

## 1 Introduction

In this document, we will recapitulate our code and provide further explanations on selected excerpts of our code

## 2 Mesh Generator

We start of with our mesh generator 'mesh\_generate'. Our domain  $\Omega = [0, 1]^2$  will not be subject to change for this task, therefore our only input will be the mesh size  $h$ . The result will be two matrices, namely *vertex\_matrix* and *cell\_matrix*. *vertex\_matrix* will store the coordinates of each vertex, while *cell\_matrix* only stores the numbers of the four vertices of each cell.

*Example:*  $h = 0.5$

$$vertex\_matrix = \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \\ 1 & 0 \\ 0 & 0.5 \\ 0.5 & 0.5 \\ 1 & 0.5 \\ 0 & 1 \\ 0.5 & 1 \\ 1 & 1 \end{pmatrix}, \quad cell\_matrix = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 2 & 3 & 5 & 6 \\ 4 & 5 & 7 & 8 \\ 5 & 6 & 8 & 9 \end{pmatrix}$$

## 3 Shape Functions

An essential concept for finite element methods would be shape functions. We have to set up certain points, so called *nodes* on a reference cell  $\hat{T} = [0, 1]^2$  such that our shape functions  $p_i$  are uniquely defined by two properties:

1.  $p_i(x_j) = \delta_{ij}$ ,  $x_j$  node
2.  $p_i$  is a polynomial on each edge.

## 4 Weak Formulation

In order to apply some of the results from the lecture, we need to derive the weak formulation of the given problem

$$\text{Find } u \in C^2(\Omega) : -\Delta u + u = \cos(\pi x) \cos(\pi y) \quad \text{in } \Omega \quad (1)$$

$$\partial_n u = 0 \quad \text{on } \partial\Omega. \quad (2)$$

Multiplying with an arbitrary  $v \in C^2(\Omega)$  and integrating over  $\Omega$  gives us

$$-\int_{\Omega} \Delta u v \, d\mathbf{x} + \int_{\Omega} u v \, d\mathbf{x} = \int_{\Omega} f v \, d\mathbf{x}$$

where  $f = \cos(\pi x) \cos(\pi y)$  and  $\mathbf{x} = (x, y)$ . Using Green's first formula and (2) we can obtain the weak formulation

$$\int_{\Omega} \nabla u \nabla v \, d\mathbf{x} + \int_{\Omega} u v \, d\mathbf{x} = \int_{\Omega} f v \, d\mathbf{x}$$

From now on, we will denote the left hand side of the equation by  $a(u, v)$  and the right hand side by  $F(v)$ . Thus, we obtain the weak formulation

$$\text{Find } u \in H^1(\Omega) : \quad a(u, v) = F(v) \quad \forall v \in H^1(\Omega) \quad (3)$$

## 5 Existence and Uniqueness of a Solution

We can already see, that our bilinear form  $a(\cdot, \cdot)$  is the inner product associated with the norm on our function space  $H^1(\Omega)$ . We want to use the Riesz representation theorem to prove existence and uniqueness of a solution. In order to do so, it remains to show that our functional  $F(\cdot)$  is linear and bounded. Linearity follows from the properties of integration. Using Hoelder's inequality, we show that

$$\begin{aligned} F(u) &= \|fu\|_{L^1(\Omega)} \\ &\stackrel{\text{Hld.}}{\leq} \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ &\leq \|1\|_{L^2(\Omega)} (\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \\ &\leq c \|u\|_{H^1(\Omega)}, \end{aligned}$$

where  $c$  depends on our domain  $\Omega$ . For our case, we have  $\Omega = [0, 1]^2$ , in particular this means that  $\Omega$  is bounded and our constant  $c$  is finite. Therefore,  $F(\cdot)$  is a bounded, linear functional and we can apply the Riesz representation theorem.

## 6 Finding the Analytical Solution

Now that we know that a unique solution exists, we want to actually compute it. We will use the ansatz  $u = C \cos(\pi x) \cos(\pi y)$ , with its gradient  $\Delta u = 2\pi^2 u$ . Inserting in (1) gives us

$$-2C\pi^2 \cos(\pi x) \cos(\pi y) + C \cos(\pi x) \cos(\pi y) = \cos(\pi x) \cos(\pi y) \quad (4)$$

$$\Rightarrow -2C\pi^2 + C = 1 \quad (5)$$

$$\Leftrightarrow C = \frac{1}{1 - 2\pi^2} \quad (6)$$

This leaves us with the solution  $u = \frac{1}{1-2\pi^2} \cos(\pi x) \cos(\pi y)$ .

## 7 Integrals and Transformations

The transformations we have to use for our FEM-code are very simple, since our mesh will be generated uniformly. Ultimately, we will only have to scale and displace our reference cell. We decided, that we will hard-code this property, since it will heavily simplify the computation of the local stiffness-matrices and the right-hand-side of the linear system, as we will see in brief.

Let  $T$  be a cell of the mesh with vertices  $v_1, v_2, v_3, v_4$ , where  $v_1$  is the vertex on the bottom left and  $h$  is the length of every edge.  $\hat{T} = [0, 1]^2$  shall be our reference cell. Then, our transformation  $F$  will look like

$$F: \hat{T} \rightarrow T, \quad \hat{\mathbf{x}} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mapsto \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + v_1 := \begin{pmatrix} x \\ y \end{pmatrix} := \mathbf{x}$$

We immediately can see, that the jacobian  $J$  is given by

$$J(\hat{\mathbf{x}}) = J = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \quad (7)$$

in particular, it is independent of  $\hat{\mathbf{x}}$ . Consequently, we also get

$$|\det(J(\hat{\mathbf{x}}))| = |\det(J)| = h^2 \text{ and } J^{-1} = h^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (8)$$

Using the results from the lecture on the transformation of our bilinear form  $a(\cdot, \cdot)$  and using (7) and (8), we can compute the stiffness matrix in the following way

$$a_{ij}^{(T)} = \int_{\hat{T}} \nabla p_i \cdot \nabla p_j \, d\hat{\mathbf{x}} + h^2 \int_{\hat{T}} p_i p_j \, d\hat{\mathbf{x}},$$

where  $p_i$  and  $p_j$  are our shape functions.

*Remark:* Note that the  $\nabla$  in our formula refers to the gradient with respect to  $\hat{\mathbf{x}} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$ . *Remark:* Note that for affine transformations, the local stiffness matrices do not depend on the cell  $T$ , for which we want to compute

it. Therefore, every local matrix looks the same and we only have to compute it once.

For the right-hand side, we have to look at the substitution formula for higher dimension integrals. But first of all, we need to split the integral up. Let  $\phi_i$  be a basis function of our domain  $\Omega$ ,  $\mathcal{T}_h$  our meshcells and  $\mathcal{T}_i$  the mesh cells, on which  $\phi_i \neq 0$ . Then we get

$$\begin{aligned}
\int_{\Omega} f(\mathbf{x}) \phi_i(\mathbf{x}) \, d\mathbf{x} &= \sum_{T \in \mathcal{T}_h} \int_T f(\mathbf{x}) \phi_i(\mathbf{x}) \, d\mathbf{x} \\
&= \sum_{T \in \mathcal{T}_i} \int_T f(\mathbf{x}) \phi_i(\mathbf{x}) \, d\mathbf{x} \\
&= \sum_{T \in \mathcal{T}_i} \int_T f(\mathbf{x}) p_{i(T)}^{(T)}(\mathbf{x}) \, d\mathbf{x} \\
&= \sum_{T \in \mathcal{T}_i} \int_{\hat{T}} f(F^{(T)}(\hat{\mathbf{x}}) \hat{p}_{i(T)}(\hat{\mathbf{x}}) |\det(J)| \, d\hat{\mathbf{x}} \\
&= h^2 \sum_{T \in \mathcal{T}_i} \int_{\hat{T}} f(h\hat{\mathbf{x}} + v_1^{(T)}) \hat{p}_{i(T)}(\hat{\mathbf{x}}) \, d\hat{\mathbf{x}}
\end{aligned}$$