NTNU | Norwegian University of Science and Technology

PLENARY EXERCISES - TMA4145

Week 38, Wednesday 20. September 2023

Problem 1

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{C}$, and consider $\|x\|^2 = \langle x, x\rangle$, the induced norm on V.

1. Show that $\|\cdot\|$ satisfies the parallelogram law.

   $$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right),$$

   for all $x, y \in V$.
2. For $n \geq 3$, let ω be a n^{th} root of unity, i.e. $\omega^n = 1$ and $\omega^k \neq 1$ for $k < n$. Show that

   $$\langle x, y\rangle = \frac{1}{n}\sum_{k=1}^{n} \omega^k \|x + \omega^k y\|^2.$$

3. Show that

   $$\langle x, y\rangle = \int_0^1 e^{2\pi i\varphi}\|x + e^{2\pi i\varphi} y\|^2 d\varphi.$$

Hint:

1. Write the norms in terms of inner products, and expand.
2. Recall that a finite geometric series, with $a \neq 1$, can be written as

   $$\sum_{k=1}^{n} a^k = \frac{a(1-a^n)}{1-a}.$$

Problem 2

Let $(W, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product space, and $U, V \subseteq W$ be a subspace of W.

1. Show that
$$(U + V)^\perp = U^\perp \cap V^\perp$$
2. Show that U and V are direct, i.e. $U + V = U \oplus V$, if and only if $U^\perp + V^\perp = W$.
3. Show that $W = U \oplus V$ if and only if $W = U^\perp \oplus V^\perp$.

Hint:

1. $x \in U^\perp$ if $\langle x, u \rangle = 0$ for all $u \in U$.
2. $(U^\perp)^\perp = U$.

Problem 3 (Riesz representation Theorem)

Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space, and let $T : V \to \mathbb{K}$ be linear. Show that there exists $u \in V$ such that

$$T(v) = \langle v, u\rangle, \qquad \text{for all } v \in V.$$

Hint: There are several ways to solve the problem, here are a few options:

1. What is the singular value decomposition of T?
2. Recall that for any $x \in V$ can be written on the form $x = \sum_{i=1}^{n} \langle x, e_i\rangle e_i$, where $\{e_i\}_{i=1}^{n}$ is a orthonormal basis of V.
3. Choose a basis of V. What is the corresponding matrix representation of V?