TIØ4146 Finance for Science and Technology Students

Chapter 8 - Option Pricing in Continuous Time

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Logarithmic stock returns

Recall from second chapter the difference between discretely and continuously compounded returns

- discretely compounded returns:

$$r = (S_t - S_{t-1})/S_{t-1}$$

  - r is return over period $t-1$ to t
  - S_t, S_{t-1} are stock prices end - begin period.
  - End prices calculated as:

$$S_T = S_0(1+r)^T$$

  S_T, S_0 are stock price time T, and now

- Discretely compounded stock returns
  - are *easily aggregated across investments:*
    - attractive in portfolio analysis
    - return portfolio = weighted average stock returns

$$\frac{S_0^A}{S_0^A + S_0^B} \times \frac{S_1^A - S_0^A}{S_0^A} + \frac{S_0^B}{S_0^A + S_0^B} \times \frac{S_1^B - S_0^B}{S_0^B} = \frac{S_1^A - S_0^A}{S_0^A + S_0^B} + \frac{S_1^B - S_0^B}{S_0^A + S_0^B}$$

  - but *non-additive over time:*
    - 5% p. year over 10 years = 62,9% return (1.05^{10}) not 50%
- Option pricing uses individual returns over time
  - makes continuously compounded returns convenient

- Continuously compounded returns calculated as:

$$\frac{S_T}{S_0} = e^{rT} \quad or \quad S_T = S_0 e^{rT} \quad and \quad \ln\frac{S_T}{S_0} = \ln e^{rT} = rT$$

  - Log returns additive over time:
    - $\ln\left(\frac{S_1}{S_0} \times \frac{S_2}{S_1}\right) \Rightarrow \ln\frac{S_1}{S_0} + \ln\frac{S_2}{S_1} = \ln e^{r_1} + \ln e^{r_2} = r_1 + r_2$
    - convenient to use in continuous time models
  - But: non-additive across investments:
    - log is non-linear $\rightarrow$ ln of sum $\neq$ sum of ln's

- How to describe return behaviour over time?
  - log returns are independently and identically distributed (iid)
    - iid assumption means we can invoke Central Limit Theorem: *sum of n iid variables is $\pm$ normally distributed*

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Consequences of normally distributed returns:

- returns = ln stock prices
  - if returns $\backsim$ N $\Rightarrow$ stock prices $\backsim$ log N.

$$f_R(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right), \quad x \in \mathbb{R}$$

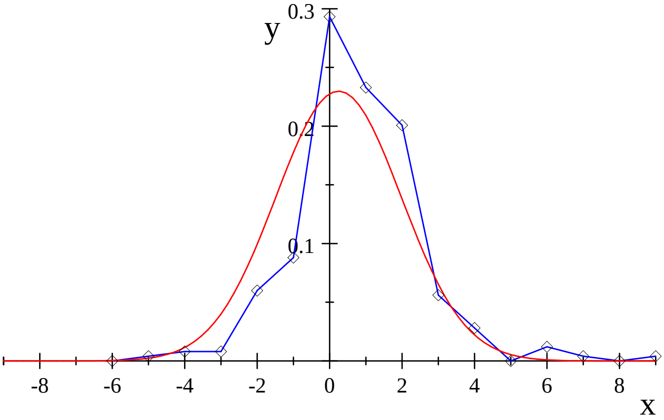
$$f_S(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{log(x)-\mu}{\sigma}\right)^2\right), \quad x \in (0, +\infty)$$

- sum 2 indep. normal variables is also normal with
  - mean = sum 2 means
  - variance = sum 2 variances

- extend to many time periods $\Rightarrow$ mean & variance grow linearly with time
  - so $R_T \backsim N(\mu T, \sigma^2 T)$
  - $R_T =$ continuously compounded return time $[0, T]$
  - expectation $E[R_T] = \mu T$
  - variance $var[R_T] = \sigma^2 T$
  - instantaneous return = μ
- Some more consequences:
  - iid returns follow a random walk
  - random walks have *Markov property* of memorylessness
    - past returns & patterns useless to predict future returns
    - means market is weak form efficient.

- Assumptions & consequences fit the real world well but real life stock returns have:
  - fatter tails
  - more skewness,
  - more kurtosis

  than normal distribution
  Fatter tails give underpricing of financial risks



Frequency distribution of daily returns Apple (blue) and normal distribution with same mean and variance (red)

Modelling stock returns: Brownian motion

- In discrete time - variables:
  - we list all possibilities as:
    - states of the world or
    - values in binomial tree
- In continuous time - variables:
  - infinite number of possibilities, cannot be listed
  - have to express in probabilistic way;
  - Standard equipment: *stochastic process*.

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Most used process is *Brownian motion*, or Wiener process

- Discovered ± 1825 by botanist Robert Brown
- looked through microscope at pollen floating on water
- observed pollen moving around

Physics described by Albert Einstein in 1905

Mathematical process described by Norbert Wiener in 1923

We use the

- term *Brownian motion*
- and the symbol W or $\widetilde{W}$ (for Wiener)

Definition:

Process $\widetilde{W}$ is standard Brownian motion if:

- $\widetilde{W}_t$ is continuous and $\widetilde{W}_0 = 0$,
- has independent increments
- increments $\widetilde{W}_{s+t} - \widetilde{W}_s \backsim N(0, t)$, which implies:
- increments are stationary: only function of length of time interval t, not of location s.

From definition follows:

- Brownian motion has Markov property

Discrete representation over short period δt :

- $\epsilon\sqrt{\delta t}$, ϵ = random drawing from standard normal distribution

Brownian motion has remarkable properties:

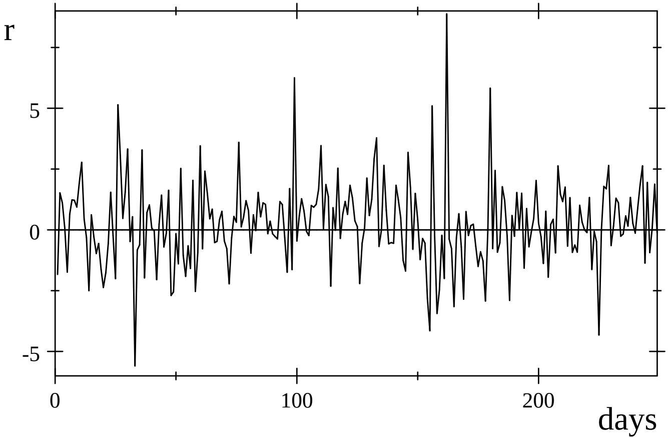
- wild: no upper - lower bounds, will eventually hit any barrier
- continuous everywhere, differentiable nowhere:
  - never 'smooths out' if scale is compressed or stretched
  - that why special, stochastic calculus is required
- is a fractal

Standard Brownian motion poor model of stock price behaviour:

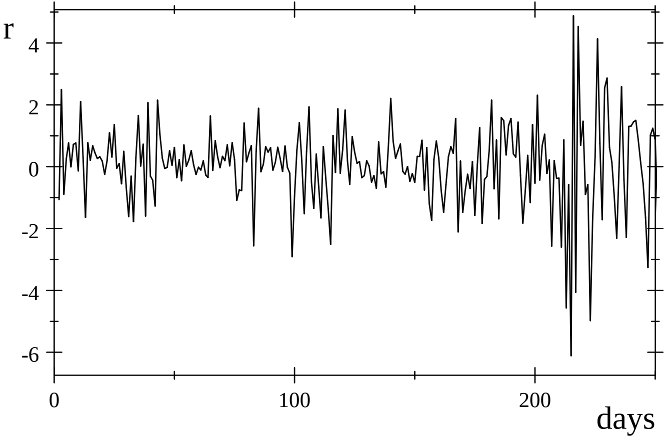
- Catches only random element
- Misses individual parameter for stock's volatility
- Misses expected positive return (positive *drift*)
- Misses proportionality: changes should be in % not in amounts

Missing elements expressed by adding:

- deterministic drift term for expected return
- parameter for stock's volatility
- proportionality: return and random movements (or volatility) in proportion to stock's value



Daily returns Apple (%) from 1 Sept. 2011 to 28 Aug. 2012



Daily returns Nasdaq-100 index for 252 days from 4 October 2010 to 30 September 2011

Standard model is *geometric Brownian motion* in a stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t d\widetilde{W}_t \qquad (1)$$
$$S_0 > 0$$

- $d =$ next instant's incremental change
- $S_t =$ stock price at time t
- $\mu =$ drift coefficient, exp. instantaneous stock return
- $\sigma =$ diffusion coefficient, stock's volatility (stand. dev. returns), 'scales' random term
- $\widetilde{W} =$ standard Brownian motion, stochastic disturbance term
- $S_0 =$ initial condition (a process has to start somewhere)
- μ, σ are assumed to be constants

Geometric Brownian motion has all the properties we set out to model
But is also restricted:

- constant volatility
- no jumps or ’catastrophes’

Formula (1) is stochastic differential equation (sde)

- is a differential equation with a stochastic process in it
- Need a special, stochastic calculus to manipulate sdes

Re-write a bit to clarify:

$$\frac{S_{t+dt} - S_t}{S_t} = \mu dt + \sigma d\widetilde{W}_t$$

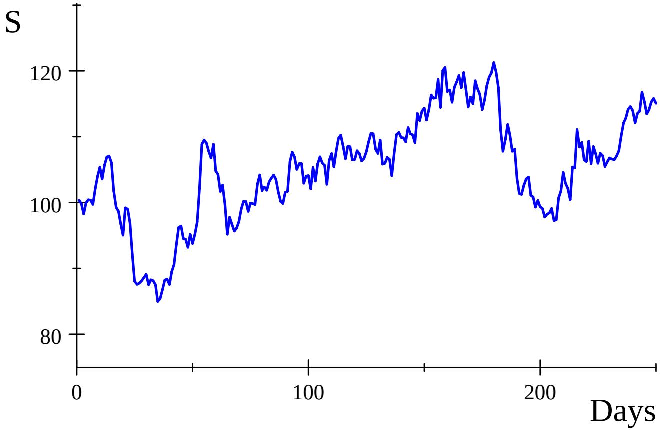
- $(S_{t+dt} - S_t) / S_t$ stock return over time interval dt
- μdt continuously compounded expected return over time interval dt (comparable with (1+r)^T in discrete time)
- $d\widetilde{W}_t$ is random element ('surprise') in stock return
- σ stock's volatility (s.d.) to scale the random element (some stocks are more volatile than others)

Financial market also contains risk free debt, D

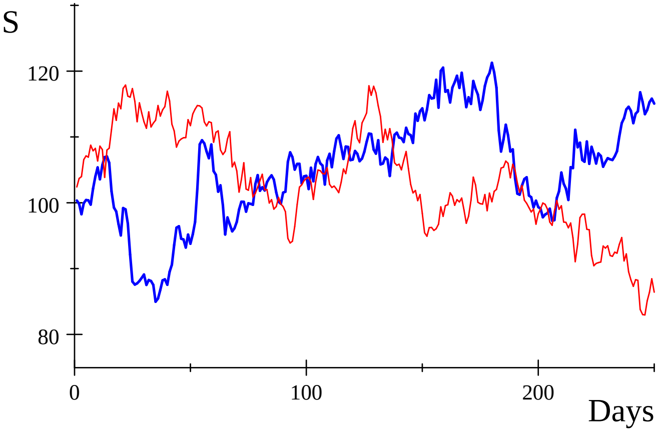
- defined in similar, but simpler, manner:

$$dD_t = rD_t dt \tag{2}$$

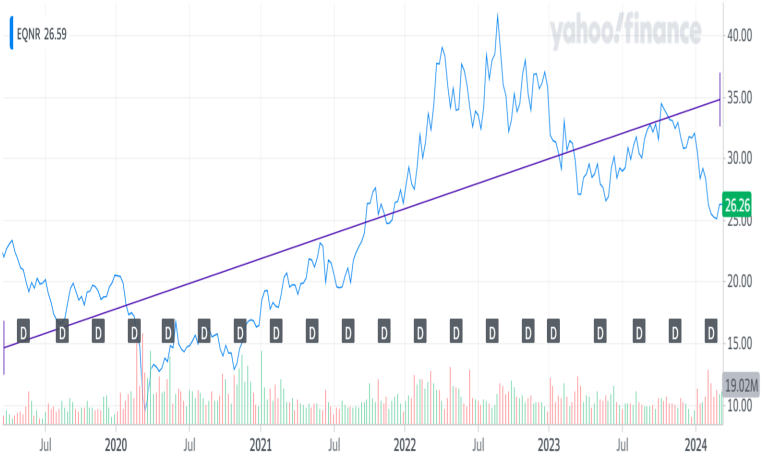
- r is short for r_f, risk free rate (also called money market account or bond)
- risk free $\rightarrow$ no stochastic disturbance term
- natural interpretation for r is short interest rate
- r is assumed to be constant



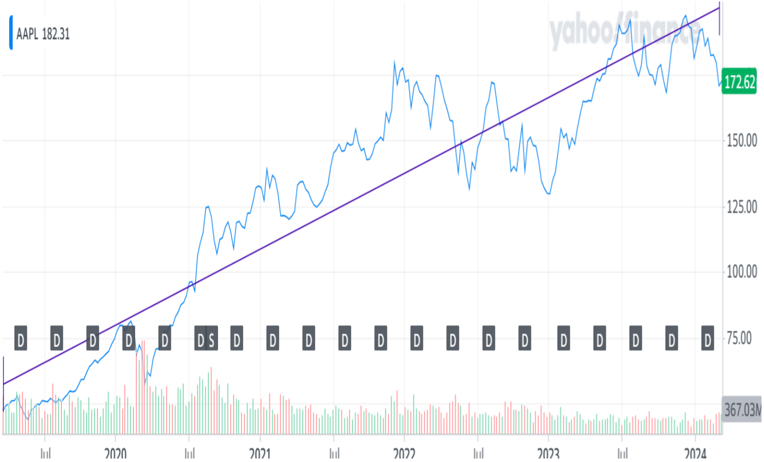
Sample path of geometric Brownian motion with $\mu = 0.15$, $\sigma = 0.3$ and T=250



Sample paths of geometric Brownian motion with $\mu = 0.15$, $\sigma = 0.3$ and T=250



Equinor returns in the past 5 years. Picture from Yahoo Finance



Apple
returns in the past 5 years. Picture from Yahoo Finance

How to price an option using this continuous setting?

- Constructing a replication portfolio
  1. Black & Scholes original work uses partial differential equations (outline in appendix)
  2. Cox, Ross Rubinstein used a replication method to construct the binomial and shows that this approach converges to B&S formula
- Simply changing from the physical probability measure to the equivalent risk neutral probability measure
  1. Martingale method (outline in appendix)

Solving the sde

Under the martingale risk measure we have a dynamic process with drift of risk free rate and, under measure P, BM disturbance term:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{3}$$

- sdes are notoriously difficult to solve
- Deterministic equivalent of (3) simplified by taking logarithms
- Try same transformation here
  - that is how it is done, trial & error

Have to use stochastic calculus (Ito's lemma), result:

$$d(\ln S_t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t \tag{4}$$

changes ln(stock price) follow BM, drift $(r - \frac{1}{2}\sigma^2)$, diffusion σ

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Recall: increments Brownian motion normally distributed
and notice: drift and diffusion of

$$d(\ln S_t) = (r - \frac{1}{2}\sigma^2)dt + \sigma\, dW_t$$

are constants $\Rightarrow d(\ln S_t)$ also normally distributed:

$$\ln S_T - \ln S_0 \sim N((r - \frac{1}{2}\sigma^2)T,\ \sigma^2 T)$$

$$or \quad \ln S_T \sim N(\ln S_0 + (r - \frac{1}{2}\sigma^2)T,\ \sigma^2 T)$$

We use this property later on

Constant drift and diffusion make process for $d(\ln S_t)$ very simple sde can be integrated directly over time interval $[0, T]$, result:

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma W_T} \tag{5}$$

- since $\ln S_T$ is normally distributed
- S_T must be lognormally distributed

$E[S_t]$ follows from properties lognormal distribution:

- expectation of lognormally distributed variable is

$$e^{m + \frac{1}{2}s^2}$$

- m and s are mean and variance of corresponding normal distribution

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We have

$$\ln S_T \sim N(\ln S_0 + (r - \frac{1}{2}\sigma^2)T,\ \sigma^2 T)$$

So expectation of S_T is:

$$E[S_T] = e^{\ln S_0 + (r - \frac{1}{2}\sigma^2)T + \frac{1}{2}\sigma^2 T} = S_0 e^{rT}$$

$$E[S_T] = S_0 e^{rT} \text{ means } e^{-rT} E[S_T] = S_0$$

discounted future exp. stock price = current stock price under prob. measure P

- risky assets can be discounted with risk free rate
- as long as expectations are under measure P

The exact equivalent of Binomial model

Problem:

- price now (t=0) of European call option $O^E_{c,0}$,
  - exercise price X,
  - matures at time T,
  - written on non-dividend paying stock

Using martingale method:

$$O_{c,0} = e^{-rT} E\left[O_{c,T}\right] \tag{6}$$

r is the risk free rate

Option's payoff at maturity:

$$O_{c,T} = \begin{cases} S_T - X & \text{if} \quad S_T > X \\ 0 & \text{if} \quad S_T \leq X \end{cases}$$

can be written as:

$$O_{c,T} = (S_T - X)1_{S_T > X} \tag{7}$$

$1_{S_T > X}$ is step function:

$$1_{S_T > X} = \begin{cases} 1 & \text{if} \quad S_T > X \\ 0 & \text{if} \quad S_T \leq X \end{cases}$$

Substituting step function (7) into option value (6):

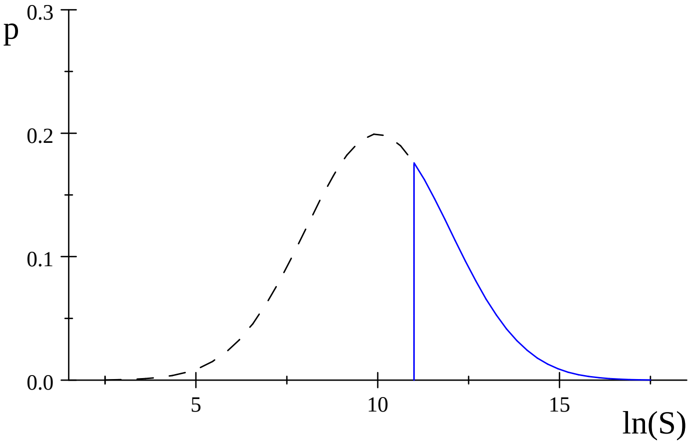
$$O_{c,0} = e^{-rT} E\left[(S_T - X) 1_{S_T > X}\right] \tag{8}$$

To prepare for rest of derivation, we write option value (8) as:

$$O_{c,0} = e^{-rT} E\left[(e^{\ln S_T} - e^{\ln X}) 1_{\ln S_T > \ln X}\right] \tag{9}$$

We use two key elements:

1. $\ln S_T$ is normally distributed,
   mean = $(\ln S_0 + (r - \frac{1}{2}\sigma^2)T)$, var.= $\sigma^2 T$
2. We can regard step function as truncation of distribution of S_T on left: values $< X$ replaced by zero
   (truncated distributions are well researched, formula for truncated normal distribution available)



Lognormally distributed stock price ($ln(S) \backsim N(10, 2)$, dashed), and its left truncation at $ln(S) = 11$ (solid)

We use following step function for normally distributed variable Y with mean M and variance v^2 truncated at A:

$$E\left[\left(e^Y - e^A\right) 1_{Y>A}\right] = e^{M+\frac{1}{2}v^2} N\left(\frac{M+v^2-A}{v}\right) - e^A N\left(\frac{M-A}{v}\right) \tag{10}$$

$N(.)$ is cum. standard normal distr.
Has same form as (9), apply to option pricing problem :
$M = \ln S_0 + (r - \frac{1}{2}\sigma^2)T$
$v^2 = \sigma^2 T \rightarrow v = \sigma\sqrt{T}$
$Y = \ln S_T$
$A = \ln X$

Substituting:

- Details of our problem (M, v^2, Y, A) into formula (10) for the expectation of truncated distribution
- that expectation formula in our option pricing formula

and collecting terms we get the famous Black and Scholes formula:

$$\begin{aligned} O_{c,0} = S_0 N & \left(\frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \\ & - Xe^{-rT} N \left(\frac{\ln(S_0/X) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \end{aligned} \tag{11}$$

Defining, as is commonly done:

$$d_1 = \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{12}$$

and

$$d_2 = \frac{\ln(S_0/X) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{13}$$

we get the usual form of the *Black & Scholes option pricing formula*:

$$O_{c,0} = S_0 N(d_1) - Xe^{-rT} N(d_2) \tag{14}$$

with the corresponding value of a European put:

$$O_{p,0} = Xe^{-rT} N(-d_2) - S_0 N(-d_1) \tag{15}$$

Interpretation:

$$O_{c,0} = \underbrace{(S_0)}_{\substack{\text{stock}\\ \text{price}}} \underbrace{N(d_1)}_{\substack{\text{option}\\ \text{delta}}} - \underbrace{\left(Xe^{-rT}\right)}_{\substack{\text{PV}\\ \text{(exerc.p.)}}} \underbrace{N(d_2)}_{\substack{\text{prob. of}\\ \text{exercise}}}$$

$N(d_1)$ = option delta, has different interpretations:

- *hedge ratio*: # shares needed to replicate option
- *sensitivity:* of option price for changes in stock price
- technical: partial derivative w.r.t. stock price: $\partial O_{c,0} / \partial S_0 = N(d_1)$

What is *not* in the Black and Scholes formula:

- real drift parameter μ
- investors' attitudes toward risk
- other securities or portfolios

Greediness, in $max[]$ expressions, implicit in analysis.

Reflects conditional nature of B&S:
As the binomial model, B&S only translates existing security prices on a market into prices for additional securities.

Determinants of option prices

In B&S, stock price + four other variables
Option price sensitivity for these 4 derived in same way as Δ (partial derivatives), called 'the Greeks'

Determinant	Greek	Effect on call option	Effect on put option
Exercise price		< 0	> 0
Stock price	Delta	$0 < \Delta_c < 1$	$-1 < \Delta_p < 0$
Volatility	Vega	$\nu_c > 0$	$\nu_p > 0$
Time to maturity	Theta	$-\Theta_c < 0$	$-\Theta_p <> 0$
Interest rate	Rho	$\rho_c > 0$	$\rho_p < 0$
	Gamma	$\Gamma_c > 0$	$\Gamma_p > 0$

'The Greeks' is a bit of a misnomer

- X is determinant without Greek
- Vega is not a Greek letter
- Gamma is Greek without determinant, gamma is:
  - effect of increase in stock price on delta
  - second derivative option price w.r.t. stock price

Generally, option value increases with time to maturity

- American options always do
- European call on dividend paying stock may decrease with time to maturity if dividends are paid in 'extra' time.
- Value of deep in the money European puts can decrease with time to maturity: means longer waiting time before exercise money is received

An example:

Calculate value of at the money European call

- matures in one year
- strike price of 100
- underlying stock pays no dividends
- has annual volatility of 20%
- risk free interest rate is 10% per year.

We have our five determinants:
$S_0 = 100$, $X = 100$, $r = .1$, $\sigma = .2$ and $T = 1$.

$$d_1 = \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$= \frac{\ln(100/100) + (.1 + \frac{1}{2}.2^2)1}{.2\sqrt{1}} = .6$$

$$d_2 = d_1 - \sigma\sqrt{T} = .6 - .2\sqrt{1} = .4$$

Areas under normal curve for values of d_1 and d_2 can be found:

- table in compendium (good enough for this course), calculator, spread sheet, etc.:

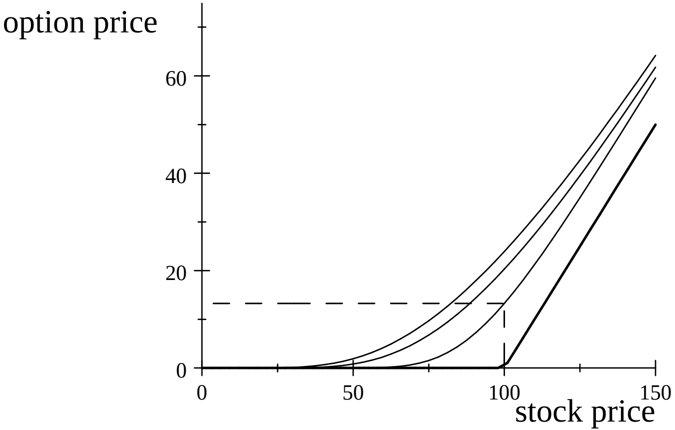
d=	0	0.01	0.02	0.03	0.04	0.05	0.06	0.09
0	0.500	0.504	0.508	0.512	0.516	0.520	0.524	0.536
0.1	0.540	0.544	0.548	0.552	0.556	0.560	0.564	0.575
0.2	0.579	0.583	0.587	0.591	0.595	0.599	0.603	0.614
0.3	0.618	0.622	0.626	0.629	0.633	0.637	0.641	0.652
0.4	0.655	0.659	0.663	0.666	0.670	0.674	0.677	0.688
0.5	0.691	0.695	0.698	0.702	0.705	0.709	0.712	0.722
0.6	0.726	0.729	0.732	0.736	0.739	0.742	0.745	0.755
0.7	0.758	0.761	0.764	0.767	0.770	0.773	0.776	0.785
0.8	0.788	0.791	0.794	0.797	0.800	0.802	0.805	0.813
0.9	0.816	0.819	0.821	0.824	0.826	0.829	0.831	0.839
1	0.841	0.844	0.846	0.848	0.851	0.853	0.855	0.862
2.5	0.994	0.994	0.994	0.994	0.994	0.995	0.995	0.995

NormalDist$(.6) = 0.72575$, NormalDist$(.4) = 0.65542$,
Option price becomes:

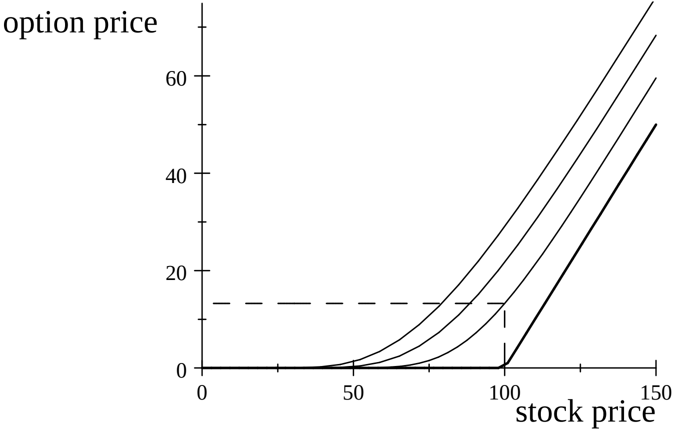
$$O_{c,0} = 100 \times (0.72575) - 100e^{-.1}(0.65542) = 13.27$$

Value put option calculated with equation or the put call parity:

$$\begin{aligned} O_{p,0} &= O_{c,0} + Xe^{-rT} - S_0 \\ &= 13.27 + 100e^{-.1} - 100 = 3.75 \end{aligned}$$



Call option prices for $\sigma = 0.5$ (top), 0.4 and 0.2 (bottom)



Call option prices for $T = 3$ (top), 2 and 1 (bottom)

Dividends

Black & Scholes assumes

- European options
- on non dividend paying stocks

Can be adapted to allow for deterministic (non-stochastic) dividends (can be predicted with certainty)

Dividends:

- stream of value out of the stock
- stream accrues to stockholders
- not option holders

Stock price for stockholders has:

- stochastic part (stock without dividends)
- deterministic part (PV dividends)

Stock price for option holders:

- only stochastic part relevant

Adaptation Black & Scholes formula:

- subtract PV(dividends) from stock price (S_0)
- dividends certain $\rightarrow$ discount with risk free rate
- (implicitly redefines volatility parameter σ for stochastic part only)

Other determinants (X, T and r) unaffected by dividends

Example:

- same stock used before
- pays semi-annual dividends of 2.625
  - first after 3 months
  - then after 9 months

Stock price = 100, volatility 20%, risk free interest rate 10% per year.

What is value European call, maturity 1 year,
strike price = 100?

$S_0 = 100$, $X = 100$, $r = .1$, $\sigma = .2$ and $T = 1$.

Start by calculating PV dividends:

- $2.625e^{-.25\times.1} + 2.625e^{-.75\times.1} = 5.$
- makes adjusted stock price $S_0 = 100 - 5 = 95$

Then we can proceed as before:

$$d_1 = \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$= \frac{\ln(95/100) + (.1 + \frac{1}{2}.2^2)1}{.2\sqrt{1}} = 0.34353$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.34353 - .2\sqrt{1} = 0.14353$$

Areas under normal curve for values d_1 and d_2 are:

- $\text{NormalDist}(0.34353) = 0.6344$ and
- $\text{NormalDist}(0.14353) = 0.5571$.

So the option price becomes:

$$O_{c,0} = 95 \times (0.6344) - 100e^{-.1}(0.5571) = 9.86$$

- value call lowered by dividends
- from 13.27 to 9.86

Value of a put (same specifications):

$$O_{p,0} = Xe^{-rT} N(-d_2) - S_0 N(-d_1)$$

- Just calculated that $d_1 = 0.34353$ and $d_2 = 0.14353$
  - $\text{NormalDist}(-0.34353) = 0.3656$ and
  - $\text{NormalDist}(-0.14353) = 0.44294$
- To use the table, remember the symmetric property $N(-d) = 1 - N(d)$
- Value of the put is:

$$O_{p,0} = 100 \times e^{-.1} (0.44294) - 95 \times (0.3656) = 5.35$$

- value of put increased by dividends
- from 3.75 to 5.35

Matching discrete and continuous time volatility

We have expressed volatility in 2 ways:

- In binomial model:
  - difference between up and down movement
- In Black and Scholes model:
  - volatility parameter σ used to scale $\widetilde{W}$

If we want to switch models

- we have match the parameters
- recalculate μ and σ into $u,\ d$ and p

Looking at small time interval δt

- we can equate the return expressions:

$$e^{r\delta t} = pu + (1-p)d$$

r = risk free rate and p = risk neutral probability

- we can also equate variance expressions:

$$\sigma^2\delta t = pu^2 + (1-p)d^2 - [pu + (1-p)d]^2$$

notice:

- continuous variance increases with time (δt)
- discrete variance uses definition:
  variance of a variable A is $E(A^2) - [E(A)]^2$

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This gives us 2 expressions:

- 1 for return, 1 for variance
- for 3 unknowns: $p, \ u$ and d
- need additional assumption for third equation

Most common assumption is:

$$u = \frac{1}{d}$$

three equations give (after much algebra):

$$u = e^{\sigma\sqrt{\delta t}}, \quad d = e^{-\sigma\sqrt{\delta t}} \quad and \quad p = \frac{e^{r\delta t} - d}{u - d}$$

Same definition of p we found in binomial model

Implied volatility

Black & Scholes formula has 5 determinants of option prices:

- X, T, S, r, σ are model inputs
- 6 if dividends are included

4 of them are easy to obtain:

- X, T, S, r are, at least in principle, observable:
  - X and T are determined in option contract
  - S and r are market determined

σ is not observable

There are 2 ways of obtaining numerical value for σ:

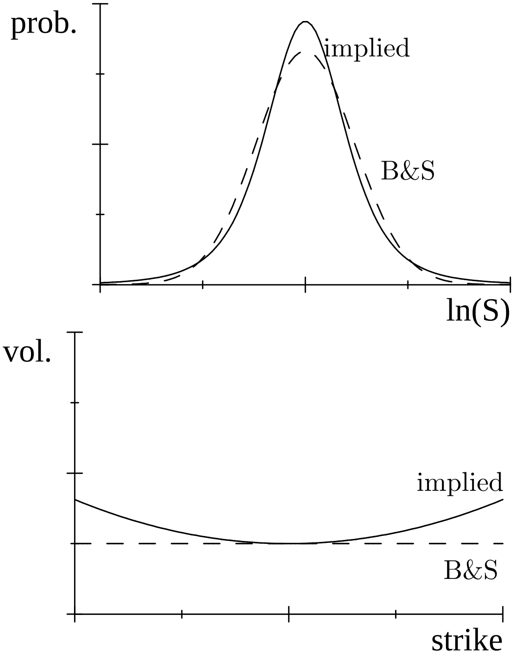
1. Estimate from historical values and extrapolate into future;
   1.1 assumes, like Black & Scholes, that volatility is constant
   1.2 known not to be the case
   (volatility peaks around events as quarterly reports)

2. Estimate from prices of other options;
   2.1 given X, T, S, r each value for σ corresponds to 1 B&S price and vice-versa
   2.2 for given price, run B&S in reverse (numerically) and find σ
   2.3 called *implied volatility*

- Implied volatility is commonly used:
    - option traders quote option prices in volatilities
    - not $ or € amounts.
- Can also be used to test validity of B&S model
- How do you use implied volatility to test B&S?
    - Black & Scholes assumes constant volatility:
    - Options with different X and T should give same implied volatility.

Implied volatility typically not constant:

- far in- and out-of the money options give higher implied volatilities than at the money options
  - called *volatility smile* after its graphical representation
  - implies more kurtosis (peakedness) of stock prices than lognormal distribution
  - also fatter tails, but intermediate values less likely
- Stock options may also imply volatility skewness:
  - far out of the money calls priced lower than far out of the money puts (or far in the money calls)
  - implies skewed distribution of stock prices
  - left tail fatter than right tail
- Implied volatility may also increase with time to maturity


prob.
implied
B&S
ln(S)
vol.
implied
B&S
strike