90x - 30y = 0$$
$$\Rightarrow -270x - 90x = -30y - 90y$$
$$\Rightarrow -360x = -120y$$
$$\Rightarrow 360x = 120y$$
$$\Rightarrow x = 1 \Rightarrow y = 3$$
$$\underline{q_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

Orthonormal eigenbasis of $A^H A$:

$$\frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \frac{1}{\sqrt{10}}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$
$$\Rightarrow P = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$$

Finding Q:

$$A = \hat{Q}\hat{\Sigma}\hat{P}^H$$
$$A\hat{P} = \hat{Q}\hat{\Sigma}$$
$$= \begin{pmatrix} 10 & 10 \\ -1 & 7 \\ 5 & 5 \\ -2 & 14 \end{pmatrix}\begin{pmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{pmatrix}$$
$$= \begin{pmatrix} 4\sqrt{10} & -2\sqrt{10} \\ 2\sqrt{10} & \sqrt{10} \\ 2\sqrt{10} & -\sqrt{10} \\ 4\sqrt{10} & 2\sqrt{10} \end{pmatrix}$$

$\Rightarrow$ 1st column of Q

$$\begin{pmatrix} 4\sqrt{10}/20 \\ 2\sqrt{10}/20 \\ 2\sqrt{10}/20 \\ 4\sqrt{10}/20 \end{pmatrix} = \begin{pmatrix} \sqrt{2/5} \\ 1/\sqrt{10} \\ 1/\sqrt{10} \\ \sqrt{2/5} \end{pmatrix}$$

2nd column of Q

$$\begin{pmatrix} -2\sqrt{10}/10 \\ \sqrt{10}/10 \\ -\sqrt{10}/10 \\ 2\sqrt{10}/10 \end{pmatrix} = \begin{pmatrix} -\sqrt{2/5} \\ 1/\sqrt{10} \\ -1/\sqrt{10} \\ \sqrt{2/5} \end{pmatrix}$$

3rd column of Q

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Problem 4.

(a) Skew-Hermitian $T^* = -T$

Proof

$$\langle Tx, x\rangle = \langle x, T^* x\rangle$$
$$= -\langle x, Tx\rangle$$
$$= -\langle x, \lambda x\rangle$$
$$= -\lambda\langle x, x\rangle$$
$$= \bar{\lambda}\langle x, x\rangle$$
$$\Rightarrow \bar{\lambda} = -\lambda$$
$\Rightarrow \lambda$ imaginary

(b) Proof:

Problem 5.

$\langle A, B\rangle = tr(A^H B)$

$T: Mat_{m,n}(K) \to Mat_{l,n}(K)$

$A \mapsto T(A) := MA$

(a) Find T^*

$$\langle TA, B\rangle_{Mat_{m,n}(K)} = \langle A, T^* B\rangle_{Mat_{l,n}(K)}$$

$$\langle TA, B\rangle = \langle MA, B\rangle$$
$$= tr((MA)^H B)$$
$$= \sum_{i=1}^{n} (MA)^H B_{ii}$$
$$= \sum_{i=1}^{n} A^H M^H B_{ii}$$
$$= \sum_{i=1}^{n} A^H (M^H B)_{ii}$$
$$= tr(A(M^H B))$$
$$= \langle A, M^H B\rangle$$

$$\Rightarrow T^*: Mat_{l,n}(K) \to Mat_{m,n}(K)$$
$$A \mapsto T^*(A) = M^H A$$

(b) Self adjoint

$T = T^*$

$l = m$

$(\Rightarrow)$

Assume $T = T^*$

$$\Rightarrow \langle TA, B\rangle = \langle A, T^* B\rangle$$
$$= \langle A, TB\rangle$$
$$= \langle A, M^H B\rangle$$
$$= \langle A, MB\rangle$$
$$\Rightarrow M = M^H$$

$(\Leftarrow)$

Assume $M = M^H$

$$\langle A, T^* B\rangle = \langle A, M^H B\rangle$$
$$= \langle A, MB\rangle$$
$$= \langle A, TB\rangle$$
$$\Rightarrow T = T^*$$

(c) $(\Rightarrow)$

Assume $\langle TA, TB\rangle = \langle A, B\rangle$:

$$\langle TA, TB\rangle = \langle MA, MB\rangle$$
$$= tr((MA)^H MB)$$
$$= \sum_{i=1}^{n} (A^H M^H MB)_{ii}$$
$$= \sum_{i=1}^{n} (A^H B)_{ii} = \langle A, B\rangle$$
$$\Rightarrow M^H M = I$$

$(\Leftarrow)$

Assume $M^H M = I$

$$\langle TA, TB\rangle = \sum_{i=1}^{n} (AM^H MB)_{ii}$$
$$= \sum_{i=1}^{n} (AB)_{ii}$$
$$= \langle A, B\rangle$$
$\Rightarrow T$ unitary