Exercise 8

Problem 1.

Determine if the following functions $f: \mathbb{R}^3 \to \mathbb{R}$ are norms on $\mathbb{R}^3$:

(a) $f(x_1, x_2, x_3) = |x_1| + |x_2|$

- Pos. def.:

  $f(x_1, x_2, x_3) \geq 0$ clear

  $f(x_1, x_2, x_3) = 0$

  $\Rightarrow |x_1| + |x_2| = 0 \Rightarrow x_1 = x_2 = 0, \forall x_3$

Not a norm since $f(x_1, x_2, x_3) = 0 \not\Leftrightarrow (x_1, x_2, x_3) = 0.$

(b) $f(x_1, x_2, x_3) = |x_1| + (|x_2|^2 + |x_3|^2)^{\frac{1}{2}}$

- Pos. def.:

  $f(x_1, x_2, x_3) \geq 0$ clear

  $f(x_1, x_2, x_3) = 0$

  $\Rightarrow |x_1| + (|x_2|^2 + |x_3|^2)^{\frac{1}{2}} = 0$

  $\Rightarrow (x_1, x_2, x_3) = 0$

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- Abs. hom.

  $$\begin{aligned} f(cx_1, cx_2, cx_3) &= |cx_1| + (|cx_2|^2 + |cx_3|^2)^{\frac{1}{2}} \\ &= |c| \cdot |x_1| + (|c|^2 |x_2|^2 + |c|^2 |x_3|^2)^{\frac{1}{2}} \\ &= |c| \cdot |x_1| + |c| (|x_2|^2 + |x_3|^2)^{\frac{1}{2}} \\ &= |c| f(x_1, x_2, x_3) \end{aligned}$$

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- Triangle ineq.:

  $$\begin{aligned} f(x_1 + y_1, x_2 + y_2, x_3 + y_3) &= |x_1 + y_1| + (|x_2 + y_2|^2 + |x_3 + y_3|^2)^{\frac{1}{2}} \\ &\leq |x_1| + |y_1| + (| \end{aligned}$$

(c) $f(x_1, x_2, x_3) = \max\{|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1|\}$

- Pos. def.:

  $f(x_1, x_2, x_3) \geq 0$ clear

  $f(x_1, x_2, x_3) = 0$

  $\Rightarrow |x_1 - x_2| = |x_2 - x_3| = |x_3 - x_1| = 0$

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- Abs. hom.

  $$\begin{aligned} f(cx) &= \max\{|cx_1 - cx_2|, |cx_2 - cx_3|, |cx_3 - cx_1|\} \\ &= \max\{|c||x_1 - x_2|, |c||x_2 - x_3|, |c||x_3 - x_1|\} \\ &= |c| f(x) \end{aligned}$$

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- Triangle ineq.:

  $$\begin{aligned} f(x+y) &= \max\{|x_1 + y_1 - x_2 - y_2|, |x_2 + y_2 - x_3 - y_3|, |x_3 + y_3 - x_1 - y_1|\} \\ &\leq \max\{|x_1 - x_2| + |y_1 - y_2|, |x_2 - x_3| + |y_2 - y_3|, |x_3 - x_1| + |y_3 - y_1|\} \\ &\leq f(x) + f(y) \end{aligned}$$

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It is a norm.

Problem 2.

Assume that U is a vector space and that $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on U and define $\|u\|_c := (\|u\|_a^2 + \|u\|_b^2)^{\frac{1}{2}}$

(a) Show that $\|\cdot\|_c$ defines a norm on U.

Proof:

- Pos. def.

  $\|u\|_c \geq 0$ clear

  $\|u\|_c = 0$

  $\Rightarrow (\|u\|_a^2 + \|u\|_b^2)^{\frac{1}{2}} = 0$

  $\Rightarrow \|u\|_a^2 + \|u\|_b^2 = 0$

  $\Rightarrow u = 0$

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- Abs. hom.:

  $$\begin{aligned} \|ku\|_c &= (\|ku\|_a^2 + \|ku\|_b^2)^{\frac{1}{2}} \\ &= (|k|^2 \|u\|_a^2 + |k|^2 \|u\|_b^2)^{\frac{1}{2}} \\ &= |k| \|u\|_c \end{aligned}$$

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- Triangle ineq.:

  $$\begin{aligned} \|u+v\|_c &= (\|u+v\|_a^2 + \|u+v\|_b^2)^{\frac{1}{2}} \\ &= (\|u\|_a^2 + 2\|u\|_a \|v\|_a + \|v\|_a^2 + \|u\|_b^2 + 2\|u\|_b \|v\|_b + \|v\|_b^2)^{\frac{1}{2}} \\ &\leq (\|u\|_a^2 + \|u\|_b^2)^{\frac{1}{2}} + (\|v\|_a^2 + \|v\|_b^2)^{\frac{1}{2}} + (2\|u\|_a \|v\|_a + 2\|u\|_b \|v\|_b)^{\frac{1}{2}} \end{aligned}$$

(b) $\{u_n\}_{n \in \mathbb{N}} \subset U$ sequence.

($\Rightarrow$)

$\|u_n - u\|_c \xrightarrow{n \to \infty} 0$

$\|u_n - u\|_c = (\|u_n - u\|_a^2 + \|u_n - u\|_b^2)^{\frac{1}{2}} \xrightarrow{n \to \infty} 0$ if $\|u_n - u\|_a \xrightarrow{n \to \infty} 0$ and $\|u_n - u\|_b \xrightarrow{n \to \infty} 0$

($\Leftarrow$)

$\|u_n - u\|_a \xrightarrow{n \to \infty} 0$

$\|u_n - u\|_b \xrightarrow{n \to \infty} 0$

$\Rightarrow (\|u_n - u\|_a^2 + \|u_n - u\|_b^2)^{\frac{1}{2}} = \|u_n - u\|_c \xrightarrow{n \to \infty} 0 \Rightarrow \|u_n - u\|_c \xrightarrow{n \to \infty} 0$

Problem 3.

For $n \in \mathbb{N}$ define

$x^{(n)} := (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots)$

which is an element of ℓ^p $\forall 1 \leq p \leq \infty$.

(a) Show that the sequence $\{x^{(n)}\}_{n \in \mathbb{N}}$ converges in ℓ^∞.

Proof.

Need to show $\|x\| := \sup_{n \in \mathbb{N}} |x^{(n)}| < \infty$.

This is clear since $x^{(n)} \xrightarrow{n \to \infty} 0$.

(b) Does the sequence $\{x^{(n)}\}_{n \in \mathbb{N}}$ converge in ℓ^1?

Have to check if $\|x\|_1 := \sum_{n=1}^{\infty} |x^{(n)}| < \infty$:

This is clear since $x^{(n)} \to 0$

(c) Does the sequence $\{x^{(n)}\}_{n \in \mathbb{N}}$ converge in ℓ^2?

Since $\ell^1 \subset \ell^2 \Rightarrow$ it does converge

Problem 4.

Show that the space $(\ell^\infty, \|\cdot\|_\infty)$ is complete.

Proof.

Assume $\{x^{(n)}\}_{n \in \mathbb{N}}$ is cauchy.

$\Rightarrow \exists \varepsilon > 0$ s.t. $\|x^{(n)} - x^{(m)}\|_\infty = \sup_k |x_k^{(n)} - x_k^{(m)}| < \varepsilon \quad \forall n, m \geq N \in \mathbb{N}.$

Then for each $k \in \mathbb{N}$, $|x_k^{(n)} - x_k^{(m)}| < \varepsilon$

And thus $\{x_k^{(n)}\}_{n \in \mathbb{N}} \in \mathbb{K}$ is cauchy $\forall k \in \mathbb{N}$

Then $\{x_k^{(n)}\}_{n \in \mathbb{N}}$ converges to x_k

Letting $x = (x_1, x_2, \dots)$

$\Rightarrow x = \{x_k\}_{k \in \mathbb{N}} \in \ell^\infty$

$$\begin{aligned} \Rightarrow \lim_{m \to \infty} \|x^{(n)} - x^{(m)}\|_\infty &= \lim_{m \to \infty} \sup_k |x_k^{(n)} - x_k^{(m)}| \\ &= \sup_k |x_k^{(n)} - x_k| < \varepsilon \end{aligned}$$

$\Rightarrow \ell^\infty$ is complete

Problem 5.

Problem 6.

Denote by $C^1([0,1])$ the space of continuously differentiable functions $f: [0,1] \to \mathbb{R}$.

On $C^1([0,1])$, we consider the following two norms:

$\|f\|_{1,1} := |f(0)| + \int_0^1 |f'(x)| dx$

$\|f\|_\infty := \max_{0 \leq x \leq 1} |f(x)|$

(a) Show that $\|\cdot\|_{1,1}$ is a norm on $C^1([0,1])$.

Proof:

- Pos. def.:

  $\|f\|_{1,1} \geq 0$ clear.

  $\|f\|_{1,1} = 0 \Rightarrow |f(0)| + \int_0^1 |f'(x)| dx = 0$

  $\Rightarrow |f(0)| = -\int_0^1 |f'(x)| dx$

  $\Rightarrow f = 0$

- Abs. hom.:

  $$\begin{aligned} \|cf\|_{1,1} &= |cf(0)| + \int_0^1 |cf'(x)| dx \\ &= |c| |f(0)| + |c| \int_0^1 |f'(x)| dx \\ &= |c| \|f\|_{1,1} \end{aligned}$$

- Triangle-ineq.:

  $$\begin{aligned} \|f+g\|_{1,1} &= |f(0) + g(0)| + \int_0^1 |f'(x) + g'(x)| dx \\ &\leq |f(0)| + |g(0)| + \int_0^1 |f'(x)| dx + \int_0^1 |g'(x)| dx \\ &\leq \|f\|_{1,1} + \|g\|_{1,1} \end{aligned}$$

(b) Assume $\|f_n - f\|_{1,1} \xrightarrow{n \to \infty} 0$

$\Rightarrow |f_n(0) - f(0)| \xrightarrow{n \to \infty} 0$

$$\begin{aligned} \int_0^1 |f_n'(x) - f'(x)| dx &= \left| \int_0^1 f_n'(x) - f'(x) dx \right| \\ &= |f_n(x) - f(x)| \xrightarrow{n \to \infty} 0 \end{aligned}$$

$\|f_n - f\|_\infty = \max_{x \in [0,1]} |f_n(x) - f(x)| \xrightarrow{n \to \infty} 0$

(c)