1 Introduction

2 Weak Formulation

In order to apply some of the results from the lecture, we need to derive the weak formulation of the given problem

$$\text{Find } u \in C^2(\Omega) : -\Delta u + u = \cos(\pi x)\cos(\pi y) \qquad \text{in } \Omega \tag{1}$$

$$\partial_n u = 0 \qquad \text{on } \partial\Omega. \tag{2}$$

Multiplying with an arbitary $v \in C^2(\Omega)$ and integrating over Ω gives us

$$-\int_\Omega \Delta u v \,\mathrm{d}\boldsymbol{x} + \int_\Omega uv \,\mathrm{d}\boldsymbol{x} = \int_\Omega fv \,\mathrm{d}\boldsymbol{x}$$

where $f = \cos(\pi x)\cos(\pi y)$ and $\boldsymbol{x} = (x, y)$. Using Green's first formula and (2) we can obtain the weak formulation

$$\int_\Omega \nabla u \nabla v \,\mathrm{d}\boldsymbol{x} + \int_\Omega uv \,\mathrm{d}\boldsymbol{x} = \int_\Omega fv \,\mathrm{d}\boldsymbol{x}$$

From now on, we will denote the left hand side of the equation by $a(u, v)$ and the right hand side by $F(v)$. Thus, we obtain the weak formulation

$$\text{Find } u \in H^1(\Omega) : \quad a(u, v) = F(v) \quad \forall v \in H^1(\Omega) \tag{3}$$

3 Existence and Uniqueness of a Solution

We can already see, that our bilinear form $a(\cdot\,,\cdot)$ is the inner product associated with the norm on our function space $H^1(\Omega)$. We want to use the Riesz representation theorem to prove existence and uniqueness of a solution. In order to do so, it remains to show that our functional $F(\cdot)$ is linear and bounded. Linearity follows from the properties of integration. Using Hoelder's inequality, we show that

$$\begin{aligned} F(u) &= \|fu\|_{L^1(\Omega)} \\ &\overset{\text{Hld.}}{\leq} \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ &\leq \|1\|_{L^2(\Omega)} \left(\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right) \\ &\leq c\,\|u\|_{H^1(\Omega)}, \end{aligned}$$

where c depends on our domain Ω. For our case, we have $\Omega = [0, 1]^2$, in particular this means that Ω is bounded and our constant c is finite. Therefore, $F(\cdot)$ is a bounded, linear functional and we can apply the Riesz representation theorem.

4 Finding the Analytical Solution

Now that we know that a unique solution exists, we want to actually compute it. We will use the ansatz $u = C\cos(\pi x)\cos(\pi y)$, with its gradient $\Delta u = 2\pi^2 u$. Inserting in (1) gives us

$$-2C\pi^2\cos(\pi x)\cos(\pi y) + C\cos(\pi x)\cos(\pi y) = \cos(\pi x)\cos(\pi y) \tag{4}$$

$$\Rightarrow -2C\pi^2 + C = 1 \tag{5}$$

$$\Leftrightarrow C = \frac{1}{1-2\pi^2} \tag{6}$$

This leaves us with the solution $u = \frac{1}{1-2\pi^2}\cos(\pi x)\cos(\pi y)$.

5 Integrals and Transformations

The transformations we have to use for our FEM-code are very simple, since our mesh will be generated uniformly. Ultimately, we will only have to scale and displace our reference cell. We decided, that we will hard-code this property, since it will heavily simplifiy the computation of the local stiffness-matrices and the right-hand-side of the linear system, as we will see in brief.

Let T be a cell of the mesh with vertices v_1, v_2, v_3, v_4, where v_1 is the vertex on the bottom left and h is the length of every edge. $\hat{T} = [0,1]^2$ shall be our reference cell. Then, our transformation F will look like

$$F\colon \hat{T} \to T, \quad \hat{\boldsymbol{x}} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mapsto \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + v_1 := \begin{pmatrix} x \\ y \end{pmatrix} := \boldsymbol{x}$$

We immediately can see, that the jacobian J is given by

$$J(\hat{\boldsymbol{x}}) = J = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \tag{7}$$

in particular, it is independet of $\hat{\boldsymbol{x}}$. Consequently, we also get

$$|\det(J(\hat{\boldsymbol{x}}))| = |\det(J)| = h^2 \text{and } J^{-1} = h^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

Using the results from the lecture on the transformation of our bilinear form $a(\cdot\,,\cdot)$ and using (7) and (8), we can compute the stiffness matrix in the following way

$$a_{ij}^{(T)} = \int_{\hat{T}} \nabla p_i \cdot \nabla p_j \,\mathrm{d}\hat{\boldsymbol{x}} + h^2 \int_{\hat{T}} p_i p_j \,\mathrm{d}\hat{\boldsymbol{x}},$$

where p_i and p_j are our shape functions.

Remark: Note that the ∇ in our formula refers to the gradient with respect to $\hat{\boldsymbol{x}} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$.

For the right-hand side, we have to look at the substitution formula for higher dimension integrals. But first of all, we need do split the intagral up. Let ϕ_i be a basis function of our domain Ω, $\mathcal{T}_h$ our meshcells and $\mathcal{T}_i$ the mesh cells, on which $\phi_i = 0$. Then we get

$$\begin{aligned}
\int_\Omega f(\boldsymbol{x})\phi_i(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} &= \sum_{T\in\mathcal{T}_h} \int_T f(\boldsymbol{x})\phi_i(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} \\
&= \Sigma \int_T f(\boldsymbol{x})\phi_i(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} \\
&= \Sigma \int_T f(\boldsymbol{x})p_{i(T)}^{(T)}(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} \\
&= \Sigma \int_{\hat{T}} f(F^{(T)}(\hat{\boldsymbol{x}})\hat{p}_{i(T)}(\hat{\boldsymbol{x}})\ \mathrm{d}\hat{\boldsymbol{x}} \\
&= \Sigma \int_{\hat{T}} f(h\hat{\boldsymbol{x}} + v_1^{(T)})\hat{p}_{i(T)}(\hat{\boldsymbol{x}})\ \mathrm{d}\hat{\boldsymbol{x}}
\end{aligned}$$