Exercise #3

September 06, 2023

Problem 1. (Generalised eigenspaces)

Find all eigenvalues and corresponding generalised eigenvectors of the following linear transformations:

a) The linear transformation $T\colon \mathbb{C}^2 \to \mathbb{C}^2$, $(w,z) \mapsto (-z,w)$.

b) The linear transformation $T\colon \mathbb{C}^2 \to \mathbb{C}^2$, $(w,z) \mapsto (z,0)$.

c) The linear transformation $T\colon \mathbb{C}^3 \to \mathbb{C}^3$, $(u,w,z) \mapsto (3u+w,-u+w,2z)$.

Solution.

a) We consider the linear transformation $T : \mathbb{C}^2 \to \mathbb{C}^2$ given by $(w,z) \mapsto (-z,w)$ We can represent the transformation as the matrix A given by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let us start by finding any eigenvalues of A.

$$0 = \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = (\lambda - i)(\lambda + i).$$

This shows that $\lambda_1 = i$ and $\lambda_2 = -i$ are the eigenvalues of A. To find the first eigenvector, we Gauss eliminate the matrix corresponding to λ_1

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \sim \begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix}$$

which gives the eigenvector

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Similarly for the eigenvector corresponding to λ_2.

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \sim \begin{pmatrix} i & -1 \\ 0 & 0 \end{pmatrix}$$

which gives the eigenvector

$$v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

This gives that the eigenvectors of T are given by

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

b) We consider the linear transformation $T : \mathbb{C}^2 \to \mathbb{C}^2$ given by $(w, z) \mapsto (z, 0)$ We can represent the transformation as the matrix A given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let us start by finding any eigenvalues of A.

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda^2.$$

This shows that $\lambda_1 = \lambda_2 = \lambda = 0$ are the eigenvalues of A. To find the the eigenvector corresponding to $\lambda = 0$, we consider

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which gives the eigenvector

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $\lambda_1 = \lambda_2 = 0$ and the eigenspace of corresponding to 0 is one-dimensional, we need to also find a generalised eigenvector for T.

Consider the matrix $(A - \lambda I)^2 = A^2$, which is given by

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then we see that the vector

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

satisfies

$$A^2 v_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad Av_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which shows that v_2 is a generalised eigenvector of the linear transformation T. Since v_1 and v_2 spans all of $\mathbb{C}^2$, we can conclude that all the eigenvalues of T is $\lambda = 0$ with generalised eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

as any ordinary eigenvector is also a generalised eigenvector.

c) We consider the linear transformation $T : \mathbb{C}^3 \to \mathbb{C}^3$ given by $(u, w, z) \mapsto (3u + 2, -u + 2, 2z)$ We can represent the transformation as the matrix A given by

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Let us start by finding any eigenvalues of A.

$$0 = \det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)\left((3-\lambda)(1-\lambda)+1\right) = (2-\lambda)\left(\lambda^2 - 4\lambda + 4\right) = -(\lambda-2)^3.$$

This shows that $\lambda = 2$ is an eigenvalue of A with multiplicity 3. To find the eigenvectors corresponding to $\lambda = 2$, we consider

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which gives the eigenvectors

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since the eigenspace corresponding to $\lambda = 2$ is two dimensional, we need to find a generalised eigenvector. Consider,

$$(A - \lambda I)^2 = (A - 2I)^2 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider the vector

$$v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

which is linear independent to v_1 and v_2. Then

$$(A - 2I)^2 v_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (A - 2I)v_3 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which shows that v_3 is a generalised eigenvector of T. We can therefore conclude that the eigenvalues of T is given by $\lambda = 2$ with multiplicity 3 and the generalised eigenvectors are

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Problem 2. (Minimal and characteristic polynomials)

In each of the following cases, find a linear operator $T \colon \mathbb{C}^4 \to \mathbb{C}^4$ that has the given polynomial as a minimal polynomial. In addition, find the corresponding characteristic polynomial.

a) The polynomial $p(x) = (x + 2)$.

b) The polynomial $p(x) = (x-1)(x-2)(x-4)$.

c) The polynomial $p(x) = (x-3)^2(x+2)^2$.

d) The polynomial $p(x) = (x+1)^2(x-1)$.

Solution.

The idea to this problem is that we are given the eigenvalues of the operator T as the zeros of the minimal polynomial.

a) For $p(x) = (x+2)$ we know that the eigenvalues are all equal -2 with multiplicity 4. Thus, an example of such an operator is

$$T = -2I,$$

or on matrix form

$$A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

In this case the characteristic polynomial becomes $p_c(x) = (x+2)^4$.

b) For $p(x) = (x-1)(x-2)(x-4)$ we know that the eigenvalues are all equal $1, 2, 4$ with multiplicity 2 for one of them. Thus, an example of such an operator T can be represented by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

In this case the characteristic polynomial becomes $p_c(x) = (x-1)^2(x-2)(x-4)$.

c) For $p(x) = (x-3)^2(x+2)^2$ we know that the eigenvalues are all equal 3 and -2, both with multiplicity 2. Moreover, the exponent tells us the size of the largest Jordan block corresponding to the eigenvalue. Thus, an example of such an operator T can be represented by the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

In this case the characteristic polynomial becomes $p_c(x) = (x-3)^2(x+2)^2 = p(x)$, which is just equal to the minimal polynomial.

d) For $p(x) = (x+1)^2(x-1)$ we know that the eigenvalues are all equal 1 and -1, with some multiplicity for some of them. Moreover, we know that the largest Jordan block corresponding to $\lambda = -1$ is of size 2×2. Thus, an example of such an operator T can be represented by the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case the characteristic polynomial becomes $p_c(x) = (x+1)^3(x-1)$.

Problem 3. (Jordan normal forms)

Let V be a complex vector space with $\dim(V) = 4$, and let $T\colon V \longrightarrow V$ be linear. Assume moreover that the transformation T has precisely the two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$.

a) Find all possibilities for the Jordan normal form of T.

b) In each of the different cases, determine the characteristic polynomial of T and the minimal polynomial of T.

Solution.

a) We have a linear transformation on a four dimensional vector space V. Moreover, it has only two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. We then want to find all the Jordan normal form. The possibilities are

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$
$$A_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad A_7 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad A_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad A_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad A_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

b) We want to find the characteristic, p_c, and the minimal polynomial, p_m, of T.

- For A_1 we have $p_c(x) = (x-1)^3(x-2)$ and $p_m(x) = (x-1)(x-2)$.
- For A_2 we have $p_c(x) = (x-1)^3(x-2)$ and $p_m(x) = (x-1)^2(x-2)$.
- For A_3 we have $p_c(x) = (x-1)^3(x-2)$ and $p_m(x) = (x-1)^3(x-2)$.
- For A_4 we have $p_c(x) = (x-1)^2(x-2)^2$ and $p_m(x) = (x-1)(x-2)$.
- For A_5 we have $p_c(x) = (x-1)^2(x-2)^2$ and $p_m(x) = (x-1)^2(x-2)$.
- For A_6 we have $p_c(x) = (x-1)^2(x-2)^2$ and $p_m(x) = (x-1)(x-2)^2$.
- For A_7 we have $p_c(x) = (x-1)^2(x-2)^2$ and $p_m(x) = (x-1)^2(x-2)^2$.
- For A_8 we have $p_c(x) = (x-1)(x-2)^3$ and $p_m(x) = (x-1)(x-2)$.
- For A_9 we have $p_c(x) = (x-1)(x-2)^3$ and $p_m(x) = (x-1)(x-2)^2$.

- For A_{10} we have $p_c(x) = (x-1)(x-2)^3$ and $p_m(x) = (x-1)(x-2)^3$.

Problem 4. (Inverse of an operator)

Assume that V is a finite-dimensional vector space and that $T\colon V \longrightarrow V$ is bijective.

a) Show that the constant term in the minimal polynomial is non-zero.

b) Show that there exists a polynomial p such that

$$T^{-1} = p(T). \tag{1}$$

c) What is the smallest possible degree of a polynomial p such that (1) holds?

Solution.

a) The map T is bijective if it is both injective and surjective. If T is injective, then we know that $\ker(T) = \{0\}$ and 0 cannot be an eigenvalue of T. In particular, all eigenvalues are non-zero. Let N be the number of different eigenvalues, then the minimal polynomial of T can be written as

$$p_m(x) = \prod_{j=1}^{N} (x-\lambda_i)^{s_i} = (x-\lambda_1)^{s_1} \dots (x-\lambda_N)^{s_N}.$$

Expanding this out gives that the constant term is $c_0 = \lambda_1^{s_1} \dots \lambda_N^{s_N} \neq 0$, as non of the eigenvalues are zero. This shows that the constant term c_0 of the minimal polynomial is non-zero.

b) We want to show that $T^{-1} = p(T)$ for some polynomial p. If this holds, then the polynomial $q(T) = Tp(T) - I = TT^{-1} - I = 0$. An example of such a q is the minimal polynomial p_m. Let p_m be the minimal polynomial corresponding to T. Then we know that

$$0 = p_m(T) = \sum_{j=0}^{N} c_j T^j,$$

where $T^0 = I$. We know from a) that the constant term $c_0 \neq 0$. Thus, if we scale the equation by $-c_0^{-1}$ we see that

$$0 = -I - \sum_{j=1}^{N} \frac{c_j}{c_0} T^j.$$

By rearranging the equation, we end up with

$$I = -\sum_{j=1}^{N} \frac{c_j}{c_0} T^j.$$

If we let p be the polynomial given by

$$p(x) = -\sum_{j=1}^{N} \frac{c_j}{c_0} x^{j-1},$$

then we see that

$$Tp(T) = -\sum_{j=1}^{N} \frac{c_j}{c_0} T T^{j-1} = -\sum_{j=1}^{N} \frac{c_j}{c_0} T^j = I, \quad p(T)T = -\sum_{j=1}^{N} \frac{c_j}{c_0} T^{j-1} T = \sum_{j=1}^{N} \frac{c_j}{c_0} T^j = I.$$

This shows that $p(T) = T^{-1}$.

c) From b) we can note that the polynomial p has degree one lower than the minimal polynomial of T. This is the smallest degree it possibly can have. To see why, let p be a polynomial such that $p(T) = T^{-1}$. Then the polynomial $q(T) = Tp(T) - I = 0$ has degree one higher. By the minimality of the minimal polynomial $\deg p_m \leq \deg q = \deg p + 1$. Which means that $\deg p \geq \deg p_m - 1$.