Exericse 1

(1) Simulate and plot 50 timesteps of the following model,

$$x_{t+1} = w_t + w_{t-1}, \quad x_1 = 0, \quad w_t \sim \mathcal{N}(0,1) \text{ are iid.}$$

Plot the theoretical and sample autocorrelation functions of $(x_t)_{t\geq 1}$. Using Property 1.2 in the book, assess the peaks in the sample autocorrelation. Repeat this process for larger samples such as 1000 and 10000. Derive and plot 95% confidence intervals for x_t. Check numerically if x_t appears stationary. What does the initial condition x_1 have to be in order for x_t to be (a snippet of) a stationary process $(x_t)_{t\in\mathbb{Z}}$?

Solution. To get the ACF of (x_t) we compute directly for some $h > 0$,

$$\begin{aligned} E[x_{t+h}x_t] &= E[(w_{t+h} + w_{t+h-1})(w_t + w_{t-1})] \\ &= E[w_{t+h}w_t + w_{t+h}w_{t-1} + w_{t+h-1}w_t + w_{t+h-1}w_{t-1}] \\ &= \delta_{h,0} + \delta_{|h|,1} + \delta_{h,0} \\ &= 2\delta_{h,0} + \delta_{|h|,1} \end{aligned}$$

where $\delta_{h,n} = 1$ if $h = n$ and 0 otherwise. Therefore the autocovariance function is,

$$\gamma(h) = \begin{cases} 2, & h = 0, \\ 1 & |h| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and the autocorrelation function is $\rho(h) = \gamma(h)/\gamma(0) = \gamma(h)/2$.

For the expression for $\gamma(h)$ above to actually be correct, we have need $t > 1$. This is because $x_1 = 0$, and so $E[x_1^2] = 0$ and $E[x_1x_2] = 0$, which does not work with the covariance function above.

To "make (x_t) into a stationary process" we need to choose x_1 so that (i) $E[x_1^2] = 2$ and (ii) $E[x_1x_2] = 1$, and (iii) $E[x_1x_k] = 0$ for $k > 2$. We can satisfy (ii) by letting $x_1 = w_1 + \alpha w_0$, for some $w_0 \sim \mathcal{N}(0,1)$, independent of $w_t, t > 0$. To satisfy (i), we see that also $\alpha = 1$. (iii) follows, since both w_0 and w_1 are independent of $w_t, t > 1$.

To get 95% confidence intervals for the value of x_t, we note x_t is Gaussian, has mean 0, and variance 2. Therefore, a 95% confidence interval for x_t would be $\pm 1.96\sqrt{2}$.

(2) Simulate and plot 50 timesteps of the following model,

$$x_{t+1} = 0.5x_t + w_{t+1}, \quad x_1 = 0, \quad w_t \sim \mathcal{N}(0,1) \text{ are iid.}$$

Plot the theoretical and sample autocorrelation functions of $(x_t)_{t\geq 1}$. Derive and plot 95% confidence intervals for x_t. Check numerically if x_t is stationary. What does the initial condition x_1 have to be in order for x_t to be (a snippet of) a stationary process $\{x_t\}_{t\in\mathbb{Z}}$?

Solution. If (x_t) was an AR(1) models that is stationary we could compute the theoretical autocorrelation function for $x_t = \alpha x_{t-1} + w_t$, $\alpha = 0.5$ as follows: take some $h > 0$, and note that,

$$\begin{aligned}\gamma(h) = E[x_{t+h}x_t] &= E[(\alpha x_{t+h-1} + w_{t+h})x_t] \\ &= E[\alpha x_{t+h-1}x_t] + E[w_{t+h}x_t] \\ &= \alpha E[x_{t+h-1}x_t] + 0 \\ &= \alpha\gamma(h-1) \\ &= \alpha^h\gamma(0).\end{aligned}$$

We can find $\gamma(0)$ by computing directly,

$$\begin{aligned}\gamma(0) = E[x_t^2] &= E[(\alpha x_{t-1} + w_t)^2] \\ &= E[\alpha^2 x_{t-1}^2 + 2\alpha x_{t-1}w_t + w_t^2] \\ &= \alpha^2 E[x_{t-1}^2] + 2\alpha E[x_{t-1}w_t] + E[w_t^2] \\ &= \alpha^2\gamma(0) + 2\cdot 0 + 1 \\ &= \alpha^2\gamma(0) + 1.\end{aligned}$$

Solving for $\gamma(0)$, we get $\gamma(0) = 1/(1-\alpha^2)$. Therefore,

$$\gamma(h) = \alpha^h\gamma(0) = \frac{\alpha^h}{1-\alpha^2}, \quad \alpha = 0.5,$$

and the autocorrelation function is $\rho(h) = \alpha^h$.

But since $x_1 = 0$, it turn out that (x_t) is not stationary, and we get instead, for some $h > 0$,

$$\begin{aligned}cov(x_{t+h}, x_t) &= E[x_{t+h}x_t] \\ &= E[\left(\alpha^{t+h-1}x_1 + \sum_{k=2}^{t+h}\alpha^{t+h-k}w_k\right)\left(\alpha^{t-1}x_1 + \sum_{k=2}^{t}\alpha^{t-k}w_k\right)] \\ &= E[\left(\sum_{k=2}^{t}\alpha^{t+h-k}w_k\right)\left(\sum_{k=2}^{t}\alpha^{t-k}w_k\right) + \left(\sum_{k=t+1}^{t+h}\alpha^{t+h-k}w_k\right)\left(\sum_{k=2}^{t}\alpha^{t-k}w_k\right)] \\ &= E[\left(\sum_{k=2}^{t}\alpha^{t+h-k}w_k\right)\left(\sum_{k=2}^{t}\alpha^{t-k}w_k\right)] + 0 \\ &= E[\sum_{i=2}^{t}\sum_{j=2}^{t}\alpha^{t+h-i+t-j}w_iw_j] \\ &= \sum_{i=2}^{t}\sum_{j=2}^{t}\alpha^{t+h-i+t-j}E[w_iw_j] \\ &= \sum_{i=2}^{t}\sum_{j=2}^{t}\alpha^{t+h-i+t-j}\delta_{i,j} \\ &= \sum_{i=2}^{t}\alpha^{h+2t-2i} \\ &= \alpha^h\sum_{k=2}^{t}\alpha^{2(t-k)} = \alpha^h\sum_{k=0}^{t-2}\alpha^{2k}.\end{aligned}$$

Note that the expression above tends to $\alpha^h/(1-\alpha^2)$ as $t \to \infty$ (provided $|\alpha| < 1$). If we assume that x_1 is not 0 but is independent of $w_t, t > 1$, we would get (convince yourself that this is true),

$$cov(x_{t+h}, x_t) = \alpha^{2t+h-2} E[x_1^2] + \alpha^h \sum_{k=0}^{t-2} \alpha^{2k}.$$

We are asked to choose an x_1 which "makes (x_t) stationary". By this we mean that we need to try to find an x_1 so that,

$$cov(x_{t+h}, x_t) = \alpha^{2t+h-2} E[x_1^2] + \alpha^h \sum_{k=0}^{t-2} \alpha^{2k} = \frac{\alpha^h}{1-\alpha^2}.$$

Dividing this expression by α^h, and using that $\sum_{k=0}^{n} x^n = (1-x^{n+1})/(1-x)$ we get,

$$\begin{aligned} \alpha^{2t-2} E[x_1^2] &= \frac{1}{1-\alpha^2} - \sum_{k=0}^{t-2} \alpha^{2k} \\ &= \frac{1}{1-\alpha^2} - \frac{1-\alpha^{2t-2}}{1-\alpha^2} \\ &= \frac{\alpha^{2t-1}}{1-\alpha^2}, \end{aligned}$$

we see that we can choose x_1 so that $var(x_1) = 1/(1-\alpha^2)$.

To get the confidence interval for x_t, we note that it is Gaussian with zero mean, and with variance,

$$var(x_t) = \sum_{k=0}^{t-2} \alpha^{2k}.$$

Therefore, a 95%-confidence interval can be found by taking $\pm 1.96\sqrt{var(x_t)}$. Plotting this, it starts at 0 for $t = 1$, and goes towards $\pm 1.96\sqrt{1/(1-\alpha^2)}$.

(3) Problem 1.6 in the textbook.

(4) Problem 1.8 in the textbook.

(5) Sample data from the following bivariate time series. Compute the sample autocorrelations and the cross-correlation for x and y and plot along with expected intervals from Properties 1.2 and 1.3. Consider these results and compare with the prominent peaks of the autocorrelations in univariate cases above.

$$x_{t+1} = 0.5x_t + 0.3y_t + w_t, \quad x_1 = 0, \quad w_t \sim \mathcal{N}(0,1) \text{ are iid.}$$
$$y_{t+1} = 0.2x_{t-5} + 0.4y_t + z_t, \quad x_1 = 0, \quad z_t \sim \mathcal{N}(0,1) \text{ are iid.}$$

(6) Let $X = (X_1, \ldots, X_n)$ be an $\mathbb{R}^n$-valued random variable, and Y be an $\mathbb{R}$-valued random variable. Express the best linear predictor of Y as MX, where $M \in \mathbb{R}^{1\times n}$ is a matrix.

Solution. We have from the "best-linear-predictor-condition",

$$\hat{Y} \text{ is the best linear predictor of if } Y, \text{ if } E[(Y-\hat{Y})X_k] = 0, \ k = 1, \ldots, n.$$

If we write $\hat{Y}$ as MX, and insert this into the equation above, we get,

$$E[(Y-MX)X_k] = 0, \quad \text{or} \quad E[(Y-MX)X^T] = (0, \ldots, 0).$$

We want to solve the equation above for some $M \in \mathbb{R}^{1\times n}$, and it turns out that,

$$M = E[YX^T]E[XX^T]^{-1}.$$