

Lecture notes  
Quantum Theory of Solids

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## Foreword to the digitalized lecture notes

Digitalized lecture notes for the course “TFY4210 - Quantum Theory of Many-Particle Systems” held by Prof. Asle Sudbø spring 2020. These notes follow that of the hand written lecture notes, which are based upon the lecture notes for the course “FY8302 - Quantum Theory of Solids”, written in 1996 or so.

There may be a few extra comments to the text and / or figures that is based on the lectures themselves.

Course website: <https://www.ntnu.edu/studies/courses/TFY4210>

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## 1 Introduction

## 2 Many-particle states, fermions

### 2.1 N-particle vacuum state

### 2.2 Completeness relation

### 2.3 Operators

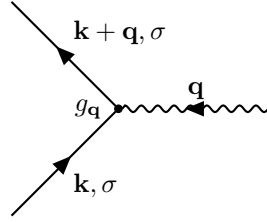


Figure 1: Diagram of electron-phonon vertex

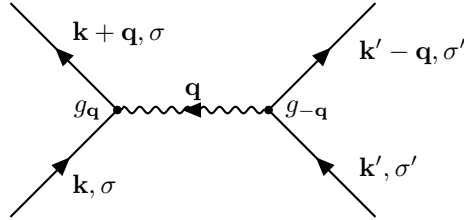


Figure 2: Diagram of the effective electron-phonon interaction

### 3 Quasi-particles in interacting electron-systems. Fermi-liquids.

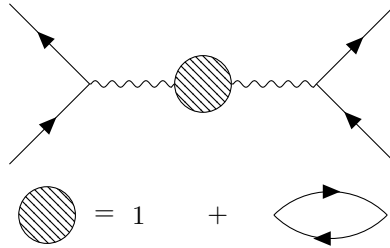
#### 3.1 Fermi-liquids

#### 3.2 Screening of the Coulomb-interaction

#### 3.3 Phonon mediated electron-electron interaction

Due to the electron-phonon coupling depicted in fig. 1, we will get an effective phonon-mediated interaction between electrons, depicted in fig. 2

This is an exchange of a virtual phonon. The above diagram is the effective interaction to second order in  $g_q$  if we regard the wavy line  $\sim$  as a bare phonon Green's function. We could also imagine that we replaced this by



which would include an effective interaction computed correctly up to order  $\mathcal{O}(g^4)$ . In fact, we might replace  $D_0 + D_0\Pi D_0 + \dots$  by  $D$ ! Thus computing

the effective interaction up to infinite order in  $g$ . Another, often used approach would be to replace  $\sim\sim\sim$  by **TODO: SETT INN DIAGRAM**

Here, we have resummed a subset of diagrams to infinite order in  $g$  in order to get an effective interaction between electrons. Under the assumption that  $g$  is weak, we will keep terms only to  $\mathcal{O}(g^2)$ .

$$V_{\text{eff}}(q, \omega) = |g_q|^2 \frac{2\omega_q}{\omega^2 - \omega_q^2} \quad (3.1)$$

Thus, the interaction part of the Hamiltonian becomes

$$\mathcal{H} = \sum_{\substack{k, k', q \\ \sigma, \sigma'}} V_{\text{tot}}(q, \omega) c_{k+q, \sigma}^\dagger c_{k', \sigma'}^\dagger c_{k', \sigma'} c_{k, \sigma} \quad (3.2)$$

$$V_{\text{tot}}(q, \omega) = \frac{e^2}{4\pi\epsilon q^2} + V_{\text{eff}}(q, \omega), \quad (3.3)$$

where the first term is the Coulomb-interaction. Furthermore,  $\omega$  is the energy transfer between scattering electrons when they exchange a phonon

$$\omega = \epsilon_{k+q} - \epsilon_k \quad (3.4)$$

**TODO: Sett inn figur**

Note the singularities in  $V_{\text{tot}}$  when  $|\omega| \rightarrow \omega_q$ . In particular, note the negative singularity when  $|\omega| \rightarrow \omega_q^-$ . This singularity persists when Coulomb-repulsion is included. For most frequencies, the Coulomb-interaction completely dominates. However, in a narrow  $\omega$ -region close to  $\omega_q$ , the extremely weak electron-phonon coupling will always beat the Coulomb-interaction! This frequency is slightly smaller than  $\omega_q$ . For small  $\omega$ ,  $V_{\text{tot}}$  is repulsive. For large  $\omega$ ,  $V_{\text{tot}}$  is repulsive. For  $|\omega| \lesssim \omega_q$ ,  $V_{\text{tot}}$  is attractive.

Let us try to give a physical picture for this: When an electron moves past an ion, they interact. The electron pulls slightly on the heavy, positively charged ion. Electrons are light, and move much faster than the heavy ions. The electron this moves quickly out of the scattering zone, while the ion relaxes slowly back to its equilibrium position. The ion in its out-of-equilibrium position represents excessive positive charge in that position, which can pull another electron towards it. This is effectively a charge-dipole interaction. If the second electron “waits” a little for the first electron to get away (thus reducing Coulomb-repulsion) but does not wait for too long (such that the ion has relaxed back to its equilibrium position), then the second electron can be attracted the scattering region. Effectively, the second electron is attracted to the scattering region because the first electron was there. This is an effective electron-electron attraction. It only works if the electron waits a little, but not for too long. A minimum time corresponds to a maximum frequency, while a maximum time corresponds to a minimum frequency. This implies that  $V_{\text{tot}}$  is attractive if  $\omega_{\text{min}} < \omega < \omega_{\text{max}}$ , as depicted in **TODO: Sett in figur og referanse til figuren (s.7)** We may view the effective electron-electron attraction as a result of an electron locally

deforming an elastic medium. Think of a rubber membrane that you put a little metal sphere on. The membrane is stretched, dipping down where you put the first sphere. If you put another little sphere on the membrane, it will fall into the dip, i.e. it will be attracted to the first particle. This is also how gravity works: A mass deforms space-time (an elastic medium) and thus attracts another mass.

Disclaimer: The above two analogs are classical. There will be an important quantum effect coming into play here, which we will come back to. here, it will suffice to not that, classically, one can keep adding particles to the dip, such that all particles will be gathered in the same one, forming a large heavy object. This is not how it works quantum mechanically with fermions. Note also that in  $V_{\text{tot}}$ , and the two different simplified models for  $\bar{V}$ , they are only attractive up to a maximum  $\omega$ , i.e. only after a minimum amount of time. The second particle has to wait a minimum amount of time for the interaction to be attractive. This is called retardation.

The electrons avoid the Coulomb-interactions by avoiding each other, not in space, but in time.

### 3.4 Magnon mediated electron-electron interaction

We have seen how a boson (a phonon) with a linear coupling to electrons could give an effective attractive interaction among electrons. What if we couple the electrons linearly to other bosons? One obvious thing to investigate, is to consider the coupling of electrons to magnons. For simplicity, we consider itinerant electrons coupled to spin-fluctuations in a ferromagnetic insulator. (FMI) The question is if the spin-fluctuations of the FMI can give rise to an attractive interaction among electrons. We therefore consider a system of itinerant electrons with Hamiltonian

$$\mathcal{H}_{\text{el}} = \sum_{k,\sigma} (\varepsilon_k - \mu) c_{k\sigma}^\dagger c_{k\sigma}. \quad (3.5)$$

In this system, we envisage a regular lattice of localized spins with ferromagnetic coupling, with Hamiltonian

$$\mathcal{H}_{\text{spin}} = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j. \quad (3.6)$$

The localized spins are denoted by capital letter  $\mathbf{S}$ . The coupling between the localized spins (FMI) and the itinerant electron spins  $\mathbf{s}_i$  (lower case) is given by

$$\mathcal{H}_{\text{el-spin}} = -J_{sd} \sum_i \mathbf{S}_i \cdot \mathbf{s}_i. \quad (3.7)$$

As a minimal model, we have assumed that the electrons are hopping around on the same regular lattice that the localized spins are located. Using the



Holstein-Primakoff transformation, ignoring the classical ground-state energy, and expressing operators in momentum space, we have

$$\mathcal{H}_{\text{spin}} = \sum_q \omega_q a_q^\dagger a_q \quad (3.8)$$

$$\omega_q = 2JS(z - \gamma(\mathbf{q})) \quad (3.9)$$

$$\gamma(\mathbf{q}) = \sum_\delta e^{i\mathbf{q} \cdot \delta}, \quad (3.10)$$

where  $\delta$  connects site  $i$  to all its nearest neighbors. One important fact to make note of at once, is that  $\omega_q \sim q^2$  for small  $q$ . For the phonon-case, with acoustical phonons,  $\omega_q \sim q$ . Thus  $\omega_q$  for small  $q$  is much smaller for ferromagnetic magnons than acoustical phonons. We will return to this point. Consider next the electron-spin coupling:

$$\mathcal{H}_{\text{el-spin}} = -J_{sd} \sum_i \mathbf{S}_i \cdot \mathbf{s}_i \quad (3.11)$$

$$= -J_{sd} \sum_i (S_{iz}s_{iz} + S_{ix}s_{ix} + S_{iy}s_{iy}) \quad (3.12)$$

$$= -J_{sd} \sum_i \left( S_{iz}s_{iz} + \frac{1}{2} (S_{i+}s_{i-} + S_{i-}s_{i+}) \right), \quad (3.13)$$

where  $S_{i\pm} = S_{ix} \pm iS_{iy}$ ,  $S_{iz} = S - a_i^\dagger a_i$ ,  $S_{i+} = \sqrt{2S}a_i$ ,  $S_{i-} = \sqrt{2S}a_i^\dagger$ .  $\mathbf{s}_i = \frac{1}{2}c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta}$  with implicit summation over repeated indices  $\alpha, \beta$ .

$$\Rightarrow s_{iz} = \frac{1}{2}(c_{i\uparrow}^\dagger c_{i\uparrow} - c_{i\downarrow}^\dagger c_{i\downarrow}) = \frac{1}{2} \sum_\sigma \sigma c_{i\sigma}^\dagger c_{i\sigma}$$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

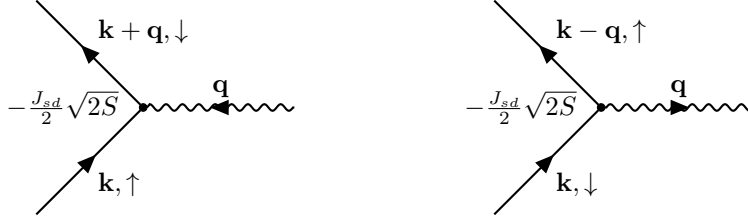
$$\sigma^\pm = \sigma^z \pm i\sigma^y \quad \sigma^+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

Thus, we have

$$\begin{aligned} \mathcal{H}_{\text{el-spin}} = & -J_{sd}S \sum_i i, \sigma \sigma c_{i\sigma}^\dagger c_{i\sigma} + J_{sd} \sum_{i,\sigma} \sigma a_i^\dagger a_i c_{i\sigma}^\dagger c_{i\sigma} \\ & - \frac{J_{sd}\sqrt{2S}}{2} \sum_i \left( a_i c_{i\downarrow}^\dagger c_{i\uparrow} + a_i^\dagger c_{i\uparrow}^\dagger c_{i\downarrow} \right) \end{aligned} \quad (3.14)$$

For the remainder of the calculation, we focus on the linear coupling of magnons to electrons, and ignore the second term. Thus, we focus on el-el interaction mediated by the vertices in fig. 3.

**TODO:** *Sett inn figurer*



(a) Spin-1 magnon is dumped into electron, flipping  $\downarrow \rightarrow \uparrow$  (b) Spin-1 magnon is excited, taking from it a spin-1, flipping  $\uparrow \rightarrow \downarrow$

Figure 3: The two interaction vertices of interest

## 4 The Cooper-problem

## 5 The Bardeen-Cooper-Scheiffer theory of superconductivity

This is essentially the many-particle version of the Cooper-problem. Superconductivity: **TODO: Sett inn figur**

Note that a non analytic function like this usually suggests that there is some phase-transition in the system, so we are essentially looking at a phase transition of the electron gas.  $T_C$ : A sharply defined temperature

$$\rho(T) = \begin{cases} 0 & \text{if } T < T_C \\ \text{nonzero} & \text{if } T > T_C \end{cases} \quad (5.1)$$

$T_C$  is denoted the critical temperature. Superconductivity was discovered experimentally in 1911 by Heike Kammerlingh Onnes in Leiden, measuring low- $T$   $\rho(T)$  in ultra pure Mercury (Hg). This was 15 years before the discovery of quantum mechanics. It turns out that the phenomena is purely a quantum effect. So in 1911, there was no hope of giving a correct explanation for what is happening. It took 46 years to figure out what is going on. The most important reasons for this, is that apart from having to invent quantum mechanics first, completely novel and radical ideas had to be formulated in order to solve the problem<sup>1</sup>. Historically, one important clue to figuring out what is happening, was the experimental observations that  $T_C$  varied with ion mass. (Isotope substitution on elemental superconductors). This indicated that lattice-vibrations somehow were involved in the early discovered superconductors. (Recall that electron-phonon-coupling  $\sim \frac{1}{\sqrt{M}}$ ). This “isotope-effect” was announced in 1950 on elemental Mercury, and the measured shift in  $T_C$  was 0.01K, something that

<sup>1</sup>From in the lecture: illustration of the Meissner effect. The Higgs providing a mass to the em-field in the metal blob drawn is the expectation value of the Cooper pair operator. Superconductivity is that the photon acquires a mass through the Higgs field, which is a cooper pair. Lots of analogs to the standard model.

required very careful and precise measurements. We may guess what will happen with  $T_C$  by appealing to what we found in the Cooper-problem, where we surmised a Cooper-pair dissociation temperature  $T^* \sim \Delta$  and

$$\Delta = 2\hbar\omega_0 e^{-\frac{1}{\lambda}} \quad (5.2)$$

$\omega_0$ : A typical phonon-frequency, if we assume that the effective attractions originates with e-ph coupling.  $\omega_0 \sim \frac{1}{\sqrt{M}} \rightarrow T^* \sim \frac{1}{\sqrt{M}}$ . This means that  $\sqrt{M}T_C = \text{constant!}$  This relation is validated very well in experiments on elemental superconductors such as Hg, Sn, and Tl. Previously, we have derived an effective e-e interaction, including Coulomb-interactions and e-ph-e interactions,  $\tilde{V}_{\text{eff}}$ .

$$\begin{aligned} \mathcal{H} = & \sum_{k,\sigma} (\varepsilon_k - \mu) c_{k\sigma}^\dagger c_{k\sigma} \\ & + \sum_{\substack{k,k',q \\ \sigma,\sigma'}} \tilde{V}_{\text{eff}} c_{k+q,\sigma}^\dagger c_{k'-q,\sigma'}^\dagger c_{k',\sigma'} c_{k,\sigma} \end{aligned} \quad (5.3)$$

Notice the global U(1)-symmetry of this Hamiltonian. This is on the standard form for a second-quantized electron-gas, now including the (potentially singular) effects of e-ph- coupling

$$\tilde{V}_{\text{eff}} = \frac{2|g_q|^2\omega_q}{\omega^2 - \omega_q^2} + V_{\text{Coulomb}}(q) \quad (5.4)$$

**TODO: Set in figure**

$\omega$  : Energy-transfer in scattering.  $\omega = \varepsilon_{k+q} - \varepsilon_k$ ,  $\varepsilon_{k'} = \varepsilon_{k'-q} + \omega$ . The effect of the repulsive interaction can be calculated perturbatively. In any case, this repulsion is not a singular perturbation. We therefore set it aside for the moment, and consider

$$\tilde{V}_{\text{eff}} = \frac{2|g_q|^2\omega_q}{\omega^2 - \omega_q^2}. \quad (5.5)$$

This interaction as attractive ( $< 0$ ) if

$$(\varepsilon_{k+q} - \varepsilon_k)^2 < \omega_q^2$$

or

$$(\varepsilon_{k'-q} - \varepsilon_{k'})^2 < \omega_q^2$$

We now focus on those scattering processes that give attraction between electrons. The processes giving repulsion do nothing more than what the Coulomb interaction does. We will include these effects later on. We now simplify this in a series of steps. The scattering caused by the weak e-ph-e coupling can only take place in a thin shell around the Fermi-surface. Thus  $\varepsilon_k, \varepsilon_{k'}, \varepsilon_{k+q}, \varepsilon_{k'-q}$  must all lie within a thin shell around the Fermi surface. Let us take a look at the relevant kinematics seen in fig. 4. We see that in general, the state with

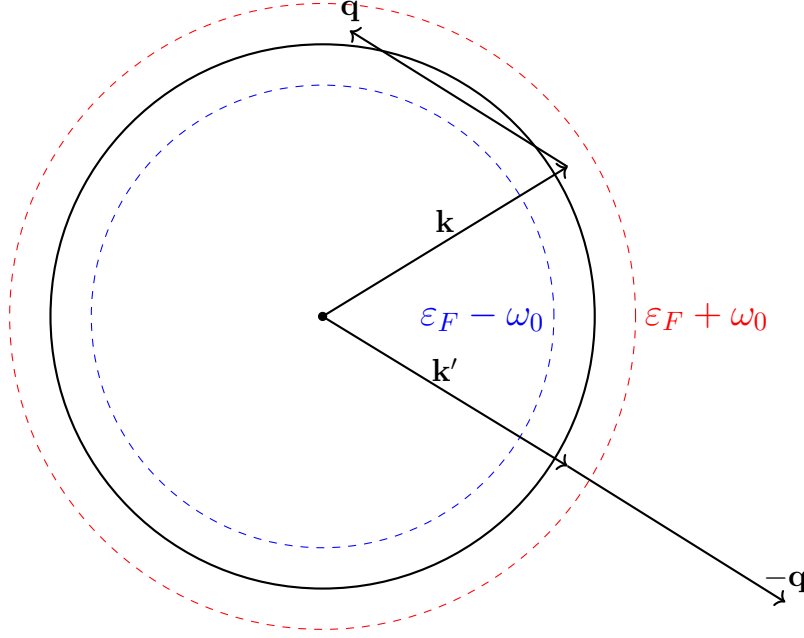


Figure 4: Thin shell around the Fermi surface.

momenta  $k' - q$  will lie outside the shell, even if  $\varepsilon_k, \varepsilon_{k'}, \varepsilon_{k+q}$  lie within the shell. There is an important special case where  $\varepsilon_{k'-q}$  will always lie within shell if  $\varepsilon_k, \varepsilon_{k'}, \varepsilon_{k+q}, \varepsilon_{k'-q}$  is within shell, namely the case when  $k' = -k$ . This choice will maximize the scattering phase-space for attractive interactions. We will retain only such terms:  $k' = -k$ .

A second simplification:  $\sigma' = -\sigma$ . The spatial extent of attractive interaction is small. We may essentially think of it (in real space) as an attractive Hubbard-interaction. Thus, we end up with the following simplified Hamiltonian

$$\mathcal{H} = \sum_{k,\sigma} (\varepsilon_k - \mu) c_{k\sigma}^\dagger c_{k\sigma} + \sum_{k,q,\sigma} \tilde{V}_{\text{eff}} c_{k+q,\sigma}^\dagger c_{-(k+q),-\sigma}^\dagger c_{-k,-\sigma} c_{k\sigma}. \quad (5.6)$$

Now redefine variables  $k \rightarrow k', \quad k + q \rightarrow k, \quad \tilde{V}_{\text{eff}} \rightarrow V_{k,k'}/2$  (spin independent interaction). Thus we can write eq. (5.6) as

$$\mathcal{H} = \sum_{k,\sigma} (\varepsilon_k - \mu) c_{k\sigma}^\dagger c_{k\sigma} + \sum_{k,k'} V_{k,k'} c_{k,\uparrow}^\dagger c_{-k,\downarrow}^\dagger c_{-k',\downarrow} c_{k,\uparrow}, \quad (5.7)$$

with  $V_{k,k'}$  being attractive if  $k, k'$  lie in a small vicinity of the Fermi-surface, and zero otherwise. eq. (5.7) is the so called BCS-model of superconductivity. Although it has been motivated by an attractive e-ph-e interaction, the above model is in fact more general than that, and can be applied to any system

with an effective (somehow) attractive electron-electron interaction. This model in spirit is very much like the model we looked at for the Cooper-problem. The difference is that  $V_{k,k'}$  in the BCS-model works between all electrons in a thin shell around the Fermi-surface, while the Cooper-problem only considered interactions between two such electrons. The Hamiltonian can not be treated exactly. Moreover, from the Cooper-problem, there is every reason to believe that in order to get correct eigenvalues, we cannot use perturbations theory. We must therefore treat  $\mathcal{H}$  both approximately and non-perturbatively. This is what we will do next. We will transform this many-body problem to a self-consistent one-particle problem. This is done very much like what we do when we perform a mean-field approximation on spin-systems:

$$\begin{aligned} c_{-k\downarrow}c_{k\uparrow} &= \underbrace{\langle c_{-k\downarrow}c_{k\uparrow} \rangle}_{\equiv b_k} + \underbrace{c_{-k\downarrow}c_{k\uparrow} - \langle c_{-k\downarrow}c_{k\uparrow} \rangle}_{\delta b_k} \\ &= b_k + \delta b_k. \end{aligned} \quad (5.8)$$

Here,  $b_k$  is a statistical average<sup>2</sup>. Note that giving the  $b$ 's a finite expectation value breaks the  $\mathrm{U}(1)$ -symmetry of the system. There is no way to gradually break this symmetry, it either happens or not. Note also that these expectations values are not on the usual  $\langle c^\dagger c \rangle$ -form, and the question now is to answer whether such expectation values can exist or not. Now insert the definitions in eq. (5.8) and the Hermitian conjugate into eq. (5.7) and ignore terms  $\mathcal{O}((\delta b)^2)$  Consider the interaction term:

$$\begin{aligned} \sum_{k,k'} V_{k,k'} c_{k,\uparrow}^\dagger c_{-k,\downarrow}^\dagger c_{-k',\downarrow} c_{k',\uparrow} &= \sum_{k,k'} V_{k,k'} (b_k^\dagger + \delta b_k^\dagger) (b_{k'} + \delta b_{k'}) \\ &= \sum_{k,k'} V_{k,k'} (b_k^\dagger b_{k'} + b_k^\dagger \delta b_{k'} + \delta b_k^\dagger b_{k'}) + \mathcal{O}((\delta b)^2) \\ &\simeq \sum_{k,k'} V_{k,k'} (b_k^\dagger b_{k'} + b_k^\dagger c_{-k'\downarrow} c_{k'\uparrow} + b_{k'} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger - 2b_k^\dagger b_{k'}). \end{aligned}$$

Next, define

$$\Delta_k \equiv - \sum_{k'} V_{kk'} b_{k'} \quad (5.9a)$$

$$\Delta_k^\dagger \equiv - \sum_k V_{kk'} b_k^\dagger. \quad (5.9b)$$

Inserting these in definitions into the Hamiltonian gives

$$\begin{aligned} \mathcal{H} &= \sum_{k,\sigma} (\varepsilon_k - \mu) c_{k\sigma}^\dagger c_{k\sigma} - \sum_k \left[ \Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \Delta_k^\dagger c_{-k\downarrow} c_{k\uparrow} \right] \\ &\quad + \sum_k \Delta_k b_k^\dagger. \end{aligned} \quad (5.10)$$

<sup>2</sup>with respect to the “correct” Hamiltonian.

This is the mean-field approximation to the BCS-model, where  $b_k$  (and hence  $\Delta_k$ ) must be determined self-consistently, by minimizing the free energy of the system. We return to that below, but first we must diagonalize the Hamiltonian in eq. (5.10). Note that we now have terms like  $c^\dagger c$ ,  $c^\dagger c^\dagger$ , and  $cc$  in  $\mathcal{H}$ , reminiscent of what we had with boson-operators for the case of quantum antiferromagnets. We will now proceed along a similar route, but with the important difference that we are now considering fermions. Introduce new fermion-operators

$$\eta_k = u_k c_{k\uparrow} + v_k c_{-k\downarrow}^\dagger \quad (5.11a)$$

$$\gamma_k = u_k c_{-k\downarrow} - v_k c_{k\uparrow} \quad (5.11b)$$

These operators are fermionic quasi-particles as linear combinations of spin-up and spin-down particles. Thus spin is not a correct quantum number for the new fermions. Note the minus sign in eq. (5.11b). This transformation is required to preserve fermionic commutation relations, for instance

$$\{\eta_k, \eta_{k'}^\dagger\} = \delta_{kk'}, \quad (5.12)$$

$\eta_k, \gamma_k$  anticommute,

$$\{\eta_k, \gamma_{k'}^\dagger\} = 0$$

$$u_k u_{k'} \delta_{kk'} + v_k v_{k'} \delta_{kk'} = \delta_{kk'},$$

with the  $+$ -sign originating in anti-commutation relations. Thus  $u_k^2 + v_k^2 = 1$ . We reach the same conclusion with  $\{\gamma_k, \gamma_{k'}\} = \delta_{kk'}$ . This relation is the reason for the minus sign in front of  $v_k$  in eq. (5.11b)!

$$\begin{pmatrix} \eta_k \\ \gamma_k \end{pmatrix} = \overbrace{\begin{pmatrix} u_k & v_k \\ -v_k & u_k \end{pmatrix}}^{\equiv M} \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix} \quad (5.13)$$

With these signs,  $\det M = u_k^2 + v_k^2 = 1$

$$M = \begin{pmatrix} u_k & v_k \\ -v_k & u_k \end{pmatrix} \quad M^T = \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix} \quad M^{-1} = \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix} \quad (5.14)$$

Thus,  $M$  is a unitary transformation with the constraint  $u_k^2 + v_k^2 = 1 \implies |u_k|, |v_k| \leq 1$ . This is very different from the “squeezing” transformation we

used in the quantum AFM-case. Going back to eq. (5.11), we have

$$\begin{pmatrix} \eta_k \\ \gamma_k \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ -v_k & u_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix} \quad (5.15a)$$

$$\begin{pmatrix} \eta_k^\dagger \\ \gamma_k^\dagger \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ -v_k & u_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow}^\dagger \\ c_{-k\downarrow} \end{pmatrix} \quad (5.15b)$$

$$\begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} \eta_k \\ \gamma_k \end{pmatrix} \quad (5.15c)$$

$$\begin{pmatrix} c_{k\uparrow}^\dagger \\ c_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} \eta_k^\dagger \\ \gamma_k^\dagger \end{pmatrix} \quad (5.15d)$$

Insert this into the Hamiltonian eq. (5.10),

$$\begin{aligned} \mathcal{H} = & \sum_k \left\{ (\varepsilon_k - \mu) (u_k \eta_k^\dagger - v_k \gamma_k^\dagger) (u_k \eta_k - v_k \gamma_k) \right. \\ & + (\varepsilon_k - \mu) (v_k \eta_k + u_k \gamma_k) (v_k \gamma_k^\dagger + u_k \gamma_k^\dagger) \\ & - \Delta_k (u_k \eta_k^\dagger - v_k \gamma_k^\dagger) (v_k \eta_k + u_k \gamma_k) \\ & \left. - \Delta_k^\dagger (u_k \eta_k^\dagger + u_k \gamma_k^\dagger) (u_k \eta_k - v_k \gamma_k) + \Delta_k b_k^\dagger \right\} \end{aligned} \quad (5.16)$$

As in the quantum antiferromagnet-case, we now collect terms of different types:

$$\eta_k^\dagger \eta_k : (\varepsilon_k - \mu) u_k^2 - u_k v_k (\Delta_k + \Delta_k^\dagger) \quad (5.17a)$$

$$\gamma_k^\dagger \gamma_k : (\varepsilon_k - \mu) v_k^2 + u_k v_k (\Delta_k + \Delta_k^\dagger) \quad (5.17b)$$

$$\eta_k \eta_k^\dagger : (\varepsilon_k - \mu) v_k^2 \quad (5.17c)$$

$$\gamma_k \gamma_k^\dagger : (\varepsilon_k - \mu) u_k^2 \quad (5.17d)$$

$$\gamma_k^\dagger \eta_k : -2(\varepsilon_k - \mu) u_k v_k + \Delta_k v_k^2 - \Delta_k^\dagger u_k^2 \quad (5.17e)$$

$$\eta_k^\dagger \gamma_k : -2(\varepsilon_k - \mu) u_k v_k - \Delta_k u_k^2 + v_k^2 \Delta_k^\dagger \quad (5.17f)$$

Using the anticommutation relations in eq. (5.12), and the corresponding for  $\gamma_k$ , we may express eqs. (5.17a) and (5.17b) as

$$\eta_k^\dagger \eta_k : (\varepsilon_k - \mu) (u_k^2 - v_k^2) - u_k v_k (\Delta_k + \Delta_k^\dagger) \quad (5.18a)$$

$$\gamma_k^\dagger \gamma_k : (\varepsilon_k - \mu) (v_k^2 - u_k^2) + u_k v_k (\Delta_k + \Delta_k^\dagger) \quad (5.18b)$$

These are the same, except opposite sign. Adjust  $u_k, v_k$  such that the coefficients in front of eq. (5.17e) and ?? are zero. Fortunately, these two equations are just

complex conjugate of each other, so if one is fulfilled, so is the other. If we set these two to 0, we have

$$-2(\varepsilon_k - \mu) u_k v_k = u_k^2 \Delta_k - v_k^2 \Delta_k^\dagger \quad (5.19)$$

$$-2(\varepsilon_k - \mu) u_k v_k = u_k^2 \Delta_k^\dagger - v_k^2 \Delta_k. \quad (5.20)$$

By adding these two equations, we get

$$\begin{aligned} -4 \underbrace{(\varepsilon_k - \mu)}_{\equiv \tilde{\varepsilon}_k} u_k v_k &= (u_k^2 - v_k^2) (\Delta_k + \Delta_k^\dagger) \\ \Delta_k + \Delta_k^\dagger &= 2 \operatorname{Re}\{\Delta_k\} \equiv 2\tilde{\Delta}_k \\ -2\tilde{\varepsilon}_k u_k v_k &= (u_k^2 - v_k^2) \tilde{\Delta}_k \end{aligned}$$

Since we have  $u_k^2 + v_k^2 = 1$ , we may write

$$\begin{aligned} u_k &= \cos \theta \\ v_k &= \sin \theta \\ -\tilde{\varepsilon}_k \sin 2\theta &= \tilde{\Delta}_k \cos 2\theta \\ \tan 2\theta &= -\frac{\tilde{\Delta}_k}{\tilde{\varepsilon}_k} \end{aligned} \quad (5.21)$$

This is an equation for  $\theta$ , and thus  $u_k, v_k$ , which gives coefficients of  $\gamma_k^\dagger \eta_k, \eta_k^\dagger \gamma_k$  equal to zero. Choose  $\tilde{\Delta}_k \geq 0$

$$\begin{aligned} \tan 2\theta &< 0; \quad \tilde{\varepsilon} > 0 \\ \tan 2\theta &> 0; \quad \tilde{\varepsilon} < 0 \\ \frac{\sin^2(2\theta)}{\cos^2(2\theta)} &= \left( \frac{\tilde{\Delta}_k}{\tilde{\varepsilon}_k} \right)^2 \equiv b^2 \\ \cos^2(2\theta) &= \frac{1}{1+b^2} \\ \cos(2\theta) &= \begin{cases} \frac{-1}{\sqrt{1+b^2}}; & \tilde{\varepsilon} > 0 \\ \frac{1}{\sqrt{1+b^2}}; & \tilde{\varepsilon} < 0 \end{cases} \end{aligned}$$



Coefficient in front of  $\eta_k^\dagger \eta_k$ :

$$\begin{aligned}
\tilde{\varepsilon}_k \cos 2\theta - \tilde{\Delta}_k \sin 2\theta &= \cos(2\theta) \left( \tilde{\varepsilon}_k + \left( \frac{\tilde{\Delta}_k}{\tilde{\varepsilon}_k} \right)^2 \right) \\
&= -\frac{\text{sign} \tilde{\varepsilon}_k}{\tilde{\varepsilon}_k} \frac{(\tilde{\varepsilon}_k^2 + \tilde{\Delta}_k^2)}{(1+b^2)^{\frac{1}{2}}} \\
&= -\frac{\text{sign} \tilde{\varepsilon}_k}{\frac{\tilde{\varepsilon}_k}{|\tilde{\varepsilon}_k|}} \left( \tilde{\varepsilon}_k^2 + \tilde{\Delta}_k^2 \right)^{\frac{1}{2}} \\
&= -\left( \tilde{\varepsilon}_k^2 + \tilde{\Delta}_k^2 \right)^{\frac{1}{2}}
\end{aligned}$$

Coefficient in front of  $\gamma_k^\dagger \gamma_k$ :  $\left( \tilde{\varepsilon}_k^2 + \tilde{\Delta}_k^2 \right)^{\frac{1}{2}}$ . Thus, we have finally diagonalized the Hamiltonian

$$\mathcal{H} = \sum_k \left[ 2(\varepsilon_k - \mu) + \Delta_k b_k^\dagger + E_k \left( \gamma_k^\dagger \gamma_k - \eta_k^\dagger \eta_k \right) \right], \quad (5.22)$$

where the summation over spins has been made, and with

$$E_k \equiv \left( \tilde{\varepsilon}_k^2 + \tilde{\Delta}_k^2 \right)^{\frac{1}{2}}. \quad (5.23)$$

$b_k$  and  $\tilde{\Delta}_k$  are as yet undetermined. They will have to be determined by minimizing the free energy of this system. The long-lived fermionic excitation are described by  $(\eta_k, \eta_k^\dagger)$ ,  $(\gamma_k, \gamma_k^\dagger)$ . **TODO: Sjekkk at fortegnene er riktige på eta/gamma**

$$\mathcal{H} = E_0 + \sum_k E_k \left( \gamma_k^\dagger \gamma_k - \eta_k^\dagger \eta_k \right) \quad (5.24)$$

$$E_0 = \sum_k \left[ 2(\varepsilon_k - \mu) + \Delta_k b_k^\dagger \right] \quad (5.25)$$

If we have a fermionic system with a Hamiltonian

$$\mathcal{H} = \sum_k (\varepsilon_k - \mu) c_k^\dagger c_k,$$

the grand canonical partition function is given by

$$\mathcal{Z}_g = \prod_k \left( 1 + e^{-\beta(\varepsilon_k - \mu)} \right). \quad (5.26)$$

In the present case, this gives

$$\mathcal{Z}_g = e^{-\beta E_0} \prod_k \left( 1 + e^{\beta E_k} \right) \left( 1 + e^{-\beta E_k} \right). \quad (5.27)$$

In the limit of large number of particles, all ensembles are equivalent. To expedite the computations, we will consider  $\mathcal{Z}_g$  to be equal to  $\mathcal{Z} = e^{-\beta F}$ , where  $F$  is the Helmholtz free energy for a system.

$$F = E_0 - \frac{1}{\beta} \sum_k [\ln(1 + e^{-\beta E_k}) + \ln(1 + e^{\beta E_k})] \quad (5.28)$$

We now minimize with respect to  $\Delta_k$  or  $b_k^\dagger$  for a particular value of  $k$ . It does not matter which one of these we use. We look for

$$\frac{\partial F}{\partial \Delta_k} = 0, \quad (5.29)$$

or

$$b_k^\dagger - \frac{1}{\beta} \left( \frac{1}{1 + e^{-\beta E_k}} (-\beta) \frac{\partial E_k}{\partial \Delta_k} e^{-\beta E_k} + \frac{1}{1 + e^{\beta E_k}} (\beta) \frac{\partial E_k}{\partial \Delta_k} e^{\beta E_k} \right) = 0$$

$$\begin{aligned} b_k^\dagger &= \frac{\partial E_k}{\partial \Delta_k} \left( \frac{e^x}{1 + e^x} - \frac{e^{-x}}{1 + e^{-x}} \right) \\ &= \frac{\partial E_k}{\partial \Delta_k} \cdot \tanh\left(\frac{x}{2}\right); \quad x = \beta E_k \\ \frac{\partial E_k}{\partial \Delta_k} &= \frac{\Delta_k}{\sqrt{\tilde{\varepsilon}_k^2 + \Delta_k^2}}, \end{aligned}$$

where we have set  $\Delta_k$  to be real, such that  $\tilde{\Delta}_k = \Delta_k$ .

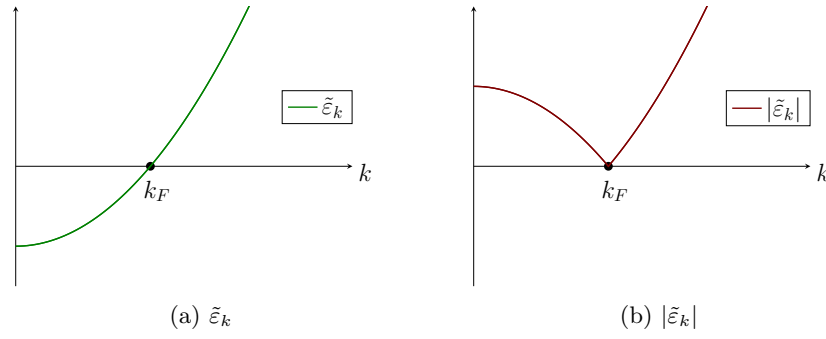
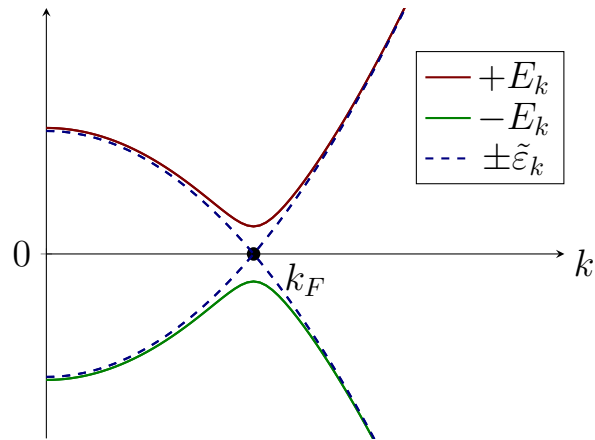
$$b_k^\dagger = \frac{\Delta_k}{\sqrt{\tilde{\varepsilon}_k^2 + \Delta_k^2}} \tanh\left(\frac{\beta E_k}{2}\right) \quad (5.30)$$

We close this to an equation for  $\Delta_k$  by using the definitions in eq. (5.9) to obtain

$$\Delta_k = - \sum_{k'} V_{kk'} \Delta_{k'} \chi_{k'} \quad (5.31)$$

$$\chi_k = \frac{1}{(\tilde{\varepsilon}_k^2 + \Delta_k^2)^{\frac{1}{2}}} \tanh\left(\frac{\beta E_k}{2}\right) \quad (5.32)$$

This is the so-called BCS gap-equation. The reason for this name is that it is an equation for  $\Delta_k$ , which represents a gap in the excitation-spectrum. To see this, we first consider the excitation spectrum at  $\Delta_k = 0$ , depicted in fig. 5. Note that on the Fermi-surface ( $k = k_F$ ), there is zero gap in the excitation spectrum  $\Delta_k = 0$ . From fig. 6, we see that  $\Delta_k$  represents a gap on the Fermi-surface in the excitation spectrum of the Bogoliubov fermions. We will now solve eq. (5.31) for the same model for  $V_{kk'}$  that we used in the Cooper problem, namely a


 Figure 5: Typical excitation spectrum at  $\Delta_k = 0$ .

 Figure 6:  $\Delta_k \neq 0$  represents a gap in the excitation spectrum at the Fermi surface, here greatly exaggerated. The difference in the minimum of  $+E_k$  and the maximum of  $-E_k$  is  $2\Delta_k$ . This gap always tracks the Fermi surface.

constant attractive potential in a thin shell around the Fermi-surface. With the understanding that  $k, k'$  lie within this thin shell, we have

$$\Delta_k = V \sum_{k'} \Delta_{k'} \chi_{k'} \quad (5.33)$$

This means that  $\Delta_k$  is independent of  $k$ , and we can divide by  $\Delta = \Delta_k$  and get

$$1 = V \sum_{k'} \frac{1}{(\tilde{\varepsilon}_{k'}^2 + \Delta^2)^{\frac{1}{2}}} \tanh\left(\frac{\beta E_{k'}}{2}\right). \quad (5.34)$$

Now use

$$\sum_k g(\tilde{\varepsilon}_k) = \int_{-\infty}^{\infty} d\varepsilon \sum_k \delta(\varepsilon - \tilde{\varepsilon}_k) g(\varepsilon) = \int_{-\infty}^{\infty} d\varepsilon N(\varepsilon) g(\varepsilon), \quad (5.35)$$

$$1 = V \int_{-\omega_0}^{\omega_0} d\varepsilon \frac{N(\varepsilon)}{\sqrt{\varepsilon^2 + \Delta^2}} \tanh\left(\frac{\beta \sqrt{\varepsilon^2 + \Delta^2}}{2}\right) \quad (5.36)$$

Note:  $\varepsilon$  must lie within a thin energy-shell around the Fermi surface! We have denoted the width of this shell  $2\omega_0$

$$-\omega_0 < \varepsilon < \omega_0$$

when energies  $\varepsilon$  are measured relative  $\mu$ . Consider now  $N(\varepsilon)$  a rather slowly varying function of  $\varepsilon$  in the thin shell around the Fermi-surface, such that  $N(\varepsilon) \simeq N(\varepsilon_F) = N(\mu)$ . Introduce  $\lambda \equiv VN(\varepsilon_F)$ , such that

$$1 = \lambda \int_{-\omega_0}^{\omega_0} d\varepsilon \frac{1}{\sqrt{\varepsilon^2 + \Delta^2}} \tanh\left(\frac{\beta \sqrt{\varepsilon^2 + \Delta^2}}{2}\right). \quad (5.37)$$

eq. (5.37) is difficult to solve, but we will look at two special cases, at  $T = 0$  and  $T = T_C$ .

i)  $T = 0$

At  $T = 0$ ,  $\tanh\left(\frac{\beta \sqrt{\varepsilon^2 + \Delta^2}}{2}\right) = 1$ , such that <sup>3</sup>

$$1 = \lambda \int_0^{\omega_0} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}} \quad (5.38)$$

$$= \lambda \int_0^{\frac{\omega_0}{\Delta}} \frac{dx}{\sqrt{x^2 + 1}} \quad (5.39)$$

$$= \lambda \sinh^{-1}\left(\frac{\omega_0}{\Delta}\right) \quad (5.40)$$

$$= \lambda \ln\left(\frac{\omega_0}{\Delta} + \sqrt{\left(\frac{\omega_0}{\Delta}\right)^2 + 1}\right) \quad (5.41)$$

$$\Rightarrow \frac{\omega_0}{\Delta} = \sinh\left(\frac{1}{\lambda}\right) = \frac{1}{2} \left(e^{\frac{1}{\lambda}} - e^{-\frac{1}{\lambda}}\right) \quad (5.42)$$

---

<sup>3</sup>The subsequent  $\lambda$  is now redefined by a factor 2, not any more considering per-spin density of states.

Now consider  $\lambda \ll 1 \implies \frac{2\omega_0}{\Delta} \simeq e^{\frac{1}{\lambda}}$ . Reinserting  $\hbar$  we obtain

$$\Delta = 2\hbar\omega_0 e^{\frac{-1}{\lambda}}. \quad (5.43)$$

Notice the similar expression in eq. (5.43) **TODO: Sett inn uttrykk for  $\Delta$  fra Cooper-problemet.**

## ii) The critical temperature

We see from the gap-equation that as  $T$  increases, the tanh-factor decreases. Thus, to obtain a solution to the gap-equation requires smaller  $\Delta$ . As  $T$  increases,  $\Delta$  decreases. Sooner or later, we will reach a temperature where  $\Delta \rightarrow 0^+$ . This will be the critical temperature,  $T_C$ , which is determined by the equation

$$1 = \lambda \int_0^{\omega_0} d\varepsilon \frac{\tanh\left(\frac{\beta\varepsilon}{2}\right)}{\varepsilon}. \quad (5.44)$$

Introducing  $x = \frac{\beta\varepsilon}{2}$ ,

$$\begin{aligned} \frac{1}{\lambda} &= \int_0^{\frac{\beta\omega_0}{2}} \frac{dx}{x} \tanh(x) \\ &= \ln(x) \tanh(x) \Big|_0^{\frac{\beta\omega_0}{2}} - \int_0^{\frac{\beta\omega_0}{2}} dx \frac{\ln(x)}{\cosh^2(x)}. \end{aligned}$$

Again, we consider  $\lambda \ll 1 \implies \frac{\beta\omega_0}{2} \gg 1$ . In the last term, we may replace the upper limit by  $\infty$ .

$$\frac{1}{\lambda} \simeq \ln\left(\frac{\beta\omega_0}{2}\right) - \underbrace{\int_0^\infty dx \frac{\ln(x)}{\cosh^2(x)}}_{=\ln\left(\frac{\pi}{4}e^{-\gamma}\right) \equiv \ln(C)}, \quad (5.45)$$

where  $\gamma \simeq 0.5772156649$  is the Euler-Mascheroni constant.

$$\frac{1}{\lambda} = \ln\left(\frac{\beta\omega_0}{2C}\right) \quad (5.46)$$

$$\frac{\beta\omega_0}{2C} = e^{\frac{1}{\lambda}} \quad (5.47)$$

$$k_B T_C = \frac{2}{\pi} e^{\gamma} \omega_0 e^{\frac{-1}{\lambda}} \quad (5.48)$$

Note how extremely sensitive both  $\Delta$  and  $k_B T_C$  is to  $\lambda$ ! Note also that the depend on  $\lambda$  in the same way. So if we take their ratios, this dependence cancels, as does the prefactor.

$$\frac{2\Delta(T=0)}{k_B T_C} = \frac{4\omega_0 e^{\frac{-1}{\lambda}}}{\frac{2}{\pi} e^{\gamma} \omega_0 e^{\frac{-1}{\lambda}}} \quad (5.49)$$

$$= 2\pi e^{-\gamma} \simeq 3.53, \quad (5.50)$$

which is a universal number! This prediction of the theory turned out to be remarkably accurate for real superconductors where  $\lambda \ll 1$ . Examples are Hg, Al, Sn. For Pb, the ratio is somewhat larger, and this is attributed to the fact that  $\lambda$  is larger than in Hg, Al, Sn. The opening up of a gap on the Fermi-surface means that the electron many-body state is protected from scattering by this gap. As a result, the resistivity of the electron gas drops abruptly to 0 at  $T = T_C$ .

Superconductivity-onset is thus a phase-transition associated with the opening of a gap  $\Delta_k$  on the Fermi-surface.

Often, this gap  $\Delta_k$  is referred to as an order-parameter of the system. The original Hamiltonian is invariant under the transformation

$$c_{k\sigma} \rightarrow e^{i\theta} c_{k\sigma}, \quad (5.51)$$

where  $\theta$  is a global phase factor. This is therefore a symmetry of the problem, a global  $U(1)$ -symmetry as mentioned above.  $\Delta_k$  is nonzero if and only if  $b_k$  is nonzero. The quantity

$$\langle b_k \rangle = \langle c_{-k\downarrow} c_{k\uparrow} \rangle \quad (5.52)$$

is, however, not invariant under this global  $U(1)$ -transformation. Thus, in the superconducting state, the  $U(1)$ -symmetry of  $\mathcal{H}$  is spontaneously broken.

## 6 The Meissner Effect