1 Introduction

2 Weak Formulation

In order to apply some of the results from the lecture, we need to derive the weak formulation of the given problem

$$\text{Find } u \in C^2(\Omega) : -\Delta u + u = \cos(\pi x)\cos(\pi y) \qquad \text{in } \Omega \tag{1}$$
$$\partial_n u = 0 \qquad \text{on } \partial\Omega. \tag{2}$$

Multiplying with an arbitary $v \in C^2(\Omega)$ and integrating over Ω gives us

$$-\int_\Omega \Delta uv \, \mathrm{d}\boldsymbol{x} + \int_\Omega uv \, \mathrm{d}\boldsymbol{x} = \int_\Omega fv \, \mathrm{d}\boldsymbol{x}$$

where $f = \cos(\pi x)\cos(\pi y)$ and $\boldsymbol{x} = (x, y)$. Using Green's first formula and (2) we can obtain the weak formulation

$$\int_\Omega \nabla u \nabla v \, \mathrm{d}\boldsymbol{x} + \int_\Omega uv \, \mathrm{d}\boldsymbol{x} = \int_\Omega fv \, \mathrm{d}\boldsymbol{x}$$

From now on, we will denote the left hand side of the equation by $a(u, v)$ and the right hand side by $F(v)$. Thus, we obtain the weak formulation

$$\text{Find } u \in H^1(\Omega) : \quad a(u, v) = F(v) \quad \forall v \in H^1(\Omega) \tag{3}$$

3 Existence and Uniqueness of a Solution

We can already see, that our bilinear form $a(\cdot\,,\cdot)$ is the inner product associated with the norm on our function space $H^1(\Omega)$. We want to use the Riesz representation theorem to prove existence and uniqueness of a solution. In order to do so, it remains to show that our functional $F(\cdot)$ is linear and bounded. Linearity follows from the properties of integration. Using Hoelder's inequality, we show that

$$\begin{aligned} F(u) &= \|fu\|_{L^1(\Omega)} \\ &\overset{\text{Hld.}}{\leq} \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ &\leq \|1\|_{L^2(\Omega)} \left(\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right) \\ &\leq c\, \|u\|_{H^1(\Omega)}, \end{aligned}$$

where c depends on our domain Ω. For our case, we have $\Omega = [0, 1]^2$, in particular this means that Ω is bounded and our constant c is finite. Therefore, $F(\cdot)$ is a bounded, linear functional and we can apply the Riesz representation theorem.

4 Finding the Analytical Solution

Now that we know that a unique solution exists, we want to actually compute it. We will use the ansatz $u = C\cos(\pi x)\cos(\pi y)$, with its gradient $\Delta u = 2\pi^2 u$. Inserting in (1) gives us

$$-2C\pi^2\cos(\pi x)\cos(\pi y) + C\cos(\pi x)\cos(\pi y) = \cos(\pi x)\cos(\pi y) \tag{4}$$

$$\Rightarrow -2C\pi^2 + C = 1 \tag{5}$$

$$\Leftrightarrow C = \frac{1}{1-2\pi^2} \tag{6}$$

This leaves us with the solution $u = \frac{1}{1-2\pi^2}\cos(\pi x)\cos(\pi y)$.