Quantum Theory of Solids
Lecture notes - Spring 2020

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Digitialized lecture notes for the course "TFY4210 - Quantum Theory of Many-Particle Systems" held by Prof. Asle Sudbø spring 2020. These notes follow the pdf containing the hand written lecture notes, which are based upon the lecture notes for the course "FY8302 - Quantum Theory of Solids"

Website: https://www.ntnu.edu/studies/courses/TFY4210

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3.1 Fermi-liquids

3.2 Screening of the Coulomb-interaction

3.3 Phonon mediated electron-electron interaction

Due to the electron-phonon coupling depicted in **TODO:** *SETT INN DIAGRAM*, we will get an effective phonon-mediated interaction between electrons.

This is an exchange of a virtual phonon. The above diagram is the effective interaction to second order in g_q if we regard the wavy line as a bare phonon Green's function. We could also imagine that we replaced this line by **TODO:** *SETT INN DIAGRAM* which would include an effective interaction computed correctly up to order $\mathcal{O}(g^4)$. In fact, we might replace $D_0 + D_0 \Pi D_0 + \dots$ by D! Thus computing the effective interaction up to infinite order in g. Another, often used approach would be to replace by **TODO:** *SETT INN DIAGRAM*

Here, we have resummed a <u>subset</u> of diagrams to infinite order in g in order to get an effective interaction between electrons. Under the assumption that g is weak, we will keep terms only to $\mathcal{O}(g^2)$.

$$V_{\text{eff}}(q,\omega) = |g_q^2| \frac{2\omega_q}{\omega^2 - \omega_q^2} \tag{3.1}$$

Thus, the interaction part of the Hamiltonian becomes

$$\mathcal{H} = \sum_{\substack{k,k',q \\ \sigma,\sigma'}} V_{\text{tot}}(q,\omega) c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k',\sigma'} c_{k,\sigma} \tag{3.2}$$

$$V_{\text{tot}}(q,\omega) = \frac{e^2}{4\pi\varepsilon q^2} + V_{\text{eff}}(q,\omega), \tag{3.3}$$

where the first term is the Coulomb-interaction. Furthermore, ω is the energy transfer between scattering electrons when they exchange a phonon

$$\omega = \varepsilon_{k+q} - \varepsilon_k \tag{3.4}$$

TODO: *Sett inn figur*

Note the singularities in V_{tot} when $|\omega| \to \omega_q$. In particular, note the <u>negative</u> singularity when $|\omega| \to \omega_q^-$. This singularity persists when Coulomb-repulsion is included. For most frequencies, the Coulomb-interaction completely dominates. However, in a narrow ω-region close to ω_q, the extremely weak electron-phonon coupling will always beat the Coulomb-interaction! This frequency is slightly smaller than ω_q. For small ω, V_{tot} is repulsive. For large ω, V_{tot} is repulsive. For $|\omega| \lesssim \omega_q$, V_{tot} is <u>attractive</u>.

Let us try to give a physical picture for this: When an electron moves past an ion, they interact. The electron pulls slightly on the heavy, positively charged ion. Electrons are light, and move much faster than the heavy ions. The electron this moves quickly out of the scattering zone, while the ion relaxes slowly back to its equilibrium position. The ion in its out-of-equilibrium position represents excessive positive charge in that position, which can pull another electrn towards it. This is effectively a charge-dipole interaction. If the second electron "waits" a little for the first electron to get away (thus reducing Coulomb-repulsion) but does not wait for too long (such that the ion has relaxed back to its equilibrium position), then the second electron can be attracted the scattering region. Effectively, the second electron is attracted to the scattering region because the first electron was there. This is an effective electron-electron attraction. It only works if the electron waits a little, but not for too long. A minimum time corresponds to a maximum frequency, while a maximum time corresponds to a minimum frequency. This implies that V_{tot} is attractive if $\omega_{\text{min}} < \omega < \omega_{\text{masx}}$, as depicted in **TODO:** *Sett in figurer og referanse til figuren (s.7)* We may view the effective electron-electron attrraction as a result of an electron locally deforming an elastic medium. Think of a rubber membrane that you put a little metal sphere on. The membrane is stretched, dipping down where you put the first sphere. If you put another little sphere on the membrane, it will fall into the dip, i.e. it will be attracted to the first particle. This is also how gravity works: A mass deforms space-time (an elastic medium) and thus attracts another mass.

Disclaimer: The above two analogs are classical. There will be an important quantum effect coming into play here, which we will come back to. here, it will suffice to not that, classically, one can keep adding particles to the dip, such that all particles will be gathered in the same one, forming a large heavy object. This is not how it works quantum mechanically with fermions. Note also that in V_{tot}, and the two different simplified models for $\bar{V}$, they are only attractive up to a maximum ω, i.e. only after a minimum amount of time. The second particle has to wait a minimum amount of time for the interaction to be attractive. This is called retardation.

> The electrons avoid the Coulomb-interactions by avoiding each other, not in space, but in time.

3.4 Magnon mediated electron-electron interaction

We have seen how a boson (a phonon) with a linear coupling to electrons could give an effective attractive interaction aming electrons. What is we couple the electrons linearly to other bosons? One obvious thing to investigate, is to consider the coupling of electrons to magnons. For simplicity, we consider itinerant electrons coupled to spin-fluctuations in a ferromagnetic insulator. (FMI) The question is if the spin-fluctuations of the FMI can give rise to an attractive interaction aming electrons. We therefore consider a system of itinerant electrons with Hamiltonian

$$\mathcal{H}_{\text{el}} = \sum_{k,\sigma} (\varepsilon_k - \mu) c^\dagger_{k\sigma} c_{k\sigma}. \tag{3.5}$$

In this system, we envisage a regular lattice of localized spins with ferromagnetic coupling, with Hamiltonian

$$\mathcal{H}_{\text{spin}} = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j. \tag{3.6}$$

The localized spins are denoted by capital letter $\mathbf{S}$. The coupling between the localized spins (FMI) and the itinerant electron spins $\mathbf{s}_i$ (lower case) is given by

$$\mathcal{H}_{\text{el-spin}} = -J_{sd} \sum_i \mathbf{S}_i \cdot \mathbf{s}_i. \tag{3.7}$$

As a minimal model, we have assumed that the electrons are hopping around on the same regular lattice that the localized spins are located. Using the Holstein-Primakoff transformation, ignoring the classical ground-state energy, and expressing operators in momentum space, we have

$$\begin{aligned}
\mathcal{H}_{\text{spin}} &= \sum_q \omega_q a^\dagger_q a_q &(3.8)\\
\omega_q &= 2JS(z - \gamma(\mathbf{q}))) &(3.9)\\
\gamma(\mathbf{q}) &= \sum_\delta \mathrm{e}^{i\mathbf{q}\cdot\delta}, &(3.10)
\end{aligned}$$

where δ connects site i to all its nearest neighbors. One important fact ato make note of at once, is that $\omega_q \sim q^2$ for small q. For the phonon-case, with acoustical phonons, $\omega_q \sim q$. Thus ω_q for small q is much smaller for ferromagnetic magnons than acoustical phonons. We will return to this point. Consider next the electron-spin coupling:

$$\begin{aligned}
\mathcal{H}_{\text{el-spin}} &= -J_{sd} \sum_i \mathbf{S}_i \cdot \mathbf{s}_i &(3.11)\\
&= -J_{sd} \sum_i (S_{iz}s_{iz} + S_{ix}s_{ix} + S_{iy}s_{iy}) &(3.12)\\
&= -J_{sd} \sum_i \left(S_{iz}s_{iz} + \frac{1}{2}\left(S_{i+}s_{i-} + S_{i-}s_{i+}\right) \right), &(3.13)
\end{aligned}$$

where $S_{i\pm} = S_{ix} \pm iS_{iy}$, $S_{iz} = S - a^\dagger_i a_i$, $S_{i+} = \sqrt{2S} a_i$, $S_{i-} = \sqrt{2S} a^\dagger_i$. $\mathbf{s}_i = \frac{1}{2} c^\dagger_{i\alpha} \vec{\sigma}_{\alpha\beta} c_{i\beta}$ with implicit summation over repeated indices α, β.

$$\implies s_{iz} = \frac{1}{2}(c^\dagger_{i\uparrow} c_{i\uparrow} - c^\dagger_{i\downarrow} c_{i\downarrow}) = \frac{1}{2} \sum_\sigma \sigma c^\dagger_{i\sigma} c_{i\sigma}$$

$$\sigma^x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \qquad \sigma^y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right) \qquad \sigma^z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

$$\sigma^{\pm} = \sigma^z \pm i\sigma^y \qquad \sigma^+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

Thus, we have

$$\begin{aligned} \mathcal{H}_{\text{el-spin}} &= -J_{sd}S \sum i, \sigma\sigma c^\dagger_{i\sigma} c_{i\sigma} + J_{sd} \sum_{i,\sigma} \sigma a^\dagger_i a_i c^\dagger_{i\sigma} c_{i\sigma} \\ &- \frac{J_{sd}\sqrt{2S}}{2} \sum_i \left(a_i c^\dagger_{i\downarrow} c_{i\uparrow} + a^\dagger_i c^\dagger_{i\uparrow} c_{i\downarrow} \right) \end{aligned} \tag{3.14}$$

For the remainder of the calculation, we focus on the linear coupling of magnons to electrons, and ignore the second term. Thus, we focus on el-el interaction mediated by the vertices **TODO:** *Sett inn figurer*

4 The Cooper-problem