Exercise 7

Problem 1.

Assume that (X, d_X), (Y, d_Y), and (Z, d_Z) are metric spaces and that $f: X \to Y$ and $g: Y \to Z$ are functions.

(a) Assume that f, g are continous.

Show that $g \circ f: X \to Z$ is continous.

Proof:

Since f is continous, there exists $\delta > 0$ for every $\varepsilon > 0$ such that
$$d_Y(f(x), f(\hat{x})) < \varepsilon$$
whenever $d_X(x, \hat{x}) < \delta$ at a point $x \in X$

Since g is continous, there exists $\delta > 0$ for every $\varepsilon > 0$ such that
$$d_Z(g(y), g(\hat{y})) < \varepsilon$$
whenever $d_Y(y, \hat{y}) < \delta$ at a point $y \in Y$

Selecting $x \in X$ such that $f(x) = y$ and $g(y)$ exists.
$$d_Z(g(f(x)), g(f(\hat{x}))) = d_Z(g(y), g(\hat{y})) < \varepsilon$$
when $d_X(x, \hat{x}) < \delta$

(b) Assume that f, g are Lipschitz continous.

Show that $g \circ f: X \to Z$ is Lipschitz continous.

Proof:

Since f is Lipschitz continous, there exists $L \geq 0$ such that
$$d_Y(f(x), f(y)) \leq L d_X(x, y) \quad \forall x, y \in X.$$
Since g is Lipschitz continous, there exists $L \geq 0$ such that
$$d_Z(g(x), g(y)) \leq L d_Y(x, y) \quad \forall x, y \in Y.$$
Selecting $x, y \in X$,
$$\begin{aligned} d_Z(g(f(x)), g(f(y))) &\leq L_1 d_Y(f(x), f(y)) \\ &\leq L_1 \cdot L_2 d_X(x, y) \\ &= L_3 d_X(x, y), \quad L_3 \leq L_1 \cdot L_2 \end{aligned}$$

Problem 2.

We consider the space $X = \mathcal{C}([0,1])$ with the metric $d_\infty(f, g) := \max_{x \in [0,1]} |f(x) - g(x)|$.

Define the mapping $T: X \to X$ by
$$(Tf)(t) := \int_0^t s f(s)\, ds$$
for $f \in X$ and $0 \leq t \leq 1$.

Show that T is a contraction on X and that 0 is the unique fixed point of T.

Proof:

T contraction:
$$\begin{aligned} d_\infty(T(f(t)), T(g(t))) &= \max_{t \in [0,1]} |T(f(t)) - T(g(t))| \\ &= \max_{t \in [0,1]} \left| \int_0^t s f(s)\, ds - \int_0^t s g(s)\, ds \right| \\ &= \max_{t \in [0,1]} \left| \int_0^t s (f(s) - g(s))\, ds \right| \\ &\leq \max_{t \in [0,1]} \int_0^t |s|\, |f(s) - g(s)|\, ds \\ &\leq d_\infty(f, g) \cdot \max_{t \in [0,1]} \int_0^t |s|\, ds \\ &= d_\infty(f, g) \cdot \max_{t \in [0,1]} \int_0^t s\, ds \\ &= d_\infty(f, g) \cdot \max_{t \in [0,1]} \tfrac{1}{2} t^2 \\ &= \tfrac{1}{2} d_\infty(f, g) \end{aligned}$$

Showing 0 is a fixed point of T:
$$\begin{aligned} T(0)(t) &= \int_0^t s \cdot 0\, ds \\ &= 0 \end{aligned}$$

Since $(\mathcal{C}([0,1]), d_\infty(f, g))$ is complete, this is unique.

Problem 3.

Assume that (X, d_X) is a complete metric space and that $f: X \to X$ is a function.

Assume that there exists $N \in \mathbb{N}$ such that $f^N: X \to X$ is a contraction.

(a) Show that the mapping f has a unique fixed point $x^* \in X$.

Proof:

$f^N(x^*) = x^*$ by Banach fixed point thm.

Since $f^N(x)$ is a contraction
$$d_X(f^N(x), f^N(y)) \leq L d_X(x, y), \quad L < 1.$$
Then,
$$\begin{aligned} d_X(f^N(f(x)), f^N(f(y))) &\leq L d_X(f(x), f(y)) \\ &\leq L d_X(x, y), \quad L < 1. \end{aligned}$$
$\Rightarrow$ f is a contraction, therefore $f(\hat{x}) = \hat{x}$.

So, $f^n(\hat{x}) \xrightarrow{n \geq N} \hat{x}$

$\Rightarrow f(\hat{x}) = \hat{x}$

And since x^* is unique, $f(\hat{x}) = \hat{x} \Rightarrow f(x^*) = x^*$

(b) Show that the sequence given by $x_{n+1} = f(x_n)$ converges for each $x_0 \in X$ to x^*.

Proof:

Since f is a contraction $f(x_n) \xrightarrow{n \to \infty} x^*$.

Problem 4.

Show that there exists a unique continous function $f: [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$ such that
$$f(x) = \begin{cases} \frac{1}{2} f(3x) & , 0 \leq x \leq \frac{1}{3} \\ \frac{1}{2} & , \frac{1}{3} \leq x \leq \frac{2}{3} \\ 1 - \frac{1}{2} f(3(1-x)) & , \frac{2}{3} \leq x \leq 1 \end{cases}$$