Exercise 3

Problem 1.

(a) $T: \mathbb{C}^2 \to \mathbb{C}^2$:

$(w,z) \mapsto (-z, w)$

Want to find λ s.t. $T\lambda = \lambda v$

$T(w,z) = (-z, w)$

$= \lambda(w,z)$

$\Rightarrow \lambda w = -z, \lambda z = w$

$\Rightarrow \lambda(-\lambda w) = w$

$\Rightarrow -\lambda^2 w = w$

$\Rightarrow \lambda^2 = -1$

$\Rightarrow \lambda = \pm i$

The eigenvalues of T are i and $-i$.

Want to find v s.t. $(T - \lambda Id)^j v = 0$, where $j = \dim(\mathbb{C}^2) = 2$.

$(T - \lambda Id)(w,z) = (-z - \lambda, w - \lambda)$

$(T - \lambda Id)^2 (w,z) = (-w + \lambda, -z - \lambda)$

For $\lambda = i$

$(T - i Id)^2 (w,z) = (-w + i, -z - i) = (0,0)$

$\Rightarrow -w + i = 0, -z - i = 0$

$\Rightarrow v = (i, -i)$

For $\lambda = -i$

$(T - i Id)^2 (w,z) = (-w - i, -z + i) = (0,0)$

$\Rightarrow -w - i = 0, -z + i = 0$

$\Rightarrow v = (-i, i)$

The generalized eigenvector for $\lambda = i$ is $v = (i, -i)$, and for $\lambda = -i$ is $v = (-i, i)$.

(b) $T: \mathbb{C}^2 \to \mathbb{C}^2$:

$(w,z) = (z, 0)$

Want to find λ s.t. $T\lambda = \lambda v$

$T(w,z) = (z, 0)$

$= \lambda(w,z)$

$\Rightarrow z = \lambda w, 0 = \lambda z$

$\Rightarrow 0 = \lambda(\lambda w)$

$\Rightarrow 0 = \lambda^2 w$

$\Rightarrow \lambda = 0$

The eigenvalue of T is $\lambda = 0$.

Want to find v s.t. $(T - \lambda Id)^j v = 0$, where $j = \dim(\mathbb{C}^2) = 2$.

$(T - \lambda Id)(w,z) = (z, 0)$

$(T - \lambda Id)^2 (w,z) = (0,0)$

The generalized eigenvectors for $\lambda = 0$ are all $v \in \mathbb{C}^2$.

(c) $T: \mathbb{C}^3 \to \mathbb{C}^3$:

$(u,w,z) = (3u + w, -u + w, 2z)$

Want to find λ s.t. $T\lambda = \lambda v$

$T(u,w,z) = (3u + w, -u + w, 2z)$

$= \lambda(u,w,z)$

$\Rightarrow 3u + w = \lambda u, -u + w = \lambda w, 2z = \lambda z$

$\Rightarrow \lambda = 2$

The eigenvalue of T is $\lambda = 2$.

Want to find v s.t. $(T - \lambda Id)^j v = 0$, where $j = \dim(\mathbb{C}^2) = 3$.

$(T - \lambda Id)(u,w,z) = (3u + w - \lambda, -u + w - \lambda, 2z - \lambda)$

$(T - \lambda Id)^2 (u,w,z) = (3(3u + w - \lambda) + (-u + w - \lambda) - \lambda, -(3u + w - \lambda) + (-u + w - \lambda) - \lambda, 2(2z - \lambda) - \lambda)$

$= (9u + 3w - 3\lambda - u + w - \lambda - \lambda, -3u - w + \lambda - u + w - \lambda - \lambda, 4z - 4\lambda - \lambda)$

$= (8u + 4w - 5\lambda, -4u - \lambda, 4z - 5\lambda)$

$(T - \lambda Id)^3 (u,w,z) = (3(8u + 4w - 5\lambda) + (-4u - \lambda) - \lambda, -(8u + 4w - 5\lambda) + (4u - \lambda) - \lambda, 2(4z - 5\lambda) - \lambda)$

$= (24u + 12w - 15\lambda - 4u - \lambda - \lambda, -8u - 4w + 5\lambda + 4u - \lambda - \lambda, 8z - 10\lambda - \lambda)$

$= (20u + 12w - 17\lambda, -4u - 4w + 3\lambda, 8z - 11\lambda)$

For $\lambda = 2$:

$(T - 2Id)^3 (u,w,z) = (20u + 12w - 34, -4u - 4w + 6, 8z - 22)$

$= (0, 0, 0)$

$\Rightarrow 20u + 12w - 34 = 0, -4u - 4w + 6 = 0, 8z - 22 = 0$

$\Rightarrow z = \frac{11}{4}$

$\Rightarrow u = -w + \frac{3}{2} \Rightarrow 20(-w + \frac{3}{2}) + 12w - 34 = 0$

$\Rightarrow -20w + 30 + 12w - 34 = 0$

$\Rightarrow -8w - 4 = 0$

$\Rightarrow w = -\frac{1}{2}$

$\Rightarrow u = -\frac{1}{2} + \frac{3}{2} = 1$

$\Rightarrow (u,w,z) = (1, -\frac{1}{2}, \frac{11}{4})$

The generalized eigenvector for $\lambda = 2$ is $v = (1, -\frac{1}{2}, \frac{11}{4})$.

Problem 2.

(a) $p_m(x) = (x+2)$

Eigenvalue $\lambda = -2$

$A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

$p_m(A) = (A + 2I) = 0$ ✓

$\Rightarrow T(z_1, z_2, z_3, z_4) = (-2z_1, -2z_2, -2z_3, -2z_4)$

$\chi_T = (x+2)^4$

(b) $p_m(x) = (x-1)(x-2)(x-4)$

$A = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&4 \end{pmatrix}$, $B = \begin{pmatrix} 1&0&0&0 \\ 0&2&0&0 \\ 0&0&2&0 \\ 0&0&0&4 \end{pmatrix}$, $C = \begin{pmatrix} 1&0&0&0 \\ 0&2&0&0 \\ 0&0&4&0 \\ 0&0&0&4 \end{pmatrix}$

$T_A(z_1, z_2, z_3, z_4) = (z_1, z_2, 2z_3, 4z_4)$

$T_B(z_1, z_2, z_3, z_4) = (z_1, 2z_2, 2z_3, 4z_4)$

$T_C(z_1, z_2, z_3, z_4) = (z_1, 2z_2, 4z_3, 4z_4)$

$\chi_{T_A} = (x-1)^2(x-2)(x-4)$

$\chi_{T_B} = (x-1)(x-2)^2(x-4)$

$\chi_{T_C} = (x-1)(x-3)(x-4)^2$

(c) $p_m(x) = (x-3)^2(x+2)^2$

$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

$T(z_1, z_2, z_3, z_4) = (3z_1 + z_2, 3z_2 + z_3, -2z_3 + z_4, -2z_4)$

$\chi_T = p_m(x)$

(d) $p_m(x) = (x+1)^2(x-1)$

$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$T_A(z_1, z_2, z_3, z_4) = (-z_1 + z_2, -z_2, -z_3, z_4)$

$T_B(z_1, z_2, z_3, z_4) = (-z_1 + z_2, -z_2, z_3, z_4)$

$\chi_{T_A} = (x+1)^3(x-1)$

$\chi_{T_B} = (x+1)^2(x-1)^2$

Problem 3.

(a) $A_1 = \begin{pmatrix} 1&1&0&0 \\ 0&1&1&0 \\ 0&0&1&0 \\ 0&0&0&2 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1&1&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&2 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1&0&0&0 \\ 0&1&1&0 \\ 0&0&1&0 \\ 0&0&0&2 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&2 \end{pmatrix}$

$B_1 = \begin{pmatrix} 1&1&0&0 \\ 0&1&0&0 \\ 0&0&2&1 \\ 0&0&0&2 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1&1&0&0 \\ 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&2 \end{pmatrix}$, $B_3 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&2&1 \\ 0&0&0&2 \end{pmatrix}$, $B_4 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&2 \end{pmatrix}$

$C_1 = \begin{pmatrix} 1&0&0&0 \\ 0&2&1&0 \\ 0&0&2&1 \\ 0&0&0&2 \end{pmatrix}$, $C_2 = \begin{pmatrix} 1&0&0&0 \\ 0&2&1&0 \\ 0&0&2&0 \\ 0&0&0&2 \end{pmatrix}$, $C_3 = \begin{pmatrix} 1&0&0&0 \\ 0&2&0&0 \\ 0&0&2&1 \\ 0&0&0&2 \end{pmatrix}$, $C_4 = \begin{pmatrix} 1&0&0&0 \\ 0&2&0&0 \\ 0&0&2&0 \\ 0&0&0&2 \end{pmatrix}$

(b) Char. poly. for the A's is $\chi_{T_A} = (x-1)^3(x-2)$

Min. poly. for A_1 is $p_m(x) = (x-1)^3(x-2)$

— " — A_2 is $p_m(x) = (x-1)^2(x-2)$

— " — A_3 is $p_m(x) = (x-1)^2(x-2)$

— " — A_4 is $p_m(x) = (x-1)(x-2)$

Char. poly. for the B's is $\chi_{T_B} = (x-1)^2(x-2)^2$

Min. poly. for B_1 is $p_m(x) = (x-1)^2(x-2)^2$

— " — B_2 is $p_m(x) = (x-1)^2(x-2)$

— " — B_3 is $p_m(x) = (x-1)(x-2)^2$

— " — B_4 is $p_m(x) = (x-1)(x-2)$

Char. poly. for the C's is $\chi_{T_C} = (x-1)(x-2)^3$

Min. poly. for C_1 is $p_m(x) = (x-1)(x-2)^3$

— " — C_2 is $p_m(x) = (x-1)(x-2)^2$

— " — C_3 is $p_m(x) = (x-1)(x-2)^2$

— " — C_4 is $p_m(x) = (x-1)(x-2)$

Problem 4.

Assume V finite dimensional vector space

$T: V \to V$ bijective

(a) The constant term a_0 in the minimal polynomial $p_m(x) = a_n x^n + \dots + a_1 x + a_0$ is non-zero iff $p_m(0) = a_0$.
In other words, a_0 is non-zero iff 0 is not a root to $p_m(x)$, i.e. 0 cannot be an eigenvalue of T.
Since T is bijective, we have that $T - 0I$ is bijective, so 0 is not an eigenvalue.
So $a_0 \neq 0$

(b) Since T is bijective, it is also injective and surjective, so the inverse T^{-1} exists.

$T^{-1} = p(T) \quad | \cdot T$

$I = T \cdot p(T)$

$Tp(T) - I = 0$

$\chi_T(T) = 0$

$Tp(T) - I = \chi_T(T)$

$p(T) = T^{-1}\chi_T(T) + T^{-1}$ exists

(c) The smalles degree p can have is the same degree as the minimal polynomial of T.