Exercise 2

Problem 1

$T: P_5 \to P_5$

$p(x) \mapsto xp'(x) - p''(x)$

(a) We have $M := \{1, x, x^2, x^3, x^4, x^5\}$

$1 \mapsto 0$

$x \mapsto x$

$x^2 \mapsto x \cdot 2x - 2 = 2x^2 - 2$

$x^3 \mapsto x \cdot 3x^2 - 6x = 3x^3 - 6x$

$x^4 \mapsto x \cdot 4x^3 - 12x^2 = 4x^4 - 12x^2$

$x^5 \mapsto x \cdot 5x^4 - 20x^3 = 5x^5 - 20x^3$

$$\Rightarrow A = \begin{pmatrix} 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 & 0 & 0 \\ 0 & 0 & 2 & 0 & -12 & 0 \\ 0 & 0 & 0 & 3 & 0 & -20 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

(b) $Tv = \lambda v$

$(Tv - \lambda v) = (A - \lambda I)v$

$$(A - \lambda I) = \begin{pmatrix} -\lambda & & & & & \\ & 1-\lambda & & & & \\ & & 2-\lambda & & & \\ & & & 3-\lambda & & \\ & 0 & & & 4-\lambda & \\ & & & & & 5-\lambda \end{pmatrix}$$

$\det(A - \lambda I) = (-\lambda)(1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda)(5-\lambda) = 0$

$\Rightarrow$ The eigenvalues are $0, 1, 2, 3, 4, 5$

2.

$\dim(V) = n \geq 2$

$T: V \to V$

(a) Show that $\exists u \in V$ and a linear mapping $l: V \to \mathbb{K}$ s.t. $Tv = l(v)u \ \forall v \in V$

Know $\operatorname{ran}(T) = \operatorname{span}(u)$:

$Tv = \operatorname{ran}(T),\ v \in V$

$= \operatorname{span}(u)$

$= l(v)u$

$\operatorname{ran}(T) = \{Tv \mid v \in V\}$

$\operatorname{span}(u) = \{a \cdot u \mid \alpha \in \mathbb{K}\}$

$\alpha = l(v) \in \mathbb{K}$

Prove that $l(\alpha v) = \alpha l(v)$ and $l(v_1 + v_2) = l(v_1) + l(v_2)$, $l(v) = Tvu^{-1}$

$l(\alpha v) = T(\alpha v)u^{-1}$

$= \alpha T(v)u^{-1}$

$= \alpha(T(v)u^{-1})$ ✓

$l(v_1 + v_2) = T(v_1 + v_2)u^{-1}$

$= T(v_1)u^{-1} + T(v_2)u^{-1}$

$= l(v_1) + l(v_2)$ ✓

(b) Show that $\ker(T - \lambda I) \neq \{0\}$ iff $\lambda = 0$, $\lambda = l(u)$

$T - \lambda I =$

$\dim(V) \geq 2$

$\dim(\operatorname{ran}(T)) = 1$

$\ker(T - \lambda I) = \{u \in V \mid (T - \lambda I)u = 0\}$

$Tu = \lambda u \Rightarrow Tu - \lambda u = 0 \quad (T - \lambda)u = 0$

$Tv = l(v)u$

$Tu = l(u)u$

$Tu - l(u)u = 0 \Rightarrow (T - l(u))u = 0$

$\operatorname{ran}(T - \lambda I) = (T - \lambda I)u,\ u \in V$

$\lambda = 0$:

$\ker(T - 0 \cdot I) = \{u \in V \mid (T - 0 \cdot I)u = 0\} \neq \{0\} \ \forall u \in V$ ✓

$\Rightarrow 0$ is an eigenvalue

$\lambda = l(u)$

Since we know $Tu = l(u)u$, then $l(u)$ is an eigenvalue

$\lambda \neq 0,\ \lambda \neq l(u)$

$\ker(T - \lambda I) = \{u \in V \mid (T - \lambda I)u = 0\}$

Must show that this implies $u = 0$

(c) Since T have eigenvalues 0 and $l(u)$, the sum

$E(0, T) + E(l(u), T)$

is direct

$\Rightarrow V = E(0, T) \oplus E(l(u), T)$

$\Rightarrow T$ is diagonalizable

Problem 3.

$T: \operatorname{Mat}_n(\mathbb{C}) \to \operatorname{Mat}_n(\mathbb{C})$

$A \mapsto DAD^{-1}$

$$D = \begin{pmatrix} \alpha_1 & & 0 \\ & \alpha_2 & \\ 0 & \ddots & \\ & & \alpha_n \end{pmatrix}$$

(a) Need to show $T(A_1 + A_2) = T(A_1) + T(A_2)$:

$T(A_1 + A_2) = D(A_1 + A_2)D^{-1}$

$= (DA_1 + DA_2)D^{-1}$

$= DA_1D^{-1} + DA_2D^{-1}$

$= T(A_1) + T(A_2)$

Need to show $T(\alpha A) = \alpha T(A)$:

$T(\alpha A) = D\alpha AD^{-1}$

$= \alpha DAD^{-1}$

$= \alpha T(A)$

The mapping T is linear.

(b) T is bijective since if D stay the same, and $A, B \in \operatorname{Mat}_n(\mathbb{C})$

$\Rightarrow DAD^{-1} = DBD^{-1} \Rightarrow A = B$

$T^{-1}: \operatorname{Mat}_n(\mathbb{C}) \to \operatorname{Mat}_n(\mathbb{C})$

$DAD^{-1} \mapsto A$

$\Rightarrow DAD^{-1} \mapsto D^{-1}DAD^{-1}D = A$

(c) $E_{k,l}^{(i,j)} = \begin{cases} 1, & k = i,\ l = j \\ 0, & \end{cases}$

$TE^{(i,j)} = \lambda E^{(i,j)}$

$DE^{(i,j)}D^{-1} = \frac{\alpha}{\det(D)}E^{(i,j)}$

$\Rightarrow \lambda = \frac{\alpha}{\det(D)}$