

Numerical methods for partial differential equations

Programming Exam

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1 Introduction

The programming exam states the following task.

Write a finite element code in your preferred programming language solving the boundary value problem:

$$\text{Find } u \in C^2(\Omega) : -\Delta u + u = \cos(\pi x) \cos(\pi y) \quad \text{in } \Omega \quad (1)$$

$$\partial_n u = 0 \quad \text{on } \partial\Omega. \quad (2)$$

Here, $\Omega = [0, 1]^2$.

2 Weak Formulation

In order to apply some of the results from the lecture, we need to derive the weak formulation of the given problem

$$\text{Find } u \in C^2(\Omega) : -\Delta u + u = \cos(\pi x) \cos(\pi y) \quad \text{in } \Omega \quad (3)$$

$$\partial_n u = 0 \quad \text{on } \partial\Omega. \quad (4)$$

Multiplying with an arbitrary $v \in C^2(\Omega)$ and integrating over Ω gives us

$$-\int_{\Omega} \Delta u v \, d\mathbf{x} + \int_{\Omega} u v \, d\mathbf{x} = \int_{\Omega} f v \, d\mathbf{x}$$

where $f = \cos(\pi x) \cos(\pi y)$ and $\mathbf{x} = (x, y)$. Using Green's first formula and (2) we can obtain the weak formulation

$$\int_{\Omega} \nabla u \nabla v \, d\mathbf{x} + \int_{\Omega} u v \, d\mathbf{x} = \int_{\Omega} f v \, d\mathbf{x}$$

From now on, we will denote the left hand side of the equation by $a(u, v)$ and the right hand side by $F(v)$. Thus, we obtain the weak formulation

$$\text{Find } u \in H^1(\Omega) : a(u, v) = F(v) \quad \forall v \in H^1(\Omega) \quad (5)$$

3 Existence and Uniqueness of a Solution

We can already see, that our bilinear form $a(\cdot, \cdot)$ is the inner product associated with the norm on our function space $H^1(\Omega)$. We want to use the

Riesz representation theorem to prove existence and uniqueness of a solution. In order to do so, it remains to show that our functional $F(\cdot)$ is linear and bounded. Linearity follows from the properties of integration. Using Hoelder's inequality, we show that

$$\begin{aligned}
F(u) &= \|fu\|_{L^1(\Omega)} \\
&\stackrel{\text{Hld.}}{\leq} \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\
&\leq \|1\|_{L^2(\Omega)} (\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \\
&\leq c \|u\|_{H^1(\Omega)},
\end{aligned}$$

where c depends on our domain Ω . For our case, we have $\Omega = [0, 1]^2$, in particular this means that Ω is bounded and our constant c is finite. Therefore, $F(\cdot)$ is a bounded, linear functional and we can apply the Riesz representation theorem.

4 Finding the Analytical Solution

Now that we know that a unique solution exists, we want to actually compute it. We will use the ansatz $u = C \cos(\pi x) \cos(\pi y)$, with its gradient $\Delta u = 2\pi^2 u$. Inserting in (1) gives us

$$-2C\pi^2 \cos(\pi x) \cos(\pi y) + C \cos(\pi x) \cos(\pi y) = \cos(\pi x) \cos(\pi y) \quad (6)$$

$$\Rightarrow -2C\pi^2 + C = 1 \quad (7)$$

$$\Leftrightarrow C = \frac{1}{1 - 2\pi^2} \quad (8)$$

This leaves us with the solution $u = \frac{1}{1-2\pi^2} \cos(\pi x) \cos(\pi y)$.

5 Integrals and Transformations

The transformations we have to use for our FEM-code are very simple, since our mesh will be generated uniformly. Ultimately, we will only have to scale and displace our reference cell. We decided, that we will hard-code this property, since it will heavily simplify the computation of the local stiffness-matrices and the right-hand-side of the linear system, as we will see in brief.

Let T be a cell of the mesh with vertices v_1, v_2, v_3, v_4 , where v_1 is the vertex

on the bottom left and h is the length of every edge. $\hat{T} = [0, 1]^2$ shall be our reference cell. Then, our transformation F will look like

$$F: \hat{T} \rightarrow T, \quad \hat{\mathbf{x}} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mapsto \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \mathbf{v}_1 := \begin{pmatrix} x \\ y \end{pmatrix} := \mathbf{x}$$

We immediately can see, that the jacobian J is given by

$$J(\hat{\mathbf{x}}) = J = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \quad (9)$$

in particular, it is independent of $\hat{\mathbf{x}}$. Consequently, we also get

$$|\det(J(\hat{\mathbf{x}}))| = |\det(J)| = h^2 \text{ and } J^{-1} = h^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (10)$$

Using the results from the lecture on the transformation of our bilinear form $a(\cdot, \cdot)$ and using (7) and (8), we can compute the stiffness matrix in the following way

$$a_{ij}^{(T)} = \int_{\hat{T}} \nabla p_i \cdot \nabla p_j \, d\hat{\mathbf{x}} + h^2 \int_{\hat{T}} p_i p_j \, d\hat{\mathbf{x}},$$

where p_i and p_j are our shape functions.

Remark: Note that the ∇ in our formula refers to the gradient with respect to $\hat{\mathbf{x}} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$. *Remark:* Note that for affine transformations, the local stiffness matrices do not depend on the cell T , for which we want to compute it. Therefore, every local matrix looks the same and we only have to compute it once.

For the right-hand side, we have to look at the substitution formula for higher dimension integrals. But first of all, we need to split the integral up. Let ϕ_i be a basis function of our domain Ω , \mathcal{T}_h our meshcells and \mathcal{T}_i the mesh

cells, on which $\phi_i \neq 0$. Then we get

$$\begin{aligned}
\int_{\Omega} f(\mathbf{x}) \phi_i(\mathbf{x}) \, d\mathbf{x} &= \sum_{T \in \mathcal{T}_h} \int_T f(\mathbf{x}) \phi_i(\mathbf{x}) \, d\mathbf{x} \\
&= \sum_{T \in \mathcal{T}_i} \int_T f(\mathbf{x}) \phi_i(\mathbf{x}) \, d\mathbf{x} \\
&= \sum_{T \in \mathcal{T}_i} \int_T f(\mathbf{x}) p_{i(T)}^{(T)}(\mathbf{x}) \, d\mathbf{x} \\
&= \sum_{T \in \mathcal{T}_i} \int_{\hat{T}} f(F^{(T)}(\hat{\mathbf{x}})) \hat{p}_{i(T)}(\hat{\mathbf{x}}) |\det(J)| \, d\hat{\mathbf{x}} \\
&= h^2 \sum_{T \in \mathcal{T}_i} \int_{\hat{T}} f(h\hat{\mathbf{x}} + v_1^{(T)}) \hat{p}_{i(T)}(\hat{\mathbf{x}}) \, d\hat{\mathbf{x}}
\end{aligned}$$