

# **CISS451: Cryptography**

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# Chapter 1

## Classical ciphers

### 1.1 Classical ciphers: instructions

1. At my website, in the Tutorials section, you'll find `latex.pdf`. Post  $\LaTeX$  questions in CCCS discord.
2. In `thispreamble.tex`, change `AUTHOR` and `SHORTAUTHOR` to your name.
3. To speed up compilation, in `chap-classical-ciphers.tex`, you might want to comment out some sections using `%`.
4. Rewrite the contents of this chapter in your own words, otherwise your book is considered plagiarized. (You probably want to make a copy of this directory.) Note that you need not rewrite the questions in the exercises. You may retain the chapter and section organization (and their titles).
5. Every cipher in my notes must be present in your notes. You can add extra ciphers not found in my notes. (Example: enigma, playfair, etc.)
6. For each cipher, have a complete definition of each cipher and then have at least one example on encryption and decryption. Include definitions of terms. Your example(s) must be different from the examples in my notes.
7. You must write in proper English and using proper mathematical style.
8. Think of your notes as the only notes you can use in an open-book test or open-book final exam. Therefore you need not include historical or pedagogical remarks (but that's up to you).
9. Solve as many exercises as you can. The exercises are stored in directory `exercises`. For instance if you see `\input{exercises/abc/main.tex}`, this means the question of this exercise is stored in `\input{exercises/abc/question.tex}` and the answer should be written in `\input{exercises/abc/answer.tex}`
10. In terms of writing style, technically speaking, in formal writings, you should not use personal noun like "I". Instead, "we" should be used.

For instance instead of saying

“I will now prove my theorem.”

you should write

“We will now prove the (or our) theorem.”

I use “I” just to make my notes informal. For your book, you should use the formal writing style.

11. When you are done with this chapter, comment out this section of instructions.

The subtitle: Stuff that you should not use anymore.

However some of these old stuff is important because their ideas are used in modern-day cryptography.

Here we go ...

## 1.2 Shift cipher debug: classical-cipher-shift-cipher.tex

This is one of the earliest cryptosystems and apparently Julius Caesar used it (high tech, eh?). To encrypt a message, you simply do the following:

$$\alpha \mapsto \delta, \dots$$

or in our alphabet system:

$$\begin{array}{l} a \mapsto d \\ b \mapsto e \\ c \mapsto f \\ \vdots \quad \vdots \end{array}$$

i.e.  $a$  is replaced by  $d$ ,  $b$  is replaced by  $e$ , etc. And of course you “go around in a circle”:  $x$  is replaced by  $a$ ,  $y$  is replaced by  $b$ ,  $z$  is replaced by  $c$ .

$$\begin{array}{l} a \mapsto d \\ b \mapsto e \\ c \mapsto f \\ \vdots \quad \vdots \\ x \mapsto a \\ y \mapsto b \\ z \mapsto c \end{array}$$

This is known as the **Caesar cipher**.

Caesar cipher

Here’s a simple Python code to encrypt a character:

```
def E(x):
    i = ord(x) - ord('a')
    i = (i + 3) % 26
    return chr(ord('a') + i)
```

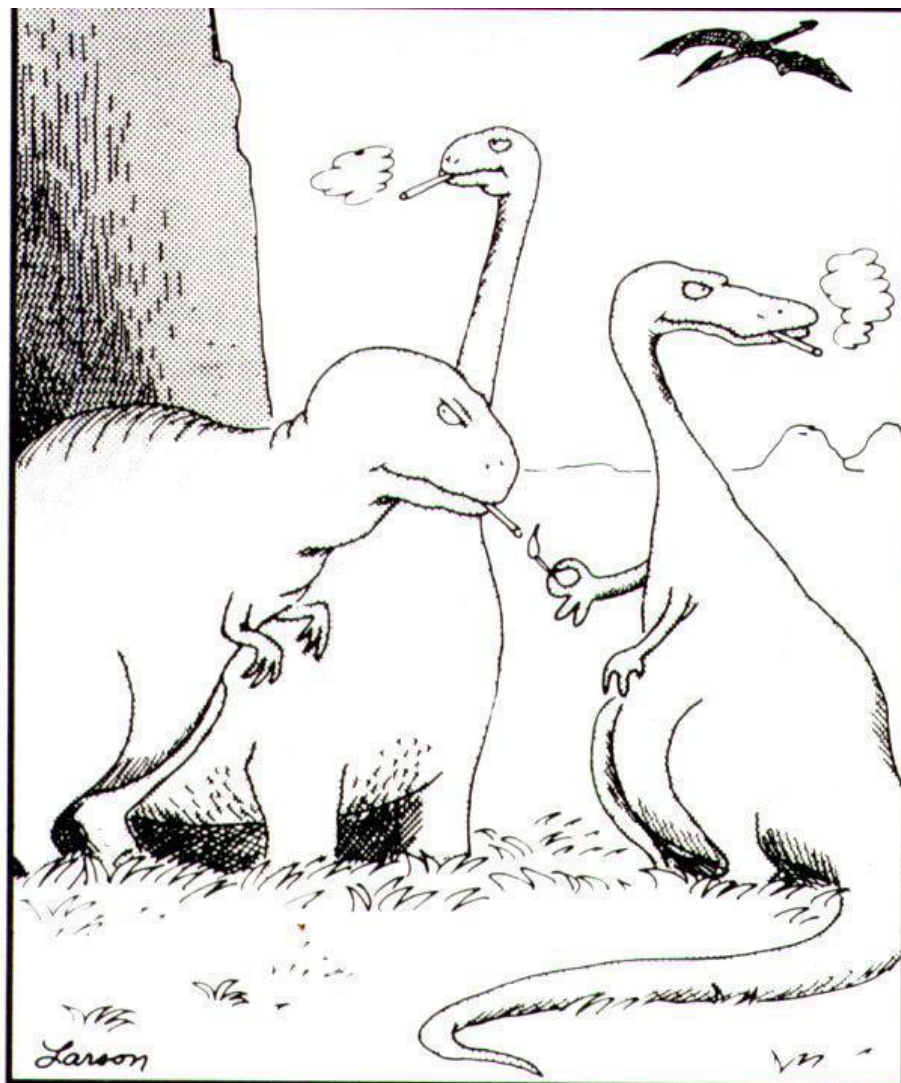
and for C++:

```
char E(char x)
{
    return (x - 'a' + 3) % 26 + 'a';
}
```

Of course there’s no reason why you must “shift by 3”. You can also shift by 7. Right? Note that the encryption input is only a single character. Of course for

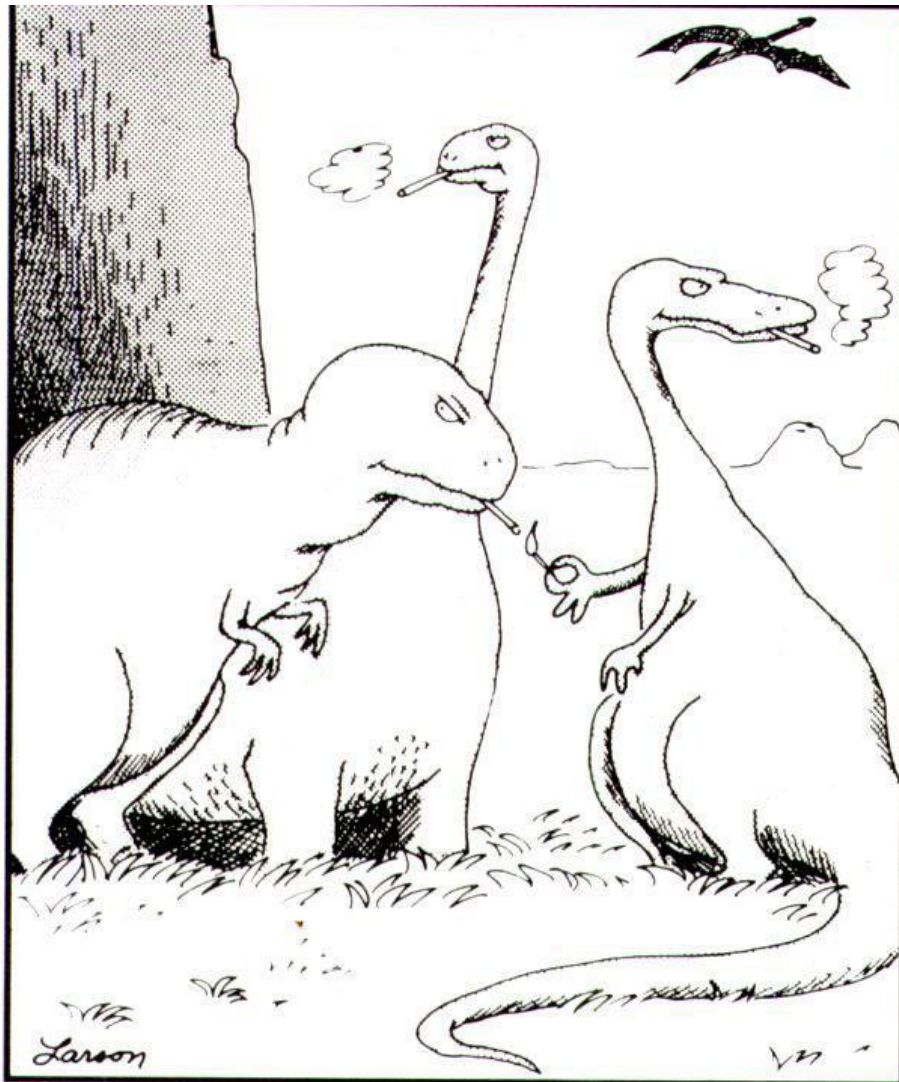
a string, you just encrypt character-by-character. Also, uppercase is replaced by lowercase. Furthermore, anything not *a-z* (example: punctuations, spaces) is thrown away.

Easy, right? OK. Break the following the code:



wkhuhdouhdvrqglqrvdxuvehfdphhawlqfw

\*\*\* WARNING: SPOILERS ON THE NEXT PAGE \*\*\*



The real reason dinosaurs became extinct

**Exercise 1.2.1.** Do you think Caesar cipher is secure? Do you think Julius Caesar was smart? ([Go to solution](#), page 14) □

debug:  
exercises/shift-  
0/question.tex

Let  $P$  and  $C$  be two sets. A **cipher** is just a pair of functions  $E : P \rightarrow C$  and  $D : C \rightarrow P$  where  $E$  is called the **encryption** and  $D$  is called the **decryption** such that

cipher  
encryption  
decryption

$$D(E(x)) = x \quad \text{for all } x \in P$$

In other words if you decrypt what you have encrypted, you get back the same



data. (It'd better be so!) An element of  $P$  is called a **plaintext** – it's what you encrypt. An element of  $C$  is called a **ciphertext** – it's what you get when you encrypt.

plaintext  
ciphertext

Caesar cipher is an example of a cipher. For Caesar cipher  $P = C = \{a, b, c, \dots, z\}$ . Also, although the encryption function maps one character to another, it's understood that if you want to encrypt a string, you simply encrypt each character of the string and join them up into a string.

But if you allow the shift amount in the Caesar cipher to change, then the encryption and decryption depends on the shift amount – the key. A general principle in cryptography is the following concept due to [Auguste Kerckhoffs](#):

**Kerckhoffs' principle** (1883): A secure cipher should not depend on the secrecy of the encryption and decryption algorithm, but rather on the secrecy of the key.

Kerckhoffs' principle

The opposite and a really bad idea is called **security through obscurity**, i.e., it's the hope that your encrypted messages are safe as long as the encryption and decryption algorithm are kept secret.

security through  
obscurity

Why is this important? Because it's easy to change the key while changing the encryption and decryption algorithm might not be that easy. If a worker who performs the encryption or decryption process is captured, then he/she can be made (tortured?) to reveal the algorithm. On the other hand, if the key is stolen, then we can simply change the key. So in cryptography, it is always assumed that the algorithms (the cipher) cannot be kept secret for long.

In fact in modern cryptography, once a cipher is designed, the cryptography researcher(s) is expected to publish the cipher so that other researchers can check if the cipher is actually secure.

So we just need to modify the definition of our cipher:

**Definition 1.2.1.** A **cipher** is a pair of functions  $E : K \times P \rightarrow C$  and  $D : K \times C \rightarrow P$  such that if  $k \in K$ ,

cipher

$$D(k, E(k, x)) = x \quad \text{for all } x \in P$$

$P$  is the set of plaintexts,  $C$  is the set of ciphertexts, and  $K$  is the set of keys. Notice that in the above the key used for encryption  $k$  is the same as the key used for decryption.

Instead of writing  $E(k, x)$  and  $D(k, x)$ , it's also common to write  $E_k(x)$  and  $D_k(x)$ . Depending on which book you read, the encryption and decryption functions can also be written  $e$  instead of  $E$  and  $d$  instead of  $D$ .

Humans have used ciphers for thousands of years. The early ciphers always use the same key for encryption and decryption. It was only very recently in 1970 that [James H. Ellis](#), asked if it's possible to have a cipher that uses two distinct keys, one for encryption and one for decryption. Ellis was a British cryptographer at the [GCHQ](#) (UK Government Communications Headquarters). If this is possible, then only the decryption key has to be kept secret. Why do you want to use such a cipher?

Well, I can publish the encryption key for such a cipher on a website, you encrypt with the encryption key and send me the ciphertext by email. On receiving the ciphertext, I decrypt it using the decryption key. Note that I can make the encryption key public, but I must keep the decryption key a secret. On the other hand for a symmetric key cipher, we would have to meet secretly and decide on the common key.

Surprisingly such a cipher exists.

So now I have to modify our definition of ciphers ...

**Definition 1.2.2.** A **symmetric cipher** is a pair of functions  $E : K \times P \rightarrow C$  and  $D : K \times C \rightarrow P$  such that if  $k \in K$ ,

symmetric cipher

$$D_k(E_k(x)) = x \text{ for all } x \in P$$

$P$  is the set of plaintexts,  $C$  is the set of ciphertexts, and  $K$  is the set of keys. A symmetric cipher is also called a **private key cipher** because the key used must be kept private.

private key cipher

And of course we also must have

**Definition 1.2.3.** An **asymmetric cipher** is a cipher where there are two distinct keys, one for encryption and one for decryption. An asymmetric cipher is also called a **public key cipher** because the encryption key can be made public (but the decryption key has to be kept secret). In this case, if  $k, k'$  are the encryption and decryption keys, then the cipher must satisfy

asymmetric cipher

public key cipher

$$D_{k'}(E_k(x)) = x$$

for  $x \in P$ . The encryption key  $k$  is called the **public key** (because it can be made public) while the decryption key is called the **private key**.

public key  
private key

**Exercise 1.2.2.** Is the Caesar's cipher symmetric or asymmetric? (Duh) ([Go to solution](#), page 15) □

debug:  
exercises/shift-  
1/question.tex

Public key ciphers use quite a bit of math. So we won't see public key ciphers for a while.

Let's go back to our Caesar cipher. You can think of the Caesar cipher as a special case of a symmetric cipher that uses the key 3:

- encryption is "shift forward by 3"
- decryption is "shift backward by 3".

In other words, generalizing the Caesar cipher, we get the shift cipher:

- encryption is "shift forward by  $k$ "
- decryption is "shift backward by  $k$ ".

where  $k$  is the key. I hope it's clear that the shift cipher with key 27 is the same as the shift cipher with key 1. Effectively speaking there are only 26 shifts, including the very bad key of 0. Hence for the shift cipher,  $K = \{0, 1, 2, \dots, 25\}$ .

For classical ciphers, assuming we're only interested in English, the plaintexts are strings involving  $a$ - $z$ . I will write  $\{a, b, c, \dots, z\}^*$  for the set of all strings with characters from  $\{a, b, c, \dots, z\}$ . If  $n$  is a positive integer, I will also write  $\{a, b, c, \dots, z\}^n$  for the set of strings with length  $n$  and with characters from the set  $\{a, b, c, \dots, z\}$ . For instance

$$\{a, b, c, \dots, z\}^2 = \{aa, ab, ac, \dots, zx, zy, zz\}$$

**Exercise 1.2.3.** How many strings are there in  $\{a, b, c, \dots, z\}^2$ ? How many strings are there in  $\{a, b, c, \dots, z\}^n$ ? ([Go to solution](#), page 16) □

debug:  
exercises/shift-  
2/question.tex

In modern day cryptography, we frequently work with bit strings. The set of all bit strings is denoted by  $\{0, 1\}^*$ . Bit strings of length exactly 8 is denoted by  $\{0, 1\}^8$  – these would be bytes. For instance you might have heard of the SHA2 family of hash function. SHA256 takes in bit strings and spits out bit strings of length 256. So SHA256 is a function of type

$$\{0, 1\}^* \rightarrow \{0, 1\}^8$$

(Technically speaking SHA256 inputs do have a maximum limit in length, but it's so huge that for practical purposes it's as good as all possible bit strings.)

You know this is coming ... we'll be using *lots* of math to do encryption and decryption. In particular, for this notes, we associate letters  $a$  to  $z$  with numbers. The encryption and decryption function will work with either numbers or letters. Specifically we have the following correspondence:

$$\begin{aligned} a &\leftrightarrow 0 \\ b &\leftrightarrow 1 \\ c &\leftrightarrow 2 \\ &\vdots \\ z &\leftrightarrow 25 \end{aligned}$$

In math, "correspondence" is the same as "1-1 correspondence" which is the same as "bijection". Remember bijection? It's time to check your discrete math notes

if  $E$  encrypts  $a$  to  $c$ , I will say either

$$E(a) = c$$

or

$$E(0) = 2$$

Now you might say ... "so what's the big deal? Why rewrite  $a$  as 0,  $b$  as 1, etc. I can also come up with some secret encoding for instance why can't I rewrite  $a$  as a square,  $b$  as a triangle, etc.?"

Well ... the reason is because  $0, 1, 2, \dots$  are numbers ... and ... *they have operations (addition, subtraction, multiplication, division).*

With the above in mind, instead of describing the shift cipher as functions on  $a-z$ , I'll describe it as function on  $\mathbb{Z}/26$ :

**Definition 1.2.4.** The **shift cipher**  $(E, D)$  is given by

$$E_k(x) = x + k \pmod{26}$$

and

$$D_k(x) = x - k \pmod{26}$$

shift cipher

It's clear that for the shift cipher  $(E, D)$ ,

$$D_k(E_k(x)) \equiv x \pmod{26}$$

And of course the shift cipher with key  $k = 3$  is the **Caesar cipher**.

Caesar cipher

## Solutions

Solution to Exercise [1.2.1](#).

Solution not provided.

debug:  
exercises/shift-  
0/answer.tex

Solution to Exercise [1.2.2](#).

The shift cipher is a symmetric cipher, i.e., it's private key cipher.

debug:  
exercises/shift-  
1/answer.tex

Solution to Exercise [1.2.3](#).

Solution not provided.

debug:  
exercises/shift-  
2/answer.tex



## 1.3 Modular arithmetic

debug: classical-cipher-modular-arithmetic.tex

Number theory is basically the study of whole numbers. That however is an over-simplification. The study of number theory involves almost all areas of mathematics. In fact many areas of mathematics were created just to study certain problems in number theory.

Number theory is an extremely huge area of study in Mathematics and Computer Science. It is also extremely fascinating. Many problems in number theory can be stated very simply so that even a high school student can understand the statement of the problem. And yet the techniques used to solve some of these problems require the mathematical tools from almost every area of Math. Gauss once said that “Mathematics is the queen of sciences and number theory is the queen of mathematics”.

Although there are many branches within Number Theory, right now we only need to know a little bit about Elementary Number Theory. “Elementary” here does not mean simple (although it will be for us since you’re only seeing a small part of Elementary Number Theory). It means we are studying Number Theory using only properties of whole numbers (integers). Research in Number Theory requires real numbers, complex numbers, calculus, geometry, complex analysis, etc.

This will be a very short introduction to the vast area of Number Theory. In fact this is only a tiny fraction of Elementary Number Theory. This is one of the oldest areas of Mathematics and one of the most fascinating because of its history. If you want to learn more about number theory, just let me know. I can easily find a project for you to work on.

For now, we will look at modular arithmetic. Besides cryptography, modular arithmetic is also used in data compression and error correction codes.

The set of integers  $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$  denoted  $\mathbb{Z}$  has two operations  $+$  and  $\cdot$ . In terms of the algebraic structure (i.e. the operations),  $\mathbb{Z}$  is known as a **commutative ring**. Basically a commutative ring is a set of “things” with two operations, addition and multiplication, with rules that look like the addition and multiplication rules for  $\mathbb{Z}$ . For instance one such rule in  $\mathbb{Z}$  is

$$a(b + c) = ab + ac$$

This same rule holds true for  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  and polynomials with coefficients in  $\mathbb{Z}$ .

The reason why mathematicians even bother defining this concept of “commutative ring” is more or less the same reason why we write functions in

programs: for re-use. There are *many* naturally occurring rings. So if while developing the theory for  $\langle \text{blah}_1 \rangle$  and  $\langle \text{blah}_2 \rangle$  and they are both rings, then it's enough to prove a general fact that applies to both and quote the fact. This is also related to the concept of inheritance and abstract base classes. You can think of  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  as subclasses of `CommutativeRing`. Therefore if you have a function

```
void f(CommutativeRing & r)
{
    ...
}
```

then `f` can work with `x` if `x` is a object of  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ .

The mathematician develop general results while the programmer write general functions working on abstract base class objects. The idea is the same. The reason for generality is efficiency.

Here's a very important advice on studying rings, groups, fields, math, etc. (in fact this applies to any area of study where there is a great deal of generalization): Although the definition and theorems are general, you *always* keep a couple of standard examples in your mind as you read the statements. While reading them, mentally substitute your examples into the facts so that you can associate it to something more concrete. This is not just a learn technique for undergraduate students. Even researchers do that when they read research papers. For the definition of ring below, think of the ring of integers.

I will try to be as informal as possible for now so that you can develop some feel/intuition for what we need for now. Later I'm going to come back to this topic and redo the whole thing rigorously. The focus for now is to understand modulo 26 arithmetic.

Anyway, a **commutatative ring**  $R$  is a set of "things" with two operations abstractly denoted by  $\oplus$  and  $\odot$  ("addition" and "multiplication") Furthermore there are two special "things" in  $R$  which we will call  $0_R$  and  $1_R$ . The properties satisfied by  $R, 0_R, 1_R, \oplus, \odot$  (of course there must be something satisfied by them!) are as follows. For  $\oplus$  the properties are:

- (a)  $r \oplus s$  is in  $R$  for all  $r, s$  in  $R$
- (b)  $(r \oplus s) \oplus t = r \oplus (s \oplus t)$  for all  $r, s, t$  in  $R$
- (c)  $r \oplus 0_R = r = 0_R \oplus r$  for all  $r$  in  $R$
- (d) For all  $r$  in  $R$  there is something in  $R$  which we will call  $r'$  such that  $r \oplus r' = 0_R = r' \oplus r$

For  $\odot$  the properties are:

Example: the integer  $\mathbb{Z}, \dots$  and  $\cdot$  of  $\mathbb{Z} \dots$

0 and 1 of  $\mathbb{Z} \dots$

$i + j$  is in  $\mathbb{Z}$  for integers  $i, j, \dots$   
 $(i + j) + k = i + (j + k)$  for integers  $i, j, k \dots$   
 $i + 0 = i = 0 + i$  for integer  $i \dots$   
 If  $i$  is an integer, then  $-i$  is also an integer and  $i + (-i) = 0 = (-i) + i$ .

- (a)  $r \odot s$  is in  $R$  for all  $r, s$  in  $R$
- (b)  $(r \odot s) \odot t = r \odot (s \odot t)$  for all  $r, s, t$  in  $R$
- (c)  $r \odot 1_R = r = 1_R \odot r$  for all  $r$  in  $R$
- (d)  $r \odot s = s \odot r$  for all  $r, s$  in  $R$

$ij = ji$  for integer  $i, j, \dots$   
 $(ij)k = i(jk)$  for integers  $i, j, k$   
 $i1 = i = 1i$  for integer  $i$  ...  
 $ij = ji$  for integers  $i, j$

The property involving both  $\oplus$  and  $\odot$  is

- (a)  $r \odot (s \oplus t) = r \odot s \oplus r \odot t$

$i(j+k) = ij + ik$  for integers  $i, j, k$ . Phew!  
 So  $\mathbb{Z}$  is a commutative ring.

Just remember this: A ring is a set of things with addition and multiplication. And when you're lost just think of the set of integers and its operations.

Now we'll be working with the alphabet a,b,...,z. We'll call them by their new identities: 0, 1, ..., 25. This is not exactly all of  $\mathbb{Z}$ . The formulas for the encryption and decryption of Caesar's cipher involves addition and subtraction. What if we go beyond? What is  $3 + 25$  (i.e., d + z)? No problem, we will take remainders mod 26. So instead of  $3 + 25$  we think of  $(3 + 25) \bmod 26$  instead. Of course the remainders are 0, 1, ..., 25 which is exactly what we want.

In C++-speak, we take `%26`.

To indicate that we're only interested in remainders or more accurately, we ignore multiples of 26, we write

$$\begin{aligned} 3 + 25 &= 28 \\ &\equiv 2 \pmod{26} \end{aligned}$$

In general, if  $x$  and  $y$  are integers, we write

$$x \equiv y \pmod{26}$$

if 26 divides  $x - y$ . It does not mean that  $x$  is equal to  $y$ . It means that  $x$  and  $y$  are the same if you ignore additive multiples of 26, i.e.,

$$x \equiv y \pmod{26}$$

is the same as saying

$$x = y + (\dots \text{some multiple of } 26 \dots)$$

This is the same as saying the remainder when  $x$  is divided by 26 is the same as the remainder when  $y$  is divided by 26.

If two numbers differ by a multiple of 26, we say that they are **congruent mod 26**.

congruent

Note that

$$\begin{aligned} 26 &\equiv 0 \pmod{26} \\ 27 &\equiv 1 \pmod{26} \\ 28 &\equiv 2 \pmod{26} \\ &\dots \end{aligned}$$

and

$$\begin{aligned} -1 &\equiv 25 \pmod{26} \\ -2 &\equiv 24 \pmod{26} \\ -3 &\equiv 23 \pmod{26} \\ &\dots \end{aligned}$$

So in the mod 26 world, since you are ignoring multiples of 26, *in some sense* there are only 26 numbers:

$$0, 1, 2, 3, \dots, 25$$

I'll write  $\mathbb{Z}/26$  for this world of 26 values. Remember that in this world, you can write the symbol

$$28$$

but this is the same as 2 in  $\mathbb{Z}/26$ :

$$28 \equiv 2 \pmod{26}$$

Of course in  $\mathbb{Z}$ , these symbols, i.e. 28 and 2, are different.

Most of the algebraic rules involving  $+$ ,  $-$ ,  $*$ ,  $0$ ,  $1$  applies when working with integers mod 26. For instance suppose

$$x \equiv y \pmod{26}$$

where  $x$  and  $y$  are integers (i.e.,  $x$  differs from  $y$  by a multiple of 26), then

$$x + z \equiv y + z \pmod{26}$$

where  $z$  is an integer. Likewise from

$$x \equiv y \pmod{26}$$

we get

$$xz \equiv yz \pmod{26}$$

It's also true that

$$0 + x \equiv x \pmod{26}$$

and

$$1 \cdot x \equiv x \pmod{26}$$

To be more precise,  $\mathbb{Z}/26$  is a commutative ring. It's a finite commutative ring with 26 values. Let me rewrite the axioms for a commutative ring for  $\mathbb{Z}/26$ .

For  $+$  on  $\mathbb{Z}/26$ , the properties are:

- (a)  $(r + s) \pmod{26}$  is in  $\mathbb{Z}/26$  for all  $r, s$  in  $\mathbb{Z}/26$
- (b)  $(r + s) + t \equiv r + (s + t) \pmod{26}$  for all  $r, s, t$  in  $\mathbb{Z}/26$
- (c)  $r + 0 \equiv r \equiv 0 + r \pmod{26}$  for all  $r$  in  $\mathbb{Z}/26$ . In fact  $r'$  is just  $(26 - r) \pmod{26}$ . For instance for  $r = 2 \pmod{26}$ ,  $r' = 26 - 2 = 24 \pmod{26}$ .
- (d) For all  $r$  in  $\mathbb{Z}/26$  there is something in  $\mathbb{Z}/26$  which we will call  $r'$  such that  $r + r' \equiv 0 \equiv r' \oplus r \pmod{26}$

For  $\cdot$  the properties are:

- (a)  $r \cdot s \pmod{26}$  is in  $\mathbb{Z}/26$  for all  $r, s$  in  $\mathbb{Z}/26$
- (b)  $(r \cdot s) \cdot t \equiv r \cdot (s \cdot t) \pmod{26}$  for all  $r, s, t$  in  $\mathbb{Z}/26$
- (c)  $r \cdot 1 \equiv r \equiv 1 \cdot r \pmod{26}$  for all  $r$  in  $\mathbb{Z}/26$
- (d)  $r \cdot s \equiv s \cdot r \pmod{26}$  for all  $r, s$  in  $\mathbb{Z}/26$

The property involving both  $+$  and  $\cdot$  is

- (a)  $r \cdot (s + t) \equiv r \cdot s + r \cdot t \pmod{26}$

It's also not too surprising that you can talk about mod  $n$  for any positive integer  $n$ .

If I don't say so, when I want you write some  $x \pmod{26}$ , I mean the  $x$  such that  $0 \leq x < 26$ . Of course there's no difference in mod 26 between 2 and 28. But in mod 26, the values in  $[0, 26)$  is the preferred range. Also, when I say simplify  $27 \pmod{26}$ , I mean  $1 \pmod{26}$ .

Note that since  $\mathbb{Z}/26$  is *finite*, you can always solve any equation in mod 26. This is similar to boolean values: there are only two. To solve a boolean equation, you just try all possible boolean values. So to solve a  $\mathbb{Z}/26$  equation, you just try all the 26 possible values on all the variables that appear in the equation.

**Exercise 1.3.1.**

debug:  
exercises/exercise-  
0/question.tex

- (a) True or false:  $100 \equiv 74 \pmod{26}$
- (b) True or false:  $-3 \equiv 133 \pmod{26}$
- (c) True or false:  $-20 \equiv 21 \pmod{26}$
- (d) True or false:  $7 \equiv 3 \pmod{3}$
- (e) True or false:  $17 \equiv -3 \pmod{5}$
- (f) True or false:  $21 \equiv 11 \pmod{8}$
- (g) True or false:  $42 \equiv 0 \pmod{7}$

([Go to solution](#), page 38)



**Exercise 1.3.2.**

debug:  
exercises/exercise-  
1/question.tex

- (a) Simplify  $100 \pmod{26}$ .
- (b) If you have very simple calculator with  $+$ ,  $-$ ,  $*$ ,  $/$  how would simplify  $131246845 \pmod{26}$ ? You have 10 seconds ... the clock is ticking ...
- (c) Simplify  $33 \pmod{26}$ .
- (d) Solve  $2x + 1 \equiv 0 \pmod{26}$  by brute force. Use Python or C++.
- (e) Solve  $10x + 20 \equiv 4x + 78 \pmod{26}$  by brute force. Use Python or C++.
- (f) Simplify  $10x + 20 \equiv 4x + 78 \pmod{26}$  first, and then solve it by brute force. Do you get the same results as in the previous part?
- (g) Solve  $42x^5 + 10x + 1 \equiv 73 \pmod{3}$ .
- (h) Solve  $(1000000x + 2)^3 \equiv 2 \pmod{3}$ .

([Go to solution](#), page 39)



**Exercise 1.3.3.**

debug:  
exercises/exercise-  
2/question.tex

- (a) If  $x \equiv 1 \pmod{7}$  and  $x \equiv 5 \pmod{13}$ , what can you tell me about  $x$ ?
- (b) If  $x \equiv 1 \pmod{7}$  and  $x \equiv 5 \pmod{13}$ , what can you tell me about  $x \pmod{7 \cdot 13}$ ?
- (c)  $x \equiv 2 \pmod{7 \cdot 13}$  what can you tell me about  $x \pmod{7}$  and  $x \pmod{13}$ ?
- (d) True or false:  $x^3 \equiv x^0 \pmod{3}$  since the 3 in the exponent can be replaced by 0.
- (e) In  $\mathbb{Z}$ , is it true that  $(x + y)^2 = x^2 + y^2$ ? Try some values for  $x$  and  $y$ .
- (f) In  $\mathbb{Z}/2$ , is it true that  $(x + y)^2 = x^2 + y^2 \pmod{2}$ ? Try all values for  $x$  and  $y$ .
- (g) In  $\mathbb{Z}$ , is it true that  $(x + y)^3 = x^3 + y^3$ ? Try some values for  $x$  and  $y$ .
- (h) In  $\mathbb{Z}/3$ , is it true that  $(x + y)^3 \equiv x^3 + y^3 \pmod{3}$ ? Try all values for  $x$  and  $y$ .
- (i) In  $\mathbb{Z}/n$ , is it true that  $(x + y)^n \equiv x^n + y^n \pmod{n}$ ? Can you prove your

claim.

([Go to solution](#), page 40)

□

**Exercise 1.3.4.** Just like for boolean values, you can write down the complete behavior of the boolean and and boolean or and boolean not operators (these are called truth tables), you can also complete specify the complete behavior of addition in mod 26, “negative of” in mod 26, multiplication in mod 26, and also multiplicative inverse mod 26. The multiplicative inverse of  $x$  in mod 26 is just the number  $y$  mod 26 such that

$$xy \equiv 1 \pmod{26}$$

The multiplicative inverse of  $x \pmod{26}$  is written  $x^{-1} \pmod{26}$  – this is an integer mod 26!!! It’s not a fraction in  $\mathbb{R}$ !!! Sometimes  $x^{-1} \pmod{26}$  might not exist. In that case write None. Write down these 4 tables.

Addition table for  $\mathbb{Z}/26$ :

debug:  
exercises/exercise-  
3/question.tex

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0																										
1																										
2																										
3																										
4																										
5																										
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In the above, when I write 5, I meant of course 5 (mod 26).

Multiplication table for  $\mathbb{Z}/26$ :



×	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0																										
1																										
2																										
3																										
4																										
5																										
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Negative of table for  $\mathbb{Z}/26$ :

$x \pmod{26}$	$-x \pmod{26}$
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	
20	
21	
22	
23	
24	
25	

Multiplicative inverse table for  $\mathbb{Z}/26$ :

$x \pmod{26}$	$x^{-1} \pmod{26}$
0	None
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	
20	
21	
22	
23	
24	
25	

It should be clear that  $0 \pmod{26}$  does not have an inverse.

([Go to solution](#), page 41)

□

**Exercise 1.3.5.** Solve

$$3x = 8$$

debug:  
exercises/exercise-  
4/question.tex

in  $\mathbb{Z}$ . Can you? Now solve

$$3x \equiv 8 \pmod{26}$$

([Go to solution](#), page 45)

□

**Exercise 1.3.6.**

debug:  
exercises/exercise-  
5/question.tex

(a) Solve

$$3x = 1$$

in  $\mathbb{Z}$ . Of course you can't! Now solve

$$3x = 1$$

in  $\mathbb{Q}$ . Of course you can! We would say that  $\frac{1}{3}$  is the multiplicative inverse of 3 in  $\mathbb{Q}$ . Also, we would say that, in  $\mathbb{Z}$ , 3 is not invertible. What about

$$3x \equiv 1 \pmod{26}$$

Can you? If you can then you have found a multiplicative inverse of 3 (mod 26).

(b) Now try to solve

$$ax \equiv 1 \pmod{26}$$

for  $a \equiv 0, 1, 2, \dots, 25 \pmod{26}$ . Which  $a$ 's have multiplicative inverses? Is there a pattern to  $a$ 's with multiplicative inverses mod 26 and who which do not?

(c) Now try

$$ax \equiv 1 \pmod{N}$$

where  $N$  is a positive integer and  $a \equiv 0, 1, 2, \dots, N-1 \pmod{N}$ ; say you try  $N = 4, 5, 6, 7, 8$ . Notice something? See a pattern? Is there a pattern to  $a$ 's with multiplicative inverses mod 26 and who which do not?

([Go to solution](#), page 46)

□

**Exercise 1.3.7.** Solve

$$3x = 13$$

in  $\mathbb{Z}$ . Of course you can't! Now solve

$$3x = 13$$

debug:  
exercises/exercise-  
6/question.tex

in  $\mathbb{Q}$ . Of course you can! Do you realize that you solved it using the multiplicative inverse of 3 in  $\mathbb{Q}$ ? What about

$$3x \equiv 13 \pmod{26}$$

Are you going to use brute force and try all values (mod 26)? (HINT: You had better not.) ([Go to solution](#), page 47)  $\square$

**Exercise 1.3.8.** Solve

debug:  
exercises/exercise-  
7/question.tex

$$3x - 5 \equiv 10x + 20 \pmod{26}$$

by brute force. Next, simplify the above first before solving it. Do you get the same solutions? ([Go to solution](#), page 48)  $\square$

**Exercise 1.3.9.** Solve

debug:  
exercises/exercise-  
8/question.tex

$$3x + 12 \equiv 10x + 23 \pmod{26}$$

by brute force. Next, simplify the above first before solving it. (Are you sure it's really simplified? HINT: Multiplicative inverse.) Do you get the same solutions? ([Go to solution](#), page 49)  $\square$

**Exercise 1.3.10.**

debug:  
exercises/exercise-  
9/question.tex

(a) Can you solve

$$x^2 \equiv 1 \pmod{26}$$

Of course one solution is 1 (mod 26). Can you find the other one in 1 second? Are there just two? Or are there more than two solutions? These are (of course) square roots of 1, but in mod 26 and not  $\mathbb{R}$ .

(b) What about

$$x^2 \equiv 2 \pmod{26}$$

(c) Keep going ... try to solve

$$x^2 \equiv a \pmod{26}$$

for all cases. Draw a table for the square roots of  $a \pmod{26}$  for all  $a$ 's.

([Go to solution](#), page 50)  $\square$

**Exercise 1.3.11.** Continuing the previous exercise ...

debug:  
exercises/exercise-  
10/question.tex

- (a) Next, try to solve

$$x^2 \equiv a \pmod{N}$$

for all  $a \pmod{N}$  for at least 3 values of  $N$ . See any patterns?

- (b) The above is probably too tough. What if you try

$$x^2 \equiv 2 \pmod{p}$$

for primes  $p$ ? Try a few primes (maybe 20-30) and make a table. Notice a pattern?

([Go to solution](#), page 51)

□

**Exercise 1.3.12.**

debug:  
exercises/exercise-  
11/question.tex

- (a) What are all the possible values of  $x^2 \pmod{4}$ ?  
(Note: The key thing to note is that  $x \pmod{4}$  can take all the values 0, 1, 2, 3 mod 4. But the squares  $x^2 \pmod{4}$  can only certain values mod 4.)
- (b) What are all the possible values of  $x^2 \pmod{5}$ ?
- (c) What about squares mod 7?
- (d) What about squares mod 15? (Be efficient.)
- (e) Prove that there are no integer solutions to

$$5x^2 - 8y^2 = 3$$

(HINT: Prove this by contradiction. Take mod 4 ... and ... Checkmate!)  
Without modular arithmetic, there's no clear way to solve this problem!  
Go number theory!)

([Go to solution](#), page 52)

□

**Exercise 1.3.13.**

debug:  
exercises/exercise-  
12/question.tex

- (a) Solve

$$5x^2 + y^2 = 3$$

(HINT: You don't really need number theory for this one. Why? But if you want to, imitate the solution for the previous problem.)

- (b) Solve

$$11y^2 - 5x^2 = 3$$

(HINT: This is just a slight change from the previous problem. *But* now you need number theory. Try mod 4. If it does not work, try mod 5. Repeat.)

(c) Solve

$$y^2 - 5x^2 = 2$$

(d) What about this one:

$$x^2 - 5y^2 = 1$$

[ASIDE. Integer solutions to  $x^2 - dy^2 = 1$  has been studied since at least 400BC. This equation appear the Cattle Problem of Archimedes:

If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, the third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls went to pasture together. Now the dappled in four parts<sup>8</sup> were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shall thou be numbered among the wise. But come, understand also all these conditions regarding the cows of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

If  $W, X, Y, Z$  represents the number of white, black, yellow, dappled bulls, you will get a systems of 7 linear equations, the first two being

$$W = (1/2 + 1/3)X + Z$$

$$X = (1/4 + 1/5)Y + Z$$

together with some constraints such as  $W + X$  must be a square. After some manipulations, it can be shown that the equation to solve looks like

$$x^2 - 410286423278424y^2 = 1$$

What was Archimedes thinking? You are find information on the Archimedes Cattle Problem on the web.]

([Go to solution](#), page 53)

□

**Exercise 1.3.14.** Consider

$$5x^2 - 8y^2 = 3z^2$$

debug:  
exercises/exercise-  
13/question.tex

where we are only interested in finding integer solutions. Of course  $(0, 0, 0)$  is a solution, but that's easy. (Right?) In general, an equation like

$$ax^p + by^q = cz^r$$

where  $p, q, r > 0$  always has  $(0, 0, 0)$  as a solution.  $(0, 0, 0)$  is sometimes called the trivial solution. So we might as well assume  $(x, y, z) \neq (0, 0, 0)$ , i.e., assume they are not all zero. By the way the polynomial

$$5x^2 - 8y^2 - 3z^2$$

is a sum of terms where the number of variables appearing in each term is the same, i.e., 2. Such polynomial are said to be homogeneous of degree 2.

How many solutions can you find? Is there any at all? Is there a finite number of solutions? If there are infinitely many solutions, can you write them down? If not all of them, maybe an infinite family of them. For instance it would be nice to say: "For any integer  $n$ ,  $x = n, y = 2n, z = n + 5$  is a solution". This would be a 1-parameter family of solutions.

- (a) What's the first thing you should do?
- (b) Prove that if  $x, y, z$  is a solution, then  $nx, ny, nz$  is also a solution for any integer  $n$ .
- (c) In case you can't see a solution right away, do the following. If  $(x, y, z)$  are solutions, show that  $x$  and  $z$  must be even. (HINT: mod 4.) This is helpful since we won't have to check the odd  $x$  or odd  $z$  cases. That cuts down a brute force search down by  $3/4$ .
- (d) Continuing the above: let  $x = 2a$ ,  $y = b$ , and  $z = 2c$ , substitute, and you'll get a new equation in  $a, b, c$ .
- (e) The equation in  $a, b, c$  is easier than the one in  $x, y, z$ . Why? Because a solution in the equation in  $a, b, c$  for the above question would correspond to a solution  $x, y, z$  in the original equation and  $a, b, c$  solution is smaller. Can you now find some solutions for the equation in  $a, b, c$ ?
- (f) I won't go further. But for those who want to solve this probably complete, let me just say that once you have one solution, you can parametrize all solutions using two integer parameters.

([Go to solution](#), page 54)

□

**Exercise 1.3.15.** Find all the integer solutions to  $3x^2 + 4y^2 = 5z^2$ . (Hint: mod 3.) ([Go to solution](#), page 55)

□

debug:  
exercises/exercise-  
14/question.tex



**Exercise 1.3.16.** Find all solutions to

debug:  
exercises/exercise-  
15/question.tex

$$y^2 = x^3 + x + 1 \pmod{26}$$

(No this is not a random, idle question. It's actually very important.) ([Go to solution](#), page 58)  $\square$

**Exercise 1.3.17.** (This “idle” problem is way more important than you think.)

debug:  
exercises/exercise-  
16/question.tex

(a) Compute

$$2^n \pmod{26}$$

for  $n = 0, 1, 2, 3, \dots$ . What do you notice? Can you tell me what is

$$2^{1000} \pmod{26}$$

- (b) Now try  $3^n \pmod{26}$  for  $n = 0, 1, 2, 3, \dots$ . Can you tell me what is  $3^{1000} \pmod{26}$ ?
- (c) Now try  $4^n \pmod{26}$  for  $n = 0, 1, 2, 3, \dots$ . Can you tell me what is  $4^{1000} \pmod{26}$ ?
- (d) Now try  $5^n \pmod{26}$  for  $n = 0, 1, 2, 3, \dots$ . Can you tell me what is  $5^{1000} \pmod{26}$ ?
- (e) Of course since  $\mathbb{Z}/26$  is finite, you would expect  $a^n \pmod{26}$  to be finite even though  $n = 0, 1, 2, \dots$  runs through an infinite set. Which  $a$  would give you most distinct value of  $a^n \pmod{26}$  as you run through all values for  $n$ ? And do you notice a pattern in the finite collection of all the  $a^n \pmod{26}$ ?

([Go to solution](#), page 59)  $\square$

**Exercise 1.3.18.** Continuing the previous question ... What about  $a^n \pmod{p}$  when  $p$  is a prime? How many values do you get as you run through  $n = 0, 1, 2, \dots$ ? First try it for  $a = 2$ : try about 20 to 50 primes  $p$ . Do you see a pattern? Once you have spotted the pattern, try  $a = 3$ , etc. (The above is a very serious research problem for you.) ([Go to solution](#), page 60)  $\square$

debug:  
exercises/exercise-  
17/question.tex

**Exercise 1.3.19.**

debug:  
exercises/exercise-  
18/question.tex

- (a) What is  $10 \pmod{3}$ ? (when simplified).
- (b) What is  $100 \pmod{3}$ ? (when simplified).
- (c) What is  $1000 \pmod{3}$ ? (when simplified).

- (d) What is  $5342 \pmod{3}$ ? (when simplified).
- (e) What is  $534603142235187 \pmod{3}$ ? (when simplified).
- (f) What is the general fact here?

([Go to solution](#), page 61)



**Exercise 1.3.20.**

debug:  
exercises/exercise-  
19/question.tex

- (a) What is  $10 \pmod{9}$ ? (when simplified).
- (b) What is  $100 \pmod{9}$ ? (when simplified).
- (c) What is  $1000 \pmod{9}$ ? (when simplified).
- (d) What is  $9142 \pmod{9}$ ? (when simplified).
- (e) What is  $96331420351887 \pmod{9}$ ? (when simplified).
- (f) What is the general fact here?

([Go to solution](#), page 62)



**Exercise 1.3.21.**

debug:  
exercises/exercise-  
20/question.tex

- (a) What is  $10 \pmod{11}$ ? (when simplified).
- (b) What is  $100 \pmod{11}$ ? (when simplified).
- (c) What is  $1000 \pmod{11}$ ? (when simplified).
- (d) What is  $9142 \pmod{11}$ ? (when simplified).
- (e) What is  $80432440556787 \pmod{11}$ ? (when simplified).
- (f) What is the general fact here?

([Go to solution](#), page 78)



**Exercise 1.3.22.** Although the algebraic manipulations on mod  $N$  seems to be very similar to the algebraic manipulations in  $\mathbb{Z}$  (or even  $\mathbb{R}$ ), certain “bizarre” things do happen. In  $\mathbb{Z}$  (in fact in  $\mathbb{Q}$  and  $\mathbb{R}$  as well), you have the implication

debug:  
exercises/exercise-  
21/question.tex

$$xy = 0 \implies x = 0 \text{ or } y = 0$$

Is it true that

$$xy \equiv 0 \pmod{26} \implies x \equiv 0 \pmod{26} \text{ or } y \equiv 0 \pmod{26}$$

For each  $N$ , check when

$$xy \equiv 0 \pmod{N} \implies x \equiv 0 \pmod{N} \text{ or } y \equiv 0 \pmod{N}$$

holds. And when it does not, give one counterexample. Do you see a pattern?  
 ([Go to solution](#), page 79) □

**Exercise 1.3.23.** (Dr.Liow's magic formula) Suppose I want to simplify 23532 (mod 26). With a calculator or C++ or python, we see quickly that

debug:  
 exercises/exercise-  
 22/question.tex

$$23532 \equiv 2 \pmod{26}$$

I claim that you can use this magic formula: Suppose the digits of a 5-digit number is  $edcba$ . For instance  $edcba = 23532$  where  $e = 2, d = 3, c = 5, b = 3, a = 2$ . Then

$$edcba \equiv ba + 2(-2c + 6d - 5e) \pmod{26}$$

For instance when  $edcba = 23532$ , we have

$$edcba \equiv 32 + 2(-2 \cdot 5 + 6 \cdot 3 - 5 \cdot 2) \pmod{26}$$

and

$$32 + 2(-2 \cdot 5 + 6 \cdot 3 - 5 \cdot 2) \equiv 6 + 2(-2) \equiv 2 \pmod{26}$$

which is easier to work with than the much larger 23532. So ... is

$$edcba \equiv ba + 2(-2c + 6d - 5e) \pmod{26}$$

really true for any 5-digit number  $edcba$ ? Write a program to test all cases. Next, write a proof (that does not involve checking all cases like a program).

([Go to solution](#), page 65) □

**Exercise 1.3.24.** Continuing the previous question:

debug:  
 exercises/exercise-  
 23/question.tex

- (a) I further claim that if the number is an 8-digit number  $hgfedcba$ , then

$$hgfedcba \equiv ba + 2((-2c + 6d - 5e) + (2f - 6g + 5h)) \pmod{26}$$

This is also the same as saying

$$hgfedcba \equiv ba + 2((2(f - c) + 6(d - g) + 5(h - e))) \pmod{26}$$

Is this true? Write a program to check. If it's true, prove it.

- (b) Can you conjecture a general formula? Can you prove it?

([Go to solution](#), page 66) □

**Exercise 1.3.25.**

debug:  
exercises/exercise-  
24/question.tex

- (a) Solve the following

$$9x + 5y \equiv 3 \pmod{26}$$

$$5x + 7y \equiv 1 \pmod{26}$$

First solving by writing a program that performs a brute force search for solutions. Next, try to solve it algebraically by hand.

- (b) What about this one:

$$3x - y \equiv 2 \pmod{26}$$

$$2x + 19y \equiv 14 \pmod{26}$$

- (c) And this one:

$$9x + y \equiv 2 \pmod{26}$$

$$19x + 5y \equiv 7 \pmod{26}$$

- (d) Write a program that solves

$$ax + b \equiv c \pmod{26}$$

$$dx + ey \equiv f \pmod{26}$$

for  $a, b, c, d, e, f$  in  $\mathbb{Z}/26$  by brute force search. Then write a program that randomly picks  $a, b, c, d, e, f$  in  $\mathbb{Z}/26$  and ask you to solve it. Print all the cases where  $a, b, c, d, e, f$  provides a linear system of two equations or two unknowns where there is no solution. Do you notice a pattern in these degenerate cases?

([Go to solution](#), page 67)

□

**Exercise 1.3.26.** Notice that earlier on, we wrote down multiplicative inverses of elements in  $\mathbb{Z}/26$ . Some elements do not have inverses. Can you tell if there's something common among them?

debug:  
exercises/exercise-  
25/question.tex

- (a) Compute the multiplicative inverses of elements in  $\mathbb{Z}/3$ .
- (b) Compute the multiplicative inverses of elements in  $\mathbb{Z}/5$ .
- (c) Compute the multiplicative inverses of elements in  $\mathbb{Z}/7$ .
- (d) Compute the multiplicative inverses of elements in  $\mathbb{Z}/11$ .
- (e) Do you notice something special about the above cases? How many elements have multiplicative inverse?

- (f) Compute the multiplicative inverses of elements in  $\mathbb{Z}/6$ .
- (g) Compute the multiplicative inverses of elements in  $\mathbb{Z}/10$ .
- (h) Compute the multiplicative inverses of elements in  $\mathbb{Z}/14$ .
- (i) Compute the multiplicative inverses of elements in  $\mathbb{Z}/15$ .
- (j) In the above 4 cases is there something special about values which are invertible? Don't see the pattern? In the above, the modulus are all products of two distinct primes.

([Go to solution](#), page 68)

□

**Exercise 1.3.27.** Suppose you want to write a random number generator. Say the numbers are to be in the range  $[0, 256)$ . (You can try a bigger range later – don't be too ambitious for now.) Of course use (reasonable) formula.

debug:  
exercises/exercise-  
26/question.tex

- (a) Go ahead and try this one:

$$h(n) = (2n + 5) \pmod{256}$$

Starting with a seed value of 1, the next value is  $h(1) = 7$ . And the value after that is  $h(7) = 19$ . And the next is  $h(19) = 43$ . What are all the possible values you can obtain using this  $h$ ?

- (b) Now of course you do want to have lots of values. For instance you might want to use this hash function to build a hashtable. Or a cryptographic hash (which we have not covered yet). But in any case, you hope that  $h$  covers *all* the values in  $[0, 255]$ . Was the function above a good function?
- (c) What if you use a different seed value? What if you start with 2? Or 3?
- (d) Try to find a function that generates as many values in  $[0, 255]$  as possible. Can you find one that covers the whole range of  $[0, 255]$ ?

([Go to solution](#), page 69)

□

## Solutions

Solution to Exercise [1.3.1](#).

debug:  
exercises/exercise-  
0/answer.tex

- (a) True:  $26 \mid 100 - 74 = 26$
- (b) False:  $26 \nmid -3 - 133 = -136 = -130 - 6 = (-6 \cdot 26) + 20$
- (c) False:  $26 \nmid -20 - 21 = -41 = (-2) \cdot 26 + 11$
- (d) False:  $3 \nmid 7 - 3 = 4 = 1 \cdot 3 + 1$
- (e) True:  $5 \mid 17 - (-3) = 4 \cdot 5$
- (f) False:  $8 \nmid 21 - 11 = 10 = 1 \cdot 8 + 2$
- (g) True:  $7 \mid 42 - 0 = 6 \cdot 7$

Solution to Exercise [1.3.2](#).

Solution not provided.

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Solution to Exercise [1.3.3](#).

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2/answer.tex



Solution to Exercise 1.3.4.

Addition table for  $\mathbb{Z}/26$ :

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3/answer.tex

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0
2	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1
3	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2
4	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3
5	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4
6	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4	5
7	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4	5	6
8	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4	5	6	7
9	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4	5	6	7	8
10	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4	5	6	7	8	9
11	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4	5	6	7	8	9	10
12	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4	5	6	7	8	9	10	11
13	13	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4	5	6	7	8	9	10	11	12
14	14	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4	5	6	7	8	9	10	11	12	13
15	15	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
16	16	17	18	19	20	21	22	23	24	25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
17	17	18	19	20	21	22	23	24	25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
18	18	19	20	21	22	23	24	25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
19	19	20	21	22	23	24	25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
20	20	21	22	23	24	25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
21	21	22	23	24	25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
22	22	23	24	25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
23	23	24	25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
24	24	25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
25	25	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24

Multiplication table for  $\mathbb{Z}/26$ :

×	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	0	2	4	6	8	10	12	14	16	18	20	22	24	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6	9	12	15	18	21	24	1	4	7	10	13	16	19	22	25	2	5	8	11	14	17	20	23
4	0	4	8	12	16	20	24	2	6	10	14	18	22	0	4	8	12	16	20	24	2	6	10	14	18	22
5	0	5	10	15	20	25	4	9	14	19	24	3	8	13	18	23	2	7	12	17	22	1	6	11	16	21
6	0	6	12	18	24	4	10	16	22	2	8	14	20	0	6	12	18	24	4	10	16	22	2	8	14	20
7	0	7	14	21	2	9	16	23	4	11	18	25	6	13	20	1	8	15	22	3	10	17	24	5	12	19
8	0	8	16	24	6	14	22	4	12	20	2	10	18	0	8	16	24	6	14	22	4	12	20	2	10	18
9	0	9	18	1	10	19	2	11	20	3	12	21	4	13	22	5	14	23	6	15	24	7	16	25	8	17
10	0	10	20	4	14	24	8	18	2	12	22	6	16	0	10	20	4	14	24	8	18	2	12	22	6	16
11	0	11	22	7	18	3	14	25	10	21	6	17	2	13	24	9	20	5	16	1	12	23	8	19	4	15
12	0	12	24	10	22	8	20	6	18	4	16	2	14	0	12	24	10	22	8	20	6	18	4	16	2	14
13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13
14	0	14	2	16	4	18	6	20	8	22	10	24	12	0	14	2	16	4	18	6	20	8	22	10	24	12
15	0	15	4	19	8	23	12	1	16	5	20	9	24	13	2	17	6	21	10	25	14	3	18	7	22	11
16	0	16	6	22	12	2	18	8	24	14	4	20	10	0	16	6	22	12	2	18	8	24	14	4	20	10
17	0	17	8	25	16	7	24	15	6	23	14	5	22	13	4	21	12	3	20	11	2	19	10	1	18	9
18	0	18	10	2	20	12	4	22	14	6	24	16	8	0	18	10	2	20	12	4	22	14	6	24	16	8
19	0	19	12	5	24	17	10	3	22	15	8	1	20	13	6	25	18	11	4	23	16	9	2	21	14	7
20	0	20	14	8	2	22	16	10	4	24	18	12	6	0	20	14	8	2	22	16	10	4	24	18	12	6
21	0	21	16	11	6	1	22	17	12	7	2	23	18	13	8	3	24	19	14	9	4	25	20	15	10	5
22	0	22	18	14	10	6	2	24	20	16	12	8	4	0	22	18	14	10	6	2	24	20	16	12	8	4
23	0	23	20	17	14	11	8	5	2	25	22	19	16	13	10	7	4	1	24	21	18	15	12	9	6	3
24	0	24	22	20	18	16	14	12	10	8	6	4	2	0	24	22	20	18	16	14	12	10	8	6	4	2
25	0	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

Negative of table for  $\mathbb{Z}/26$ :

$x$	$-x \pmod{26}$
0	0
1	25
2	24
3	23
4	22
5	21
6	20
7	19
8	18
9	17
10	16
11	15
12	14
13	13
14	12
15	11
16	10
17	9
18	8
19	7
20	6
21	5
22	4
23	3
24	2
25	1

Multiplicative inverse table for  $\mathbb{Z}/26$ :

$x \pmod{26}$	$x^{-1} \pmod{26}$
0	None
1	1
2	None
3	9
4	None
5	21
6	None
7	15
8	None
9	3
10	None
11	19
12	None
13	None
14	None
15	7
16	None
17	23
18	None
19	11
20	None
21	5
22	None
23	17
24	None
25	25

Solution to Exercise [1.3.5](#).

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Solution to Exercise [1.3.6](#).

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Solution to Exercise [1.3.7](#).

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Solution to Exercise [1.3.8](#).

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Solution to Exercise [1.3.9](#).

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Solution to Exercise [1.3.10](#).

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9/answer.tex

Solution to Exercise [1.3.11](#).

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10/answer.tex

Solution to Exercise 1.3.12.

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11/answer.tex

(a)  $0^2, 1^2, 2^2, 3^2 \equiv 0, 1, 4, 9 \equiv 0, 1, 0, 1 \pmod{4}$  (respectively). Therefore a square mod 4 is either 0 or 1 mod 4.

(b)  $0^2, 1^2, 2^2, 3^2, 4^2 \equiv 0, 1, 4, 9, 16 \equiv 0, 1, 4, 4, 1 \pmod{4}$  (respectively). Therefore a square mod 5 is either 0 or 1 or 4 mod 5.

(c)  $0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2 \equiv 0, 1, 4, 9, 16, 25, 36 \equiv 0, 1, 4, 2, 2, 4, 1 \pmod{4}$  (respectively). Therefore a square mod 7 is either 0 or 1 or 2 or 4 mod 7.

(c) Note that  $(n - i)^2 \equiv (0 - i)^2 \equiv (-i)^2 \equiv i^2 \pmod{n}$ . Therefore  $14^2 \equiv 1, 13^2 \equiv 2, \dots, 8^2 \equiv 7^2 \pmod{15}$ .  $0, 1, 4, 9, 16, 25, 36, 49 \equiv 0, 1, 4, 9, 11, 10, 6 \pmod{15}$  (respectively). Therefore a square mod 15 is 0, 1, 4, 6, 9, 10 or 11 mod 15.

(e) I claim that there are no integer solutions to the given equation. Assume on the contrary that  $(x, y)$  is an integer solution to

$$5x^2 - 8y^2 = 3$$

Therefore

$$\begin{aligned} 5x^2 - 8y^2 &\equiv 3 \pmod{4} \\ \therefore 1 \cdot x^2 - 0 \cdot 8y^2 &\equiv 3 \pmod{4} \\ \therefore x^2 &\equiv 3 \pmod{4} \end{aligned}$$

which is a contradiction since squares mod 4 are either 0 or 1 mod 4. Therefore  $5x^2 - 8y^2 = 3$  does not have any integer solution.

Solution to Exercise [1.3.13](#).

Solution not provided.

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Solution to Exercise [1.3.14](#).

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Solution to Exercise 1.3.15.

Of course  $(x, y, z) = (0, 0, 0)$  is a solution since, for instance, the polynomial is homogeneous. If any two of  $x, y, z$  are 0s, then the third must also be 0.

Now consider the case where exactly one of  $x, y, z$  is 0.

- (a) If  $x = 0$ , then  $4y^2 = 5z^2$ . We can assume that all common factors between  $y, z$  are removed so that  $\gcd(y, z) = 1$ . (Suppose  $g = \gcd(y, z)$ . Let  $y' = y/g$  and  $z' = z/g$ . We again would arrive at  $4y'^2 = 5z'^2$ .) Since  $5 \mid 5z^2$ , we have  $5 \mid 4y^2$ , and hence  $5 \mid y$ . Therefore  $y = 5y'$ . Hence  $4(5y')^2 = 5z^2$ , i.e.,  $4(5)y'^2 = z^2$ , which implies that  $5 \mid z$ . This contradicts  $\gcd(y, z) = 1$ .
- (b) If  $y = 0$ , then  $3x^2 = 5z^2$ . We can assume that all common factors between  $x, z$  are removed so that  $\gcd(x, z) = 1$ . Since  $3 \mid 3x^2$ , we obtain  $3 \mid 5z^2$ , which implies that  $3 \mid z$ . Therefore  $z = 3z'$  and hence  $3x^2 = 5(3z')^2$ . This implies that  $x^2 = 5(3)z'^2$ . Therefore  $3 \mid x^2$ , and hence  $3 \mid x$ . This contradicts  $\gcd(x, z) = 1$ .
- (c) If  $z = 0$ , then  $3x^2 + 4y^2 = 0$ . Since  $3 \mid 3x^2$ , we get  $3 \mid 4y^2$ , which implies that  $3 \mid y$ . Let  $y = 3y'$ . Then  $3x^2 + 4(3y')^2 = 0$  and hence  $x^2 = 4(3)y'^2$ . Therefore  $3 \mid x^2$  and hence  $3 \mid x$ . This contradicts  $\gcd(x, y) = 1$ .

Now suppose  $(x, y, z) \neq (0, 0, 0)$ .

Let  $x, y, z$  be a solution with  $x > 0, y > 0, z > 0$ . We may assume  $\gcd(x, y, z) = 1$ .

METHOD 1. Taking mod 3,

$$y^2 \equiv 2z^2 \pmod{3}$$

Squares in mod 3 are 0 or 1 mod 3. If  $z^2 \equiv 1 \pmod{3}$ , then  $y^2 \equiv 2 \pmod{3}$  which is impossible. Hence  $z^2 \equiv 0 \pmod{3}$  and therefore  $y^2 \equiv 0 \pmod{3}$ . This implies that  $3 \mid z^2$  and  $3 \mid y^2$ . Therefore

$$3 \mid y, \quad 3 \mid z$$

Let  $y = 3y'$  and  $z = 3z'$ . Then

$$3x^2 + 4(3y')^2 = 5(3z')^2$$

i.e.,

$$x^2 + 12y'^2 = 15z'^2$$

which implies that  $3 \mid x$ . This is a contradiction since  $\gcd(x, y, z) = 1$ .

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14/answer.tex

METHOD 2. Taking mod 4, we get

$$3x^2 \equiv z^2 \pmod{4}$$

Then  $2 \mid x$  and  $2 \mid z$ . Let  $x = 2x'$  and  $z = 2z'$ . Then

$$3(2x')^2 + 4y^2 = 5(2z')^2$$

i.e.,

$$3x'^2 + y^2 = 5z'^2$$

Taking mod 4,

$$3x'^2 + y^2 \equiv z'^2 \pmod{4}$$

Then

- Assume  $z'$  is even. Then either  $x', y$  are even or  $x'^2 \equiv 1 \equiv y^2 \pmod{4}$ . If  $x', y$  are even and  $z'$  is also even, then  $\gcd(x, y, z) \neq 1$  which is a contradiction. For the other case

$$3(4m+1)^2 + (4n+1)^2 = (2z'')^2$$

i.e.,

$$3(16m^2 + 8m + 1) + (16n^2 + 8n + 1) = 4z''^2$$

i.e.,

$$12m^2 + 6m + 4n^2 + 2n + 1 = z''^2$$

All we can say is  $z''$  is odd. There's no clear path forward.

- Assume  $z'^2 \equiv 1 \pmod{4}$ . Then  $2 \mid x'$  and  $y'^2 \equiv 1 \pmod{4}$ .

There are more cases to consider and this seems to be a bad direction to take.

METHOD 3. Taking mod 5, we get

$$3x^2 + 4y^2 \equiv 0 \pmod{5}$$

i.e.

$$3x^2 \equiv y^2 \pmod{5}$$

Squares in mod 5 are  $0, 1, 4 \pmod{5}$ . If  $x^2 \equiv 1 \pmod{5}$ , then  $y^2 \equiv 3x^2 \equiv 3 \pmod{5}$  which is impossible. If  $x^2 \equiv 4 \pmod{5}$ , then  $y^2 \equiv 3x^2 \equiv 12 \equiv 2 \pmod{5}$  which is again impossible. Hence  $x^2 \equiv 0 \pmod{5}$  and hence  $y^2 \equiv 0 \pmod{5}$ . Altogether we have

$$5 \mid x^2, \quad 5 \mid y^2$$



and therefore

$$5 \mid x, \quad 5 \mid y$$

Let  $x = 5x'$  and  $y = 5y'$ . Then we have

$$3(5x')^2 + 4(5y')^2 = 5z^2$$

i.e.,

$$3(5)x'^2 + 4(5)y'^2 = z^2$$

which implies that  $5 \mid z^2$ . Therefore  $5 \mid z$ . This contradicts  $\gcd(x, y, z) = 1$ .

□

NOTES.

- What about  $9x^2 + 4y^2 = 5z^2$ ? How far can you go using the method above?
- What about  $3x^2 + 4y^2 = 25z^2$ ? How far can you go using the method above?
- What about  $9x^2 + 4y^2 = 25z^2$ ?
- What about  $3x^2 + 2y^2 = 5z^2$ ?
- What about  $3x^3 + 4y^3 = 5z^4$ ?

Solution to Exercise [1.3.16](#).

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15/answer.tex

Solution to Exercise [1.3.17](#).

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16/answer.tex

Solution to Exercise [1.3.18](#).

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Solution to Exercise [1.3.19](#).

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Solution to Exercise [1.3.20](#).

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Solution to Exercise [1.5.1](#).

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Solution to Exercise [1.5.2](#).

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Solution to Exercise [1.3.23](#).

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Solution to Exercise [1.3.24](#).

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Solution to Exercise [1.3.25](#).

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Solution to Exercise [1.3.26](#).

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Solution to Exercise [1.3.27](#).

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## 1.4 Attacks debug: classical-cipher-attacks.tex

OK. Now it's time to attack some of the classical cryptosystems. Before we do that we want to know exactly what is it we want to achieve.

Recall that a cryptosystems is made up of the encryption and decryption function  $E$  and  $D$ .

Suppose Alice wants to send a message  $m$  to Bob. She encrypts the message to get  $E_k(m)$  and delivers  $E_k(m)$  to Bob. Bob decrypts  $E_k(m)$  by applying  $D_k$  and get  $D_k(E_k(m)) = m$ .

Now let's think about the rogue agent Eve. What does she want?

The most common mode of attack assumes Eve receives a copy of  $E_k(m)$  and she wants to derive  $m$ . It's even better if she can derive  $k$  because in that case she actually has  $D_k$  and hence can decrypt all future ciphertexts. (Don't forget we assume that the encryption and decryption algorithm is known. That means if Eve has  $k$ , she has  $D_k$  and in fact also  $E_k$ . Only the key(s) is secret). Not only that. If she has  $k$ , she also has  $E_k$  and therefore she can actually impersonate Alice!

The various assumptions of what Eve has are called **attack models** or **attack modes** Here are some standard ones.

attack models  
attack modes

- (a) The **known ciphertext attack** is an attack where the ciphertext of a plaintext is available to the attacker. In other words, this attack involves computing  $k$  from  $E_k(m)$ :

known ciphertext  
attack

$$E_k(m) \mapsto k$$

This include the case when there is more than one ciphertext.

- (b) The **known plaintext attack** is an attack where the Eve has some messages and their ciphertext. So her goal is this:

known plaintext  
attack

$$m_1, m_2, \dots, m_n, E_k(m_1), E_k(m_2), \dots, E_k(m_n) \mapsto k$$

This does happen. For instance the breaking of Germany's Enigma ciphers during WWII is based on this assumption.

- (c) The **chosen plaintext attack** is an attack where the Eve can actually encrypt messages/plaintexts that she chooses. Make sure you see that this is different from known plaintext.
- (d) The **chosen ciphertext attack** is an attack where the Eve has access to the decryptions of ciphertexts that she chooses. Her goal is to compute

chosen plaintext  
attack

chosen ciphertext  
attack

the key. Make sure you see that this is different from known plaintext. For instance suppose Eve chooses (a meaningless) ciphertext  $x$  and sends it to the decryption machine. Suppose the decryption of  $x$  is  $y$ . It's very likely that  $y$  is totally meaningless. Bob then calls Alice and ask her why she sends the gibberish  $y$  message. On listening to their phone conversation, she now knows that

$$D_k(x) = y$$

Eve then attempts to find  $k$ .

## 1.5 Attacking the shift cipher debug: classical-cipher-attacking-shift-cipher.tex

Recall that for the shift cipher, we have

$$\begin{aligned}E_k(x) &\equiv (x + k) \pmod{26} \\D_k(x) &\equiv (x - k) \pmod{26}\end{aligned}$$

Since  $0 \leq k \leq 25$ , there are not many possible values for  $k$ . So you can try to apply *all* the possible decryptions  $D_0, D_1, D_2, \dots, D_{25}$ . Assuming the message is something Eve can read, all that is required is that Eve has enough time to read 26 messages. Easy! Since Eve is trying all keys, this is a brute force attack.

But in fact Eve can do better, by using a heuristic approach based on probability. If the message is long enough, based on letter frequencies, heuristically, she can try to tabulate the frequencies of all the characters and then assume the most common occurring character is decrypted as **e** which is statistically the most commonly occurring letter in English. Of course knowing how to decrypt to **e** is sufficient for Eve to decrypt all the letters right?

Too bad if the  
message is in Russian.



The following is a table of probabilities for each letter used in English.

Letter	Probability
e	0.127
t	0.091
a	0.082
o	0.075
i	0.070
n	0.067
s	0.063
h	0.061
r	0.060
d	0.043
l	0.040
c	0.028
u	0.028
m	0.024
w	0.023
f	0.022
g	0.020
y	0.020
p	0.019
b	0.015
v	0.010
k	0.008
j	0.002
x	0.001
q	0.001
z	0.001

Of course these are probabilities. It does not mean that the second most frequently occurring letter *must* be *t*! I've divided up the probabilities into groups according to jumps in the values.

It is also useful to know that besides commonly occurring letters, which *pairs* of letters occurring frequently next to each other. These are called **digrams** (or 2-grams). For three, they are called **trigrams** (or 3-grams). The following is a table of commonly occurring digrams and trigrams listed in decreasing order of frequencies:

digrams  
trigrams

$n$	$n$ -grams (in decreasing order)
2	th he in er an re ed on es st en at to nt ha nd ou ea ng as or ti is et it ar te se hi of
3	the ing and her ere ent tha nth was eth for dth

Here are the frequencies of the 2-grams (of course not all 2-grams, just the top few):

2-gram	Probability
th	0.0271
he	0.0233
in	0.0203
er	0.0178
an	0.0161
re	0.0141
es	0.0132
on	0.0132
st	0.0125
nt	0.0117
en	0.0113
at	0.0112
ed	0.0108
nd	0.0107
to	0.0107
or	0.0106
ea	0.0100
ti	0.0099
ar	0.0098
te	0.0098
ng	0.0089
al	0.0088
it	0.0088
as	0.0087
is	0.0086
ha	0.0083
et	0.0076
se	0.0073
ou	0.0072
of	0.0071

and the 3-grams:

3-gram	Probability
the	0.0181
and	0.0073
ing	0.0072
ent	0.0042
ion	0.0042
her	0.0036
for	0.0034
tha	0.0033
nth	0.0033
int	0.0032
ere	0.0031
tio	0.0031
ter	0.0030
est	0.0028
ers	0.0028
ati	0.0026
hat	0.0026
ate	0.0025
all	0.0025
eth	0.0024
hes	0.0024
ver	0.0024
his	0.0024
oft	0.0022
ith	0.0021
fth	0.0021
sth	0.0021
oth	0.0021
res	0.0021
ont	0.0020

and the 4-grams:

4-gram	Probability
tion	0.31
nthe	0.27
ther	0.24
that	0.21
ofth	0.19
fthe	0.19
thes	0.18
with	0.18
inth	0.17
atio	0.17
othe	0.16
tthe	0.16
dthe	0.15
ingt	0.15
ethe	0.15
sand	0.14
sthe	0.14
here	0.13
thec	0.13
ment	0.12
them	0.12
rthe	0.12
thep	0.11
from	0.10
this	0.10
ting	0.10
thei	0.10
ngth	0.10
ions	0.10
andt	0.10

So a slight improve to brute force search of trying  $k = 0, 1, 2, \dots, 25$ , is to try encrypt **e** to the most commonly occurring letter, the second, the third, etc.

(Some authors use  $M$  for their set of plaintexts instead of  $P$ . In that case they might call their plaintexts messages.)

### Exercise 1.5.1.

debug:  
exercises/exercise-  
20/question.tex

- (a) What is  $10 \pmod{11}$ ? (when simplified).
- (b) What is  $100 \pmod{11}$ ? (when simplified).
- (c) What is  $1000 \pmod{11}$ ? (when simplified).
- (d) What is  $9142 \pmod{11}$ ? (when simplified).
- (e) What is  $80432440556787 \pmod{11}$ ? (when simplified).
- (f) What is the general fact here?

([Go to solution](#), page 78)

□

**Exercise 1.5.2.** Although the algebraic manipulations on  $\text{mod } N$  seems to be very similar to the algebraic manipulations in  $\mathbb{Z}$  (or even  $\mathbb{R}$ ), certain “bizarre” things do happen. In  $\mathbb{Z}$  (in fact in  $\mathbb{Q}$  and  $\mathbb{R}$  as well), you have the implication

debug:  
exercises/exercise-  
21/question.tex

$$xy = 0 \implies x = 0 \text{ or } y = 0$$

Is it true that

$$xy \equiv 0 \pmod{26} \implies x \equiv 0 \pmod{26} \text{ or } y \equiv 0 \pmod{26}$$

For each  $N$ , check when

$$xy \equiv 0 \pmod{N} \implies x \equiv 0 \pmod{N} \text{ or } y \equiv 0 \pmod{N}$$

holds. And when it does not, give one counterexample. Do you see a pattern?

([Go to solution](#), page 79)

□

## Solutions

Solution to Exercise [1.5.1](#).

Solution not provided.

debug:  
exercises/exercise-  
20/answer.tex

Solution to Exercise [1.5.2](#).

Solution not provided.

debug:  
exercises/exercise-  
21/answer.tex

## 1.6 Affine cipher debug: classical-cipher-affine-cipher.tex

Suppose we use  $\mathbb{Z}/26$  instead of 'a' to 'z' again. Recall that the shift cipher is

$$E(k, x) = (x + k) \pmod{26}$$

and

$$D(k, x) = (x - k) \pmod{26}$$

The benefit of translating our encryption/decryption “shift up” and “shift down” into mathematical operations in  $\mathbb{Z}/26$ , is that we can now generalize and use different mathematical formulas!

Here's the affine cipher. The encryption algorithm for the affine cipher looks like this:

$$E((a, b), x) = (ax + b) \pmod{26}$$

Note that the key is not one single number – the key is  $(a, b) \pmod{26}$  which is made up of two numbers. Why is that important?

Because this means that there are more key values! Which means that Eve has to try more key!!!! Get it??

But what is the decryption function? Of course we know that

$$D((a, b), E((a, b), x)) = x \pmod{26}$$

This means that

$$D((a, b), ax + b) = x \pmod{26}$$

Suppose I make a guess ... When I look at the shift cipher, I notice that the decryption function is similar in form to the encryption function. Maybe the decryption function for the affine cipher is similar in form to the encryption function???

Let's try

$$D((a, b), x) = cx + d \pmod{26}$$

In that case, from

$$D((a, b), ax + b) \equiv x \pmod{26}$$

we would get

$$c(ax + b) + d \equiv x \pmod{26}$$

which gives us

$$cax + cb + d \equiv x \pmod{26}$$



Now what? Well maybe

$$cax \equiv x \pmod{26}$$

and

$$cb + d \equiv 0 \pmod{26}$$

The first condition

$$cax \equiv x \pmod{26}$$

is achieved if we have

$$ca \equiv 1 \pmod{26}$$

Can this be done?

**Exercise 1.6.1.** Suppose  $a = 1$ . What  $c$ 's would make  $ca \equiv 1 \pmod{26}$ ? What if  $a = 2$ ? What about  $a = 3$ ? Etc. ([Go to solution](#), page 85)  $\square$

debug:  
exercises/exercise-  
30/question.tex

Given an integer  $a$ , if  $c$  satisfies

$$ca \equiv 1 \pmod{26}$$

we say that  $c$  is a **multiplicative inverse** of  $a \pmod{26}$ . We usually write  $c$  as  $a^{-1} \pmod{26}$ . Note that  $a^{-1} \pmod{26}$  is NOT a fraction!!! It's a whole number. If  $a$  has a multiplicative inverse mod 26, we say that  $a$  is **invertible** mod 26.

multiplicative inverse

invertible

Recall from a previous section, you already have a table of  $a^{-1} \pmod{26}$  for all  $a$ 's in mod 26.

**Exercise 1.6.2.** How many integers 0, 1, 2, ..., 25 have multiplicative inverses mod 26? ([Go to solution](#), page 86)  $\square$

debug:  
exercises/exercise-  
31/question.tex

Therefore to satisfy

$$ca \equiv 1 \pmod{26}$$

we can't just pick any  $a$ . We have to pick an  $a$  with a multiplicative inverse mod 26.

After we have chosen a good  $a$ , what do we do? We then have

$$D((a, b), x) = cx + d \pmod{26}$$

where  $c$  is the multiplicative inverse of  $a \pmod{26}$ . But what about  $d$ ???

Remember we still have the condition

$$cb + d \equiv 0 \pmod{26}$$

Writing  $a^{-1} \pmod{26}$  for  $c$ , we get

$$a^{-1}b + d \equiv 0 \pmod{26}$$

we get

$$d \equiv -a^{-1}b \pmod{26}$$

Therefore we have the following: The affine cipher is

$$E((a, b), x) = (ax + b) \pmod{26}$$

where  $a$  is invertible mod 26 and

$$\begin{aligned} D((a, b), x) &\equiv a^{-1}x - a^{-1}b \pmod{26} \\ &\equiv a^{-1}(x - b) \pmod{26} \end{aligned}$$

**Exercise 1.6.3.** Write down the (simplified) encryption and decryption function for the affine cipher when the key is  $(3, 12)$ . Encrypt `gollum` and then decrypt to check that you get back `gollum`. ([Go to solution](#), page 87)  $\square$

debug:  
exercises/exercise-  
32/question.tex

The above however uses a lot of “iffy” math. For instance we used the fact

$$a(b + c) \equiv ab + ac \pmod{26}$$

(where?) We also use the fact that if

$$a + b \equiv 0 \pmod{26}$$

then

$$a \equiv -b \pmod{26}$$

We seem to be treated math in mod 26 like math in  $\mathbb{Z}$  and  $\mathbb{R}$ !!! Is that justifiable? It turns out that the above algebra is actually correct. I’ll have to come back to that later otherwise people will think we are rambling nonsense and making things up.

**Exercise 1.6.4.** What is the size of the key space for the affine cipher? In the worse case how many tries must Eve attempt before breaking an affine cipher?

debug:  
exercises/exercise-  
33/question.tex

Compare this with the shift cipher. ([Go to solution](#), page 88) □

Now let's look at attacking the affine cipher.

Recall the encryption and decryption of the affine cipher looks like

$$E_{a,b}(x) \equiv (ax + b) \pmod{26}, \quad D_{a,b}(x) \equiv a^{-1}(x - b) \pmod{26}$$

Note that the key is  $(a, b)$ . Note also that  $a$  must be invertible mod 26.

Therefore (by the multiplication principle in discrete mathematics), the total numbers of keys is

$$\phi(26) \cdot 26 = 312$$

This is not that big, but it's definitely bigger than the number of keys for the shift cipher (which is 26). This means that to carry out a brute force attack on an affine cipher, assume the attacker has the cipher, he/she must try 312 possible keys.

**Exercise 1.6.5.** This is easy: Show that  $\phi(26) \cdot 26 = 312$ . ([Go to solution](#), page 89) □

debug:  
exercises/affine-  
0/question.tex

Again you can do a brute force search for  $a, b$ . After all there are not that many possibilities for  $a$  and  $b$ . But we can do better if we use letter (1-gram) frequencies again. Again suppose you have computed the frequencies of the letters of the ciphertext and say that **g** is the most common letter. So you assume **e** is encrypted as **g**. This is the same as saying 4 is encrypted as 6, i.e.,

$$E_{a,b}(4) = 6$$

right? Now using the formula for  $E_{a,b}$  we get

$$4a + b = 6$$

To be accurate the equation should be

$$4a + b \equiv 6 \pmod{26}$$

Now suppose the second most common letter in the ciphertext is **y**. So you assume that **t** is encrypted as **y**. This means

$$20a + b \equiv 24 \pmod{26}$$

Right? Yes, no? Think about it. So you can solve for  $a$  and  $b$  from the linear

equations

$$\begin{aligned} a + b &\equiv 6 \pmod{26} \\ 20a + b &\equiv 24 \pmod{26} \end{aligned}$$

**Exercise 1.6.6.** Solve the above for  $a$  and  $b$ . ([Go to solution](#), page 90)  $\square$

debug:  
exercises/affine-  
1/question.tex

**Exercise 1.6.7.** Now let's abstract the above. Suppose  $A, B, C, D$  are numbers and  $a$  and  $b$  satisfy the equations

debug:  
exercises/affine-  
2/question.tex

$$\begin{aligned} Aa + b &= B \\ Ca + b &= D \end{aligned}$$

Solve for  $a$  and  $b$  in terms for  $A, B, C, D$ . If this can be done, then of course you can write a C++ function to return  $a$  and  $b$  directly. ([Go to solution](#), page 91)  $\square$

**Exercise 1.6.8.** But wait!  $a$  is not arbitrary! Remember that  $a$  must be invertible in mod 26. Recall that this means there is some integer  $c$  such that  $ac \equiv 1 \pmod{26}$ . Therefore it would be helpful if you have a function that will determine if  $a$  is invertible mod 26. How would you do that? ([Go to solution](#), page 92)  $\square$

debug:  
exercises/affine-  
3/question.tex

## Solutions

Solution to Exercise [1.6.1](#).

Solution not provided.

debug:  
exercises/exercise-  
30/answer.tex

Solution to Exercise [1.6.2](#).

Solution not provided.

debug:  
exercises/exercise-  
31/answer.tex

Solution to Exercise [1.6.3](#).

Solution not provided.

debug:  
exercises/exercise-  
32/answer.tex

Solution to Exercise [1.6.4](#).

Solution not provided.

debug:  
exercises/exercise-  
33/answer.tex



Solution to Exercise [1.6.5](#).

Solution not provided.

debug:  
exercises/affine-  
0/answer.tex

Solution to Exercise [1.6.6](#).

Solution not provided.

debug:  
exercises/affine-  
1/answer.tex

Solution to Exercise [1.6.7](#).

Solution not provided.

debug:  
exercises/affine-  
2/answer.tex

Solution to Exercise [1.6.8](#).

Solution not provided.

debug:  
exercises/affine-  
3/answer.tex

## 1.7 Vigenère cipher debug: classical-cipher-vigenere-cipher.tex

The shift, affine, substitution ciphers are called **monoalphabetic** ciphers – monoalphabetic each letter can be mapped to only one letter.

A **polyalphabetic** ciphers is the opposite of monoalphabetic ciphers. polyalphabetic

The Vigenere cipher is basically a collection of shifts. Here's an example.

My key is going to be the word **fun**. Note that when I translate **fun** to numbers (with  $a \rightarrow 0, b \rightarrow 1, c \rightarrow 2, \dots$ ) I get 5, 20, 13. So all I need to do is to encrypt the characters by doing

- shift by 5
- shift by 20
- shift by 13
- shift by 5
- shift by 20
- shift by 13
- ...

For instance suppose the plaintext is

It's a dangerous business, Frodo, going out your door.

I change everything to lowercase and throwing away non a-z and our plaintext  $x$  is:

itsadangerousbusinessfrodogoingoutyourdoor

Suppose the encrypted text is

$$y = y_1 y_2 y_3 y_4 y_5 y_6 \dots$$

Then

- $y_1 = \text{shift } i \text{ by } 5 = n$
- $y_2 = \text{shift } t \text{ by } 20 = n$
- $y_3 = \text{shift } s \text{ by } 13 = f$
- $y_4 = \text{shift } a \text{ by } 5 = f$
- $y_5 = \text{shift } d \text{ by } 20 = x$
- $y_6 = \text{shift } a \text{ by } 13 = n$
- $y_4 = \text{shift } n \text{ by } 5 = s$
- $y_5 = \text{shift } g \text{ by } 20 = a$
- $y_6 = \text{shift } e \text{ by } 13 = r$
- ...

i.e.,

$$y = \text{nnffxnsar...}$$

**Exercise 1.7.1.** What is the size of the set of all possible keys for the Vigenere cipher? Is this better than the shift cipher? ([Go to solution](#), page 107)  $\square$

debug:  
exercises/vigenere-  
0/question.tex

Now let's attack the Vignere cipher.

Suppose the length of the key is  $m$ . Then you can think of your message as being chopped up into  $m$  strings and each is encrypted by the shift of a character and then everything is put together. For instance if the ciphertext is

$$y = y_1 y_2 y_3 \dots$$

and the key is `fun`, i.e.  $m$  is 3, then the 3 strings are

$$z_1 = y_1 y_4 y_7 \dots$$

$$z_2 = y_2 y_5 y_8 \dots$$

$$z_3 = y_3 y_6 y_9 \dots$$

Now, the above of course is the encryption process and you know that the key has length 3, therefore you are three different shifts. If you do know the key has length 3, then each of the above  $z_i$  is encrypted using the same shift and therefore you can use frequency analysis to decrypt each of them separately.

The problem is when you do *not* have the length of the key. So the first step is always to figure out the length of the key, i.e.  $m$ .

**METHOD 1.** Here's a strategy. Look at the substrings of the ciphertext of length 3. Think about it. Suppose you see the following in your ciphertext:

...etu...etu...

What can you do? One might guess that if the distance between the first `e` to the next is  $d$ , then  $m$  divides  $d$  right? Think about it. So you need to scan for all possible substrings of length 3 and look for such distances. Suppose you have a bunch of such distances  $d_1, d_2, \dots, d_k$ . Then  $m$  must divide all these  $d_i$ 's. This means that  $m$  divides  $\gcd(d_1, d_2, \dots, d_k)$ . Obviously the more  $d_i$ 's you have the better.

**Exercise 1.7.2.** Write a function (in your favorite programming language). It returns a hashtable of frequencies of strings of length 1. ([Go to solution](#),

debug:  
exercises/vigenere-  
1/question.tex

page 108)

□

**Exercise 1.7.3.** Generalize the above: Write a function (in your favorite programming language) that accepts a string  $s$  and returns a hashtable of where the key-value pair  $(k, v)$  has a string length  $k$  for key and the value  $v$  is the count of occurrences of  $k$  in  $s$ . (Go to solution, page 109) □

debug:  
exercises/vigenere-  
2/question.tex

**Exercise 1.7.4.** Now modify the above: Write a function (in your favorite programming language) that accepts a string  $s$  and returns a hashtable of where the key-value pair  $(k, v)$  has a string length  $k$  for key and the value  $v$  is a list of index positions in  $s$  where the  $k$  was found. (Go to solution, page 110) □

debug:  
exercises/vigenere-  
3/question.tex

**Exercise 1.7.5.** Using the above, write a function that computes a probabilistic guess on the length of the key when given a string encrypted using the Vigenere cipher. (Go to solution, page 111) □

debug:  
exercises/vigenere-  
4/question.tex

Once you have a probabilistic guess of the length, say  $m$ , you break up the encrypted string into  $m$  pieces. Each piece is a shift ciphertext. So you just use whatever we talked about in the sections on shift cipher.

Done!

**METHOD 2.** Here's another technique. It involves some simple counting from discrete math. Suppose again the length of the key is  $m$  and you cut up your string into  $m$  pieces so that the characters in each substring is shifted by the same amount. Clearly the probabilities of the letters for each substring is preserved. In other words if **e** is encrypted as **g** for the first string and to **n** for the second, etc., then the probability of **g** in the first string and of **n** in the second string must be approximately the same as the probability of **e**. On the other hand, if we assumed that the length wrongly, say  $m + 1$ , then the probabilities must be different. As a matter of fact the substrings would appear very random. OK. So let's do some math.

Suppose the frequencies of the letters in a string  $s$  are  $f_0, f_1, \dots, f_{25}$ , i.e.,  $f_0$  is the frequency of **a**, etc. Suppose the length of the  $s$  is  $n$ . Therefore

$$\sum_{i=0}^{25} f_i = n$$

In this string  $s$ , the probability  $p_0$  that a randomly chosen character of  $s$  is

the character `a` is just  $f_0/n$ . Etc. In the following I might write  $f(0)$  instead of  $f_0$ . So in the following discussion if you see  $p_0$  you can think of  $f_0/n$ .

In python or C++, for statistical computation of frequencies of characters (or substrings), you can use a hashtable. (Details in assignment.) For instance in python you can do this. A hashtable in python is called a python dictionary. In C++ a hashtable is called an unordered map. Try this for python:

```
s = "tobeornottobethatisthequestionandtheanswerisfortytwo"
f = {}
for c in s:
    if c not in f:
        f[c] = 1
    else:
        f[c] += 1
for c,count in f:
    print(c, count)
```

or

```
s = "tobeornottobethatisthequestionandtheanswerisfortytwo"
f = {}
for c in s:
    f[c] = f.get(c, 0) + 1
for c,count in f.items():
    print(c, count)
```

Make sure you run the above. (For those of you who have taken CISS350, check my notes on hashtables and review `std::unordered_map`.) A lot more information on python dictionary is found in the assignment. Moving on ...

Now the probability of two randomly chosen characters being the same must be

$$I(s) = \frac{\binom{f_0}{2} + \dots + \binom{f_{25}}{2}}{\binom{n}{2}} = \frac{\sum_{i=0}^{25} f_i(f_i - 1)}{n(n - 1)}$$

$\binom{n}{k}$  is the  $n$ -choose- $k$  binomial coefficient, i.e.  $n!/((n - k)!k!)$ . In particular  $\binom{n}{2} = n(n - 1)/2$ .

One thing nice about the above formula is that if any substitution has been applied to the string, *the above value remains the same*.

**Exercise 1.7.6.** Approximate  $I(s) = \frac{\sum_{i=0}^{25} f_i(f_i - 1)}{n(n - 1)}$  where  $p_i = f_i/n$ 's are the probabilities taken from our table of letter probabilities. (Go to solution,

debug:  
exercises/vigenere-  
5/question.tex



page [112](#))

□

So  $I(s)$  is approximately  $\sum_{i=0}^{25} p_i^2$  where  $p_i$  is the probability of a character of the string being the  $i$ -th letter.

On the other hand if there is no pattern and everything is random, for instance you cut up your ciphertext, which was encrypted by a substitution with a string of length  $m$ , into  $m + 1$  substrings, then the probabilities of the characters would be almost the same since the string is gibberish. In other words, the probabilities would be about  $f_i/n = 1/26$ .

**Exercise 1.7.7.** Suppose  $s$  is a random string, i.e.  $f_i/n = 1/26$ . Compute  $I(s) = \frac{\sum_{i=0}^{25} f_i(f_i-1)}{n(n-1)}$ .

debug:  
exercises/vigener-  
6/question.tex

([Go to solution](#), page [113](#))

□

**Exercise 1.7.8.** Write a function `I` that accepts a string of lowercase letters or a list of numbers 0..25 and computes the  $I$ -value as described above. ([Go to solution](#), page [114](#))

debug:  
exercises/vigener-  
7/question.tex

□

So the question is this: Is the  $I$  value sufficiently different for a meaningful string and a random string? Is so, then we have an algorithm for determining  $m$ :

Test  $m = 1$ : You get one piece (the complete string) from the ciphertext  $y$ . Compute  $I(y)$ . If  $I(y)$  is approximately 0.065,  $m = 1$ .

Test  $m = 2$ : Cut up your ciphertext  $y$  into two pieces  $z_1, z_2$ . Compute  $I(z_1)$ ,  $I(z_2)$ . If they are approximately 0.065 (you can use the average of  $I(z_1)$  and  $I(z_2)$ ), then  $m = 2$ .

Test  $m = 3$ : Cut up your ciphertext  $y$  into three pieces  $z_1, z_2, z_3$ . Compute  $I(z_1)$ ,  $I(z_2)$ ,  $I(z_3)$ . If they are approximately 0.065 (you can use the average of all three), then  $m = 3$ .

Etc.

So now we know how to compute the length of the key. Suppose the plaintext is  $x$  and the  $m$  pieces of  $x$  are  $x_1, x_2, x_3, \dots, x_m$ . Recall  $y$  is the ciphertext and the  $m$  pieces of  $y$  are  $z_1, z_2, z_3, \dots, z_m$ . Suppose the shifts are by  $k_1, k_2, \dots, k_m$ . So  $z_1$  is a shift of  $x_1$  by  $k_1$ ,  $z_2$  is a shift of  $x_2$  by  $k_2$ , etc.

Suppose  $p_0$  is the probability of **a**,  $p_1$  is the probability of **b**, etc. in unencrypted English. Look at  $y_1$ . If you look at the probability of **a** in  $z_1$ , then you're really looking at the  $p_\alpha$  where  $\alpha$  is encrypted as **a**. For instance suppose the shift is by 3. In other words **x** is encrypted as **a**. So the probability of **a** in  $z_1$  is the same as the probability of **x** which is  $p_{26-3}$ . For simplicity, from now on all the indices will be considered mod 26. So when I write  $p_{-3}$  I really mean  $p_{26-3}$ . In general, if the shift is  $k$ , then the probability must be  $p_{-k}$ . Similarly, the probability of **b** in  $y_1$  must be  $p_{1-k}$ . In general, the probability of the  $i$ -th character in  $z_1$  must be  $p_{i-k}$ .

Now we make the following observation: What is the probability that a randomly chosen character from  $z_1$  and a randomly chosen character from  $z_2$  are both **a**? It is approximately

$$M(y_i, y_j) = \sum_{\ell=0}^{25} p_{\ell-k_1} p_{\ell-k_2}$$

For any  $z_i$  and  $z_j$ , this is then

$$\sum_{\ell=0}^{25} p_{\ell-k_i} p_{\ell-k_j}$$

You now notice that the above is the same as

$$\sum_{\ell=0}^{25} p_{\ell} p_{\ell-k_j+k_i}$$

or

$$\sum_{\ell=0}^{25} p_{\ell} p_{\ell+k_i-k_j}$$

The number  $k_i - k_j$  is called the **relative shift** of  $z_i$  and  $z_j$ .

relative shift

**Exercise 1.7.9.** Using the probabilities from our table of letter probabilities,

debug:  
exercises/vigenere-  
8/question.tex

complete the following table:

Relative shift	$M$
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	

([Go to solution](#), page 115)

□

You notice that when there is no shift, the  $M$  value is approximately 0.065. Otherwise it is much smaller, around 0.04. So this what you can do. Take  $z_1$  and  $z_2$  as examples. Compute the  $M$  value for  $z_1$  and  $z_2$ . Next you shift  $z_2$  by 1 to get  $E_1(z_2)$  ( $E_1$  is the encryption for the shift cipher). Now compute the  $M$  value for  $y_1$  and  $E_1(y_2)$ , i.e.  $M(y_1, E_1(y_2))$ . Next you compute  $M(y_1, E_2(y_2))$ . Continue until you have the  $M$  value of  $y_1$  and  $E_{25}(y_2)$ . If the  $M$  value for  $y_1$  and  $E_{17}(y_2)$  is approximately 0.065, then the relative shift is very likely 17. In other words

$$k_1 - k_2 = 17$$

Now perform the same for different pairs of  $y_i, y_j$ . You have  $\binom{m}{2}$  pairs and you hope to get as many equations as possible.

You should be able to write your shifts in terms of one shift, say the first  $k_1$ . For instance the second shift might be  $k_1 + 3$ , the third is  $k_1 + 2$ , etc. You still have the unknown  $k_1$ .

But that's only one unknown!!!

You can then easily solve it with 26 different values for  $k_1$ . Alternatively, for a Vigenere cipher, the shifts come from a sequence of letters and the letters usually make up a meaningful word. You can scan for meaningful words using your sequence of shifts. For instance if the shifts are  $(k_1, k_1 + 17, k_1 + 4, k_1 +$

21,  $k_1 + 10$ ), then when  $k_1 = 9$  you get the word **janet**.

Notice that the cryptanalysis as above already hints at the fact that algebra and probability theory are extremely important in this area of study. In particular, there is an area of study called information theory that studies the amount of randomness or lack of information in random variables encoded within something called the entropy of random variables. Using information theory one can prove that a substitution ciphertext has a unique decryption if the string has a length of at least 25.

In case the above proof hurts your head too much, let me give you an informal proof.

Suppose I have a sequence of numbers:

$$[0.1, 0.2, 0.3, 0.4]$$

Think of these as the probabilities of a language that involves only 4 letters, say the probabilities of the symbols  $a, b, c, d$ . Now I rotate the numbers in a circle by 1 step to get this:

$$[0.4, 0.1, 0.2, 0.3]$$

and by 3 steps to get this:

$$[0.2, 0.3, 0.4, 0.1]$$

These corresponds to a shift cipher of 1 and of 3. The relative shift between the 1st and 2nd is 2 steps: you can see that 0.1 is 2 steps away between the two sequences. One way to compute the relative shift (i.e. 2) is to compute the sum of products of the corresponding terms:

- Relative shift the second by 0:

$$\begin{array}{c} [0.4, 0.1, 0.2, 0.3] \\ [0.2, 0.3, 0.4, 0.1] \end{array}$$

and compute:

$$0.4 \times 0.2 + 0.1 \times 0.3 + 0.2 \times 0.4 + 0.3 \times 0.1 = 0.2200...$$

- Relative shift the second by -1:

$$[0.4, 0.1, 0.2, 0.3]$$

$$[0.3, 0.4, 0.1, 0.2]$$

and compute:

$$0.4 \times 0.3 + 0.1 \times 0.4 + 0.2 \times 0.1 + 0.3 \times 0.2 = 0.2399...$$

- Relative shift the second by -2

$$[0.4, 0.1, 0.2, 0.3]$$

$$[0.4, 0.1, 0.2, 0.3]$$

and compute

$$0.4 \times 0.4 + 0.1 \times 0.1 + 0.2 \times 0.2 + 0.3 \times 0.2 = 0.3000...$$

- Relative shift the second by -3:

$$[0.4, 0.1, 0.2, 0.3]$$

$$[0.1, 0.2, 0.3, 0.4]$$

and compute

$$0.4 \times 0.1 + 0.1 \times 0.2 + 0.2 \times 0.3 + 0.3 \times 0.4 = 0.2399...$$

You see that the largest value gives you the correct relative shift.

Of course if the probabilities were from ciphertexts, the numbers will only be *close* to 0.1, 0.2, 0.3, 0.4 if the lengths are large enough.

(By the way, the reason why you get the largest value when the numbers line up nicely is because the above sum of product of corresponding term is called the inner product and the inner product is largest when the two sequence of positive numbers, think vectors, are pointing in the same direction.)

**Exercise 1.7.10.** Here's another example. Say this is the plaintext:

debug:  
exercises/vigenere-  
9/question.tex

This is a test. This tutorial is aimed at getting familiar with the bare bones of LaTeX. First, ensure that you have LaTeX installed on your computer (see Installation for instructions of what you will need). We will begin with creating the actual source LaTeX, and then take you through how to feed this through the LaTeX system to produce quality output, such as postscript or PDF.

We use the key of fun to obtain the ciphertext

```
ybvxcffnrnxngmcfyogtlvffvxuvryqfntjngnhtkuznfvfljnnuy
brguejvbsyftzyfnrczvwmghfzlrbybnysbzbnyyfnrccaxnnqf
riiadihwbrjhyeyxrnhyuyqugniakienhfylhnhvthftzjmug
dihbcyqhrjxjjqvqfojavsqvybpwynycalnujupyonqmbzlpjfny
ykhqybrsnpyltogmlbzaumiyyisjyqybvxnuihlbgmyyfnrcm
lxnrrnbulbiopjkhffvysbznznfzwufmctmgxwenjgtlciz
```

The key fun gives us the shifts (5, 20, 13). Let's try to re-discover the shifts.

The (sorted) incidence values  $I$  for up to length 6 are

Average I-values	keylength
-----	-----
0.06467025914470374	3
0.06262443438914027	6
0.04968484284445197	1
0.04938482570061517	4
0.04908514013749339	5
0.048510313216195575	2

Therefore we conjecture that the length is 3. (It shouldn't be surprising that multiples of 3 will also give high  $I$ -values, right?) In more details, for the case of testing a keylength of 3, the 3 strings are:

```
yxfxmytfxrfjknfnnygjstfcwjzyyzafcxqidwryxnyqnknyhtm
dbqjjqjsywylyjqzjyfysptmzmyjyxwlmfcxruijfyzzzftxnti

bcnncolfuynnhuflnbuvyznzmhlbsbyncnfiiwjyyhuuiihlnhzu
ichxqfaqbycnuomlfyhbnyolaiiybnibynmnnlokfsnnwmmwjlz

vfrgfgvvvtgtztvjurebfyrvgfrnbnyranrahbherfygaefhvfjg
hyrjvovvpnaupnbpnkqrnlgbujsqvuhgyrlrbpbhvbcbfucgegc
```

with respective  $I$ -values

```
0.0651056539121
0.0687226346849
0.0601824888371
```

The  $I$ -values are computed with

$$\frac{\sum_{i=0}^{25} f(i)f(i-1)}{n(n-1)}$$

where  $f$  is the frequency function for the string (for instance  $f(0)$  is the frequency of **a**) and  $n$  is the length of the string. The average of the above 3  $I$ -values for keylength 3 is 0.0646702591447.

Next we compute the  $M$ -values. This means that for each distinct pair of strings (from the 3), let  $f_1$  and  $f_2$  be their frequencies and  $n, n'$  be their length, we compute

$$\frac{\sum_{i=0}^{25} f_1(i)f_2((i-g) \bmod 26)}{nn'}$$

with all possible relative shifts  $g = 0, 1, 2, \dots, 25$ . ( $f(0)$  means the frequency of **a** in the first string, etc.)

M-values	Index of 1st string	Index of 2nd string	Relative shift g
0.0498633235932	1	2	0
0.0344990102743	1	2	1
0.0310114054105	1	2	2
0.0330851164106	1	2	3
0.0452446036384	1	2	4
0.0342162315016	1	2	5
0.0333678951833	1	2	6
0.0332736355924	1	2	7
0.0331793760015	1	2	8
0.0291262135922	1	2	9
0.0398718069564	1	2	10
0.0707889527759	1	2	11
0.044584786502	1	2	12
0.0329908568197	1	2	13
0.0328023376379	1	2	14
0.0415684795928	1	2	15
0.0334621547742	1	2	16
0.0401545857291	1	2	17
0.0454331228202	1	2	18
0.031576962956	1	2	19
0.0286549156377	1	2	20
0.0325195588651	1	2	21
0.0441134885475	1	2	22
0.0446790460929	1	2	23
0.0419455179565	1	2	24
0.0379866151381	1	2	25
0.0355035217971	1	3	0
0.0272225395012	1	3	1
0.0371216447744	1	3	2
0.0509232819341	1	3	3
0.046925566343	1	3	4
0.0324576432515	1	3	5
0.0341709499334	1	3	6
0.0503521797068	1	3	7
0.0397867885018	1	3	8
0.0293165810013	1	3	9
0.0390253188654	1	3	10

0.0426422996383	1	3	11
0.0329335617742	1	3	12
0.0278888254331	1	3	13
0.0438796877974	1	3	14
0.0348372358652	1	3	15
0.028745478774	1	3	16
0.0399771559109	1	3	17
0.0667237768894	1	3	18
0.0402627070246	1	3	19
0.029602132115	1	3	20
0.0323624595469	1	3	21
0.0442604226156	1	3	22
0.0345516847516	1	3	23
0.0413097277746	1	3	24
0.037216828479	1	3	25
0.0413097277746	2	3	0
0.035313154388	2	3	1
0.0286502950695	2	3	2
0.0434989529792	2	3	3
0.0369312773653	2	3	4
0.0317913573196	2	3	5
0.0440700552065	2	3	6
0.0655815724348	2	3	7
0.0368360936608	2	3	8
0.031505806206	2	3	9
0.0355987055016	2	3	10
0.0369312773653	2	3	11
0.037216828479	2	3	12
0.0395012373882	2	3	13
0.0356938892062	2	3	14
0.0318865410242	2	3	15
0.0336950314106	2	3	16
0.0343613173425	2	3	17
0.055396916048	2	3	18
0.0403578907291	2	3	19
0.04016752332	2	3	20
0.0357890729107	2	3	21
0.0484485056158	2	3	22
0.0339805825243	2	3	23
0.0326480106606	2	3	24
0.0328383780697	2	3	25

We sort and list 6 rows with the highest M-values:

M-values	Index of 1st string	Index of 2nd string	Relative shift g
0.0707889527759	1	2	11
0.0667237768894	1	3	18
0.0655815724348	2	3	7
0.055396916048	2	3	18
0.0509232819341	1	3	3
0.0503521797068	1	3	7

For the first row, if the  $M$ -value using index 1 and index 2 string (i.e.,  $z_1, z_2$ ) with a relative shift of 11. The computed  $M$ -value is 0.0707.... In other words if  $f_1$  and  $f_2$  denote the frequency data of  $z_1$  and  $z_2$  respectively, then the



$M$ -value shown in the first row above is

$$\frac{\sum_{i=0}^{25} f(i)f'((i-11) \bmod 26)}{n_1 n_2}$$

where  $n_1$  and  $n_2$  are the lengths of  $z_1, z_2$  respectively and  $f_1(0)$  is the frequency of  $a$  in  $z_1$ ,  $f_1(1)$  is the frequency of  $b$  in  $z_1$ , etc. From the top 2 entries of  $M$ -values:

M-values	Index of 1st string	Index of 2nd string	Relative shift $g$
0.0707889527759	1	2	11
0.0667237768894	1	3	18

we get

$$\begin{aligned} k_1 - k_2 &\equiv 11 \pmod{26} \\ k_1 - k_3 &\equiv 18 \pmod{26} \end{aligned}$$

or

$$\begin{aligned} k_2 &\equiv k_1 - 11 \equiv k_1 + 15 \pmod{26} \\ k_3 &\equiv k_1 - 18 \equiv k_1 + 8 \pmod{26} \end{aligned}$$

At this point, even though we don't have the 3 shifts yet, we at least know that the shifts are probably of the form

$$(k_1, (k_1 + 15) \bmod 26, (k_1 + 8) \bmod 26)$$

There is now *one* unknown. Now trying different values of  $k_1$ , we see that when  $k_1 = 5$ , we get

$$k_1 = 5, \quad k_2 = 20, \quad k_3 = 13$$

which corresponds to the word **fun**. Vóila! ... we have discovered the key.

Note that without any of the above techniques, the key space can be any string, i.e., the size of the key space is infinite. The first step was finding (probabilistically) the length of the key. Once this is found, say the (probable) length of the key is  $m$ , then the (probable) size of the key space becomes  $26^m$ . The second step reduces the search down to search for  $k_1$  (basically the search for the first letter of the key) which is in a search space of size 26. (Go to [solution](#), page 116)  $\square$

**Exercise 1.7.11.** (Go to [solution](#), page 117)

$\square$  debug:  
exercises/vigenere-  
10/question.tex

**Exercise 1.7.12.** Let  $x = \langle x_1, x_2, \dots, x_n \rangle$   $y = \langle y_1, y_2, \dots, y_n \rangle$  be two sequences of real numbers. Let

debug:  
exercises/vigenere-  
11/question.tex

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$$

and likewise for  $\|y\|$ . Also, let

$$x \cdot y = \sum_{i=1}^n x_i y_i$$

Note that

$$x \cdot x = \|x\|^2$$

(a) Show that

$$|x \cdot y| \leq \|x\| \cdot \|y\|$$

and equality holds iff  $y = cx = \langle cx_1, \dots, cx_n \rangle$  for some  $c \in \mathbb{R}$ .

(b) In particular if  $x$  is made up of real numbers  $\geq 0$  which are not all zeros and not all the same and  $y$  is a permutation of  $x$ , then  $x \cdot y$  is maximum iff  $y = x$  (i.e., the permutation used to create  $y$  is the identity permutation that does not move the values of  $x$ ).

The above is the justification of the method used in breaking Vigenère's cipher.  
([Go to solution](#), page 118) □

## Solutions

Solution to Exercise [1.7.1](#).

Solution not provided.

debug:  
exercises/vigenere-  
0/answer.tex

Solution to Exercise [1.7.2](#).

Solution not provided.

debug:  
exercises/vigenere-  
1/answer.tex

Solution to Exercise [1.7.3](#).

Solution not provided.

debug:  
exercises/vigenere-  
2/answer.tex

Solution to Exercise [1.7.4](#).

Solution not provided.

debug:  
exercises/vigenere-  
3/answer.tex

Solution to Exercise [1.7.5](#).

Solution not provided.

debug:  
exercises/vigenere-  
4/answer.tex

Solution to Exercise [1.7.6](#).

0.065

debug:  
exercises/vigenere-  
5/answer.tex



Solution to Exercise [1.7.7](#).

0.038

debug:  
exercises/vigenere-  
6/answer.tex

Solution to Exercise [1.7.8](#).

Solution not provided.

debug:  
exercises/vigenere-  
7/answer.tex

Solution to Exercise [1.7.9](#).

Solution not provided.

debug:  
exercises/vigenere-  
8/answer.tex

Solution to Exercise [1.7.10](#).

Solution not provided.

debug:  
exercises/vigenere-  
9/answer.tex

Solution to Exercise [1.7.11](#).

Solution not provided.

debug:  
exercises/vigenere-  
10/answer.tex

Solution to Exercise [1.7.12](#).

Solution not provided.

debug:  
exercises/vigenere-  
11/answer.tex

## 1.8 Substitution cipher debug: classical-cipher-substitution-cipher.tex

OK folks, things are getting more fun (or worse, depending on your point of view). If you've read your notes and following the readings you should know why the substitution cipher is not that easy to break. Right?

Because of the gap in the probability between **e** and the next letter, you can usually figure out what **e** encrypts to. The problem is that next group of letters have frequencies which are rather close together.

Here's a suggestion. If you know (or think you do) that **e** is encrypted to **r**, then you should use the digrams. Basically the idea is the same as solving crossword puzzles. You want to decrypt the letters near to those letters which are already decrypted, just like in crossword puzzles you want to work near to whose rows and columns that are already solved. So look at all the places where **r** are decrypted as **e**. Look at the letter before and after it and hope that you can now decrypt by looking at digrams in the ciphertext of the form **?r** and **r?**

**Example 1.8.1.** The goal is to decrypt the following ciphertext (i.e., you need to compute the plaintext) and also to discover the key used. The substitution cipher is used.

```
amqtijxtheitijxnvatjhumajxltuhetlitukmoajvaotqvewm
hmlojiaueecqrtgqtxlljtnhtrqtqjmximhmlajqqmtiqgbxhe
bxlieftvmrmajqtggmthtxvmrtlrmntqgjtqghmgthmlfehq
ermiajxnqihtxnubixefehqeewmhgeomhxnrtgmhnextkjis
tqiajqjiotqajqqjpmoajvaieecexmquhmtiatotsajqqjpmtx
lajqjrgeqjxngmhqmxvmajqamtlotqmxeheqbiamkthnmqija
twmmwmhqmmbxgbextabrtxumjxnjtrqbhmiatiajqiegatiatlj
mwmhwmxbhmlielexjioebklatwmqkjggmlewmhrmmxijhmkst
xlhmqimlexrsqaebkmlhqamatliamftvmtxlumthloajvajtqq
evjtimojiatxtqqshjtxubkkiamfehrmhfkehjl iamktiimhqe
uktvtctqtkreqiieatwmtqbqgjvjexefukbmqgtlmaqatgmltxlh
jggkjxnleoxewmhajqvamqiamatjhotqgmvbkjthgktqimhml
leoxjxfhexijxtkexnvbhwxnojqgewmhajqrtqqjwmfehmamt
liammsmqomhmukbmnhstbxmlhnhmtiuktvcibfiqwmhsvkmthw
mhsvhjijvtktxlwmhsrtqimhfbktabnmqghmtlefqaebklmhqt
xltvamqikjcmtuthhmkomhmiameiamhgthiqefajroajvatggm
thmltuewmiamitukmqtwmfehioemxeheqbqatxlqvewmhmloji
akexnuktvcatjhiajqtxltumkeojxnhethjxnhbrukjxnwejv
mrtlmbgrsfjhqijrghmqjexefiamxeiehjebqghefmqqehvat
kkmxnmh
```







```
mrtlmbgrsfjhqijrghmqjxexefiamxeiehjbqghefmqqehvatkkmxnmh
e---e-----r-t---re-----t-e--t-r-----r-e--r-----e--er
cipertext
1grams: m:116 t:95 q:75 h:74 j:68 e:65 i:59 a:59 x:53 l:40 k:34 g:30
2grams: mh:25 ia:20 hm:20 aj:19 wm:18 at:16 am:16 tq:15 mq:15 xn:14
        tx:14 mt:14 jx:14 xl:12 qi:12 jq:12 th:11 ml:10 ex:10 eh:10
3grams: wmh:11 ajq:11 jxn:10 iam:10 txl:9 hml:7 ewm:7 mhm:6 otq:5 mhq:5
        feh: 5 twm: 4 qqj: 4 oaj: 4 mth:4 mqi:4 mia:4 jva:4 imh:4 iat:4
plaintext
1grams: e t a o i n s h r
2grams: th he in er an re ed on es st en at to nt ha nd
        ou ea ng as or ti is et it ar te se hi of
3grams: the ing and her ere ent tha nth was eth for dth
e->m, r->h, t->i, h->a
```

ent is also a common plaintext trigram. This is encrypted as m?i. The only one that fits is mqi but the frequency of this is only 4 – so this is probably wrong.

Another high frequency plaintext 3-gram is tha. This would encrypt as ia?. We notice that iam has a high frequency. So perhaps a->m.

Now let's look at pairs of digrams.

es,st is a high frequency digram. This is encrypted as m?,?i. The only possibility is mq,qi. So we suspect s->q. This is what we have now:

```
amqtijxtheitijxnvatjhumajxltuhetlitukmoajvaotqvewmhmljoiaueecqrtgqxlljtnhtrqtqjmximhmlajqqmtiqgbxhe
hes-t---r-t-t---h-r-eh-----r---t---e-h-h--s---ere---th---s---s-----r-s-s-e-tere-h-sse-ts---r-
bxlietvmrmajqtggmthtxvmtlrmmtgqjotqghgmthmlfehqmriaixnqihntxmubixiefheqewmhgeomhixntgmhextkjis
---t---e-h-s---e-r---e---e-e-s---s-re-re---rs---eth---str---e-t-t-t-rs---er---er---ers---t-
tqiajqjotqajqqjpmoajvaeicexmquhmtiatotsajqqjpmtxlajqjrgeqjxngmhmvmxvmajqamtlotqmxehebqiamkthnmqija
-sth-s-t-sh-ss--e-h--ht-----es-re-th---h-ss--e---h-s---s---rese--eh-she---se-r---sthe-r-est-h
twmmwmhqmmbxgextabrtxumjxnjtrqbhmiaiajgiegatiatljmwmmhwmxihbmlielexjioebklatwmqjggmlewmhrmxiijhmkst
--ee-ersee-----h-----e---s-s-reth-th-st--h-th--e-er-e-t-re-t-----t---h-es---e---er-ee-t-re--
xlhmqimlexrsqaebkmlmqamatliamftvmtxlumthloajvajtqqevjtimojiatxtqqshjtxubkiamfehmrhmfkehjliamktiimhqe
--reste-----sh-----ersheh--the---e---e-r-h--h-ss---te--th---ss-r-----the-r-er---r-the--tters-
uktvtctkreqieatwmtqbgqjvjexefukbmqgtlmqatgmltxlhjggkixnleoxewmhajqvamqiamatjhotqgmvbkjthgktqimhml
-----s---stt-h--e-s-s-----es---esh--e---r-----erh-s-hesttheh--r--s-e-----r---stere-
leoxjxfhexijxteknvbwjxnojqgewmhajqrtqqjwmfehmtliamsmqomhmukbmhntsbxlmhnhmtiuktvcibfiqwmhsvkmtwh
-----r-t-----r-----s---erh-s-ss--e--rehe--thee-es-ere---e-r-----er-re-t-----ts-er---e-r-
mhsvhjijvtktxlwmhertqimhfkbtabnmqghmtlefqaebkmlmqxtlvtamqikjcmthhmkomhmiameiamhgtthiqefajroajvatggm
er--r-t-----er---ster---h--es-re---sh---ers---hest--e---rre--erethe-ther--rts--h---h--h--e
thmltuewmiमितकुमटम्वफीओमखेरेबगतखलव्वमहमलोआखेनकुटवकटजहआजखलकुमकेओजखनेथजखनहब्रुकजखनेजव
-re---ethet---es--e-rt--e-r--sh---s---ere---th-----h--rth-s---e-----r--r--r-----
mrtlmbgrsfjhqijrghmqjxexefiamxeiehjbqghefmqqehvatkkmxnmh
e---e-----rst---ress---the--t-r---s-r-ess-r-h--e--er
cipertext
1grams: m:116 t:95 q:75 h:74 j:68 e:65 i:59 a:59 x:53 l:40 k:34 g:30
2grams: mh:25 ia:20 hm:20 aj:19 wm:18 at:16 am:16 tq:15 mq:15 xn:14
        tx:14 mt:14 jx:14 xl:12 qi:12 jq:12 th:11 ml:10 ex:10 eh:10
3grams: wmh:11 ajq:11 jxn:10 iam:10 txl:9 hml:7 ewm:7 mhm:6 otq:5 mhq:5
        feh: 5 twm: 4 qqj: 4 oaj: 4 mth:4 mqi:4 mia:4 jva:4 imh:4 iat:4
plaintext
1grams: e t a o i n s h r
2grams: th he in er an re ed on es st en at to nt ha nd
        ou ea ng as or ti is et it ar te se hi of
3grams: the ing and her ere ent tha nth was eth for dth
e->m, r->h, t->i, h->a, s->q
```

Note that `ti,is` is a common plaintext 2-gram. When encrypted, this is `i?,?q`. Unfortunately we can't find this pattern.

We now have enough substitutions to consider multiple cases of pairs of digrams.

Consider the common plaintext digram `aj,jq`. With what we have at this point, the encryption is `h?,?s -> aj,jq`. The possibilities for `h?,?s` are

- `he,es`: Therefore `e->j`, but `e` is already encrypted as `m`.
- `ha,as`: Therefore `a->j`.
- `hi,is`: Therefore `i->j`.

So we have `a->j` or `i->j`. Before we make a choice, let's consider more digrams.

Consider `h?,e?`. `h?,e?` might be encrypted as `at,mt`. Possibilities for `h?,e?`

- `hi,ei`: But `ei` is not common.
- `ha,ea`: Therefore `a->t`.

Therefore `a->t`.

Consider `h?,?r`. `h?,?r` might be encrypted to `at,th`. The only possibilities for `h?,?r` are

- `ha,ar`: Therefore `a->t`.
- `hi,ir`: But `ir` is not common.

Therefore `a->t`.

Consider `?s,e?`. `?s,e?` might be encrypted as `tq,mt`. The only possibility for `?s,e?` is `as,ea` which implies `a->t`.

`?s,e?` might be encrypted as `tq,th`. The only possibility for `?s,e?` is `as,ea` which implies `a->t`.

`e?,?r` might be encrypted to `mt,th`. The possibilities for `e?,?r` are

- `ed,rd`: But `rd` is not common.
- `es,sr`: But `sr` and `os` not common.
- `en,nr`: But `nr` is not common.
- `ea,ar`: Therefore `a->t`.
- `et,tr`: But `tr` is not common.

All in all, this case implies `a->t`.

From all the above cases, it seems that `a->t` and `i->j`. (The argument for `a->t` is stronger.) We now have

```
amqtijsxtheitijxnvatjhumajxltuhetlitukmoajvaotqvewmhmlojiaueecqrtgqtxlljtnhtrqtqjmximhmlajqqmtiqgbxhe
hesati-ar-tati---hair-ehi--a-r-a-ta--e-hi-h-as---ere--ith----s-a-sa---ia-ra-sasie-tere-hisseats---r-
```

```

b x l i e f t v m r m a j t g g m t h t x v m t l m r m n t g j o t q g h m g t h m l f e h q e r m i a j x n q i h t x n m u b i x e i f e h q e e w m h g e o m h j x n t g m h q e x t k j i s
--t--a-e-ehisa--e-a-e-e-as-l-as-re-are--rs--ethi--stra--e--t--t--rs--er--er--eri--a-ers--a-it-

t q i a j j i o t q a j q j p m o a j v a i e e c e x m q u h m t i a t o t s a j q j p m t x l a j q j r g e q j x n g h m q m x v m a j q a m t l o t q m x e h r e b q i a m k t h m q i j a
a s t h i s i t - a s h i s s i - e - h i - h t - - - - e s - r e a t h a - a - h i s s i - e a - h i s i - - - - s i - - - - r e s e - - e h i s h e a - - a s e - - r - - - - s t h e - a r - e s t i h

t w m m w m h q m m x b g e x t a b r t x u m j x n j t r q b h m i a t i a j q i e g a t i a t l j m w m h m x i b h m l i e l e x j i o e b k l a t w m q k j g g m l e w m h r m m x i j h m k s t
a - e e - e r s e e - - - - a h - - - - e i - - - - i a - s - r e t h a t h i s t - - h a t h a - i e - e r - e - t - r e - t - - - - i t - - - - h a - e s - i - - - - e r - e e - t i r e - - a

x l h m q i m l e x r s q a e b k l m h q a m a t l i a m f t v m t x l u m t h l o a j v a j t q q e v j t i m o j i a t x t q q s h j t x u b k k i a m f e h r m h f k e h j l i a m k t i i m h q e
--reste--sh--ersheha-the-a-ea--ear--hi-hiass--iate-itha-ass-ria--the--r--er--ri-the-atters-

u k t v c t q t k r e q i e a t w m t q b g j v j e x e f u k b m g t l m q a t g m l t x l h j g g k j x n l e o x e w m h a j q v a m q i a m a t j h o t q g m v b k j t h g k t q i m h m l
--a--asa--stt-ha-eas-s-i-i--es-a-esha-e-a--ri--i--erhis-hestthehair-as-e--iar--astere-

l e o x j f h e x i j x t k e x n v b h w j x n o j g g e w m h a j r t q j j w m f e h m a m t l i a m s m q o m h u k b m n h t s b x l m n h m t i u k t v c i b f i q w m h s v k m t h w
---i--r--ti-a-----r-i--is--erhis-assi-e--rehea-thee-es-ere--e-ra--er-reat--a--t--ts-er--ear-

m h s v h j i j v t k t x l w m h s r t q i m h f k t a b n m q g h m t l e f q a e b k l m h q t x l t v a m q i k j c m t u t h m k o m h m i a e i a m h g t h i q e f a j r o a j v a t g g m
e r - r i t i - a - a - - e r - a s t e r - - - a h - e s - r e a - - - s h - - - e r s a - - a - h e s t - i - e a - a r r e - - e r e t h e - t h e r - a r t s - - h i - - h i - h a - e

t h m l t u e w i a m i t u k m q t w m f e h i o e m x e h r e b q a t x l q v e w m h m l o j i a k e x n u k t v c a t j h i a j q t x l t u m k e o j x n h e t h j x n h b r u k j x n w e j v
a r e - a - - e t h e t a - - e s a - e - r t - - e - - r - - - s h a - - - - e r e - - i t h - - - - - a - - h a i r t h i s a - a - e - - - i - r - a r i - r - - - i - - - i -

m r t l m b g r s f j b h q i j r g h m q j j e x e f i a m x e i e h j e b g q h e f m q q e h v a t k k m x n m h
e - a - e - - - - i r s t i - - r e s s i - - - - t h e - t - r i - s - r - - e s s - r - h a - e - e - r

ciphertext
1grams: m:116 t:95 q:75 h:74 j:68 e:65 i:59 x:53 l:40 k:34 g:30
2grams: mh:25 ia:20 hm:20 aj:19 wm:18 at:16 am:16 tq:15 mq:15 xn:14
        tx:14 mt:14 jx:14 xl:12 qi:12 jq:12 th:11 ml:10 ex:10 eh:10
3grams: vmh:11 ajq:11 jxn:10 iam:10 txl:9 hml:7 ewm:7 mhm:6 otq:5 mhq:5
        feh: 5 twm: 4 qqj: 4 oaj: 4 mth:4 mqi:4 mia:4 jva:4 imh:4 iat:4
plaintext
1grams: e t a o i n s h r
2grams: t h h e i n e r a n r e e d o n e s e t e n a t o n t h a n d
        o u e a n g a s o r t i i s e t i t a r t e s e h i o f
3grams: t h e i n g a n d h e r e r e e n t t h a n t h w a s e t h f o r d t h

e->m, r->h, t->i, h->a, s->q, a->t, i->j

```

At this point we can already see (possibly) “he sat” at line one and “that his” at line 4 and “his -hest the hair” at line 4 – perhaps “chest” is the second word?

Next, we try tx,jx. a?,i? might be encrypted as tx,jx. The possibilities for a?,i? are

- an,in: Therefore n->x.
- ar,ir: But ir is not common.

We get

```

a m q t i j x t h e i t i j x n v a t j h u m a j x l t u h e t l i t u k m o a j v a o t q v e w m h m l o j i a u e e c r t g q t x l l j t n h t r t q j m x i m h l a j q q m t i q g b x h e
h e s a t i n a r - t a t i n - - h a i r - e h i n - a - r - a - t a - - e - h i - h - a s - - - e r e - - i t h - - - - s - a - s a n - - i a - r a - s a s i e n t e r e - h i s s e a t s - - n r -

b x l i e f t v m r m a j t g g m t h t x v m t l m r m n t g j o t q g h m g t h m l f e h q e r m i a j x n q i h t x n m u b i x e i f e h q e e w m h g e o m h j x n t g m h q e x t k j i s
-n-t--a-e-ehisa--earan-e-a-e-e-as-i-as-re-are--rs--ethin--stran--e--tn-t--rs--er--er--erin-a-ers-na-it-

t q i a j j i o t q a j q j p m o a j v a i e e c e x m q u h m t i a t o t s a j q j p m t x l a j q j r g e q j x n g h m q m x v m a j q a m t l o t q m x e h r e b q i a m k t h m q i j a
a s t h i s i t - a s h i s s i - e - h i - h t - - - - n e s - r e a t h a - a - h i s s i - e a n - h i s i - - - - s i n - - r e s e n - e h i s h e a - - a s e n - r - - - - s t h e - a r - e s t i h

t w m m w m h q m m x b g e x t a b r t x u m j x n j t r q b h m i a t i a j q i e g a t i a t l j m w m h m x i b h m l i e l e x j i o e b k l a t w m q k j g g m l e w m h r m m x i j h m k s t
a - e e - e r s e e n - - n a h - - a n - e i n - i a - s - r e t h a t h i s t - - h a t h a - i e - e r - e n t - r e - t - - - n i t - - - - h a - e s - i - - - - e r - e e n t i r e - - a

x l h m q i m l e x r s q a e b k l m h q a m a t l i a m f t v m t x l u m t h l o a j v a j t q q e v j t i m o j i a t x t q q s h j t x u b k k i a m f e h r m h f k e h j l i a m k t i i m h q e
n-reste--n--sh--ersheha-the-a-ean--ear--hi-hiass--iate-ithanass-rian--the--r--er--ri-the-atters-

u k t v c t q t k r e q i e a t w m t q b g j v j e x e f u k b m g t l m q a t g m l t x l h j g g k j x n l e o x e w m h a j q v a m q i a m a t j h o t q g m v b k j t h g k t q i m h m l
--a--asa--stt-ha-eas-s-i-i--es-a-esha-e-a--ri--in--n--erhis-hestthehair-as-e--iar--astere-

l e o x j f h e x i j x t k e x n v b h w j x n o j g g e w m h a j r t q j j w m f e h m a m t l i a m s m q o m h u k b m n h t s b x l m n h m t i u k t v c i b f i q w m h s v k m t h w
--nin-r-ntina--n--r-in--is--erhis-assi-e--rehea-thee-es-ere--e-ra--n--er-reat--a--t--ts-er--ear-

m h s v h j i j v t k t x l w m h s r t q i m h f k t a b n m q g h m t l e f q a e b k l m h q t x l t v a m q i k j c m t u t h m k o m h m i a e i a m h g t h i q e f a j r o a j v a t g g m
e r - r i t i - a - a n - e r - a s t e r - - - a h - e s - r e a - - - s h - - - e r s a n - a - h e s t - i - e a - a r r e - - e r e t h e - t h e r - a r t s - - h i - - h i - h a - e

```

```

thmltuewmiamitukmqtwmfheioemxehrebqatxlqvewmhmllojiakexnuktvcattjhiajqtxtltumkkojxnhethjxnhbrukjxnwejv
are-a---etheta--esa-e--rt--en-r---shan-s---ere--ith--n---a---hairthisan-a-e-----in-r-arin-r-----in--i-

mrtlmbgrsfjhhqjrhgmqqjexefiamxeiehjebqghefmqqehvatkkmxnmh
e-a-e-----irsti--ressi-n--then-t-ri--s-r--ess-r-ha--en-er

ciphertext
1grams: m:116 t:95 q:75 h:74 j:68 e:65 i:59 a:59 x:53 l:40 k:34 g:30
2grams: mh:25 ia:20 hm:20 aj:19 wm:18 at:16 am:16 tq:15 mq:15 xn:14
      tx:14 mt:14 jx:14 xl:12 qi:12 jq:12 th:11 ml:10 ex:10 eh:10
3grams: wmh:11 ajq:11 jxn:10 iam:10 txl:9 hml:7 ewm:7 mhm:6 otq:5 mhq:5
      feh: 5 twm: 4 qqj: 4 oaj: 4 mth:4 mqi:4 mia:4 jva:4 imh:4 iat:4
plaintext
1grams: e t a o i n s h r
2grams: th he in er an re ed on es st en at to nt ha nd
      ou ea ng as or ti is et it ar te se hi of
3grams: the ing and her ere ent tha nth was eth for dth

e->m, r->h, t->i, h->a, s->q, a->t, i->j, n->x

```

The beginning of the plaintext now reads “he sat in a.” We now look at xl,ml and ex,eh.

n?,e? might be encrypted as xl,ml. The possibilities for n?,e? are

- nt,et: But t is already assigned.
- nd,ed: Therefore implies d->l.
- ng,eg: But eg is not common.

?n,?r might be encrypted as ex,eh. The possibilities for ?n,?r are

- in,ir: But i is already assigned.
- an,ar: But a is already assigned.
- on,or: This implies o->e.
- en,er: But e is already assigned.

Adding d->l and o->e, we get

```

amqtijxtheitijxnvatjhumaajxtuhetlitukmoajvaotqvewmhmllojiaueecrtgqtxlljtnhtrqtqjmximhmlajqqmtiqgxbhe
hesatinarotatin--hair-ehinda-roadta--e-hi-h-as-o-ered-ith-oo-s-a-sanddia-ra-sasieredhiseseats--nro

bxlietvmrmatjqtggmthtxvmtlrmmtqgjtqgghmgthmlfehqmriaixnqihntxmubixefehqeevmhgeomhxnrtgmhgehtkjis
-ndto-a-e-ehisa--earan-e-ade-e-as-i-as-re-ared-orso-ethin-stran-e--tnot-orsoo-er-o-erin-a-ersona-it-

tqiajqjotqajqqjpmoajvaieecexmqumhtiatotsajqqjpmxtlajqrgeqjxngmqmxvmajqamtlotqmxehebqiamkthnmqija
asthisit-ashissi-e-hi-htoo-ones-reatha-a-hissi-eandhisi--osin--resen-ehishead-asenor-o-sthe-ar-estih

twmmwmhqmmbxgextabrtxumjxnjtrqbhmiaiajiegatiatljmwmmhwmxbhmliexjioebklatwmqkjggmlewmhmmxijhmkst
a-ee-erseen--onah--an-ein-ia-s-rethathisto-hathadie-er-ent-redtodonit-o--dha-es-i--edo-er-eentire--a

xlhmqimlexrsqaeblmhqamatliamftvmtxlumthloajvajtqqevjtimojiatxtqqshjtxubkkiamfehrmhfkhejliamktiimhqe
ndrestedon--sho--dershehadthe-a-eand-eard-hi-hiasso-iate-ithanass-rian----the-or-er--oridthe-aterso

uktvtctkreqieatwmtqbgjvjexefukbmqgtlmqatgmltxlhjggkixnleoxewmhajqvamqiamatjhotqgmvbktjthgktqimhml
--a--asa--osttoha-eas-s-i-iono----es-adesha-edandri---in-do-no-erhis-hestthehair-as-e---iar--astered

leoxjxfhexijxtkexnvbhwxnojqgwmhajtqrqqjwmfehmamtliaamsmqomhukbmhntsbxlmhnmhtiuktvcibfiqwmhsvkmthw
do-nin-rontina-on--r-in--is-o-erhis-assi-e-oreheadthee-es-ere---e-ra-nder-reat--a--t--ts-er---ear-

mhsvhjijvtktxlwmhrtqimhfktabnmqghmtlefqaeblmhqtxltvamqikjcmthhmkomhmiameiamhgthiqefajroajvatggm
er--riti-a-and-er--aster---ah--es-reado-sho--dersanda-hest-i-ea-arre--eretheother-artso-hi--hi-ha--e

thmltuewmiamitukmqtwmfheioemxehrebqatxlqvewmhmllojiakexnuktvcattjhiajqtxtltumkkojxnhethjxnhbrukjxnwejv
areda-o-etheta--esa-e-ort-oenor-o-shands-o-ered-ith-on---a---hairthisanda-e--o-in-roarin-r-----in--oi-

mrtlmbgrsfjhhqjrhgmqqjexefiamxeiehjebqghefmqqehvatkkmxnmh

```

```
e-ade-----irsti--ressiono--thenotorio-s-ro-essor-ha--en-er

cipertext
1grams: m:116 t:95 q:75 h:74 j:68 e:65 i:59 a:59 x:53 l:40 k:34 g:30
2grams: mh:25 ia:20 hm:20 aj:19 wm:18 at:16 am:16 tq:15 mq:15 xn:14
        tx:14 mt:14 jx:14 xl:12 qi:12 jq:12 th:11 ml:10 ex:10 eh:10
3grams: vmh:11 ajq:11 jxn:10 iam:10 txl:9 hml:7 ewm:7 mhm:6 otq:5 mhq:5
        feh: 5 twm: 4 qqj: 4 oaj: 4 mth:4 mqi:4 mia:4 jva:4 imh:4 iat:4
plaintext
1grams: e t a o i n s h r
2grams: th he in er an re ed on es st en at to nt ha nd
        ou ea ng as or ti is et it ar te se hi of
3grams: the ing and her ere ent tha nth was eth for dth

e->m, r->h, t->i, h->a, s->q, a->t, i->j, n->x, d->l, o->e
```

The beginning reads “he sat in a rotatin--hair-ehinda-road...” which is very likely “he sat in a rotating chair-ehinda-road...”, giving us g->n and c->v. This gives us

```
amqtijxtheitijxnvatjhumajxltuhtlitukmoajvaotqvewmhmlojiaueecqrtgqtxlljtnhtrqtqjmximhmlajqqmtiqgbxhe
hesatinarotatingchair-ehinda-roadta--e-hich-asco-ered-ith-oo-s-a-sanddiagra-sasienteredhiseseats--nro

bxlieftvmrajqtggmthtxvmtlrmmtqgjtqghmgthmlfehqermiajxnqihntxmubixefehqeevmhgeomhjxntgmhgxextkjis
-ndto-ace-ehisa--earance-ade-egas-i-as-re-ared-orso-ethingstrange--tnot-orsoo-er-o-eringa-ersona-it-

tqiajqjotqajqqjpmoajvaieecexmquhmtiatotsajqqjpmtxlaajqrgeqjxngmhmvmajqamtlotqmxehebrebqiamkthnmqija
asthisit-ashissi-e-hichtoo-ones-reatha-a-hissi-eandhisi--osing-resencehishead-asenor-o-s-the-argestih

twmmvmhqmmbxgbextabrtxumxnjtrqbhmiaiajqiegatlatlwmvmhmxbhmliexjioebklatwmqjggmlewmhmmxijhmkst
a-ee-erseen--onah--an-eingia-s-rethathisto-hathadie-er-ent-redtodonit-o--dha-es-i--edo-er-eentire--a

xlhmqimlexrsqaeblmhqamatliamtvtmxtlumthloajvajtqqevjtimojiatxtqqshjtxubkkiamfehrmhfkhejliamktiinhqe
ndrestedon--sho--dershehadthe-aceand-eard-hichiassociate-ithanass-rian----the-or-er--oridthe-attherso

uktvtctqtkreqieatwmtqbqjvjexefukbmqgtlmaqatgmltxlhjggkxjnlxewmhajqvamqiamatjhotqgmvbkjthgktqimhml
--ac-asa--osttoha-eas-s-iciono----es-adesha-edandri---ingdo-no-erhischestthehair-as-ec--iar--astered

leoxjxfhexijxtkexnbvhwjxnojqgwmhbjrtqqjwmfemhamtliamsmqomhmukbmhntsbxlmhnhmtiuktvcibfiqwmhsvkmtw
do-nin-rontina-ongc-r-ing-is-o-erhis-assi-e-oreheadthee-es-ere---egra--ndergreat--ac-t--ts-er-c-ear-

mshvbjjvjtktxlwmhrtqimhfktabnmqghmtlefqaeblmhqtxltvamqikjcmvtuthhmkomhmiameiamhgthiqefajroajvatggm
er-critica-and-er--aster---ah-ges-reado-sho--dersandachest-i-ea-arre--eretheother-artso-hi--hicha--e

thmltuewmiamitukmgtwmfehoemxehrebqatxlqvewmhmlojiakexnuktvcatjhiajqtltumkkojxnhehthjxnbrukjxnwejv
areda-o-etheta--esa-e-ort-oenor-o-shandsco-ered-ith-ong--ac-hairthisanda-e--o-ingroaringr----ing-oic

mrtlmbgrsfjhqijrghmqjexefiamxeiehjbqghefmqqehvatkkmxnmh
e-ade-----irsti--ressiono--thenotorio-s-ro-essorcha--enger

cipertext
1grams: m:116 t:95 q:75 h:74 j:68 e:65 i:59 a:59 x:53 l:40 k:34 g:30
2grams: mh:25 ia:20 hm:20 aj:19 wm:18 at:16 am:16 tq:15 mq:15 xn:14
        tx:14 mt:14 jx:14 xl:12 qi:12 jq:12 th:11 ml:10 ex:10 eh:10
3grams: vmh:11 ajq:11 jxn:10 iam:10 txl:9 hml:7 ewm:7 mhm:6 otq:5 mhq:5
        feh: 5 twm: 4 qqj: 4 oaj: 4 mth:4 mqi:4 mia:4 jva:4 imh:4 iat:4
plaintext
1grams: e t a o i n s h r
2grams: th he in er an re ed on es st en at to nt ha nd
        ou ea ng as or ti is et it ar te se hi of
3grams: the ing and her ere ent tha nth was eth for dth

e->m, r->h, t->i, h->a, s->q, a->t, i->j, n->x, d->l, o->e, g->n, c->v
```

Near the middle of the first line, “-hich-asco-ered-ith” is probably “which-asco-eredwith” giving us w->o.

On the second line “orso-ethingstrange-” is probably “or something strange-”, giving us m->r.

On the third line “sthe-argestiha-ee-erseen” is probably “s the largest

i have ever seen.” This gives us l->k and v->w.

At this point we have

```
amqti jxtheitijxnvatjhumajxltuhetlitukmoajvaotqvewmhmljoiaueecqrtgqtxlljtnhtrqtqjmximhmlajqqmtiqgbxhe
hesatinarotatingchair-ehinda-roadta-lewhichwascoveredwith-oo-sma-sanddiagramsasienteredhisseat--nro

bxliefertvmrajtgqgmthtxvmtlrmmtgqjotqghmgthmlfehqmria jxnqihntxmubixefehqeevwmgheomh jxntgmhextkjis
-ndto-acemehisa--earancemademegas-iwas-re-ared-orsomethingstrange--tnot-orsoover-oweringa-ersonalit-

tqiajqjiotqajqqjpmoajvaieecexmqumhtiatotsajqqjpmtxlajqjrgeqjxngmqmxvmajqamtlotqmxehrebqiamkthnmqija
asthisitwashissi-ewhichtoo-ones-reathawa-hissi-eandhisim-osing-resencehisheadwasenormo-sthelargestih

twmmwmhqmmbxgextabrtxumjxnjtrqbhmiaiajqiegatiatljmwvmhwmxbhmliexjioebklatwmqjggmlewmhmmxijhmkst
aveeverseen--onah-man-eingiams-rethathisto-hathadievertent-redtodonitwo-ldhavesli--edovermeentirel-a

xlhmqimlexrsqaeblmhmamatliamtvtmxtlumthloajvajtqqevjtimojiatxtqsshjtxubkkiamfehrmhfkhejl iamktiinhqe
ndrestedonm-sho-ldershehadthe-acend-eardwhichiassociatewithanass-rian--llthe-ormer-loridthelatterso

uktvtctqtkrequeiatwmtqbqgjvjexefukbmqgtlmgatgmltxlhjggk jxnleoxewmhajqvamqiamatjhotqgm vbkjtghktqimhml
-lac-asalmosttohaveas-siciono--l-es-adesha-edandri--lingdownoverhischestthehairwas-ec-liar-lastered

leoxjxfhexijxtkexnvbhwxnojqgwmhbjqrtqqjwmfehmamtlia msmqomhmukbmhntsbxlmhnmhtiuktvcibfiqwmhsvkmtw
downin-rontinalongc-rvingwis-overhismassive-oreheadthee-eswere-l-egra--ndergreat-lac-t--tsver-clearv

mhsvhjijvtktxlwmhrtqimhfktabnmqghmtlefqaeblmhmqt xltvamqikjcm tuthhmkomhmiameiamhgthiqefajroajvatggm
er-criticalandver-master--lah-ges-reado-sho-ldersandachestli-ea-arrelweretheother-artso-himwhicha--e

thmltuewmiamitukmtqwmfheioemxehrebqatxlqvewmhmljoiakexnuktvcatjhiajqt xltumkkojxn hethjxn hbrukjxnwejv
areda-ovetheta-lesave-ortwoenormo-shandscoveredwithlong-lac-hairthisanda-ellowingroaringr-m-lingvoic

mrtlmbgrsfj hqijrghmqjexefiamxeiehjbqghefmqqehvatkkmxnmh
emade--m--irstim-ressiono-thenotorio-s-ro-essorchallenger

e->m, r->h, t->i, h->a, s->q, a->t, i->j, n->x, d->l, o->e, g->n, c->v, w->o, m->r,
l->k, v->w
```

At the third line “hisim-osing-resencehisheadwasenormo-sthelargestihaveeverseen” is probably “his imposing presence his head was enormous the largest i have ever seen” giving us p->g and u->b.

At line 7, “hismassive-orehead” is “his massive forehead” giving us f->f.

```
amqti jxtheitijxnvatjhumajxltuhetlitukmoajvaotqvewmhmljoiaueecqrtgqtxlljtnhtrqtqjmximhmlajqqmtiqgbxhe
hesatinarotatingchair-ehinda-roadta-lewhichwascoveredwith-oo-smapsanddiagramsasienteredhisseatspunro

bxliefertvmrajtgqgmthtxvmtlrmmtgqjotqghmgthmlfehqmria jxnqihntxmubixefehqeevwmgheomh jxntgmhextkjis
undtofacemehisappearanceademegaspiwaspreparedforsomethingstrange-utnotforsooverpoweringapersonalit-

tqiajqjiotqajqqjpmoajvaieecexmqumhtiatotsajqqjpmtxlajqjrgeqjxngmqmxvmajqamtlotqmxehrebqiamkthnmqija
asthisitwashissi-ewhichtoo-ones-reathawa-hissi-eandhisimposingpresencehisheadwasenormousthelargestih

twmmwmhqmmbxgextabrtxumjxnjtrqbhmiaiajqiegatiatljmwvmhwmxbhmliexjioebklatwmqjggmlewmhmmxijhmkst
aveeverseenuponahuman-eingiamsurethathistophathadievertentredtodonitwouldhaveslippedovermeentirel-a

xlhmqimlexrsqaeblmhmamatliamtvtmxtlumthloajvajtqqevjtimojiatxtqsshjtxubkkiamfehrmhfkhejl iamktiinhqe
ndrestedonm-shouldershehadthefaceand-eardwhichiassociatewithanass-rian-ulltheformerfloridthelatterso

uktvtctqtkrequeiatwmtqbqgjvjexefukbmqgtlmgatgmltxlhjggk jxnleoxewmhajqvamqiamatjhotqgm vbkjtghktqimhml
-lac-asalmosttohaveasuspicionof-luespadeshapedandriplingdownoverhischestthehairwaspeculiarplastered

leoxjxfhexijxtkexnvbhwxnojqgwmhbjqrtqqjwmfehmamtlia msmqomhmukbmhntsbxlmhnmhtiuktvcibfiqwmhsvkmtw
downinfrontinalongcurvingwispovertismassiveforeheadthee-eswere-luegra-undergreat-lac-tuftsver-clearv

mhsvhjijvtktxlwmhrtqimhfktabnmqghmtlefqaeblmhmqt xltvamqikjcm tuthhmkomhmiameiamhgthiqefajroajvatggm
er-criticalandver-masterfulahugespreadofshouldersandachestli-ea-arrelweretheotherpartsofhimwhichappe

thmltuewmiamitukmtqwmfheioemxehrebqatxlqvewmhmljoiakexnuktvcatjhiajqt xltumkkojxn hethjxn hbrukjxnwejv
areda-ovetheta-lesavefortwoenormoushandscoveredwithlong-lac-hairthisanda-ellowingroaringrum-lingvoic

mrtlmbgrsfj hqijrghmqjexefiamxeiehjbqghefmqqehvatkkmxnmh
emadeupm-firstimpressionofthenotoriousprofessorchallenger

e->m, r->h, t->i, h->a, s->q, a->t, i->j, n->x, d->l, o->e, g->n, c->v, w->o, m->r,
l->k, v->w, p->g, u->b, f->f
```

The beginning “hesatinarotatingchair-ehinda-roadta-le” is “he sat in a rotating chair behind abroad table” giving us b->u.

At line 5, “restedonm-shoulder” is “rested on my shoulder” giving us y->s.

At line 8, “shouldersandachestli-ea-arrel” is “shoulders and a chest like a barrel, giving us k->c.

At line 3, “itwashissi-ewhichtookonesbreathaway” is “it was his size which took ones breath away” giving us z->p.

We now have

```
amqtijxtheitijxnvatjhumaixltuhetlitukmoajvaotqvewmhmljiaueecqrtgqtxlljtnhtrqtqjmximhmlajqqmtiqgbxhe
hesatinarotatingchairbehindabroadtablewhichwascoveredwithbooksmapsanddiagramsasienteredhisseatspunro

bxlietvmrmajqtggmthtxvmtlrmmtqgjtqghmgthmlfehqmiajxnqihntxmubixiefheqewmhgeomhixntgmhextkjis
undtofacemehisappearancemademegaspiwaspreparedforsomethingstrangebutnotforsooverpoweringapersonality

tqiajqjotqajqqjpmoajvaeicexmqumhtiatotsajqqjpmtxlajqjrgeqjxngmhmvmajqamtlotqmxehebqiamkthnmqija
asthisitwashissizewhichtookonesbreathawayhissizeandhisimposingpresencehisheadvasenormousthelargestih

twmmvmhqmmbxgextabrtxumjxnjtrqbhmiaiajqiegatlatljmvmhvmxibhmliexjioebklatwmqkjggmlewmhrmxiijhmkst
aveeverseenuponahumanbeingiamsurethathistophathadieverventuredtodonitwouldhaveslippedovermeentirelya

xlhmqimlexrsqaeblmqamatliamftvmtxlumthloajvajtqqevjtimojiatxtqqshjtxubkkiamfehrmhfkhejljamktiimhqe
ndrestedonmyshouldershehadthefaceandbeardwhichiassociatewithanassyrianbulltheformerfloridthelatterso

uktvtctkreqieatvmtqbqgvjvexefukbmqgtlmaqatgmltxlhjggkixnleoxewmhajqvamqiamatjhotqgmvbkjthgktqimhml
blackasalmosttohaveasuspicionofbluespadeshapedandrippingdownoverhischestthehairwaspeculiarplastered

leoxjxfhexijxtkexnvbhvjxnojqgewmhajqrtqqjwmfehmamtliaamsmqomhmukbmhntsbxlmhnmhtiuktvcibfiqwmhsvkmtw
downinfrontinalongcurvingwispovertismassiveforetheadtheeyeswerebluegrayundergreatblacktuftsveryclearv

mshvhiijvtktxlwmhertqimhfkbtabnmqghmtlefqaebklmbqxtlvtamqikjcmthhmkomhmiameiamhgtiqefajroajvatggm
erycriticalandverymasterfulahugespreadofshouldersandachestlikeabarrelweretheotherpartsofhimwhichappe

thmltuewmiमितुमकुतम्वफीओमखेहरेबकतल्वेवमहमलोजाकेखनुक्तवज्जिहजितलुमकेोजनहथिजनहब्रुकजनवेज्
aredabovethetablesavefortwoenormoushandscoveredwithlongblackhairthisandabellowingroaringgrumblingvoic

mrtlmbgrsfjbjqjrgmqqjexefiamxeiehjbqghefmqqehvatkkmxnmh
emadeupmyfirstimpressionofthenotoriousprofessorchallenger

e->m, r->h, t->i, h->a, s->q, a->t, i->j, n->x, d->l, o->e, g->n, c->v, w->o, m->r,
l->k, v->w, p->g, u->b, f->f, b->b, y->s, k->c, z->p
q->q, x->x, j->j
```

Note that q,x,j were not used in the plaintext. We have added q->q, x->x, j->j to the substitution key. The following is the plaintext with spaces inserted (puncuations not restored):

he sat in a rotating chair behind a broad table which was covered with books maps and diagrams as i entered his seat spun round to face me his appearance made me gasp i was prepared for something strange but not for so overpowering a personality as this it was his size which took ones breath away his size and his imposing presence his head was enormous the largest i have ever seen upon a human being i am sure that his tophat had i ever ventured to don it would have slipped over me entirely and rested on my shoulders he had the face and beard which i associate with



an assyrian bull the former florid the latter so black as almost to have a suspicion of blue spade shaped and rippling down over his chest the hair was peculiar plastered down in front in a long curving wisp over his massive forehead the eyes were blue gray under great black tufts very clear very critical and very masterful a huge spread of shoulders and a chest like a barrel were the other parts of him which appeared above the table save for two enormous hands covered with long black hair this and a bellowing roaring rumbling voice made up my first impression of the notorious professor challenger

□

The following programs are helpful (so go ahead and write them):

- (a) Code to print the top 1-grams, 2-grams, 3-grams. The 1-grams will help determine the character that `e` is encrypted to. The trigrams might help determine what `the` is encrypted to.
- (b) Given a character `c`, code that computes character(s) `d` such that `cd` and `dc` occurs most frequently. So if you suspect `e` is encrypted to `r`, your code will print all `x` such that `rx` and `xr` is common will be helpful in decoding `x`.
- (c) Given a collection of common 2-grams (in plaintext), a partially specified substitution, compute pairs of commonly occurring digrams of the form `xy,xz` or `xy,zx` or `xy,yz` or `xy,zy` (i.e., there are three distinct characters in the pairs of digrams) where two of the characters have already been decrypted and the remaining one has not and has not been assigned to a plaintext character.
- (d) Instead of the above where there are two digrams, listing 4-grams where the decryption of 3 are known and one is unknown is also useful.

Here are some exercises for you:

**Exercise 1.8.1.** Find the key and the plaintext of the following ciphertext encrypted using the substitution cipher:

debug:  
exercises/substitution-  
cipher-0/question.tex

```
psxdltuxtcauuxvifgtzwacspstpcvxqtdgcvxqtdgrvxbxd
pwzxdgxadtdipsxvtvlspstpwiiijxgihpinxvtblxdxbiwajx
tralphvxihpifteiijteihpaptwzpstpxnxvzidxbevxtpsqtb
ptjxdtqtztdgpsxzbaurwzqtwjxgfivqtvgbawxdptdgbptvad
ctbsivptnxdhxiflrvxbxbwxbgqagxdadctbapqxdppitutve
wxpxvvtlxpstpwtzevitgtdgqsapxadpsxbhdwacspssxlsawg
vxdewadjadcwxtxdgpxsavtvubidpsxevitgfwtpetwhbpvtgx
```

tdgctmxgauuxgatpxwzexwiqpsxuqtbttwtjxyhbpwajxtwtjxa  
dpsxexthpaxbifaptwzttwtjxqapsbqtdbtdgtdabwtgdtdgqxx  
radcqawwiqbexzidgapqxvxcvxxdbwirxbgippxgqapscvinxb  
ifpvxxbtdgtuagpsxpvxwbcwxtuxgpsxqsapxwauebifbptphx  
btctadbptwappwxawwpipsxwxftpqbtvthdgqsapxehawgadc  
qapsrawwtvbtgdgipsxvacstqtpxvftwltuxphuewadcgigd  
tuidcuibbzbpidxbpibrwtbsadpipsxwtjxbpxrbfxgfviupsx  
pxvvtlxpipsxqtpxvtdgipsxvbpixrbpipsxcvxxdwtqdbexbag  
xaptqtztlvibbpsxcvtbbzbwirxbgxxvqxvxfxxgadctdgadps  
xgabptdlxqsxvpsxcvinxbifpvxxbpsaljxdxgadpiqstpwii  
jxgtwuibptfivxpbqxvxxdivuihbbsrxbifcvxzbpidxwajxd  
ipsadcpstppsxlsawgvxdstgxnxbxxdxfivx

([Go to solution](#), page 132)



**Exercise 1.8.2.** Here's another one that is harder:

debug:  
exercises/substitution-  
cipher-1/question.tex

oznftyomrrrtqlnlzftqlqlmxemftlnyoozjtzyvqlzfzvgmnzm  
drsjmrrlxtqlxyjtynmvxqlhlmnfjmkmvxpyhvlglvhqlvdznj  
qzvpemrlomjtynfhqzjqqlxylfzvtlnezttlvtrstqnyipqyit  
tqlxmsynmttlvxzvpmjnzjaltemtjqoynmrrhlavyhqlhlmnft  
qlezvdlx

([Go to solution](#), page 133)



**Exercise 1.8.3.** And another:

debug:  
exercises/substitution-  
cipher-2/question.tex

xsdwoddnuskapbgxaayrpecuevbsxjjuskyaqddpdxvrolrhh  
qsyqmsdnvxumxsnsxsgxfuskbdahjrbbdbbursrhgxsddmnxme  
dnxmrrqsnevdedsxsnjxqbdnusevdumgunbebrgdcvxevdxdn  
cuevvubvxbcd

([Go to solution](#), page 134)



Once you have solved a few substitution ciphers by hand, you are ready to write a program to automate your process. Instead of a perfect solver, your program should aim to print a list of possible decryptions, ordered by likelihood (of being the original plaintext).

You can learn more about breaking substitution ciphers by doing a google search. Clearly the process of breaking a substitution cipher involves trying substitutions. This leads to search algorithms. Starting with the traditional

backtracking search, you will be led to other heuristic search algorithms such as local search algorithms, genetic algorithms, etc. Studying substitution ciphers will also lead to studies of probability theory and specifically markov chains. A search on google will reveal many research papers (many of which are very recent) on substitution cipher, AI search algorithms, markov chains, etc.

## Solutions

Solution to Exercise [1.8.1](#).

ANSWER: Then came a glimmer of daylight that grew and grew, and presently ended in another arch that looked out over a scene so like a picture out of a book about Italy that everyone's breath was taken away, and they simply walked forward silent and staring. A short avenue of cypresses led, widening as it went, to a marble terrace that lay broad and white in the sunlight. The children, blinking, leaned their arms on the broad, flat balustrade and gazed. Immediately below them was a lake just like a lake in "The Beauties of Italy" a lake with swans and an island and weeping willows; beyond it were green slopes dotted with groves of trees, and amid the trees gleamed the white limbs of statues. Against a little hill to the left was a round white building with pillars, and to the right a waterfall came tumbling down among mossy stones to splash into the lake. Steps fed from the terrace to the water, and other steps to the green lawns beside it. Away across the grassy slopes deer were feeding, and in the distance where the groves of trees thickened into what looked almost a forest were enormous shapes of grey stone, like nothing that the children had ever seen before.

debug:  
exercises/substitution-  
cipher-0/answer.tex

Solution to Exercise [1.8.2](#).

ANSWER: First of all there is the Headmaster of Fiction. He is invariably called "The Doctor," and he wears cap and gown even when birching malefactors—which he does intermittently throughout the day—or attending a cricket match. For all we know he wears them in bed.

debug:  
exercises/substitution-  
cipher-1/answer.tex

Solution to Exercise [1.8.3](#).

ANSWER: An exceedingly small boy with snapping blue eyes, a shock of sunburned hair and an amazing self-possession of manner, darted around the tent and paused in their midst, somewhat heated with his haste. ”’

debug:  
exercises/substitution-  
cipher-2/answer.tex

## 1.9 Permutation cipher debug: classical-cipher-permutation-cipher.tex

In the above ciphers, a character is replaced by another. The **permutation cipher** is different: each character is *moved* to a different *location* in the plaintext to form the ciphertext.

permutation cipher

Here's an example. Look at this:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix}$$

It's just the function where a value in the top row maps to the corresponding value at the bottom row:

$$\begin{aligned} \pi(1) &= 3 \\ \pi(2) &= 1 \\ \pi(3) &= 2 \\ &\vdots \\ \pi(6) &= 6 \end{aligned}$$

Let  $\pi$  is called a permutation of  $\{1, 2, 3, 4, 5, 6\}$ . A **permutation** is simply a bijection (1–1 and onto) function from  $\{1, 2, 3, \dots, n\}$  to  $\{1, 2, 3, \dots, n\}$ . As a shorthand, the above permutation

permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix}$$

is also written as

$$\pi = (1 \ 3 \ 2)(4 \ 5)(6)$$

This is called the **cycle notation** of  $\pi$ . In the above example, we say that the permutation  $\pi$  has **length** 6.

cycle notation  
length

**Exercise 1.9.1.** What is the cycle notation of

debug:  
exercises/permutation-  
0/question.tex

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 7 & 2 & 5 & 8 & 3 & 6 \end{pmatrix}$$

([Go to solution](#), page 145)

□

How does a permutation give you an encryption? For the permutation  $\pi = (1 \ 3 \ 2)(4 \ 5)(6)$ , you do this:

1. character at position 1 goes to position 3,
2. character at position 2 goes to position 1,
3. etc.

In the example above, the  $n$  is 6. For instance for the string `marvin` and the permutation  $\pi = (1\ 3\ 2)(4\ 5)(6)$ , the since in the permutation  $1 \mapsto 3$ , the character at position 1 goes to position 3:

1	2	3	4	5	6
m	a	r	v	i	n
		m			

since  $3 \mapsto 2$ , the character at position 3 goes to position 2:

1	2	3	4	5	6
m	a	r	v	i	n
	r	m			

Etc. Altogether I get

$$E(\pi, \text{marvin}) = \text{armivn}$$

This assumes your string has a length which is a multiple of 6. So you might need to add some dummy data at the end of your string.

Note that in the above example, you have to encrypt substrings of length 5. Therefore if the original plaintext does not have a length that is a multiple of 5, you would need to pad it with some characters until the length is a multiple of 5.

**Exercise 1.9.2.** How would you choose to pad the plaintext? You might want to finish this section before you answer this question. (I.e., you want to know what Eve will do and design your padding to make it difficult for her.) ([Go to solution](#), page 146) □

debug:  
exercises/permutation-  
1/question.tex

The decryption is the same algorithm as the encryption except that the permutation is read in the “opposite direction”. If

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$$



then the inverse of  $\pi$ , denoted by  $\pi^{-1}$  is the “opposite” of the above:

$$\pi^{-1} = \begin{pmatrix} 3 & 1 & 2 & 5 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

(do you see the row switch?) and to make things look nice, you arrange the top row:

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

Using the cycle notation,

$$\pi^{-1} = (1 \ 2 \ 3)(4 \ 5)$$

So

$$D(\pi, y)$$

is the same as

$$E(\pi^{-1}, y)$$

So

$$\begin{aligned} D((1 \ 3 \ 2)(4 \ 5), \text{armivn}) &= E((1 \ 3 \ 2)(4 \ 5)^{-1}, \text{armivn}) \\ &= E((1 \ 2 \ 3)(4 \ 5), \text{armivn}) \\ &= \text{marvin} \end{aligned}$$

### Exercise 1.9.3.

debug:  
exercises/permutation-  
2/question.tex

1. Encryption "Where there's life there's hope ... and need of vittles."  
using the permutation cipher with key  $\pi = (1 \ 5)(2 \ 6 \ 3 \ 4)$ . Pad with  
z if necessary.
2. What is the inverse of  $\pi$ ?
3. Decrypt your ciphertext from the first part.

([Go to solution](#), page 147)

□

**Exercise 1.9.4.** What is the size of the key space of the permutation cipher if you know that the permutation is a permutation on 1, 2, 3, 4, 5? What if it's a permutation on 1, 2, 3, 4, 5, 6, 7, 8, 9, 10? ([Go to solution](#), page 148)

debug:  
exercises/permutation-  
3/question.tex

□

**Exercise 1.9.5.** Why is the permutation cipher a special case of the Hill cipher? To try it out, write down an example of Hill cipher and re-interpret

debug:  
exercises/permutation-  
4/question.tex

it as a permutation cipher. You'll see that the matrix has a special form – it has only 0s and 1s arranged in a certain way. Such a matrix is called a **permutation matrix**. ([Go to solution](#), page 149)  $\square$

permutation matrix

## BREAKING THE PERMUTATION CIPHER

Frequency analysis obviously won't help: each letter is not replaced by another. It's the position of a letter that is changed.

Now suppose the permutation is

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$$

Suppose you know the length of the permutation is 5. After breaking your ciphertext into substrings of length 5, suppose one of these substrings is

hatoe

We know that **th** is a commonly occurring digram. Then it's natural to suspect that this comes from the encryption of 5 letters containing **th**, either

```
plaintext : ..... th... ...
ciphertext: ..... hatoe ...
```

or

```
plaintext : ..... .th.. ...
ciphertext: ..... hatoe ...
```

or

```
plaintext : ..... ..th. ...
ciphertext: ..... hatoe ...
```

or

```
plaintext : ..... ...th ...
ciphertext: ..... hatoe ...
```

Whereas for a substring of length 5, there are  $5! = 120$  possible permutations, once two is fixed, there are  $3! = 6$  possible permutations. This might cut down on the search for the key and help break the ciphertext. If **th** is not found or the above analysis involving **th** does not work, you go on to the next commonly occurring digram.

Of course you do not know if the length of the permutation is length 5. You therefore have to do a loop over all possible permutation lengths. Of course the key length cannot be 1 since that would be doing nothing! For key of size two, the permutation must be (1, 2). For a key of size three, there are  $3! = 6$  permutations: There are 6 possible permutations: (1)(2)(3), (1)(2, 3), (2)(1, 3), (3)(1, 2), (1, 2, 3), (1, 3, 2). Since there are not many permutations of length 3, for this case, you might want to simply try all permutations. In general, you try key lengths dividing the length of the ciphertext.

Note that it's also possible that the **th** in the plaintext is broken up with the **t** is one substring of length 5 and **h** in the following substring. For instance, here is a case:

```
plaintext : canth efish
ciphertext: acnht feihs
```

Of course it's also OK if you do not wish to consider cases where the digram is split across two substrings, since in the above case, you would have picked up the **he** when you consider a key of length (say) 10.

By looking at lots of digrams and trigrams, if the ciphertext is long enough, you should be able to break the code.

**Example 1.9.1.** Let me break the following permutation ciphertext

imahsnrwnesotanfusefvitieedoslswma

SOLUTION.

The length is 35. Since the length of the permutation must divide 35, it must be 1, 5, 7, 35.

LENGTH 1. Of course if the length is 1, then the ciphertext is the same as the plaintext. But the plaintext is meaningless. Therefore the length cannot be 1.

LENGTH 5. Assume the key length is 5. I break up the string above into substrings of length 5 to make it easier to read:

imahs nrwne sotan fusef vitie edosl wswma

LENGTH 5. DIGRAM TH. I don't see any th in each substrings or consecutive substrings.

LENGTH 5. DIGRAM HE. I do see he:

imahs nrwne sotan fusef vitie edosl wsma

If you assume these two characters come from he, then it has to come from this:

```
plaintext:  ....h e.... ..... ..... ..... .....
ciphertext: imahs nrwne sotan fusef vitie edosl wsma
```

which means the permutation contains  $1 \mapsto 5, 5 \mapsto 4$ . If that's the case, we have

```
plaintext:  s...h e...n n...a f...e e...i l...s a...m
ciphertext: imahs nrwne sotan fusef vitie edosl wswma
```

Look at just the first group of 5 characters the s...h can only be simah, siamh, saimh, samih, smiah, or smaih. The only promising one is siamh (maybe "Siam has cats ...?"). Using this we get

```
plaintext:  siamh enwrn nstoa ffsue evtii leods awwsm
ciphertext: imahs nrwne sotan fusef vitie edosl wswma
```

which is meaningless.

[force newpage]

LENGTH 5. DIGRAM **IN**. The next digram to try is in:

```
plaintext:  ....i n.... ..... ..... ..... .....
ciphertext: imahs nrwne sotan fusef vitie edosl wswma
```

or

```
plaintext:  ....i n.... ..... ..... ..... .....
ciphertext: imahs nrwne sotan fusef vitie edosl wswma
```

The first is not possible. (Why?) The second gives us

```
plaintext:  h...i n...n a...s e...f i...v s...e m...w
ciphertext: imahs nrwne sotan fusef vitie edosl wswma
```

Looking at just the first 5 characters, the only possible decryptions are hmasi, hmsai, hamsi, hasmi, hsami, hsmi. Only hamsi, hasmi, are promising. From hamsi,

```
plaintext:  hamsi nwren atons esuff itiev sodle mwsaw
ciphertext: imahs nrwne sotan fusef vitie edosl wswma
```

which is meaningless. From hasmi,

```
plaintext:  hasmi nwern atnos esfuf iteiv solde mwasw
ciphertext: imahs nrwne sotan fusef vitie edosl wswma
```

which is meaningless.

LENGTH 5. DIGRAM **ER**. The next digram to try is er:

- From

```
plaintext:  .... er... ..... ..... ..... .....
ciphertext: imahs nrwne sotan fusef vitie edosl wswma
```

we get

```
plaintext:  sm... er... no... fu... ei... ld... as...
ciphertext: imahs nrwne sotan fusef vitie edosl wswma
```

The first 5 characters can only be smahs, smash, smhas, smhsa, smsah, smsa. Only smash seems to form a word. (smhas might be for instance a chasm has separated us, but the smhas is the *beginning* of the plaintext.) Using that, we get

```
plaintext:  smash erwen notna fusfe eitei ldols aswam
```

ciphertext: imahs nrwne sotan fusef vitie edosl wswma  
which is not meaningful.

- From

plaintext: ..... .er.. ..... ..... .....  
ciphertext: imahs nrwne sotan fusef vitie edosl wswma

we get

plaintext: .sm.. .er.. .no.. .fu.. .ei.. .ld.. .as..  
ciphertext: imahs nrwne sotan fusef vitie edosl wswma

The first 5 characters can only be ismah, ismha, asmih, asmhi, hsmia, hsmia, none of which is meaningful, except possibly for ismah (example: is mahi mahi a fish?), ismha (example: a schism happened overnight), asmih (example: protoplasm i have), asmhi (example: protoplasm have i) but you can check that these do not lead to anything meaningful.

- From

plaintext: ..... ..er. .... ..er. .... ..er. .... ..er.  
ciphertext: imahs nrwne sotan fusef vitie edosl wswma

we get

plaintext: ..sm. ..er. ..no. ..fu. ..ei. ..ld. ..as..  
ciphertext: imahs nrwne sotan fusef vitie edosl wswma

The first 5 characters can only be iasmh, ihsma, aismh, ahsmi, hisma, hasmi. Only ahsmi, hisma and hasmi seem promising. For ahsmi, we get

plaintext: ahsmi wnern tanos sefuf tieiv oslde wmasw  
ciphertext: imahs nrwne sotan fusef vitie edosl wswma

which is meaningless. For hisma, we get

plaintext: hisma nnerw asnot effus iveit seldo mwasw  
ciphertext: imahs nrwne sotan fusef vitie edosl wswma

which finally gives us the plaintext

hismannerwasnoteffusiveitseldomwasw

This gives us

His manner was not effusive. It seldom was.

The last w is redundant (it's a padding).

The key (i.e., permutation) is  $(1, 4, 2)(3, 5)$  with a key length of 5. □

**Exercise 1.9.6.** Of course the process is tedious and error prone. So you know what to do:

debug:  
exercises/permutation-  
5/question.tex

- Write a program to encrypt and decrypt messages for the permutation cipher. For instance you can try this permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$$

by entering 2,3,5,4,1. Note that (mathematically speaking) permutations are written as bijection between sets  $\{1, 2, 3, \dots, m\}$ . You might want to start with 0 instead. In that case, you should tell your user.

- From the above experience of computation-by-hand, we see that for breaking a permutation cipher, the program accepts a ciphertext and ask you for a permutation. Furthermore your program should allow you to enter a permutation partially. For instance to enter this permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & 3 & & & 1 \end{pmatrix}$$

the user enters ?,3,?,?,1 in your program. You also want to allow the user to modify a permutation.

- You also want the program to list commonly occurring digrams that appears in the ciphertext, especially including those that appear within substrings of length  $m$  (the length of the permutation).
- With all the above, you can then automate the process to imitate what I did in the previous example.

([Go to solution](#), page 150)

□

**Exercise 1.9.7.** Decrypt this permutation ciphertext:

rhetibbalhotnweerstthigalontakeinuntolfemsorywae

and write down the key. It's advisable to do some of these decryption by hand.

([Go to solution](#), page 151)

□

**Exercise 1.9.8.** Decrypt this permutation ciphertext:

uyashonlpotlsasdaskeathrsshcdp

and write down the key. It's advisable to do some of these decryption by hand.

([Go to solution](#), page 152)

□

debug:  
exercises/permutation-  
6/question.tex

debug:  
exercises/permutation-  
7/question.tex

**Exercise 1.9.9.** Decrypt this permutation cipher:

debug:  
exercises/permutation-  
8/question.tex

konusyowairduathsiatirttleistmkewihsihenatprdmpevn  
oanigoliorwackatmhfianboemlrteseudganubtodatofeao  
hispaxtinyioenedpipstcheiaatlalairywdhlitsisdntehe  
owytmhmaotorlteedmnhieswwanogutyf

and write down the key. It's advisable to do some of these decryption by hand.  
([Go to solution](#), page 153) □

**Exercise 1.9.10.** Decrypt this permutation ciphertext:

debug:  
exercises/permutation-  
9/question.tex

wshiattesotbtfeistamiweseostwrhttmsoiefd

and write down the key. It's advisable to do some of these decryption by hand.  
([Go to solution](#), page 154) □

**Exercise 1.9.11.** You have been intercepting ciphertexts sent between Alice and Bob. You know that they use the permutation cipher. You notice this fact: All the 1000 ciphertexts have length 720. Why is that? Or is that just coincidence? ([Go to solution](#), page 155) □

debug:  
exercises/permutation-  
10/question.tex



## Solutions

Solution to Exercise [1.9.1](#).

Solution not provided.

debug:  
exercises/permutation-  
0/answer.tex

Solution to Exercise [1.9.2](#).

Solution not provided.

debug:  
exercises/permutation-  
1/answer.tex

Solution to Exercise [1.9.3](#).

Solution not provided.

debug:  
exercises/permutation-  
2/answer.tex

Solution to Exercise [1.9.4](#).

Solution not provided.

debug:  
exercises/permutation-  
3/answer.tex

Solution to Exercise [1.9.5](#).

Solution not provided.

debug:  
exercises/permutation-  
4/answer.tex

Solution to Exercise [1.9.6](#).

Solution not provided.

debug:  
exercises/permutation-  
5/answer.tex

Solution to Exercise [1.9.7](#).

plaintext: therabbitholewentstraightonlikeatunnelforsomeway

the rabbit-hole went straight on like a tunnel for some way

(3,1,2,0)

debug:  
exercises/permutation-  
6/answer.tex

Solution to Exercise [1.9.8](#).

youshallnotpassdaserkthaspscdh

yousha llnotp assdas erktha spscdh

(1,5,0,3,4,2)

debug:  
exercises/permutation-  
7/answer.tex



Solution to Exercise [1.9.9](#).

youknowsaidarthuritsattimeslikethiswhenimtrappedinavogonairlockwithama  
nfrombetelgeuseandabouttodieofasphyxiationindeepspacethatireallywishid  
listenedtowhatmymothertoldmewheniwasyoung

(5,1,3,0,2,6,7,4)

debug:  
exercises/permutation-  
8/answer.tex

Solution to Exercise [1.9.10](#).

It was the best of times, it was the worst of times

itwasthebestoftimesitwastheworstoftimesd

itwasthe bestofti mesitwas theworst oftimesd

(3,6,0,4,1,5,2,7)

ehtitsaw itfbeots sawmetis tsrthows dseofmit

debug:  
exercises/permutation-  
9/answer.tex

Solution to Exercise [1.9.11](#).

Solution not provided.

debug:  
exercises/permutation-  
10/answer.tex

## 1.10 Hill cipher debug: classical-cipher-hill-cipher.tex

**Hill's cipher** was announced in 1929. This cipher works with a block of  $n$  characters at a time. For both encryption and decryption, you treat each block of  $n$  characters (or mod 26 integers) as a column vector and we multiply it with an  $n$ -by- $n$  matrix. Of course everything is done in mod 26. Here are more details.

Hill's cipher

Recall how to multiply a  $n$ -by- $n$  matrix with a column vector of size  $n$ . I'll do this for  $n = 2$ .

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

(If you have not seen this before take the computer graphics class CISS380. You can't call yourself a serious computer scientist if you can't do matrix multiplication. Most top CS programs actually has a linear algebra requirement.) That's matrix multiplication where all numbers are in  $\mathbb{R}$ . The (multiplicative) **inverse matrix** of

inverse matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The expression  $ad - bc$  is called the **determinant** of  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and is written

determinant

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

If a matrix has an inverse, we say that the matrix is **invertible**.

invertible

Not only can you multiply a 2-by-2 matrix with a 2-by-1, you can actually also multiply a 2-by-2 matrix with a 2-by-2. Here's how you do it:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} aA + bB & aC + bD \\ cA + dB & cC + dD \end{pmatrix}$$

**Exercise 1.10.1.** The following refers to matrices with  $\mathbb{R}$  values. Do all these exercises:

debug: exercises/hill-0/question.tex

1. Compute

$$\begin{pmatrix} 2 & 0 \\ 4 & 12 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 23 & 7 \end{pmatrix}$$

2. Compute

$$\begin{pmatrix} 0 & 2 \\ 23 & 7 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 4 & 12 \end{pmatrix}$$

3. If  $M, N$  are 2-by-2 matrices, is it always true that  $MN = NM$ ?
4. Prove the following theorem: Let  $A, B, C$  be three 2-by-2 matrices. Prove that

$$(AB)C = A(BC)$$

□

([Go to solution](#), page 165)

□

Define the following:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is called the (2-by-2) **identity matrix**.

identity matrix

**Exercise 1.10.2.** The following refers to matrices with  $\mathbb{R}$  values.

debug: exercises/hill-1/question.tex

1. Show that

$$I \begin{pmatrix} 2 & 0 \\ 4 & 12 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 12 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 12 \end{pmatrix} I$$

2. Show that

$$I \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

3. Compute

$$\begin{pmatrix} 2 & 0 \\ 4 & 12 \end{pmatrix}^{-1}$$

Then show that

$$\begin{pmatrix} 0 & 2 \\ 23 & 7 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 23 & 7 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 23 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 2 \\ 23 & 7 \end{pmatrix}$$

□

([Go to solution](#), page 166)

□

**Exercise 1.10.3.** The following refers to matrices with  $\mathbb{R}$  values. Let  $M$  be

debug: exercises/hill-2/question.tex

a 2-by-2 matrix.

1. Is every matrix invertible?
2. Prove that if  $M$  is a 2-by-2 matrix,

$$MM^{-1} = I = M^{-1}M$$

3. Prove that if  $M$  is a 2-by-2 matrix, Prove that

$$IM = M = MI$$

4. Prove that if  $v$  is a 2-by-1 matrix,

$$Iv = v$$

□

([Go to solution](#), page 167)

□

The set of  $n$ -by- $n$  matrices with  $\mathbb{R}$  values is usually denoted by  $M_n(\mathbb{R})$ . The set of invertible matrices is usually denoted by  $GL_n(\mathbb{R})$ . This is called the **general linear group** of  $n$ -by- $n$  matrices over  $\mathbb{R}$ .

general linear group

It turns out that everything works just as fine if the numbers are in mod 26. You just need to be careful with the determinant of the inverse computation. Recall that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for values in  $\mathbb{R}$ . This is the same for values in  $\mathbb{Z}/26$  as long as you think of

$$\frac{1}{ad - bc}$$

as the multiplicative inverse of  $ad - bc$  in  $\mathbb{Z}/26$ .

**Exercise 1.10.4.** The following refers to matrices with  $\mathbb{Z}/26$  values. All the values in the matrices should be simplified to a value in the range 0..25. Do all these exercises:

debug: exercises/hill-3/question.tex

1. Compute

$$\begin{pmatrix} 2 & 0 \\ 4 & 12 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 23 & 7 \end{pmatrix}$$

2. Compute

$$\begin{pmatrix} 0 & 2 \\ 23 & 7 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 4 & 12 \end{pmatrix}$$

3. If  $M, N$  are 2-by-2 matrices, is it always true that  $MN = NM$ ?  
 4. Prove the following theorem: Let  $A, B, C$  be three 2-by-2 matrices. Prove that

$$(AB)C = A(BC)$$

(Look at the facts below on breaking the Hill's cipher. Why did I ask you to prove this theorem? Where is this fact used below?)

5. Does

$$\begin{pmatrix} 0 & 2 \\ 23 & 7 \end{pmatrix}$$

have an inverse? If it does compute it. If not explain why.

6. Does

$$\begin{pmatrix} 1 & 2 \\ 23 & 7 \end{pmatrix}$$

have an inverse? If it does compute it. If not explain why.

7. Compute

$$\begin{pmatrix} 5 & 2 \\ 23 & 7 \end{pmatrix}^{-1}$$

and show that

$$\begin{pmatrix} 5 & 2 \\ 23 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 2 \\ 23 & 7 \end{pmatrix} = I = \begin{pmatrix} 5 & 2 \\ 23 & 7 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 23 & 7 \end{pmatrix}^{-1}$$

([Go to solution](#), page 168)

□

**Exercise 1.10.5.** Say the key is

debug: exercises/hill-4/question.tex

$$M = \begin{pmatrix} 11 & 8 \\ 3 & 7 \end{pmatrix}$$

Here  $n = 2$ , i.e., the block size is 2.

1. What is the encryption of  $x = \text{fortytwo}$ ? Call the ciphertext  $y$ .
2. What is the inverse matrix  $M^{-1}$  of  $M$ ? (Note: we are in mod 26.) Check that  $MM^{-1} = I = M^{-1}M$ .
3. Decrypt your  $y$  and make sure you get  $x$ .
4. Look for the probability of 2-grams. Take a long enough string, encrypt it using Hill cipher and with the above key. Look at the probability of 2-grams in the ciphertext. Compare the two probabilities of the 2-grams

from the plaintext and from the ciphertext.

([Go to solution](#), page 169)

□

This should not be surprising: The set of  $n$ -by- $n$  matrices with mod 26 values is usually denoted by  $M_n(\mathbb{Z}/26)$ . The set of invertible matrices is usually denoted by  $GL_n(\mathbb{Z}/26)$ .

**Exercise 1.10.6.** Let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

a mod 26 matrix. Prove that

$$MI = M = IM$$

for any 2-by-2 mod 26 matrix  $M$ .

□ ([Go to solution](#), page 170) □

Note that in the above, since the block size is 2, you would need to encrypt a string with a string length that is a multiple of 2. If it's not, you can (for instance) pad with some character.

The important thing to observe is that if

$$E(M, (x, y)) = (x', y')$$

( $M$  is a matrix key) note that  $x$  takes part in the value  $x'$  and  $y'$ . Same thing for  $y'$ . This means that if I change one character in my plaintext, *two* character in the ciphertext is changed. Informally this means that the statistical behavior of  $x$  is spread out across multiple characters. Note only that  $x'$  also depends on several values of the key (the matrix). This makes it much hard to analyze the key one part at a time when studying ciphertexts. This makes it a lot harder to break Hill's cipher, especially when the matrix size is larger, say the matrix is 32-by-32.

The above two important observations were later formalized and studied by Claude Shannon in 1945.

1. The fact that a character of the ciphertext depends on several parts of the key is the concept of **confusion**
2. the fact that one character in the plaintext influences multiple characters in the ciphertext is called the concept of **diffusion**

confusion

diffusion

More on that in a much later chapter.



**Exercise 1.10.7.** Write a function `matmult` so that if `M` is a 2D array of size 2-by-2 and `v` is a column vector, i.e., a 2D array of size 2-by-1, then `matmult(M, v)` will return the product `Mv` as described above. Remember to mod by 26 all the values. (If you know your linear algebra – either from computer graphics CISS380 or linear algebra math class, you can figure out how to complete this function for `M` and `v` of sizes  $n$ -by- $n$  and  $n$ -by-1 respectively) (Go to solution, page 171)  $\square$

debug: exercises/hill-6/question.tex

**Exercise 1.10.8.** Use a 2D array for 2-by-2 matrices. Write a function `matdet` so that `matdet(M)` return the determinant of `M`. Remember that we are in mod 26.

debug: exercises/hill-7/question.tex

(Go to solution, page 172)  $\square$

**Exercise 1.10.9.** Use a 2D array for 2-by-2 matrices. Write a function `matinv` so that `matinv(M)` return the inverse matrix of `M`. Return `None` if `M` is not invertible. Remember that we are in mod 26. (Go to solution, page 173)  $\square$

debug: exercises/hill-8/question.tex

**Exercise 1.10.10.** Let

debug: exercises/hill-9/question.tex

$$M = \begin{pmatrix} 3 & 7 \\ 5 & 2 \end{pmatrix}$$

1. Encrypt  $x = \text{solongandthanksforthefish}$  using Hill's cipher with key  $M$ . Call the resulting string  $y$ .
2. Compute  $M^{-1}$ .
3. Decrypt  $y$ . You should get  $x$ .

(Go to solution, page 174)  $\square$

**Exercise 1.10.11.**

debug: exercises/hill-10/question.tex

1. What is the size of the key space assuming the matrix is 2-by-2? (I mean for mod 26 of course.) This means: how many invertible mod 26 matrices are there? Or: What is  $|\text{GL}_2(\mathbb{Z}/26)|$ ? You can write down a few invertible mod 26 matrices. But after a while you might want to write a program.
2. See if you can find a plausible formula for the case of  $n = 2$ . The formula involves 2 and 13 because  $26 = 2 \times 13$  and because  $n = 2$ . (Yes, there's a formula.) It also works if mod 26 is replaced by mod  $N$  for any positive integer  $N \geq 1$ . You can try to work with matrices with values in  $\mathbb{Z}/5$

then  $\mathbb{Z}/8$ , etc. and see if you can find a formula for  $|\text{GL}_2(\mathbb{Z}/N)|$ . (The formula works when  $n$  is any integer  $\geq 1$ . But you would need to know about matrices of size larger than 2-by-2.)

□

([Go to solution](#), page 175)

□

Collecting up all the information above, formally, the Hill's cipher of block size  $n$  (just think of  $n = 2$ ) is defined as follows: Let

$$P = C = \{0, \dots, 25\}^n = (\mathbb{Z}/26)^n$$

and

$$K = \text{GL}_n(\mathbb{Z}/26)$$

The encryption function  $E$

$$E : K \times P \rightarrow C$$

is defined to be

$$E(M, x) = Mx$$

and

$$D : K \times C \rightarrow P$$

is defined to be

$$D(M, x) = M^{-1}x$$

And  $(E, D)$  is a cipher because

$$D(M, E(M, x)) = D(M, Mx) = M^{-1}Mx = Ix = x$$

**Exercise 1.10.12.** What about 3-by-3 matrices? How do you compute the inverse? What is the size of the key space for 3-by-3? What about  $n$ -by- $n$ ? ([Go to solution](#), page 176)

□

debug: exercises/hill-11/question.tex

## BREAKING HILL'S CIPHER

Now how do you break the Hill's cipher?

Let's stick to a 2-by-2 key. Suppose you have the following: you have the two plaintexts  $x_1 = (A, B)$  and  $x_2 = (C, D)$  and their encryptions  $y_1 = (A', B')$  and  $y_2 = (C', D')$ . Therefore

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} aA + bB \\ cA + dB \end{pmatrix} = \begin{pmatrix} A' \\ B' \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} aC + bD \\ cC + dD \end{pmatrix} = \begin{pmatrix} C' \\ D' \end{pmatrix}$$

In the above, the unknowns are  $a, b, c, d$ , i.e., the key. These two equations can be combined to get an equation involving the product of 2-by-2 matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} aA + bB & aC + bD \\ cA + dB & cC + dD \end{pmatrix} = \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix}$$

You now have this matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix}$$

Note that you have the values of  $A, B, C, D, A', B', C', D'$ . The unknowns are  $a, b, c, d$ . What do you do? You multiply on the right by the inverse matrix of  $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$  to get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix}^{-1} = \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix}^{-1}$$

which gives you

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} I = \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix}^{-1}$$

which finally gives you the key

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix}^{-1}$$

There's an assumption above: we have to assume that  $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$  is invertible. It's possible the first 4 characters of the original message does not form an

invertible matrix. But if you have enough pairs of plaintext and corresponding ciphertext blocks, you hope to be able to get an invertible matrix and perform the above computation. Therefore we are assuming the known plaintext attack model. Or we are assuming the chosen plaintext attack model.

## Solutions

Solution to Exercise [1.10.1](#).

Solution not provided.

debug: exercises/hill-  
0/answer.tex

Solution to Exercise [1.10.2](#).

Solution not provided.

debug: exercises/hill-  
1/answer.tex

Solution to Exercise [1.10.3](#).

Solution not provided.

debug: exercises/hill-  
2/answer.tex

Solution to Exercise [1.10.4](#).

Solution not provided.

debug: exercises/hill-  
3/answer.tex



Solution to Exercise [1.10.5](#).

Solution not provided.

debug: exercises/hill-  
4/answer.tex

Solution to Exercise [1.10.6](#).

Solution not provided.

debug: exercises/hill-  
5/answer.tex

Solution to Exercise [1.10.7](#).

Solution not provided.

debug: exercises/hill-  
6/answer.tex

Solution to Exercise [1.10.8](#).

Solution not provided.

debug: exercises/hill-  
7/answer.tex

Solution to Exercise [1.10.9](#).

Solution not provided.

debug: exercises/hill-  
8/answer.tex

Solution to Exercise [1.10.10](#).

Solution not provided.

debug: exercises/hill-  
9/answer.tex

Solution to Exercise [1.10.11](#).

Solution not provided.

debug: exercises/hill-  
10/answer.tex

Solution to Exercise [1.10.12](#).

Solution not provided.

debug: exercises/hill-  
11/answer.tex



## 1.11 One-time pad cipher debug: classical-cipher-one-time-pad-cipher.tex

The one time pad is easy: Suppose Bob wants to send a message. Bob will need to translate this into bits. (For instance use the ASCII code or some other agreed upon format.) Say the plaintext (in bits) is 00101011.

The key is a very long sequence of random bits that Bob and Alice has agreed upon. Suppose the sequence is 0101101011101110001010101100110101001.

Bob takes his message 00101011 and exclusive-or with the first eight bits of the key 0101101011101110001:

```
00101011
0101101011101110001010101100110101001
01110001
```

He then removes the 8 bits used:

```
0101101011101110001010101100110101001
```

and sends the ciphertext (in bits) 01110001 to Alice.

Once Alice received 01110001, she exclusive-or with her key:

```
01110001
0101101011101110001010101100110101001
00101011
```

which is the plaintext. She also removes the bits in her key used in the decryption: ~~01011010~~11101110001010101100110101001.

The next time they communicate, they will start with the remaining bits of their key.

That's it.

It's a very simple cipher. We don't have enough math (yet) to prove it, but you can sense that this cipher is extremely secure. Why?

**Exercise 1.11.1.** Suppose the ciphertext is zsdeasheir. (Of course you have to convert this to bits, say using ASCII code. Find a key so the above is decrypted to killatfour. Find another key so that the above is decrypted to anapplepie. (Go to solution, page 179)  $\square$

debug: exercises/otp-0/question.tex

There's a rumor that during the Cold War, Washington D.C. communicates with Moscow using one time pad.

To be secure the key must be a random sequence. Furthermore, the key cannot be reused. The other problem is that the key is really long.

(Using the concept of entropy of information theory, Claude Shannon can prove that the ciphertext contains no information about the plaintext, other than the length.)

## Solutions

Solution to Exercise [1.11.1](#).

Solution not provided.

debug: exercises/otp-  
0/answer.tex

## 1.12 Block and stream ciphers debug:

classical-cipher-block-and-stream-cipher.tex

Recall that in Caesar cipher, you only define the encryption and decryption for single characters.

The encryption/decryption is then extended to a whole string by encrypting/decrypting character-by-character.

So if you're encrypting the string `cat` with the encryption function  $E_K$ , you just do

$$E_K(\text{cat}) = E_K(\text{c})E_K(\text{a})E_K(\text{t})$$

Another thing you should know is that, instead of writing

`cat`

might write

`c||a||t`

i.e., it's common to use `||` to denote concatenation of data.

A **block** cipher is a cipher system where more than one character (a block) is encrypted together at the time same.

But why encrypt a block of characters at a time?

In the case of shift and affine cipher, the frequency of the plaintext character is the same as the frequency of the corresponding ciphertext character. Sure, an `a` might become a `t` which looks different from `a`. But guess what? `a`'s fingerprint – its frequency – follows it to the ciphertext.

The same idea is the same for Vigenere as long as you have the length of the key.

The next cipher, Hill cipher, is different. We're going to encrypt several characters at the same time. For instance Hill cipher (depending on the key) can encrypt 2 characters at the same time to produce 2 new characters.

**Exercise 1.12.1.** Is Caesar's cipher a block or stream cipher?

## 1.13 Linear feedback shift register debug: classical-cipher-lfsr.tex

Recall that the one time pad uses exclusive-or to operator on bit sequences. Note that the exclusive-or is the same as addition mod 2!!! In other words addition *in mod 2*

$$\begin{aligned} 0 + 0 &\equiv 0 \pmod{2} \\ 0 + 1 &\equiv 1 \pmod{2} \\ 1 + 0 &\equiv 1 \pmod{2} \\ 1 + 1 &\equiv 0 \pmod{2} \end{aligned}$$

is the same as exclusive-or operation *on bits*:

$$\begin{aligned} 0 \oplus 0 &= 0 \\ 0 \oplus 1 &= 1 \\ 1 \oplus 0 &= 1 \\ 1 \oplus 1 &= 0 \end{aligned}$$

(I'm using  $\oplus$  for exclusive-or bit operator – that's pretty standard.)

Now I'm going to this: first I define the following bit sequence of length 5:

$$x_1 x_2 x_3 x_4 x_5 = 10110$$

which is the same as defining integer  $x_1, \dots, x_5$  in  $\mathbb{Z}/2$ . Then I define

$$x_{n+6} \equiv x_{n+1} + x_{n+2} + x_{n+4} \pmod{2}$$

for  $n \geq 0$ . For instance

$$x_6 \equiv x_1 + x_2 + x_4 \equiv 1 + 0 + 1 \equiv 0 \pmod{2}$$

Here are next about 15:

$$\begin{aligned}
 x_6 &\equiv x_1 + x_2 + x_4 \equiv 1 + 0 + 1 \equiv 0 & (\text{mod } 2) \\
 x_7 &\equiv x_2 + x_3 + x_5 \equiv 0 + 1 + 0 \equiv 1 & (\text{mod } 2) \\
 x_8 &\equiv x_3 + x_4 + x_6 \equiv 1 + 1 + 0 \equiv 0 & (\text{mod } 2) \\
 x_9 &\equiv x_4 + x_5 + x_7 \equiv 1 + 0 + 1 \equiv 0 & (\text{mod } 2) \\
 x_{10} &\equiv x_5 + x_6 + x_8 \equiv 0 + 0 + 0 \equiv 0 & (\text{mod } 2) \\
 x_{11} &\equiv x_6 + x_7 + x_9 \equiv 0 + 1 + 0 \equiv 1 & (\text{mod } 2) \\
 x_{12} &\equiv x_7 + x_8 + x_{10} \equiv 1 + 0 + 1 \equiv 1 & (\text{mod } 2) \\
 x_{13} &\equiv x_8 + x_9 + x_{11} \equiv 0 + 0 + 1 \equiv 1 & (\text{mod } 2) \\
 x_{14} &\equiv x_9 + x_{10} + x_{12} \equiv 0 + 0 + 1 \equiv 1 & (\text{mod } 2) \\
 x_{15} &\equiv x_{10} + x_{11} + x_{13} \equiv 0 + 1 + 1 \equiv 0 & (\text{mod } 2) \\
 x_{16} &\equiv x_{11} + x_{12} + x_{14} \equiv 1 + 1 + 1 \equiv 1 & (\text{mod } 2) \\
 x_{17} &\equiv x_{12} + x_{13} + x_{15} \equiv 1 + 1 + 0 \equiv 0 & (\text{mod } 2) \\
 x_{18} &\equiv x_{13} + x_{14} + x_{16} \equiv 1 + 1 + 1 \equiv 1 & (\text{mod } 2) \\
 x_{19} &\equiv x_{14} + x_{15} + x_{17} \equiv 1 + 0 + 0 \equiv 1 & (\text{mod } 2) \\
 x_{20} &\equiv x_{15} + x_{16} + x_{18} \equiv 0 + 1 + 1 \equiv 0 & (\text{mod } 2)
 \end{aligned}$$

More generally, after defining  $x_1, \dots, x_5$  (the initial conditions) you can generate the sequence  $x_i$  using

$$x_{n+6} \equiv c_1x_{n+1} + c_2x_{n+2} + c_3x_{n+3} + c_4x_{n+4} + c_5x_{n+5} \pmod{2}$$

for  $n \geq 0$  for constants  $c_1, \dots, c_5$  in  $\mathbb{Z}/2$ . I will say that this is linear relation has **degree 5**. Even more generally, you can have any number of bits for the initial condition. Say you begin with  $x_1, \dots, x_k$  (the initial condition) and the relation is

$$x_{n+k+1} \equiv c_1x_{n+1} + c_2x_{n+2} + c_3x_{n+3} + \dots + c_kx_{n+k} \pmod{2}$$

LSRFs are very easy to implement in both hardware and software and they are extremely fast (they just access bits and XOR them). The LSRF generator itself need to remember the  $k$  bits  $c_1, \dots, c_k$  (which is fixed) and the  $k$  bits of the sequence so far  $x_{n+1}, \dots, x_{n+k}$  in order to generate the next bit  $x_{n+k+1}$ .

In the above example, the sequence from  $x_1$  to  $x_{20}$  is

10110010001111010110010001111010110...

Notice that the pattern repeats itself:

101100100011110 101100100011110 10110...

The period is 15.

The problem with the one-time pad is that you need to generate a random sequence of 0s and 1s. You can see that with 5 bits

$$x_1x_2x_3x_4x_5 = 10110$$

and the relation

$$x_{n+6} \equiv x_{n+1} + x_{n+2} + x_{n+4} \pmod{2}$$

which involves (1,1,0,1,0) (5 bits), a total of 10 bits) we can generate

101100100011110

which has length 15. The 15 bits is somewhat random – we say that the 15 bits are **pseudorandom**. Therefore LFSR can be used to generate a pseudo-random bit sequence, which can used, for instance, as a key for the one-time pad. Of course you want to find a LFSR with extremely long periods.

pseudorandom

**Exercise 1.13.1.** What if you define

debug: exercises/lfsr-0/question.tex

$$x_1x_2x_3x_4 = 1011$$

and use

$$x_{n+5} \equiv x_{n+1} + x_{n+2} + x_{n+4} \pmod{2}$$

for  $n \geq 0$  instead? (Compare with the LSRF above). What sequence do you get? What is the period? ([Go to solution](#), page 186)  $\square$

**Exercise 1.13.2.** Compute all the possible degree 1 LSRF bit sequences. How many possibilities are there? What is the period for each of them? ([Go to solution](#), page 187)  $\square$

debug: exercises/lfsr-1/question.tex

**Exercise 1.13.3.** Compute all the possible degree 2 LSRF bit sequences. How many possibilities are there? What is the period for each of them? ([Go to solution](#), page 188)  $\square$

debug: exercises/lfsr-2/question.tex

**Exercise 1.13.4.** Compute all the possible degree 3 LSRF bit sequences. How many possibilities are there? What is the period for each of them? ([Go to solution](#), page 189)  $\square$

debug: exercises/lfsr-3/question.tex

**Exercise 1.13.5.** Write a program that generated a sequence of integers with values 0 and 1 using the LFSR method. You want to have a `c` array as a parameter. Your generator also need to remember bits of the sequence  $x_1, \dots$  needed to generate the next bits. In general, for a fixed  $k$ , you want to have an array `c` of size  $k$  and an array `x` also of size  $k$ . When you call your function, the next bit is placed in the array `x`, shifting the bits of `x` so that one bit is lost. So if you want the full sequence for analysis, you need a very long array to keep the bits before it's removed from `x`. For instance the main program might look like this:

debug: exercises/lfsr-4/question.tex

```
x = [x1, x2, x3, x4, x5] # the initial bits
c = [c1, c2, c3, c4, c5] # the coefs of the linear relation
bits = [x1, x2, x3, x4, x5] # the full sequence
LSRF(c, x) # append rightmost value of x to the right of bits
LSRF(c, x) # append rightmost value of x to the right of bits
etc.
```

Besides putting the new bit into `x`, it's also a good idea to return that bit as well. Then the above becomes

```
x = [x1, x2, x3, x4, x5] # the initial bits
c = [c1, c2, c3, c4, c5] # the coefs of the linear relation
bits = [x1, x2, x3, x4, x5] # the full sequence
b = LSRF(c, x); append b to the right side of bits
b = LSRF(c, x); append b to the right side of bits
etc.
```

If you like you can also write an LFSR class. Then the above becomes

```
x = [x1, x2, x3, x4, x5] # the initial bits
c = [c1, c2, c3, c4, c5] # the coefs of the linear relation
LFRS = LRFSClass(c, x)
bits = [x1, x2, x3, x4, x5] # the full sequence
b = LFRS.run(); append b to the right side of bits
b = LFRS.run(); append b to the right side of bits
etc.
```

In the above, the code works with integers 0 and 1. For scenarios where there



is a huge number of bits, the bits are packed into a register (say of size 64 bits). ([Go to solution](#), page 190) □

**Exercise 1.13.6.** For a given initial sequence of bits and the sequence  $\mathbf{c}$ , write a function that attempts to compute the length of the period. Try lots of examples and see if you can produce cases of extremely long periods. ([Go to solution](#), page 191) □

debug: exercises/lfsr-5/question.tex

## Solutions

Solution to Exercise [1.13.1](#).

Solution not provided.

debug: exercises/lfsr-  
0/answer.tex

Solution to Exercise [1.13.2](#).

Solution not provided.

debug: exercises/lfsr-  
1/answer.tex

Solution to Exercise [1.13.3](#).

Solution not provided.

debug: exercises/lfsr-  
2/answer.tex

Solution to Exercise [1.13.4](#).

Solution not provided.

debug: exercises/lfsr-  
3/answer.tex

Solution to Exercise [1.13.5](#).

Solution not provided.

debug: exercises/lfsr-  
4/answer.tex

Solution to Exercise [1.13.6](#).

Solution not provided.

debug: exercises/lfsr-  
5/answer.tex

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