Chapter 7: Option pricing in discrete time - part 2

Exercises - solutions

1. (a) We use the put-call parity: Share + Put = Call + PV(X) or
Share + Put - Call $= 97.70 + 4.16 - 23.20 = 78.66$
and $PV(X) = 80 \times e^{-0.0315} = 77.519$.
We see that the put-call parity did not precisely hold. This probably does not mean that are any arbitrage opportunities since:

    - PCP is only valid for European options, and most traded options are American options.
    - ZX Co. may pay a dividend before expiry.
    - We ignored bid – ask spreads in trading and any other transaction costs.
    - There may have been non synchronous trading, i.e. put price may refer to a different time than the call and/or the share.

    In a perfect market the arbitrage opportunity could be exploited by selling what is expensive and buying what is cheap. Shorting the share, selling the put and buying the call would bring in 97.70+4.16-23.20=78.66. Of this, 77.52 should be put in the bank, the difference 78.66-77.52=1.14 is arbitrage profit. At maturity the share must be returned, but either the short put or the long call will end in the money. In both cases the share will be bought for the exercise price 80 and returned to the broker. The exercise price comes from the bank account, which has grown from PV(X) to X, i.e. 80, so the whole remaining position ends in zero.

2. (a) Incorrect, the risk neutral probabilities cannot be used to describe the likelihood of events, they are for pricing purposes only.

    (b) Incorrect, risk neutral probabilities can be calculated as compounded state prices, as in equation (7.8) in the book, and also as 'standardized' state prices (that sum to 1) as in equation (7.12). The higher the (real) probability is that a state occurs, the higher the state price will be. This is a direct link between the real and the risk neutral probabilities, so they are not independent.

    (c) Incorrect, the risk neutral probabilities are standardized state prices, so they incorporate the market price of risk. Calculating expectations with the risk neutral probabilities means that the 'correction for risk' is included in the expectation, so that it can be omitted from the discount rate. Hence, discounting with the risk free interest rate does not mean assuming that investors are risk neutral, but that risk is accounted for in the numerator of the discounting equation, not in the denominator.

    (d) That is correct. If investors are risk neutral, the 'correction for risk' is empty so that the real and risk neutral probabilities are the same. But investors are risk averse, not risk neutral, so the real probabilities are not the same as the risk neutral ones.

    (e) That is correct. Risk neutral probabilities are standardized state prices, i.e. transformed into a variable with values between zero and 1 that sum to 1. This makes them behave as probabilities, but their nature is that of a price.

3. (a) Since $u \neq r \neq d$ the stock and risk free debt are linearly independent, so there are as many independent securities as there are states of the world, which means that the market is complete. Since $d < r < u$ there are no arbitrage opportunities on this market.

(b) We first calculate the value tree for the stock without and with dividends:

$$
\begin{array}{ccccc}
 & & & & 156.25 \\
 & & & \nearrow & \\
 & & 125 & & \\
 & \nearrow & & \searrow & \\
100 & & & & 100 \\
 & \searrow & & \nearrow & \\
 & & 80 & & \\
 & & & \searrow & \\
 & & & & 64
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & & & 117.19 \\
 & & 125 & \nearrow & \\
 & \nearrow & 93.75 & & \\
100 & & & \searrow & \\
 & \searrow & & & 75 \\
 & & 80 & \nearrow & \\
 & & 60 & & \\
 & & & \searrow & \\
 & & & & 48
\end{array}
$$

The parameters of the binomial process are: $u = 1.25$, $d = 0.8$, $r = 1.07$, $p = (r-d)/(u-d) = (1.07-.8)/(1.25-.8) = 0.6$

i. European call, no dividends:
$O_{uu} = \max[0, 156.25 - 110] = 46.25$
$O_{ud} = \max[0, 100 - 110] = 0$
$O_{dd} = \max[0, 64 - 110] = 0$
$O_u = (.6 \times 46.25 + .4 \times 0)/1.07 = 25.935$
$O_d = (.6 \times 0 + .4 \times 0)/1.07 = 0$
$O = (.6 \times 25.935 + .4 \times 0)/1.07 = 14.54$

ii. European call, 25% dividends:
$O_{uu} = \max[0, 117.195 - 110] = 7.195$
$O_{ud} = \max[0, 75 - 110] = 0$
$O_{dd} = \max[0, 48 - 110] = 0$
$O_u = (.6 \times 7.195 + .4 \times 0)/1.07 = 4.0346$
$O_d = (.6 \times 0 + .4 \times 0)/1.07 = 0$
$O = (.6 \times 4.0346 + .4 \times 0)/1.07 = 2.26$

iii. American call on a non dividend paying stock is the same as a European call

iv. American call, 25% dividend:
$O_{uu} = \max[0, 117.195 - 110] = 7.195$
$O_{ud} = \max[0, 75 - 110] = 0$
$O_{dd} = \max[0, 48 - 110] = 0$
$O_u - alive = (.6 \times 7.195 + .4 \times 0)/1.07 = 4.0346$
$O_u - dead = \max[0, 125 - 110] = 15$
$O_u = \max[alive, dead] = 15$
$O_d = 0$ both dead and alive
$O - alive = (.6 \times 15 + .4 \times 0)/1.07 = 8.4112$
$O - dead = \max[0, 100 - 110] = 0$
$O = \max[alive, dead] = 8.41$

v. European put, no dividends:
$O_{uu} = \max[0, 110 - 156.25] = 0$
$O_{ud} = \max[0, 110 - 100] = 10$
$O_{dd} = \max[0, 110 - 64] = 46$
$O_u = (.6 \times 0 + .4 \times 10)/1.07 = 3.738$
$O_d = (.6 \times 10 + .4 \times 46)/1.07 = 22.804$
$O = (.6 \times 3.738 + .4 \times 22.804)/1.07 = 10.62$

vi. European put, 25% dividends:
$O_{uu} = \max[0, 110 - 117.19] = 0$
$O_{ud} = \max[0, 110 - 75] = 35$
$O_{dd} = \max[0, 110 - 48] = 62$
$O_u = (.6 \times 0 + .4 \times 35)/1.07 = 13.084$
$O_d = (.6 \times 35 + .4 \times 62)/1.07 = 42.804$
$O = (.6 \times 13.084 + .4 \times 42.804)/1.07 = 23.34$

4. Next period the stock price is either $50 \times 1.3 = 65$ or $50 \times .75 = 37.5$. The value trees for the options are:

$$\begin{array}{llllll} & & \max[0, 65 - 45] = 20 & & & \max[0, 45 - 65] = 0 \\ & \nearrow & & & \nearrow & \\ \mathrm{O}_c & & & \mathrm{O}_p & & \\ & \searrow & & & \searrow & \\ & & \max[0, 37.5 - 45] = 0 & & & \max[0, 45 - 37.5] = 7.5 \end{array}$$

(a) For a call option, the option Δ and D are:

$$\begin{aligned} \Delta &= \frac{O_u - O_d}{(u-d) \times S} = \frac{20 - 0}{(1.3 - 0.75) \times 50} = 0.72727 \\ D &= \frac{u \times O_d - d \times O_u}{(u-d) \times r} = \frac{1.3 \times 0 - 0.75 \times 20}{(1.3 - 0.75) \times 1.08} = -25.253 \end{aligned}$$

The price of the call is $O_c = S \times \Delta + D = 50 \times 0.72727 - 25.253 = 11.111$

(b) For a put option, the option Δ and D are:

$$\begin{aligned} \Delta &= \frac{O_u - O_d}{(u-d) \times S} = \frac{0 - 7.5}{(1.3 - 0.75) \times 50} = -0.272\,73 \\ D &= \frac{u \times O_d - d \times O_u}{(u-d) \times r} = \frac{1.3 \times 7.5 - 0.75 \times 0}{(1.3 - 0.75) \times 1.08} = 16.414 \end{aligned}$$

The price of the put $O_P = S \times \Delta + D = 50 \times -0.27273 + 16.414 = 2.7775$

5. The self-financing property means that, when the portfolio is rebalanced, the new portfolio has exactly the same value as the old one. So we could sell the old portfolio and use the money to buy the new one without needing any additional cash. In the two-period example on page 212-213 we start with a hedging portfolio of 0.753 shares and 212.11 in borrowing. If the stock price rises to 500 at t_1 the value of this old portfolio is:

$$0.753 \times 500 + (-212.11 \times 1.07) = 149.54$$

The amount of borrowing has increased with the risk free interest rate over the period. With this stock price the new hedge ratio becomes 1, so we need to buy $1 - 0.753 = 0.247$ of the stock at a price of 500, which costs $0.247 \times 500 = 123.50$. We borrow this amount, so the value of the new portfolio is:

$$1 \times 500 - (123.50 + 212.11 \times 1.07) = 149.54$$

exactly the same as the old portfolio.
If the stock price falls to 320 at t_1 the value of the old portfolio is:

$$0.753 \times 320 + (-212.11 \times 1.07) = 14$$

With this stock price the new hedge ratio becomes 0.174, so we have to sell $0.753 - 0.174 = 0.579$ of the stock at a price of 320. This brings in $0.579 \times 320 = 185.28$

and we use this amount to reduce the borrowing. The value of the new portfolio thus becomes:

$$0.174 \times 320 - ((212.11 \times 1.07) - 185.28) = 14$$

again exactly the same as the old portfolio.