Engineering Calculus III

Week 2: Elementary Complex Analysis

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Monday, 4 PM - 6 PM

Today's Agenda

Introduction to Complex Numbers

Definition

A **complex number** is a number of the form:

$$z = a + bi$$

where:

- a is the **real part** of z.
- b is the **imaginary part** of z.
- i is the imaginary unit, defined by $i^2 = -1$.

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Standard Form

$$z = a + bi \quad \text{where} \quad a, b \in \mathbb{R}$$

Example of a Complex Number

Example

Let $z = 3 + 4i$:

- Real part: $\Re(z) = 3$
- Imaginary part: $\Im(z) = 4$

Algebra of Complex Numbers: Addition and Subtraction

Addition and Subtraction

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$$

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Example 1: Addition

$$(2 + 3i) + (1 + 4i) = (2 + 1) + (3 + 4)i = 3 + 7i$$

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Example 1: Addition

$$(2 + 3i) + (1 + 4i) = (2 + 1) + (3 + 4)i = 3 + 7i$$

Example 2: Subtraction

$$(5 + 6i) - (2 + 3i) = (5 - 2) + (6 - 3)i = 3 + 3i$$

Algebra of Complex Numbers: Multiplication

Multiplication

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

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Example 1: Basic Multiplication

$$(2+3i)(1+4i) = (2\times1-3\times4)+(2\times4+3\times1)i = (-10)+11i = -10+11i$$

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Example 2: Multiplying with Negative Imaginary Part

$$(1-2i)(3+4i) = (1\times 3-(-2)\times 4)+(1\times 4+(-2)\times 3)i = (3+8)+(4-6)i = 11-$$

Algebra of Complex Numbers: Division

Division

To divide $z_1 = a + bi$ by $z_2 = c + di$:

$$\frac{z_1}{z_2} = \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

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Example 2: Division with Negative Denominator

$$\frac{4-i}{2-3i} = \frac{(4-i)(2+3i)}{2^2+3^2} = \frac{8+12i-2i-3i^2}{13} = \frac{8+10i+3}{13} = \frac{11+10i}{13} =$$

Conjugate and Modulus of Complex Numbers

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Modulus of a Complex Number

The **modulus** (or absolute value) of $z = a + bi$ is:

$$|z| = \sqrt{a^2 + b^2}$$

Properties of Modulus

Properties

1. $|z| \geq 0$, and $|z| = 0$ if and only if $z = 0$.
2. $|z_1 z_2| = |z_1| \cdot |z_2|$
3. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ (for $z_2 \neq 0$)

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Example 2: Modulus of a Product

Let $z_1 = 2 + i$ and $z_2 = 1 - 3i$:

$$|z_1 z_2| = |z_1| \cdot |z_2| = \sqrt{2^2 + 1^2} \cdot \sqrt{1^2 + (-3)^2} = \sqrt{5} \cdot \sqrt{10} = \sqrt{50} = 5\sqrt{2}$$

Complex Plane (Argand Plane)

Definition

The **complex plane** is a two-dimensional plane where:

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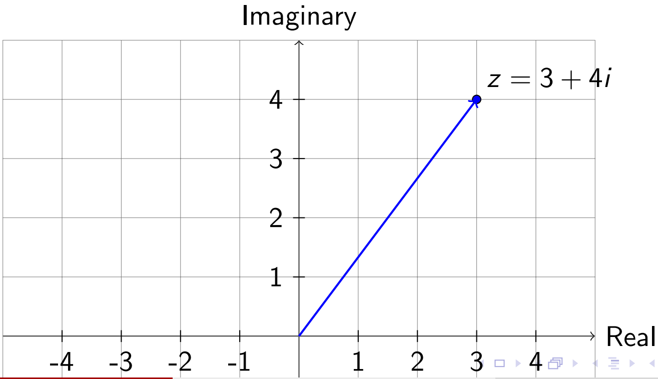

Representation of a Complex Number

Mapping to the Complex Plane

$$z = a + bi \quad \leftrightarrow \quad (a, b) \text{ in the complex plane}$$

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Example

Plotting $z = 3 + 4i$:

- Real part ($a = 3$) on the x-axis.
- Imaginary part ($b = 4$) on the y-axis.

Addition of Complex Numbers

Geometric Interpretation

Adding two complex numbers corresponds to vector addition in the complex plane.

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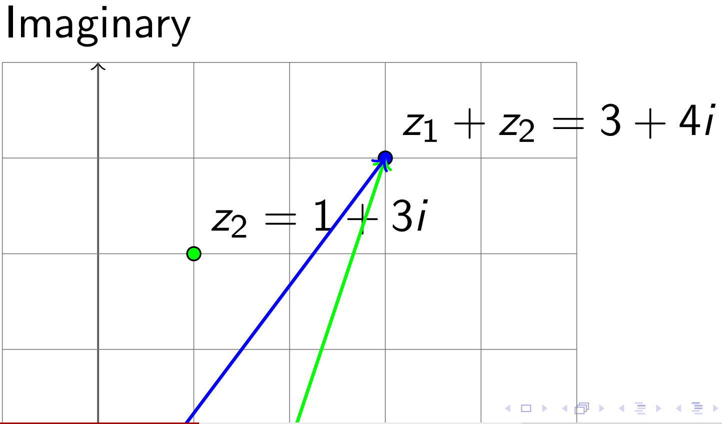

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Example

Let $z = 1 + i$. Then:

$$iz = i(1 + i) = i + i^2 = i - 1 = -1 + i$$

Geometrically, this is a 90° rotation of z in the complex plane.

Multiplication by i Visualization

Geometric Interpretation

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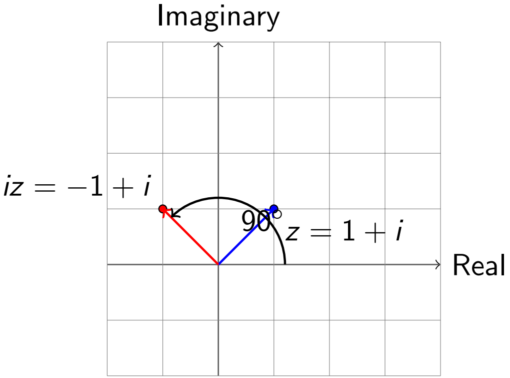

Figure: Multiplication by i Rotates z by 90°

Interactive Activity

Activity: Exploring Rotations

Objective: Understand the effect of multiplying complex numbers by i, -1, and $-i$.

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Discussion Points

- How does each multiplication affect the position of z in the complex plane?
- What patterns do you observe in the transformations?

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Guiding Questions

- What is the result of multiplying z by i? By -1? By $-i$?

Additional Example Problems

Example 3: Conjugate and Division

$$\frac{z}{\overline{z}} \quad \text{where} \quad z = 4 + 3i$$

$$\frac{4+3i}{4-3i} = \frac{(4+3i)(4+3i)}{4^2+3^2} = \frac{16+24i+9i^2}{25} = \frac{16+24i-9}{25} = \frac{7+24i}{25} = \frac{7}{25}$$

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Example 4: Modulus and Division

Compute $\frac{z_1}{z_2}$ where $z_1 = 5 + 12i$ and $z_2 = 3 + 4i$.

$$|z_1| = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

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$$\frac{z_1}{z_2} = \frac{5+12i}{3+4i} = \frac{(5+12i)(3-4i)}{3^2+4^2} = \frac{15-20i+36i-48i^2}{25} = \frac{15+16i+48}{25}$$

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- **Algebra of Complex Numbers** includes operations such as addition, subtraction, multiplication, division, conjugation, and modulus.
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Assignment

Please complete **Assignment 2** on Complex Number Operations, which will be reviewed in the next class.

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See you next Monday!