Engineering Calculus III

Week 3: Complex Exponentials

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Monday, 4 PM - 6 PM

Today's Agenda

1. Complex Exponentials
2. Logarithmic Functions
3. Trigonometric (Circular) Functions
4. Euler's Formula
5. Euler's Formula

Complex Exponentials: Definition

Definition

The complex exponential function is defined as:

$$\begin{aligned} e^z &= e^{a+bi} \\ &= e^a\left(\cos b + i\sin b\right) \end{aligned}$$

where $z = a + bi$, and $a, b \in \mathbb{R}$.

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Properties

1. $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$
2. $e^0 = 1$
3. $\frac{d}{dz} e^z = e^z$

Complex Exponentials: Examples

Example 1

Compute e^{2+3i}:

$$\begin{aligned} e^{2+3i} &= e^2 (\cos 3 + i \sin 3) \\ &\approx 7.389 \times (-0.98999 + i \cdot 0.14112) \\ &\approx -7.328 + 1.045i \end{aligned}$$

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Example 2

Simplify $e^{i\pi}$:

$$\begin{aligned} e^{i\pi} &= \cos\pi + i\sin\pi \\ &= -1 + 0i \\ &= -1 \end{aligned}$$

Example 3

Compute e^{2+3i}:

$$e^{2+3i} = e^2 (\cos 3 + i \sin 3)$$

First, calculate e^2:

$$= 7.389 (\cos 3 + i \sin 3)$$

Next, compute $\cos 3$ and $\sin 3$ (in radians):

$$\approx 7.389 (-0.98999 + i \cdot 0.14112)$$

Finally, multiply e^2 with the computed cosine and sine values:

$$\approx -7.328 + 1.045i$$

Complex Exponentials: Additional Worked Examples

Example 4

Compute $e^{-1+\frac{\pi}{2}i}$:

$$\begin{aligned} e^{-1+\frac{\pi}{2}i} &= e^{-1}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\ &= \frac{1}{e}(0 + i \cdot 1) \\ &= \frac{i}{e} \approx 0 + 0.3679i \end{aligned}$$

Complex Exponentials: Additional Worked Examples

Example 4

Compute $e^{-1+\frac{\pi}{2}i}$:

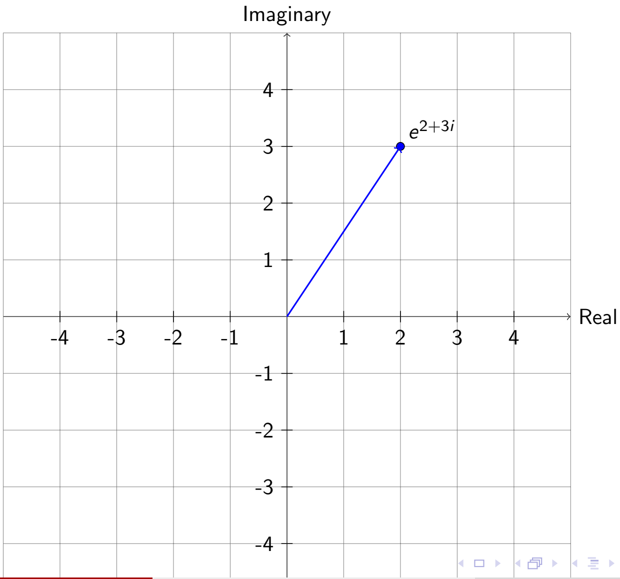
$$\begin{aligned} e^{-1+\frac{\pi}{2}i} &= e^{-1}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\ &= \frac{1}{e}(0 + i\cdot 1) \\ &= \frac{i}{e} \approx 0 + 0.3679i \end{aligned}$$

Example 5

Simplify e^{3-2i}:

$$\begin{aligned} e^{3-2i} &= e^{3}(\cos(-2) + i\sin(-2)) \\ &= e^{3}(\cos 2 - i\sin 2) \\ &\approx 20.086 \times (-0.4161 - i\cdot 0.9093) \end{aligned}$$

Complex Exponentials: Graphical Representation



Complex Exponentials: Interactive Activity

Activity: Calculating Complex Exponentials

Objective: Practice computing complex exponentials using Euler's formula.

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1. Compute $e^{1+\frac{\pi}{4}i}$.
2. Simplify e^{-2i}.
3. Calculate $e^{3-\frac{\pi}{2}i}$.

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Objective: Practice computing complex exponentials using Euler's formula.

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Guiding Questions

- How does the real part a affect the magnitude of e^{a+bi}?
- What is the relationship between the imaginary part b and the angle in the complex plane?

Example 6

Simplify $e^{i\pi}$:

$$e^{i\pi} = \cos\pi + i\sin\pi$$

Evaluate $\cos\pi$ and $\sin\pi$:

$$= -1 + i \cdot 0$$

Therefore, the simplified form is:

$$= -1$$

Logarithmic Functions: Definition

Definition

The complex logarithmic function is the inverse of the complex exponential function:

$$\log z = \ln|z| + i \arg(z)$$

where $z = re^{i\theta}$, $r > 0$, and θ is the argument of z.

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Properties

1. $\log(z_1 z_2) = \log z_1 + \log z_2$
2. $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$
3. $\log(z^n) = n \log z$ for integer n

Logarithmic Functions: Branch Cuts

Multivaluedness

The complex logarithm is a **multivalued** function because the argument θ can be expressed as $\theta + 2k\pi$ for any integer k.

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The **principal branch** of the logarithm restricts the argument to $-\pi < \theta \leq \pi$, denoted as $\mathrm{Arg}(z)$.

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Branch Cut

To define a single-valued branch, a **branch cut** is introduced, typically along the negative real axis:

$$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$$

Logarithmic Functions: Worked Examples

Example 1: Computing log(1)

$$\begin{aligned}\log(1) &= \ln|1| + i\arg(1)\\ &= 0 + i\cdot 0\\ &= 0\end{aligned}$$

Logarithmic Functions: Worked Examples

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$$\begin{aligned} \log(1) &= \ln|1| + i\arg(1) \\ &= 0 + i \cdot 0 \\ &= 0 \end{aligned}$$

Example 2: Computing $\log(-1)$ using the Principal Branch

$$\begin{aligned} \log(-1) &= \ln|-1| + i\arg(-1) \\ &= 0 + i\pi \\ &= i\pi \end{aligned}$$

Logarithmic Functions: Additional Worked Examples

Example 3: Computing $\log\left(e^{i\frac{\pi}{3}}\right)$

$$\begin{aligned}
\log\left(e^{i\frac{\pi}{3}}\right) &= \ln\left|e^{i\frac{\pi}{3}}\right| + i\arg\left(e^{i\frac{\pi}{3}}\right) \\
&= \ln(1) + i\cdot\frac{\pi}{3} \\
&= 0 + i\cdot\frac{\pi}{3} \\
&= i\frac{\pi}{3}
\end{aligned}$$

Logarithmic Functions: Additional Worked Examples

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Example 4: Computing $\log\left(2e^{-i\frac{\pi}{4}}\right)$

$$\log\left(2e^{-i\frac{\pi}{4}}\right) = \ln|2e^{-i\frac{\pi}{4}}| + i\arg\left(2e^{-i\frac{\pi}{4}}\right)$$

Logarithmic Functions: Worked Examples Continued

Example 5: Simplifying $\log\left(\frac{1+i}{1-i}\right)$

Problem: Simplify $\log\left(\frac{1+i}{1-i}\right)$

Solution:

$$\log\left(\frac{1+i}{1-i}\right) = \log(1+i) - \log(1-i)$$

Compute $\log(1+i)$:

$$\begin{aligned}
&= \ln|1+i| + i\arg(1+i) \\
&= \ln\sqrt{1^2+1^2} + i\cdot\frac{\pi}{4} \\
&= \ln\sqrt{2} + i\frac{\pi}{4} \\
&= \frac{1}{2}\ln 2 + i\frac{\pi}{4}
\end{aligned}$$

Logarithmic Functions: Complex Logarithm Properties

Property 1: Logarithm of a Product

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

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Property 2: Logarithm of a Quotient

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$$

Property 3: Logarithm of a Power

$$\log(z^n) = n \log z \quad \text{for integer } n$$

Logarithmic Functions: Interactive Activity

Activity: Evaluating Complex Logarithms

Objective: Practice evaluating logarithms of complex numbers using the principal branch.

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1. Compute $\log\left(e^{i\frac{\pi}{3}}\right)$.
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Guiding Questions

- How does the magnitude and argument of a complex number affect its logarithm?
- What challenges arise due to the multivalued nature of the complex logarithm?

Trigonometric Functions: Overview

Definition

The primary trigonometric functions are:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

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Circular Functions

Trigonometric functions relate to the unit circle in the complex plane, where θ represents the angle from the positive real axis.

Trigonometric Functions: Graphical Representation

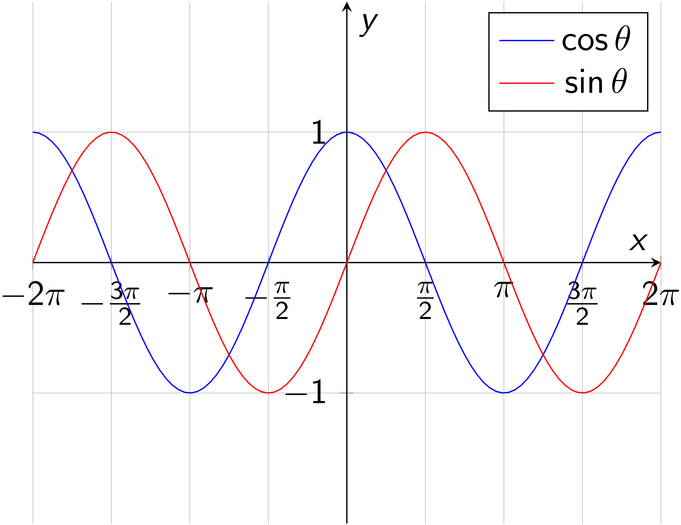

Figure: Graphs of $\cos\theta$ and $\sin\theta$

Trigonometric Functions: Properties

Key Properties

1. **Periodic Functions:**

$$\cos(\theta + 2\pi) = \cos\theta$$
$$\sin(\theta + 2\pi) = \sin\theta$$

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$$\cos(-\theta) = \cos\theta \quad \text{(Even)}$$
$$\sin(-\theta) = -\sin\theta \quad \text{(Odd)}$$

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3. **Pythagorean Identity:**

$$\cos^2\theta + \sin^2\theta = 1$$

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Angle Addition Formulas

Trigonometric Functions: Worked Examples

Example 1: Computing $\cos(\pi)$ and $\sin(\pi)$

$$\cos(\pi) = -1$$
$$\sin(\pi) = 0$$

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Example 2: Evaluating $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{3}\right)$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$
$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Trigonometric Functions: Additional Worked Examples

Example 3: Simplifying Trigonometric Expressions using Euler's Formula

Problem: Simplify $\cos\theta + i\sin\theta$.

Solution:

$$\cos\theta + i\sin\theta = e^{i\theta} \quad \text{(Using Euler's Formula)}$$

Trigonometric Functions: Additional Worked Examples

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Example 4: Expressing Trigonometric Functions in Terms of Complex Exponentials

Problem: Express $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.
Solution:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Trigonometric Functions: Complex Trigonometric Identities

Product-to-Sum Identity

$$\cos\alpha\cos\beta = \frac{e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)}}{4} + \frac{e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)}}{4}$$

$$\sin\alpha\sin\beta = \frac{e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}}{4i^2} - \frac{e^{i(\alpha-\beta)} - e^{-i(\alpha-\beta)}}{4i^2}$$

$$= \frac{\cos(\alpha+\beta) - \cos(\alpha-\beta)}{2}$$

Trigonometric Functions: Complex Trigonometric Identities

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Sum-to-Product Identity

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

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Activity: Exploring Trigonometric Identities

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1. Show that $\cos^2\theta + \sin^2\theta = 1$ using Euler's formula.
2. Prove the angle addition formula for cosine:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

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Guiding Questions

- How can Euler's formula simplify the derivation of trigonometric identities?
- What role do the real and imaginary parts of complex exponentials play in these identities?

Euler's Formula: Introduction

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where θ is a real number.

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Significance

Euler's Formula beautifully connects the exponential function with trigonometric functions, providing a powerful tool in complex analysis.

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Significance

Euler's Formula elegantly connects the exponential function with trigonometric functions, providing a foundational tool in complex analysis and engineering applications.

Euler's Formula: Derivation Using Taylor Series

Taylor Series Expansions

The Taylor series expansions of $e^{i\theta}$, $\cos\theta$, and $\sin\theta$ around $\theta = 0$ are:

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\
&= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots \\
\cos\theta &= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} \\
&= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \\
\sin\theta &= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}
\end{aligned}$$

$\theta^3 \quad \theta^5 \quad \theta^7$

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\sin\theta &= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}
\end{aligned}
$$

$\theta^3 \quad \theta^5 \quad \theta^7$

Euler's Formula: Separating Real and Imaginary Parts

Expanding $e^{i\theta}$

Substitute the powers of i into the Taylor series expansion of $e^{i\theta}$:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots \\ &= 1 + i\theta + \frac{(-1)\theta^2}{2!} + \frac{(-i)\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \end{aligned}$$

Euler's Formula: Separating Real and Imaginary Parts

Expanding $e^{i\theta}$

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Identifying Series

Notice that the real and imaginary parts correspond to the Taylor series of $\cos\theta$ and $\sin\theta$ respectively:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

Euler's Formula: Conclusion

Combining the Results

From the previous steps, we have:

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

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Combining the Results

From the previous steps, we have:

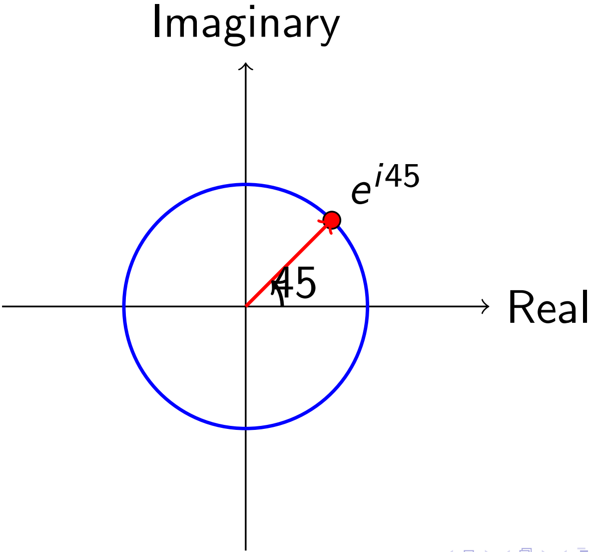
$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Final Statement

Therefore, Euler's Formula is proven:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Euler's Formula: Graphical Interpretation



Euler's Formula: Applications

Applications in Engineering

1. **Signal Processing:** Representing sinusoidal signals as complex exponentials simplifies analysis and manipulation.
2. **Electrical Engineering:** Analyzing AC circuits using phasors and impedance calculations.
3. **Control Systems:** Solving differential equations with complex eigenvalues.

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Applications in Mathematics

1. **Fourier Series:** Expressing periodic functions as sums of sines and cosines using complex exponentials.
2. **Differential Equations:** Simplifying solutions of linear differential equations with constant coefficients.
3. **Complex Analysis:** Facilitating contour integration and residue calculations.

Euler's Formula: Additional Worked Examples

Example 5: Using Euler's Formula to Express $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$

Problem: Express $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$ in terms of $e^{i\theta}$.
Solution:

$$\cos\left(\frac{\pi}{4}\right) = \frac{e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}}{2}$$

Using Euler's Formula for $e^{i\theta}$ and $e^{-i\theta}$:

$$= \frac{\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)}{2}$$

Simplify the expression by combining like terms:

$$= \frac{2\cos\left(\frac{\pi}{4}\right)}{2}$$

Euler's Formula: Additional Worked Examples

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Solution:

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Using Euler's Formula for $e^{i\theta}$ and $e^{-i\theta}$:

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Simplify the expression by combining like terms:

$$= \frac{2\cos\left(\frac{\pi}{4}\right)}{2}$$

Euler's Formula: Complex Trigonometric Identities

Expressing $\sin\theta$ and $\cos\theta$ in Terms of $e^{i\theta}$ and $e^{-i\theta}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Euler's Formula: Complex Trigonometric Identities

Expressing $\sin\theta$ and $\cos\theta$ in Terms of $e^{i\theta}$ and $e^{-i\theta}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
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Deriving Trigonometric Identities Using Euler's Formula

Example: Prove that $\cos^2\theta = \frac{1+\cos 2\theta}{2}$
Solution:

$$\cos^2\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2$$

Expand the squared term:

Euler's Formula: Interactive Activity

Activity: Exploring Euler's Formula

Objective: Understand and apply Euler's Formula to solve complex problems.

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1. Use Euler's Formula to express $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$ in terms of $e^{i\theta}$.
2. Verify Euler's Formula by computing $e^{i\frac{\pi}{2}}$ and comparing it to $\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$.
3. Express $\sin\theta$ and $\cos\theta$ solely in terms of $e^{i\theta}$ and $e^{-i\theta}$.

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1. Use Euler's Formula to express $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$ in terms of $e^{i\theta}$.
2. Verify Euler's Formula by computing $e^{i\frac{\pi}{2}}$ and comparing it to $\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$.
3. Express $\sin\theta$ and $\cos\theta$ solely in terms of $e^{i\theta}$ and $e^{-i\theta}$.

Guiding Questions

- How does Euler's Formula simplify the representation of trigonometric functions?
- In what ways can complex exponentials be utilized to solve real-world engineering problems?

Summary

- **Complex Exponentials** bridge exponential functions and trigonometric functions, providing a powerful tool in complex analysis.
- **Logarithmic Functions** in the complex plane are multivalued and require careful consideration of branch cuts.
- **Trigonometric Functions** can be expressed using complex exponentials, simplifying many mathematical operations and proofs.
- **Euler's Formula** elegantly connects the exponential and trigonometric functions, underpinning many applications in engineering and mathematics.

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We will delve into **Applications of Euler's Formula** in solving differential equations and Fourier analysis.

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Assignment

Please complete **Assignment 3** on Complex Exponentials and Euler's Formula, which will be discussed in the next class.

Questions & Discussion

Any questions or clarifications?

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Discussion Points

- Understanding the connection between complex exponentials and trigonometric functions.
- Challenges in handling multivalued logarithmic functions.
- Practical applications of Euler's Formula in engineering contexts.

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- Understanding the connection between complex exponentials and trigonometric functions.
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Reminder

Office Hours: Monday & Wednesday (2:00 PM - 4:00 PM), Friday (10:00 AM - 12:00 PM)

Thank You!

Thank You!

Thank You!

Thank You!

See you next Monday!