1 Introduction

2 Weak Formulation

In order to apply some of the results from the lecture, we need to derive the weak formulation of the given problem

$$\text{Find } u \in C^2(\Omega) : -\Delta u + u = \cos(\pi x)\cos(\pi y) \qquad \text{in } \Omega \tag{1}$$
$$\partial_n u = 0 \qquad \text{on } \partial\Omega. \tag{2}$$

Multiplying with an arbitary $v \in C^2(\Omega)$ and integrating over Ω gives us

$$-\int_\Omega \Delta uv \,\mathrm{d}\boldsymbol{x} + \int_\Omega uv \,\mathrm{d}\boldsymbol{x} = \int_\Omega fv \,\mathrm{d}\boldsymbol{x}$$

where $f = \cos(\pi x)\cos(\pi y)$ and $\boldsymbol{x} = (x, y)$. Using Green's first formula and (2) we can obtain the weak formulation

$$\int_\Omega \nabla u \nabla v \,\mathrm{d}\boldsymbol{x} + \int_\Omega uv \,\mathrm{d}\boldsymbol{x} = \int_\Omega fv \,\mathrm{d}\boldsymbol{x}$$

From now on, we will denote the left hand side of the equation by $a(u, v)$ and the right hand side by $F(v)$. Thus, we obtain the weak formulation

$$\text{Find } u \in H^1(\Omega) : \quad a(u, v) = F(v) \quad \forall v \in H^1(\Omega) \tag{3}$$

3 Existence and Uniqueness of a Solution

We can already see, that our bilinear form $a(\cdot\,,\cdot)$ is the inner product associated with the norm on our function space $H^1(\Omega)$. We want to use the Riesz representation theorem to prove existence and uniqueness of a solution. In order to do so, it remains to show that our functional $F(\cdot)$ is linear and bounded. Linearity follows from the properties of integration. Using Hoelder's inequality, we show that

$$\begin{aligned} F(u) &= \|fu\|_{L^1(\Omega)} \\ &\overset{\text{Hld.}}{\leq} \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\leq \|1\|_{L^2(\Omega)} \left(\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right) \\ &\leq c\ \|u\|_{H^1(\Omega)}, \end{aligned}$$

where c depends on our domain Ω. For our case, we have $\Omega = [0, 1]^2$, in particular this means that Ω is bounded and our constant c is finite. Therefore, $F(\cdot)$ is a bounded, linear functional and we can apply the Riesz representation theorem.

4 Finding the Analytical Solution

Now that we know that a unique solution exists, we want to actually compute it. We will use the ansatz $u = C\cos(\pi x)\cos(\pi y)$, with its gradient $\Delta u = 2\pi^2 u$. Inserting in (1) gives us

$$-2C\pi^2\cos(\pi x)\cos(\pi y) + C\cos(\pi x)\cos(\pi y) = \cos(\pi x)\cos(\pi y) \tag{4}$$

$$\Rightarrow -2C\pi^2 + C = 1 \tag{5}$$

$$\Leftrightarrow C = \frac{1}{1-2\pi^2} \tag{6}$$

This leaves us with the solution $u = \frac{1}{1-2\pi^2}\cos(\pi x)\cos(\pi y)$.

5 Integrals and Transformations

The transformations we have to use for our FEM-code are very simple, since our mesh will be generated uniformly. Ultimately, we will only have to scale and displace our reference cell. We decided, that we will hard-code this property, since it will heavily simplifiy the computation of the local stiffness-matrices and the right-hand-side of the linear system, as we will see in brief.

Let T be a cell of the mesh with vertices v_1, v_2, v_3, v_4, where v_1 is the vertex on the bottom left and h is the length of every edge. $\hat{T} = [0,1]^2$ shall be our reference cell. Then, our transformation F will look like

$$F\colon \hat{T} \to T, \quad \hat{\boldsymbol{x}} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mapsto \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + v_1 := \begin{pmatrix} x \\ y \end{pmatrix} := \boldsymbol{x}$$

We immediately can see, that the jacobian J is given by

$$J(\hat{\boldsymbol{x}}) = J = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \tag{7}$$

in particular, it is independet of $\hat{\boldsymbol{x}}$. Consequently, we also get

$$|\det(J(\hat{\boldsymbol{x}}))| = |\det(J)| = h^2 \text{and } J^{-1} = h^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

Using the results from the lecture on the transformation of our bilinear form $a(\cdot\,,\cdot)$ and using (7) and (8), we can compute the stiffness matrix in the following way

$$a_{ij}^{(T)} = \int_{\hat{T}} \nabla p_i \cdot \nabla p_j \,\mathrm{d}\hat{\boldsymbol{x}} + h^2 \int_{\hat{T}} p_i p_j \,\mathrm{d}\hat{\boldsymbol{x}},$$

where p_i and p_j are our shape functions.

Remark: Note that the ∇ in our formula refers to the gradient with respect to $\hat{\boldsymbol{x}} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$. *Remark:* Note that for affine transformations, the local stiffness matrices do not depend on the cell T, for which we want to compute it. Therefore, every local matrix looks the same and we only have to compute it once.

For the right-hand side, we have to look at the substitution formula for higher dimension integrals. But first of all, we need do split the intagral up. Let ϕ_i be a basis function of our domain Ω, $\mathcal{T}_h$ our meshcells and $\mathcal{T}_i$ the mesh cells, on which $\phi_i \neq 0$. Then we get

$$
\begin{aligned}
\int_\Omega f(\boldsymbol{x})\phi_i(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} &= \sum_{T\in\mathcal{T}_h} \int_T f(\boldsymbol{x})\phi_i(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} \\
&= \sum_{T\in\mathcal{T}_i} \int_T f(\boldsymbol{x})\phi_i(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} \\
&= \sum_{T\in\mathcal{T}_i} \int_T f(\boldsymbol{x}) p^{(T)}_{i^{(T)}}(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} \\
&= \sum_{T\in\mathcal{T}_i} \int_{\hat{T}} f(F^{(T)}(\hat{\boldsymbol{x}})\hat{p}_{i^{(T)}}(\hat{\boldsymbol{x}})|\det(J)|\ \mathrm{d}\hat{\boldsymbol{x}} \\
&= h^2 \sum_{T\in\mathcal{T}_i} \int_{\hat{T}} f(h\hat{\boldsymbol{x}} + v_1^{(T)})\hat{p}_{i^{(T)}}(\hat{\boldsymbol{x}})\ \mathrm{d}\hat{\boldsymbol{x}}
\end{aligned}
$$