# Finans for teknisk naturvitenskapelige studenter - Sammendrag

Finans for teknisk-naturvitenskapelige studenter (Norges teknisk-naturvitenskapelige universitet)

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# Contents

# 1. Introdution

## 1.1 Finance as a science

Finance studies how people choose between uncertain future values. In general we look at is as a social science that investigates how people allocate scarce resources. that have alternative uses, among competing goals.

### 1.1.1 How does finance work

As in many other sciences, the main tools in finance are mathematical modelling and empirical testing. What makes finance special as a social science is how well financial markets lend themselves to modelling.

Research in finance typically has an actual problem as its starting point. We simplify, translate into mathematical terms, and use this to formulate predictions or hypotheses. The predictions are tested with real-world data, and rejected if our theories are wrong. **See the figure below to understand the cycles of scientific and applied research in finance.**

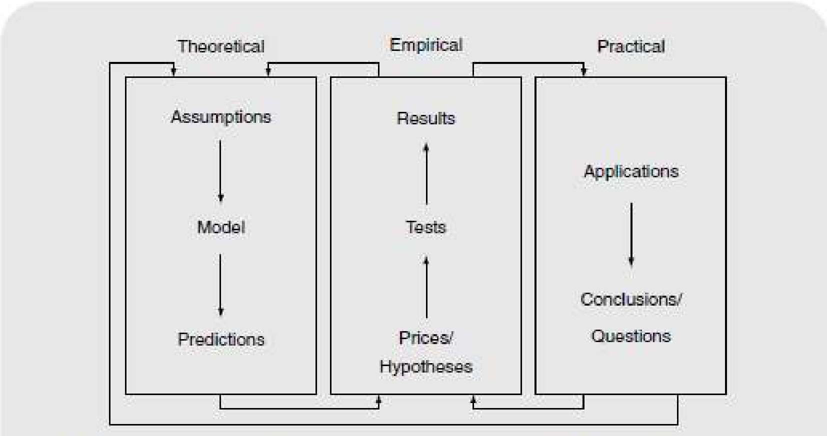



**Figure 1.1** The interlocking cycles of scientific and applied research

## 1.2 Valuation of assets - A central issue

A central issue in finance is the valuation of assets such as investment projects, firms, stock options etc. The value might not be what you paid for it when you bought it, or what some bookkeeper has written down somewhere. We generally value what the expected future cash flows from the assets is worth today.

This means the value depends on how risky cash flows are and how far in the future they will be generated. Value has a time and an uncertainty dimension; these dimensions determine the value of cash flow.

We say that the time value of money is expressed in the risk free interest rate. The rate moves risk less cash flows in time: discount future cash flow to the present and compound present cash flow to the future. We express this with a general *present-value formula* where t stand for time:

$$Value = \sum_t \frac{Exp[Cashflows_t]}{(1 + discountrate_t)^t} \tag{1.1}$$

The numerator contains the expected cash flow in each period, and if the cash flow is risk less the future amount is always the same no matter what happens.

We also have three different ways to account for risk in the valuation procedure:

1. Adjust the discount rate to a risk-adjusted discount rate that reflects not only the time value of money but also the riskiness of the cash flows. For this we have Capital Asset Pricing Model (CAPM). A equally elegant and more general but less precise method is Arbitrage Pricing Theory (APT).

2. Adjust risky cash flow so that they become cartain cash flows that have the same value as the risky ones. These *certainty equivalent cash flows can be calculated with the CAPM or with derivative securities such as futures.*

3. Redefine the probabilities that are incorporated in the expectations operator, in such a way that they contain pricing information. Risk is then embedded in probabilities and the expectation calculated with them can be discounted at the risk free interest rate. This is the essence of the Black-Scholes-Merton Option Pricing Theory.

## 1.3 Difference with the natural sciences

The natural sciences generally study phenomena that can be precisely measured.

Finance is a social science: it studies human behavior. Controlled experiements are impossible. Financial economist cannot keep firms in an isolated experiment, controll all economic variables etc. We can only observe noisy real-life data that changes based on a million different factor.

# 2. Fundamental concepts and techniques

## 2.1 The Time Value of Money

### 2.1.1 Sources of time value

Simplified: \$ 1 now has a higher value than \$1 later.

Time value of money comes from two sources, time preference and productive investment opportunities. Time preference means that we prefer current consumption rather than later. This is not just due to impatience, but also due to necessity.

Productive investment means using the money on an investment that generates more than the original amount.

We usually express the time value of money as a positive risk-free interest rate. In free markets this comes from supply and demand, which in turn comes from factors such as amounts of money held by people and businesses, availability of productive investments etc.

A consequence of the time value of money is that we cannot directly compare money now to money then. We have to use some sort of function to move money to time. Money moved forwards in time is called compounding while money moved backwards is called discounting.

### 2.1.2 Compounding and discount

In its simplest form compounding takes place after the period for which the interest is set. A yearly interest is compounded after a year etc. The following formula, with $T$ as time (years), $r$ as interest, $FV_T$ as future value at T and $PV$ as present value, gives us future value of present money:

$$FV_T = PV(1+r)^T \tag{2.1}$$

We can also solve for present value or interest rate:

$$PV = \frac{FV_T}{(1+r)^T} \tag{2.2}$$

$$r = \sqrt[T]{\frac{FV_T}{PV}} - 1 \tag{2.3}$$

We can also let the time window for returning interest become infinitesimal small, letting $T \to \inf$. This would make compounding continuous, and we get a new formula for future value:

$$FV_T = PV \exp^{rT} \tag{2.4}$$

### 2.1.3 Annuities and perpetuities

An Annuity is a series of equal payments at regular time intervals. The series can be written as:

$$PV = \frac{A}{(1+r)} + \frac{A}{(1+r)^2} + ... + \frac{A}{(1+r)^n} \tag{2.5}$$

The series sums up to:

$$PV = \frac{A}{1+r} \frac{1 - (\frac{1}{1+r})^n}{1 - \frac{1}{1+r}} \tag{2.6}$$

Or if we define the annuity such that is starts today:

$$PV = A \frac{1 - (\frac{1}{1+r})^n}{1 - \frac{1}{1+r}} \tag{2.7}$$

Perpetuities are annuities with an infinite number of payments. In technical terms this means that the $n$ mentioned above become infinite. This makes our calculations very easy, as we simply get.

$$PV = \frac{A}{r} \tag{2.8}$$

Because they are so easy to calculate, perpetuities are often included as exam questions.

## 2.2 Accounting representation of the firm

Records two things, the flow of goods and money through a firm, as well as the effect these flows has on the firms assets.

### 2.2.1 Financial statement

Reports the firms revenues, costs and profits over a particular period.

### 2.2.2 The balance sheet

Gives the firms assets (possessions, what it is invested in) and the claims against these assets (liabilities and equity, or from which source the capital was raised). Meant to give insight into the resources at the firms disposal and its financial structure. The combined value of the claims has to be the same as the combined value of the assets, so we have the balance sheet identity:

$$totalassets = equity + liabilities \tag{2.9}$$

## 2.3 Utility and risk aversion

Finance studies people preferences. A person can prefer some good A to B, or Bundle to bundle 1. Using utility, we can define value based on preference. If someone prefers good A to B, we say that the utility of A is higher than B.

$$A \succ B \Longleftrightarrow U(A) \succ U(B) \tag{2.10}$$

In finance, we need to define some laws of utility to analyse markets, and so we make three assumptions:

1. People are greedy: The prefer more of a good to less.
2. Each additional unit of a good gives less utility than its predecessor.
3. Peoples preferences are well-behaved: asymmetric (if $a \succ b$ then not $b \succ a$), transitive (if $a \succ b$ and $b \succ c$ then $a \succ c$)

The last one means we can express preferences in a *utility function*.

### 2.3.1 Indifference curves

Depiction of choices that have the same utility. Constructed by plotting a two-dim utility function with the utility as a function of the two choices.

As we see from the form of the curve, the utility increases away from (0,0) which we would presume. In addition, the rounded shape means utility per step decreases as we move towards the axis. This is due to our assumption that the utility of something decreases strictly for each new one we get.

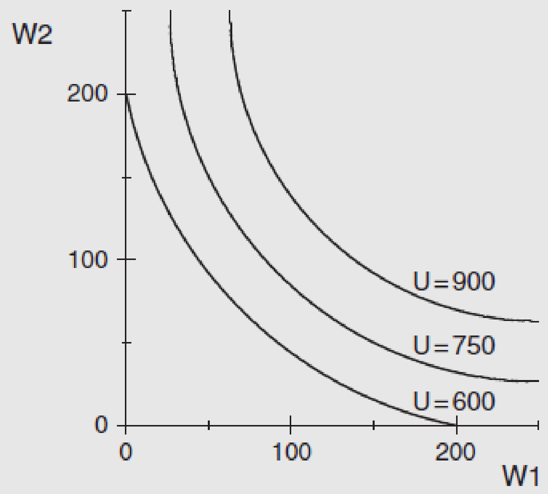



**Figure 2.3** Indifference curves

**Figure 2.1:** Indifference curves

### 2.3.2 Risk aversion

In laymans terms: a safe £1 has a higher value than a risky £1. We can display risk through uncertainty within the utility function. Assume a utility function $U = 5W - 0.01 \times 100^2$ where W is wealth. With a wealth of 100 we would get

$$U(100) = 500 - 0.01 \times 100^2 = 400 \tag{2.11}$$

If we instead add uncertainty, either wealth is 50 or 100, we get

$$\begin{aligned} U(50) &= 250 - 0.01 \times 25^2 = 118,75 \\ U(175) &= 875 - 0.01^2 = 568.75 \\ E[U(W)] &= (118.75 + 568.75)/2 = 343.75 \end{aligned}$$

So the uncertainty decreases our utility. Our risk aversion means that we would rather have 100 wealth safe, then the uncertain choice, even though they have the same expected value.

## 2.4 The role of financial markets

### 2.4.1 Fishers model

The setting of Fishers analysis is very simple. It involves decisions regarding two periods without uncertainty, so that we can represent the problem graphically. Its purpose is to investigate how individuals allocate their budgets to consumption, saving and investments in two periods.

A financial market gives the opportunity to borrow and lend, and so the opportunity to move money in time.

In *prefect markets* there are no transaction costs and all assets are infinitely and costlessly divisible. All investors have access to the same information and can unrestricted borrow and lend at the same rate.

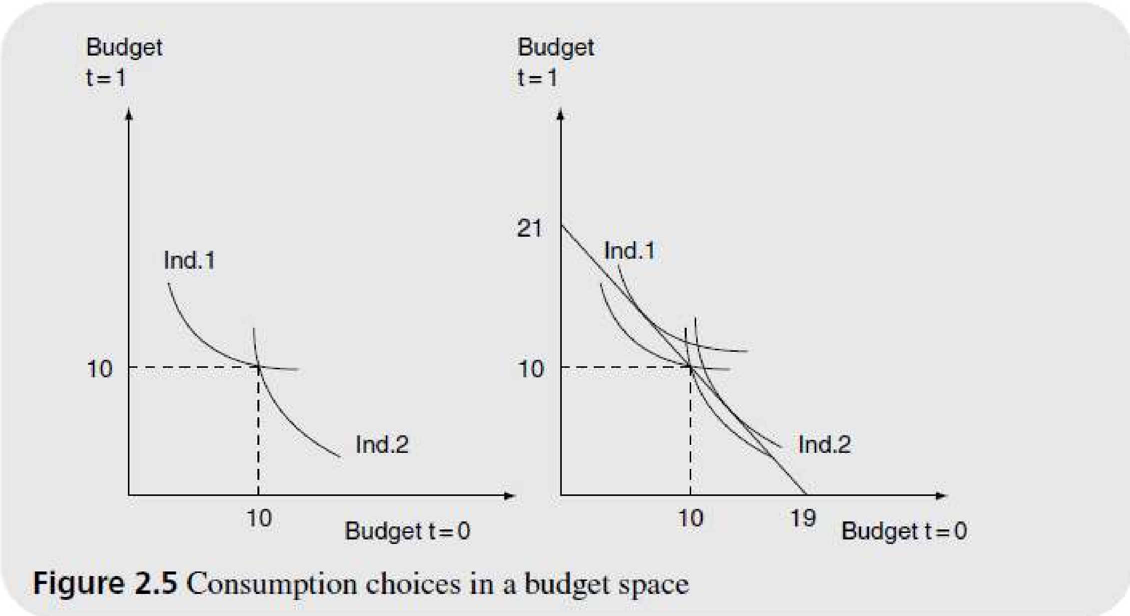



**Figure 2.2:** Difference between market with and without lending and borrowing

The figure above shows how, by simulating a perfect market, it is possible to move money through time. The y axis is one step into the future, usually a year, while the x axis is the current time step. By borrowing we can have 19 dollars now instead of our allotted 10, or by saving we can have 21 dollars next year instead of our 10 now (assuming rent of .1)

### 2.4.2 Productive investment opportunities

We now add the option of investing money.

# 3. Modern Portfolio Theory

## 3.1 What is risk?

Can be characterized in different ways:

- A function of our ignorance (Theory of error): If we were smart enough, risk would disappear.
- As a function of frequency: We know of often an event occurs but not where in the coming sequence.
- As a function of complexity: Length of the shortest formula that computes a sequence. A random sequence most complex, a constant least complex.

Two ways to model future time and uncertain future values

1. Discreetly enumerate all points in time, and outcome variablilites in each time step.
2. Continuously using dynamic process with infinitesimally small time steps.

### 3.1.1 What does risk mean for our choice problems?

The result of most of our chocies cannot be predicated with certainty.

Risk of investment can be depcited in different ways.

- We can look at prices of securities in financial markets.
- We can transform prices plus dividens in returns:

$$r_{it} = \frac{P_{i,t+1} - P_{it} + Div_{t+1}}{P_{it}} \tag{3.1}$$

- We can look at the distributional properties of returns

There are many quantative risk measures, but the ones most often used is *variance* and *root standard deviation*. This measures deviation from mean or expectation. Can be easily calculated.. Disadvantages: upwards and downwards deviations treated equally, ignores higher moments, and sometimes fails.

### Calculating portfolio risk and return

Risk of portfolio often lower than any investment in it. This comes from diversification, and shows up the portfolio variance.

For some set of investments, expected return is the probability weighted sums over scenarios.

$$E[r_i] = \sum_{n=1}^{N} \pi_n r_{ni} \tag{3.2}$$

where assets are indexed with i, scenarios with n and $\pi_n$ is the probability of a given scenario.

Asset variances are the probability weighted sums of squared deviations from expected return.

$$\sigma^2 = \sum_{n=1}^{N} \pi_n (r_{ni} - E[r_i])^2 \tag{3.3}$$

Expected portfolio return is equal parts of the assets in the portfolio.

$$E[r_p] = \sum_{i=1}^{I} x_i E[r_i] \tag{3.4}$$

where $x_i$ are the weighted asset weights (how much of the portfolio is in each assets.

When calculating the portfolio, we see the diversification effect.

- Portfolio variance is not weighted average of asset variance.
- Would ignore correlation effects.
- Combining two assets makes portfolio variance lower than any two assets.

This comes from, when writing it out, that portfolio is the weighted sum of asset variance plus covariances.

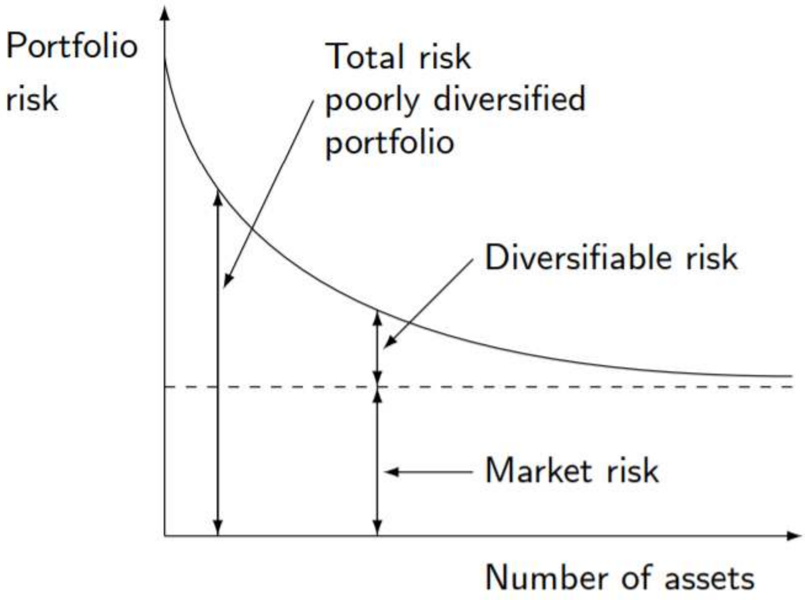
$$\sigma^2 = x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2 + 2 x_1 x_2 \sigma_{1,2} \tag{3.5}$$

Where the final part is the covariance, which measures how assets move togheter through scenarios or over time.

$$\sigma_{ij} = \sum_{n=1}^{N} \pi_n (r_{ni} - E[r_i])(r_{nj} - E[r_j]) \tag{3.6}$$

We can see that this grows exponentially by setting up portfolio variance in a *variance-covariance* matrix.

$$\begin{array}{ccc} x_1^2\sigma_1^2 & x_1x_2\sigma_{1,2} & Asset1 \\ x_1x_2\sigma_{1,2} & x_2^2\sigma_2^2 & Asset2 \\ Asset1 & Asset2 & \sum = \sigma_p^2 \end{array} \tag{3.7}$$




Diversification effect

Risk that disappears through diversification is called diversifiable risk, while risk that remains is called market risk.

This means that risk of an asset is not its variance but its contribution to the portfolio risk taking covariance into account.

We have an formulae for this, the relative contribution of an asset i to the portfolia variance:

$$\frac{contr_1}{\sigma_p^2} = x_i\beta_i \tag{3.8}$$

where

$$\beta_i = \frac{\sigma_{ip}}{\sigma_p^2} \tag{3.9}$$

The risk of an assets is reduced to a single variable $\beta$. It only measures systematic risk, not risk that disappears through diversification.

$\beta$ is also used as a measure of sensitivity. Stocks with $beta > 1$ change more than proportionally with changes in portfolio returns, Stocks with $beta < 1$ change less than proportionally with changes in portfolio.

$\beta$ is an objective measure, and all who calculate it will get the same results for the same data. It also adds linearly unlike variance.

Covariance is often standardized by standard deviations. This is called the correlation coefficient $\rho$, and is limited by -1 and 1.

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \times \sigma_j} \tag{3.10}$$

## 3.2 Capital asset pricing model (CAPM)

*The Capital Market line:*

$$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m}\sigma_p \tag{3.11}$$

The capital market line is only valid for efficient portfolios, ie combinations of risk free asset and market portfolie M, where all risks comes from market portfolio.

This means that individual stocks and inefficient portfolios dont lie on the CML, so we need a different model for that.

CAPM is a more general model. If we consider a two asset portfolie, with stock i/weight $x_i$ and the other asset being the market M. This portfolio is only efficient if $x_i = 0$.

Looking further, the portfolio has

$$\frac{\partial E(r_p)}{\partial x} = E(r_i) - E(r_m) \tag{3.12}$$

Not completed:

$$\frac{\partial \sigma_p}{\partial x} = \frac{1}{2}\left[x^2\sigma_i^2 + (1-x)^2\sigma_m^2 + 2x(1-x)\sigma_{i,m}\right]^{-\frac{1}{2}} \prod [] \tag{3.13}$$

the above uncompleted equation can be simplified to:

$$\frac{\partial \sigma_p}{\partial x} = \frac{x(\sigma_i^2 + \sigma_m^2 - 2\sigma_{i,m}) + \sigma_{i,m} - \sigma_m^2}{\sigma_p} \tag{3.14}$$

At the previously mentioned point M, all funds are invested in the market such that $x = 0$ and $\sigma_p = \sigma_m$. This gives a simplified marginal risk:

$$\frac{\partial \sigma_p}{\partial x} \mid_{x=0} = \frac{\sigma_{i,m} - \sigma_m^2}{\sigma_p} \tag{3.15}$$

We know that at point M, the slope of the CML and of the risk-return tradoff is equal. Using these two assumptions, we can solve for $E[r_i]$ and find the Capital Asset Pricing Model.

$$E(r_i) = r_f + (E(r_m) - r_f)\beta_i \tag{3.16}$$

The grapical representation of CAPM is known as the Security Market Line. The model is a pricing relation for the entire investment universe, inefficient portfolios and individual assets.

For CAPM, $(E(r_m) - r_f$ is the price of risk, while $\beta_i$ is the measure of risk. This formalization of the risk-return relationship allows well-diversified investors to value their assets according to its contribution to portfolio risk:

If asset i increases portfolie risk we have $E(r_i) > E(r_p)$ and on the other hand, if asset i decreases risk we have $E(r_i) < E(r_p)$.

The SML offers 4 important insights that are missing in the CML:

1. **Systematic and unsystematic risk:**

   If we compare the SML (CAPM line) with the CML, we get the following (With $\beta_p$ extracted from previous equation using definiton):

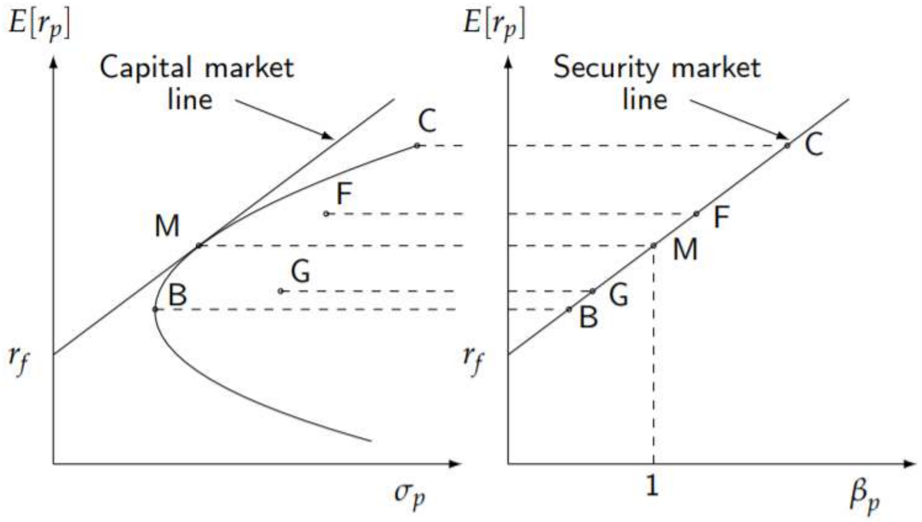
   CML:

   $$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m}\sigma_p \tag{3.17}$$

   SML:

   $$E(r_p) = r_f + \frac{E(r_m) - r_f}{\sigma_m}\sigma_p\rho_{p,m} \tag{3.18}$$

   We see that the difference is SML prices the systematic risk, and is therefore valid for every investment object, while CML prices all risk and is so only valid when all risk is systematic risk (efficient portfolio).




Systematic and unsystematic risk

2. **CAPM and discount rates:** Recall general valuation formula for investments:

$$Value = \sum_t \frac{Exp[Cashflows_t]}{(1 + discountrate_t)^t} \tag{3.19}$$

We remember that uncertainty can be accounted for in three ways: Adjust discount rate to risk adjusted discount rate, adjust cash flow to certainty equivalent risk flow, adjust probabilities from normal to risk neutral.

We can easily use CAPM as risk adjusted discount rate.

$$E(r_p) = r_f + (E(r_m) - r_f)\beta_p \tag{3.20}$$

But as we know return is also

$$E(r_p) = \frac{E(V_{p,T}) - V_{p,0}}{V_{p,0}} \tag{3.21}$$

The discount rate we are looking for is a mapping from end-of period value $E(V_{p,T})$ to value now $V_{p,0}$. Setting the two equations above equal to each other allows us to calculate value now:

$$V_{p,0} = \frac{E(V_{p,T})}{1 + r_f + (E(r_m) - r_f)\beta_p} \tag{3.22}$$

In the last expression we calculate value now with $r_f$ as the time value of money (how moneys value changes over time due to the risk free rent) and $(E(r_m) - r_f)\beta_p$ is the adjustment for risk. So we now have the risk adjusted discount rate.

3. **Certainty equivalent formulation:**

   The second way to account for risk. Adjust for uncertain cash flow to certainty equivalent. Can and should be discounted with risk free rate.

   Again using the same two equations as in last point, and omitting the calculations, gives us

$$V_{p,0} = \frac{E(V_{p,T}) - \lambda cov(V_{p,T,r_m})}{1 + r_f} \tag{3.23}$$

   This is the certainty equivalent formulation of CAMP. Uncertain end-of-period value is adjusted by

$$= \frac{E(r_m) - r_f}{\sigma_m^2} \tag{3.24}$$

   and is then multiplied by the volumn of risk, in this case the covariance of end-of-period value and return on market portfolio.

   The resulting certainty equivalent value is discounted at the risk free rate to find present value.

4. **Performance measures:**

   CML and SML relate expacted return to risk. This can be reformulated as an ex post (after the fact, based on the actual response) performance measure that relates realised return to observed risks.

   Sharpe did this based on CML:

$$\frac{E(r_p) - r_f}{\sigma_p} = \frac{E(r_m) - r_f}{\sigma_m} \tag{3.25}$$

   The left hand side of this is called the return-to-variability or Sharpe-ratio.

   If we translate to ex post formulation we get:

$$\text{Sharpe ratio:} SR_P = \frac{\bar{r_p} - \bar{r_f}}{\hat{\sigma_p}} \tag{3.26}$$

$\bar{r_p} = \sum_t r_{pt}/T$ is the portfolios historical mean return, $\bar{r_f}$ is the average risk free interest rate, and $\hat{\sigma_p} = \sqrt{\sum_t (r_{pt} - \bar{r_p})^2/T}$ is the standard deviation of the portfolio returns. $T$ is number of periods.

The Sharp ratio is widely used to evaluate portfolios.

The **Treyner Ratio** is similar but instead of $\hat{\sigma_p}$ is uses $\hat{\beta}_p$, meaning it is based on SML.

### 3.2.1 Assumptions for CAPM

CAPM assumes financial markets are perfect (no limitations on short selling and burrowing/lending at same rate) and competetive (Nobody is large enough to influence prices).

It also assumes that investors maximize expected utility based on mean-variance charecteristics of investments over a single holding period and that investors have homogenous expectations of returns.

## 3.3 Arbitrage

A strategy to profit from mispricing in the market. Either costs nothing today and gives payoff later, or payoff today and no obligations later.

### 3.3.1 Arbitrage Pricing theory

Does not assume investors maximize utility based on stocks mean-variance characteristics. Instead assumes stock returns are generated by a multi-factor process. More general than CAPM, give room for more than one risk factor. To introduce it we must first look at single-index models.

### 3.3.2 Single-Index Model

Instead of looking at variance-covariance matrices, which give very large and complicated calculation, we assume that there are only one reason that stocks vary: they response to the market as a whole. Stocks still respond in different ways, measured by $\beta$, but do not respond to unsystematic changes in other stocks values.

Can be formalized as:

$$r_i = \alpha_i + \beta_i r_m + \beta_i \tag{3.27}$$

where $r_i, r_m$ is return of the stock and the market, $\alpha$ is expected value non-marked related return, $\epsilon$ is random element of non-marked related return ($E(\epsilon) = 0$, variance $= \sigma_\epsilon^2$ and $\beta$ is sensitivity to marked.

The single-index model makes two assumptions:

- $cov(r_m, \epsilon_i) = 0$, meaning random element of non-marked return not correlated with market return.
- $cov(e_i, e_j) = 0$ for all $i \neq j$, meaning random elements of non-marked related returns are not correlated.

Means that the variance and covariance of a stocks is

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 \sigma_{\epsilon i}^2 \qquad \sigma_{i,j} = \beta_i \beta_j \sigma_m^2 \tag{3.28}$$

This simplifies analysis of larger portfolios drastically: $3I + 2$ instead of $I + \frac{1}{2}I(I-1)$ for full var-covar.

For 100 stocks we get 302 calculations instead of covars 5050 calculations.

We can extend this with more factors, accounting for different industries and general economic factors.

The expression for stock return then becomes:

$$r_i = \alpha_i + b_{1i}F_1 + b_{2i}F_2 + ... + \epsilon_i \tag{3.29}$$

Where $b_{1i}$ is sensitivity of stock i for changes in factor $F_1$, and $F_1$ is return on factor 1.

This multi-factor model assumes:

- $cov(F_m, F_k) = 0$, meaning all factors are unrelated
- $cov(F_k, \epsilon_j) = 0$: residuals uncorrelated.
- Residuals of different stocks uncorrelated: $cov(\epsilon_i, \epsilon_j) = 0$ for all $i \neq j$

### 3.3.3 Arbitrage Pricing Theory

Build on a multi-factor model, and disinguishes between the expected part of stock returns and the unexpected part.

Unexpected part (risk) consists of systematic (market) risk and unsystematic risk.

From the multifactor model we can calculate return:

$$r_i = E(r_i) + \sim_{k=1}^{K} (F_K - E(F_k)) + \epsilon_i \tag{3.30}$$

Where the terms after $E(r_i)$ are the error parts of process. They describe deviation from expected return, We can expeand this for portfolios, using

$$r_p = \sum_{i=1}^{I} x_i r_i \tag{3.31}$$

APTs equilibrium condition is that, when you make a well diversified portfolie that requires no net investment and involves no risk, the expected return must be zero.

# 4. Market efficiency

## 4.1 The efficiency concept

Finance hold the view: Randomly changing prices is the hallmark of properly functioning markets.

On such markets:

- Investors use all available information to determine prices.
- Price discovery process aggregates all that information
- This means prices only react to new information
- But new information is random both in timing and nature

This means that markets *must change randomly* if markets function properly. This gives us the following definition:

*A market in which prices always "fully reflect" available information is called "efficient"*

Market efficiency is a simple concept, but has substantial consequences. If all available information is already included in the price, then it cannot be used to make predictive financial analysis, develop investment strategies that systematically earn more than fair, risk adjusted return,

What then makes project value and return fluctuate? It is the information not included in calculations, i.e. the unpredictable or unknown information.

We formalize the concept "fully reflect" as the difference between realized and expected return given information set:

$$\epsilon_{i,t+1} = r_{i,t+1} - E[r_{i,t+1}|\mathbf{\Phi}_t] \tag{4.1}$$

Where $\epsilon$ is excess return, $r$ is return and $\Phi$ is the information set.

We can model "fully reflect" in three ways:

1. **Fair game model:**

   In a fair game, payoffs are equal to chance odds. Outcomes occur according to probability theory. Deviations from expected values become zero in the long run:

   $$E[\epsilon_{i,t+1}|\mathbf{\Phi}_t] = 0 \tag{4.2}$$

   which means $\Phi$ cannot be used to systematically earn excess return.

2. **Martingale model:**

   The expected future price, properly discounted, is equal to the current price.

   Formalized, this means:

$$\epsilon_t > 0 \tag{4.3}$$
$$E(\epsilon_{t+1}|\mathbf{\Phi_t}) = 0 \tag{4.4}$$

   and next periods price becomes

$$E[P_{i,t+1}|\mathbf{\Phi_t}] = P_{i,t}(1 + E[r_{i,t+1}|\mathbf{\Phi_t}])\text{or} \tag{4.5}$$
$$E[P_{i,t+1}|\mathbf{\Phi_t}]/(1 + E[r_{i,t+1}|\mathbf{\Phi_t}]) = P_{i,t} \tag{4.6}$$

3. **Random walk model:**

   Fair game and Martingale models only expectation. Random walk model considers entire distribution. Excess return follows random walk if they are independently and identically distributed (iid).

   The random walk model has the markov property of memorylessness.

   Applied to market efficiency: Expected returns are constant (drift), and excess returns zero expected.

We have three contents of the information set:

- Weak form market efficiency: All past prices are fully reflected in current prices.
- Semi-strong form of market efficiency: Prices fully reflect all publicly available information. Implies also weak form.
- Strong form efficiency: All information reflected in current prices, including private and inside information. Implies also semi-strong form

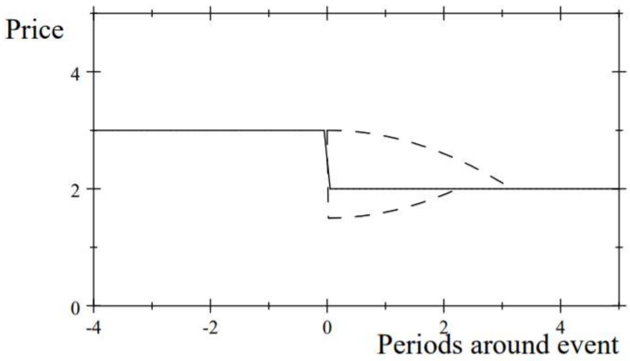
If the markets are efficient, returns cannot be systematically increased without increasing risk. This means we cannot systematically earn positive excess returns. Does not mean that one cannot get lucky.

Market efficieny has 4 implications:

1. **No autocorrelation in excess returns:**

   Autocorrelation is correlation with itself N periods ago.

   EMH implies this is zero.




Efficient (solid) and inefficient (dashed) price adjustments

2. **Investment strategies give no positive excess return.**
3. **Investment fund and investors do not systematically differ in excess returns.**
4. **Prices adjust to new information in an new way:**

   No predicatable pattern after news become known. No underreaction (News slowly incorperated over several periods) or overreactions (First reaction too strong, corrected in later periods).

### 4.1.1 Empirical tests

Use a less strict version of EMH: "information set cannot be used to make excess returns, adjusted for risk and net of all costs".

Market efficiency in strictest sense is impossible, requires zero trading- and information costs. i.e. costs that are necessary to make prices reflect all information.

Summary of subsection: **Autocorrelation, investement strategies, does not reject EMH**

### 4.1.2 Event studies

Test speed and precision of reaction to news. To do this, we need to seperate news and the event from the rest. This is called event studies.

The **Market model** most commonly used in event studies: Calculate normal return given market developments, then difference with realized return is abnormal return. The abnormal return is attributed to the event.

Market efficieny tested by pattern of abnormal returns: In an effecient market, this only occurs in the event period, and has no predictable pattern afterwards.

**The Market model** uses simple, empirical regression relation:

$$r_{it} = \hat{\gamma_{0i}} + \hat{\gamma_{1i}} r_{mt} + \epsilon_{it} \tag{4.7}$$

where $r_{it}$ is the return of individual security i, $r_{mt}$ is the return on market portfolie (broad index), $\gamma_{0,1}$ are estimated coefficients, and $\epsilon_{it}$ is the error term.

The market model has no theoretical background, no assumptions on var-covar, and is estimated in estimatsion window before the event is studied (event window).

The estimated normal return are the market models out-of-sample predictions, given return on market.

$$E(r_{it}) = \hat{r_{it}} = \hat{\gamma_{0i}} + \hat{\gamma_{1i}} r_{mt} \tag{4.8}$$

Where the coefficients are estimated over prior window and $r_{mt}$ is observed in event window.

This allows us to calculate abnormal return $ar_{it}$:

$$ar_{it} = r_{it} - E(r_{it}) = \epsilon_{it} \tag{4.9}$$

Conclusions are then made based on abnormal return over prediction period, giving us cumulative abnormal return:

$$car_i = \sum_t ar_{it} \tag{4.10}$$

# 5. Capital Structure and Dividends

## 5.1 Dimensions of securities

The general dimensions of securities:

- Riskiness: Profit dependant or predetermined return. Counterparty or default risk.
- Maturity: short - long - permanent.
- Property rights attached: Deposited or promised - financing or guarantee function.
- Spot (direct delivery) or future delivery.
- Secured or unsecured.
- Underlying or derivative.

Some examples:

Common stocks (shares): Permanent investment, profit dependent return, ownership. Low priority, no upper limit. Usually deposited, can be stock market traded. Unsecured, can be underlying for derivatives.

Bonds and bank loans: Temporary investment, predetermined return, no property rights. High priority, fixed maximum return. Deposited, bonds can be stock market traded. Can be secured, bonds can be underlying for derivatives.

The wide variety of shares:

- Common shares.
- Shares with limited voting rights.
- Preferred shares: priority claim on profits, priority voting rights for some decisions.
- Repayable shares.
- Convertible shares: from preferred to common, from certificate to common.

Many different bonds:

- Ordinary bonds: corporate or government.
- Income bonds: Only pay interest if profits allow it.
- Index bods: Interest dependent on e.g. government bonds.

- With or without regular interest payments: Ordinary coupon bonds (x% paid every year), zero-coupon bonds (1 final payment of interst + repayment).
- Junk bonds: high interest rate, high default risk.
- Convertible bond: Can be converted into share.
- Catastrophe bond: Reduced payment if specific catastrophe happens.

Debt can be insured with:

- Priority claims on certain assets: Mortage, inventories, account recivables, assets outside of the firm.
- 'Me first' rules, seniority i.e. priority.
- Ratio clauses (e.g. current ratio ¿ 2)
- Action clauses (Not allowed to sell assets)

Variety of derivatives is even larger. The value of a derivative is dependent on the value of another security. Underlying can be almost anything, usually stock, bond, currency, commodity, etc.

- Options: puts and calls, europen or american, plain vanilla or exotic.
- Forwards and futures: An agreement to buy an sell assets at a future time. A forward contract is a private and customizable agreement that settles at the end of the agreement and is traded over-the-counter. A futures contract has standardized terms and is traded on an exchange, where prices are settled on a daily basis until the end of the contract.
- Swaps: two parties exchange the cash flows or liabilities from two different financial instruments.

### 5.1.1 Financial engineering

Constructing new securities from existing ones. Destribute risk and return in other ways, make securities available for investors.

Illustrate process with mortgages and mortgage-backed securities, has played a major part in finance. Simplest situation: Bank gives mortgage loan to customer. Bank collects interest and repayments. Bank bears the risk of the loan (prepayment risk or default risk).

Mortgages can be bundled in a special firm (legal entities) called 'special purpose vehicle' (SPV).

- Bank sells its mortgage to SPV, receives a fee.
- SPV collects interest and repayment.
- SPV bears risk of the loan.
- SPV issues bonds to finance its operation.
- Parent company SPV (usually investment bank) creates secondary market for bonds so that bondholders can buy and sell. Makes bonds more attractive.

Construction is an example of *securitization* (turning internal debt into publicly traded debt):

- Mortgages that were privatly held by banks become available to investors.
- Bank gets money to issue new mortgages.
- Construction pools risk of mortgages: Safety in large numbers, risk not otherwise transformed.

Risk can be transformed by structuring bonds: Divide bonds in tranches (parts) with different risk and return called structured products.

This type of construction used to be very common before financial crisis of 2008. However, construction creates a large distance between mortgage issuer and investor who bears default risk. This requires additional risk managment tools, and not enough were put in place. Instead old averages and parameters were used, from a time when mortgages were as safe as a house.

With securitezed mortgages, banks get a fee for selling mortgage, and risk is borne by investors who buys bond of SPV. This gave the banks an incentive to lower standars, and this effect was missed by the rating agencies that assess the quality of bonds.

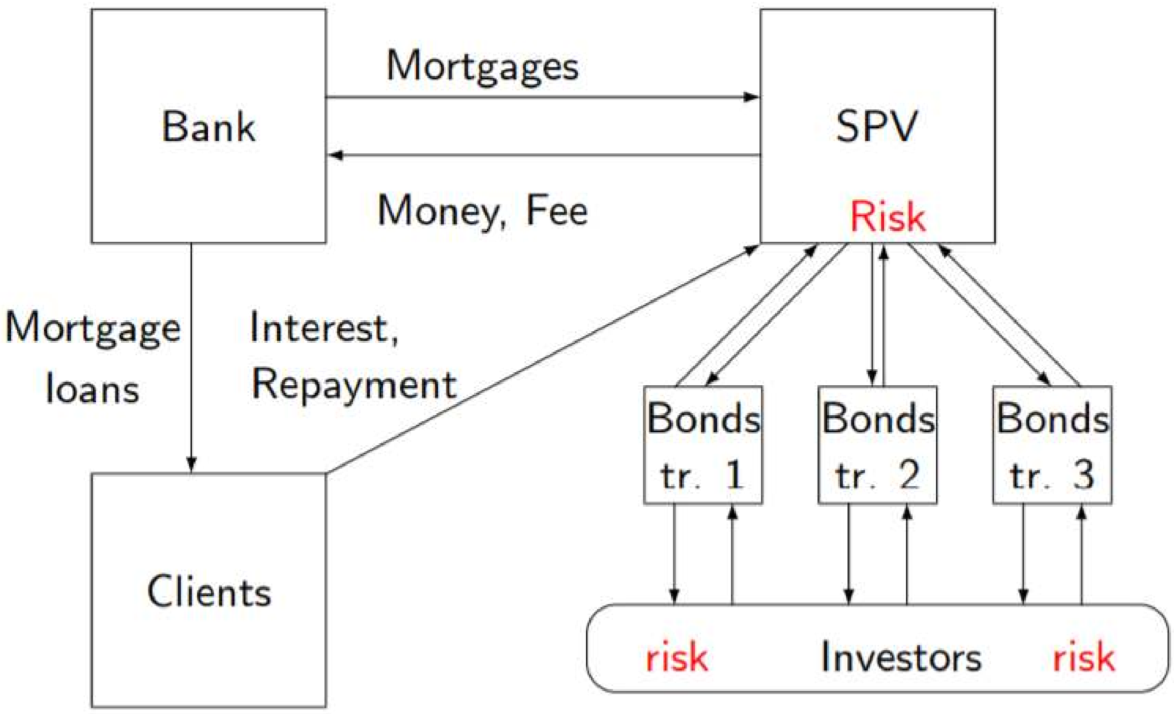



Figure: Structured products 3: bond structured in tranches

## 5.2 Capital Structure

Modigliani-Miller: The cost of capital, corporation finance and the theory of investment (1958).

Definition: **Capital structure** is the combination of capital categories a firm uses to finance its operations.

The study of capital structure is important: Connection with the cost of capital, hence capital budgeting. It also tells us which projects can carry much debt, i.e. are easy to finance. Its a major factor in the evaluation of firms.

### 5.2.1 Perfect Capital Markets

We start our analysis with *perfect capital markets*: No taxes, no transaction costs, limits on short selling. Everybody can borrow and lend unlimited amounts at the same rate.

Methodology: compare 2 firms with identical assets, giving same profits, that only differ in capital structure (one levered, one unlevered firm). We analyse two things for these firms, the value of the firm and the risk-return equity.

This gives us the Modigliani-Miller Proposition 1: The value of the levered firm is the same as the unlevered firm. This proposition is based on the arbitrage argument: In a perfect market all investors can unde and redo all capital structure decisions free of charge. This means any price difference will be arbitraged away.

Conclusion: In a perfect capital market, levered firms cannot sell at a premium or a discount, as this would create the possibility for home made levering/unlevering. This means mangagers cannot change the value of a firm by changing capital structure. This means capital structure is irrelevant.

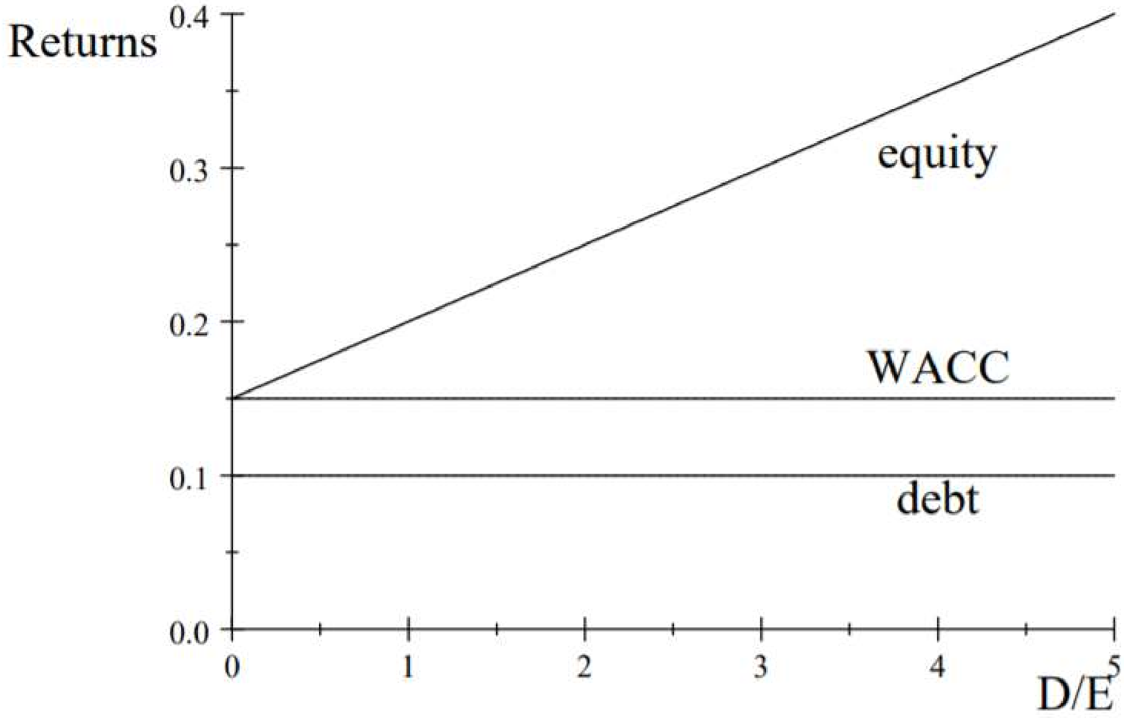
### 5.2.2 Risk, Return and Leverage

Can be shown that leverage makes equity riskier. This is done using the formula for WACC (Weigthed Average Cost of Capital):

$$WACC = (D/V)r_d + (E/V)r_a \tag{5.1}$$

where $D$ is debt, $V$ is total volum (debt + equity), $r_d$ is interest rate on debt, $E$ is equity and $r_a$ is the return of the equity/assets.

This gives us Modigliani-Miller Proposition 2: The return of equity is equal to the return of assets on $r_a$ plus a premium related to financial risk equal to the debt-to-equity ratio times the spread between $r_a$ and $r_d$.




Returns and D/E ratios

This results in the cost of equity increasing with leverage such that the weighted cost of capital is constant. As the proportion of cheap debt increases, the required rate of equity also increases, and WACC remains constant.

### 5.2.3 Market imperfections

Can make Capital structure relevant.

- Taxes: interest rate deductible, dividens not deductible.
- Limited liability / Default risk.
- Cost of financial destress: if firm gets into truble, costly measures are necessary.
- Agency costs.
- Information asymmetry: Managers know more than outside investors.

### After tax cost of Capital (After tax WACC)

With taxes there is a third party, tax collector, that claims part of corporate value. This reduces the value of the firm. If interest rate is tax deductible while dividens are not, the tax collectors part decreases with debt. This resutls in the sum of levered equity + debt ¿ unlevered equity.

This can be generlized:

- Call firms cash flow (EBIT) $x$.
- Corporate tax rate $\tau$.
- The value of an unlevered firm is then:

$$V_u = \frac{(1 - \tau x)}{r_a} \tag{5.2}$$

- The cash flow to investors in the levered firm consits of 2 parts:

  To shareholders: $(1 - \tau)(x - r_d D)$

  To debtholders: $r_d D$

  The first part should be discounted with $r_{e,l}$, the second with $r_d$.

- This gives us the cash flow to unlevered equity plus the tax advantage of debt:

$$\begin{aligned} V_l &= \frac{(1 - \tau x)}{r_a} + \frac{\tau r_d D}{r_d} && (5.3) \\ &= V_u + \tau D && (5.4) \end{aligned}$$

  This is the Modigliani-Miller Proposition 1 with taxes.

- Modligiani-Miller Proposition 1 with taxes: Value of the levered firm is the value of the unlevered firm plus the tax advantage of debt.
- Modligiani-Miller Proposition 2 with taxes:

$$r_e = r_a + (1 - \tau)(r_a - r_d)\frac{D}{E} \tag{5.5}$$

  With taxes as only market imperfection 100% debt financing becomes the optimal solution.

#### Effect of default risk

In MM analyses debt is free of default risk. This is only true when equity holders have unlimited liability and enough money to always pay debt, regardless of how low cash flow is.

Relaxing either of these assumptions introduces the risk of default.

With limited liability:

- Debt and interest can only be repaid from firm uncertain cash flows.
- Default risk is the probability that cash flow ¡ debt obligations.
- Default occurs if firm cannot pay its interest bills.

This means default risk increases with $D/E$ ratio: Larger debt means larger risk.

Default risk also increases with risk variablility: Larger variablilty increases the chance that a firms cash flow is to small to repay debt and interst.

The result of increasing default risk is that debtholders bear more of the buisness risk, $r_d$ goes up, $r_e$ goes down,

The introduction of default risk does not change MM conclusions. Cash flows and debt payments become uncertain, but WACC still decreases with leverage and the value of a firm still increases with leverage.

Instead we must account for the extra cost of financing such as cost of finacial distress. This, combined with taxes and default risk, gives *trade-off theory of optimal capital structure*

### 5.2.4 Trade-Off theory: Capital structures with tax, default risk and costs of finacial distress

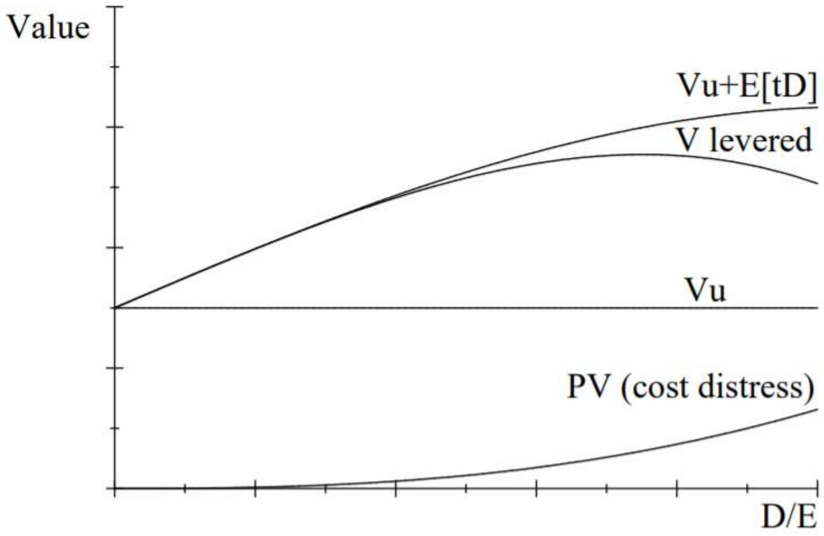
If a firm cannot pay its bills, costly emergancy measures are necessary. Collectivly called cost of financial distress: Customers and key personel run away, supplier dont deliver or require cash payments, costly refinancing may be necessary, expensive advisors have to be hired.

Costs of financial distress are ”fourth-party” claiming part of corporate value.

Expected distress (”fourth part”) increases with leverage: incurring distress becomes more likely as debt increases. Cost themselves may also increas with size of debt.

Combined with tax advantage, this becomes trade-off theory of capital structures:

- Debt will be increased as long as marginal tax advantage outweighs marginal expected cost of financial distress.




Optimal capital structure

- Optimal capital structure reached as both are equal.
- graphical represetnation:

As we can see from the graph, the expected tax advantage decreases with leverage even though the interest rate is increasing. This comes from the increasing probability of loss: With increasing leverage, it becomes increasingly likely that the debt obligations are larger than the cash flow. This means there will be no profit, hence no tax, hence no advantage. So with increasing tax leverage it becomes increasingly likely that the tax advantage will not be realised.

### 5.2.5 Information asymmetry

Models other market imperfactions than trade-off theory.

Asymmetric information: Managers know more about firms prospect than outside investors. This information may be costly or impossible to recreate.

Conflicts of interest: Managers serve interest of existing shareholders. They use their information to issue the type of security that benefits (transfers wealth to) existing shareholders. (Debt when good news is coming up, equity when bad news in coming up.)

Examples of wealth transfer:

- All equity company, valued at £400 million, 40 million outstanding shares at £10 per share.

- Comes across lucrative project (aquistion): Requires immediate investment of £40 million. Will generate safe cash flows, £60 million after 1 year.
- Risk free interest rate is 7%:

  PV cash flow is $60/1.07 = 56$

  NPV cash flow is $56 - 40 = 16$
- Project is good news so value will increase. Precise effect depends on timing of the news and the projects financing.

Internal financing:

- Project financed with internal funds, cash from retained earning: No new investors, so no wealth transfer. This means timing of announcement not relevant.
- Since NPV is £16 million, stock value will increase to $416/40 = \$10.40$

External equity financing, no information symmetry:

- No information symmetry, news can be shared freely: Stock price increases immidiatly to £10.40, and subsequently new stocks are issued $40/10.40 = 3.8642$ million new shares.
- After emission:

  Old shareholders own $(40/43.8462) \times 456 = 416$. Whole NPV accrues to them.

  New shareholders own $(3.8462/43.8462) \times 456 = 40$ , and get what they pay for.
- No information symmetry implies no wealth transfer. Managers and shareholder are therefore indifferent between internal/external financing.

External equity financing with information asymmetry:

- Good news cannot be shared equally, for example to avoid bidding war. This implies equity is undervalued and the firm must issue equity anyway (if they have no cash). Issues $40/10 = 4$ million new shares.
- After emmision, the deal is done, and value increases to £456 million.

  Old shareholders now own $(40/44) \times 456 = \$414.55$. The NPV only partially accrues to them.

  New shareholder now own $(4/44) \times 456 = \$41.45$ million, they get more than they paid for.

- Wealth is therefore transfered from old to new shareholders. This would make managers reluctant to issue equity when good news is expected.

Another example but with safe external debt:

- Project is risk free, borrow against 7% ($r_f$).
- After one year, need to pay back $40 \times 1.07 = 42.8$.
- Shareholders own $400 + (60 - 42.8)/1.07 = 416$.
- Whole NPV accrues to shareholders.

We can summarize the effect of information asymmetry:

- It gives a pecking order in financing alternatives:

  When good news are expected, internal financing is preferred to external financing, and debt is preferred to equity.

  Similar ranking when bad news is expected.
- Gives adverse selection effect: Issue debt when good news are coming up, and issue equity when bad news are coming up.

This behavior is recognized and anticipated in the market:

- Stock exchanged for debt (more stock) $\rightarrow stockpricefalls.Debtexchangeforstock(lessstock) \rightarrow stockpricesrises.$

- Stock issued $\rightarrow Stockpricefalls.Stockrepurchased \rightarrow Stockpricerises.$
- Large debt issues do not lead to significantly lower stock prices.
- Result: If new equity is issued, new stockholders require a discount. Basically, nobody wants to belong to a club that will accept them as member.

This means managers who need funds face a dilemma: Either they must issue equity at a price they know is to low, or forgo positivt NPV investment oppertunites.

Debt is in principle troubled by same problems, but to much lesser extent. Debt has a higher priority claim, can be secured, and can have shorter maturities.

This situation can be avoided if manager build up financial slack, meaning they build up cash, marketable securities or reserve burrowing power.

The pecking order also has other implications: Firms adapt dividens to their investment needs, and level out fluctiations through financial slack. This means there is no optimal debt theory, it instead depends on the financing needs. Financial slack also becomes a valuable commodity.

### 5.2.6 Empirical evidence

Trade off and pecking order theory share two predicitons: Leverage decreases with earnings volatility, and leverage decreases with growth potential.

They also predict opposite effects of profitability: Trade-off says leverage increases with profitability, while pecking order says leverage decreases with profitability.

In general, trade-off theory is supported except for profitability, where pecking order seems to be right.

## 5.3 Dividens

Dividens (payout to stock owners) are paid in various forms: In cash as ordinary dividend or special after e.g. sale of division. As a stock repurchase, or in stock (stock dividend).

Stock dividends are similar to stock splits, except on smaller scale. It does not produce value, but instead divides value over more shares.

Dividends are announced before they are paid. They are called "cum dividend" untill a preset date, and "ex dividend" after that.

**Value of dividends**

Before Modligiani-Miller, common opinions was that dividends increase shareholder wealth. Modligiani-Miller proposes that both dividens and capital structure are irrelevant in a perfect capital market.

This is due to value not being affected by how we divide it. More concretly, we say that in a perfect capital market investors can undo or redo managments dividend and capital structure decisions free of charge.

It is now accepted as a correct analysis, but not an accurate description of the situation.

**Market imperfections** can make dividend policy relevant:

- Taxes: Different tax ratio on dividends and capital gains.
- Transaction costs: Paying dividens can be cheaper than selling shares. This creates clientele effets, but does not necessarily change value if there is a large supply of dividend paying and non dividend paying stocks.
- Information asymmtery/signaling: May be amibious, no dividends can either mean that the company needs every penny or that is has no money. Dividend increase is generally considered a good sign, decrease is a bad sign.
- Agency cost: Internal financing slips control of market.

# 6. Valuing levered projects

## 6.1 Finance - Investment interactions

Structure of decision problem for investments:

1. Accept project if NPV $> 0$
2. To calculate NPV we need cash flows and discount rates.
3. Discount rates depends on:

   Business risk, or equivalently the OCC (opportunity cost of capital. Calculated from existing operations if risk is the same. Otherwise we estimate from other companies.

   Financial risk, division over debt and equity. Depends on dept ratio and financing rule

Basic elements of this was introduced in derivation of MM (Modigliani-Miller) proposition 2 with taxes: Cost of debt and equity increases with leverage. Taxes influence the cost of capital.

Some important concepts:

- Business risk: Uncertainty of cash flow generated by firms assets.
- OCC - Opportunity costs of capital: Reward for bearing business risk is what shareholders expect if all equity is financed. Must set prices such that expected return equals OCC.
- Financial risk: If cash flow splits in low-risk and high-risk return parts, debtholders have priority over shareholders. Shareholders therefore bear extra financial risk.

### 6.1.1 Discount rates

- $r = r_a = OCC$ expected rate of return for equivalent risk all equity financed firms.
- $r' = WACC$ after tax weighted average cost of capital. WACC calls for unlevered after tax cash flows. WACC is valid for assets with same risk and debt ratio.

$$WACC = r_e \frac{E}{V} + r_d(1-\tau)\frac{D}{V} \tag{6.1}$$

- $r_d$ = cost of debt.
- $r_e$ = cost of equity. Subscript u,l for unlevered.
- $\tau$ = corporate tax rate.

Two different cases:

1. The new project has a different business risk than the parent company:

   First calculate projects OCC / asset $\beta$ based on other companies in the same line of business taking leverage into account.

   Then calculate project WACC and NPV, or Adjusted Present Value (APV) - which is NPV adjusted for side effects.

2. The new project has same business as parent company, but different debt ratio:

   Project's OCC can be calculated from parent company.

   Then calculate WACC AND NPV, or APV.

### 6.1.2 APV - How to calculate

1. First calculate base case value as if the project is entirely equity financed and without side effects.

2. Seperatly calculate value of side effects and sum results.

Side effects can be anything: Tax shields, issue cost, effects on other projects, agency costs, fees to stock exchange.

Looking at side effect of taxes: First calculate the value if equity is financed, then calculate the value of the tax shields.

### 6.1.3 Value of Tax Shields

Depends on the following financing rule:

1. Money amount of debt predetermined, i.e. following a schedule. This means repayments and interest follow a schedule, and the tax shields are tied to interest payements. Implies risk is the same, and the cost of debt is an appropriate discount rate.

2. Debt rebalanced to a constant fraction of future project values: Money amount of debt goes up and down with project value. The same does the tax shields. If the tax shields are tied to the project value, this implies equal risk, and OCC is an appropriate discount rate.

Some examples of working with APV:

A project gives perpetual risky cash flow (EBIT) of 1562.5 per year. The project requires an investment of 8000. The tax rate is 20%, and the risk of assets require a return of 15%. $r = r_a = r_{e,u} = 0.15$.

This gives a value of unlevered cash flow at

$$\frac{(1-0.2)\times 1562.5 = 1250}{0.15} = 8333 \tag{6.2}$$

And we can calculate base case $NPV = 8333 - 8000 = 333$

**Issue Costs** Looking further at Issue Costs (Firm issues equity to finance project), and set them at 7.5%.

We then have to issue $\frac{100}{92.5} \times 8000 = 8649$ to collect 8000.

This gives an issue cost of 649, resulting in $APV = 333 - 649 = -316$.

**Tax shields** We know that the project is optimally financed with 50% debt. Take a perpetual loan of 4000 with an interest rate of 10%, giving a yearly interest rate charge of 400. The tax advantage is then $0.2 \times 400 = 80$.

Since our debt is predetermined, we discount with $r_d$ which means our tax shields are valued at $\frac{80}{0.1} = 800$.

Accounting for issue cost again we get $4000 \times \frac{1000}{92.5} = 4324$, or an issue cost of 324.

$APV = 333 - 324 + 800 = 809$

If we instead rebalance our debt each year to a constant fraction of the future project value, we must discount with OCC instead of $r_d$ .

Må skjønne den shitten her wtf

### 6.1.4 Adjusting the discount rate (WACC)

The structure of the problem is simple: We know what elements are on the balance sheets, and need to use these to express unknown returns as a function of known returns. Since tax shields can have two returns/discount rates, we get a set of functions.

The starting point for the balance sheet:

| | | | | | |
|---|---|---|---|---|---|
| $r_a$ | Value assets | $= V_a$ | debt | $= D$ | $r_d$ |
| $r_d$ or $r_a$ | Value tax shields | PV(TS) | equity | $= E$ | $r_e$ |
| | total value | $= V$ | total value | $= V$ | |

We have again predetermined and rebalanced debt as two options:

**Predetermined:** The risk of tax shield is the risk of debt, we discount tax advantages at $r_d$ and we use MM formula, MM Tax case.

Predetermined tax amount mean that the tax shields are just as risky as the debt itself, and so we discount at $r_d$. Since $r_d < r_a$, larger $PV(TS)/V$ means lower WACC.

We can rewrite the balance sheet in terms of weighted average across costs of capital:

$$r_a V_a + r_d PV(TS) = r_e R + r_d D \tag{6.3}$$

We can rearrange this for $r_a$ and $r_e$

$$r_a = r_e \frac{E}{V_a} + r_d \frac{D - PV(TS)}{V_a} \tag{6.4}$$

$$r_e = r_a + (r_a - r_d) \frac{D - PV(TS)}{E} \tag{6.5}$$

These general equations can be used for projects of limited life. But they have a limited use, as they need the value of Tax Shields. This is the majority part of APV, and so its more practical to continue this calculation. Only exception is when debt is permanent.

If debt is *permanent and predetermined* we can easily calculate the PV of tax shields as:

$$PV(TS) = \frac{\tau(r_d D)}{r_d} = \tau D \tag{6.6}$$

Substituting this in for two equations $r_e$ and $r_a$ above gives:

$$r_e = r_a + (r_a - r_d) \frac{D - \tau D}{E} = r_a + (r_a - r_d)(1 - \tau) \frac{D}{E} \tag{6.7}$$

Which we see is MM proposition 2 with taxes.

and for $r_a$:

$$r_a \frac{V}{V_a} = r_e \frac{E}{V} + r_d (1 - \tau) \frac{D}{V} = WACC = r' \tag{6.8}$$

The WACC formula above can be re-written in two ways:

1. Gives an explicite relationship between $r_a$ and $r'$ (exact for fixed and permanent values):

$$r_a \frac{V_a}{V} = WACC = r' \tag{6.9}$$

$$r_a \frac{V - \tau D}{V} = r' \tag{6.10}$$

$$r_a \left(1 - \tau \frac{D}{V}\right) = r' \tag{6.11}$$

If we define $L = D/V$, the debt-value ratio, we get Modligiani-Miller formula:

$$WACC = r' = r_a(1 - \tau L) \tag{6.12}$$

The MM formula can be used to lever and relever. If we are given the WACC, the formula can be used to calculate the opportunity cost of capital, $r_a$. In most situations $r_e$, $r_d$ and $\tau$ are observable, $r_a$ is not.

2. The second way to rewrite WACC gives alternate expression for $r_a$:

$$r_a = r_d(1 - \tau)\frac{D}{V - \tau D} + r_e \frac{E}{V - \tau D} \tag{6.13}$$

This same analysis can be done for $\beta$.

$$\beta_a = \beta_d(1 - \tau)\frac{D}{V - \tau D} + \beta_e \frac{E}{V - \tau D} \tag{6.14}$$

**Rebalanced:** We now look at the adjusting the discount rate for rebalanced debt. The risk of tax shield is now the risk of assets. We discount tax advantages at $r_a$. To calculate, we use the Miles-Ezzel formula, MM no-tax case,

Since we now only have $r_a$ on the left hand side, $PV(TS)/V$ does not effect WACC. We get an expression equal to MM without taxes.

$$r_a \frac{V_a}{V} + r_A \frac{PV(TS)}{V} = r_a = r = r_d \frac{D}{V} + r_e \frac{E}{V} \tag{6.15}$$

Can rewrite with regards to $r_e$ and $\beta$.

$$r_e = r + (r - r_d)\frac{D}{E} \tag{6.16}$$

$$\beta_e = \beta_a + (\beta_a - \beta_d)\frac{D}{E} \tag{6.17}$$

**Periodical rebalancing**

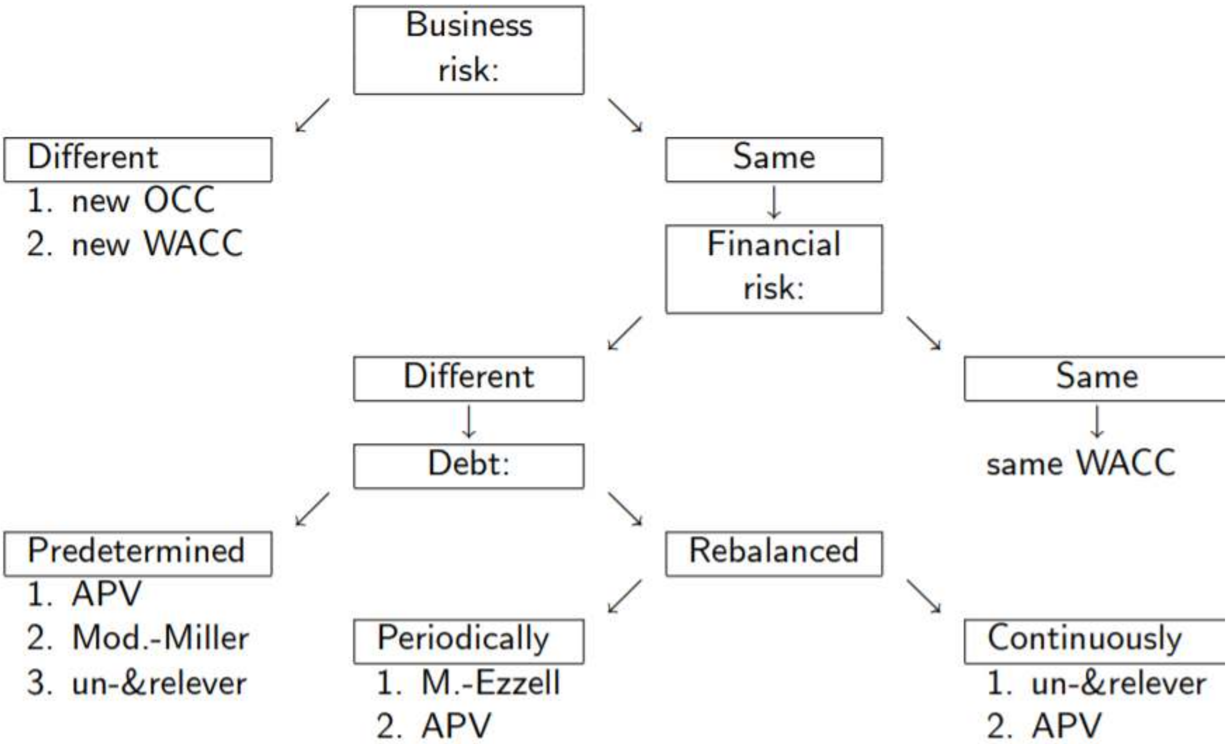
If tax shields are rebalanced once every period we can do some adjustsments to our formula. See book for equations, but it gives us Miles-Ezzell:

$$r' = WACC = r_a - \frac{D}{V} r_d \tau \left( \frac{1 + r_a}{1 + r_d} \right) \tag{6.18}$$

For discount discount rate of discretely re balanced debt. Given WACC we can use it to calculate $r_a$, and given $r_a$ we can use it to calculate WACC for a different debt ratio.

### 6.1.5 Project values with different debt ratios

This requires business risk to remain the same. The method depends on the project characteritics. The main difference is on whether debt is predetermined or varies with the project value. If we have a long life span of the project, we can use perpetuties.




explanation of procedure:

1. We know

Returns: $r_e$ and $r_d$

Relative size: $V_e/V$ and $V_d/V$

Debt and equity in existing operations.

Firms financial policy (rebalanced or predetermined)

The interest rate and $D/E$ for the new project.

2. We can use this to calculate project value

   By adjusting WACC

   Or by using APV

Altough the procedure looks complicated, the rationale is simple:

1. To calculate project value, we need WACC.
2. To calculate WACC, we need cost of equity $r_e$.
3. To calculate cost of equity, we need OCC $r = r_a$
4. OCC can be calculated from existing operations, since business risk is the same.

So we start by calculating OCC.

Three ways to rebalance debt:

1. Stepwise adjust WACC (requires continuous rebalancing):

   Unlever: Calculate oppertunity cost of capital from the existing operations.

$$r = \mathbf{r_d}\frac{\mathbf{D}}{\mathbf{V}} + \mathbf{r_e}\frac{\mathbf{E}}{\mathbf{V}} \tag{6.19}$$

   Use this OCC plus projects cost of debt and debt ratio to calculate cost of equity:

$$r_e = r + (r - r_d)\frac{D}{E} \tag{6.20}$$

   *These first steps can also be done with regard to* $\beta, usingCAPMtocalculatereturns.$

   Relever: Calculate after tax WACC using projects cost and weights.

$$WACC = r_e\frac{E}{V} + r_d(1-\tau)\frac{D}{V} \tag{6.21}$$

2. Adjust WACC using Miles-Ezzel (Requires discrete rebalancing):

   Unlever: Use data from existing operations to calculate OCC by solving Miles-Ezzel for r.

$$\mathbf{r'} = r - \tau \mathbf{r_d} \frac{\mathbf{D}}{\mathbf{V}} \left( \frac{\mathbf{1+r}}{\mathbf{1+r_d}} \right) \tag{6.22}$$

   Relever: Use Miles-Ezzel and OCC plus projects cost of debt and debt ratio to calculate projects WACC:

$$r' = r + \tau r_d \frac{D}{V} \left( \frac{1+r}{1+r_d} \right) \tag{6.23}$$

3. Use Adjusted Present Value (APV):

   Calculate OCC using one of the methods above.

   Discount projects cash flow to find base case NPV.

   Discount tax shield at opportunity cost of capital.

   Multiply PV with $(1+r)/(1+r_d)$ if debt is rebalanced periodically.

Same three ways for debt predetermined:

1. Stepwise adjust WACC (requires continuous rebalancing):

   Unlever: Calculate oppertunity cost of capital from the existing operations $r = r_a$.

$$r_a = \mathbf{r_e} \frac{\mathbf{E}}{\mathbf{V_a}} + \mathbf{r_d} \frac{\mathbf{D - PV(TS)}}{\mathbf{V_a}} \tag{6.24}$$

   or

$$r_a \frac{V_a}{V} = \mathbf{r_e} \frac{\mathbf{E}}{\mathbf{V}} + \mathbf{r_d} \frac{\mathbf{D - PV(TS)}}{\mathbf{V}} \tag{6.25}$$

   This is not very practical, used only under assumption that cash flows are perpetuities.

$$r = r_a = \mathbf{r_d}(1-\tau) \frac{\mathbf{D}}{\mathbf{V-}} + r_e \frac{E}{V - \tau D} \tag{6.26}$$

   Use OCC and projects cost of debts and debt ratio to calculate cost of equity:

$$r_e = r_a + (r_a - r_d) \frac{D - PV(TS)}{E} \tag{6.27}$$

or under MM assumption:

$$r_e = r + (r - r_d)(1 - \tau)\frac{D}{E} \tag{6.28}$$

Previous steps can also be done using $\beta$, with CAPM to calculate returns.

Relever: Calculate after tax WACC using projects cost and weights.

$$WACC = r_e\frac{E}{V} + r_d(1 - \tau)\frac{D}{V} \tag{6.29}$$

2. Adjust WACC using MM formula (Requires MM assumptions):

   Unlever: Use data from existing operations to calculate OCC by solving MM for $r_a$.

$$\mathbf{r}' = r_a(1 - \tau\mathbf{L}) \tag{6.30}$$

   Relever: Use MM and OCC ($r_a$) plus projects debt-to-value ratio to calculate projects WACC:

$$r' = r_A(1 + \tau L) \tag{6.31}$$

3. Use Adjusted Present Value (APV):

   Calculate OCC using one of the methods above.

   Find base case NPV.

   Use predetermined schedule for interest payments.

   Discount tax shields at cost of debt

# 7. Option pricing in dicrete time

## 7.1 Options vs Securities

### 7.1.1 Options

Options are a financial contract that gives their holder the right, but not obligation, to buy or sell something on a future date at a price determined today.

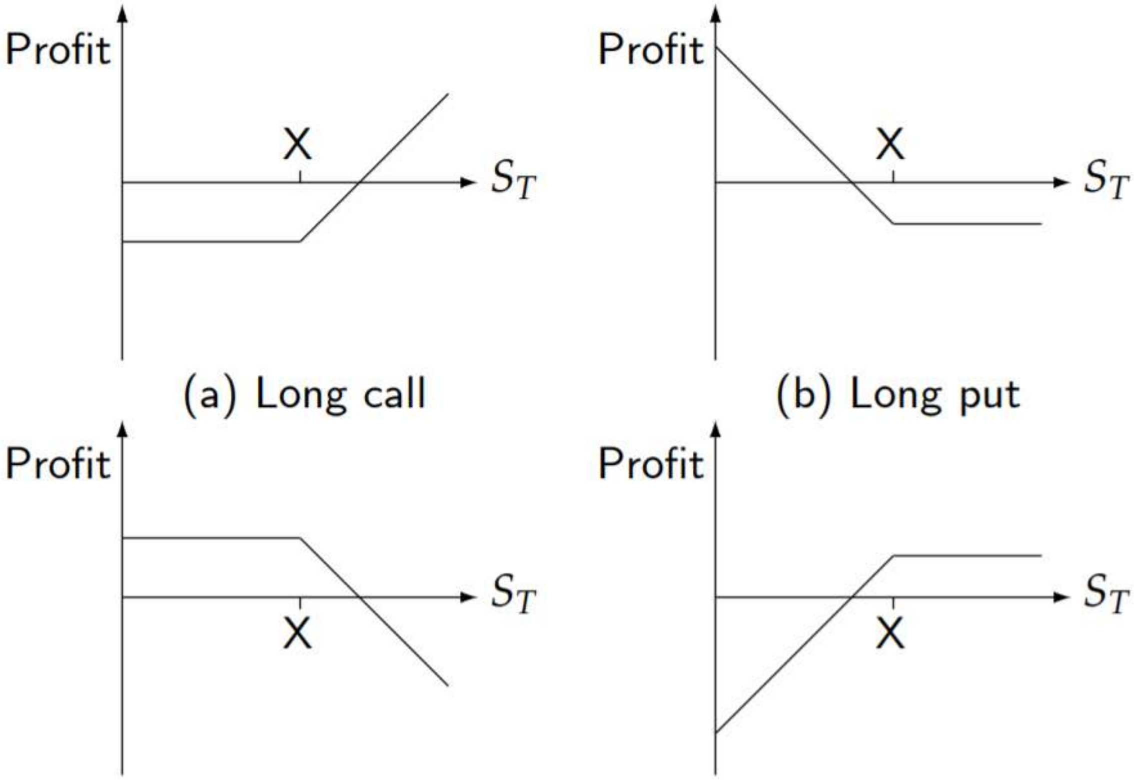
Some terminologi:

- Call option: The right to buy something at a specified price (= exercise or strike price) on a specified date (exercise date or maturity) in europe / before a specified date in USA.
- Put Option: Same just for sell.
- Price of option is called a premium.
- To sell an option is called to write an option.
- Moneyness describes the value of the option if it were exercised immediatly.

Main economic characteristics:

- A long option is a limited liability investment: gives the right but not the obligation to sell.
- Economically options represent flexibility: Possibility to chose best option and walk away from bad outcomes.
- Options are almost always riskier than underlying values.
- Redistribute risk at market prices.
- Options are zero-sum games: You earn or lose money off some else's loses or earnings

Option positions:



(a) Long call (b) Long put

(c) Short call (d) Short put

**Combined option prices:** Stradles are combinations of options that are constructed as bets on volatility.

- Long stradle is a long put + long call with the same strike. This gives profits when there are large price changes
- Short stradle is a short put + short call on the same strike. This profits from small price changes

**Spreads:**

Limited bets on stock price movements.

- A bull spread bets on increasing stock price: Has a long call and a short call with higher strike on the same stock. If the short call is cheaper the initial balance is negative.
- A bear spread bets on a decrease in stock price: Long and short positions are reveresed, the lower strike is sold and the higher is bought.

For spreads the payoff is limited on the up- and downside.

**Bounds on option prices** Simple arbitrage arguments limit the range of option prices. The only assumption made is that investors are greedy, so the bounds are independent from the pricing model. But all pricing models must stay within the bounds to be acceptable. Bounds are formulated using stock options (S, X, T). Some bounds have been proven with arbitrage theory.

- Bound 1: A call cannot be worth more than a stock.

  Intuition - Calls give the right to buy a stock, so cannot be worth more then the stock itself.

- Bound 2: A put option cannot be worth more than the strike price.

  Intuition - A Put gives the right to sell a stock for the strike price. This means it cannot be worth more than the strike price.

- A European put (Matures at specific date) cannot be worth more than the PV of the strike price.

  Intuition - Put at maturity cannot be worth more than its strike. European puts cannot be exercised early, hence its value cannot be higher than PV(strike).

- Bound 4: The minimum value of an European call option, $O_c$, on a stock that pays no dividends is $max[0, S - PV(X)]$. See book for this proof.

- Bound 5: The minimum value of an American call option on a stock is $max[0, S - X]$.

  Intuition - American call can be exercised immediately.

- Bound 6: An American call option is worth at least as much as a comparable European call option.

  Intuition - With an American option you can do everything an european option can, plus exercise early. And the right to exercise early cannot have an negative value.

- Bound 7: American call options that pay no dividend will not be exercised before maturity.

  Intuition - Exercising now means paying now, and that means we give up interest on X while ending up with the same share. It is instead always more valuable to keep the option alive.

## 7.2 Option Pricing Foundations in State-Preference Theory

### 7.2.1 State-Preference Theory

Time modelled as dicrete point in which uncertainty over the previous period is resolved, and new decisions are made. Between time periods nothing happens.

The uncertainty in variables is modelled as a dicrete number of states of the world that can occur at a future time. Each of these states introduce different numerical values for the variables we are considering.

The state of the world can be defined in different ways. Examples:

- General economic cirmumstances, such as recession or expansion.
- Results of a specific action, like drilling for oil. Can either get a large well, a medium-sized well or an large well.

Numerical example, with 1 period, 2 points in time and 3 future states of the world:

$$W = \begin{bmatrix} w_1 \\ w_2 \\ w3 \end{bmatrix} = \begin{bmatrix} bust \\ normal \\ boom \end{bmatrix} \tag{7.1}$$

The states have probabilities of occuring:

$$prob(w_i) = \begin{bmatrix} 0.3 \\ 0.45 \\ 0.25 \end{bmatrix} \tag{7.2}$$

There are three investment opportunities , $Y_1$, $Y_2$, $Y_3$:

$$Y_1(W) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \tag{7.3}$$

Sum up $Y_i$:

$$\Psi = \begin{bmatrix} 4 & 1 & 2 \\ 5 & 7 & 4 \\ 6 & 10 & 16 \end{bmatrix} \tag{7.4}$$

Can use these values to calculate present value of $Y$ by calculating expected value of pay offs, and discounting them at an appropriate rate:

$$prob^T\Psi = \begin{bmatrix} 0.3 \\ 0.45 \\ 0.25 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 5 & 7 & 4 \\ 6 & 10 & 16 \end{bmatrix} = \begin{bmatrix} 4.95 & 5.95 & 6.4 \end{bmatrix} \tag{7.5}$$

With required return on investment as:

$$\begin{bmatrix} 10\% & 13\% & 16\% \end{bmatrix} \tag{7.6}$$

We get our present value returns:

$$v = \begin{bmatrix} 4.5 & 5.25 & 5.5 \end{bmatrix} \tag{7.7}$$

### 7.2.2 Risk free and state securities

On perfect capital markets, investments are costlessly and infinitely divisible, which means that they can be combined in all possible ways to create the payoff pattern we want. An obvious desired pattern is the same payoff in all states of the world, i.e. riskless security.

We can use $\Psi$ from above to show this. Simple force each row to equal one (equal payoffs) and solve for the three different weights. These weights define riskless security, but also allows us to calculate the interest rate.

$$PV(\text{investment}) \times weights = PV(\text{riskless security}) \tag{7.8}$$

If we want to find all securities that are one in one state and zero in all other, we can use matrix algebra:

$$\Psi X = I \tag{7.9}$$

where I is indentity matrix. Solved we get:

$$X = \Psi^{-1} I = \Psi^{-1} \tag{7.10}$$

Which for our example would contain three portfolios, each having as said a return of one in one state and zero in all others. We call these types of securities state securities, or pure securities, or primitive securities, or Arrow-Debrau.

The prices of state securities can be found by multiplying the weight matrix and the present value vector.

### 7.2.3 Market completness

State securities allow construction of any payoff pattern. It simply involves combining the state securities.

The state securities could be constructed because the existing securities span all states. There are noe states without a payoff. **A market where this is the case is said to be complete.**

It is complete because no new securities can be constructed. We can also define it the other way around: If state securities can be constructed for all state, the market has to be complete.

On complete markets, all additional securities are linear combinations of the original ones. All additional securities are called redundant securities.

In the examples so far, both the risk free security and state securities are redundant, they are formed as combinations of existing securities.

A market can only be complete if the number of different (not redundant) securities = number of states. In examples, this means we must have three equations with three unknowns.

### 7.2.4 Arbitrage Free markets

Complete markets imply that any payoff pattern can be constructed, while Arbitrage free markets imply patterns are properly priced. Since proper prices offer no arbitrage opportunities, this absence of arbitrage implies that:

**State prices have to be positive**

Negative state prices would mean that you receive money by buying state securities with negative price, and that it is still possible to get a payoff off one later (if the state occurs). Net negative investment now and non-negative profit later is as we know arbitrage.

The Arbitrage Theorem: A more formal definition of the no arbitrage condition. Given a payoff matrix $\Psi$ there are no arbitrage opportunities if and only if there is a strictly positive state price vector $\psi_{1,2}$ such that the security price vector $[DS]$ satisfies

$$\begin{bmatrix} D & S \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{bmatrix} (1+r_f)D & (1+u)S \\ (1+r_f)D & (1+d)S \end{bmatrix} \tag{7.11}$$

where D is debt, S is a stock, and u/d being future state return if the stock either goes up or down.

This can be solved to give the no arbitrage condition:

$$(1+d) < (1+r_f) < (1+u) \tag{7.12}$$

Or in laymans terms: The risk free interest rate must be between the low and the high stock return.

This is intutively right. If $(1+r_f) < (1+d)$, then we can burrow risk free and invest in stock for a sure profit. Likewise for $(1+u) < (1+r_f)$, we can short sell stock and invest risk free for a sure profit.

### 7.2.5 Pricing with risk neutral probabilites

We extend our analysis into a very important pricing relation. Looking again at Equation 7.11,

$$1 = \psi_1(1+r_f) + \psi_2(1+r_f) \tag{7.13}$$

We can define:

$$p_1 = \psi_1(1+r_f) \quad \text{and} \quad p_2 = \psi_2(1+r_f) \tag{7.14}$$

With this definition, and the no arbitrage condition, $p_{1,2}$ behave as probabilities.

- $0 < p_{1,2} < 1$
- $p_1 + p_2 = 1$

These probabilities are different from real probabilities, and are called risk neutral probabilities or equivalent martingale probabilities.

The risk neutral probalilities are a product of the state price and the time value of money, so they contain the pricing information in this market.

We can use the risk neutral probabilites to obtain the following results (with some calculations):

$$S = \frac{p_1(1+u)S + p_2(1+d)S}{1+r_f} \tag{7.15}$$

This important result says that: **The expected payoff for a risky asset, discounted at a risk free interest rate, gives the true asset value if the expected payoff is calculated with the risk neutral probabilites.**

Looking further at this result - In risk neutral valuation (also called arbitrage pricing):

- We dont adjust discount rates with a risk premium, we adjust the probabilities.
- Market price of risk is embedded in the probability term.
- Discounting is done with the risk free rate, easily observable.
- Enables pricing for assets where it is difficult to calculate risk adjusted discount rates, such as options.

Notice also what doesnt appear in the formula:

- The original or real probabilities of upward or downward movement.
- The investors attitude towards risk.
- Reference to other securities or portfolies, like market portfolio.

The reason for this is:

- Risk neutral valuation is not an equilibrium model such as CAPM. There is no matching of supply and demand, but it does have the absence of arbitrage opportunities.
- Equilibrium model produce a set of market clearing equilibrium prices as a function of investors preferences, demand for securities etc. The equilibirum prices are explained by supply and demand,
- Risk neutral valuation does not explain prices of existing securities on a complete and arbitrage free market, but instead takes them as given, then translates them into prices for additional redundant securities.
- So it is a relative, or conditional, pricing approach.

### 7.2.6 Return equalization

Under the risk netural probabilities, all securities earn the same expected riskless return, and all returns are equalized.

The book proves this, but in general the outcome is this:

**Expected return** $[D, S] = r_f$ **under risk neutral probabilities.** This is trivial for the risk free debt, but not for the stock.

## 7.3 Binomial option pricing

The binomial option pricing model was introduced in 1979:

- It is an easy and elegant way of demonstrating the economic intuition behind option pricing and its principal techniques.
- The model is not an simple approximation tool, it is an valuable asset for valuing general derivative securities.
- Can be used when no analytical closed form solution to the continuous time problem exists.
- In the limit, when the discrete time steps converges to a continuum, the model converges to an exact formula of continuous time.

The setting of the model is similar to State-Preference theory, but a bit more specific:

- The binomial methods uses discrete time and variables.
- Time modelled as a series of points in time, where new decisions are made an uncertainty over the period is resolved
- Uncertainty in variables resolved by distinguishing only 2 different future states: up and down.
- Both states have an return factor for the underlying variable, for which u and d are used as symbols.
- Customary to use return and interest factors, not rates: interest rate of 8% is expressed as $r = 1.08$
- State up occurs with $q$ prob, so down occurs with $(1 - q)$ prob.
- Since $u \times d = d \times u$, the result is a recombining binomial tree (lattice) and the underlying variable follows a multiplicative binomial process

It might appear restrictive to model uncertainty by only two discrete changes, but:

- Many variables move in small, discrete, steps. Stock change in tics, minimum allowed amounts, and changes in interest rate are expressed in discrete basic points of one hundreth of a precent.
- The number of states increase with the number of steps.

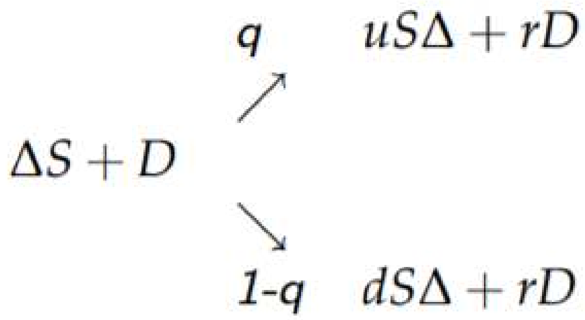
### 7.3.1 Cox Ross Rubenstein approach to pricing options

Construct a replicating portfolio of existing and thus priced securities that gives the same pay-offs as the option. The option price then has to be the same as the price of the portfolio, otherwise there are arbitrage opportunities.

We form a portfolio with

- A fraction $\Delta$ of the stock
- A risk free loan of $D$
- both can be postive or negative: positions can be long or short.

Gives following payoff tree for portfolio:




Lattice 3: Binomial tree for the replicating portfolio

We have to chose $\Delta$ and $D$ such that they make the end of period value of the portfolio equal to the end of the period value of the option:

$$uS\Delta + rD = O_u \tag{7.16}$$
$$dS\Delta + rD = O_d \tag{7.17}$$

These equations can be solved for D and $\Delta$:

$$\Delta = \frac{O_u - O_d}{(u-d)S} \tag{7.18}$$

$$D = \frac{uO_d - dO_u}{(u-d)r} \tag{7.19}$$

$\Delta$ is often used in finance. Its the number of shares needed to replicate an option. Called the hedge ratio or the option delta. We measure it as the spread in option values divided by the spread in stock values.

Portfolios with $\Delta$ and D are called hedging portfolios or the option equivalent portfolio. They are equivalent to options, so they generate the same payoffs as options. It also means that they must have the same price:

$$O = \Delta S + D \tag{7.20}$$

There is an derivation in the book, with substitiution for *Delta* and D, and $p_{1,2}$, but we skip this and just look at the exact formula for option price:

$$O = \frac{pO_u + (1-p)O_d}{r} \tag{7.21}$$

Where $O_u$ and $O_d$ is the end of period value of the option for stock going up and down, $p = (r-d)/(u-d)$, and r is as defined the interest factor. ($u$ and $d$ are return factors for up and down)

Characteristcs of the model:

- Risk accounted for by adjusting probabilities, not discount rate or cash flows.
- What does not appear in the formula: Investors attitude towards risk, other securities or portfolios, the real probabilities $q$ or $(1-q)$
- The reason for this is conditional nature of the pricing approach:

  The option pricing model does not explain prices of existing securties, like CAPM. Instead it translates prices of existing securities into option prices.

  The price of a stock must be dependant of the stock price, but the real probabilites are not needed as all investors agree on the value of an option relative to the share.

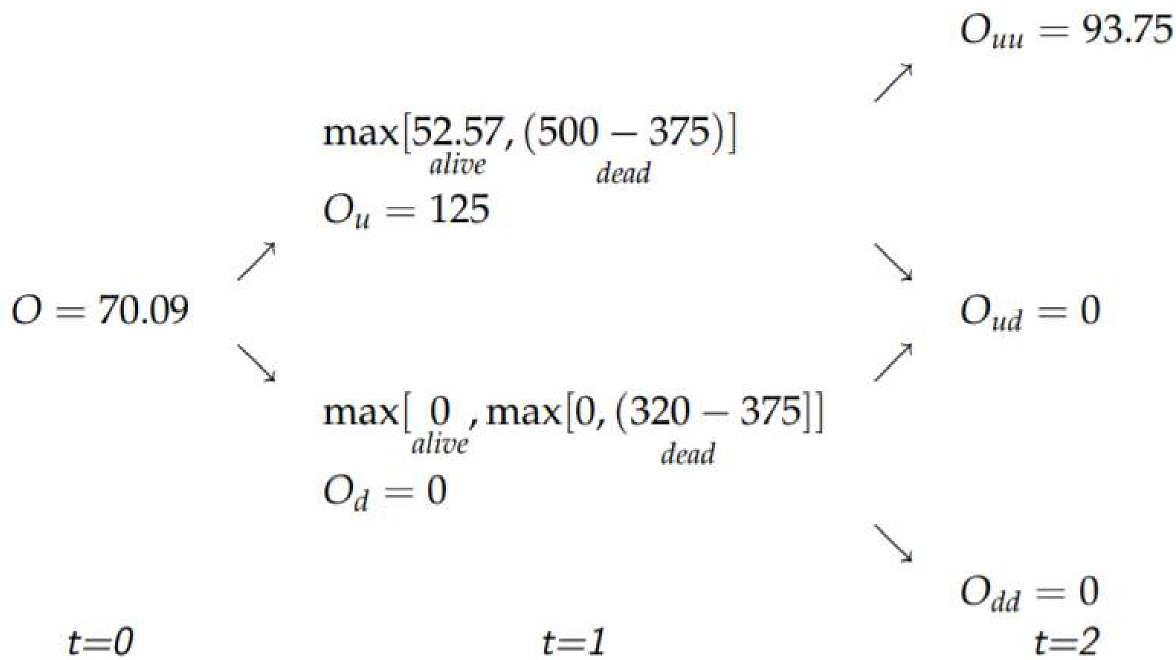
### 7.3.2 Dividends

We have previously show that in perfect markets, dividens are irrelevant for stock holder. But for the option holder, it can be very relevant. In the extreme case, a firm sells all its assets, paying out dividens, and leaves the option holder with the right to buy a worthless stock. It can be shown that if the stock value drops with the dividend amount right after payment, the value of the option is substantially lower.

### 7.3.3 American Calls

American calls can be exercised early. Without dividens, early exercise of a call will not be profitable. With dividends however, early exercise might be optional. This means we have to test in the binomial model if the option should be exercised or not. We look at max[exercising, keeping]/max[dead, alive].

Looking at the nodes, it is not possible to keep at maturity so these nodes remain the same. So we start from the back and calculate the values of the option in maturity-1 timestep, and compare to exercising at this time step.




Lattice 8 Binomial tree for an option

### 7.3.4 Put options

Binomial pricing works just as well for put options. We just formulated the payoff as

$$O_{u,d} = max[0, S_{u,d} - X] \tag{7.22}$$

Which can be changed to

$$O_{u,d} = max[0, X - S_{u,d}] \tag{7.23}$$

and gives us puts without changing derivation.q

# 8. Option pricing in continuous time

## 8.1 Black and Scholes

The idea: write value of europen option now as difference between value now of what you can expect to get upon exercise, and value now of what you can expect to pay upon exercise.

This approach is called the martingale method. We get prices by directly calculating expectation under equivalent martingale probability measure. For this we need to solve the discount rate problem: Under equivalent martingale measure all returns are equalized to $r_f$.

Building blocks of the Black and Scholes formula:

$$\begin{array}{ccccc}
\text{Modelling stock returns} & & \text{Transforming probabilities} & & \text{Partial expectation} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Geometric Brownian motion} & & \text{Measure transformation} & & \text{Conditional expectation} \\
& \searrow & \downarrow & \swarrow & \\
& & \text{Martingale method: Black \& Scholes formula} & &
\end{array}$$

### 8.1.1 Modelling stock returns: logarithmic returns

Recall from second chapter how to discretly compound returns:

$$r = \frac{S_t - S_{t-1}}{S_{t-1}} \tag{8.1}$$

These discretly compunded returns are easily aggragated across investments (attractive in portfolio analysis), but are non-additive over time (5% per year over 10 years = 62.5% not 50%)

Since option prices uses individual returns over time, it makes continously compounded returns convenient.

These are calculated as:

$$\frac{S_T}{S_0} = e^{rT} \tag{8.2}$$

Taking the natural log gives the log returns:

$$ln\frac{S_T}{S_0} = e^{rT} = rT \tag{8.3}$$

These log returns are additive over time, and so are convenient to use in continuous time models.

### 8.1.2 Properties of log returns

To describe the return behavior over time we have to make one critical assumption:*Log return are independently and identically distributed (iid)*

This has several consequences:

- Return = ln stock price: If return are approx N, the stock prices are approx log N
- The sum of 2 independent normal variables is also normal with:

  mean = sum 2 means

  variance = sum 2 variances
- Extended to many (T) time periods, so the mean and variance grows linearly over time.

  This means $R_T \sim N(\mu T, \sigma^2 T)$, where $R_T$ is the continuously compounded returns over time, $E[R_T] = \mu T$, $var[R_T] = \sigma^2 T$, and instantaneous return is $\mu$

It is also worth noting that iid follows a random walk, and that random walks have the Markov property of memorylessness: past returns and patterns are useless to predict future returns. So the market is a weak form of efficient.

### 8.1.3 Modelling stock returns: Brownian notion

For this we have to model the property of stock return in a forward looking way.

For discrete time, we list all possibilities as states of the world or values in a binomial tree.

For continuous time, variables have an infinite number of possibilities, and have to be expressed in a probabilistic way.

Standard equipment for this is the continuous time stochastic process.

We use the term brownian motion, and the symbol $W$ or $\tilde{W}$.

Standard brownian motion is contiuous time analogue of random walk, and can be thought of as a series of very small steps each drawn randomly from normal distribution.

Definiton: Process $\tilde{W}$ is standard brownian motion if:

- $\tilde{W}_t$ is continuous and $\tilde{W}_o = 0$
- It has independent increments.
- Increments $\tilde{W}_{s+t} - \tilde{W}_s \sim N(0, \sqrt{t})$, which implies that we have a stationary process, $W$ is a function of time but not location.

From the definition it also follows that Brownian motion has the Markov property.

The dicrete representation over a short time period $\delta t$ is $\epsilon\sqrt{\delta t}$, where $\epsilon$ is the random drawing from normal distribution.

The Brownian motion process has some remarkable properties: No upper or lower bounds, continuous every and never differentiable, its fractial.

This standard Brownian motion is a poor model of stock price behavior: It catches only random elements, neglects individual volatility for each stock, neglects the positive drift in stock prices, and it misses proportionality (changes should be in % not amounts).

We handle this by adding:

- Deterministic drift
- A parameter for volatility
- Proportionality: return and movement in % of stocks value

This standard model is geometric Brownian motion in a stochatic differntial equation:

$$dS_t = \mu S_t dt + \sigma S_t d\tilde{W}_t, \quad S_0 > 0 \tag{8.4}$$

where

- d = next instant incremental change.
- $S_t$ = stock price at time t.
- $\mu$ = drift coefficient

- $\sigma$ = diffusion coefficient / stock volatility.
- $S_0$ = initial condition.
- $\sigma, \mu$ are considered constant.

The financial market also contains debt, defined in similar matter:

$$dD_t = rD_t dt \tag{8.5}$$

Where $r = r_f$, meaning it is risk free, and so there is no stochastic disturbance term. We assume r to be constant.

### 8.1.4 Technique of changing measure

We want to change our model of stock behavior

$$dS_t = \mu S_t dt + \sigma S_t d\tilde{W}_t \tag{8.6}$$

such that the market price of risk is embedded in the probabilities $d\tilde{W}_t$, and all assets earn the risk free rate.

The mathematical insturment we use to do this is Girsanovs theorem:

- Transform the stochastic process (Brownian motion) under one probability measure.
- into another stochastic process that is a Brownian measure under another probability measure.

Expression for the Girsanov Kernel is:

$$d\tilde{W}_t = \tag{8.7}$$

Where $W$ is the transformed process, and $\theta$ is Girsanov Kernel.

Inserting this kernel into our original model gives:

$$dS_t = \mu S_t dt + \sigma S_t(\theta dt + dW_t) \tag{8.8}$$

We know the desired result from the definiton: The process should contain pricing information, so that the proper discount rate = drift = risk free rate r. The solution to this is to define $\theta$ as:

$$\theta = -\frac{\mu - r}{\sigma} \tag{8.9}$$

This definition means the Girsanov kernel is very simple. It is deterministic and constant (all variables are defined as constant).

If we insert this into the model and do some calculations we get

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{8.10}$$

### 8.1.5 Solving the SDE

Baiscly trial and error and stochastic calculus, and get

$$d(lnS_t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t \tag{8.11}$$

Where changes in ln(stock prices) follows BM, drift $(r - \frac{1}{2}\sigma^2$, and diffusion $\sigma$.

We also notice that Browian motion is normally distributed and that drift in constant, so $d(lnS_t)$ is also normally distributed.

Constant drift and diffusion also makes the process easy to handle and we can integrate over the time interval to get:

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma W_T} \tag{8.12}$$

Which must be lognormally distributed as $lnS_T$ is normally distributed.

This also gives us the expected value:

$$E[S_T] = S_0 e^{rT} \tag{8.13}$$

Giving us the following results:

- discounted future exp. stock price = current stock price.
- Discounted future exp. stock price = martingale.
- Which in turn means that risky assets can be discounted with risk free rate.
- The exact equivalent of the binomial model!

### 8.1.6 Partial and conditional expectation

Partial expectation: the expectation of part of a probability distribution. Calculated as the sum or integral of values in that part multiplied by their probability.

Conditional expectation: Is the partial expectation divided by the probability that the outcomes are in that part. This dividing rescalles the partial expectation so that they sum to 1.

### 8.1.7 Black and Scholes formula

The problem: We want to know the price now of an european call option $O^E_{c,0}$ with an exercise price of X, maturity at T and written on non dividend paying stock.

Using the martingale method:

$$O_{c,0} = e^{-rT} E[O_{c,T}] \tag{8.14}$$

where r is risk free interest rate.

Calculating the probability that an option will be exercised:

$$P(ln(S_T) > ln(X)) \tag{8.15}$$

For this we have to express the difference between mean $\mu$ and ln(X) in terms of standard deviation $\sigma$:

$$d = \frac{\mu - ln(X)}{\sigma} \tag{8.16}$$

We can then use the standard distribution function $\Phi$ to calculate probability that $ln(S_T)$ is no more than d standard deviation above or bellow $ln(X)$:

$$P(ln(S_T) > ln(X)) = \Phi\left(\frac{\mu - ln(X)}{\sigma}\right) \tag{8.17}$$

# 9. Real options analysis

Real options have underlying assets in investments in real assets, not financial securities such as stocks and bonds.

A analogy between real and financial options are given in the table below.

| Determinant | Stock option | Real option |
|---|---|---|
| underlying | stock | project revenue |
| strike | exercise price | investment |
| time to maturity | maturity | license validity |
| volatility | stock $\sigma$ | price volatility |
| interest rate | $r_f$ | $r_f$ |

**Table 9.1:** Stock-real option analogy

For financial options the value originates from the right to buy/sell assets at a fixed price. This can also be the case for real options, by extending/terminating a project. But often real options are not written contracts, rather investment opportunities with an varying degree of exclusiveness. Value often come from the advantage the firm has over it's competitors. Sources of value may be

- patents or copyrights
- mineral extraction rights (Oil field)
- firm's skills and knowledge (Know-how)
- firms' reputation and market position (Customer know-how: Windows OS)
- market opportunities

Real-option value tend to have an erosive nature. With competition catching up with similar products and skills.

**NB!** Follow-up projects are not necessarily real options. In mature, established markets competitors may perform same project at equal cost and generate same revenue. In these circumstances the real-option has no value. Financial equivalent; the option to sell/buy at market prices.

## 9.1 Interaction between real-options

Exercising a financial option no effect on the value of other financial options. This may not be the case for real-options.

Value of option changed by action of competition. Leading to game theory. Rule of thumb, this (+ erosion of value) leads to early exercise of option.

The combined value of options are less then the value of the option separatly. Example, if a contract is terminated, the option to extend contract is worthless. The total interaction effect is strongest if options are of the same type (call/puts)

# Bibliography