NTNU | Norwegian University of Science and Technology

PLENARY EXERCISES - TMA4145

Week 40, Wednesday 04. October 2023

Problem 1

Let (X, d) be a metric space, and let $A_1, \dots A_n$ be subsets of X.

1. Show that

$$\bigcup_{i=1}^{n} \overline{A_i} = \overline{\bigcup_{i=1}^{n} A_i}.$$

Hint:

1. $\overline{A}$ is the smallest closed subset which contains A?
2. $\overline{A}$ contains A and all its limit points.
3. What do we know of the subsequences of convergent sequences?

Problem 2

Consider the linear map $T : C([0,1]) \to \mathbb{R}$ given By

$$Tf = f\left(\frac{1}{2}\right).$$

1. Is $T : (C([0,1]), d_\infty) \to (\mathbb{R}, |\cdot|)$ continuous?
2. Is $T : (C([0,1]), d_1) \to (\mathbb{R}, |\cdot|)$ continuous?

Hint:

1. $d_\infty(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$.
2. $d_1(f,g) = \displaystyle\int_0^1 |f(x) - g(x)| dx$.
3. T is continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $Tg \in B_\varepsilon(Tf)$ whenever $g \in B_\delta(f)$.

Problem 3

Let (X, d) be a metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence.

1. Show that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded.

Hint:

1. A subset $A \subset X$ is called bounded if there exists a constant $M > 0$ such that $\sup_{x,y \in A} d(x, y) \leq M$.
2. We need to show that $d(x_n, x_m) \leq M$ for all $n, m \in \mathbb{N}$ and some $M > 0$.
3. Start by writing the definition of a Cauchy Sequence.

Problem 4 - Old exam problem

Consider the metric space $(\mathbb{R}, d)$ with the metric

$$d(x, y) = \frac{|x-y|}{|x-y|+1}.$$

1. Is this a complete metric space?

Hint:

1. You do not have to prove that d is a metric on $\mathbb{R}$
2. What upper bounds do we have on d?
3. We are interested in small values of ε for convergence of limits.