Week 8: Integration of Functions of Complex Variables

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Monday, 2 Hours

Outline

Introduction to Cauchy's Integral Theorem

Overview

Cauchy's Integral Theorem is a fundamental result in complex analysis which states that the integral of an analytic function over a closed contour is zero, provided the function is analytic within and on the contour.

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Cauchy's Integral Theorem is a fundamental result in complex analysis which states that the integral of an analytic function over a closed contour is zero, provided the function is analytic within and on the contour.

Importance

This theorem is foundational for many other results in complex analysis, including Cauchy's Integral Formula and the Residue Theorem.

Statement of Cauchy's Integral Theorem

Theorem

Let $f(z)$ be analytic within and on a simple closed contour γ in the complex plane. Then:

$$\oint_\gamma f(z)\,dz = 0$$

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- $f(z)$ must be analytic on and inside the contour γ.

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$$\oint_{\gamma} f(z)\, dz = 0$$

Conditions

- $f(z)$ must be analytic on and inside the contour γ.
- The contour γ must be simple (no self-intersections) and closed.

Proof Overview of Cauchy's Integral Theorem

Assumptions

- $f(z)$ is analytic (holomorphic) on and inside γ.

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- γ is a positively oriented, simple closed contour.

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Key Idea

Express $f(z)$ as its power series expansion within the domain of analyticity and integrate term by term.

Proof Details of Cauchy's Integral Theorem

Express $f(z)$ as a Power Series

Since $f(z)$ is analytic, it can be expressed as:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where z_0 is a point inside γ.

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where z_0 is a point inside γ.

Integrate Term by Term

$$\oint_\gamma f(z)\, dz = \oint_\gamma \sum_{n=0}^{\infty} a_n (z - z_0)^n\, dz = \sum_{n=0}^{\infty} a_n \oint_\gamma (z - z_0)^n\, dz$$

Proof Completion of Cauchy's Integral Theorem

Evaluate Each Integral

$$\oint_\gamma (z - z_0)^n \, dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

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Evaluate Each Integral

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Apply to the Series

Since $n \geq 0$ in the power series, none of the integrals have $n = -1$.
Therefore:

$$\oint_\gamma f(z) \, dz = \sum_{n=0}^{\infty} a_n \cdot 0 = 0$$

Proof Completion of Cauchy's Integral Theorem

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Conclusion

$$\oint_\gamma f(z) \, dz = 0$$

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Statement

If $f(z)$ is analytic within and on a simple closed contour γ, and z_0 is a point inside γ, then:

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \, dz$$

Applications of Cauchy's Integral Formula

Key Applications

- **Evaluation of Contour Integrals:** Direct computation of integrals involving analytic functions.

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- **Estimation of Function Values:** Determining function values inside contours based on boundary behavior.

Applications of Cauchy's Integral Formula

Key Applications

- **Evaluation of Contour Integrals:** Direct computation of integrals involving analytic functions.
- **Derivation of Taylor Series:** Expansion of analytic functions around a point.
- **Estimation of Function Values:** Determining function values inside contours based on boundary behavior.
- **Proving Uniqueness Theorems:** Showing that analytic functions are uniquely determined by their values on a contour.

Proof of Cauchy's Integral Formula

Assumptions

- $f(z)$ is analytic within and on γ.

Proof of Cauchy's Integral Formula

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- z_0 is inside γ.

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- z_0 is inside γ.

Utilize Cauchy's Integral Theorem

Consider the function:

$$\frac{f(z)}{z - z_0}$$

Since $f(z)$ is analytic and z_0 is inside γ, this function has a simple pole at $z = z_0$.

Proof Completion of Cauchy's Integral Formula (Part 1)

Apply the Residue Theorem

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i \cdot \text{Res}\left(\frac{f(z)}{z - z_0}, z_0\right)$$

The residue at $z = z_0$ is:

$$\text{Res}\left(\frac{f(z)}{z - z_0}, z_0\right) = f(z_0)$$

Proof Completion of Cauchy's Integral Formula (Part 1)

Apply the Residue Theorem

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i \cdot \mathrm{Res}\left(\frac{f(z)}{z - z_0}, z_0\right)$$

The residue at $z = z_0$ is:

$$\mathrm{Res}\left(\frac{f(z)}{z - z_0}, z_0\right) = f(z_0)$$

Substitute the Residue

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i \cdot f(z_0)$$

Proof Completion of Cauchy's Integral Formula (Part 2)

Solve for $f(z_0)$

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \, dz$$

Proof Completion of Cauchy's Integral Formula (Part 2)

Solve for $f(z_0)$

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} \, dz$$

Conclusion

This completes the proof of **Cauchy's Integral Formula**.

Introduction to Evaluation of Contour Integrals

Overview

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Key Techniques

- Parametrization of Paths

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Key Techniques

- Parametrization of Paths
- Cauchy's Integral Theorem and Formula

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Overview

Contour integrals involve integrating complex functions along specified paths in the complex plane. Evaluating these integrals is essential for various applications in physics and engineering.

Key Techniques

- Parametrization of Paths
- Cauchy's Integral Theorem and Formula
- Residue Theorem

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Key Techniques

- Parametrization of Paths
- Cauchy's Integral Theorem and Formula
- Residue Theorem
- Laurent Series Expansion

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Contour integrals involve integrating complex functions along specified paths in the complex plane. Evaluating these integrals is essential for various applications in physics and engineering.

Key Techniques

- Parametrization of Paths
- Cauchy's Integral Theorem and Formula
- Residue Theorem
- Laurent Series Expansion
- Contour Deformation

Parametrization of Paths

Methodology

To evaluate a contour integral, one often parametrizes the path γ by a parameter t in an interval $[a, b]$:

$$z(t) = x(t) + iy(t), \quad a \leq t \leq b$$

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To evaluate a contour integral, one often parametrizes the path γ by a parameter t in an interval $[a, b]$:

$$z(t) = x(t) + iy(t), \quad a \leq t \leq b$$

Integral Representation

The contour integral becomes:

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t)) \cdot z'(t)\,dt$$

Techniques Overview

Common Techniques

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2. **Laurent Series:** Useful for expanding functions around singular points.
3. **Contour Deformation:** Allows changing the path of integration without altering the integral's value, provided certain conditions are met.

Techniques Overview

Common Techniques

1. **Residue Theorem:** Powerful for integrals involving singularities inside the contour.
2. **Laurent Series:** Useful for expanding functions around singular points.
3. **Contour Deformation:** Allows changing the path of integration without altering the integral's value, provided certain conditions are met.
4. **Jordan's Lemma:** Assists in evaluating integrals involving exponential functions over large semicircular contours.

Example 1: Applying Cauchy's Integral Formula

Problem

Compute the integral:

$$\oint_\gamma \frac{e^z}{z-1}\, dz$$

Where γ is the circle $|z| = 3$ traversed counterclockwise, and $z_0 = 1$ lies inside γ.

Solution Overview

1. **Verify** that $f(z) = e^z$ is analytic within and on γ.

Example 1: Applying Cauchy's Integral Formula

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Solution Overview

1. **Verify** that $f(z) = e^z$ is analytic within and on γ.
2. **Apply** Cauchy's Integral Formula.

Example 1: Step 1 - Verify Analyticity

Verify Analyticity

The function $f(z) = e^z$ is entire, meaning it is analytic everywhere in $\mathbb{C}$, including within and on the contour γ.

Example 1: Step 2 - Apply Cauchy's Integral Formula

Apply Cauchy's Integral Formula

According to Cauchy's Integral Formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \, dz$$

Solving for the integral:

$$\oint_\gamma \frac{f(z)}{z - z_0} \, dz = 2\pi i \cdot f(z_0)$$

Example 1: Step 2 - Apply Cauchy's Integral Formula

Apply Cauchy's Integral Formula

According to Cauchy's Integral Formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz$$

Solving for the integral:

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i \cdot f(z_0)$$

Substitute the Values

Here, $f(z_0) = e^1 = e$, therefore:

Example 1: Conclusion

Conclusion

$$\oint_{\gamma} \frac{e^z}{z-1}\, dz = 2\pi i e$$

Interpretation

The integral evaluates to $2\pi i e$, showcasing the direct application of Cauchy's Integral Formula.

Example 2: Using the Residue Theorem

Problem

Compute the integral:

$$\oint_{\gamma} \frac{z}{z^2+1}\, dz$$

Where γ is the circle $|z| = 2$ traversed counterclockwise.

Solution Overview

1. **Identify** the singularities of $f(z) = \frac{z}{z^2+1}$.

Example 2: Using the Residue Theorem

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Compute the integral:

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Where γ is the circle $|z| = 2$ traversed counterclockwise.

Solution Overview

1. **Identify** the singularities of $f(z) = \frac{z}{z^2+1}$.
2. **Determine** which singularities lie inside γ.

Example 2: Using the Residue Theorem

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Compute the integral:

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Solution Overview

1. **Identify** the singularities of $f(z) = \frac{z}{z^2+1}$.
2. **Determine** which singularities lie inside γ.
3. **Find** the residues at these singularities.

Example 2: Using the Residue Theorem

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Example 2: Step 1 - Identify Singularities

Identify Singularities

The function $f(z) = \frac{z}{z^2+1}$ has singularities where the denominator is zero:

$$z^2 + 1 = 0 \quad \Rightarrow \quad z = i \quad \text{and} \quad z = -i$$

Example 2: Step 2 - Determine Singularities Inside γ

Determine Singularities Inside γ

The contour γ is the circle $|z| = 2$. Both $z = i$ and $z = -i$ satisfy:

$$|i| = |-i| = 1 < 2$$

Therefore, both singularities lie inside γ.

Example 2: Step 3 - Find Residues

Find Residues at $z = i$ and $z = -i$

Since both singularities are simple poles, the residues can be found using:

$$\operatorname{Res}\left(\frac{z}{z^2+1}, z_k\right) = \lim_{z \to z_k} (z - z_k)\frac{z}{z^2+1}$$

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At $z = i$**:**

$$\text{Res}\left(\frac{z}{z^2+1}, i\right) = \lim_{z \to i} \frac{z(z-i)}{(z-i)(z+i)} = \lim_{z \to i} \frac{z}{z+i} = \frac{i}{2i} = \frac{1}{2}$$

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At $z = -i$:

$$\text{Res}\left(\frac{z}{z^2+1}, -i\right) = \lim_{z \to -i} \frac{z(z+i)}{(z-i)(z+i)} = \lim_{z \to -i} \frac{z}{z-i} = \frac{-i}{-2i} = \frac{1}{2}$$

Example 2: Step 4 - Apply the Residue Theorem

Step 1: Express the Integral

$$\oint_{\gamma} \frac{z}{z^2+1}\,dz = 2\pi i\left(\operatorname{Res}\left(\frac{z}{z^2+1}, i\right) + \operatorname{Res}\left(\frac{z}{z^2+1}, -i\right)\right)$$

Example 2: Step 4 - Apply the Residue Theorem

Step 1: Express the Integral

$$\oint_\gamma \frac{z}{z^2+1}\,dz = 2\pi i\left(\text{Res}\left(\frac{z}{z^2+1}, i\right) + \text{Res}\left(\frac{z}{z^2+1}, -i\right)\right)$$

Step 2: Substitute Residues

$$= 2\pi i\left(\frac{1}{2}+\frac{1}{2}\right)$$
$$= 2\pi i$$

Example 2: Conclusion

Conclusion

$$\oint_\gamma \frac{z}{z^2+1}\, dz = 2\pi i$$

Interpretation

The integral evaluates to $2\pi i$, demonstrating the application of the Residue Theorem for functions with multiple singularities within the contour.

Example 3: Evaluation Using Cauchy's Integral Formula

Problem

Compute the integral:

$$\oint_{\gamma} \frac{e^z}{z}\, dz$$

Where γ is the unit circle $|z| = 1$ traversed counterclockwise.

Solution Overview

1. **Identify** the singularity of $f(z) = \frac{e^z}{z}$.

Example 3: Evaluation Using Cauchy's Integral Formula

Problem

Compute the integral:

$$\oint_{\gamma} \frac{e^z}{z}\, dz$$

Where γ is the unit circle $|z| = 1$ traversed counterclockwise.

Solution Overview

1. **Identify** the singularity of $f(z) = \frac{e^z}{z}$.
2. **Determine** if the singularity lies inside γ.

Example 3: Evaluation Using Cauchy's Integral Formula

Problem

Compute the integral:

$$\oint_{\gamma} \frac{e^z}{z}\, dz$$

Where γ is the unit circle $|z| = 1$ traversed counterclockwise.

Solution Overview

1. **Identify** the singularity of $f(z) = \frac{e^z}{z}$.
2. **Determine** if the singularity lies inside γ.
3. **Apply** Cauchy's Integral Formula.

Example 3: Step 1 - Identify Singularities

Identify Singularities

The function $f(z) = \frac{e^z}{z}$ has a singularity at $z = 0$, which is a simple pole.

Example 3: Step 2 - Determine Singularity Inside γ

Determine Singularity Inside γ

The contour γ is the unit circle $|z| = 1$. Since $|0| = 0 < 1$, the singularity at $z = 0$ lies inside γ.

Example 3: Step 3 - Apply Cauchy's Integral Formula

Apply Cauchy's Integral Formula

According to Cauchy's Integral Formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz$$

Here, $f(z) = e^z$ and $z_0 = 0$, so:

$$e^0 = 1 = \frac{1}{2\pi i} \oint_\gamma \frac{e^z}{z}\, dz$$

Solving for the integral:

$$\oint \frac{e^z}{z}\, dz = 2\pi i \cdot 1 = 2\pi i$$

Example 3: Conclusion

Conclusion

$$\oint_{\gamma} \frac{e^z}{z}\, dz = 2\pi i$$

Interpretation

The integral of $\frac{e^z}{z}$ around the unit circle is $2\pi i$, illustrating the use of Cauchy's Integral Formula for functions with singularities inside the contour.

Example 4: Contour Integral with Multiple Singularities

Problem

Compute the integral:

$$\oint_\gamma \frac{1}{z(z-2)}\, dz$$

Where γ is the circle $|z| = 3$ traversed counterclockwise.

Solution Overview

1. **Identify** the singularities of $f(z) = \frac{1}{z(z-2)}$.

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$$\oint_{\gamma} \frac{1}{z(z-2)}\, dz$$

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Solution Overview

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2. **Determine** which singularities lie inside γ.

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Solution Overview

1. **Identify** the singularities of $f(z) = \frac{1}{z(z-2)}$.
2. **Determine** which singularities lie inside γ.
3. **Find** the residues at these singularities.

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2. **Determine** which singularities lie inside γ.
3. **Find** the residues at these singularities.
4. **Apply** the Residue Theorem.

Example 4: Step 1 - Identify Singularities

Identify Singularities

The function $f(z) = \frac{1}{z(z-2)}$ has singularities at:

$$z = 0 \quad \text{and} \quad z = 2$$

Example 4: Step 2 - Determine Singularities Inside γ

Determine Singularities Inside γ

The contour γ is the circle $|z| = 3$. Both $z = 0$ and $z = 2$ satisfy:

$$|0| = 0 < 3 \quad \text{and} \quad |2| = 2 < 3$$

Therefore, both singularities lie inside γ.

Example 4: Step 3 - Find Residues

Find Residues at $z = 0$ and $z = 2$

Since both singularities are simple poles, the residues can be found using:

$$\text{Res}\left(\frac{1}{z(z-2)}, z_k\right) = \lim_{z \to z_k} (z - z_k)\frac{1}{z(z-2)}$$

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At $z = 0$:

$$\text{Res}\left(\frac{1}{z(z-2)}, 0\right) = \lim_{z \to 0} \frac{1}{z-2} = \frac{1}{-2} = -\frac{1}{2}$$

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At $z = 2$:

$$\text{Res}\left(\frac{1}{z(z-2)}, 2\right) = \lim_{z \to 2} \frac{1}{z} = \frac{1}{2}$$

Apply the Residue Theorem

$$\oint_{\gamma} \frac{1}{z(z-2)}\,dz = 2\pi i \left(\mathrm{Res}\left(\frac{1}{z(z-2)}, 0\right) + \mathrm{Res}\left(\frac{1}{z(z-2)}, 2\right)\right) = 2\pi$$

Example 4: Conclusion

Conclusion

$$\oint_\gamma \frac{1}{z(z-2)}\, dz = 0$$

Interpretation

The integral evaluates to zero, demonstrating how residues can cancel each other out when their sum is zero.

Example 5: Evaluating an Integral with a Higher-Order Pole

Problem

Compute the integral:

$$\oint_{\gamma} \frac{1}{(z-1)^2}\, dz$$

Where γ is the circle $|z| = 3$ traversed counterclockwise.

Solution Overview

1. **Identify** the singularity of $f(z) = \frac{1}{(z-1)^2}$.

Example 5: Evaluating an Integral with a Higher-Order Pole

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Compute the integral:

$$\oint_{\gamma} \frac{1}{(z-1)^2}\, dz$$

Where γ is the circle $|z| = 3$ traversed counterclockwise.

Solution Overview

1. **Identify** the singularity of $f(z) = \frac{1}{(z-1)^2}$.
2. **Determine** if the singularity lies inside γ.

Example 5: Evaluating an Integral with a Higher-Order Pole

Problem

Compute the integral:

$$\oint_\gamma \frac{1}{(z-1)^2}\,dz$$

Where γ is the circle $|z| = 3$ traversed counterclockwise.

Solution Overview

1. **Identify** the singularity of $f(z) = \frac{1}{(z-1)^2}$.
2. **Determine** if the singularity lies inside γ.
3. **Find** the residue at the singularity.

Example 5: Evaluating an Integral with a Higher-Order Pole

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Compute the integral:

$$\oint_{\gamma} \frac{1}{(z-1)^2}\, dz$$

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1. **Identify** the singularity of $f(z) = \frac{1}{(z-1)^2}$.
2. **Determine** if the singularity lies inside γ.
3. **Find** the residue at the singularity.
4. **Apply** the Residue Theorem.

Example 5: Step 1 - Identify Singularities

Identify Singularities

The function $f(z) = \frac{1}{(z-1)^2}$ has a singularity at $z = 1$, which is a pole of order 2.

Example 5: Step 2 - Determine Singularity Inside γ

Determine Singularity Inside γ

The contour γ is the circle $|z| = 3$. Since $|1| = 1 < 3$, the singularity at $z = 1$ lies inside γ.

Example 5: Step 3 - Find Residue

Find Residue at $z = 1$

For a pole of order n, the residue can be found using:

$$\text{Res}\,(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right]$$

Here, $n = 2$ and $z_0 = 1$:

$$\text{Res}\left(\frac{1}{(z-1)^2}, 1 \right) = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} \left[(z-1)^2 \cdot \frac{1}{(z-1)^2} \right] = \lim_{z \to 1} \frac{d}{dz}[1] = 0$$

Example 5: Step 4 - Apply the Residue Theorem

Apply the Residue Theorem

$$\oint_\gamma \frac{1}{(z-1)^2}\,dz = 2\pi i \cdot \mathrm{Res}\left(\frac{1}{(z-1)^2}, 1\right) = 2\pi i \cdot 0 = 0$$

Example 5: Conclusion

Conclusion

$$\oint_{\gamma} \frac{1}{(z-1)^2}\, dz = 0$$

Interpretation

The integral evaluates to zero, highlighting that higher-order poles contribute to the integral based on their residues.

Summary

Key Takeaways

- **Cauchy's Integral Theorem:** The integral of an analytic function over a closed contour is zero.

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- **Cauchy's Integral Theorem:** The integral of an analytic function over a closed contour is zero.
- **Cauchy's Integral Formula:** Provides a method to evaluate integrals and derive properties of analytic functions.
- **Residue Theorem:** A powerful tool for evaluating contour integrals, especially with multiple singularities.

Summary

Key Takeaways

- **Cauchy's Integral Theorem:** The integral of an analytic function over a closed contour is zero.
- **Cauchy's Integral Formula:** Provides a method to evaluate integrals and derive properties of analytic functions.
- **Residue Theorem:** A powerful tool for evaluating contour integrals, especially with multiple singularities.
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Further Reading

Questions and Discussion

Any Questions?