Exercise #1

August 23, 2023

Problem 1. (Orthogonal complement and direct sums)

Denote $U := \text{Span}\{(1,0,0,0),(0,1,0,0)\} \subset \mathbb{R}^4$.

a) Find the orthogonal complement $U^\perp$ of U, and show explicitly that $U \oplus U^\perp = \mathbb{R}^4$.

b) Find a subspace V of $\mathbb{R}^4$ different from $U^\perp$ that satisfies $U \oplus V = \mathbb{R}^4$.

Solution.

a) We want to determine $U^\perp$ and show that $U \oplus U^\perp = \mathbb{R}^4$. Note that for any $x \in \mathbb{R}^4$ it can be written as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad x_1, x_2, x_3, x_4 \in \mathbb{R}.$$

Let $y \in U$. Then it can be written as

$$y = \begin{pmatrix} y_1 \\ y_2 \\ 0 \\ 0 \end{pmatrix}, \quad y_1, y_2 \in \mathbb{R}.$$

Recall that $U^\perp = \{x \in \mathbb{R}^4 : x \cdot y = 0, \forall y \in U\}$. If $x \in U^\perp$, then by the definition of the orthogonal complement

$$0 = x \cdot y = \sum_{i=1}^{4} x_i y_i = x_1 y_1 + x_2 y_2.$$

Since this must hold for any $y_1, y_2 \in \mathbb{R}$ it follows that $x_1 = x_2 = 0$. We can therefore characterize the orthogonal complement as

$$U^\perp = \{x \in \mathbb{R}^4 : x_1 = x_2 = 0\} = \text{span}\{(0,0,1,0)^T, (0,0,0,1)^T\}.$$

In order to show that $\mathbb{R}^4 = U \oplus U^\perp$ we need that for every $x \in \mathbb{R}^4$ there exists $y \in U$ and $z \in U^\perp$ such that $x = y + z$, and that $U \cap U^\perp = \{0\}$.

By the linearity of vector addition, we have for any $x \in \mathbb{R}^4$,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} = y + z,$$

where $y \in U$ and $z \in U^\perp$.

Now assume that $x \in U \cap U^\perp$. Then

$$\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} = x = \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix},$$

and so $x = 0$. This shows that $\mathbb{R}^4 = U \oplus U^\perp$.

b) We want to find another subspace V which is different from $U^\perp$ such that $\mathbb{R}^4 = U \oplus U$. The idea is to find two linearly independent vectors y, z which together with the basis of U form a basis of $\mathbb{R}^4$, but are not orthogonal.

For instance, consider

$$V = \operatorname{span}\{(1, 0, 1, 0)^T, (0, 1, 0, 1)^T\}.$$

We claim that $\mathbb{R}^4 = U \oplus V$. To see this, we note that we can add zeros to the first two components. Namely, for any $x \in \mathbb{R}^4$,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 + x_3 \\ x_2 - x_4 + x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 \\ x_2 - x_4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \\ x_3 \\ x_4 \end{pmatrix} = y + z,$$

where $y \in U$ and $z \in V$. Moreover, assume that $x \in U \cap V$. Then

$$\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} = x = \begin{pmatrix} x_3 \\ x_4 \\ x_3 \\ x_4 \end{pmatrix}.$$

In particular, by looking at components we note that $x_1 = x_3 = 0$ and $x_2 = x_4 = 0$, and so $x = 0$. This shows that $\mathbb{R}^4 = U \oplus V$.

Problem 2. (Even and odd polynomials)

Let $\mathcal{P}$ be the vector space of polynomials with real coefficients. Denote moreover by U the subspace of even polynomials, and by V the subspace of odd polynomials.

Show that $\mathcal{P} = U \oplus V$.

Solution.

Let $\mathcal{P}$ denote the space of polynomials with real coefficients. We want to show that $\mathcal{P} = U \oplus V$ where U is the subspace of all even polynomials, and V the subspace of all odd polynomials. That is, we want to show that any polynomial can be written as a sum of even and odd polynomials.

Any polynomial p can be written as

$$p(x) = \frac{p(x) + p(-x)}{2} + \frac{p(x) - p(-x)}{2},$$

which is a sum of an even and odd polynomial. Moreover, if p is both an even and odd polynomial we must have

$$p(x) = p(-x) = -p(x) \quad \Rightarrow \quad p(x) = 0.$$

This shows that $\mathcal{P} = U \oplus V$.

Problem 3. (Differentiation and integration of polynomials)

Let $\mathcal{P}$ be the vector space of polynomials with real coefficients. We consider the following linear transformations of $\mathcal{P}$:

$$\begin{aligned} S \colon \mathcal{P} &\to \mathcal{P}, \\ p &\mapsto p', \end{aligned} \qquad \text{and} \qquad \begin{aligned} T \colon \mathcal{P} &\to \mathcal{P}, \\ p &\mapsto \int_0^x p(t)\,dt. \end{aligned}$$

(That is, S is the differentiation operator; T an integration operator.)

a) Find the kernel and the range of the transformations S and T.

b) Show that the composition $S \circ T$ is the identity operator on $\mathcal{P}$, but that $T \circ S$ is different from the identity operator.

Solution.

a) We want to find the kernel and range of the transformations S and T. Recall that the kernel $\ker(S) := \{p \in \mathcal{P} : S(p) = 0\}$, while $\operatorname{Im}(S) := \{p \in \mathcal{P} : p = S(q) \text{ for some } q \in \mathcal{P}\}$.

   Since both S and T are linear transformations it suffices to investigate the basis elements. One choice of a basis for $\mathcal{P}$ is given by $\{x^n\}_{n=0}^{\infty}$. To see why it suffices to consider the basis elements, we let $p \in \mathcal{P}$. Then there exists $N \in \mathbb{N} \cup \{0\}$ such that

$$p(x) = \sum_{n=0}^{N} c_n x^n.$$

   Now, applying S to the polynomial p gives

$$Sp(x) = \left(\sum_{n=0}^{N} c_n x^n\right)' = \sum_{n=0}^{N} c_n (x^n)' = \sum_{n=0}^{N} c_n S(x^n),$$

   and similarly for T.

Let us start by considering the transformation S. Let $n \in \mathbb{N} \cap \{0\}$. Then

$$S(x^n) = (x^n)' = \begin{cases} nx^{n-1}, & n \geq 1 \\ 0, & n = 0. \end{cases}$$

which shows that $\ker(S) = \{p \in \mathcal{P} : p(x) = c, c \in \mathbb{R}\}$. That is, the kernel of S contains all the constants functions.

For the image of S, we claim that $\mathrm{Im}(S) = \mathcal{P}$. Namely, for any $p \in \mathcal{P}$, it can be written as

$$p(x) = \sum_{n=0}^{N} c_n x^n,$$

for some $N \in \mathbb{N} \cup \{0\}$. Now consider the polynomial $q_p \in \mathcal{P}$ given by

$$q_p(x) = \sum_{n=0}^{N} \frac{c_n}{n+1} x^{n+1}.$$

Then it follows that $S(q_p) = p$. To see this, we use the linearity of S to get

$$S(q_p)(x) = \left(\sum_{n=0}^{N} \frac{c_n}{n+1} x^{n+1} \right)' = \sum_{n=0}^{N} \frac{c_n}{n+1} \left(x^{n+1}\right)' = \sum_{n=0}^{N} c_n x^n = p(x).$$

Since p was an arbitrary polynomial, it follows that any polynomial $p \in \mathcal{P}$ can be written as $S(q)$ for some polynomial $q \in \mathcal{P}$. This shows that $\mathrm{Im}(S) = \mathcal{P}$.

Let us now consider the transformation $T : \mathcal{P} \to \mathcal{P}$. Again, since this is a linear transformation, it suffices to consider how it acts on the basis elements. For any $n \in \mathbb{N} \cap \{0\}$, it follows that

$$T(x^n) = \int_0^x t^n \, dt = \frac{t^{n+1}}{n+1}\bigg|_0^x = \frac{x^{n+1}}{n+1}.$$

This means that if $p \in \ker(T)$, then

$$0 = T(p)(x) = T\left(\sum_{n=0}^{N} c_n x^n \right) = \sum_{n=0}^{N} c_n T(x^n) = \sum_{n=0}^{N} \frac{c_n}{n+1} x^{n+1},$$

which can only hold if $c_0 = \ldots = c_N = 0$. That is $p \equiv 0$, which means that $\ker(T) = \{0\}$.

Let us now consider the image of T. Let $p \in \mathrm{Im}(T)$. Then there exists a polynomial $q \in \mathcal{P}$ such that $T(q) = p$. We may assume that q is on the form

$$q(x) = \sum_{n=0}^{N} \frac{c_n}{x}^{n}.$$

However, this means that the polynomial p has to be on the form

$$p(x) = Tq(x) = \int_0^x \sum_{n=0}^{N} c_n t^n \, dt = \sum_{n=0}^{N} \frac{c_n}{n+1} x^{n+1} = \sum_{n=1}^{N+1} b_n x^n.$$

In particular p does not contain any constant terms. Since T increases the degree of the polynomial, there does not exists any polynomial q such that $T(q) = c$ for some $c \in \mathbb{R}$. Thus, we can conclude that $\mathrm{Im}(T) = \mathcal{P} \backslash \{c : c \in \mathbb{R}\}$. That is, the range of T consists of all polynomials without a constant term, meaning that they vanishes at $x = 0$.

b) We want to show that $S \circ T$ is the identity operator on $\mathcal{P}$, while $T \circ S$ is an operator different from the identity.

To show that $S \circ T$ is the identity, we must show that $S \circ T(p) = p$ for all polynomials $p \in \mathcal{P}$. Let $p \in \mathcal{P}$ be given as

$$p(x) = \sum_{n=0}^{N} c_n x^n.$$

Then we can apply the operator $S \circ T$ to p. This gives us,

$$\begin{aligned} S \circ T(p)(x) = S\left(Tp(x)\right) = S\left(\int_0^c \sum_{n=0}^{N} c_n t^n \, dt\right) &= \sum_{n=0}^{N} \frac{c_n}{n+1} S\left(x^{n+1}\right) \\ &= \sum_{n=0}^{N} \frac{c_n}{n+1} \left(x^{n+1}\right)' \\ &= \sum_{n=0}^{N} c_n x^n = p(x). \end{aligned}$$

Since p was arbitrarily chosen from $\mathcal{P}$, it follows that $S \circ T(p) = p$ for all $p \in \mathcal{P}$ and thus is the indentity operator.

Now, consider the operator $T \circ S$, and let $p \in \mathcal{P}$ be given by

$$p(x) \sum_{n=0}^{N} c_n x^n.$$

Then we have

$$\begin{aligned} T \circ S(p)(x) = T\left(S\left(\sum_{n=0}^{N} c_n x^n\right)\right) = T\left(\sum_{n=0}^{N} c_n n x^{n-1}\right) &= \sum_{n=1}^{N} c_n n T\left(x^{n-1}\right) \\ &= \sum_{n=1}^{N} c_n n \int_0^x t^{n-1} \, dt \\ &= \sum_{n=1}^{N} c_n x^n, \end{aligned}$$

which differs from p if $c_0 \neq 0$. Namely, the operator $T \circ S$ removes all constant terms from the polynomials, and is therefore not the indentity operator on $\mathcal{P}$.

Problem 4. (Invariant subspaces)

Let V be a vector space, and let $T\colon V \longrightarrow V$ be a linear operator.

a) Show that the kernel $\ker(T)$ and the range $\operatorname{ran}(T)$ of T are T-invariant subspaces of V.

b) Let $U_1, U_2 \subset V$ be T-invariant subspaces of V. Show that $U_1 \cap U_2$ and $U_1 + U_2$ are also T-invariant subspaces of V.

Solution.

a) We want to show that the kernel of T, and the image of T are T-invariant. A subset S is called T-invariant if $T(S) \subset S$.

   Recall that the kernel of T is defined as

   $$\ker(T) = \{x \in X : T(x) = 0\}.$$

   If $x \in \ker(T)$, then $T(x) = 0$. We claim that $T(0) = 0$, and hence $0 \in \ker(T)$. By the linearity of T, we have

   $$T(0) = T(0 + 0) = T(0) + T(0) = 2T(0) \quad \Rightarrow \quad T(0) = 0.$$

   This implies that $0 \in \ker(T)$, and thus $T(x) \in \ker(T)$ for every $x \in \ker(T)$. We have therefore shown that the kernel of T is T-invariant.

   The image of T is defined as

   $$\operatorname{im}(T) = \{y \in X : y = T(x) \text{ for some } x \in X\}.$$

   Note that for any $x \in X$, $T(x) \in \operatorname{im}(T)$. In particular, if $x \in \operatorname{im}(T)$, then $T(x) \in \operatorname{im}(T)$ by the definition of the image. However, this implies that the image of T is T-invariant.

b) Assume that U_1 and U_2 are T-invariant subspaces. We then want to show that $U_1 \cap U_2$ and $U_1 + U_2$ are T invariant subspaces.

   Let $x \in U_1 \cap U_2$. Since U_1 is a T-invariant subspace, it follows that $T(x) \in U_1$. On the other hand, U_2 is also a T-invariant subspace. Since $x \in U_2$ it also follows that $T(x) \in U_2$. Hence $T(x) \in U_1 \cap U_2$, and since $x \in U_1 \cap U_2$ was arbitrary we can conclude that $T(U_1 \cap U_2) \subset U_1 \cap U_2$. That is, $U_1 \cap U_2$ is T-invariant.

   For the second part of the problem, recall that

   $$U_1 + U_2 = \{x \in X : x = u_1 + u_2,\ u_1 \in U_1, u_2 \in U_2\}.$$

   Let $x = u_1 + u_2 \in U_1 + U_2$, and define $y = T(u_1)$ and $z = T(u_2)$. By the T-invariance of U_1 and U_2 it follows that $y \in U_1$ and $z \in U_2$. Hence by the linearity of T we have

   $$T(x) = T(u_1 + u_2) = T(u_1) + T(u_2) = y + z \in U_1 + U_2.$$

   Since x was arbitrarily chosen from $U_1 + U_2$ it follows that $T(U_1 + U_2) \subset U_1 + U_2$, and thus T-invariant.

Problem 5. (Span of a union and an intersection of sets)

Assume that V is a vector space.

For each of the following statements, decide whether it is true or false. If it is true, find a proof; if it is false, find a counterexample.

a) For all subsets $S \subset V$ and $T \subset V$ we have
$$\mathrm{Span}(S \cup T) = \mathrm{Span}(S) + \mathrm{Span}(T).$$

b) For all subsets $S \subset V$ and $T \subset V$ we have
$$\mathrm{Span}(S \cap T) = \mathrm{Span}(S) \cap \mathrm{Span}(T).$$

c) For all subsets $S \subset V$ and $T \subset V$ we have
$$\mathrm{Span}(S + T) = \mathrm{Span}(S) + \mathrm{Span}(T).$$

Solution.

a) True.

We may assume that the sets are non-empty. To see why, let $T = \emptyset$. Then $S \cup T = S \cup \emptyset = S$, and thus
$$\mathrm{Span}(S \cup T) = \mathrm{Span}(S) = \mathrm{Span}(S) + \{0\} = \mathrm{Span}(S) + \mathrm{Span}(\emptyset) = \mathrm{Span}(S) + \mathrm{Span}(T).$$
This also holds if $S = \emptyset$.

We therefore assume that S and T are two non-empty, possibly infinite, subsets of V. Recall that the definition of the linear span of S is the set of all finite linear combinations of elements in S. Namely,
$$\mathrm{Span}(S) = \left\{ \sum_{n=1}^{N} \alpha_n s_n : N \in \mathbb{N}, \alpha_n \in \mathbb{F}, s_n \in S \right\}.$$

The way to prove that
$$\mathrm{Span}(S \cup T) = \mathrm{Span}(S) + \mathrm{Span}(T),$$
is to first prove $\mathrm{Span}(S \cup T) \subset \mathrm{Span}(S) + \mathrm{Span}(T)$, and then to prove $\mathrm{Span}(S \cup T) \supset \mathrm{Span}(S) + \mathrm{Span}(T)$, as this would imply that any element of one set is also an element of the other.

Let $x \in \mathrm{Span}(S \cup T)$. Then by the definition of the linear span, there exists $N \in \mathbb{N}$, a finite sequence $\{\alpha_n\}_{n=1}^{N} \subset \mathbb{F}$ and a sequence of vectors $\{x_n\}_{n=1}^{N} \subset V$ where $x_n \in S$ or $x_n \in T$ such that
$$x = \sum_{n=1}^{N} \alpha_n x_n.$$

Since $x_n \in S$ or $x_n \in T$, we can find two subsequences $\{n_j\}_{j=1}^{N_1}$ and $\{n_l\}_{l=1}^{N_2}$ with $N_1 + N_2 = N$ and $\{x_{n_j}\}_{j=1}^{N_1} \subset S$ and $\{x_{n_l}\}_{l=1}^{N_2} \subset T$. If $N_1 = 0$ or $N_2 = 0$, we use the convention $\{x_n\}_{n=1}^{0} = \emptyset$. Using the subsequences, we can write

$$x = \sum_{n=1}^{N} \alpha_n x_n = \sum_{j=1}^{N_1} \alpha_{n_j} x_{n_j} + \sum_{l=1}^{N_2} \alpha_{n_l} x_{n_l} = \sum_{j=1}^{N_1} a_j s_j + \sum_{l=1}^{N_2} b_l t_l,$$

where $a_j = \alpha_{n_j}$, $s_j = x_{n_j}$, $b_l = \alpha_{n_l}$ and $t_l = x_{n_l}$. Since $s_j \in S$ and $t_l \in T$ for all $1 \le j \le N_1$ and $1 \le l \le N_2$, it follows that

$$x = \sum_{j=1}^{N_1} a_j s_j + \sum_{l=1}^{N_2} b_l t_l \in \operatorname{Span}(S) + \operatorname{Span}(T).$$

Since x was an arbitrary element of $\operatorname{Span}(S \cup T)$, it follows that $\operatorname{Span}(S \cup T) \subset \operatorname{Span}(S) + \operatorname{Span}(T)$.

To show the other direction, we assume that $x \in \operatorname{Span}(S)+\operatorname{Span}(T)$. From the definition of the Minkowski sum of sets, we can write $x = v + w$ where $v \in \operatorname{Span}(S)$ and $w \in \operatorname{Span}(T)$. Thus, by the definition of the linear spans, we have

$$x = \sum_{n=1}^{N} a_n s_n + \sum_{m=1}^{M} b_m t_m = \sum_{j=1}^{N+M} \alpha_j x_j,$$

where the coefficients α_j are given by

$$\alpha_j = \begin{cases} a_j, & 1 \le j \le N, \\ b_j, & N+1 \le j \le N+M, \end{cases}$$

while the vectors x_j are given by

$$x_j = \begin{cases} s_j, & 1 \le j \le N, \\ t_j, & N+1 \le j \le N+M. \end{cases}$$

However, this shows that x can be written as a finite linear combinations of elements $x_j \in S \cup T$, and so $x \in \operatorname{Span}(S \cup T)$. Since x was arbitrarily chosen from $\operatorname{Span}(S) + \operatorname{Span}(T)$, we have that $\operatorname{Span}(S \cup T) \supset \operatorname{Span}(S) + \operatorname{Span}(T)$. Thus, we can conclude that

$$\operatorname{Span}(S \cup T) = \operatorname{Span}(S) + \operatorname{Span}(T).$$

b) False.

This can be seen by considering two linearly dependent vecotrs. Let $v \in V$ be a non-zero vector, and let $\alpha \in \mathbb{F}\backslash\{0\}$, be such that $v \neq \alpha v$. Then consider the two sets $S = \{v\}$ and $T = \{\alpha v\}$. Since $v \neq \alpha v$, it follows that $S \cap T = \emptyset$. Thus, $\operatorname{Span}(S \cap T) = \operatorname{Span}(\emptyset) = \{0\}$. However, since αv is a scaling of v it follows that $\operatorname{Span}(S) = \operatorname{Span}(v) = \operatorname{Span}(\alpha v) = \operatorname{Span}(T)$. In particular,

$$\operatorname{Span}(S \cap T) = \{0\} \neq \operatorname{Span}(v) = \operatorname{Span}(S) \cap \operatorname{Span}(T),$$

by the assumption that v is a non-zero vector.

c) False.

Consider two non-zero linearly independent vectors $v, w \in V$, and consider the sets $S = \{v\}$ and $T = \{w\}$. Then by definition

$$S + T := \{x \in V : x = s + t, s \in S, t \in T\} = \{v + w\}.$$

This means that

$$\mathrm{Span}(S+T) = \{x \in V : x = \alpha(v+w), \alpha \in \mathbb{F}\}.$$

Now, consider the vector $x = v - w$. Then, since $v \in \mathrm{Span}(v) = \mathrm{Span}(S)$ and $-w \in \mathrm{Span}(w) = \mathrm{Span}(T)$, we have that $x \in \mathrm{Span}(S) + \mathrm{Span}(T)$. On the other hand $x = v - w \neq \alpha(v+w)$ for any $\alpha \in \mathbb{F}$ as v and w have different signs. This means that $x \notin \mathrm{Span}(S+T)$, and hence

$$\mathrm{Span}(S+T) \neq \mathrm{Span}(S) + \mathrm{Span}(T).$$

Here we used that v and w are linearly independent to ensure that $\mathrm{Span}(S) \neq \mathrm{Span}(T)$.