Exercise 1

Problem 1:

$U := \mathrm{Span}\{(1,0,0,0),(0,1,0,0)\} \subset \mathbb{R}^4$

(a) $U^\perp = \{\underline{x} \mid \underline{x}\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=0 \text{ and } \underline{x}\cdot\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\}$

$x_1 = 0$

$x_2 = 0$

Solve:

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\end{pmatrix}$$

$\Rightarrow x_3 \in \mathbb{R},\ x_4 \in \mathbb{R}$

$\Rightarrow \underline{\underline{U^\perp := \mathrm{Span}\{(0,0,1,0),(0,0,0,1)\}}}$

Since

$\mathrm{Span}\{(1,0,0,0),(0,1,0,0)\} = \{a\cdot(1,0,0,0)+b\cdot(0,1,0,0) \mid a,b\in\mathbb{R}\}$

$\mathrm{Span}\{(0,0,1,0),(0,0,0,1)\} = \{c\cdot(0,0,1,0)+d\cdot(0,0,0,1) \mid c,d\in\mathbb{R}\}$

$\underline{\underline{\Rightarrow u\in U,\ u^\perp \in U^\perp,\ \Rightarrow u+u^\perp\in\mathbb{R}^4 \Rightarrow U\oplus U^\perp = \mathbb{R}^4}}$

(b) If we take $V = \mathrm{Span}\{(0,1,1,0),(0,0,0,1)\} \subset \mathbb{R}^4$

<u>Then we have $U\oplus V = \mathbb{R}^4$, and $V \neq U^\perp$</u>

Problem 2.

U is the subspace of even polynomials and V is the subspace of odd polynomials

$P = U\oplus V$ means that every $p\in P$ can be written as $p = u+v,\ u\in U,\ v\in V$

Since P is the vector space of polynomials with real coefficients, it is a combination of odd and even parts.

<u>Therefore $P = U\oplus V$.</u>

Problem 3.

$S: P \to P$

$p \to p'$

$T: P \to P$

$p \to \int_0^x p(t)\,dt$

(a) $\ker(S) = \{p\in P \mid S(p)=0\} = \{p\in P \mid p'=0\}$

$\underline{\underline{\Rightarrow \ker(S) = \{p \mid p \text{ is a constant}\}}}$

$\mathrm{range}(S) = \{S(p) \mid p\in P\} = S(P)$

$\underline{\underline{\Rightarrow \mathrm{range}(S) = P}}$

$\ker(T) = \{p\in P \mid T(p)=0\} = \{p\in P \mid \int_0^x p(t)\,dt = 0\}$

$\underline{\underline{\Rightarrow \ker(T) = \{p \mid p=0\}}}$

$\mathrm{range}(T) = \{T(p) \mid p\in P\} = T(P)$

$\underline{\underline{\Rightarrow \mathrm{range}(T) = P}}$

(b) Since $S\circ T$ means that T is used first and then S is used we have

$\frac{d}{dx}\left(\int_0^x p(t)\,dt\right) = \frac{d}{dx}\left(P(x)-P(0)\right)$

$= p(x) = p(t)$

So $S\circ T$ is the identity operator.

But, $T\circ S$ means

$\int_0^x \frac{d}{dt}p(t)\,dt = \int_0^x p'(t)\,dt$

$= p(x)-p(0) \neq p(t)$

<u>So $T\circ S$ is different from the identity operator.</u>

Problem 4.

(a) Since $\ker(T) := \{v\in V \mid Tv=0\}$

$\Rightarrow$ If $u\in\ker(T)$, then $Tu=0 \Rightarrow Tu\in\ker(T)$

<u>Thus $\ker(T)$ is T-invariant</u>

Since $\mathrm{ran}(T) := \{v\in V \mid \exists u\in V \text{ s.t. } Tu=v\}$

$\Rightarrow$ if $w\in\mathrm{ran}(T)$, then $Tw\in\mathrm{ran}(T)$

<u>Thus $\mathrm{ran}(T)$ is T-invariant</u>

(b) $U_1, U_2 \subset V$ be T-invariant subspaces of V

Show for $U_1\cap U_2$:

Let $u_1\in U_1,\ u_2\in U_2$

$Tu_1\in U_1,\ Tu_2\in U_2$

Need to show

$T(u_1\cap u_2)\in(U_1\cap U_2)$

<u>Since $u_1\cap u_2$ is in $U_1\cap U_2$, this must mean that $U_1\cap U_2$ is T-invariant</u>

Show for U_1+U_2:

$\underline{\underline{T(u_1+u_2) = T(u_1)+T(u_2)\in(U_1+U_2)}}$

Problem 5.

(a) $\mathrm{Span}(S\cup T) = \mathrm{Span}(S)+\mathrm{Span}(T)$

Let $s_i\in S,\ t_j\in T,\ v\in\mathrm{Span}(S\cup T)$

$v = \sum_{i=1}^{n} a_i s_i + \sum_{j=1}^{m} b_j t_j \in \mathrm{Span}(S)+\mathrm{Span}(T)$

$\Rightarrow \mathrm{Span}(S\cup T) \subseteq \mathrm{Span}(S)+\mathrm{Span}(T)$

Need to show $\mathrm{Span}(S)\subseteq\mathrm{Span}(S\cup T)$ and $\mathrm{Span}(T)\subseteq\mathrm{Span}(S\cup T)$

This is trivial, since one can see that they are true

<u>So, $\mathrm{Span}(S\cup T) = \mathrm{Span}(S)+\mathrm{Span}(T)$ is true if S and T are independent</u>

(b) $\mathrm{Span}(S\cap T) = \mathrm{Span}(S)\cap\mathrm{Span}(T)$

$S\cap T = \{v \mid v\in S \text{ and } v\in T\}$

$\mathrm{Span}(S\cap T) = \{a_1v_1+\dots+a_nv_n \mid a_1,\dots,a_n\in F\}$

Letting $v = s\cap t$

$\Rightarrow \mathrm{Span}(S\cap T) = \{a_1(s_1\cap t_1)+\dots+a_n(s_n\cap t_n) \mid a_1,\dots,a_n\in F\}$

$= \{a_1s_1+a_1t_1-a_1(s_1\cup t_1)+\dots+a_ns_n+a_nt_n-a_n(s_n\cup t_n) \mid a_1,\dots,a_n\in F\}$

$= \mathrm{Span}(S)+\mathrm{Span}(T)-\mathrm{Span}(S\cup T)$

<u>So $\mathrm{Span}(S\cap T) = \mathrm{Span}(S)\cap\mathrm{Span}(T)$ is false</u>

(c) $\mathrm{Span}(S+T) = \mathrm{Span}(S)+\mathrm{Span}(T)$

$\mathrm{Span}(S+T) = \{a_1v_1+\dots+a_nv_n \mid a_1,\dots,a_n\in F\}$

$v_1 = s_1+t_1$

$\Rightarrow \mathrm{Span}(S+T) = \{a_1s_1+a_1t_1+\dots+a_ns_n+a_nt_n \mid a_1,\dots,a_n\in F\}$

<u>$= \mathrm{Span}(S)+\mathrm{Span}(T)$ True</u>