NTNU | Norwegian University of Science and Technology

PLENARY EXERCISES - TMA4145

Week 35, Wednesday 30. August 2023

Problem 1

Let X, Y and Z be sets. Show that $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

Hint:

1. $x \in X \cap Y$ if $x \in X$ and $x \in Y$.
2. $x \in X \cup Y$ if $x \in X$ or $x \in Y$.
3. Two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$.

Problem 2

Let X, Y, Z be sets, and consider maps $f : X \to Y$ and $g : Y \to Z$. Which of the following statements are true? Give a proof if its true, and a counterexample if not.

a) If $g \circ f : X \to Z$ is injective, then
  - **i)** f is injective.
  - **ii)** g is injective.

b) If $g \circ f : X \to Z$ is surjective, then
  - **i)** f is surjective.
  - **ii)** g is surjective.

c) If $g \circ f : X \to Z$ is bijective, what can we say about f and g?

Hint:

1. A map $f : X \to Y$ is called injective if for any $x, y \in X$ $f(x) = f(y)$ implies $x = y$.
2. A map $f : X \to Y$ is called surjective if for any $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
3. A map $f : X \to Y$ is called bijective if it is both injective and surjective.
4. Let $X = Z = \{x, y\}$ be a two point set.

Problem 3

Let $f : X \to Y$ be a function, and let A be a subset of X. Prove that

$$A \subseteq f^{-1}(f(A)),$$

and if f is injective then equality holds. Show by example that equality need not hold if f is not injective.

Hint:

1. $f(A) := \{y \in Y : y = f(x) \text{ for some } x \in A\}$.
2. $f^{-1}(B) := \{x \in X : f(x) \in B\}$.
3. A map $f : X \to Y$ is called injective if for any $x, y \in X$ such that $f(x) = f(y)$, then $x = y$.
4. Consider the set $A = \{1\} \subset \mathbb{R}$

Problem 4

Let $f : X \to Y$ be a function, and let B be a subset of Y. Prove that

$$f(f^{-1}(B)) \subseteq B,$$

and if f is surjective then equality holds. Show by example that equality need not hold if f is not surjective.

Hint:

1. $f(A) := \{y \in Y : y = f(x) \text{ for some } x \in A\}$.
2. $f^{-1}(B) := \{x \in X : f(x) \in B\}$.
3. A map $f : X \to Y$ is called surjective if for any $y \in Y$ there exists $x \in X$ such that $f(x) = y$.
4. Consider the set $B = \{-1, 1\} \subset \mathbb{R}$