Exercise #2

August 30, 2023

Problem 1. (Eigenvalues for a transformation of polynomials)

Denote by $\mathcal{P}_5$ the space of all polynomials of degree ≤ 5 with complex coefficients. Define moreover the transformation

$$\begin{aligned} T \colon \mathcal{P}_5 &\longrightarrow \mathcal{P}_5, \\ p(x) &\mapsto xp'(x) - p''(x). \end{aligned}$$

a) Find the matrix of the transformation T with respect to the monomial basis $M := \{1, x, x^2, x^3, x^4, x^5\}$ of $\mathcal{P}_5$.

b) Find all the eigenvalues of T.

Solution.

a) Let $p \in \mathcal{P}_5$. Then we can write p in terms of the monomial basis $M = \{1, x, x^2, x^3, x^4, x^5\}$ as

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5.$$

Let us consider how T acts on the polynomial p. This gives

$$\begin{aligned} Tp(x) = xp'(x) - p''(x) =& x(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^5) - 2c_2 - 6c_3 x - 12c_4 x^2 - 20c_5 x^3 \\ =& 2c_2 + (c_1 - 6c_3)x + (2c_2 - 12c_4)x^2 + (3c_3 - 20c_5)x^3 + 4c_4 x^4 + 5c_5 x^5. \end{aligned} \tag{1}$$

To find the matrix representation of the transformation T, we recall that $\mathcal{P}_5 \cong \mathbb{C}^6$ through the map

$$1 \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \dots \quad x^5 \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

If we denote the matrix representation of T by A and use (1), we can then write the polynomial p and Tp

as the column vectors,

$$x_p = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix}, \quad Ax_p = \begin{pmatrix} 2c_2 \\ c_1 - 6c_3 \\ 2c_2 - 12c_4 \\ 3c_3 - 20c_5 \\ 4c_4 \\ 5c_5 \end{pmatrix}.$$

This gives the matrix

$$A = \begin{pmatrix} 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 & 0 & 0 \\ 0 & 0 & 2 & 0 & -12 & 0 \\ 0 & 0 & 0 & 3 & 0 & -20 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

b) From the matrix representation we see that A is a upper triangular matrix, and so the eigenvalues are given by the diagonal elements. This means that the eigenvalues of A, and thus the eigenvalues of T, are $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2, \lambda_4 = 3, \lambda_5 = 4$, and $\lambda_6 = 5$.

Problem 2. (Linear transformations of rank one)

Assume that V is a finite dimensional vector space with $\dim(V) = n \geq 2$, and that $T\colon V \to V$ is a linear mapping of rank one.

(Recall that the rank of T is defined as the dimension of the range of T. In particular, T is of *rank one*, if $\dim(\operatorname{ran} T) = 1$.)

a) Show that there exist $u \in V$ and a linear mapping $\ell\colon V \to \mathbb{K}$ such that

$$Tv = \ell(v)\, u \qquad \text{for all } v \in V.$$

(Hint: Write $\operatorname{ran} T = \operatorname{Span}(u)$ *for some* $u \in V$ *and conclude that there exists a mapping* $\ell\colon V \to \mathbb{K}$ *such that* $Tv = \ell(v)u$*. Then it remains to show that* ℓ *is linear.)*

b) Show that T has two eigenvalues, namely 0 and $\ell(u)$. Find the corresponding eigenspaces.

(Hint: What is the dimension of $\ker(T)$ *and what is the connection between the kernel and eigenvalues?)*

c) Show that the mapping T is diagonalisable if $\ell(u) \neq 0$.

Solution.

a) We want to show that there exist $u \in V$ and a linear mapping $\ell : V \to \mathbb{K}$ such that $T(v) = \ell(v)u$ for all $v \in V$, under the assumption that T is a mapping of rank one.

Since T is a mapping of rank one, we know that $\dim(\operatorname{ran} T) = 1$, and so there exists some some non-zero vector $u \in V$ such that $\operatorname{ran} T = \operatorname{Span}(u)$. This means that for any $v \in V$ we have $T(v) = c_v u$ for some $c_v \in \mathbb{K}$. Define the mapping $\ell : V \longrightarrow \mathbb{K}$ as $\ell(v) = c_v$. It remains to show that ℓ is a linear mapping from V to $\mathbb{K}$.

Since T is a linear mapping it follows that for any $v, w \in V$, and $\alpha \in \mathbb{K}$, we have

$$\ell(v+w)u = T(v+w) = T(v) + T(w) = \ell(v)u + \ell(w)u = (\ell(v) + \ell(w))u.$$

It then follows that

$$(\ell(v+w) - \ell(v) - \ell(w))\, u = 0,$$

and therefore $\ell(v+w) = \ell(v) + \ell(w)$ as u is a non-zero element of V. Similarly, we have

$$\ell(cv)u = T(cv) = cT(v) = c\ell(v)u,$$

and therefore $\ell(cv) = c\ell(v)$, as u is a non-zero element of V. This proves that $\ell : V \longrightarrow \mathbb{K}$ is a linear map.

b) We want to show that T has two eigenvalues, namely 0 and $\ell(u)$.

Let us first ensure that 0 and $\ell(u)$ are eigenvalues. By the rank-nullity theorem, we have

$$\dim(\operatorname{ran} T) + \dim(\ker T) = \dim V = n, \quad \dim(\ker T) = n - \dim(\operatorname{ran} T) \geq n - 1 \geq 1.$$

This means that $\ker T$ is strictly bigger than $\{0\}$. We can therefore find a non-zero vector $v \in \ker(T)$. Thus,

$$T(v) = 0 = 0v,$$

which shows that 0 is an eigenvalue of T. Likewise, note that $T(v) = \ell(v)u$ for all $v \in V$. In particular, for $v = u$, we have $T(u) = \ell(u)u$. Note that $u \neq 0$, or $\operatorname{ran}(T)$ would be zero dimensional. This shows that $\ell(u)$ is also an eigenvalue of T.

Assume now there is some other eigenvalue $\lambda \in \mathbb{K} \backslash \{0, \ell(u)\}$, and let v be the corresponding eigenvector. Then

$$\lambda v = T(v) = \ell(v)u,$$

which shows that v is linear dependent on u, and thus $v \in \operatorname{Span}(u)$. In particular, $v = \alpha u$ for some $\alpha \in \mathbb{K} \backslash \{0\}$. However, this means that

$$\lambda\alpha u = \lambda v = T(v) = T(\alpha u) = \alpha T(u) = \alpha\ell(u)u \quad \Longrightarrow \quad \ell(u) = \lambda,$$

as $\alpha \neq 0$, which is a contradiction. Therefore, the only eigenvalues are 0 and $\ell(u)$.

If $\ell(u) = 0$, then $u \in \ker(T)$ and 0 is the only eigenvalue of T. Moreover, the dimension of the kernel is $n - 1$ as T is of rank one. Then, for any $v \in V$, we have

$$T^2(v) = T(\ell(v)u) = \ell(v)T(u) = 0,$$

and so T is nilpotent. In particular, T is not diagonalisable.

If $\ell(u) \neq 0$, then the eigenspaces span all of V. Since any non-zero element of $\ker T$ is an eigenvector of T with eigenvalue 0, we have that the eigenspace of T corresponding to the eigenvalue 0 is $\ker T$. By the rank-nullity theorem, $\dim(\ker T) = n - 1$. Likewise, $\operatorname{Span}(u) = \operatorname{ran} T$ is the eigenspace corresponding to the eigenvalue $\ell(u) \neq 0$, which has dimension 1. Since $\ker(T) \cap \operatorname{Span}(u) = \{0\}$ and $\dim(\ker T) + \dim(\operatorname{Span} u) = n = \dim V$, we see that $V = \operatorname{Span}(u) \oplus \ker(T)$. This means that any vector $v \in V$ can be written as $v = c_v u + w$ where $c_v \in \mathbb{K}$ and $w \in \ker(T)$, and since the direct sum of these two eigenspace gives all of V, there cannot be any other eigenspaces.

c) We want to show that T is diagonalisable under the assumption that $\ell(u) \neq 0$. This is equivalent to the eigenvectors of T being a basis of V, which is equivalent to V being the direct sum of the eigenspaces. This we already showed in b).

A map is diagonalisable if there exists a basis $\mathcal{B} = \{v_1, \ldots v_n\}$ of V consiting of the eigenvectors of T. Thus for any $v \in V$ we can expand it as $v = \sum_{i=1}^{n} c_i v_i$ such that

$$T(v) = T(c_1 v_1 + \ldots + c_n v_n) = \lambda_1 c_1 v_1 + \ldots + \lambda_n c_n v_n.$$

Since $V = \ker(T) + \operatorname{Span}(u)$, we have a basis for V given by $\mathcal{B} = \{u\} \cup \mathcal{M}$, where $\mathcal{M} = \{w_1, \ldots, w_{n-1}\}$ is any basis of $\ker T$. In particular, any $v \in V$ can then be written as $v = c_1 u + c_2 w_1 + \ldots c_n w_{n-1}$. Then by linearity of T, we are left with

$$\begin{aligned} T(v) = T(c_1 u + c_2 w_1 + \ldots c_n w_{n-1}) = & c_1 T(u) + + c_2 T(w_1) + \ldots c_n T(w_{n-1}) \\ = & \ell(u) c_1 u \\ = & \ell(u) c_1 u + 0 \left(c_2 w_1 + \ldots c_n w_{n-1}\right). \end{aligned}$$

This shows that T is diagonalisable as the eigenvalues of T are precisely $\ell(u)$ and 0.

Problem 3. (Eigenvalues and eigenvectors of a matrix transformation)

Denote by $\mathrm{Mat}_n(\mathbb{C})$ the space of $(n \times n)$-dimensional complex matrices.

Let $\alpha_1, \dots, \alpha_n \in \mathbb{C} \setminus \{0\}$ be fixed and denote by $D \in \mathrm{Mat}_n(\mathbb{C})$ the diagonal matrix with diagonal entries $\alpha_1, \dots, \alpha_n$, that is,

$$D = \begin{pmatrix} \alpha_1 & 0 & \cdots & \cdots & 0 \\ 0 & \alpha_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \alpha_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & \alpha_n \end{pmatrix}.$$

We define the linear transformation

$$T\colon \mathrm{Mat}_n(\mathbb{C}) \longrightarrow \mathrm{Mat}_n(\mathbb{C}),$$
$$A \longmapsto DAD^{-1}.$$

a) Verify that the mapping T is linear.

b) Show that T is bijective, and find the inverse mapping T^{-1}.

c) Recall that the (i, j)-th *elementary matrix* $E^{(i,j)} \in \mathrm{Mat}_n(\mathbb{C})$ for $1 \leq i, j \leq n$ is given by its entries

$$E^{(i,j)}_{k,\ell} = \begin{cases} 1 & \text{if } k = i \text{ and } \ell = j, \\ 0 & \text{else.} \end{cases}$$

Show that each elementary matrix $E^{(i,j)}$ is an eigenvector of T and find the corresponding eigenvalue.

Solution.

a) Note that D is invertible as it is a diagonal matrix with non-zero entries on the diagonal. Thus D^{-1} is the diagonal matrix with α_j^{-1} on the diagonal.

   To verify that T is linear, we need to show that $T(cA) = cT(A)$ and $T(A + B) = T(A) + T(B)$ for all $c \in \mathbb{C}$, and $A, B \in \mathrm{Mat}_n(\mathbb{C})$.

   Let $c \in \mathbb{C}$ and $A \in \mathrm{Mat}_n(\mathbb{C})$. Then

   $$T(cA) = D(cA)D^{-1} = (cD)AD^{-1} = cDAD^{-1} = cT(A),$$

   where we used the linearity of matrix multiplication, namely $D(cx) = cDx$. If $A, B \in \mathrm{Mat}_n(\mathbb{C})$, then

   $$T(A + B) = D(A + B)D^{-1} = DAD^{-1} + DBD^{-1} = T(A) + T(B),$$

   by the associative properties of matrix multiplication. This shows that T is a linear map.

b) We want to show that T is a bijection. That is, we need to show that T is injective and surjective, or you can find the inverse map directly.

Let us start by showing that T is surjective. That means that for each $A \in \mathrm{Mat}_n(\mathbb{C})$, there exists $B = \mathrm{Mat}_n(\mathbb{C})$ such that $A = T(B)$. Let $A \in \mathrm{Mat}_n(\mathbb{C})$, and consider $B = D^{-1}AD$. Then

$$T(B) = D(D^{-1}AD)D^{-1} = A.$$

Since $A \in \mathrm{Mat}_n(\mathbb{C})$ was arbitrary, we can conclude that T is surjective.

injective. Namely, let $A, B \in \mathrm{Mat}_n(\mathbb{C})$, and assume that $T(A) = T(B)$. This gives

$$0 = T(A) - T(B) = DAD^{-1} - DBD^{-1} = D(A - B)D^{-1}.$$

Multiplying by D^{-1} on the left, and D on the right, we see that

$$A - B = D^{-1}(D(A - B)D^{-1})D = D^{-1}(0)D = 0,$$

which shows that $A = B$, and so T is injective.

From the above calculation, we see that a candidate for T^{-1} is given by the map $A \mapsto D^{-1}AD$. Let us call this map S for now. To verify that S actually is the inverse, we note that

$$S(T(A)) = D^{-1}(D(A)D^{-1})D = A, \quad T(S(A)) = D(D^{-1}AD)D^{-1} = A,$$

which shows that $S = T^{-1}$.

c) We want to verify that $E^{(i,j)}$ is an eigenvector of T, and find the corresponding eigenvalue.

Let $D_{k,l}$ and $D^{-1}_{k,l}$ denote the entries of D and D^{-1} respectively. That is

$$D_{k,l} = \begin{cases} \alpha_k, & k = l, \\ 0, & \text{else}, \end{cases} \qquad D^{-1}_{k,l} = \begin{cases} \alpha_k^{-1}, & k = l, \\ 0, & \text{else}. \end{cases}$$

We can start by writing out the matrix multiplication. Recall that for two matrices $A = (a_{i,j}), B = (b_{l,k})$ the elements of the matrix AB through matrix multiplication is given by

$$(AB)_{k,l} = \sum_{j=1}^{n} a_{k,j} b_{j,l}.$$

This gives

$$(E^{(i,j)}D^{-1})_{k,l} = \sum_{m=1}^{n} E^{(i,j)}_{k,m} D^{-1}_{m,l} = E^{(i,j)}_{k,j} D^{-1}_{j,l} = \alpha_l E^{(i,j)}_{k,l} = \begin{cases} \alpha_j^{-1}, & l = j, k = i, \\ 0 & \text{else}. \end{cases}$$

Thus, if we look at $T(E^{(i,j)})$ we see that

$$(T(E^{(i,j)}))_{k,l} = (DE^{(i,j)}D^{-1})_{k,l} = \sum_{m=1}^{n} \alpha_l^{-1} D_{k,m} E^{(i,j)}_{m,l} = \alpha_j^{-1} \alpha_i E^{(i,j)}_{k,l} = \begin{cases} \alpha_i \alpha_j^{-1}, & i = k, j = l, \\ 0 & \text{else}. \end{cases}$$

This shows that $T(E^{(i,j)}) = \alpha_i \alpha_j^{-1} E^{(i,j)}$, and so is an eigenvector of T with eigenvalue $\alpha_i \alpha_j^{-1}$.