Week 6: Cauchy-Riemann's Equations

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Monday, 2 Hours

Outline

Introduction to Cauchy-Riemann Equations

Overview

The **Cauchy-Riemann Equations** are a set of two partial differential equations which, together with continuity, provide a necessary and sufficient condition for a complex function to be differentiable (and hence analytic) at a point.

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Importance

These equations are fundamental in complex analysis, allowing us to determine whether a complex function is differentiable and to explore its analytic properties.

Complex Differentiability

Definition

A complex function $f(z)$ is **differentiable** at a point $z_0 = x_0 + iy_0$ if the following limit exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

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Expressing in Terms of x and y

Let $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, where u and v are real-valued functions representing the real and imaginary parts of f, respectively.

Derivation of Cauchy-Riemann Equations

Step 1: Expand the Difference Quotient

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{[u(x, y) + iv(x, y)] - [u(x_0, y_0) + iv(x_0, y_0)]}{(x + iy) - (x_0 + iy_0)}$$

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Step 2: Simplify the Denominator

$$(x+iy) - (x_0+iy_0) = (x - x_0) + i(y - y_0)$$

Derivation of Cauchy-Riemann Equations (Continued)

Step 3: Rationalize the Denominator

Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{[u(x,y)-u(x_0,y_0)]+i[v(x,y)-v(x_0,y_0)]}{(x-x_0)+i(y-y_0)}\cdot\frac{(x-x_0)-i(y-y_0)}{(x-x_0)-i(y-y_0)}$$

Derivation of Cauchy-Riemann Equations (Continued)

Step 3: Rationalize the Denominator

Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{[u(x,y) - u(x_0,y_0)] + i[v(x,y) - v(x_0,y_0)]}{(x-x_0) + i(y-y_0)} \cdot \frac{(x-x_0) - i(y-y_0)}{(x-x_0) - i(y-y_0)}$$

Step 4: Perform the Multiplication

$$\frac{[(u(x,y) - u(x_0,y_0))(x-x_0) + (v(x,y) - v(x_0,y_0))(y-y_0)]}{(x-x_0)^2 + (y-y_0)^2} + i\,[(v(x, y$$

Derivation of Cauchy-Riemann Equations (Final Steps)

Step 5: Take the Limit as $z \to z_0$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \left[\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right]$$

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Step 6: Equate Real and Imaginary Parts

For the limit to exist and be independent of the path of approach, the following must hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

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Conclusion

These are the **Cauchy-Riemann Equations**, providing necessary

Analytic Functions

Definition

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Importance of Analyticity

Analytic functions possess powerful properties, including infinite differentiability and the ability to be represented by convergent power series within their domain of analyticity.

Using Cauchy-Riemann Equations to Determine Analyticity

Procedure

To determine if a function $f(z) = u(x, y) + iv(x, y)$ is analytic at $z_0 = x_0 + iy_0$:

1. Verify that u and v have continuous first-order partial derivatives in a neighborhood around z_0.

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2. Check if the Cauchy-Riemann equations are satisfied at z_0.

Conclusion

If both conditions are met, $f(z)$ is analytic at z_0.

Example 1: Determining Analyticity

Problem

Determine whether the function $f(z) = z^2$ is analytic at $z_0 = 1 + i$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.

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Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.
2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.

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3. Verify the Cauchy-Riemann equations at $z_0 = 1 + i$.

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3. Verify the Cauchy-Riemann equations at $z_0 = 1 + i$.
4. Conclude whether $f(z)$ is analytic at z_0.

Example 1: Step 1

Step 1: Express $f(z) = z^2$ in Terms of x and y

Let $z = x + iy$, then:

$$f(z) = (x + iy)^2 = x^2 - y^2 + i(2xy)$$

Therefore:

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy$$

Example 1: Step 2

Step 2: Compute Partial Derivatives

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y \quad ; \quad \frac{\partial v}{\partial y} = 2x$$

Example 1: Step 3

Step 3: Verify Cauchy-Riemann Equations at $z_0 = 1 + i$

Substitute $x = 1$ and $y = 1$:

$$\frac{\partial u}{\partial x} = 2(1) = 2 \quad ; \quad \frac{\partial v}{\partial y} = 2(1) = 2$$

$$\frac{\partial u}{\partial y} = -2(1) = -2 \quad ; \quad \frac{\partial v}{\partial x} = 2(1) = 2$$

Check the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \Rightarrow \quad 2 = 2 \quad \text{(True)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad -2 = -2 \quad \text{(True)}$$

Example 1: Conclusion

Conclusion

Since the Cauchy-Riemann equations are satisfied at $z_0 = 1 + i$ and the partial derivatives are continuous, the function $f(z) = z^2$ is **analytic** at $z_0 = 1 + i$.

Implications

Being analytic implies that $f(z)$ is differentiable in some neighborhood around z_0, possessing an infinite number of derivatives and being representable by a convergent power series within that neighborhood.

Example 2: Determining Analyticity

Problem

Determine whether the function $f(z) = \overline{z}$ is analytic at $z_0 = 0$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.

Example 2: Determining Analyticity

Problem

Determine whether the function $f(z) = \overline{z}$ is analytic at $z_0 = 0$.

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1. Express $f(z)$ in terms of x and y, where $z = x + iy$.
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3. Verify the Cauchy-Riemann equations at $z_0 = 0$.

Example 2: Determining Analyticity

Problem

Determine whether the function $f(z) = \overline{z}$ is analytic at $z_0 = 0$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.
2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.
3. Verify the Cauchy-Riemann equations at $z_0 = 0$.
4. Conclude whether $f(z)$ is analytic at z_0.

Example 2: Step 1

Step 1: Express $f(z) = \overline{z}$ in Terms of x and y

Let $z = x + iy$, then:

$$\overline{z} = x - iy$$

Therefore:

$$u(x, y) = x \quad \text{and} \quad v(x, y) = -y$$

Example 2: Step 2

Step 2: Compute Partial Derivatives

$$\frac{\partial u}{\partial x} = 1 \quad ; \quad \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = 0 \quad ; \quad \frac{\partial v}{\partial y} = -1$$

Example 2: Step 3

Step 3: Verify Cauchy-Riemann Equations at $z_0 = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \Rightarrow \quad 1 = -1 \quad \text{(False)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad 0 = 0 \quad \text{(True)}$$

Since the first equation is not satisfied, the Cauchy-Riemann equations fail at $z_0 = 0$.

Example 2: Conclusion

Conclusion

Since the Cauchy-Riemann equations are **not satisfied** at $z_0 = 0$, the function $f(z) = \bar{z}$ is **not analytic** at $z_0 = 0$.

Implications

The failure of the Cauchy-Riemann equations implies that $f(z)$ does not possess a complex derivative at $z_0 = 0$, and hence, is not analytic at that point.

Introduction to Harmonic Functions

Definition

A **harmonic function** is a twice continuously differentiable function $u(x, y)$ that satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

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Properties

- Harmonic functions arise naturally in physics, particularly in the study of gravitational and electric potentials.
- The real and imaginary parts of any analytic function are harmonic.
- Harmonic functions enjoy the **mean value property** and **maximum principle**.

Relationship Between Analytic and Harmonic Functions

Cauchy-Riemann Equations Imply Harmonicity

If $f(z) = u(x, y) + iv(x, y)$ is analytic, then both u and v satisfy Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

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Conversely

If u is harmonic, there exists a harmonic conjugate v such that $f(z) = u + iv$ is analytic.

Example 3: Identifying Harmonic Functions

Problem

Determine whether the function $u(x, y) = x^2 - y^2$ is harmonic.

Solution Overview

1. Compute the second partial derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$.

Example 3: Identifying Harmonic Functions

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Determine whether the function $u(x, y) = x^2 - y^2$ is harmonic.

Solution Overview

1. Compute the second partial derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$.
2. Sum the second partial derivatives and check if the result equals zero.

Example 3: Step 1

Step 1: Compute Second Partial Derivatives

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -2$$

Example 3: Step 2

Step 2: Sum the Second Partial Derivatives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + (-2) = 0$$

Example 3: Conclusion

Conclusion

Since:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

the function $u(x, y) = x^2 - y^2$ is **harmonic**.

Implications

Being harmonic implies that u satisfies Laplace's equation and possesses properties like the mean value property and the maximum principle.

Example 4: Constructing Harmonic Conjugates

Problem

Given the harmonic function $u(x, y) = x^2 - y^2$, find its harmonic conjugate $v(x, y)$ such that $f(z) = u + iv$ is analytic.

Solution Overview

1. Apply the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

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2. Integrate to find $v(x, y)$.

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2. Integrate to find $v(x, y)$.
3. Ensure consistency in the integration constants.

Example 4: Step 1

Step 1: Apply Cauchy-Riemann Equations

Given $u(x, y) = x^2 - y^2$:

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 2y$$

Example 4: Step 2

Step 2: Integrate to Find $v(x, y)$

Integrate $\frac{\partial v}{\partial y} = 2x$ with respect to y:

$$v(x, y) = 2xy + g(x)$$

Here, $g(x)$ is an arbitrary function of x.

Example 4: Step 3

Step 3: Determine $g(x)$

Use $\frac{\partial v}{\partial x} = 2y$:

$$\frac{\partial v}{\partial x} = 2y + g'(x) = 2y$$

Therefore:

$$g'(x) = 0 \quad \Rightarrow \quad g(x) = C$$

where C is a constant. For simplicity, take $C = 0$.

Example 4: Conclusion

Conclusion

The harmonic conjugate of $u(x, y) = x^2 - y^2$ is:

$$v(x, y) = 2xy$$

Thus, the analytic function is:

$$f(z) = u + iv = (x^2 - y^2) + i(2xy) = z^2$$

Implications

The existence of $v(x, y)$ satisfying the Cauchy-Riemann equations confirms that $f(z) = z^2$ is analytic.

Harmonic Conjugate

Given a harmonic function $u(x, y)$, a **harmonic conjugate** $v(x, y)$ is a function that satisfies the Cauchy-Riemann equations with u. Together, $f(z) = u + iv$ forms an analytic function.

Harmonic Conjugates and Analytic Functions

Harmonic Conjugate

Given a harmonic function $u(x, y)$, a **harmonic conjugate** $v(x, y)$ is a function that satisfies the Cauchy-Riemann equations with u. Together, $f(z) = u + iv$ forms an analytic function.

Example of Harmonic Conjugates

If $u(x, y) = x^2 - y^2$, then its harmonic conjugate is $v(x, y) = 2xy$, forming the analytic function $f(z) = z^2$.

Example 5: Finding Harmonic Conjugate

Problem

Find the harmonic conjugate $v(x, y)$ of the harmonic function $u(x, y) = e^x \cos(y)$.

Solution Overview

1. Apply the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Example 5: Finding Harmonic Conjugate

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Solution Overview

1. Apply the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

2. Compute the partial derivatives of $u(x, y)$.

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2. Compute the partial derivatives of $u(x, y)$.
3. Integrate to find $v(x, y)$.

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2. Compute the partial derivatives of $u(x, y)$.
3. Integrate to find $v(x, y)$.
4. Ensure consistency in the integration constants.

Example 5: Step 1

Step 1: Apply Cauchy-Riemann Equations

Given $u(x, y) = e^x \cos(y)$:

$$\frac{\partial u}{\partial x} = e^x \cos(y) = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -e^x \sin(y) = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = e^x \sin(y)$$

Example 5: Step 2

Step 2: Integrate to Find $v(x, y)$

Integrate $\frac{\partial v}{\partial y} = e^x \cos(y)$ with respect to y:

$$v(x, y) = e^x \sin(y) + g(x)$$

Here, $g(x)$ is an arbitrary function of x.

Example 5: Step 3

Step 3: Determine $g(x)$

Use $\frac{\partial v}{\partial x} = e^x \sin(y)$:

$$\frac{\partial v}{\partial x} = e^x \sin(y) + g'(x) = e^x \sin(y)$$

Therefore:

$$g'(x) = 0 \quad \Rightarrow \quad g(x) = C$$

For simplicity, take $C = 0$.

Example 5: Conclusion

Conclusion

The harmonic conjugate of $u(x, y) = e^x \cos(y)$ is:

$$v(x, y) = e^x \sin(y)$$

Thus, the analytic function is:

$$f(z) = u + iv = e^x \cos(y) + ie^x \sin(y) = e^{x+iy} = e^z$$

Implications

The function $f(z) = e^z$ is analytic everywhere in $\mathbb{C}$, as it satisfies the Cauchy-Riemann equations and has continuous partial derivatives.

Example 6: Non-Existence of Harmonic Conjugate

Problem

Determine whether the harmonic function $u(x, y) = |z|^2 = x^2 + y^2$ has a harmonic conjugate.

Solution Overview

1. Apply the Cauchy-Riemann equations to find $v(x, y)$.

Example 6: Non-Existence of Harmonic Conjugate

Problem

Determine whether the harmonic function $u(x, y) = |z|^2 = x^2 + y^2$ has a harmonic conjugate.

Solution Overview

1. Apply the Cauchy-Riemann equations to find $v(x, y)$.
2. Check for consistency in the resulting equations.

Example 6: Non-Existence of Harmonic Conjugate

Problem

Determine whether the harmonic function $u(x, y) = |z|^2 = x^2 + y^2$ has a harmonic conjugate.

Solution Overview

1. Apply the Cauchy-Riemann equations to find $v(x, y)$.
2. Check for consistency in the resulting equations.
3. Conclude whether a harmonic conjugate exists.

Example 6: Step 1

Step 1: Apply Cauchy-Riemann Equations

Given $u(x, y) = x^2 + y^2$:

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = 2y = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = -2y$$

Example 6: Step 2

Step 2: Integrate to Find $v(x, y)$

Integrate $\frac{\partial v}{\partial y} = 2x$ with respect to y:

$$v(x, y) = 2xy + g(x)$$

Example 6: Step 3

Step 3: Determine $g(x)$

Use $\frac{\partial v}{\partial x} = -2y$:

$$\frac{\partial v}{\partial x} = 2y + g'(x) = -2y$$

Therefore:

$$2y + g'(x) = -2y \quad \Rightarrow \quad g'(x) = -4y$$

However, $g'(x)$ should be a function of x alone, but the right-hand side depends on y. This inconsistency implies that no such $g(x)$ exists.

Example 6: Conclusion

Conclusion

The function $u(x, y) = x^2 + y^2$ does **not** have a harmonic conjugate because the Cauchy-Riemann equations lead to an inconsistency.

Implications

Without a harmonic conjugate, there is no analytic function $f(z) = u + iv$ associated with $u(x, y) = x^2 + y^2$.

Summary

Key Takeaways

- **Cauchy-Riemann Equations:** Provide necessary conditions for a complex function to be differentiable and analytic.

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- **Derivation:** Derived from the definition of complex differentiability by expressing $f(z)$ in terms of x and y.

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Key Takeaways

- **Cauchy-Riemann Equations:** Provide necessary conditions for a complex function to be differentiable and analytic.
- **Derivation:** Derived from the definition of complex differentiability by expressing $f(z)$ in terms of x and y.
- **Analyticity:** A function is analytic at a point if it satisfies the Cauchy-Riemann equations and has continuous partial derivatives in a neighborhood around that point.

Summary

Key Takeaways

- **Cauchy-Riemann Equations:** Provide necessary conditions for a complex function to be differentiable and analytic.
- **Derivation:** Derived from the definition of complex differentiability by expressing $f(z)$ in terms of x and y.
- **Analyticity:** A function is analytic at a point if it satisfies the Cauchy-Riemann equations and has continuous partial derivatives in a neighborhood around that point.
- **Harmonic Functions:** The real and imaginary parts of analytic functions are harmonic, satisfying Laplace's equation.

Summary

Key Takeaways

- **Cauchy-Riemann Equations:** Provide necessary conditions for a complex function to be differentiable and analytic.
- **Derivation:** Derived from the definition of complex differentiability by expressing $f(z)$ in terms of x and y.
- **Analyticity:** A function is analytic at a point if it satisfies the Cauchy-Riemann equations and has continuous partial derivatives in a neighborhood around that point.
- **Harmonic Functions:** The real and imaginary parts of analytic functions are harmonic, satisfying Laplace's equation.
- **Harmonic Conjugates:** Given a harmonic function, finding its harmonic conjugate allows the construction of an analytic function.

Summary

Key Takeaways

- **Cauchy-Riemann Equations:** Provide necessary conditions for a complex function to be differentiable and analytic.
- **Derivation:** Derived from the definition of complex differentiability by expressing $f(z)$ in terms of x and y.
- **Analyticity:** A function is analytic at a point if it satisfies the Cauchy-Riemann equations and has continuous partial derivatives in a neighborhood around that point.
- **Harmonic Functions:** The real and imaginary parts of analytic functions are harmonic, satisfying Laplace's equation.
- **Harmonic Conjugates:** Given a harmonic function, finding its harmonic conjugate allows the construction of an analytic function.

Further Reading

Questions and Discussion

Any Questions?