Exercise 4

Problem 1.

U finite dimensional vector space.

V inner product space with inner product $\langle \cdot, \cdot \rangle_V$.

$T: U \to V$ linear and injective.

Define $\langle \cdot, \cdot \rangle_U : U \times U \to \mathbb{K}$:

$$\langle u, v \rangle_U := \langle Tu, Tv \rangle_V$$

An inner product must satisfy:

Linearity in the first component:

$$\langle \lambda u + \mu v, w \rangle = \lambda \langle u, w \rangle + \mu \langle v, w \rangle$$
$$\forall u, v, w \in U$$
$$\lambda, \mu \in \mathbb{K}$$

Conjugate symmetry:

$$\langle u, v \rangle = \overline{\langle v, u \rangle}$$

Positive definiteness

$$\langle u, u \rangle \in \mathbb{R}_{\geq 0}$$
$$\forall u \in U$$
$$\langle u, u \rangle = 0 \Leftrightarrow u = 0$$

Proof:

Linearity in the first component:

$$\begin{aligned}\langle \lambda u + \mu v, w \rangle_U &= \langle T(\lambda u + \mu v), Tw \rangle_V \\ &= \langle \lambda Tu + \mu Tv, Tw \rangle_V \\ &= \lambda \langle Tu, Tw \rangle_V + \mu \langle Tv, Tw \rangle_V \\ &= \lambda \langle u, w \rangle_U + \mu \langle v, w \rangle_U\end{aligned}$$

Conjugate symmetry:

$$\begin{aligned}\overline{\langle v, u \rangle_U} &= \overline{\langle Tv, Tu \rangle_V} \\ &= \langle Tu, Tv \rangle_V \\ &= \langle u, v \rangle_U\end{aligned}$$

Positive definiteness:

$$\begin{aligned}\langle u, u \rangle_U &= \langle Tu, Tu \rangle_V \\ &\geq 0\end{aligned}$$

Since T is linear we have:

$$\begin{aligned}T(0) &= 0 \cdot T(0) \\ &= 0\end{aligned}$$

Since T is injective, this is unique

So $\langle u, u \rangle_U = 0$ iff $u = 0$

Problem 2.

$\mathcal{P}_n$ space of polynomials of degree $\leq n$ w. complex coeffs

$X := \{x_0, x_1, \ldots, x_n\} \subset \mathbb{R}$ distinct.

$T: \mathcal{P}_n \to \mathbb{C}^{n+1}$

$$Tp := \begin{pmatrix} p(x_0) \\ p(x_1) \\ \vdots \\ p(x_n) \end{pmatrix}$$

(a) Proof:

Linearity:

$$\begin{aligned}T(\alpha p + \beta q) &= \begin{pmatrix} \alpha p(x_0) + \beta q(x_0) \\ \alpha p(x_1) + \beta q(x_1) \\ \vdots \\ \alpha p(x_n) + \beta q(x_n) \end{pmatrix} \\ &= \begin{pmatrix} \alpha p(x_0) \\ \alpha p(x_1) \\ \vdots \\ \alpha p(x_n) \end{pmatrix} + \begin{pmatrix} \beta q(x_0) \\ \beta q(x_1) \\ \vdots \\ \beta q(x_n) \end{pmatrix} \\ &= \alpha Tp + \beta Tq\end{aligned}$$

Injectivity

(b)

Problem 3.

U finite dimensional inner product space

$U_1, U_2 \subset U$

$\pi_{U_i}: U \to U$ orthogonal projection onto U_i, $i = 1, 2$

Proof $\pi_{U_1} = \pi_{U_1} \circ \pi_{U_2} \Leftrightarrow U_1 \subset U_2$:

$(\Rightarrow)$:

$$\begin{aligned}\pi_{U_1} \circ \pi_{U_2}(u_2) &= \pi_{U_1} \pi_{U_2} u_2, \quad u_2 \in U_2 \\ &= \pi_{U_1} u_2\end{aligned}$$

$$\Rightarrow U_1 \subset U_2$$

$(\Leftarrow)$:

$$U_1 \subset U_2 \Rightarrow u \in U_1 \Rightarrow u \in U_2$$

$$\begin{aligned}\pi_{U_1} \circ \pi_{U_2}(u) &= \pi_{U_1} \pi_{U_2} u \\ &= \pi_{U_1} u\end{aligned}$$

$$\Rightarrow \pi_{U_1} \circ \pi_{U_2} = \pi_{U_1}$$

Problem 4.

(a) Proof:

Linearity in the first component:

$$\begin{aligned}\langle aA + bB, C \rangle &= tr(C^H(aA + bB)) \\ &= tr(aC^H A + bC^H B) \\ &= a\,tr(C^H A) + b\,tr(C^H B) \\ &= a \langle A, C \rangle + b \langle B, C \rangle\end{aligned}$$

Conjugate symmetry:

$$\begin{aligned}\overline{\langle B, A \rangle} &= \overline{tr(A^H B)} \\ &= \overline{tr(\overline{A}^T B)} \\ &= \overline{\sum_{i=1}^{n} (\overline{A}^T B)_{ii}} \\ &= \overline{\sum_{i=1}^{n} \sum_{j=1}^{m} \overline{A}_{ij} \cdot B_{ji}} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} \cdot \overline{B}_{ij} \\ &= \sum_{i=1}^{n} (B^H A)_{i} \\ &= tr(B^H A) \\ &= \langle A, B \rangle\end{aligned}$$

Positive definiteness

$$\begin{aligned}\langle A, A \rangle &= tr(A^H A) \\ &= \sum_{i=1}^{n} (A^H A)_{ii} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} A^H_{ij} A_{ji} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ji} A_{ji} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ji}^2 \\ &\geq 0\end{aligned}$$

(b) $U := \{A \in Mat_n(\mathbb{R}) \mid A^H = A\}$

$U^\perp := \{B \in Mat_n(\mathbb{R}) \mid \langle A, B \rangle = 0 \quad \forall A \in U\}$

$$\begin{aligned}\langle A, B \rangle &= tr(B^H A) \\ &= \sum_{i=1}^{n} (B^H A)_{ii} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \overline{B}_{ij} A_{ij}\end{aligned}$$

$\sum^{m}$