Limit, Continuity, and Differentiability...

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Monday, 2 Hours

Outline

Limits of Complex Functions

Definition

The limit of a complex function $f(z)$ as z approaches z_0 is L if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |z - z_0| < \delta$, it follows that $|f(z) - L| < \epsilon$.

$$\lim_{z \to z_0} f(z) = L$$

Properties of Limits

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Properties of Limits

- **Uniqueness:** A function can have at most one limit at a given point.
- **Arithmetic of Limits:** Limits respect addition, subtraction, multiplication, and division (provided the divisor's limit is not zero).
- **Limit Laws:** Includes laws like the Squeeze Theorem.
- **Path Independence:** If the limit exists, it is the same regardless of the path taken to approach z_0.

Example 1: Computing a Limit

Problem

Compute the limit:

$$\lim_{z \to 2+i} \frac{z^2 - (2+i)^2}{z - (2+i)}$$

Solution Overview

1. Recognize the expression as a difference quotient.

Example 1: Computing a Limit

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Solution Overview

1. Recognize the expression as a difference quotient.
2. Simplify the numerator using algebraic identities.

Example 1: Computing a Limit

Problem

Compute the limit:

$$\lim_{z \to 2+i} \frac{z^2 - (2+i)^2}{z - (2+i)}$$

Solution Overview

1. Recognize the expression as a difference quotient.
2. Simplify the numerator using algebraic identities.
3. Cancel the common factor and evaluate the limit.

Example 1: Step 1

Step 1: Recognize the Difference Quotient

Notice that:

$$\frac{z^2 - a^2}{z - a} = \frac{(z-a)(z+a)}{z-a} = z + a \quad \text{where } a = 2 + i$$

Therefore:

$$\frac{z^2 - (2+i)^2}{z - (2+i)} = z + (2+i)$$

Example 1: Step 2

Step 2: Simplify the Expression

After canceling the common factor:

$$\frac{z^2 - (2+i)^2}{z - (2+i)} = z + (2+i)$$

Example 1: Step 3

Step 3: Evaluate the Limit

Substitute $z = 2 + i$:

$$\lim_{z \to 2+i} (z + 2 + i) = (2 + i) + 2 + i = 4 + 2i$$

Example 1: Conclusion

Conclusion

$$\lim_{z \to 2+i} \frac{z^2 - (2+i)^2}{z - (2+i)} = 4 + 2i$$

Interpretation

The limit exists and equals $4 + 2i$, demonstrating the use of algebraic simplification in computing complex limits.

Example 2: Limit Does Not Exist

Problem

Determine whether the limit exists:

$$\lim_{z \to 0} \frac{\overline{z}}{z}$$

Solution Overview

1. Express z in polar form: $z = re^{i\theta}$.

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Determine whether the limit exists:

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Solution Overview

1. Express z in polar form: $z = re^{i\theta}$.
2. Substitute into the expression.

Example 2: Limit Does Not Exist

Problem

Determine whether the limit exists:

$$\lim_{z \to 0} \frac{\overline{z}}{z}$$

Solution Overview

1. Express z in polar form: $z = re^{i\theta}$.
2. Substitute into the expression.
3. Analyze the limit as $r \to 0$ for different values of θ.

Example 2: Limit Does Not Exist

Problem

Determine whether the limit exists:

$$\lim_{z \to 0} \frac{\overline{z}}{z}$$

Solution Overview

1. Express z in polar form: $z = re^{i\theta}$.
2. Substitute into the expression.
3. Analyze the limit as $r \to 0$ for different values of θ.
4. Conclude whether the limit exists.

Example 2: Step 1

Step 1: Express z in Polar Form

Let $z = re^{i\theta}$, where $r = |z|$ and $\theta = \arg(z)$.

Example 2: Step 2

Step 2: Substitute into the Expression

$$\frac{\overline{z}}{z} = \frac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta}$$

Example 2: Step 3

Step 3: Analyze the Limit

As $r \to 0$, $e^{-2i\theta}$ depends on θ. Different paths approaching 0 yield different limit values.

Example 2: Step 3

Step 3: Analyze the Limit

As $r \to 0$, $e^{-2i\theta}$ depends on θ. Different paths approaching 0 yield different limit values.

- **Along the Positive Real Axis ($\theta = 0$):**

$$e^{-2i(0)} = 1$$

- **Along the Positive Imaginary Axis ($\theta = \frac{\pi}{2}$):**

$$e^{-2i\left(\frac{\pi}{2}\right)} = e^{-i\pi} = -1$$

- **Along the Line $\theta = \frac{\pi}{4}$):**

$$e^{-2i\left(\frac{\pi}{4}\right)} = e^{-i\frac{\pi}{2}} = -i$$

Example 2: Conclusion

Conclusion

Since the limit depends on the path taken to approach 0 and yields different values for different θ, the limit:

$$\lim_{z \to 0} \frac{\overline{z}}{z}$$

does not exist.

Example 3: Computing a Complex Limit (Intermediate Difficulty)

Problem

Compute the limit:

$$\lim_{z \to 1-i} \frac{|z|^2 - |1-i|^2}{z - (1-i)}$$

Solution Overview

1. Express $|z|^2$ in terms of z and $\overline{z}$.

Example 3: Computing a Complex Limit (Intermediate Difficulty)

Problem

Compute the limit:

$$\lim_{z \to 1-i} \frac{|z|^2 - |1-i|^2}{z - (1-i)}$$

Solution Overview

1. Express $|z|^2$ in terms of z and $\overline{z}$.
2. Substitute $z_0 = 1 - i$ and simplify.

Example 3: Computing a Complex Limit (Intermediate Difficulty)

Problem

Compute the limit:

$$\lim_{z \to 1-i} \frac{|z|^2 - |1-i|^2}{z - (1-i)}$$

Solution Overview

1. Express $|z|^2$ in terms of z and $\overline{z}$.
2. Substitute $z_0 = 1 - i$ and simplify.
3. Evaluate the limit by direct substitution.

Example 3: Step 1

Step 1: Express $|z|^2$

Recall that:

$$|z|^2 = z\bar{z}$$

Therefore, the expression becomes:

$$\frac{z\bar{z} - (1+i)(1-i)}{z - (1-i)}$$

Example 3: Step 2

Step 2: Simplify the Expression

Compute $(1+i)(1-i)$:

$$(1+i)(1-i) = 1 - i^2 = 1 - (-1) = 2$$

Now, the expression is:

$$\frac{z\overline{z} - 2}{z - (1-i)}$$

Example 3: Step 3

Step 3: Evaluate the Limit

Since $z \to 1 - i$, substitute $z = 1 - i$ directly:

$$\lim_{z \to 1-i} \frac{z\bar{z} - 2}{z - (1 - i)} = \frac{(1 - i)(1 + i) - 2}{(1 - i) - (1 - i)} = \frac{2 - 2}{0}$$

This results in an indeterminate form $\frac{0}{0}$. To resolve, use the definition of the derivative.
Alternatively, express $z = 1 - i + h$, where $h \to 0$:

$$|z|^2 = |1-i+h|^2 = (1-i+h)(1+i+\bar{h}) = 2+h(1+i)+\bar{h}(1-i)+|h|^2$$

Thus:

$$\frac{|z|^2 - 2}{h} = \frac{h(1 + i) + \bar{h}(1 - i) + |h|^2}{h} = 1 + i + \frac{\bar{h}}{h}(1 - i) + |h|$$

Example 3: Conclusion

Conclusion

The limit:

$$\lim_{z \to 1-i} \frac{|z|^2 - |1-i|^2}{z - (1-i)}$$

does not exist because it depends on the path taken to approach $z_0 = 1 - i$.

Example 4: Computing a Complex Limit (Advanced Difficulty)

Problem

Compute the limit:

$$\lim_{z \to 0} \frac{e^z - 1}{z}$$

Solution Overview

1. Recall the Taylor series expansion of e^z around $z = 0$.

Example 4: Computing a Complex Limit (Advanced Difficulty)

Problem

Compute the limit:

$$\lim_{z \to 0} \frac{e^z - 1}{z}$$

Solution Overview

1. Recall the Taylor series expansion of e^z around $z = 0$.
2. Substitute the series into the expression.

Example 4: Computing a Complex Limit (Advanced Difficulty)

Problem

Compute the limit:

$$\lim_{z\to 0} \frac{e^z - 1}{z}$$

Solution Overview

1. Recall the Taylor series expansion of e^z around $z = 0$.
2. Substitute the series into the expression.
3. Simplify and evaluate the limit as $z \to 0$.

Example 4: Step 1

Step 1: Taylor Series Expansion

The Taylor series expansion of e^z around $z = 0$ is:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Example 4: Step 2

Step 2: Substitute the Series

Substitute the expansion into the limit expression:

$$\frac{e^z - 1}{z} = \frac{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots - 1}{z} = \frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}{z}$$

Simplify by dividing each term by z:

$$= 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

Example 4: Step 3

Step 3: Evaluate the Limit

As $z \to 0$, all terms containing z tend to zero:

$$\lim_{z\to 0}\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\cdots\right)=1$$

Example 4: Conclusion

Conclusion

$$\lim_{z\to 0} \frac{e^z - 1}{z} = 1$$

This result aligns with the derivative of e^z at $z = 0$.

Continuity of Complex Functions

Definition

A complex function $f(z)$ is said to be **continuous** at a point z_0 if:

$$\lim_{z \to z_0} f(z) = f(z_0)$$

In other words, the function does not have any "jumps" or "breaks" at z_0.

Properties of Continuous Functions

- **Composition:** The composition of continuous functions is continuous.

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- **Arithmetic Operations:** Sums, differences, products, and quotients (where the denominator is non-zero) of continuous functions are continuous.
- **Limits and Continuity:** If $f(z)$ is continuous at z_0 and $\lim_{z \to z_0} g(z) = z_0$, then $\lim_{z \to z_0} f(g(z)) = f(z_0)$.

Properties of Continuous Functions

- **Composition:** The composition of continuous functions is continuous.
- **Arithmetic Operations:** Sums, differences, products, and quotients (where the denominator is non-zero) of continuous functions are continuous.
- **Limits and Continuity:** If $f(z)$ is continuous at z_0 and $\lim_{z \to z_0} g(z) = z_0$, then $\lim_{z \to z_0} f(g(z)) = f(z_0)$.
- **Inverse:** If $f(z)$ is a bijective continuous function with a continuous inverse, then $f^{-1}(z)$ is also continuous.

Example 5: Checking Continuity

Problem

Determine whether the function $f(z) = \frac{1}{z}$ is continuous at $z_0 = 1$.

Solution Overview

1. Verify if z_0 is in the domain of $f(z)$.

Example 5: Checking Continuity

Problem

Determine whether the function $f(z) = \frac{1}{z}$ is continuous at $z_0 = 1$.

Solution Overview

1. Verify if z_0 is in the domain of $f(z)$.
2. Compute $f(z_0)$.

Example 5: Checking Continuity

Problem

Determine whether the function $f(z) = \frac{1}{z}$ is continuous at $z_0 = 1$.

Solution Overview

1. Verify if z_0 is in the domain of $f(z)$.
2. Compute $f(z_0)$.
3. Evaluate the limit $\lim_{z \to z_0} f(z)$.

Example 5: Checking Continuity

Problem

Determine whether the function $f(z) = \frac{1}{z}$ is continuous at $z_0 = 1$.

Solution Overview

1. Verify if z_0 is in the domain of $f(z)$.
2. Compute $f(z_0)$.
3. Evaluate the limit $\lim_{z \to z_0} f(z)$.
4. Compare the limit with $f(z_0)$.

Example 5: Step 1

Step 1: Verify Domain

The function $f(z) = \frac{1}{z}$ is defined for all $z \in \mathbb{C}$ except $z = 0$.
Since $z_0 = 1 \neq 0$, z_0 is in the domain of $f(z)$.

Example 5: Step 2

Step 2: Compute $f(z_0)$

$$f(z_0) = f(1) = \frac{1}{1} = 1$$

Example 5: Step 3

Step 3: Evaluate the Limit

$$\lim_{z \to 1} f(z) = \lim_{z \to 1} \frac{1}{z} = \frac{1}{1} = 1$$

Example 5: Step 4

Step 4: Compare Limit with $f(z_0)$

$$\lim_{z \to 1} f(z) = 1 = f(1)$$

Therefore, $f(z)$ is continuous at $z_0 = 1$.

Example 6: Continuity of $f(z) = |z|^2$

Problem

Determine whether the function $f(z) = |z|^2$ is continuous at $z_0 = 0$.

Solution Overview

1. Express $|z|^2$ in terms of z and $\overline{z}$.

Example 6: Continuity of $f(z) = |z|^2$

Problem

Determine whether the function $f(z) = |z|^2$ is continuous at $z_0 = 0$.

Solution Overview

1. Express $|z|^2$ in terms of z and $\overline{z}$.
2. Substitute $z = 0$ and compute $f(z_0)$.

Example 6: Continuity of $f(z) = |z|^2$

Problem

Determine whether the function $f(z) = |z|^2$ is continuous at $z_0 = 0$.

Solution Overview

1. Express $|z|^2$ in terms of z and $\bar{z}$.
2. Substitute $z = 0$ and compute $f(z_0)$.
3. Evaluate the limit $\lim_{z \to z_0} f(z)$.

Example 6: Continuity of $f(z) = |z|^2$

Problem

Determine whether the function $f(z) = |z|^2$ is continuous at $z_0 = 0$.

Solution Overview

1. Express $|z|^2$ in terms of z and $\overline{z}$.
2. Substitute $z = 0$ and compute $f(z_0)$.
3. Evaluate the limit $\lim_{z \to z_0} f(z)$.
4. Compare the limit with $f(z_0)$.

Example 6: Step 1

Step 1: Express $|z|^2$

Recall that:

$$|z|^2 = z\overline{z}$$

Example 6: Step 2

Step 2: Compute $f(z_0)$

$$f(z_0) = f(0) = |0|^2 = 0$$

Example 6: Step 3

Step 3: Evaluate the Limit

$$\lim_{z \to 0} |z|^2 = \lim_{z \to 0} z\bar{z} = 0$$

Example 6: Step 4

Step 4: Compare Limit with $f(z_0)$

$$\lim_{z \to 0} |z|^2 = 0 = f(0)$$

Hence, $f(z) = |z|^2$ is continuous at $z_0 = 0$.

Example 7: Continuity of a Piecewise Function

Problem

Determine whether the function $f(z)$ is continuous at $z_0 = 0$:

$$f(z) = \begin{cases} z^2 & \text{if } \operatorname{Re}(z) > 0, \\ \overline{z} & \text{if } \operatorname{Re}(z) \leq 0. \end{cases}$$

Solution Overview

1. Compute $f(0)$.

Example 7: Continuity of a Piecewise Function

Problem

Determine whether the function $f(z)$ is continuous at $z_0 = 0$:

$$f(z) = \begin{cases} z^2 & \text{if } \mathrm{Re}(z) > 0, \\ \overline{z} & \text{if } \mathrm{Re}(z) \leq 0. \end{cases}$$

Solution Overview

1. Compute $f(0)$.
2. Evaluate the limit $\lim_{z \to 0} f(z)$ from both sides of the boundary $\mathrm{Re}(z) = 0$.

Example 7: Continuity of a Piecewise Function

Problem

Determine whether the function $f(z)$ is continuous at $z_0 = 0$:

$$f(z) = \begin{cases} z^2 & \text{if } \mathrm{Re}(z) > 0, \\ \overline{z} & \text{if } \mathrm{Re}(z) \leq 0. \end{cases}$$

Solution Overview

1. Compute $f(0)$.
2. Evaluate the limit $\lim_{z \to 0} f(z)$ from both sides of the boundary $\mathrm{Re}(z) = 0$.
3. Compare the limits to determine continuity.

Example 7: Step 1

Step 1: Compute $f(z_0)$

$$f(0) = \overline{0} = 0$$

Example 7: Step 2

Step 2: Evaluate the Limit from Re(z) ¿ 0

$$\lim_{z \to 0, \mathrm{Re}(z) > 0} f(z) = \lim_{z \to 0} z^2 = 0$$

Example 7: Step 3

Step 3: Evaluate the Limit from Re(z) ≤ 0

$$\lim_{z \to 0, \mathrm{Re}(z) \leq 0} f(z) = \lim_{z \to 0} \bar{z} = 0$$

Example 7: Conclusion

Conclusion

Since the limits from both sides equal $f(0) = 0$, the function $f(z)$ is **continuous** at $z_0 = 0$.

Example 8: Discontinuity in a Complex Function

Problem

Determine whether the function $f(z) = \frac{1}{|z|}$ is continuous at $z_0 = 1$.

Solution Overview

1. Check if z_0 is in the domain of $f(z)$.

Example 8: Discontinuity in a Complex Function

Problem

Determine whether the function $f(z) = \frac{1}{|z|}$ is continuous at $z_0 = 1$.

Solution Overview

1. Check if z_0 is in the domain of $f(z)$.
2. Compute $f(z_0)$.

Example 8: Discontinuity in a Complex Function

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Determine whether the function $f(z) = \frac{1}{|z|}$ is continuous at $z_0 = 1$.

Solution Overview

1. Check if z_0 is in the domain of $f(z)$.
2. Compute $f(z_0)$.
3. Evaluate the limit $\lim_{z \to z_0} f(z)$.

Example 8: Discontinuity in a Complex Function

Problem

Determine whether the function $f(z) = \frac{1}{|z|}$ is continuous at $z_0 = 1$.

Solution Overview

1. Check if z_0 is in the domain of $f(z)$.
2. Compute $f(z_0)$.
3. Evaluate the limit $\lim_{z \to z_0} f(z)$.
4. Compare the limit with $f(z_0)$.

Example 8: Step 1

Step 1: Verify Domain

The function $f(z) = \frac{1}{|z|}$ is defined for all $z \in \mathbb{C}$ except $z = 0$. Since $z_0 = 1 \neq 0$, z_0 is in the domain of $f(z)$.

Example 8: Step 2

Step 2: Compute $f(z_0)$

$$f(z_0) = f(1) = \frac{1}{|1|} = 1$$

Example 8: Step 3

Step 3: Evaluate the Limit

$$\lim_{z \to 1} \frac{1}{|z|} = \frac{1}{|1|} = 1$$

Example 8: Conclusion

Conclusion

$$\lim_{z \to 1} \frac{1}{|z|} = 1 = f(1)$$

Therefore, $f(z)$ is **continuous** at $z_0 = 1$.

Example 9: Limit Along Different Paths (Advanced Difficulty)

Problem

Compute the limit:

$$\lim_{z \to 0} \frac{z}{\overline{z}}$$

Solution Overview

1. Express z in polar form: $z = re^{i\theta}$.

Example 9: Limit Along Different Paths (Advanced Difficulty)

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Solution Overview

1. Express z in polar form: $z = re^{i\theta}$.
2. Substitute into the expression.

Example 9: Limit Along Different Paths (Advanced Difficulty)

Problem

Compute the limit:

$$\lim_{z \to 0} \frac{z}{\overline{z}}$$

Solution Overview

1. Express z in polar form: $z = re^{i\theta}$.
2. Substitute into the expression.
3. Analyze the limit as $r \to 0$ for different values of θ.

Example 9: Limit Along Different Paths (Advanced Difficulty)

Problem

Compute the limit:

$$\lim_{z \to 0} \frac{z}{\overline{z}}$$

Solution Overview

1. Express z in polar form: $z = re^{i\theta}$.
2. Substitute into the expression.
3. Analyze the limit as $r \to 0$ for different values of θ.
4. Conclude whether the limit exists.

Example 9: Step 1

Step 1: Express z in Polar Form

Let $z = re^{i\theta}$, where $r = |z|$ and $\theta = \arg(z)$.

Example 9: Step 2

Step 2: Substitute into the Expression

$$\frac{z}{\overline{z}} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{2i\theta}$$

Example 9: Step 3

Step 3: Analyze the Limit

As $r \to 0$, $e^{2i\theta}$ depends on θ. Different paths approaching 0 yield different limit values.

Example 9: Step 3

Step 3: Analyze the Limit

As $r \to 0$, $e^{2i\theta}$ depends on θ. Different paths approaching 0 yield different limit values.

- **Along $\theta = 0$:**
$$e^{2i(0)} = 1$$
- **Along $\theta = \frac{\pi}{4}$):**
$$e^{2i\left(\frac{\pi}{4}\right)} = e^{i\frac{\pi}{2}} = i$$
- **Along $\theta = \frac{\pi}{2}$):**
$$e^{2i\left(\frac{\pi}{2}\right)} = e^{i\pi} = -1$$

Example 9: Conclusion

Conclusion

Since the limit varies depending on the path of approach (different values for different θ), the limit:

$$\lim_{z \to 0} \frac{z}{\overline{z}}$$

does not exist.

Example 10: Limit Involving Exponential Functions (Intermediate Difficulty)

Problem

Compute the limit:

$$\lim_{z \to 0} \frac{e^z - 1}{z}$$

Solution Overview

1. Recall the Taylor series expansion of e^z around $z = 0$.

Example 10: Limit Involving Exponential Functions (Intermediate Difficulty)

Problem

Compute the limit:

$$\lim_{z \to 0} \frac{e^z - 1}{z}$$

Solution Overview

1. Recall the Taylor series expansion of e^z around $z = 0$.
2. Substitute the series into the expression.

Example 10: Limit Involving Exponential Functions (Intermediate Difficulty)

Problem

Compute the limit:

$$\lim_{z \to 0} \frac{e^z - 1}{z}$$

Solution Overview

1. Recall the Taylor series expansion of e^z around $z = 0$.
2. Substitute the series into the expression.
3. Simplify and evaluate the limit as $z \to 0$.

Example 10: Step 1

Step 1: Taylor Series Expansion

The Taylor series expansion of e^z around $z = 0$ is:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Example 10: Step 2

Step 2: Substitute the Series

Substitute the expansion into the limit expression:

$$\frac{e^z - 1}{z} = \frac{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots - 1}{z} = \frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}{z}$$

Simplify by dividing each term by z:

$$= 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

Example 10: Step 3

Step 3: Evaluate the Limit

As $z \to 0$, all terms containing z tend to zero:

$$\lim_{z \to 0} \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots \right) = 1$$

Example 10: Conclusion

Conclusion

$$\lim_{z \to 0} \frac{e^z - 1}{z} = 1$$

This result aligns with the derivative of e^z at $z = 0$.

Differentiability of Complex Functions

Definition

A complex function $f(z)$ is **differentiable** at a point z_0 if the following limit exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

This limit, if it exists, is called the **derivative** of f at z_0.

Cauchy-Riemann Equations

Necessary Conditions for Differentiability

For $f(z) = u(x, y) + iv(x, y)$ to be differentiable at $z_0 = x_0 + iy_0$, the following **Cauchy-Riemann equations** must hold at (x_0, y_0):

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Sufficiency

If u and v have continuous first-order partial derivatives and satisfy the Cauchy-Riemann equations at z_0, then $f(z)$ is differentiable at z_0.

Example 11: Differentiability of $f(z) = \overline{z}$

Problem

Determine whether the function $f(z) = \overline{z}$ is differentiable at $z_0 = 0$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.

Example 11: Differentiability of $f(z) = \overline{z}$

Problem

Determine whether the function $f(z) = \overline{z}$ is differentiable at $z_0 = 0$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.
2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.

Example 11: Differentiability of $f(z) = \overline{z}$

Problem

Determine whether the function $f(z) = \overline{z}$ is differentiable at $z_0 = 0$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.
2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.
3. Check if the Cauchy-Riemann equations are satisfied at $z_0 = 0$.

Example 11: Differentiability of $f(z) = \overline{z}$

Problem

Determine whether the function $f(z) = \overline{z}$ is differentiable at $z_0 = 0$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.
2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.
3. Check if the Cauchy-Riemann equations are satisfied at $z_0 = 0$.
4. Conclude differentiability based on the results.

Example 11: Step 1

Step 1: Express $f(z)$ in Terms of x and y

Let $z = x + iy$, then:

$$f(z) = \bar{z} = x - iy$$

Therefore:

$$u(x, y) = x \quad \text{and} \quad v(x, y) = -y$$

Example 11: Step 2

Step 2: Compute Partial Derivatives

$$\frac{\partial u}{\partial x} = 1 \quad ; \quad \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = 0 \quad ; \quad \frac{\partial v}{\partial y} = -1$$

Example 11: Step 3

Step 3: Check Cauchy-Riemann Equations at $z_0 = 0$

The Cauchy-Riemann equations require:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \Rightarrow \quad 1 = -1 \quad \text{(False)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad 0 = 0 \quad \text{(True)}$$

Since the first equation is not satisfied, the Cauchy-Riemann equations fail at $z_0 = 0$.

Example 11: Conclusion

Conclusion

Since the Cauchy-Riemann equations are not satisfied at $z_0 = 0$, the function $f(z) = \overline{z}$ is **not differentiable** at $z_0 = 0$.

Implication

The failure of the Cauchy-Riemann equations implies that $f(z)$ does not possess a complex derivative at $z_0 = 0$, and hence, is not analytic at that point.

Example 12: Differentiability of an Analytic Function

Problem

Determine whether the function $f(z) = z^2$ is differentiable at $z_0 = 1 + i$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.

Example 12: Differentiability of an Analytic Function

Problem

Determine whether the function $f(z) = z^2$ is differentiable at $z_0 = 1 + i$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.
2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.

Example 12: Differentiability of an Analytic Function

Problem

Determine whether the function $f(z) = z^2$ is differentiable at $z_0 = 1 + i$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.
2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.
3. Check if the Cauchy-Riemann equations are satisfied at $z_0 = 1 + i$.

Example 12: Differentiability of an Analytic Function

Problem

Determine whether the function $f(z) = z^2$ is differentiable at $z_0 = 1 + i$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.
2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.
3. Check if the Cauchy-Riemann equations are satisfied at $z_0 = 1 + i$.
4. Conclude differentiability based on the results.

Example 12: Step 1

Step 1: Express $f(z)$ in Terms of x and y

Let $z = x + iy$, then:

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + i(2xy)$$

Therefore:

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy$$

Example 12: Step 2

Step 2: Compute Partial Derivatives

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y \quad ; \quad \frac{\partial v}{\partial y} = 2x$$

Example 12: Step 3

Step 3: Check Cauchy-Riemann Equations at $z_0 = 1 + i$

Substitute $x = 1$ and $y = 1$:

$$\frac{\partial u}{\partial x} = 2(1) = 2 \quad ; \quad \frac{\partial v}{\partial y} = 2(1) = 2$$

$$\frac{\partial u}{\partial y} = -2(1) = -2 \quad ; \quad \frac{\partial v}{\partial x} = 2(1) = 2$$

Check the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \Rightarrow \quad 2 = 2 \quad \text{(True)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad -2 = -2 \quad \text{(True)}$$

Example 12: Conclusion

Conclusion

Since the Cauchy-Riemann equations are satisfied at $z_0 = 1 + i$ and the partial derivatives are continuous, the function $f(z) = z^2$ is **differentiable** at $z_0 = 1 + i$.

Implication

The differentiability of $f(z) = z^2$ at $z_0 = 1 + i$ implies that $f(z)$ is analytic at that point.

Example 13: Differentiability of $f(z) = z^n$ for Integer $n \geq 1$

Problem

Show that the function $f(z) = z^n$, where n is a positive integer, is differentiable everywhere in $\mathbb{C}$.

Solution Overview

1. Express $f(z)$ in terms of x and y.

Example 13: Differentiability of $f(z) = z^n$ for Integer $n \geq 1$

Problem

Show that the function $f(z) = z^n$, where n is a positive integer, is differentiable everywhere in $\mathbb{C}$.

Solution Overview

1. Express $f(z)$ in terms of x and y.
2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.

Example 13: Differentiability of $f(z) = z^n$ for Integer $n \geq 1$

Problem

Show that the function $f(z) = z^n$, where n is a positive integer, is differentiable everywhere in $\mathbb{C}$.

Solution Overview

1. Express $f(z)$ in terms of x and y.
2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.
3. Verify the Cauchy-Riemann equations.

Example 13: Differentiability of $f(z) = z^n$ for Integer $n \geq 1$

Problem

Show that the function $f(z) = z^n$, where n is a positive integer, is differentiable everywhere in $\mathbb{C}$.

Solution Overview

1. Express $f(z)$ in terms of x and y.
2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.
3. Verify the Cauchy-Riemann equations.
4. Conclude differentiability and analyticity.

Example 13: Step 1

Step 1: Express $f(z) = z^n$ in Terms of x and y

Let $z = x + iy$, then:

$$f(z) = (x + iy)^n$$

Expanding using the binomial theorem, we get:

$$f(z) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are polynomials in x and y.

Step 2: Compute Partial Derivatives

For $f(z) = z^n$, the partial derivatives can be computed using standard differentiation rules.

$$\frac{\partial u}{\partial x} = (\text{derived from } u(x, y))$$

$$\frac{\partial u}{\partial y} = (\text{derived from } u(x, y))$$

$$\frac{\partial v}{\partial x} = (\text{derived from } v(x, y))$$

$$\frac{\partial v}{\partial y} = (\text{derived from } v(x, y))$$

Example 13: Step 3

Step 3: Verify Cauchy-Riemann Equations

Since $f(z) = z^n$ is a polynomial, its partial derivatives exist and are continuous everywhere.
By applying the Cauchy-Riemann equations, one can verify that they are satisfied for all $z \in \mathbb{C}$.

Example 13: Conclusion

Conclusion

Since the Cauchy-Riemann equations are satisfied everywhere in $\mathbb{C}$ and the partial derivatives are continuous, the function $f(z) = z^n$ is **differentiable everywhere in $\mathbb{C}$**.
Additionally, $f(z)$ is **analytic** throughout $\mathbb{C}$.

Implication

Analytic functions like $f(z) = z^n$ have powerful properties, including the ability to be represented by convergent power series within their domains.

Example 14: Differentiability of $f(z) = \mathrm{Re}(z) + i\mathrm{Re}(z)$

Problem

Determine whether the function $f(z) = \mathrm{Re}(z) + i\mathrm{Re}(z)$ is differentiable at $z_0 = 1$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.

Example 14: Differentiability of $f(z) = \text{Re}(z) + i\text{Re}(z)$

Problem

Determine whether the function $f(z) = \text{Re}(z) + i\text{Re}(z)$ is differentiable at $z_0 = 1$.

Solution Overview

1. Express $f(z)$ in terms of x and y, where $z = x + iy$.
2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.

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3. Check if the Cauchy-Riemann equations are satisfied at $z_0 = 1$.

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2. Compute the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.
3. Check if the Cauchy-Riemann equations are satisfied at $z_0 = 1$.
4. Conclude differentiability based on the results.

Example 14: Step 1

Step 1: Express $f(z)$ in Terms of x and y

Let $z = x + iy$, then:

$$\mathrm{Re}(z) = x$$

Therefore:

$$f(z) = x + ix = x(1 + i)$$

Thus:

$$u(x, y) = x \quad \text{and} \quad v(x, y) = x$$

Example 14: Step 2

Step 2: Compute Partial Derivatives

$$\frac{\partial u}{\partial x} = 1 \quad ; \quad \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = 1 \quad ; \quad \frac{\partial v}{\partial y} = 0$$

Example 14: Step 3

Step 3: Check Cauchy-Riemann Equations at $z_0 = 1$

The Cauchy-Riemann equations require:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \Rightarrow \quad 1 = 0 \quad \text{(False)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \Rightarrow \quad 0 = -1 \quad \text{(False)}$$

Both equations are not satisfied.

Example 14: Conclusion

Conclusion

Since the Cauchy-Riemann equations are not satisfied at $z_0 = 1$, the function $f(z) = \text{Re}(z) + i\text{Re}(z)$ is **not differentiable** at $z_0 = 1$.

Implication

The function $f(z)$ lacks a complex derivative at $z_0 = 1$, indicating that it is not analytic at that point.

Example 15: Differentiability of $f(z) = e^{az}$ for Complex a

Problem

Let a be a complex constant. Determine whether the function $f(z) = e^{az}$ is differentiable everywhere in $\mathbb{C}$.

Solution Overview

1. Recall the definition of differentiability for e^{az}.

Example 15: Differentiability of $f(z) = e^{az}$ for Complex a

Problem

Let a be a complex constant. Determine whether the function $f(z) = e^{az}$ is differentiable everywhere in $\mathbb{C}$.

Solution Overview

1. Recall the definition of differentiability for e^{az}.
2. Compute the derivative using the limit definition.

Example 15: Differentiability of $f(z) = e^{az}$ for Complex a

Problem

Let a be a complex constant. Determine whether the function $f(z) = e^{az}$ is differentiable everywhere in $\mathbb{C}$.

Solution Overview

1. Recall the definition of differentiability for e^{az}.
2. Compute the derivative using the limit definition.
3. Confirm differentiability by verifying the existence of the derivative everywhere.

Example 15: Step 1

Step 1: Recall the Definition of Differentiability

The function $f(z) = e^{az}$ is differentiable at z_0 if the derivative $f'(z_0)$ exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Example 15: Step 2

Step 2: Compute the Derivative

Using the limit definition:

$$f'(z_0) = \lim_{z \to z_0} \frac{e^{az} - e^{az_0}}{z - z_0}$$

Recognize this as the derivative of the exponential function:

$$f'(z_0) = ae^{az_0}$$

Example 15: Step 3

Step 3: Confirm Differentiability Everywhere

Since e^{az} is an entire function (analytic everywhere in $\mathbb{C}$), its derivative exists at every point $z \in \mathbb{C}$.

Example 15: Conclusion

Conclusion

The function $f(z) = e^{az}$ is **differentiable everywhere in $\mathbb{C}$**.

Implication

Being differentiable everywhere implies that $f(z) = e^{az}$ is an **analytic** function throughout the complex plane.

Cauchy's Integral Theorem

Statement

Let U be a simply connected open subset of $\mathbb{C}$, and let f be analytic on U. Then, for any closed contour γ entirely contained within U:

$$\oint_{\gamma} f(z)\, dz = 0$$

Implications

- Provides a fundamental property of analytic functions.

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$$\oint_{\gamma} f(z)\, dz = 0$$

Implications

- Provides a fundamental property of analytic functions.
- Essential for evaluating complex integrals.
- Leads to Cauchy's Integral Formula.

Summary

Key Takeaways

- **Limits:** Fundamental for defining continuity and differentiability in complex functions.

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Summary

Key Takeaways

- **Limits:** Fundamental for defining continuity and differentiability in complex functions.
- **Continuity:** Ensures no abrupt changes in the function's value; essential for differentiability.
- **Differentiability:** Governed by the Cauchy-Riemann equations; a precursor to analyticity.
- **Cauchy-Riemann Equations:** Necessary and sufficient (with continuous partial derivatives) for differentiability.

Summary

Key Takeaways

- **Limits:** Fundamental for defining continuity and differentiability in complex functions.
- **Continuity:** Ensures no abrupt changes in the function's value; essential for differentiability.
- **Differentiability:** Governed by the Cauchy-Riemann equations; a precursor to analyticity.
- **Cauchy-Riemann Equations:** Necessary and sufficient (with continuous partial derivatives) for differentiability.
- **Analytic Functions:** Differentiable in an open neighborhood; possess powerful properties like infinite differentiability and power series expansions.

Further Reading

Questions and Discussion

Any Questions?