Summary of Stochastic Process

2024 Fall

This is my summary of the course SEEM 5580, taught by professor Xuefeng Gao. The reference book is *Stochastic processes, Sheldon M. Ross (1995)*, and the problem solutions can be found at http://www.charmpeach.com/?s=stochastic.

- Basic Probability Theory:

Distribution	PDF	MGF	Mean	Variance
Poisson	$e^{-\lambda}\frac{\lambda^x}{x!}$	$\exp(\lambda(e^t-1))$	λ	λ
Geometric	pq^{n-1}	-	$1/p$	q/p
Exponential	$\lambda e^{-\lambda x}$	$\frac{\lambda}{\lambda-t}$	$1/\lambda$	$1/\lambda^2$
Normal	$\frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(x-\mu)^2}{2\sigma})$	$\exp(\mu t+\sigma^2t^2/2)$	μ	σ^2

  where the moment generating function is defined as $\psi(t)=E\left[e^{tX}\right]=\int e^{tX}dF(x)$ Probability Inequalities-Chernoff Bounds:

  $$P(X\geq a)\leq e^{-ta}M(t),\forall t>0;\ P(X\leq a)\leq e^{-ta}M(t),\forall t<0$$

- Poisson Process:
  - $N(0)=0$;
  - $P\{N(t+s)-N(s)=n\}=e^{-\lambda t}\frac{(\lambda t)^n}{n!},\quad n=0,1,\ldots;$
  - independent increment;

  $P\left\{X_1>t\right\}=P\{N(t)=0\}=e^{-\lambda t}$

  $S_n=\sum_{i=1}^n X_t,\quad n\geq 1$

  order statistics of uniform distribution: the conditional density of $S_1,\ldots,S_n$ given $N(t)=n$ is $f(t_1,\ldots,t_n)=\frac{n!}{t^n}$;

  Let $N_1(t),N_2(t)$ represent the number of type-1/2 events that occur by time t, then they are independent Poisson random variables having means λtp and $\lambda t(1-p)$, where $p=\frac{1}{t}\int_0^t P(s)ds$;

- Nonhomogeneous Poisson Process: independent, nonstationary increment.

  $m(t)=\int_0^t\lambda(s)ds$

  $P\{N(t+s)-N(t)=n\}=\exp\{-\int_t^{t+s}\lambda(t')dt'\}(\int_t^{t+s}\lambda(t')dt')^n/n!,n\geq 0$

- Compound Poisson Process: independent, stationary increment.

  Compound Poisson r.v.: $W=\sum_{i=1}^N X_t, E[W]=\lambda E[X], \mathrm{Var}(W)=\lambda E\left[X^2\right]$, $\psi_W(\theta)=\exp(\lambda\psi_X(\theta)-\lambda)$;

  Definition: $X(t):=\sum_{i=1}^{N(t)}X_i$ where X_i are i.i.d. and independent of Poisson process N.

- Conditional Poisson Process: nonindependent stationary increment.

  Definition: $\tilde{N}(t):=N(t\Lambda)$ where N is a Poisson process with rate 1 and Λ is a positive r.v. with distribution G.

  $P\{\Lambda\leq x\mid\tilde{N}(t)=n\}=\frac{\int_0^x e^{-\lambda t}(\lambda t)^n dG(\lambda)}{\int_0^\infty e^{-\lambda t}(\lambda t)^n dG(\lambda)}$;

- Discrete Time Markov Chain: $P(X_{n+1} = j|X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = P(X_{n+1} = j|X_n = i) := P_{ij}$.

  Chapman-Kolmogorov equations: $P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m$;

  $i \leftrightarrow j$: communicate, same class; *irreducible*: only one class;

  period: greatest common divisor of n s.t. $P_{ii}^n \neq 0$; *aperiodic*: $d(i) = 1$;

  $f_{ij}^n = P\left\{X_n = j, X_k \neq j, k = 1, ..., n-1 \mid X_0 = i\right\}, f_{ij} = \sum_{n=1}^{\infty} f_{ij}^n$;

  recurrent: $f_{jj} = 1 (\Leftrightarrow \sum_{n=1}^{\infty} P_{jj}^n = \infty)$; *transient*: otherwise.

  Define the expected number of transitions needed to return to the state j, $\mu_{jj} = \begin{cases} \infty & \text{if } j \text{ is transient} \\ \sum_{n=1}^{\infty} n f_{jj}^n & \text{if } j \text{ is recurrent.} \end{cases}$

  (Limiting distribution) If $i \leftrightarrow j$, then

  - If j is aperiodic, then $\lim_{n\to\infty} P_{ij}^n = \frac{1}{\mu_{jj}}$;
  - If j has period d, then $\lim_{n\to\infty} P_{jj}^{nd} = \frac{d}{\mu_{jj}}$.

  positive recurrent: $\mu_{jj} < \infty$; *null recurrent*: $\mu_{jj} = \infty$;

  class property: positive recurrent, null recurrent, recurrent, period;

  stationary: $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j \geq 0$

  (stationary distribution) irreducible aperiodic Markov chain,

  - either the states are all transient or all null recurrent; in this case, $P_{ij}^n \to 0$ and there exists no stationary distribution;
  - or else, all states are positive recurrent (*ergodic*); in this case, $\{\pi_j = \lim_{n\to\infty} P_{jj}^n\}$ is a stationary distribution and there exists no other stationary distribution.

- CTMC: We say $\{X(t), t \geq 0\}$ is CTMC if for all $s, t \geq 0$

  $$P(X(t+s) = j|X(s) = i, X(u) = x(u), 0 \leq u < s) = P(X(t+s) = j|X(s) = i).$$

  The survival time τ_i is exponentially distributed.

  In fact, the above gives us a way of constructing CTMC:

  - v_i as the rate of exponential distribution of τ_i;
  - when the process leaves state i, it will enter j with probability P_{ij}, where $\sum_{j\neq i} P_{ij} = 1$.

  Define

  $$q_{ij} = \begin{cases} v_i P_{ij} & \forall i \neq j \\ -v_i & j = i. \end{cases}$$

  We call q_{ij} the transition rate from i to j. The matrix Q is called the generator matrix.

  Let us denote by $P_{ij}(t) = P(X(t+s) = j|X(s) = i)$.

  A CTMC with states $0, 1, \dots$ for which $q_{ij} = 0$ whenever $|i - j| > 1$ is called a *birth and death process*. Let λ_i and μ_i be given by $\lambda_i = q_{i,i+1}, \mu_i = q_{i,i-1}$ and are called respectively birth rates and death rates. We see that $v_i = \lambda_i + \mu_i, P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}$.

  Kolmogorov's Backward Eqautions: $P'(t) = QP(t)$.

  Kolmogorov's Forward Eqautions. Under suitable regularity condition, $P'(t) = P(t)Q$.

  $\pi = (\pi_i)_{i\in S}$ is said to be a stationary distribution for the CTMC if

  $$0 = \pi Q, \pi_i \geq 0, \sum_{i\in S} \pi_i = 1.$$

  (M/M/1) $\pi_n = \rho^n \pi_0$. Define $\rho = \lambda/\mu$, name it *utilization* or *traffic intensity.*

  Consider an irreducible CTMC. If a stationary distribution π exists, then it is unique and $\lim_{t\to\infty} P_{ij}(t) = \pi_j$ for all $i \in S$.

- Martingale: A stochastic process $\{Z_n, n = 0, 1, \ldots\}$ is called a martingale if $\mathbb{E}[|Z_n|] < \infty$ and $\mathbb{E}[Z_{n+1}|Z_1, \ldots, Z_n] = Z_n$ for all n.

  (Stopping Time) The positive integer-valued random variable N is *random time* for the process $\{Z_n, n \geq 1\}$ if the event $\{N = n\}$ is determined by the random variables $Z_1, \ldots, Z_n$. If $P(N < \infty) = 1$, then N is *stopping time*.

  (Doob's Stopping Theorem / Sampling Theorem) If either:

  1. $\bar{Z}_n$ are uniformly bounded, or;
  2. N is bounded, or;
  3. $\mathbb{E}[N] < \infty$, and there is an $M < \infty$ such that

  $$E[|Z_{n+1} - Z_n| \mid Z_1, \ldots, Z_n] < M,$$

  then

  $$E[Z_N] = E[Z_1].$$

  (Wald's equation) If X_i are iid with $E[|X|] < \infty$ and if N is a stopping time for X_i with $E[N] < \infty$, then

  $$E[\sum_{i=1}^{N} X_i] = E[N]E[X].$$

  (Martingale convergence theorem) If $\{Z_n, n \geq 1\}$ is a martingale such that for some $M < \infty$

  $$E[|Z_n|] \leq M, \forall n$$

  then, with probability 1, $\lim_{n\to\infty} Z_n$ exists and is finite.

  How to compute a value (e.g., expected happening time, probability) related to some event:

  1. Identify the underlying random variables and identify the happening of the event as a stopping time;
  2. Define a $\{Y_n, n = 1, 2, \ldots\}$ with known value at the stopping time;
  3. Check whether $\{Y_n, n = 1, 2, \ldots\}$ is a martingale;
  4. Apply Doob's stopping theorem.

- Brownian Motions:

  1. $X(0) = 0$;
  2. stationary and independent increment;
  3. $X(t) \sim N(0, c^2 t)$.

  $T_a := \inf\{t : X(t) \geq a\}$.

  $$P(T_a \leq t) = 2P(X(t) \geq a).$$

  $\{T_a \leq t\} = \{\max_{0\leq s\leq t} X(s) \geq a\}$.