Exercise 1

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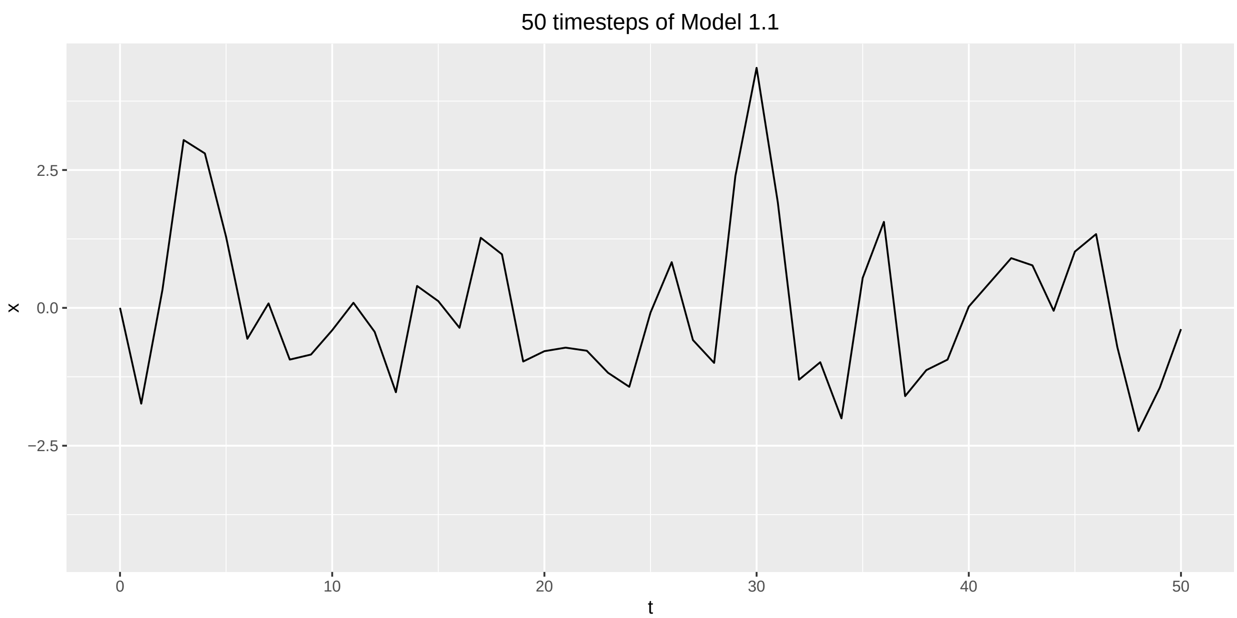
Contents

1 Problem 1

We have a linear process given by

$$x_{t+1} = w_t + w_{t-1}, \quad x_1 = 0, \quad w_t \overset{i.i.d.}{\sim} \mathcal{N}(0,1). \tag{Model 1.1}$$

```r
problem1_fun <- function(t) {
  set.seed(97)
  x <- 0
  w <- c(0, rnorm(t))
  for (i in 1:t + 1) {
    x[i] <- w[i] + w[i - 1]
  }
  return(x)
}
x_50 <- problem1_fun(50)
```



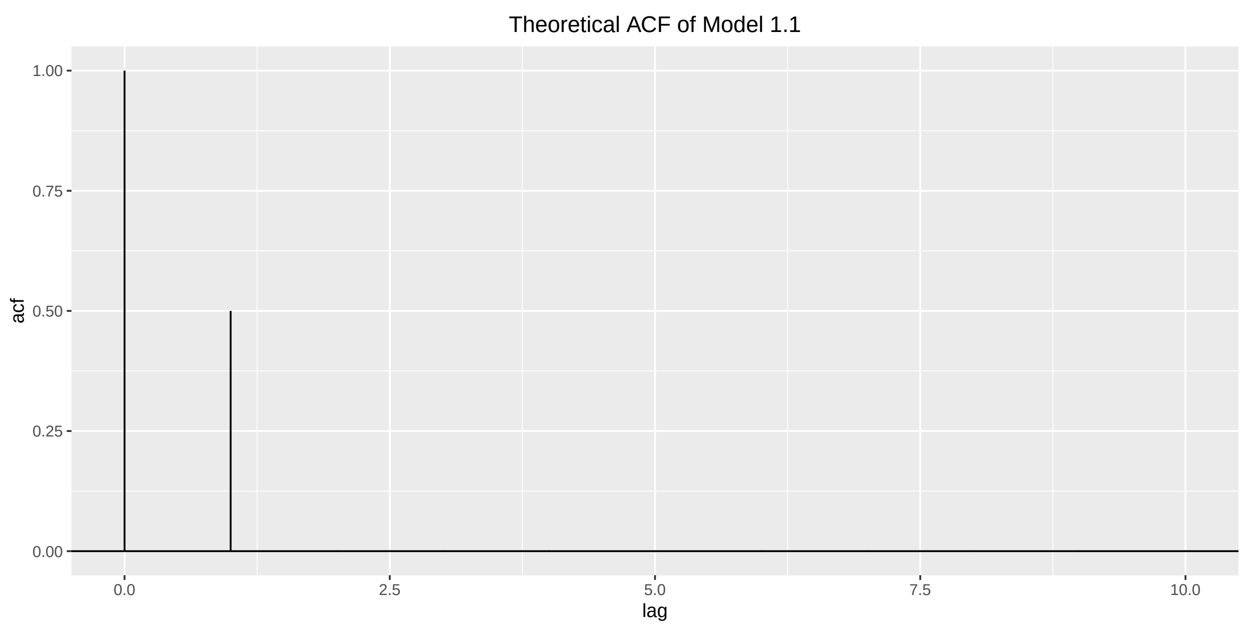
The theoretical autocorrelation function for a linear process is given by

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)},$$
$$\gamma(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_{j+h}\psi_j.$$

In our case, $\sigma_w^2 = 1$, $\psi_i = 1, \quad i = 1, 2.$ Then

$$\gamma(h) = \begin{cases} 2, & |h| = 0 \\ 1, & |h| = 1 \\ 0, & |h| \geq 2 \end{cases}$$

$$\rho(h) = \begin{cases} 1, & |h| = 0 \\ \frac{1}{2}, & |h| = 1 \\ 0, & |h| \geq 2 \end{cases}$$



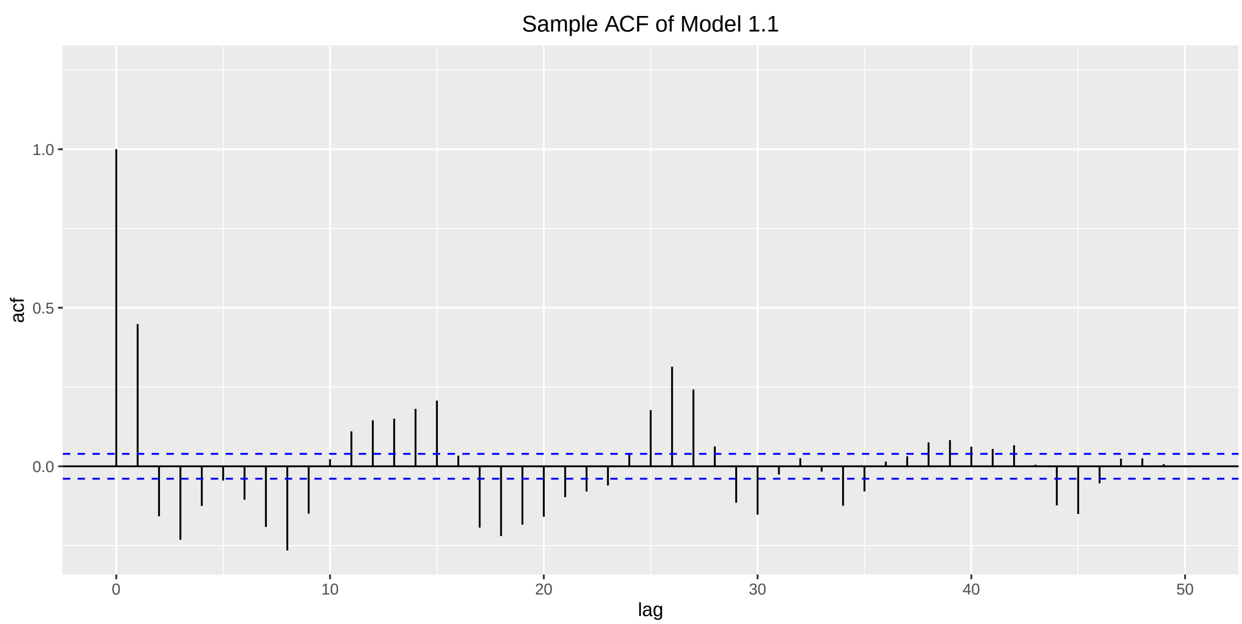
The sample autocorrelation function is given by

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)},$$

$$\hat{\gamma}(h) = n^{-1} \sum_{t=1}^{n-h} (x_{t+h} - \bar{x})(x_t - \bar{x}).$$

```
sample_acf <- function(x,h) {
  s_acovf <- vector()
  x_mean <- mean(x)
  n <- length(x)
  lag <- 0:h
  for (i in lag) {
    summ <- vector()
    for (j in 1:(n - i)) {summ[j] <- (x[j + i] - x_mean) * (x[j] - x_mean)}
    s_acovf[i + 1] <- n^(-1) * sum(summ)
  }
  s_acf <- s_acovf/s_acovf[1]

  return(s_acf)
}
s_acf_50 <- sample_acf(x_50,50)
```



In our case $n = 51$, so $\sigma_{\hat{\rho}_x}(h) = \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{51}}$. Property 1.2 says that with large n, the sample ACF will be approximately normally distributed for fixed H, where $h = 1, 2, \ldots, H$, with zero mean and standard deviation $\sigma_{\hat{\rho}_x}(h)$. We can see in the peaks that they does not seem to follow this distribution for lag larger than 1.

```
x_1000 <- problem1_fun(1000)
x_10000 <- problem1_fun(10000)

s_acf_1000 <- sample_acf(x_1000,1000)
s_acf_10000 <- sample_acf(x_10000,10000)
```