TIØ4146 Finance for Science and Technology Students

Chapter 3 - Part 2
Modern Portfolio Theory

The Capital Asset Pricing Model and Arbitrage pricing theory

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Weeks	Subject	Chapt.	Lecturer
2	Introduction, Time value of money	1/2	Carlos
3	Recap fundamentals	2	Ståle
4-5	Portfolio theory, CAPM, APT	3	Ståle
6	Market efficiency	4	Felipe
7	Capital structure	5	Felipe
8	Valuing levered projects	6	Felipe
9	Options and their pricing foundations	7	Carlos
10	Binomial option pricing	7	Carlos
11	Black and Scholes option pricing	8	Carlos
12	BS option pricing and Real options	8/9	Carlos
13-14	Easter break		
15	Real options analysis	9	Carlos
16	Summary and exam instructions		Carlos/Felipe

Plan for this lecture

Capital Asset Pricing Model

A summarizing digression

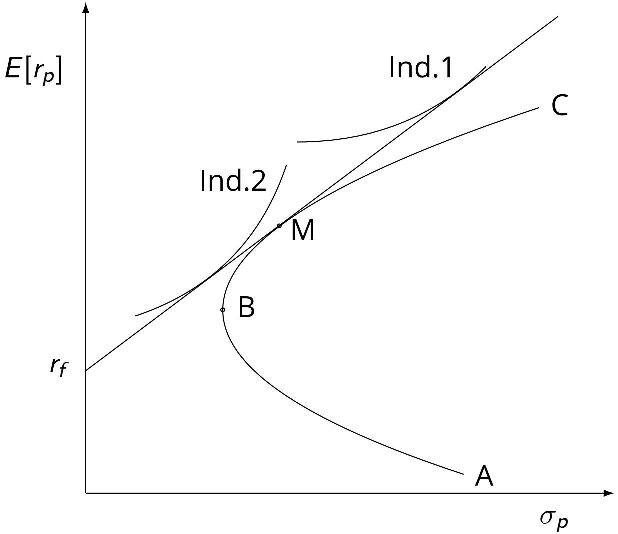
Arbitrage Pricing Theory

Recap last lecture - Capital Market Line (CML)

James Tobin (1958): Liquidity Preference as Behavior Towards Risk. The Review of Economic Studies 25(2): 65-86.

An investment choice can be broken down into two phases: first, the choice of a unique optimum combination of risky assets; and second, a separate choice concerning the allocation of funds between such a combination and a single riskless asset.

> *His main point is that by combining a diversified efficient portfolio with a riskless asset one can obtain a combination which is even more efficient than the efficient frontier*



The Capital Market Line: $E(r_p) = r_f + \frac{E(r_m)-r_f}{\sigma_m}\sigma_p$

Capital Asset Pricing Model CAPM: The history

- Jack L. Treynor (1999). Toward a Theory of Market Value of Risky Assets. Robert Korajczyk (Ed.), Asset Pricing and Portfolio Performance. London: Risk Books.
- William F. Sharpe (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. Journal of Finance, 19(3), 425-42.
- John Lintner (1965a). The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. Review of Economics and Statistics, 73, 13-37.
- - (1965b). Security Prices, Risk and Maximal Gains from Diversification. Journal of Finance, 20, 587-615.
- Jan Mossin (1966). Equilibrium in a Capital Asset Market. Econometrica, 34 (4), 768-83.

Capital Asset Pricing Model CAPM

Capital Market Line only valid for efficient portfolios

- combinations of risk free asset and market portfolio M
  - $\Rightarrow$ all risk comes from market portfolio

What about inefficient portfolios or individual stocks?

- don't lie on the CML, cannot be priced with it
  need a different model for that

> What needs to be changed in the model:
> the market price of risk $((E(r_m) - r_f)/\sigma_m)$,
> or the measure of risk σ_p?

CAPM is more general model, developed by Sharpe

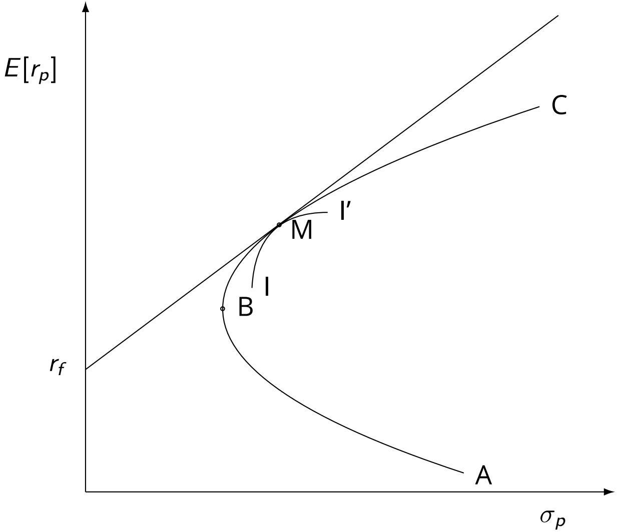
Consider a two asset portfolio:

- one asset is market portfolio M, weight $(1 - x)$
- other asset is individual stock i, weight x

Is this an efficient portfolio?

Analyse what happens if we vary proportion x invested in i

- begin in point I, 100% in i, x=1
- in point M, x=0, but asset i is included in M with its market value weight
- to point I', x$<$0 to eliminate market value weight of i



Portfolios of asset i and market portfolio M

Risk-return characteristics of this 2-asset portfolio:

$$E(r_p) = xE(r_i) + (1-x)E(r_m)$$

$$\sigma_p = \sqrt{[x^2\sigma_i^2 + (1-x)^2\sigma_m^2 + 2x(1-x)\sigma_{i,m}]}$$

The slope of the curve I-I' at any point is:

$$\frac{\partial E(r_p)}{\partial x} = \frac{\partial E(r_p)/\partial x}{\partial \sigma_p/\partial x}$$

Expected return and risk of a marginal change in x are:

$$\frac{\partial E(r_p)}{\partial x} = E(r_i) - E(r_m)$$

$$\frac{\partial \sigma_p}{\partial x} = \frac{1}{2}\left[x^2\sigma_i^2 + (1-x)^2\sigma_m^2 + 2x(1-x)\sigma_{i,m}\right]^{-\frac{1}{2}} \times \left[2x\sigma_i^2 - 2\sigma_m^2 + 2x\sigma_m^2 + 2\sigma_{i,m} - 4x\sigma_{i,m}\right]$$

First term of $\partial\sigma_p/\partial x$ is $\frac{1}{2\sigma_p}$, [1]:

$$\begin{aligned}
\frac{\partial\sigma_p}{\partial x} &= \frac{2x\sigma_i^2 - 2\sigma_m^2 + 2x\sigma_m^2 + 2\sigma_{i,m} - 4x\sigma_{i,m}}{2\sigma_p} \\
&= \frac{x\sigma_i^2 - \sigma_m^2 + x\sigma_m^2 + \sigma_{i,m} - 2x\sigma_{i,m}}{\sigma_p}
\end{aligned}$$

Isolating x gives:

$$\frac{\partial\sigma_p}{\partial x} = \frac{x(\sigma_i^2 + \sigma_m^2 - 2\sigma_{i,m}) + \sigma_{i,m} - \sigma_m^2}{\sigma_p}$$

[1] Note: error in the book at page 72

At point M all funds are invested in M so that:

- $x = 0$ and $\sigma_p = \sigma_m$

Note also that:

- i is already included in M with its market value weight
- economically x represents *excess* demand for i
- in equilibrium M excess demand is zero

This simplifies marginal risk to:

$$\left.\frac{\partial \sigma_p}{\partial x}\right|_{x=0} = \frac{\sigma_{i,m} - \sigma_m^2}{\sigma_p} = \frac{\sigma_{i,m} - \sigma_m^2}{\sigma_m}$$

So the slope of the risk-return trade-off at equilibrium point M is:

$$\left.\frac{\partial E(r_p)/\partial x}{\partial \sigma_p/\partial x}\right|_{x=0} = \frac{E(r_i) - E(r_m)}{\left(\sigma_{i,m} - \sigma_m^2\right)/\sigma_m}$$

But at point M the slope of the risk-return trade-off is also the slope of the CML, so:

$$\frac{E(r_i) - E(r_m)}{\left(\sigma_{i,m} - \sigma_m^2\right)/\sigma_m} = \frac{E(r_m) - r_f}{\sigma_m}$$

Solving for $E(r_i)$ gives:

$$\begin{aligned} E(r_i) &= r_f + (E(r_m) - r_f)\frac{\sigma_{i,m}}{\sigma_m^2} \\ &= r_f + (E(r_m) - r_f)\beta_i \end{aligned}$$

$$E(r_i) = r_f + (E(r_m) - r_f)\beta_i$$

This is the *Capital Asset Pricing Model*

- Sharpe was awarded the Nobel prize for this result
- Its graphical representation is known as the
  - *Security Market Line*
- Pricing relation for entire investment universe
  - including inefficient portfolios
  - including individual assets
- clear price of risk: $E(r_m) - r_f$
- clear measure of risk: β

CAPM formalizes risk-return relationship:

- well-diversified investors value assets according to their contribution to portfolio risk
  - if asset i increases portf. risk $E(r_i) > E(r_p)$
  - if asset i decreases portf. risk $E(r_i) < E(r_p)$
- expected risk premium proportional to β

Offers other insights as well. Look at 4 of them:

1. Systematic and unsystematic risk
2. Risk adjusted discount rates
3. Certainty equivalents
4. Performance measures

1. Systematic & unsystematic risk

- The CML is pricing relation for *efficient* portfolios:

$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m}\sigma_p$$

- $\frac{E(r_m)-r_f}{\sigma_m}$ is the price per unit of risk
- σ_p is the volume of risk.

The SML valid for all investments, incl. inefficient portfolios and individual stocks:

$$E(r_p) = r_f + (E(r_m) - r_f)\beta_p$$

we can write β as:

$$\beta_p = \frac{cov_{p,m}}{\sigma_m^2} = \frac{\sigma_p \sigma_m \rho_{p,m}}{\sigma_m^2} = \frac{\sigma_p \rho_{p,m}}{\sigma_m}$$

so that the SML becomes:

$$E(r_p) = r_f + (E(r_m) - r_f)\frac{\sigma_p \rho_{p,m}}{\sigma_m},$$

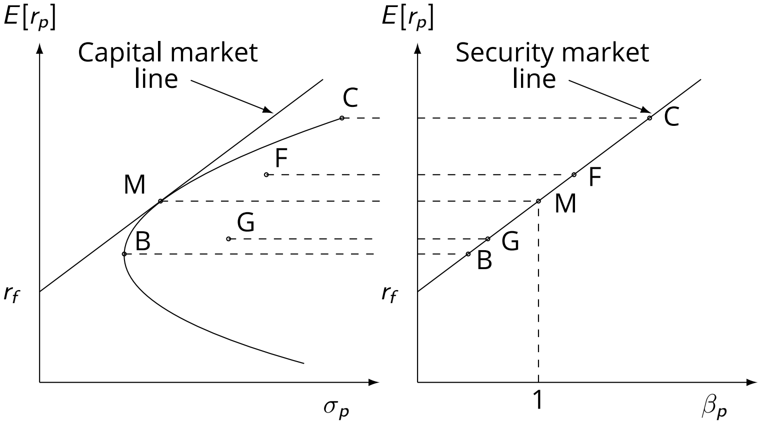
$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m}\sigma_p \rho_{p,m}$$

Compare with CML:

$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m}\sigma_p$$

The difference between CML and SML is in volume part:

- SML only prices the *systematic risk*
  - is therefore valid for all investment objects.
- CML prices *all risks*
  - only valid when all risk is systematic risks, i.e. for efficient portfolios
  - otherwise, CML uses 'wrong' risk measure
- difference is correlation term, that is ignored in CML
  - efficient portfolios only differ in proportion M in it
  - so all efficient portfolios are perfectly positively correlated: $\rho_{M,(1-x)M} = 1$
  - if $\rho_{p,m} = 1 \Rightarrow \sigma_p \rho_{p,m} = \sigma_p$ and $CML = SML$



Systematic and unsystematic risk

2. CAPM and discount rates

Recall general valuation formula for investments:

$$Value = \sum^{t} \frac{Exp\left[Cash\ flows_t\right]}{\left(1 + discount\ rate_t\right)^t}$$

Uncertainty can be accounted for in 3 different ways:

1. Adjust discount rate to *risk adjusted discount rate*
2. Adjust cash flows to *certainty equivalent cash flows*
3. Adjust probabilities (expectations operator) from normal to *risk neutral* or *equivalent martingale probabilities*

Use of CAPM as risk adjusted discount rate is easy
CAPM gives expected (=required) return on portfolio P as:

$$E(r_p) = r_f + (E(r_m) - r_f)\beta_p$$

But return is also:

$$E(r_p) = \frac{E(V_{p,T}) - V_{p,0}}{V_{p,0}}$$

Discount rate:

- links expected end-of-period value, $E(V_{p,T})$, to value now, $V_{p,0}$
- found by equating the two expressions:

$$\frac{E(V_{p,T}) - V_{p,0}}{V_{p,0}} = r_f + (E(r_m) - r_f)\beta_p$$

solving for $V_{p,0}$ gives:

$$V_{p,0} = \frac{E(V_{p,T})}{1 + r_f + (E(r_m) - r_f)\beta_p}$$

- r_f is the time value of money
- $(E(r_m) - r_f)\beta_p$ is the adjustment for risk
- together they form the risk adjusted discount rate

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3. Certainty equivalent formulation

The second way to account for risk:

- adjust uncertain cash flow to a *certainty equivalent*
- can (and should) be discounted with risk free rate

Requires some calculations, partly omitted here, but start with the expression equating formulas for portfolio return

$$\frac{E(V_{p,T}) - V_{p,0}}{V_{p,0}} = r_f + (E(r_m) - r_f)\beta_p$$

Since we know that $\beta_p = cov_{r_p,r_m}/\sigma_m^2$ and r_p can be written as $(V_{p,t} - V_{p,0})/V_{p,0} = (V_{p,t}/V_{p,0}) - 1$ and substituting in the right-hand side, we get after some calculations (see page 76):

$$V_{p,0} = \frac{E(V_{p,T}) - \lambda cov\left(V_{p,T}, r_m\right)}{1 + r_f}$$

This is the *certainty equivalent formulation* of the CAPM:

- uncertain end-of-period value is adjusted by
  - the market price of risk, λ:

$$\lambda = \frac{E(r_m) - r_f}{\sigma_m^2}$$

  - $\times$ the volume of risk, i.e. covariance (end-of-period value, return on market portfolio)
- The resulting certainty equivalent value is discounted at the risk free rate to find the present value.

4. Performance measures Portfolio management involves a trade-off

between risk and return
CML and SML relate expected return to risk

- can be reformulated as *ex post performance measures*
- relate realized returns to observed risk

Present a few performance measures:

- Sharpe ratio
- Treynor ratio
- Jensen’s alpha

Sharpe ratio

Sharpe uses slope of CML for this:

$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m}\sigma_p \Rightarrow$$

$$\frac{E(r_p) - r_f}{\sigma_p} = \frac{E(r_m) - r_f}{\sigma_m}$$

Left hand side is *return-to-variability ratio* or *Sharpe ratio*

In ex post formulation:

$$\text{Sharpe ratio:} \quad SR_p = \frac{\overline{r}_p - \overline{r}_f}{\widehat{\sigma}_p}$$

- SR_p is Sharpe ratio of portfolio p
- $\overline{r}_p$ is portfolio's historical average return $\overline{r}_p = \sum_t r_{pt}/T$
- $\overline{r}_f$ is historical average risk free interest rate
- $\widehat{\sigma}_p$ is stand. dev.portf. returns: $\widehat{\sigma}_p = \sqrt{\sum_t (r_{pt} - \overline{r}_p)^2/T}$
- T is number of observations (periods)

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Sharpe ratios widely used to:

- rank portfolios, funds or managers
- identify poorly diversified portfolios (too high $\widehat{\sigma}_p$)
- identify funds that charged too high fees ($\overline{r}_p$ too low)

Sharpe ratio can be adapted:

- measure the risk premium over other benchmark than r_f
  - also known as the *information ratio*
- measure risk as semi-deviation (downward risk)
  - known as *Sortino ratio*

	WKN	Name	Volatilität	Sharpe-Ratio	Ausgabe-Aufschlag	Mgmt.-gebühr
☐	A0DPXX	DB Platinum IV Europ...	2,00	1,88	4,00%	1,00%
☐	593060	Nordea 1 Danish Bond...	4,50	1,25	3,00%	1,35%
☐	532652	KCD-Union-Renten Plu...	3,96	1,19	0,00%	1,25%
☐	A0LCFH	Fortis L Fund Bond S...	5,09	0,97	5,00%	0,30%
☐	A0ML43	CAAM Volatility Equi...	16,23	0,95	4,50%	1,00%
☐	A0NG05	BNY Mellon Vietnam,I...	44,32	0,95	5,00%	1,00%
☐	A0MVST	pulse invest - ABSOL...	21,84	0,95	--	--
☐	A0H1AL	Dexia Sustainable Eu...	5,79	0,94	2,50%	0,60%
☐	A0J4YL	Dexia Sustainable Eu...	5,79	0,94	2,50%	0,60%
☐	A0MVSS	pulse invest - ABSOL...	21,65	0,94	--	--
☐	A0NG03	BNY Mellon Vietnam,I...	44,39	0,92	5,00%	2,00%
☐	849625	Allianz ExxonMobil-M...	5,47	0,91	0,00%	0,60%

Example from www.handelsblatt.com, Sonntag, 02.08.2009

Treynor ratio

Treynor ratio uses security market line, β as risk measure:

$$\text{Treynor ratio:} \quad TR_p = \frac{\bar{r}_p - \bar{r}_f}{\widehat{B}_p}$$

$\widehat{B}_p$ is estimated from historical returns

- Treynor ratio usually compared with risk premium market portfolio
- is TR for portfolio with β of 1

What does the CAPM predict about the TR of different assets and portfolios?
All assets lie on SML $\Rightarrow$ all have same TR

Jensen's alpha

Jensen's alpha also based on CAPM

- measures portfolio return in excess of CAPM
- found by regressing portfolio risk-premium on market portfolio's risk-premium:

$$r_{pt} - r_{ft} = \widehat{\alpha}_p + \widehat{B}_p(r_{mt} - r_{ft}) + \widehat{\varepsilon}_{pt}$$

- taking averages and re-writing gives Jensen's alpha:

$$\text{Jensen's alpha}: \quad \widehat{\alpha}_p = \bar{r}_p - (\bar{r}_f + \widehat{B}_p(\bar{r}_m - \bar{r}_f))$$

We will meet these performance measures again in market efficiency tests

Comparing performance measures

- Sharpe ratio uses total risk (σ_p)
- Treynor ratio and Jensen's alpha take only systematic risk β into account

Sharpe ratio is better for evaluating investor's total portfolio

- When a portfolio is split into subportfolios (e.g., countries, portfolio manager) Sharpe ratio will ignore correlations and overstate risk
- Treynor ratio more appropriate for well-diversified subportfolios

Jensen's alpha uses CAPM and easy to interpret - excess or below what CAPM specifies

- Difficult to compare between portfolios with different risk as Sharpe and Treynor ratios do

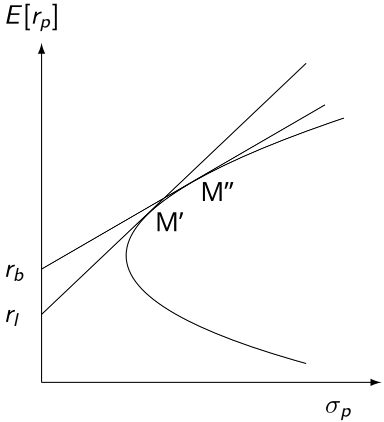
Treynor ratio and Jensen's alpha is dependent on the choice of market index or portfolio

Assumptions CAPM is based on:

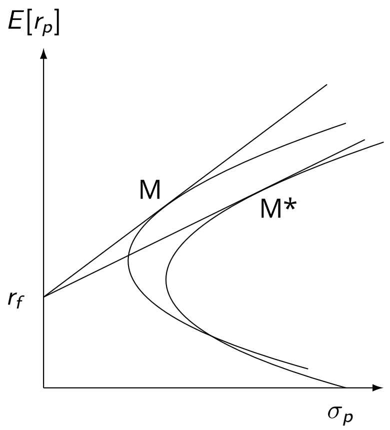
- Financial markets are perfect and competitive:
  - no taxes or transaction costs, all assets are marketable and perfectly divisible, no limitations on short selling and risk free borrowing and lending
  - large numbers of buyers and sellers, none large enough to individually influence prices, all information simultaneously and costlessly available to all investors
- Investors
  - maximize expected utility of their end wealth by choosing investments based on their mean-variance characteristics over a single holding period
  - have homogeneous expectations w.r.t. returns (i.e. they observe same efficient frontier)

Assumptions have different backgrounds and importance

- Some make modelling easy, model doesn't break down if we include phenomena now 'assumed away':
  - no taxes or transaction costs, all assets are marketable and divisible
- Another points at unresolved shortcoming of the model:
  - single holding period clearly unrealistic, real multi-period model not available
- Still others have important consequences:
  - different borrowing and lending rates invalidate same risk-return trade-off for all (see picture)
  - if investors see different frontiers, effect comparable to restriction, e.g. ethical and unethical investments (see picture)



CML with different borrowing and lending rates



CML with heterogeneous expectations

Key assumption is:

Investors maximize expected utility of their end wealth by choosing investments based on their mean-variance characteristics

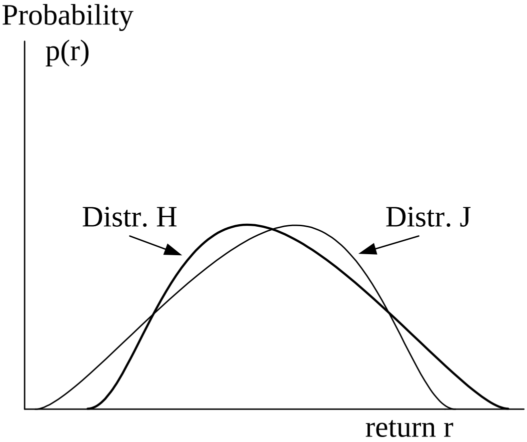
- Is the ’behavioural assumption’ (assertion):
- the behaviour (force) that drives the model into equilibrium
- Mean variance optimization *must* take place for the model to work

We did not explicitly say anything about mean-variance in utility theory. Is that special for Markowitz’ analysis? Not quite

Mean variance optimization fits in with general economic theory under 2 possible scenario's (assumptions):

1. Asset returns are jointly normally distributed
   1.1 means, variances and covariances completely describe return distributions (higher moments zero)
   1.2 no other information required for investment decisions
2. Investors have quadratic utility functions
   2.1 If $U(W) = \alpha + \beta W - \gamma W^2$; choosing a portfolio to maximize U only depends on $E[W]$ and $E[W^2]$, i.e. expected returns and their (co-)variances
   2.2 means investors only care about first 2 moments

Do investors ignore higher moments? Which would you chose?



2 mirrored distributions with identical mean and stand.dev.

Empirical tests of the CAPM

Require approximations and assumptions:

- model formulated in expectations
- has to be tested with historical data
- gives returns a function of β, not directly observable

Tested with a two pass regression procedure:

1. time series regression of individual assets
2. cross section regression of assets' βs on returns

First pass, time series regression estimates βs:

$$r_{it} - r_{ft} = \widehat{\alpha}_i + \widehat{\beta}_i(r_{mt} - r_{ft}) + \widehat{\varepsilon}_{it}$$

- regresses asset risk premia on market risk premia
- for each asset separately
- market approximated by some index
- usually short observation periods (weeks, months)
- result is called *characteristic line*
- slope coefficient is estimated beta of asset i, $\widehat{\beta}_i$

Second pass, cross section regression estimates risk premia:

$$\overline{rp}_i = \gamma_0 + \gamma_1 \widehat{\beta}_i + \gamma_{2n}(testvar_n) + \widehat{u}_i$$

- regresses average risk premia on $\widehat{\beta}$
- rp averaged over observation period $\overline{rp}_i = \sum_t (r_{it} - r_{ft})/T$
- β can also be estimated over prior period

Some more details:

- usually done with portfolios, not individual assets
- over longer periods (years)
- with rolling time window (drop oldest year, add new year)
- often includes other variables (testvars)

What does the CAPM predict about the coefficients of 2^{nd} pass regression?

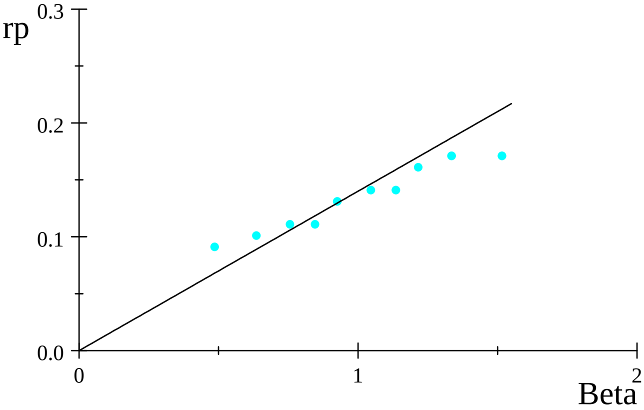
$$\overline{rp}_i = \gamma_0 + \gamma_1 \widehat{\beta}_i + \gamma_{2n}(testvar_n) + \widehat{u}_i$$

1. $\gamma_0 = 0$
2. $\gamma_1 = \overline{rp}_m$
3. $\gamma_2 = 0$
4. and relation should be linear in β
   e.g. β^2 as testvar should not be significant
5. R^2 should be reasonably high

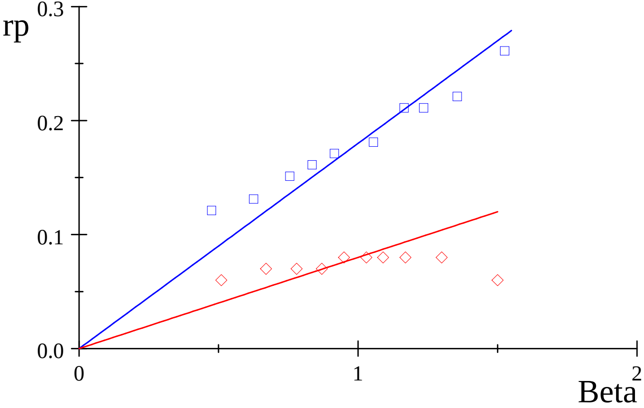
Example: Fischer Black: Return and Beta, *Journal of Portfolio Management*, vol.20 no.1, fall 1993

- uses all stocks on NYSE 1926-1991, monthly data
  - 1931: 592 stocks, 1991: 1505 stocks
- starting 1931, makes yearly β portfolios:
  - estimates individual βs over previous 60 months
  - by regressing risk premium on market risk premium
  - makes 10 portfolios, after β deciles (high - low β)
  - calculates portfolio average β, $\overline{rp}_i$ etc.
- repeats 'rolling' for 1932, 1933, etc. yearly rebalancing
- calculates portfolio averages whole period + sub-periods

For 10 portfolios, β plotted against risk premium:



Black, 1931-1991, line is $\overline{rp}_m \times \beta$



Black, 1931-1965 (blue) and 1966-1991 (red), lines are $\overline{rp}_m \times \beta$

Black's results are typical for many other studies:

1. $\gamma_0 > 0$ (i.e. too high)
2. $\gamma_1 < \overline{rp}_m$ but $\gamma_1 > 0$ (i.e. too low)
   2.1 in recent data, γ_1 is lower than before
   2.2 even close to zero ('Beta is dead')
3. linearity generally not rejected
4. other variables are significantly $\neq 0$, so other factors play a role:
   4.1 small firm effect
   4.2 book-to-market effect
   4.3 P/E ratio effect
5. R^2 ?

Roll's critique: [2] can CAPM be tested at all?

Roll argues: CAPM produces only 1 testable hypothesis:

the market portfolio is mean-variance efficient

Argument based on following elements:

- There is only 1 ex ante efficient market portfolio using the whole investment universe
- includes investments in human capital, venture idea's, collectors' items as wine, old masters' paintings etc.
- is unobservable
- tested with ex post sample of market portfolio, e.g. S&P 500 index, MSCI, Oslo Børs Benchmark Index

[2] Richard Roll (1977). A critique of the asset pricing theory's tests Part I: On past and potential testability of the theory. Journal of Financial Economics, 4 (2): 129–176

Gives rise to benchmark problem:

- sample may be mean-variance efficient, while the market portfolio is not
- or the other way around

But if sample is ex post mean-variance efficient:

- mathematics dictate that $\beta' s$ calculated relative to sample portfolio will satisfy the CAPM
- means: all securities will plot on the SML

Only test is whether portfolio we use is really the market portfolio $\Rightarrow$ untestable

A simple practical application of what we have learned so far

Suppose you are very risk averse, what would you choose:

1. A very risky share of 250 in a company you expect to perform badly in the near future
2. A risk free bond of 235

What would you chose:

1. 250 today
2. 235 today

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What do we learn from this?

- Financial markets provide information needed to value alternatives
  - nature of the bond and stock already reflected in price
  - nobody needs stocks or bonds to allocate consumption over time
  - everybody prefers more to less
- Financial decisions can be made rationally by maximizing value regardless of risk preferences or expectations
  - risky share and risk free bond have the same value for risk averse student and rich businessman
  - doesn’t matter where the money comes from
  - simply choose highest PV, reallocate later

Financial markets give the opportunity to:

- expose to risk / eliminate risk
- move consumption back and forth in time

On well functioning financial markets:

- prices are ’fair’, i.e. arbitrage free
- arbitrage brings about the ’Law of one price’:
  - same assets have same price
  - asset value comes from its cash flow pattern over time/scenario’s
  - if same pattern can be constructed with different combination of assets, price must the same
  - if not, buying what is cheap and selling what is expensive will drive prices to same level

Arbitrage

Arbitrage is strategy to profit from mispricing in markets

Formally, an arbitrage strategy:

- either requires
  - investment ≤ 0 today, while
  - all future pay-offs ≥ 0 and
  - at least one payoff > 0
- or requires
  - investment < 0 today (=profit) and
  - all future pay-offs ≥ 0

Less formally:

- either costs nothing today + payoff later
- or payoff today without obligations later

Arbitrage

Example:
If gold costs

- \$670/ounce in New York
- ¥80.000/ounce in Tokyo
- then this implies ¥119 for \$1

At ¥115/\$1 there is this arbitrage opportunity:

- buy gold in New York, costs \$670
- sell gold in Tokyo, gives ¥80.000
- change ¥80.000/115= \$696 or \$26 riskless, instantaneous arbitrage profit
- and then you do it again, and again..

In practice, you and I cannot do this, and certainly not again and again

- Deals are done electronically with very large amounts (measured in trillions - 10^9 per day) and very low transaction costs
  - makes even small price differences profitable
  - profiting makes them disappear quickly
- Real arbitrage opportunities are few and far between
  - takes a lot of research to find them (usually)
  - are not scalable (cannot do them again and again)

Ross (2005) estimates arbitrage opportunities at less than 0.1%, and many people look out for them

Power of arbitrage: a horror story

- Thursday 8 Dec. 2005, 9:27 am, a trader at Japanese brokerage unit of Mizuho Financial Group (2nd largest bank in Japan) wrongly put in an order to sell 610,000 shares of J-Com for ¥1 each.
- The intention was to sell 1 share for ¥610,000 for a client.
- Was first day of J-Com's listing. Order was 42 times larger than 14,500 outstanding J-Com shares, which had a total market value of 11.2 billion yen ($93 million).
- Within the 11 minutes before Mizuho could cancel the order, 607,957 shares traded, generating $3.5 billion of trades in a company the market valued at $93 million.
- Mizuho Securities lost about $347 million on the mistake

Arbitrage Pricing Theory

- Introduced by Ross (1976) [3]
- Does not assume that investors maximize utility based on stocks' mean-variance characteristics
- Instead, assumes stock returns are generated by a multi-index, or multi-factor, process
- More general than CAPM, gives room for more than 1 risk factor
- Widely used, e.g. Fama-French 3 factor model

Introduce with detour over *single index model*

[3] Stephen A. Ross (1976). The arbitrage theory of capital asset pricing. Journal of Economic Theory. 13 (3): 341–360.

Single index model

So far, we used whole variance-covariance matrix

- With I stocks, calls for $\frac{1}{2}$I(I-1) covariances
- Gives practical problems for large I
- plus: non marked related part of covariance low/erratic

Single index model is practical way around this:

- Assumes there is *only 1* reason why stocks covary: they all respond to changes in market as a whole
- Stocks respond in different degrees (measured by β)
- But stocks do not respond to unsystematic (not marked related) changes in other stocks' values

Can be formalized by writing return on stock i as:

$$r_i = \alpha_i + \beta_i r_m + \varepsilon_i$$

$r_i, r_m =$ return stock i, market
$\alpha =$ expected value non marked related return
$\varepsilon =$ random element of non marked related return, with $E(\varepsilon) = 0$ and variance = σ_ε^2
$\beta =$ beta coefficient (sensitivity for changes in the market)

Single factor model makes 2 assumptions:

1. $cov(r_m, \varepsilon_i) = 0$: random element of non marked related return not correlated with market return
2. $cov(\varepsilon_i, \varepsilon_j) = 0$ for all $i \neq j$: random elements of non marked related returns are uncorrelated

Means that variance, covariance of stocks is:

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon i}^2 \qquad \sigma_{i,j} = \beta_i \beta_j \sigma_m^2$$

covar determined by stocks' responses to changes in marked

Simplifies analysis of large portfolios drastically:

- have to calculate each stock's α, β and σ_ε^2
- plus r_m and σ_m^2, i.e. 3I + 2 $<$ I + $\frac{1}{2}$I(I-1)
- for 100 stock portfolio
    - full var-covar has $100 \times 99/2 =$4950 covar's + 100 var's
    - index model uses $3 \times 100 + 2 = 302$

The single index model

$$r_i = \alpha_i + \beta_i r_m + \varepsilon_i$$

can also be looked upon as a *return generating process* :
The returns on any investment consist of:

- α_i expected return not related to the return on the market
- $\beta_i r_m$ return that is related to the return on the market
- ε_i random element

Return generating process easily extended to more indices (or factors):

- 'split' market index in several industry indices (industrials, shipping, financial,....)
- general economic factors (interest rate, oil price,...)

Expression for stock returns then becomes:

$$r_i = \alpha_i + b_{1i}F_1 + b_{2i}F_2 + ... + b_{Ki}F_K + \varepsilon_i$$

b_{1i} = sensitivity of stock i for changes in factor F_1
F_1 = return on factor 1, etc.

The multi-factor (-index) model assumes that:

- factors are uncorrelated: $cov(F_m, F_k) = 0$ for all $m \neq k$
- residuals uncorrelated with factors $cov(F_k, \varepsilon_i) = 0$
- residuals of different stocks uncorrelated $cov(\varepsilon_i, \varepsilon_j) = 0$ for all $i \neq j$

Arbitrage Pricing Theory

- Arbitrage pricing theory builds on such a multi-factor return generating process
- Distinguishes between
  - *expected* part of stock returns
  - *unexpected* part
- Unexpected part (risk) consists of
  - systematic (or market) risk
  - and unsystematic (or idiosyncratic) risk
- Market risk not expressed as covar with market but as sensitivity to (any) number of risk factors

To derive pricing relation, start with return generating process:

$$r_i = \alpha_i + b_{1i}F_1 + b_{2i}F_2 + \ldots + b_{Ki}F_K + \varepsilon_i$$

taking expectations:

$$E(r_i) = \alpha_i + b_{1i}E(F_1) + b_{2i}E(F_2) + \ldots + b_{Ki}E(F_K)$$

subtracting lower from upper gives:

$$\begin{aligned} r_i - E(r_i) &= (\alpha_i + b_{1i}F_1 + \ldots + b_{Ki}F_K + \varepsilon_i) \\ &\quad - (\alpha_i + b_{1i}E(F_1) + \ldots + b_{Ki}E(F_K)) \end{aligned}$$

which can be re-written as:

$$r_i = E(r_i) + \sum_{k=1}^{K} b_{ik}(F_k - E(F_k)) + \varepsilon_i$$

- $E(r_i) =$ is expected return of stock i
- $b_{ik} =$ is sensitivity of stock i to factor k
- $F_k =$ return of factor k, with $E(F_k - E(F_k)) = 0$
  ($\Rightarrow$ fair game: expectations accurate in long run)
- $\varepsilon_i =$ idiosyncratic return stock i, $E(\varepsilon_i) = 0$

Terms after $E(r_i)$ are 'error' part of process:

- describe deviation from expected return
- b_{ik} is sensitivity for *unexpected* factor changes
- expected part included in $E(r_i)$

Next, construct portfolio, I assets, weights x_i, then portfolio return is:

$$r_p = \sum_{i=1}^{I} x_i r_i$$

substituting expression for r_i gives:

$$r_p = \sum_{i=1}^{I} x_i E(r_i) + \sum_{i=1}^{I} \sum_{k=1}^{K} x_i b_{ik}(F_k - E(F_k)) + \sum_{i=1}^{I} x_i \varepsilon_i$$

In well diversified portfolios, idiosyncratic risk (last term) disappears

APT's equilibrium condition is:

the absence of arbitrage opportunities

Means if you make a well diversified portfolio ($\sum_i x_i \varepsilon_i = 0$):

1. that requires no net investment
   - sum portfolio weights is zero: $\sum_i x_i = 0$
2. that involves no risks
   - weighted sum of all b_{ik} is zero : $\sum_i x_i b_{ik} = 0$ for all k
3. then

what?

the expected return must be zero:

- $\sum_i x_i E(r_i) = 0$

These three no-arbitrage conditions can be interpreted as orthogonality conditions from linear algebra:

1. $\sum_i x_i = 0$ means:
   - vector of weights is orthogonal to a vector of 1's
2. $\sum_i x_i b_{ik} = 0$ means:
   - vector of weights orthogonal to vectors of sensitivities
3. $\sum_i x_i E(r_i) = 0$ means:
   - vector weights orthogonal to vector expected returns

This means that the last vector, $E(r_i)$, must be a linear combination of the other 2:

$$E(r_i) = \lambda_0 + \lambda_1 b_{1i} + \lambda_2 b_{2i} + \ldots + \lambda_k b_{ki}$$

To give lambda's economic meaning:

- construct risk free portfolio:
  - earns risk free rate
  - has zero value for all b_{ij}
  - $r_f = \lambda_0 + \lambda_1 0 + .. + \lambda_k 0 \Rightarrow \lambda_0 = r_f$
- construct portfolio only sensitive to factor 1:
  - sensitivity 1 for factor 1 and zero value for all other b_{ij} :
  - earns expected return of factor 1
  - $E(F_1) = r_f + \lambda_1 1 + \lambda_1 0 + .. + \lambda_k 0 \Rightarrow \lambda_1 = E(F_1) - r_f$
- repeat for all factors

Gives usual form of APT as equilibrium relation:

$$E(r_i) = r_f + \sum_{k=1}^{K} b_{ik}(E(F_k) - r_f)$$

Example

Illustrates APT with 3 well diversified portfolios and their sensitivities to 2 factors, priced to give these returns:

	P_1	P_2	P_3
r_p	.18	.15	.12
b_1	1.5	0.5	0.6
b_2	0.5	1.5	0.3

Portfolio returns are functions of

- risk free rate and 2 factor returns (risk premia)
- portfolios' sensitivities

Example (cont.'ed)

Factor returns and r_f found by solving 3 APT equations:

$$.18 = \lambda_0 + \lambda_1 \times 1.5 + \lambda_2 \times .5$$
$$.15 = \lambda_0 + \lambda_1 \times .5 + \lambda_2 \times 1.5$$
$$.12 = \lambda_0 + \lambda_1 \times .6 + \lambda_2 \times .3$$

which gives $\lambda_0 = 0.075\,, \lambda_1 = 0.06$ and $\lambda_2 = 0.03$

Equilibrium relation $E(r_i) = .075 + .06b_{1i} + .03b_{2i}$

- defines return plane in 2 risk dimensions
- all investments must lie on this plane
- otherwise arbitrage opportunities exist

Example (cont.'ed)

Suppose you make a portfolio:

- with b_1=.75 and b_2=.7
- you figure it is somewhere between P_1 and P_2
- price it to offer a .16 return, also between P_1 and P_2

What happens?

You go bankrupt quickly! You offer this arbitrage opportunity:

- construct arbitrage portfolio of $.2P_1 + .3P_2 + .5P_3$, has:
- $b_1 = .2 \times 1.5 + .3 \times .5 + .5 \times .6 = .75$
- $b_2 = .2 \times .5 + .3 \times 1.5 + .5 \times .3 = .7$
- return of $.2 \times .18 + .3 \times .15 + .5 \times .12 = .141$

Example (cont.'ed)

Arbitrage strategy:

- buy what is cheap (your portfolio)
- sell what is expensive (arbitrage portfolio)

	Cfl_{now}	Cfl_{later}	b_1	b_2
buy your portfolio	-1	1.160	.75	.7
sell arbitrage.portfolio	1	-1,141	-.75	-.7
net result	0	.019	0	0

Profit of .019 is risk free, zero sensitivity to both factors

Empirical tests of APT

- require same assumptions & approximations as CAPM
- done with similar two pass regression procedure:
  - time series regression to estimate sensitivities
  - cross section analysis to estimate risk premia

Example: split total market in 2 industry indices:

- manufacturing (F_{man})
- trade (F_{trad})

1. First pass regression: estimate sensitivities

$$r_{it} - r_{f\ t} = \widehat{\alpha}_i + \widehat{\beta}_{man,i}(F_{man,t} - r_{ft}) + \widehat{\beta}_{trad,i}(F_{trad,t} - r_{ft}) + \widehat{\varepsilon}_{it}$$

   for all individual assets

2. Then calculate average risk premia ($\overline{rp}_i$) etc. over same/subsequent period and estimate risk factor premia in second pass regression:

$$\overline{rp}_i = \gamma_0 + \gamma_1 \widehat{\beta}_{man,i} + \gamma_2 \widehat{\beta}_{trad,i} + \widehat{u}_i$$

3. APT predictions tested by:
   - γ_0 should be zero
   - γ_1 should be $\overline{F_{man} - r_f}$
   - γ_2 should be $\overline{F_{trad} - r_f}$

Industry indices are easy:

- readily observable, also their risk premia
- describe market completely:
  market = manufacturing + trade

More difficult if we use:

- business characteristics
  - size, book-to-market value, price-earnings ratio, etc.
- general economic variables
  - interest rate, oil price, exchange rates, etc.

No observed risk premia, difficult to be 'complete'
$\Rightarrow$ omitted variable bias

Example: Fama-French three factor model [4]

- estimated on monthly data 1963-1991
- all stocks on all US exchanges (NYSE, ASE, NASDAQ)
- for each year, different portfolios are made:
  - size: small and big stocks
    - each month portfolio returns calculated
    - difference: SMB, small minus big
    - approximates premium size related risk factor
  - book-to-market: high (top 30%), middle, low (bottom 30%)
    - each month portfolio returns calculated
    - difference: HML, high minus low
    - approximates premium book-to-market related risk factor

[4] Eugene F. Fama, Kenneth R.French (1993). Common risk factors in the returns on stocks and bonds. Journal of Financial Economics. 33: 3–56.

- Third factor is total market: return value weighted portfolio of all stocks minus r_f

First pass estimates portfolios' sensitivities (factor loadings):

$$r_{it} - r_{ft} = \widehat{a}_i + \widehat{b}_i(r_{mt} - r_{ft}) + \widehat{s}_i SMB_t + \widehat{h}_i HML_t + \widehat{\varepsilon}_{it}$$

- $\widehat{b}_i$, $\widehat{s}_i$ and $\widehat{h}_i$ are sensitivities of portfolio i
- SMB, HML are risk premia (small-big, high-low), no $-r_f$

Fama-French three factor model formulated as:

$$E(r_i) - r_f \; = \widehat{a}_i + \widehat{b}_i[E(r_m) - r_f] + \widehat{s}_i E(SMB) + \widehat{h}_i E(HML)$$

APT predicts intercept, $\widehat{a}_i$, should be zero

- Fama and French find $\widehat{a}_i$ close to 0 in most cases
- also claim model explains much ($>$90%) of variance in average stock returns

Fama-French model widely used

- to calculate $E(r_p)$ when size and value effects can play a role
- see examples in market efficiency

But: more recent research shows that the model's relevance has diminished over time.

Summarizing, Arbitrage Pricing Theory:

- Rests on different assumptions than CAPM
- Is more general than CAPM
  - makes less restrictive assumptions
  - allows more factors, more realistic
- Is less precise than CAPM
  - does not give a volume of risk (what or even how many factors to use)
  - does not give a price of risk (no expression for factor risk premia, have to be estimated empirically)
- has interesting applications in risk management, default prediction, etc.