Introduction of Robust Optimization

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SINGLE-STAGE

Consider a single-stage optimization problem

$$
\begin{aligned}
\min_{x \in \mathcal{X}} \quad & f_0(x, \xi) \\
\text{s.t.} \quad & f_1(x, \xi) \leq 0.
\end{aligned}
$$

SINGLE-STAGE

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Now we consider a variant:

- ξ is revealed after decision x;
- The only information we have is $\xi \in \Xi$;
- We want to make a decision x that is feasible on any realization of ξ;
- We care about the worst outcome.

SINGLE-STAGE

Consider a single-stage robust optimization problem

$$\begin{aligned} \min_{x \in \mathcal{X}} \max_{\xi \in \Xi} \ & f_0(x, \xi) \\ \text{s.t.} \max_{\xi \in \Xi} & f_1(x, \xi) \leq 0. \end{aligned}$$

- ξ is revealed after decision x;
- The only information we have is $\xi \in \Xi$;
- We want to make a decision x that is feasible on any realization of ξ;
- We care about the worst outcome.

TWO-STAGE

Consider a two-stage robust optimization problem.
The first-stage decision is x, then ξ is revealed, and second-stage decision y is made.

$$\begin{aligned} \min_{x \in \mathcal{X}, y(\cdot):\Xi \to \mathcal{Y}} \quad & \max_{\xi \in \Xi} f_0(x, y(\xi), \xi) \\ \text{s.t.} \quad & \max_{\xi \in \Xi} f_1(x, y(\xi), \xi) \leq 0. \end{aligned} \tag{1}$$

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Because of the structure min-max-min, the two-stage robust optimization problem can be equivalently expressed as

$$\min_{x \in \mathcal{X}} \max_{\xi \in \Xi} \min_{y \in \mathcal{Y}} \{ f_0(x, y, \xi) : \; f_1(x, y, \xi) \leq 0 \}. \tag{2}$$

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Write in the epigraph formulation,

$$\begin{aligned} & \min \tau \\ & \text{s.t. } \tau \ge \max_{\xi\in\Xi} \min_{y\in\mathcal{Y}} \{f_0(x, y, \xi) : \ f_1(x, y, \xi) \le 0\} \\ & \quad\ x \in \mathcal{X}, \tau \in \mathbb{R}. \end{aligned}$$

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Write in the epigraph formulation,

$$\begin{aligned} \min \ & \tau \\ \text{s.t.} \ & \forall \xi \in \Xi, \exists y \in \mathcal{Y} : f_1(x, y, \xi) \leq 0 \text{ and } \tau \geq f_0(x, y, \xi) \\ & x \in \mathcal{X}, \tau \in \mathbb{R}. \end{aligned}$$

TWO-STAGE

Notice

$$\begin{aligned} &\min \tau \\ &\text{s.t. } \forall \xi \in \Xi, \exists y \in \mathcal{Y} : f_1(x, y, \xi) \leq 0 \text{ and } \tau \geq f_0(x, y, \xi) \\ &\quad x \in \mathcal{X}, \tau \in \mathbb{R} \end{aligned}$$

can be written in a compact form

$$\min_{x \in \mathcal{X}, \tau \in \mathbb{R}} \max_{\xi \in \Xi} \min_{y \in \mathcal{Y}} \{\tau : f_1(x, y, \xi) \leq 0, f_0(x, y, \xi) - \tau \leq 0\}. \tag{3}$$

This is the most widely used form.