Exercise 9

Problem 1.

Let $(U,\|\cdot\|_U)$ and $(V,\|\cdot\|_V)$ be normed spaces and let $T: U \to V$ be a linear transformation.

Show that T is not continuous, if and only if there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset U$ with $\|u_n\|_U = 1$ and $\|T(u_n)\|_V \geq n$, $\forall n$.

Proof:

($\Rightarrow$):

Assuming that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset U$ with the properties above.

Then, $\lim_{n\to\infty} \|u_n\|_U = 1$, but $\lim_{n\to\infty} \|Tu_n\|_V = \infty$.

So T is not bounded, hence it is not continuous.

($\Leftarrow$)

Assuming that T is not continuous.

Then there must exist a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $\|u_n - u\|_V \xrightarrow[n\to\infty]{} 0$ but $\|Tu_n - Tu\|_V \not\xrightarrow[n\to\infty]{} 0$.

Problem 2.

On the space $C^\infty([0,1])$ of arbitrarily differentiable functions $f: [0,1] \to \mathbb{R}$, we consider the norm $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$.

For each of the following mappings, decide whether it is continuous with respect to $\|\cdot\|_\infty$.

(a) The mapping $T: C^\infty[0,1] \to \mathbb{R}$, $Tf = \int_0^1 f(x)\,dx$.

$$\begin{aligned}\|Tf\|_\infty &= \left\|\int_0^1 f(x)\,dx\right\|_\infty \\ &= \max_{x\in[0,1]} \left|\int_0^1 f(x)\,dx\right| \\ &\leq \max_{x\in[0,1]} \int_0^1 |f(x)|\,dx \\ &\leq \max_{x\in[0,1]} |f(x)| \int_0^1 dx \\ &= \|f\|_\infty\end{aligned}$$

$\Rightarrow$ Continuous.

(b) The mapping $T: C^\infty([0,1]) \to C^\infty([0,1])$, $Tf = f'$.

$$\begin{aligned}\|Tf - Tg\|_\infty &= \|f' - g'\|_\infty \\ &< \delta = \varepsilon\end{aligned}$$

$\Rightarrow$ Continuous.

(c) The mapping $T: C^\infty([0,1]) \to C^\infty([0,1])$, $(Tf)(x) = \int_0^x f(y)\,dy$.

$$\begin{aligned}\|(Tf)(x)\|_\infty &= \left\|\int_0^x f(y)\,dy\right\|_\infty \\ &= \max_{x\in[0,1]} \left|\int_0^x f(y)\,dy\right| \\ &\leq \int_0^1 |f(y)|\,dy \\ &\leq \max_{x\in[0,1]} |f(x)| \int_0^1 dy \\ &= \|f\|_\infty\end{aligned}$$

$\Rightarrow$ Continuous.

(d) The mapping $T: C^\infty([0,1]) \to \ell^\infty$, $Tf = (f(1), f(\frac{1}{2}), f(\frac{1}{3}), \dots)$.

Problem 3.

Let $(U,\|\cdot\|_U)$ and $(V,\|\cdot\|_V)$ be normed spaces with $U \neq \{0\}$ and denote $L(U,V)$ the space of all bounded linear operators $T: U \to V$.

Show that $\|T\| := \sup_{u\neq 0} \frac{\|Tu\|_V}{\|u\|_U}$ defines a norm on $L(U,V)$.

Proof:

- Pos-def.:

  $\|T\| \geq 0$ clear

  $$\begin{aligned}\|T\| = 0 &\Rightarrow \sup_{u\neq 0} \frac{\|Tu\|_V}{\|u\|_U} = 0 \\ &\Rightarrow \|Tu\|_V = 0 \\ &\Rightarrow T = 0\end{aligned}$$

  O.K.

- Abs. hom.:

  $$\begin{aligned}\|cT\| &= \sup_{u\neq 0} \frac{\|cTu\|_V}{\|u\|_U} \\ &= \sup_{u\neq 0} \frac{|c|\,\|Tu\|_V}{\|u\|_U} \\ &= |c|\,\|T\|\end{aligned}$$

  O.K.

- Tri.-ineq.:

  $$\begin{aligned}\|T+S\| &= \sup_{u\neq 0} \frac{\|(T+S)u\|_V}{\|u\|_U} \\ &\leq \sup_{u\neq 0} \frac{\|Tu\|_V + \|Su\|_V}{\|u\|_U} \\ &= \sup_{u\neq 0} \frac{\|Tu\|_V}{\|u\|_U} + \sup_{u\neq 0} \frac{\|Su\|_V}{\|u\|_U} \\ &= \|T\| + \|S\|\end{aligned}$$

  O.K.

$\Rightarrow$ It defines a norm.

Problem 4.

Denote

$$c_{fin} = \{x = (x_k)_{k\in\mathbb{N}} \mid \text{there exists } K \in \mathbb{N} \text{ s.t. } x_k = 0\ \forall k > K\}$$

the space of finite sequences, and

$$c_0 = \{x = (x_k)_{k\in\mathbb{N}} \mid \lim_{k\to\infty} x_k = 0\}$$

the space of 0-sequences.

On c_{fin} and c_0 we consider the ∞-norm

$\|x\|_\infty := \sup_{k\in\mathbb{N}} |x_k|$.

(a) Show that c_0 is a closed linear subspace of ℓ^∞.

Proof:

ℓ^∞ is a sequence space whose elements are bounded;

c_0 is a sequence space whose elements converge to 0.

Thus all elements of c_0 are bounded and then c_0 must be a closed linear subspace of ℓ^∞.

(b) Show that c_{fin} is dense in c_0.

Proof: