5.4 3D angular averaging

The issue at hand is to find the mean 3D vector (trivial) and its standard deviation for a collection of unit vectors. For 2D there exists theorems, see for example Wikipedia.

The average direction is along the z-axis, chosen such that $\bar{Z} = (1/N)\sum_i^N \hat{z}.\vec{v_i}$ is maximal and $\bar{X} = (1/N)\sum_i^N \hat{x}.\vec{v_i} = 0$ and the same for $\hat{x}$.

To obtain the expression for the standard deviation, assume that the vectors are distributed around the z-axis with a Gaussian probability distribution given by

$$\mathcal{P} = (1/\mathcal{N})e^{-\theta^2/2\sigma^2} \sin\theta \, d\theta \, d\phi \; . \tag{5.44}$$

The following equations are valid in the limit of small σ, such that everywhere the small angle approximation can be used thus simplifying the integrals.

$$\mathcal{N} = \int_0^\pi e^{-\theta^2/2\sigma^2} \theta \, d\theta \, 2\pi = \pi \int_0^\infty e^{-\theta^2/2\sigma^2} d\theta^2 = 2\pi\sigma^2 \; . \tag{5.45}$$

From this we obtain

$$\begin{aligned} R = \bar{Z} &= \frac{1}{2\pi\sigma^2}\int_0^\pi \cos\theta \, e^{-\theta^2/2\sigma^2} \sin\theta \, d\theta 2\pi = \frac{1}{2\sigma^2}\int_0^\pi e^{-\theta^2/2\sigma^2} \sin 2\theta \, d\theta \\ &= \frac{1}{2\sigma^2}\int_0^\pi e^{-\theta^2/2\sigma^2} \left(2\theta - (2\theta)^3/6\right) \, d\theta = 1 - (4/3)\sigma^2 \end{aligned} \tag{5.46}$$

where R is the length of the averaged unit vectors. The standard-deviation square can thus be calculated as

$$\sigma^2 = (3/4)(1 - R) \; . \tag{5.47}$$

This is derived in the limit where $\sigma \ll 1$ and thus $0 < (1 - R) \ll 1$. To arrive at an expression that (may) also apply outside this limit we use the same approach as used to calculate the standard deviation for circular averaging where $(1 - R)$ is replaced by $-\ln R$ (which is valid in the limit where $R \approx 1$) thus arriving at

$$\sigma = \sqrt{-(3/4)\ln R} \; . \tag{5.48}$$

For polarization vectors (where + and - directions are ambiguous) the mean direction can be obtained almost as before where $\bar{Z_p} = (1/N)\sum_i^N |\hat{z}.\vec{v_i}| = R_p$ is maximal.

To obtain the standard deviation a similar approach as before could be used. The complication is that R is always (much) larger than zero since it is obtained from a sum of absolute values. In fact for very large σ we have

$$R^\infty \equiv R_p(\sigma = \infty) = \int_0^{\pi/2} \cos\theta \, d\cos\theta = 1/2 \tag{5.49}$$

For finite statistics with N vectors we will have $R_N^\infty = 1 - \frac{N-1}{2N} = \frac{N+1}{2N}$ (R_N^∞ should approach unity for $N = 1$). For small σ we should have the same expression as for the 3D case. For larger values of $(1 - R_p)$ the deviation should be quadratic where for $R_p = R_N^\infty$ the expression should give ∞. We thus arrive at

$$\sigma_p = \sqrt{-(3/4)\ln\left[R_p - R_N\,(1 - R_p)^2\frac{(2N)^2}{(N-1)^2}\right]} = \sqrt{-(3/4)\ln\left[R_p - (1 - R_p)^2\frac{2N(N+1)}{(N-1)^2}\right]} \; . \tag{5.50}$$