TIØ4146 Finance for Science and Technology Students

Chapter 8 - Option Pricing in Continuous Time

Carlos Miguel Dos Santos Oliveira
Ståle Størdal
Felipe Van de Sande Araujo

Norwegian University of Science and Technology

Black and Scholes option pricing formula models 5 (or 6) determinants of option prices:

1. Stock price S_0
2. Exercise price X
3. Time to maturity T
4. Risk free interest rate r
5. Volatility σ
6. (Dividends)

With how much does the option price change if the value of a determinant changes marginally?

That is what the Greeks show

Greeks are comparative statics of Black and Scholes model named after Greeks letters used as symbol:

Option price determinants and their Greeks

Determinant	Greek	symbol
Exercise price	-	-
Stock price	Delta	Δ
Volatility	Vega	ν
Time to maturity	Theta	$-\Theta$
Interest rate	Rho	ρ
	Gamma	Γ

Terminology is sloppy:

- Exercise price is determinant
  - has neither name nor symbol
  - sometimes called ’no-name’
- Vega is not a Greek letter
  - Greek letter nu (ν) is used instead
- Gamma has no option price determinant
  - is comparative static for delta
  - shows how much delta changes if stock price changes

How are the Greeks calculated?

- By taking partial derivative of Black and Scholes formula
- with respect to one determinant at a time
- Does not involve stochastic calculus
  - Black and Scholes formula does not contain Brownian motion
  - can be a bit complicated anyway, contains probability terms

Calculations can be found in the book (appendix 8B)

We will look at

- proper use
- interpretation

We use this example:

Value of at the money European call / put

- matures in one year
- strike price of 100
- underlying stock pays no dividends
- has annual volatility of 20%
- risk free interest rate is 10% per year.

Recall the Black and Scholes option pricing formulas for a European call:

$$O_{c,0} = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

and a European put:

$$O_{p,0} = Xe^{-rT} N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/X) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

We have our five determinants:

$S_0 = 100$, $X = 100$, $r = .1$, $\sigma = .2$ and $T = 1$.

$$d_1 = \frac{\ln(S_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$= \frac{\ln(100/100) + (.1 + \frac{1}{2}.2^2)1}{.2\sqrt{1}} = .6$$

$$d_2 = d_1 - \sigma\sqrt{T} = .6 - .2\sqrt{1} = .4$$

Areas under normal curve for values of d_1 and d_2 can be found:

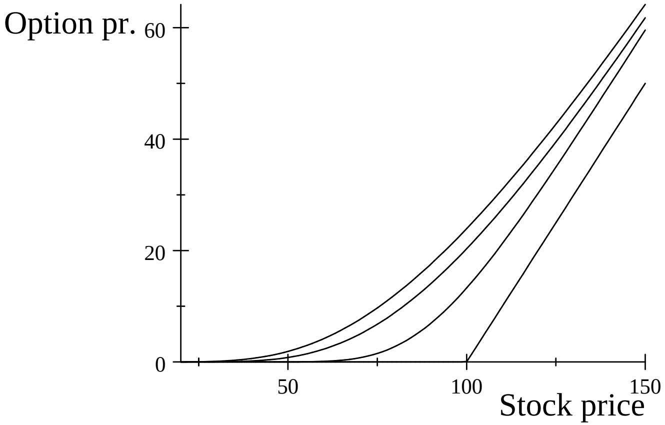
- table in appendix 8C in the book (good enough for this course), calculator, spread sheet, etc.:

$\text{NormalDist}(.6) = 0.72575$, $\text{NormalDist}(.4) = 0.65542$,
Option price becomes:

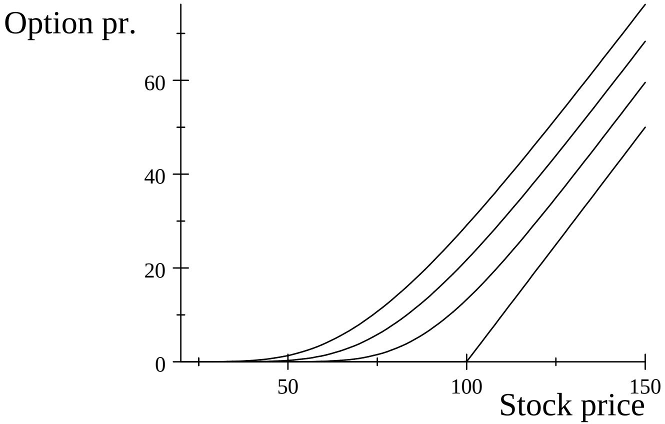
$$O_{c,0} = 100 \times (0.72575) - 100e^{-.1}(0.65542) = 13.27$$

Value put option calculated with equation or the put call parity:

$$\begin{aligned} O_{p,0} &= O_{c,0} + Xe^{-rT} - S_0 \\ &= 13.27 + 100e^{-.1} - 100 = 3.75 \end{aligned}$$



Call option prices for X=100, T=2, r=.1 and $\sigma = .5$ (top), $.4$ and $.2$ (bottom)



Call option prices forX=100, r=.1, $\sigma = .2$ and $T = 3$ (top), 2 and 1 (bottom)

Delta

Option price's sensitivity for changes in value of underlying stock

- Looked extensively at Δ in binomial model
- also at hedging
- Intuition should be clear

Does call price increase or decrease with underlying?

- gives right to buy at fixed price
- good to buy cheaply
- call price increases with value underlying
- $\Delta > 0$

- Delta calculated for a call as:

$$\Delta_c = \frac{\partial\left(S_0 N\left(d_1\right) - Xe^{-rT} N\left(d_2\right)\right)}{\partial S_0} = N\left(d_1\right)$$

  so obviously:

$$0 \leq \Delta_c \leq 1$$

- and for a put as:

$$\Delta_p = \frac{\partial\left(Xe^{-rT} N\left(-d_2\right) - S_0 N\left(-d_1\right)\right)}{\partial S_0} = N(d_1) - 1$$

  so

$$-1 \leq \Delta_p \leq 0$$

As a partial derivative:

- Delta measures the sensitivity of option prices
  - for single-unit changes
  - in the underlying value S_0
- The underlying value is in money units (€ or $)
  - so Δ gives the change in option price
  - resulting from a €1 or $1 change in the underlying value
- Delta only valid in neighbourhood of given stock price
  - numerical value changes with other parameters
  - hedge is dynamic
  - seen equivalent in binomial model

Example

What is the delta of our example call for a stock price of 80?

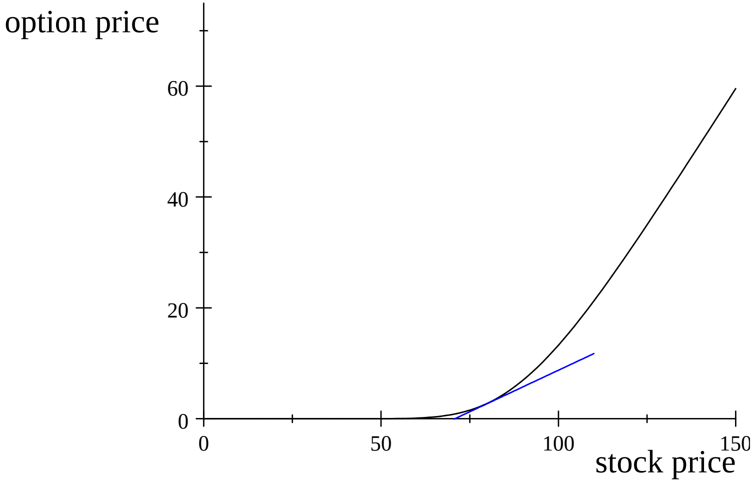
- Have to calculate $N(d_1)$

$$d_1 = \frac{\left(\ln(80/100) + (0.1 + 0.5 \times 0.2^2) \times 1\right)}{0.2\sqrt{1}} = -0.51572$$

- so the option delta is:

$$N(d_1) = \text{NormalDist}(-0.51572) = 0.3$$

Can be depicted as slope of the line tangent to the option price curve at the point where the stock price is 80



Black and Scholes option prices and option delta at S=80; T=1, r=0.1, σ=0.2, X=100

Suppose a bank has sold 50 option contracts

- of at-the-money options
- each on 100 shares.

How does the bank hedge its obligations from these contracts?

- contracts give option holders right to buy 5000 shares at maturity
    - the at-the-money option delta, $N(d_1)$, is 0.73
    - to 'delta-hedge' its position, bank has to buy $0.73 \times 5000 = 3650$ shares

What happens if stock price goes up with €1?

- options increase in value with $5000 \times 0.73 = 3650$ euro
- bank has short position in options
    - it loses €3650
- loss exactly compensated by share position
    - which gains $3650 \times 1 = 3650$ euro
- Bank has perfect hedge
    - portfolio of short calls and long shares is 'delta neutral'

What happens if stock price increases to €110?

- Delta hedging is dynamic
- bank has to adjust its hedge
- d_1 becomes:

$$d_1 = \left(\ln(110/100) + (0.1 + 0.5 \times 0.2^2) \times 1\right) / 0.2\sqrt{1} = 1.077$$

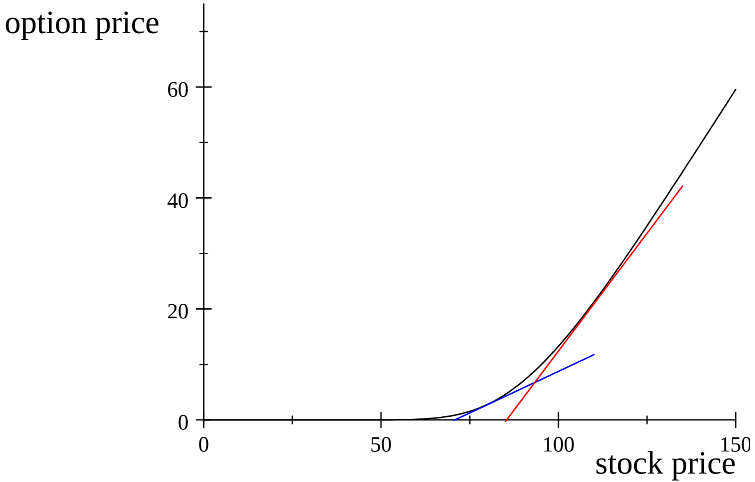
- option delta becomes $\text{NormalDist}(1.077) = 0.8593$
- bank has to buy more shares to remain delta neutral:
  - $0.8593 \times 5000 = 4296$
  - buy $4296 - 3650 = 646$ shares

Black and Scholes formula assumes continuous hedge

- Delta continuously recalculated
- Portfolio continuously rebalanced
- Is, of course, impossible:
  - continuously rebalancing: transactions $\to \infty$
  - transaction costs $\to \infty$

In practice:

- traders rebalance frequently
- are delta neutral at least once a day

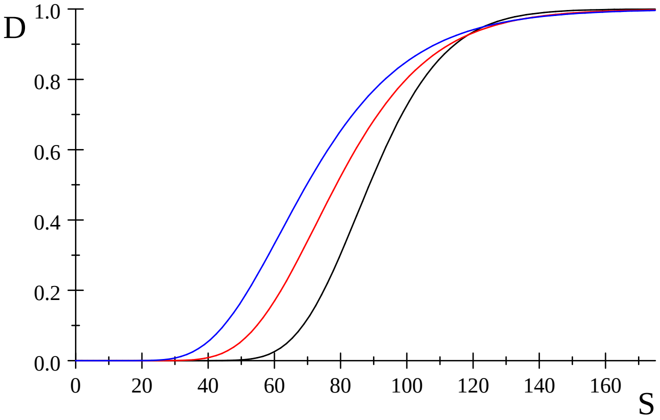


Black and Scholes option prices and option delta at S=80 and S=110; T=1, r=0.1, σ=0.2, X=100

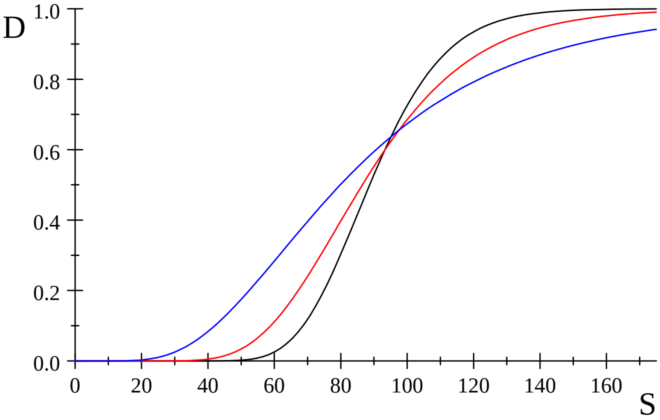
Illustrates more general relation:

- If a call is far out-of-the-money
  - it will almost certainly not be exercised, expires worthlessly
  - small changes in stock price do not matter, not exercised anyway
  - means $\Delta \approx 0$
- If a call is far in-the-money
  - it will almost certainly be exercised
  - option position $\approx$ holding stock without having paid for it yet
  - what happens to option $\approx$ what happens to stock
  - means $\Delta \approx 1$

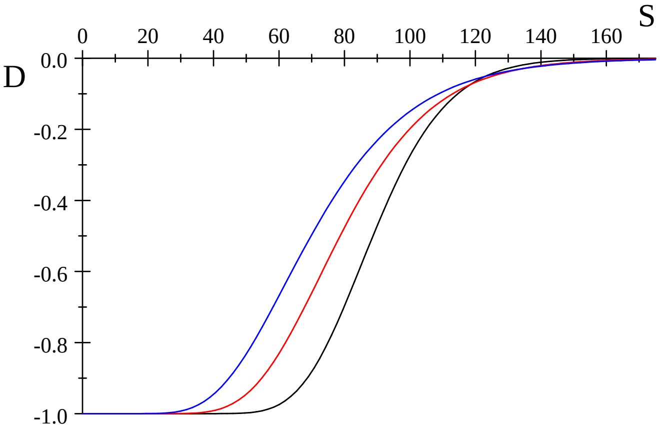
Depict delta for some parameter values



Call option delta (D) vs. stock price S for $T = 1$ (black), $T = 2$ (red) and $T = 3$ (blue); $X = 100$, $r = 0.1$ and $\sigma = 0.2$



Call option delta (D) vs. stock price S for $\sigma = 0.2$ (black), $\sigma = 0.3$ (red) and $\sigma = 0.5$ (blue); $X = 100$, $r = 0.1$ and $T = 1$



Put option delta (D) vs. stock price S for $T = 1$ (black), $T = 2$ (red) and $T = 3$ (blue); $X = 100$, $r = 0.1$ and $\sigma = 0.2$

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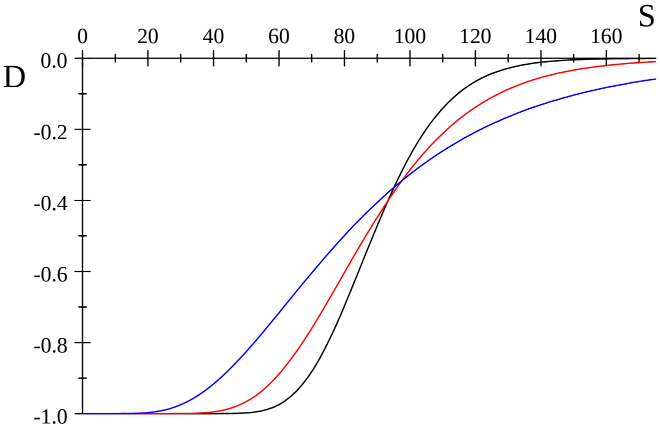

Put option delta (D) vs. stock price S for $\sigma = 0.2$ (black), $\sigma = 0.3$ (red) and $\sigma = 0.5$ (blue); $X = 100$, $r = 0.1$ and $T = 1$

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Vega

Sensitivity of option prices for single-unit changes in volatility σ

- Volatility usually measured in decimal fractions
  - 20% volatility entered into calculations as 0.2
- Dividing vega by 100 transforms single-unit change
  - into change in option price per percentage point change in volatility

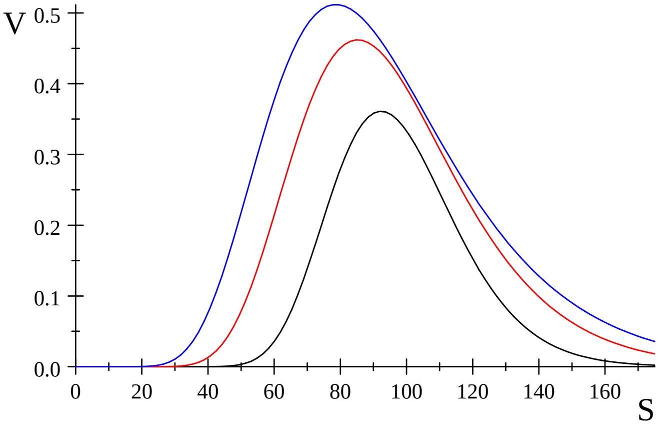
Vega of a call is calculated as:

$$\nu_c = \frac{\partial \left(S_0 N\left(d_1\right) - Xe^{-rT} N\left(d_2\right)\right)}{\partial \sigma} = S_0 n(d_1)\sqrt{T} > 0$$

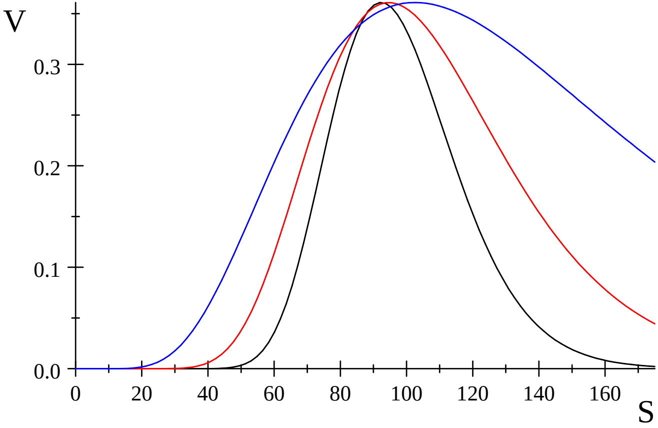
and for a put as:

$$\nu_p = \frac{\partial \left(Xe^{-rT} N\left(-d_2\right) - S_0 N\left(-d_1\right)\right)}{\partial \sigma} = S_0 n(d_1)\sqrt{T} = \nu_c > 0$$

- same for put and call
- option values increase with volatility



Vega (per percentage point) (V) vs. stock price S for $T = 1$ (black), $T = 2$ (red) and $T = 3$ (blue); $X = 100$, $r = 0.1$ and $\sigma = 0.2$



Vega (per percentage point) (V) vs. stock price S for $\sigma = 0.2$ (black), $\sigma = 0.3$ (red) and $\sigma = 0.5$ (blue); $X = 100$, $r = 0.1$ and $T = 1$

Theta

Sensitivity of option prices for single-unit changes in time T

- can be interpreted in different ways
- first give partial derivative for call
- then look at what it means

Theta of a call is calculated as:

$$\Theta_c = S_0 n(d_1) \frac{\sigma}{2\sqrt{T}} + rXe^{-rT} N(d_2) > 0$$

What does $\Theta_c > 0$ mean?

Call value increases with time to maturity

- longer time to maturity, higher value

The theta of a put is calculated in the same way:

$$\Theta_p = \frac{\partial \left(Xe^{-rT} N\left(-d_2\right) - S_0 N\left(-d_1\right)\right)}{\partial T}$$
$$= -rXe^{-rT} N\left(-d_2\right) + S_0 n(d_1) \frac{\sigma}{2\sqrt{T}} \lesseqgtr 0$$

Sign cannot be determined, but:

- options, puts included, generally increase in value with time to maturity
- except when early exercises would be profitable:
  - (far) in-the-money calls on stocks that pay high dividend
  - (far) in the money puts (will be exercised anyway, have to wait longer for X is received)

Rho

Sensitivity of option prices for single-unit changes in interest rate r

Does a call in- or decrease with r?

- call may involve paying X at later date
- PV of future payment decreases with r
- calls increase in value with r

What about puts?

- put may involve receiving X at later date
- PV of future payment decreases with r
- puts decrease in value with r

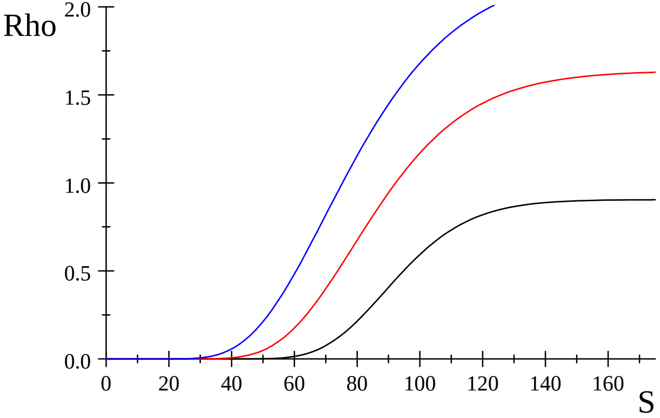
The expressions are:

$$\rho_c = XTe^{-rT} N(d_2) > 0$$

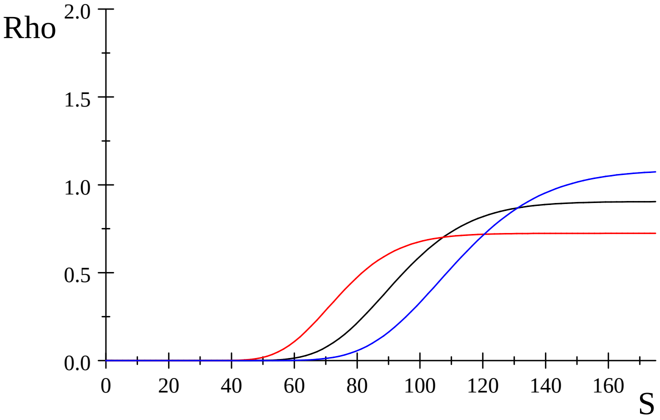
and

$$\rho_p = -XTe^{-rT} N(-d_2) < 0$$

- Rho calculated for single-unit (1) changes in interest rate r
- Dividing by 100 transforms it into change per percentage point change in interest rate



Call option rho vs. stock price S for $T = 1$ (black), $T = 2$ (red) and $T = 3$ (blue); $X = 100$, $r = 0.1$ and $\sigma = 0.2$



Call option rho vs. stock price S for $X = 100$ (black), $X = 80$ (red) and $X = 120$ (blue); $T = 1$, $r = 0.1$ and $\sigma = 0.2$

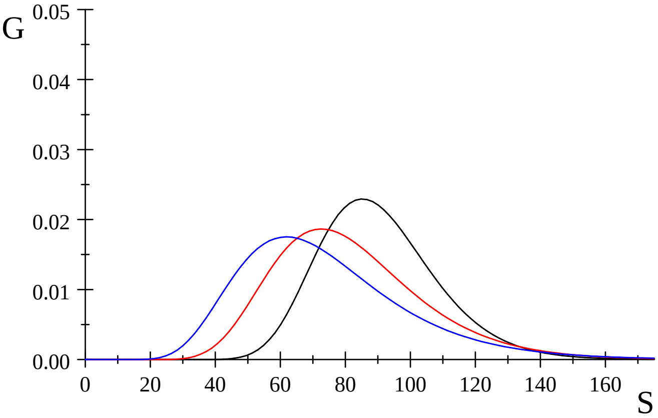
Gamma

Sensitivity of option prices for single-unit changes in delta Δ

- calculated as second derivative option pricing formula w.r.t. underlting stock S_0

$$\Gamma_c = \frac{n(d_1)}{S_0 \sigma \sqrt{T}} = \Gamma_P > 0$$

- same for put and call
- Delta changes most for options $\pm$ at-the-money
  - stable for far in- or out-of-the-money options
- Behaviour reflected in gamma



Option gamma (G) vs. stock price S for $T = 1$ (black), $T = 2$ (red) and $T = 3$ (blue); $X = 100$, $r = 0.1$ and $\sigma = 0.2$

Strike price (no-name)

Sensitivity of option prices for single-unit changes in exercise price X

- Doesn't have a Greek letter
- Is a price determinant just as other 4
- But is not a risk factor (like theta):
  - written in the contract
  - cannot change during the option's life time (unlike S, σ, T and r)
- Useful to compare options with different X

- Other things equal, if X is higher call price is lower
- Other things equal, if X is higher put price is higher
- Call may involve buying $\Rightarrow$ better to buy more cheaply
- Put may involve selling $\Rightarrow$ better to sell more expensively

The formulas are:

$$\frac{\partial O_c}{\partial X} = -e^{-rT} N(d_2) < 0$$

$$\frac{\partial O_p}{\partial X} = e^{-rT} N(-d_2) > 0$$