MMF derivation

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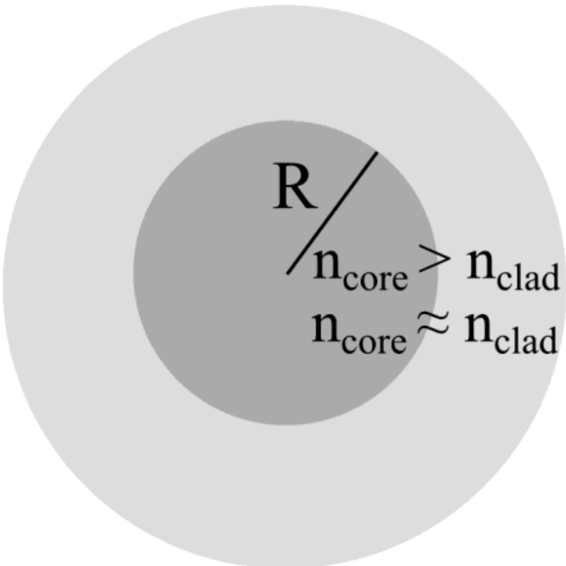
First, define the electric and magnetic fields oscillating at a certain angular frequency at a certain location as:

$\widetilde{\boldsymbol{E}}(\bar{\boldsymbol{r}},\omega) = \widetilde{\boldsymbol{E}}$
$\widetilde{\boldsymbol{H}}(\bar{\boldsymbol{r}},\omega) = \widetilde{\boldsymbol{H}}$

Our goal is to determine which distributions of the electric and magnetic fields solve the Helmholtz equation in cylindrical coordinates for a step-index fiber.

$$\nabla^2\widetilde{\boldsymbol{E}} = -n^2(\bar{\boldsymbol{r}},\omega)\frac{\omega^2}{c^2}\widetilde{\boldsymbol{E}}$$



We will assume that the core has a refractive index, which is larger than that of the cladding, but that the difference is quite small. For example if $n_{core} = 1.454, n_{clad} = 1.450$, the difference is $0.004 = 4 \cdot 10^{-3} = 0.4\%$.

First, we express the electric field in polar coordinates:

$$\widetilde{\boldsymbol{E}} = E_\rho\hat{\boldsymbol{\rho}} + E_\phi\hat{\boldsymbol{\phi}} + E_z\hat{\boldsymbol{z}}\,.$$

A similar expression can be created for the magnetic field.

This is substituted into the HHE in polar coordinates

$$k_0 = \frac{2\pi}{\lambda_{vacuum}} = \frac{\omega}{c}$$

$$\nabla^2\widetilde{\boldsymbol{E}} + n^2(\bar{\boldsymbol{r}},\omega)k_0^2\widetilde{\boldsymbol{E}} = 0$$

[Del in cylindrical and spherical coordinates - Wikipedia]

Vector Laplacian $\nabla^2\mathbf{A} \equiv \Delta\mathbf{A}$[2]	$\nabla^2 A_x \hat{\mathbf{x}} + \nabla^2 A_y \hat{\mathbf{y}} + \nabla^2 A_z \hat{\mathbf{z}}$	$\left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\varphi}{\partial \varphi}\right)\hat{\boldsymbol{\rho}} + \left(\nabla^2 A_\varphi - \frac{A_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \varphi}\right)\hat{\boldsymbol{\varphi}} + \nabla^2 A_z \hat{\mathbf{z}}$
Laplace operator $\nabla^2 f \equiv \Delta f$[1]	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$

$$0 =$$
$$\left[\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]E_\rho - \frac{E_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial\phi} + (nk_0)^2 E_\rho\right]\hat{\boldsymbol{\rho}} +$$
$$\left[\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]E_\phi - \frac{E_\phi}{\rho^2} - \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial\phi} + (nk_0)^2 E_\phi\right]\hat{\boldsymbol{\phi}} +$$
$$\left[\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]E_z + (nk_0)^2 E_z\right]\hat{\mathbf{z}} +$$

This looks complicated to say the least! How do we find both the radial, angular and linear parts of the E-field for this expression?

First, we can notice that the field component in the $\hat{\mathbf{z}}$ term only depends on E_z, so that should be the easiest to tackle first!

$$0 = \left[\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \frac{\partial^2 E_z}{\partial z^2}\right] + (nk_0)^2 E_z\right]$$

$$0 = \left[\left[\frac{1}{\rho}\left(\frac{\partial E_z}{\partial\rho} + \rho\frac{\partial^2 E_z}{\partial\rho^2}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \frac{\partial^2 E_z}{\partial z^2}\right] + (nk_0)^2 E_z\right]$$

$$0 = \frac{1}{\rho}\frac{\partial E_z}{\partial\rho} + \frac{\partial^2 E_z}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \frac{\partial^2 E_z}{\partial z^2} + (nk_0)^2 E_z$$

To solve this, we will assume that separation of variables can be used. In other words, we assume that
$E_z = A(\omega)F(\rho)T(\phi)Z(z)$

Here, A(ω), is the field amplitude, which we can define to contain all the "units" of E_z, making the other factors dimensionless and easy to work with. Now,

$$0 = ATZ\frac{1}{\rho}\frac{\partial F}{\partial\rho} + AZT\frac{\partial^2 F}{\partial\rho^2} + AFZ\frac{1}{\rho^2}\frac{\partial^2 T}{\partial\phi^2} + AFT\frac{\partial^2 Z}{\partial z^2} + (nk_0)^2 AFTZ$$

$$0 = TZ\frac{1}{\rho}\frac{\partial F}{\partial\rho} + ZT\frac{\partial^2 F}{\partial\rho^2} + FZ\frac{1}{\rho^2}\frac{\partial^2 T}{\partial\phi^2} + FT\frac{\partial^2 Z}{\partial z^2} + (nk_0)^2 FTZ$$

To simplify, divide through by FTZ

$$0 = \frac{1}{F}\frac{1}{\rho}\frac{\partial F}{\partial\rho} + \frac{1}{T}\frac{1}{\rho^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + \frac{1}{F}\frac{\partial^2 F}{\partial\rho^2} + (nk_0)^2$$

Move the Z-terms to the other side of the equality:

$$-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \frac{1}{F}\frac{\partial^2 F}{\partial \rho^2} + \frac{1}{F}\frac{1}{\rho}\frac{\partial F}{\partial \rho} + \frac{1}{T}\frac{1}{\rho^2}\frac{\partial^2 T}{\partial \phi^2} + (nk_0)^2$$

The LHS now only depends on z, while the RHS only depends on the angle and the radial distance. In other words, we can pick a random value of the linear position, z, and vary ρ and/or ϕ arbitrarily; the equation above will still be true! This implies that both sides are equal to a constant, which is independent of z, ρ, ϕ! For later convenience, we will define this constant to be:

$$-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \beta^2 = \frac{1}{F}\frac{\partial^2 F}{\partial \rho^2} + \frac{1}{F}\frac{1}{\rho}\frac{\partial F}{\partial \rho} + \frac{1}{T}\frac{1}{\rho^2}\frac{\partial^2 T}{\partial \phi^2} + (nk_0)^2$$

This is useful because we can now turn one big equation depending on 3 spatial coordinates into two smaller equations, where one of them only depends on a single spatial coordinate

$$-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \beta^2$$

$$\frac{1}{F}\frac{\partial^2 F}{\partial \rho^2} + \frac{1}{F}\frac{1}{\rho}\frac{\partial F}{\partial \rho} + \frac{1}{T}\frac{1}{\rho^2}\frac{\partial^2 T}{\partial \phi^2} + (nk_0)^2 = \beta^2$$

The Z-equation can easily be recognized as the characteristic equation for simple harmonic oscillation:

$$\frac{\partial^2 Z}{\partial z^2} = -\beta^2 Z$$

$$Z(z) = \exp(i\beta z)$$

Thus, we see that the "magical fixed constant", β, is actually the spatial frequency of the electric field as we move along the length of the fiber.

Rearranging the other equation and multiplying through by the radial distance yields:

$$\frac{1}{F}\frac{\partial^2 F}{\partial \rho^2} + \frac{1}{F}\frac{1}{\rho}\frac{\partial F}{\partial \rho} + \frac{1}{T}\frac{1}{\rho^2}\frac{\partial^2 T}{\partial \phi^2} + (nk_0)^2 - \beta^2 = 0$$

$$\rho^2\frac{1}{F}\frac{\partial^2 F}{\partial \rho^2} + \rho\frac{1}{F}\frac{\partial F}{\partial \rho} + \rho^2\left[(nk_0)^2 - \beta^2\right] = -\frac{1}{T}\frac{\partial^2 T}{\partial \phi^2}$$

Once again, we notice that the LHS only depends on the radial distance, while the RHS only depends on the angle. Since these can be varied independently while still letting the equation above be satisfied, they must be equal to the same constant!

$$\rho^2\frac{1}{F}\frac{\partial^2 F}{\partial \rho^2} + \rho\frac{1}{F}\frac{\partial F}{\partial \rho} + \rho^2\left[(nk_0)^2 - \beta^2\right] = m^2 = -\frac{1}{T}\frac{\partial^2 T}{\partial \phi^2}$$

Once again, the equation for the angular part looks like simple harmonic oscillation:

$$m^2 = -\frac{1}{T}\frac{\partial^2 T}{\partial \phi^2}$$

$$T(\phi) = \exp(im\phi)$$

Note that since we have polar symmetry, moving 360 degrees around should give us the

same value of T no matter where we start:
$T(\phi) = T(\phi + 2\pi)$

$\exp(im\phi) = \exp(im(\phi + 2\pi)$

$\exp(im\phi) = \exp(im\phi)\exp(im2\pi)$

$1 = \exp(im2\pi)$

The complex exponential is only equal to 1 when its argument is an exact multiple of 2π, which is only possible when m is an integer!

$m = \cdots - 2, -1, 0, 1, 2, \ldots$

So m essentially controls how quickly the phase of the electric field changes as we move around the center of the fiber! If m=0, the phase will be constant throughout the fiber cross section. If m=1, it will change from positive to negative once as we go around the center. For m=2 it will change twice and so on.

Finally, we can find the radial part of the field distribution:

$$\rho^2 \frac{1}{F}\frac{\partial^2 F}{\partial \rho^2} + \rho \frac{1}{F}\frac{\partial F}{\partial \rho} + \rho^2 \left[(nk_0)^2 - \beta^2\right] = m^2$$

$$\rho^2 \frac{\partial^2 F}{\partial \rho^2} + \rho \frac{\partial F}{\partial \rho} + \left[\rho^2 \left[(nk_0)^2 - \beta^2\right] - m^2\right] F = 0$$

<u>Solution in the core</u>

For convenience, let's define

$$X = \rho p = \rho\sqrt{\left[(n_{core}k_0)^2 - \beta^2\right]}.$$

$$\frac{X}{p} = \rho = \rho\sqrt{\left[(n_{core}k_0)^2 - \beta^2\right]}.$$

Here, we are assuming that $(n_{core}k_0)^2 > \beta^2$. This assumption basically says that

"*The propagation constant that light would have if it were propagating in a medium consisting only of the type of glass making up the core is larger than the actual propagation constant of light in the fiber.*"

This should make sense. The light in the fiber experiences both the refractive index of the core and the refractive index of the cladding in some combination, so we would expect that

$$(n_{clad}k)^2 < \beta^2 < (n_{core}k_0)^2.$$

In any case, this allows us to rewrite the equation above as

$$\frac{X^2}{p^2}\frac{\partial^2 F}{\partial X^2}p^2 + \frac{X}{p}\frac{\partial F}{\partial X}p + \left[\frac{X^2}{p^2}p^2 - m^2\right]F = 0$$

$$X^2 \frac{\partial^2 F}{\partial X^2} + X\frac{\partial F}{\partial X} + \left[X^2 - m^2\right]F = 0$$

Now it's easy to see that this is identical to the characteristic equation for Bessel functions:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0$$

Therefore, the solution for F is described by

$F(X) = C_1 J_m(X) + C_2 Y_m(X)$

Plots of the two types of Bessel functions are shown below:

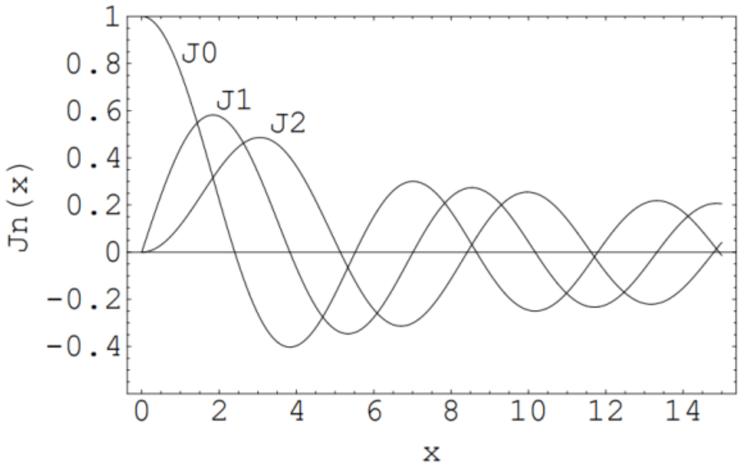


Figure 4.1: Plot of the Bessel Functions of the First Kind, Integer Order

Bessel Functions of the second kind of order **0, 1, 2** are shown in Fig. 4.2.

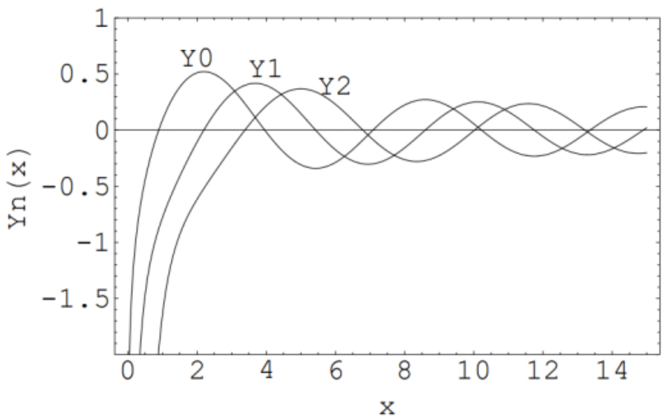


Figure 4.2: Plot of the Bessel Functions of the Second Kind, Integer Order

Note that since power density is proportional to $|F|^2$ and all Y_m have singularities at X=0 (center of the fiber), we must have $C_2 = 0$; otherwise the power in the center of the fiber would be infinitely high!

Therefore, inside the core, we have
$F(X) = C_1 J_m(X)$

Solution in the cladding

Now, we use the refractive index of the cladding:

$$\rho^2 \frac{\partial^2 F}{\partial \rho^2} + \rho \frac{\partial F}{\partial \rho} + \left[\rho^2 \left[\left(n_{clad} k_0 \right)^2 - \beta^2 \right] - m^2 \right] F = 0$$

We can introduce a normalized radial distance here as well, but note that the argument of the square root will be negative!

$$iY = \rho i q = i\rho\sqrt{\left[\beta^2 - \left(n_{clad}k_0\right)^2\right]}$$

$$\rho = \frac{Y}{q} = \frac{Y}{\sqrt{\left[\beta^2 - \left(n_{clad}k_0\right)^2\right]}}$$

since

$$\left(n_{clad}k_0\right)^2 < \beta^2$$

$$Y^2\frac{\partial^2 F}{\partial Y^2} + Y\frac{\partial F}{\partial Y} + \left[\frac{Y^2}{q^2}(-q^2) - m^2\right]F = 0$$

$$Y^2\frac{\partial^2 F}{\partial Y^2} + Y\frac{\partial F}{\partial Y} - \left[Y^2 + m^2\right]F = 0$$

This is the defining equation for the "modified Bessel functions"

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + \nu^2)y = 0$$

The solution is

$$F(\rho) = C_3 I_m(Y) + C_4 K_m(Y).$$

Plots of the two individual types are shown below:

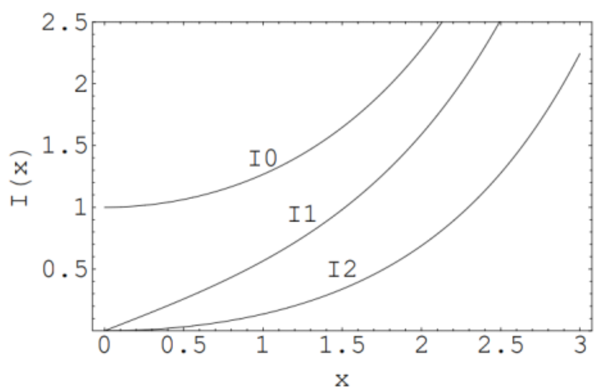


Figure 4.3: Plot of the Modified Bessel Functions of the First Kind, Integer Order

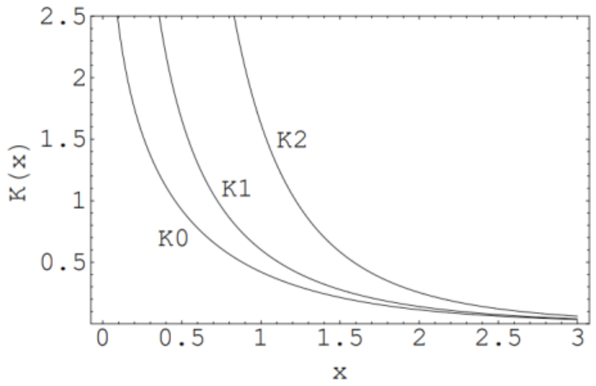


Figure 4.4: Plot of the Modified Bessel Functions of the Second Kind, Integer Order

Obviously, $C_3 = 0$. Otherwise, the power would increase to infinity when we go away from the core!

General solution for F

In summary, we can write that:

$$F(\rho) = \begin{cases} C_{core} \cdot J_m(X), X \leq Rp \\ C_{clad} \cdot K_m(Y), Y \geq Rq \end{cases}$$

Thus, inside the core:

$$E_z = C_{core} \cdot J_m(X)\, e^{im\phi} e^{i\beta z}$$
$$H_z = D_{core} \cdot J_m(X)\, e^{im\phi} e^{i\beta z}\ ,$$

While outside:

$$E_z = C_{clad} \cdot K_m(Y)\, e^{im\phi} e^{i\beta z}$$
$$H_z = D_{clad} \cdot K_m(Y)\, e^{im\phi} e^{i\beta z}\ ,$$

Summary so far

Before we go on to determine the other field components from these expressions, let's remind ourselves of the important quantities and facts that we have discovered:

β: The propagation constant for a certain electric field distribution, which is unaltered as it propagates. It both describes how the electric field changes as we move in the z-direction, but also how quickly it oscillates inside the core and how quickly it drops off when moving radially in the cladding. Currently, its value is unknown!

m: The "angular" number of the field distribution. It basically tells us how many times the sign of the electric field switches from negative to positive as we move around a given cross section of the fiber. Note that it also affects how the field changes in the radial direction through the index of the Bessel functions!

F: The transverse electric field distribution will look like oscillatory Bessel functions of the first kind inside the core and as decaying modified Bessel functions of the 2nd kind in the cladding.

Finding the other field components

To determine the other field components, we must remind ourselves that according to Maxwell's equations,

$\nabla \times \widetilde{\boldsymbol{H}} = -\mathrm{i}\omega\epsilon\widetilde{\boldsymbol{E}}$

And

$\nabla \times \widetilde{\boldsymbol{E}} = \mathrm{i}\omega\mu\widetilde{\boldsymbol{H}}$

$\mu = \mu_0$ since we are in a non-magnetizable material (glass).

There are different approaches to taking the curl of a vector field, but we will use [this one](#):

$$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial\left(\rho A_\varphi\right)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathbf{z}}$$

$$\nabla \times \widetilde{\boldsymbol{H}} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\widehat{\boldsymbol{\rho}} + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\widehat{\boldsymbol{\phi}} + \frac{1}{\rho}\left(\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi}\right)\hat{\mathbf{z}}$$

$$= -\mathrm{i}\omega\epsilon\left(E_\rho\widehat{\boldsymbol{\rho}} + E_\phi\widehat{\boldsymbol{\phi}} + E_z\hat{\mathbf{z}}\right)$$

Splitting this up into 3 equations yields

$$\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} = -\mathrm{i}\omega\epsilon E_\rho$$

$$\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = -\mathrm{i}\omega\epsilon E_\phi$$

$$\frac{1}{\rho}\left(\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi}\right) = -\mathrm{i}\omega\epsilon E_z$$

Conversely, if we use the expression for the curl of E,

$$\nabla \times \widetilde{\boldsymbol{E}} = \left(\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}\right)\widehat{\boldsymbol{\rho}} + \left(\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}\right)\widehat{\boldsymbol{\phi}} + \frac{1}{\rho}\left(\frac{\partial(\rho E_\phi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi}\right)\hat{\mathbf{z}}$$

$$= \mathrm{i}\omega\mu\left(H_\rho\widehat{\boldsymbol{\rho}} + H_\phi\widehat{\boldsymbol{\phi}} + H_z\hat{\mathbf{z}}\right),$$

this yields

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} = \mathrm{i}\omega\mu_0 \mathrm{H}_\rho$$

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = \mathrm{i}\omega\mu_0 \mathrm{H}_\phi$$

$$\frac{1}{\rho}\left(\frac{\partial(\rho \mathrm{E}_\phi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi}\right) = \mathrm{i}\omega\mu_0 \mathrm{H}_z$$

Remember that we know the z-components of the two fields. Therefore, we would like to express the angular and radial components in terms of these.

For example, consider:

$$\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} = -\mathrm{i}\omega\epsilon \mathrm{E}_\rho$$

Solve for the radial E-field component

$$\mathrm{E}_\rho = \frac{1}{-i\omega\epsilon}\left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right).$$

Take

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = \mathrm{i}\omega\mu_0 \mathrm{H}_\phi$$

And solve for the angular magnetic field component:

$$\mathrm{H}_\phi = \frac{1}{\mathrm{i}\omega\mu_0}\left(\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}\right)$$

^Substitute this into the E_ρ equation and simplify (a lot!)

$$\mathrm{E}_\rho = \frac{1}{-i\omega\epsilon}\left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial}{\partial z}\left[\frac{1}{\mathrm{i}\omega\mu_0}\left(\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}\right)\right]\right)$$

Exploit that taking the z-derivative just yields a factor of $i\beta$:

$$\mathrm{E}_\rho = \frac{1}{-i\omega\epsilon}\left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \left[\frac{1}{\mathrm{i}\omega\mu_0}\left(-\beta^2 E_\rho - \mathrm{i}\beta\frac{\partial E_z}{\partial \rho}\right)\right]\right)$$

$$\mathrm{E}_\rho = \frac{1}{-i\omega\epsilon}\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} + \frac{1}{i\omega\epsilon}\frac{1}{\mathrm{i}\omega\mu_0}\left[-\beta^2 E_\rho - \mathrm{i}\beta\frac{\partial E_z}{\partial \rho}\right]$$

$$\mathrm{E}_\rho = \frac{1}{-i\omega\epsilon}\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} + \frac{1}{\omega^2\epsilon\mu_0}\left[\beta^2 E_\rho + \mathrm{i}\beta\frac{\partial E_z}{\partial \rho}\right]$$

$$\mathrm{E}_\rho = \frac{1}{-i\omega\epsilon}\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} + \frac{1}{\omega^2\epsilon\mu_0}\beta^2 E_\rho + \frac{1}{\omega^2\epsilon\mu_0}\mathrm{i}\beta\frac{\partial E_z}{\partial \rho}$$

$$\mathrm{E}_\rho - \frac{1}{\omega^2\epsilon\mu_0}\beta^2 E_\rho = \frac{1}{-i\omega\epsilon}\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} + \frac{1}{\omega^2\epsilon\mu_0}\mathrm{i}\beta\frac{\partial E_z}{\partial \rho}$$

$$\mathrm{E}_\rho\left[1 - \frac{1}{\omega^2\epsilon\mu_0}\beta^2\right] = \frac{1}{-i\omega\epsilon}\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} + \frac{1}{\omega^2\epsilon\mu_0}\mathrm{i}\beta\frac{\partial E_z}{\partial \rho}$$

$$c = \frac{1}{\epsilon_0 \mu_0}$$

$$E_\rho \left[1 - \frac{1}{\frac{\omega^2 n^2}{c^2}} \beta^2\right] = \frac{1}{-i\omega\epsilon} \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} + \frac{1}{\omega^2 \epsilon \mu_0} i\beta \frac{\partial E_z}{\partial \rho}$$

$$E_\rho \left[1 - \frac{1}{(k_0 n)^2} \beta^2\right] = \frac{1}{-i\omega\epsilon} \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} + \frac{1}{\omega^2 \epsilon \mu_0} i\beta \frac{\partial E_z}{\partial \rho}$$

$$E_\rho \frac{1}{(k_0 n)^2} \left[(k_0 n)^2 - \beta^2\right] = \frac{1}{-i\omega\epsilon} \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} + \frac{1}{\omega^2 \epsilon \mu_0} i\beta \frac{\partial E_z}{\partial \rho}$$

$$E_\rho = \frac{(k_0 n)^2}{\left[(k_0 n)^2 - \beta^2\right]} \left(\frac{i}{(k_0 n)^2} \omega \mu_0 \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} + \frac{1}{(k_0 n)^2} i\beta \frac{\partial E_z}{\partial \rho}\right)$$

$$E_\rho = \frac{1}{\left[(k_0 n)^2 - \beta^2\right]} \left(i\omega \mu_0 \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} + i\beta \frac{\partial E_z}{\partial \rho}\right)$$

$$E_\rho = \frac{i}{\left[(k_0 n)^2 - \beta^2\right]} \left(\omega \mu_0 \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} + \beta \frac{\partial E_z}{\partial \rho}\right)$$

Now that we have E_ρ, we can find H_ϕ:

$$H_\phi = \frac{1}{i\omega\mu_0} \left(\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}\right)$$

$$H_\phi = \frac{1}{i\omega\mu_0} \left(i\beta \left[\frac{i}{\left[(k_0 n)^2 - \beta^2\right]} \left(\omega \mu_0 \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} + \beta \frac{\partial E_z}{\partial \rho}\right)\right] - \frac{\partial E_z}{\partial \rho}\right)$$

$$H_\phi = \frac{1}{i\omega\mu_0} \left(\left[\frac{-\beta}{\left[(k_0 n)^2 - \beta^2\right]} \left(\omega \mu_0 \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} + \beta \frac{\partial E_z}{\partial \rho}\right)\right] - \frac{\partial E_z}{\partial \rho}\right)$$

$$H_\phi = \frac{1}{i\omega\mu_0} \left(\left[\left(\frac{-\beta}{\left[(k_0 n)^2 - \beta^2\right]} \omega \mu_0 \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \frac{\beta}{\left[(k_0 n)^2 - \beta^2\right]} \beta \frac{\partial E_z}{\partial \rho}\right)\right] - \frac{\partial E_z}{\partial \rho}\right)$$

$$H_\phi = \frac{1}{i\omega\mu_0} \left(\frac{-\beta}{\left[(k_0 n)^2 - \beta^2\right]} \omega \mu_0 \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \frac{\beta^2}{\left[(k_0 n)^2 - \beta^2\right]} \frac{\partial E_z}{\partial \rho} - \frac{\partial E_z}{\partial \rho}\right)$$

$$H_\phi = \frac{1}{i\omega\mu_0} \left(\frac{-\beta}{\left[(k_0 n)^2 - \beta^2\right]} \omega \mu_0 \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \left[\frac{\beta^2}{\left[(k_0 n)^2 - \beta^2\right]} + 1\right] \frac{\partial E_z}{\partial \rho}\right)$$

$$H_\phi = \frac{1}{i\omega\mu_0} \left(\frac{-\beta}{\left[(k_0 n)^2 - \beta^2\right]} \omega \mu_0 \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \left[\frac{\beta^2}{\left[(k_0 n)^2 - \beta^2\right]} + \frac{\left[(k_0 n)^2 - \beta^2\right]}{\left[(k_0 n)^2 - \beta^2\right]}\right] \frac{\partial E_z}{\partial \rho}\right)$$

$$\mathrm{H}_\phi = \frac{1}{\mathrm{i}\omega\mu_0}\left(\frac{-\beta}{\left[(k_0 n)^2 - \beta^2\right]}\,\omega\mu_0 \frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{(k_0 n)^2}{(k_0 n)^2 - \beta^2}\frac{\partial E_z}{\partial \rho}\right)$$

$$\mathrm{H}_\phi = \frac{1}{\mathrm{i}\omega\mu_0((k_0 n)^2 - \beta^2)}\left(-\beta\,\omega\mu_0 \frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - (k_0 n)^2 \frac{\partial E_z}{\partial \rho}\right)$$

$$\mathrm{H}_\phi = \frac{i}{((k_0 n)^2 - \beta^2)}\left(\frac{\beta}{\rho}\frac{\partial H_z}{\partial \phi} + \frac{1}{\omega\mu_0}(k_0 n)^2 \frac{\partial E_z}{\partial \rho}\right)$$

$$\mathrm{H}_\phi = \frac{i}{((k_0 n)^2 - \beta^2)}\left(\frac{\beta}{\rho}\frac{\partial H_z}{\partial \phi} + \frac{1}{\omega\mu_0}(k_0 n)^2 \frac{\partial E_z}{\partial \rho}\right)$$

$$\frac{1}{\omega\mu_0}\ k_0^2 n^2 = \frac{1}{\omega\mu_0}\ \frac{\omega^2}{c^2} n^2 = \frac{1}{\mu_0}\ \frac{\omega}{c^2} n^2$$

$$\mathrm{H}_\phi = \frac{i}{((k_0 n)^2 - \beta^2)}\left(\frac{\beta}{\rho}\frac{\partial H_z}{\partial \phi} + \frac{1}{\mu_0}\ \frac{\omega}{c^2} n^2 \frac{\partial E_z}{\partial \rho}\right)$$

$$\mathrm{H}_\phi = \frac{i}{((k_0 n)^2 - \beta^2)}\left(\frac{\beta}{\rho}\frac{\partial H_z}{\partial \phi} + \frac{n^2 k_0}{\mu_0 c}\ \frac{\partial E_z}{\partial \rho}\right)$$

Now we can find the angular component of the E-field

$$\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = -\mathrm{i}\omega\epsilon \mathrm{E}_\phi$$

$$\mathrm{E}_\phi = \frac{1}{-\mathrm{i}\omega\epsilon}\left[\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right]$$

Substitute in the radial component of the magnetic field:

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} = \mathrm{i}\omega\mu_0 \mathrm{H}_\rho$$

$$\mathrm{H}_\rho = \frac{1}{\mathrm{i}\omega\mu_0}\left[\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}\right]$$

$$\mathrm{E}_\phi = \frac{1}{-\mathrm{i}\omega\epsilon}\left[\mathrm{i}\beta\left(\frac{1}{\mathrm{i}\omega\mu_0}\left[\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}\right]\right) - \frac{\partial H_z}{\partial \rho}\right]$$

$$\mathrm{E}_\phi = \frac{1}{-\mathrm{i}\omega\epsilon}\left[\mathrm{i}\beta\left(\frac{1}{\mathrm{i}\omega\mu_0}\left[\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \mathrm{i}\beta E_\phi\right]\right) - \frac{\partial H_z}{\partial \rho}\right]$$

$$\mathrm{E}_\phi = \frac{1}{-\mathrm{i}\omega\epsilon}\left[\frac{\beta}{\omega\mu_0}\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\beta}{\omega\mu_0}\mathrm{i}\beta E_\phi - \frac{\partial H_z}{\partial \rho}\right]$$

$$\mathrm{E}_\phi = \frac{1}{-\mathrm{i}\omega\epsilon}\left[\frac{\beta}{\omega\mu_0}\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \mathrm{i}\frac{\beta^2}{\omega\mu_0}E_\phi - \frac{\partial H_z}{\partial \rho}\right]$$

$$\mathrm{E}_\phi = \frac{1}{-\mathrm{i}\omega\epsilon}\frac{\beta}{\omega\mu_0}\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{1}{-\mathrm{i}\omega\epsilon}\mathrm{i}\frac{\beta^2}{\omega\mu_0}E_\phi - \frac{1}{-\mathrm{i}\omega\epsilon}\frac{\partial H_z}{\partial \rho}$$

$$\mathrm{E}_\phi = \frac{i}{\omega\epsilon}\frac{\beta}{\omega\mu_0}\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} + \frac{1}{\omega\epsilon}\frac{\beta^2}{\omega\mu_0}E_\phi - \frac{i}{\omega\epsilon}\frac{\partial H_z}{\partial \rho}$$

$$\mathrm{E}_\phi - \frac{1}{\omega\epsilon}\frac{\beta^2}{\omega\mu_0}E_\phi = \frac{i}{\omega\epsilon}\frac{\beta}{\omega\mu_0}\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{i}{\omega\epsilon}\frac{\partial H_z}{\partial \rho}$$

$$\mathrm{E}_\phi\left[1 - \frac{1}{\omega\epsilon}\frac{\beta^2}{\omega\mu_0}\right] = \frac{i}{\omega\epsilon}\frac{\beta}{\omega\mu_0}\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{i}{\omega\epsilon}\frac{\partial H_z}{\partial \rho}$$

$$\mathrm{E}_\phi\left[1 - \frac{\beta^2}{\omega^2\epsilon\mu_0}\right] = \frac{i}{\omega^2\epsilon}\frac{\beta}{\mu_0}\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{i}{\omega\epsilon}\frac{\partial H_z}{\partial \rho}$$

$$\mathrm{E}_\phi\left[1 - \frac{\beta^2}{\omega^2 n^2\epsilon_0\mu_0}\right] = \frac{i}{\omega^2 n^2\epsilon_0}\frac{\beta}{\mu_0}\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{i}{\omega\epsilon}\frac{\partial H_z}{\partial \rho}$$

$$\mathrm{E}_\phi\left[1 - \frac{\beta^2 c^2}{\omega^2 n^2}\right] = \frac{ic^2}{\omega^2 n^2}\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{i}{\omega\epsilon}\frac{\partial H_z}{\partial \rho}$$

$$\mathrm{E}_\phi\left[1 - \frac{\beta^2 c^2}{\omega^2 n^2}\right] = \frac{ic^2}{\omega^2 n^2}\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{i}{\omega n^2\epsilon_0}\frac{\partial H_z}{\partial \rho}$$

$$\mathrm{E}_\phi\left[1 - \frac{\beta^2}{k_0^2 n^2}\right] = \frac{i}{k_0^2 n^2}\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{i}{k_0 n^2 c\epsilon_0}\frac{\partial H_z}{\partial \rho}$$

$$\mathrm{E}_\phi\left[\frac{k_0^2 n^2 - \beta^2}{k_0^2 n^2}\right] = \frac{i}{k_0^2 n^2}\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{i\mu_0 c}{k_0 n^2}\frac{\partial H_z}{\partial \rho}$$

$$\mathrm{E}_\phi\left[\frac{k_0^2 n^2 - \beta^2}{1}\right] = \frac{i}{1}\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{i\mu_0\, c k_0^2 n^2}{k_0 n^2}\frac{\partial H_z}{\partial \rho}$$

$$\mathrm{E}_\phi\left[k_0^2 n^2 - \beta^2\right] = \mathrm{i}\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - i\mu_0 c k_0\frac{\partial H_z}{\partial \rho}$$

$$\mathrm{E}_\phi = \frac{i}{k_0^2 n^2 - \beta^2}\left[\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \mu_0 c k_0\frac{\partial H_z}{\partial \rho}\right]$$

Finally, we can find the radial part of the magnetic field

$$\mathrm{H}_\rho = \frac{1}{\mathrm{i}\omega\mu_0}\left[\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \mathrm{i}\beta\left(\frac{i}{k_0^2 n^2 - \beta^2}\left[\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \mu_0 c k_0\frac{\partial H_z}{\partial \rho}\right]\right)\right]$$

$$\mathrm{H}_\rho = \frac{1}{\mathrm{i}\omega\mu_0}\left[\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} + \frac{\beta}{k_0^2 n^2 - \beta^2}\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\beta}{k_0^2 n^2 - \beta^2}\mu_0 c k_0\frac{\partial H_z}{\partial \rho}\right]$$

$$H_\rho = \frac{1}{i\omega\mu_0}\left[\frac{\partial E_z}{\partial\phi}\left(\frac{1}{\rho} + \frac{\beta}{k_0^2 n^2 - \beta^2}\frac{\beta}{\rho}\right) - \frac{\beta}{k_0^2 n^2 - \beta^2}\mu_0 c k_0 \frac{\partial H_z}{\partial\rho}\right]$$

$$H_\rho = \frac{1}{i\omega\mu_0}\left[\frac{1}{\rho}\frac{\partial E_z}{\partial\phi}\left(1 + \frac{\beta^2}{k_0^2 n^2 - \beta^2}\right) - \frac{\beta}{k_0^2 n^2 - \beta^2}\mu_0 c k_0 \frac{\partial H_z}{\partial\rho}\right]$$

$$H_\rho = \frac{1}{i\omega\mu_0}\left[\frac{1}{\rho}\frac{\partial E_z}{\partial\phi}\frac{k_0^2 n^2}{k_0^2 n^2 - \beta^2} - \frac{\beta}{k_0^2 n^2 - \beta^2}\mu_0 c k_0 \frac{\partial H_z}{\partial\rho}\right]$$

$$H_\rho = \frac{1}{k_0^2 n^2 - \beta^2}\frac{1}{i\omega\mu_0}\left[\frac{1}{\rho}\frac{\partial E_z}{\partial\phi}k_0^2 n^2 - \beta\mu_0 c k_0 \frac{\partial H_z}{\partial\rho}\right]$$

$$H_\rho = \frac{-i}{k_0^2 n^2 - \beta^2}\frac{1}{k_0 c\mu_0}\left[\frac{1}{\rho}\frac{\partial E_z}{\partial\phi}k_0^2 n^2 - \beta\mu_0 c k_0 \frac{\partial H_z}{\partial\rho}\right]$$

$$H_\rho = \frac{-i}{k_0^2 n^2 - \beta^2}\left[\frac{1}{\rho}\frac{\partial E_z}{\partial\phi}\frac{k_0^2 n^2}{k_0 c\mu_0} - \frac{\beta\mu_0 c k_0}{k_0 c\mu_0}\frac{\partial H_z}{\partial\rho}\right]$$

$$H_\rho = \frac{i}{k_0^2 n^2 - \beta^2}\left[\beta\frac{\partial H_z}{\partial\rho} - \frac{n^2 k_0}{\mu_0 c}\frac{1}{\rho}\frac{\partial E_z}{\partial\phi}\right]$$

<u>Summary of field components</u>

$$E_\rho = \frac{i}{\left[(k_0 n)^2 - \beta^2\right]}\left(\omega\mu_0\frac{1}{\rho}\frac{\partial H_z}{\partial\phi} + \beta\frac{\partial E_z}{\partial\rho}\right)$$

$$E_\phi = \frac{i}{k_0^2 n^2 - \beta^2}\left[\frac{\beta}{\rho}\frac{\partial E_z}{\partial\phi} - \mu_0 c k_0 \frac{\partial H_z}{\partial\rho}\right]$$

$$E_z = \begin{cases} C_{core}\cdot J_m(X)\, e^{im\phi}e^{i\beta z}, \rho \le R \\ C_{clad}\cdot K_m(Y)\, e^{im\phi}e^{i\beta z}, \rho \ge R \end{cases}$$

$$H_\rho = \frac{i}{k_0^2 n^2 - \beta^2}\left[\beta\frac{\partial H_z}{\partial\rho} - \frac{n^2 k_0}{\mu_0 c}\frac{1}{\rho}\frac{\partial E_z}{\partial\phi}\right]$$

$$H_\phi = \frac{i}{\left((k_0 n)^2 - \beta^2\right)}\left(\frac{\beta}{\rho}\frac{\partial H_z}{\partial\phi} + \frac{n^2 k_0}{\mu_0 c}\frac{\partial E_z}{\partial\rho}\right)$$

$$H_z = \begin{cases} D_{core}\cdot J_m(X)\, e^{im\phi}e^{i\beta z}, \rho \le R \\ D_{clad}\cdot K_m(Y)\, e^{im\phi}e^{i\beta z}, \rho \ge R \end{cases},$$

where

$$X = \rho p = \rho\sqrt{(n_{core}k_0)^2 - \beta^2}$$
$$Y = \rho q = \rho\sqrt{\beta^2 - (n_{clad}k_0)^2}$$

Inside

Substituting the expressions for E_z and H_z for $\rho \le R$ into the equations above yields

$$E_\rho = \frac{i}{p^2}\left(D_{core}\cdot\omega\mu_0\frac{im}{\rho}J_m(\rho p) + C_{core}\cdot\beta p\, J_m'(\rho p)\right)e^{im\phi}e^{i\beta z}$$

$$E_\phi = \frac{i}{p^2}\left[C_{core}\cdot\beta\frac{im}{\rho}J_m(\rho p) - D_{core}\cdot\mu_0 c k_0 p J'_m(\rho p)\right]e^{im\phi}e^{i\beta z}$$

$$E_z = C_{core}\cdot J_m(\rho p)\, e^{im\phi}e^{i\beta z}$$

$$H_\rho = \frac{i}{p^2}\left[D_{core}\cdot\beta p J'_m(\rho p) - C_{core}\cdot\frac{n_{core}^2 k_0}{\mu_0 c}\frac{im}{\rho}J_m(\rho p)\right]e^{im\phi}e^{i\beta z}$$

$$H_\phi = \frac{i}{p^2}\left(D_{core}\cdot\beta\frac{im}{\rho}J_m(\rho p) + C_{core}\cdot\frac{n_{core}^2 k_0}{\mu_0 c}\,pJ'_m(\rho p)\right)e^{im\phi}e^{i\beta z}$$

$$H_z = D_{core}\cdot J_m(\rho p)\, e^{im\phi}e^{i\beta z}$$

Outside

$$E_\rho = \frac{-i}{q^2}\left(D_{clad}\cdot\omega\mu_0\frac{im}{\rho}K_m(\rho q) + C_{clad}\cdot\beta q K_m'(\rho q)\right)e^{im\phi}e^{i\beta z}$$

$$E_\phi = \frac{-i}{q^2}\left[C_{clad}\cdot\beta\frac{im}{\rho}K_m(\rho q) - D_{clad}\cdot\mu_0 c k_0 q\, K'_m(\rho q)\right]e^{im\phi}e^{i\beta z}$$

$$E_z = C_{clad}\cdot K_m(\rho q)\, e^{im\phi}e^{i\beta z}$$

$$H_\rho = \frac{-i}{q^2}\left[D_{clad}\cdot\beta q\, K_m'(\rho q) - C_{clad}\cdot\frac{n_{clad}^2 k_0}{\mu_0 c}\frac{im}{\rho}K_m(\rho q)\right]e^{im\phi}e^{i\beta z}$$

$$H_\phi = \frac{-i}{q^2}\left(D_{clad}\cdot\beta\frac{im}{\rho}K_m(\rho q) + C_{clad}\cdot\frac{n_{clad}^2 k_0}{\mu_0 c}\,qK_m'(\rho q)\right)e^{im\phi}e^{i\beta z}$$

$$H_z = D_{clad}\cdot K_m(\rho q)\, e^{im\phi}e^{i\beta z}$$

Let's remind ourselves why we care about this: We want all the field distributions that remain unchanged as they propagate down the length of a step-index fiber. The field is partially located in the core and partially in the cladding. Therefore, we need the field to be continuous at the core/cladding interface.

Strictly speaking, we should have $n_{core}^2 E_{\rho,inside}(R) = n_{clad}^2 E_{\rho,outside}(R)$, but since the refractive indices are almost identical we can assume that the fields on either side must simply be identical at the boundary.

Let's first consider the continuity of the z-component of the E-field at the interface:

$$C_{core}\cdot J_m(Rp)\, e^{im\phi}e^{i\beta z} = C_{clad}\cdot K_m(Rq)\, e^{im\phi}e^{i\beta z}$$

$$C_{core}\frac{J_m(Rp)}{K_m(Rq)} = C_{clad}$$

We don't know the two coefficients yet, but we can see that they must be related. For convenience, define

$$P = Rp, \qquad Q = Rq$$

and note that

$$P^2 + Q^2 = R^2\left(\left(n_{core}k_0\right)^2 - \beta^2\right) + R^2\left(\beta^2 - \left(n_{clad}k_0\right)^2\right)$$
$$P^2 + Q^2 = R^2k_0^2\left(n_{core}{}^2 - n_{clad}{}^2\right) = \left[\frac{2\pi}{\lambda_0}R\sqrt{n_{core}{}^2 - n_{clad}{}^2}\right]^2 = V^2$$

$$V^2 = k_0^2R^2(n_{core}{}^2 - n_{clad}{}^2)$$

is a <u>mode independent constant determined only by the wavelength of the launched light and the design parameters of the fiber!!!</u>

This is nice because it means that we do not have to solve for both P and Q in the process of determining β, since

$$P = R\sqrt{\left(n_{core}k_0\right)^2 - \beta^2}$$
$$Q = \sqrt{V^2 - P^2} = \sqrt{V^2 - R^2(\left(n_{core}k_0\right)^2 - \beta^2)}$$

For now, let's write:

$$C_{core}\frac{J_m(P)}{K_m(Q)} = C_{clad}$$

And do the same thing for the z-component of the magnetic field:

$$D_{core}\cdot\frac{J_m(P)}{K_m(Q)} = D_{clad}$$

Now let's look at the two angular components. For the angular electric field:

$$\frac{i}{p^2}\left[C_{core}\cdot\beta\frac{im}{R}J_m(P) - D_{core}\cdot\mu_0ck_0pJ'_m(P)\right]$$
$$= \frac{-i}{q^2}\left[C_{clad}\cdot\beta\frac{im}{R}K_m(Q) - D_{clad}\cdot\mu_0ck_0q\,K'_m(Q)\right]$$

$$\frac{1}{P^2}\left[C_{core}\cdot\beta\frac{im}{R}J_m(P) - D_{core}\cdot\mu_0ck_0pJ'_m(P)\right]$$
$$= \frac{-1}{Q^2}\left[C_{core}\frac{J_m(P)}{K_m(Q)}\cdot\beta\frac{im}{R}K_m(Q) - D_{core}\cdot\frac{J_m(P)}{K_m(Q)}\cdot\mu_0ck_0q\,K'_m(Q)\right]$$

$$C_{core}\cdot\frac{1}{P^2}\beta\frac{im}{R}J_m(P) - D_{core}\cdot\frac{1}{P^2}\mu_0ck_0pJ'_m(P)$$
$$= -C_{core}\frac{J_m(P)}{K_m(Q)}\cdot\frac{1}{Q^2}\beta\frac{im}{R}K_m(Q) + D_{core}\cdot\frac{1}{Q^2}\frac{J_m(P)}{K_m(Q)}\cdot\mu_0ck_0q\,K'_m(Q)$$

$$C_{core}\cdot\frac{1}{P^2}\beta\frac{im}{R}J_m(P) + C_{core}\frac{J_m(P)}{K_m(Q)}\cdot\frac{1}{Q^2}\beta\frac{im}{R}K_m(Q)$$
$$= D_{core}\cdot\frac{1}{Q^2}\frac{J_m(P)}{K_m(Q)}\cdot\mu_0ck_0q\,K'_m(Q) + D_{core}\cdot\frac{1}{P^2}\mu_0ck_0pJ'_m(P)$$

$$C_{core}\left(\frac{1}{P^2}\beta\frac{im}{R}\,J_m(P)+\frac{J_m(P)}{K_m(Q)}\cdot\frac{1}{Q^2}\beta\frac{im}{R}\,K_m(Q)\right)$$
$$=\;D_{core}\left(\frac{1}{Q^2}\frac{J_m(P)}{K_m(Q)}\cdot\mu_0ck_0q\,K_m'(Q)+\frac{1}{P^2}\mu_0ck_0pJ'_m(P)\right)$$

$$C_{core}\left(\frac{1}{P^2}\beta\frac{im}{R}\,J_m(P)+\frac{J_m(P)}{K_m(Q)}\cdot\frac{1}{Q^2}\beta\frac{im}{R}\,K_m(Q)\right)$$
$$=\;D_{core}\left(\frac{1}{Q^2}\frac{J_m(P)}{K_m(Q)}\cdot\mu_0ck_0q\,K_m'(Q)+\frac{1}{P^2}\mu_0ck_0pJ'_m(P)\right)$$

$$C_{core}\left(\frac{1}{P^2}\beta\frac{im}{R}\,J_m(P)+\frac{J_m(P)}{K_m(Q)}\cdot\frac{1}{Q^2}\beta\frac{im}{R}\,K_m(Q)\right)$$
$$-\,D_{core}\left(\frac{1}{Q^2}\frac{J_m(P)}{K_m(Q)}\cdot\mu_0ck_0q\,K_m'(Q)+\frac{1}{P^2}\mu_0ck_0pJ'_m(P)\right)=0$$

$$C_{core}\left(Q^2\beta im\,J_m(P)+\frac{J_m(P)}{K_m(Q)}\cdot P^2\beta im\,K_m(Q)\right)-D_{core}\left(P^2\frac{J_m(P)}{K_m(Q)}\cdot\mu_0ck_0Q\,K_m'(Q)+Q^2P\mu_0ck_0J'_m(P)\right)$$
$$=0$$

$$C_{core}\beta im\left(Q^2\,J_m(P)+P^2\frac{J_m(P)}{K_m(Q)}\,K_m(Q)\right)-D_{core}\mu_0ck_0\left(P^2Q\frac{J_m(P)}{K_m(Q)}\,K_m'(Q)+Q^2PJ'_m(P)\right)=0$$

$$C_{core}\beta imJ_m(P)(Q^2\;+P^2)-D_{core}\mu_0ck_0PQ\left(P\frac{J_m(P)}{K_m(Q)}\,K_m'(Q)+QJ'_m(P)\right)=0$$

$$C_{core}\;\cdot\beta im\,(Q^2\;+P^2)J_m(P)-D_{core}\cdot\mu_0ck_0P^2Q^2J_m(P)\left(\frac{K_m'(Q)}{QK_m(Q)}\;+\frac{J'_m(P)}{PJ_m(P)}\right)=0$$

$$C_{core}\;\cdot\beta m\,(Q^2\;+P^2)+D_{core}\cdot i\mu_0ck_0P^2Q^2\left(\frac{K_m'(Q)}{QK_m(Q)}\;+\frac{J'_m(P)}{PJ_m(P)}\right)=0$$

$$C_{core}\;\cdot\beta m\left(\frac{1}{P^2}\;+\frac{1}{Q^2}\right)+D_{core}\cdot i\mu_0ck_0\left(\frac{K_m'(Q)}{QK_m(Q)}\;+\frac{J'_m(P)}{PJ_m(P)}\right)=0$$

For the angular magnetic field:

$$\frac{i}{p^2}\left(D_{core}\,\cdot\,\beta\frac{im}{R}J_m(P)+C_{core}\cdot\frac{n_{core}^2k_0}{\mu_0c}\;pJ_m'(P)\right)$$
$$=\frac{-i}{q^2}\left(D_{clad}\,\cdot\,\beta\frac{im}{R}\;K_m(Q)+C_{clad}\cdot\frac{n_{clad}^2k_0}{\mu_0c}\;qK_m{}'(Q)\right)$$

$$\frac{1}{P^2}\left(D_{core}\cdot\;\beta\frac{im}{R}J_m(P)+C_{core}\cdot\frac{n_{core}^2k_0}{\mu_0c}\;pJ_m'(P)\right)$$
$$=\frac{-1}{Q^2}\left(D_{core}\cdot\frac{J_m(P)}{K_m(Q)}\cdot\;\beta\frac{im}{R}\;K_m(Q)+C_{core}\frac{J_m(P)}{K_m(Q)}\cdot\frac{n_{clad}^2k_0}{\mu_0c}\;qK_m{}'(Q)\right)$$

$$D_{core}\cdot\;\beta\frac{1}{P^2}imJ_m(P)+C_{core}\cdot\frac{1}{P^2}\frac{n_{core}^2k_0}{\mu_0c}\;PJ_m'(P)$$
$$=\frac{-1}{Q^2}D_{core}\cdot\frac{J_m(P)}{K_m(Q)}\cdot\;\beta im\;K_m(Q)-\frac{1}{Q^2}C_{core}\frac{J_m(P)}{K_m(Q)}\cdot\frac{n_{clad}^2k_0}{\mu_0c}QK_m{}'(Q)$$

$$D_{core}\cdot\;\beta\frac{1}{P^2}imJ_m(P)+C_{core}\cdot\frac{1}{P}\frac{n_{core}^2k_0}{\mu_0c}\;J_m'(P)$$
$$=\frac{-1}{Q^2}D_{core}\cdot\frac{J_m(P)}{K_m(Q)}\cdot\;\beta im\;K_m(Q)-\frac{1}{Q}C_{core}\frac{J_m(P)}{K_m(Q)}\cdot\frac{n_{clad}^2k_0}{\mu_0c}K_m{}'(Q)$$

$$D_{core}\cdot\ \beta\frac{1}{P^2}imJ_m(P)+\frac{1}{Q^2}D_{core}\cdot\frac{J_m(P)}{K_m(Q)}\cdot\ \beta im\,K_m(Q)$$
$$=-\frac{1}{Q}C_{core}\frac{J_m(P)}{K_m(Q)}\cdot\frac{n_{clad}^2k_0}{\mu_0c}K_m'(Q)-C_{core}\cdot\frac{1}{P}\frac{n_{core}^2k_0}{\mu_0c}J_m'(P)$$

$$D_{core}\left[\beta\frac{1}{P^2}imJ_m(P)+\frac{1}{Q^2}\cdot\frac{J_m(P)}{K_m(Q)}\cdot\ \beta im\,K_m(Q)\right]$$
$$=-C_{core}\left[\frac{1}{Q}\frac{J_m(P)}{K_m(Q)}\cdot\frac{n_{clad}^2k_0}{\mu_0c}K_m'(Q)+\frac{1}{P}\frac{n_{core}^2k_0}{\mu_0c}J_m'(P)\right]$$

$$D_{core}im\beta\left[\frac{1}{P^2}J_m(P)+\frac{1}{Q^2}\cdot\frac{J_m(P)}{K_m(Q)}\cdot\ \ K_m(Q)\right]+C_{core}\frac{n_{clad}^2k_0}{\mu_0c}\left[\frac{1}{Q}\frac{J_m(P)}{K_m(Q)}\cdot K_m'(Q)+\frac{1}{P}\frac{n_{core}^2}{n_{clad}^2}J_m'(P)\right]$$
$$=0$$

$$D_{core}im\beta J_m(P)\left[\frac{1}{P^2}+\frac{1}{Q^2}\right]+C_{core}\frac{n_{clad}^2k_0}{\mu_0c}J_m(P)\left[\frac{1}{Q}\frac{1}{K_m(Q)}\cdot K_m'(Q)+\frac{1}{P}\frac{n_{core}^2}{n_{clad}^2}\frac{J_m'(P)}{J_m(P)}\right]=0$$

$$C_{core}\cdot\frac{n_{clad}^2k_0}{\mu_0c}\left[\frac{K_m'(Q)}{QK_m(Q)}+\frac{n_{core}^2}{n_{clad}^2}\frac{J_m'(P)}{PJ_m(P)}\right]+D_{core}\cdot im\beta\left[\frac{1}{P^2}+\frac{1}{Q^2}\right]=0$$

Now, we can think about the two most recent green equations as a matrix equation:

$$\begin{pmatrix}\beta m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right) & i\mu_0ck_0\left(\frac{K_m'(Q)}{QK_m(Q)}+\frac{J'_m(P)}{PJ_m(P)}\right)\\ \frac{n_{clad}^2k_0}{\mu_0c}\left[\frac{K_m'(Q)}{QK_m(Q)}+\frac{n_{core}^2}{n_{clad}^2}\frac{J_m'(P)}{PJ_m(P)}\right] & im\beta\left[\frac{1}{P^2}+\frac{1}{Q^2}\right]\end{pmatrix}\begin{pmatrix}C_{core}\\ D_{core}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

For this to have a solution, the determinant must be zero:

$$i\left[\beta m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)\right]^2-\frac{n_{clad}^2k_0}{\mu_0c}\left[\frac{K_m'(Q)}{QK_m(Q)}+\frac{n_{core}^2}{n_{clad}^2}\frac{J_m'(P)}{PJ_m(P)}\right]i\mu_0ck_0\left(\frac{K_m'(Q)}{QK_m(Q)}+\frac{J'_m(P)}{PJ_m(P)}\right)=0$$

$$\left[\beta m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)\right]^2-n_{clad}^2k_0^2\left[\frac{K_m'(Q)}{QK_m(Q)}+\frac{n_{core}^2}{n_{clad}^2}\frac{J_m'(P)}{PJ_m(P)}\right]\left(\frac{K_m'(Q)}{QK_m(Q)}+\frac{J'_m(P)}{PJ_m(P)}\right)=0$$

<u>General equation that the propagation constant of a field distribution must satisfy to be a mode:</u>

$$\left[\frac{\beta}{n_{clad}k_0}m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)\right]^2=\left[\frac{K_m'(Q)}{QK_m(Q)}+\frac{n_{core}^2}{n_{clad}^2}\frac{J_m'(P)}{PJ_m(P)}\right]\left(\frac{K_m'(Q)}{QK_m(Q)}+\frac{J'_m(P)}{PJ_m(P)}\right)$$

<u>Weakly guided case</u>

For fibers with a small refractive index difference, $\frac{n_{core}}{n_{clad}}\approx 1$ and $\beta\approx k_0n_{core}$

$$\left[m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)\right]^2=\left[\frac{K_m'(Q)}{QK_m(Q)}+\frac{J_m'(P)}{PJ_m(P)}\right]\left(\frac{K_m'(Q)}{QK_m(Q)}+\frac{J'_m(P)}{PJ_m(P)}\right)$$

$$\pm m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)=\frac{K_m'(Q)}{QK_m(Q)}+\frac{J'_m(P)}{PJ_m(P)}$$

Note that we get two different sets of solutions depending on the choice of sign!

Our strategy for finding solutions is therefore

1) Select the MINUS case
2) Select m=0
3) For a given value of V, find all values of P and $Q=\sqrt{V^2-P^2}$, which solve the equation above.
4) Increment m by 1 and repeat 3 until no more solutions are found
5) Select the PLUS case and repeat 2) to 4).

To make finding solutions easier, we first choose the "minus case" and rewrite the equation using the following properties of Bessel functions:

me755_web.tex (uwaterloo.ca)

$$J_\nu'(x) = J_{\nu-1}(x)-\frac{\nu}{x}J_\nu(x)$$

$$K_\nu'(x) = -K_{\nu-1}(x)-\frac{\nu}{x}K_\nu(x)$$

$$-m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)=\frac{K_m'(Q)}{QK_m(Q)}+\frac{J'_m(P)}{PJ_m(P)}$$

$$-m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)=\frac{-K_{m-1}(Q)-\frac{m}{Q}K_m(Q)}{QK_m(Q)}+\frac{J_{m-1}(P)-\frac{m}{P}J_m(P)}{PJ_m(P)}$$

$$-m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)=-\frac{K_{m-1}(Q)}{QK_m(Q)}-\frac{m}{Q^2}+\frac{J_{m-1}(P)}{PJ_m(P)}-\frac{m}{P^2}$$

$$-m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)=\frac{J_{m-1}(P)}{PJ_m(P)}-\frac{K_{m-1}(Q)}{QK_m(Q)}-m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)$$

$$0=\frac{J_{m-1}(P)}{PJ_m(P)}-\frac{K_{m-1}(Q)}{QK_m(Q)}$$

To get a "nice" equation for the "plus-case", we use different (but equally valid) properties of the derivatives of Bessel functions:

$$J_\nu'(x) = \frac{\nu}{x}J_\nu(x)-J_{\nu+1}(x)$$

$$K_\nu'(x) = \frac{\nu}{x}K_\nu(x)-K_{\nu+1}(x)$$

Modified Bessel function of the first kind: Introduction to the Bessel functions (wolfram.com)

$$K_{-\nu}(z)=K_\nu(z).$$

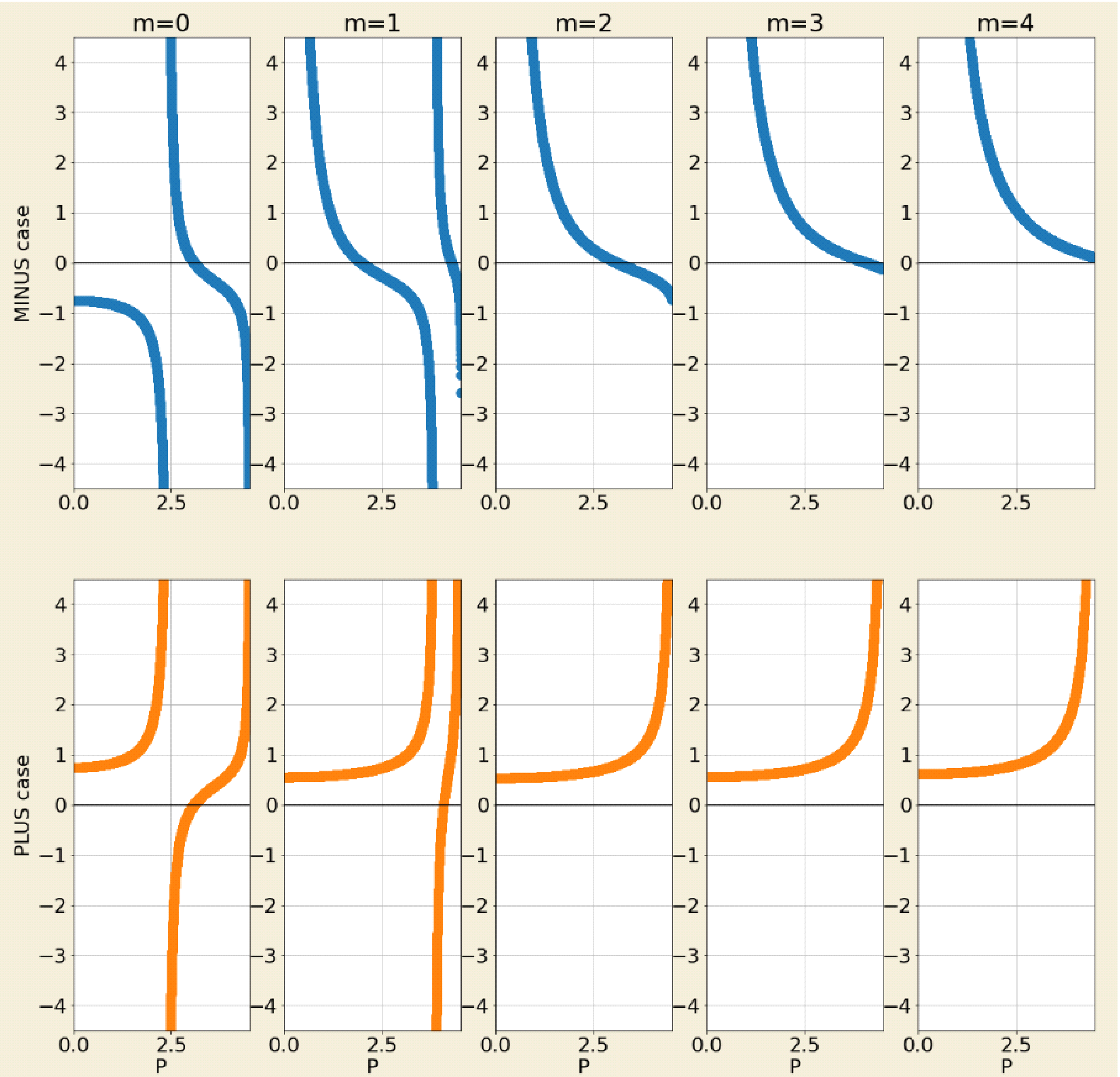
$$m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)=\frac{\left[\frac{m}{Q}K_m(Q)-K_{m+1}(Q)\right]}{QK_m(Q)}+\frac{\left[\frac{m}{P}J_m(P)-J_{m+1}(P)\right]}{PJ_m(P)}$$

$$m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)=\frac{m}{Q^2}-\frac{K_{m+1}(Q)}{QK_m(Q)}+\frac{m}{P^2}-\frac{J_{m+1}(P)}{PJ_m(P)}$$

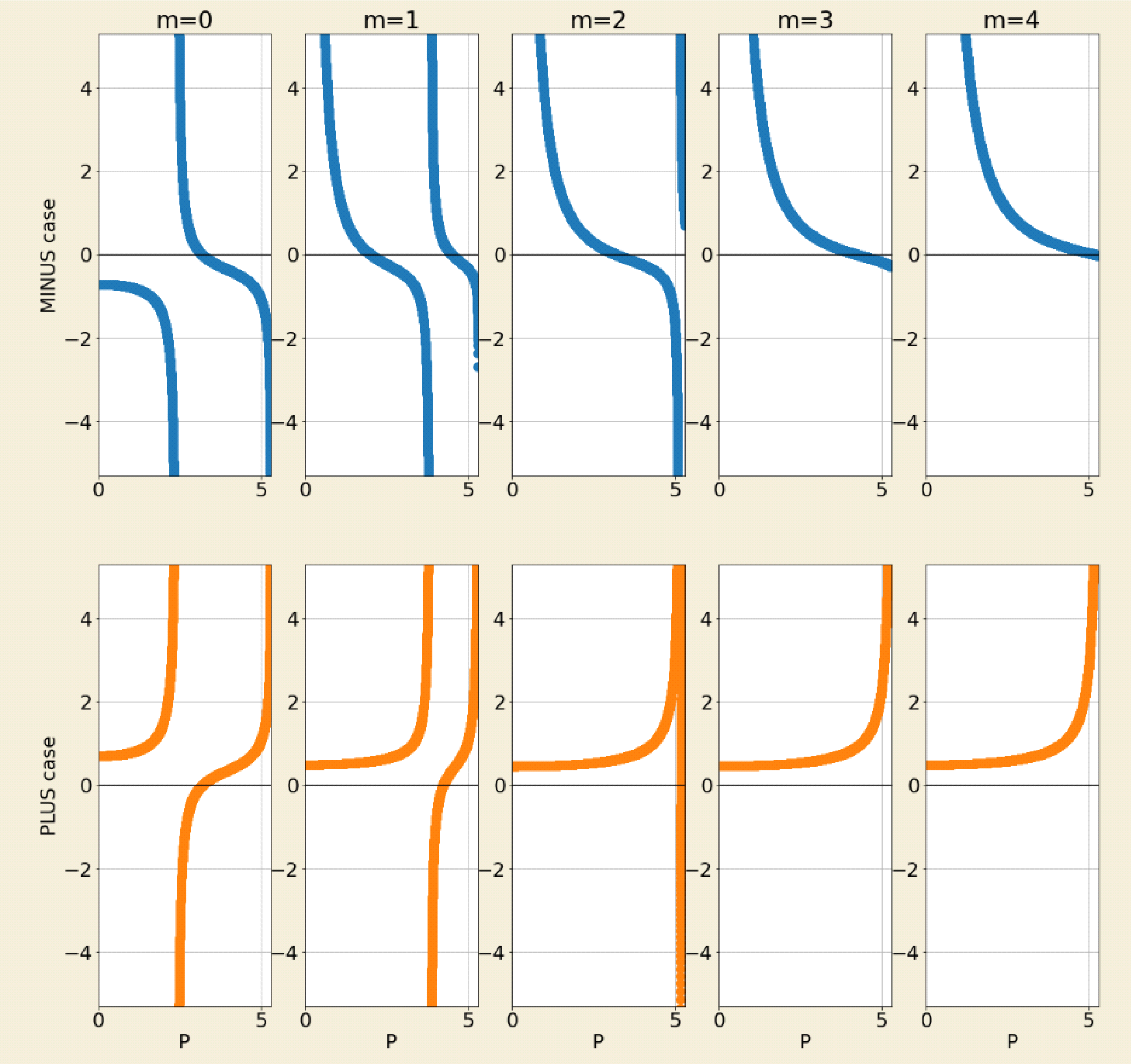
$$m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)=m\left(\frac{1}{P^2}+\frac{1}{Q^2}\right)-\frac{K_{m+1}(Q)}{QK_m(Q)}-\frac{J_{m+1}(P)}{PJ_m(P)}$$

$$0=\frac{K_{m+1}(Q)}{QK_m(Q)}+\frac{J_{m+1}(P)}{PJ_m(P)}$$

We can plot these two expressions numerically for different values of m and P and determine when they are equal to zero. Below, we see that for V=4.5 (maximum P value), we have 7 solutions.

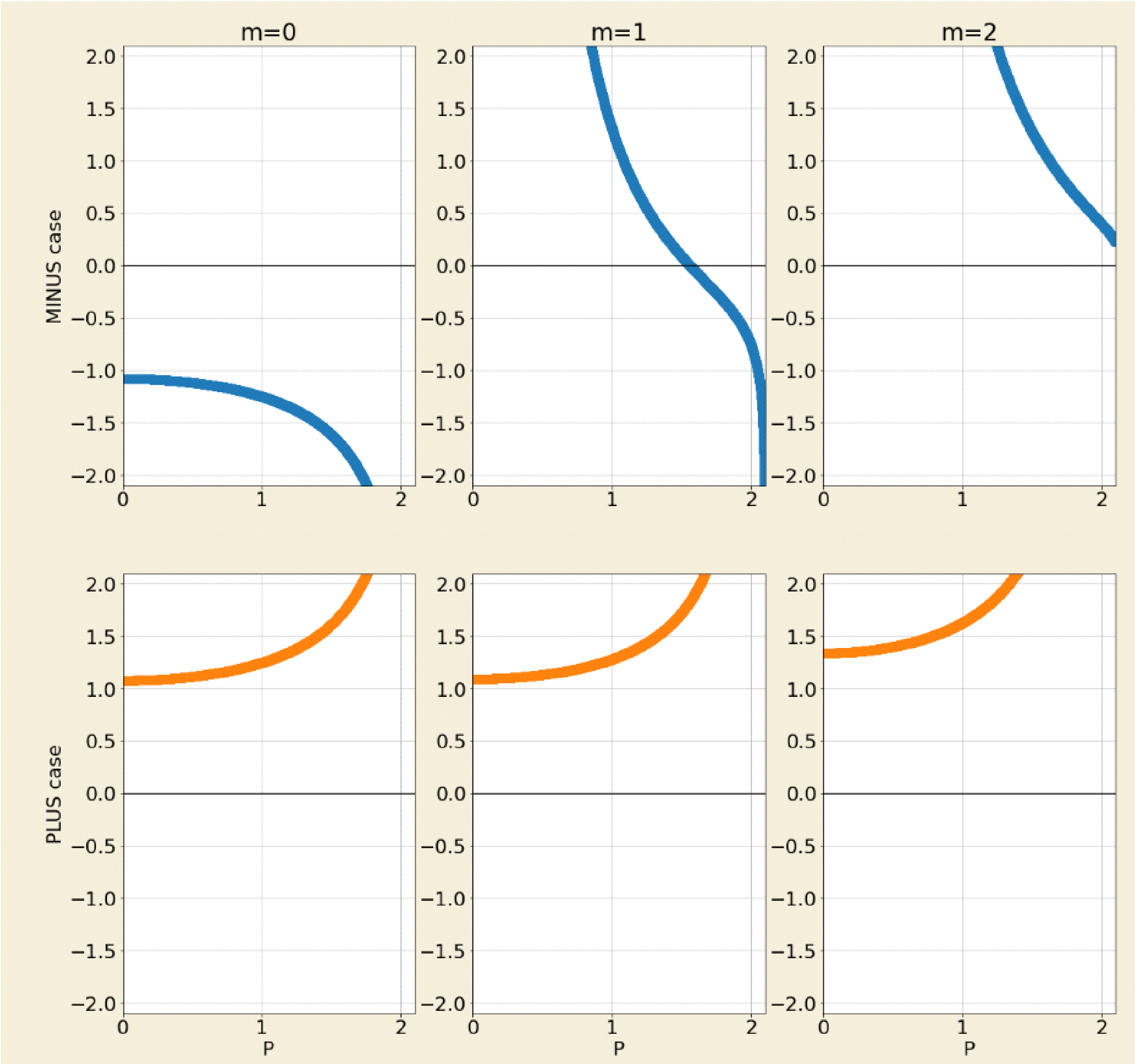


If we increase V a bit more to 5.3, we get two additional ones; one for the m=2 PLUS case and one for the m=4 MINUS case:

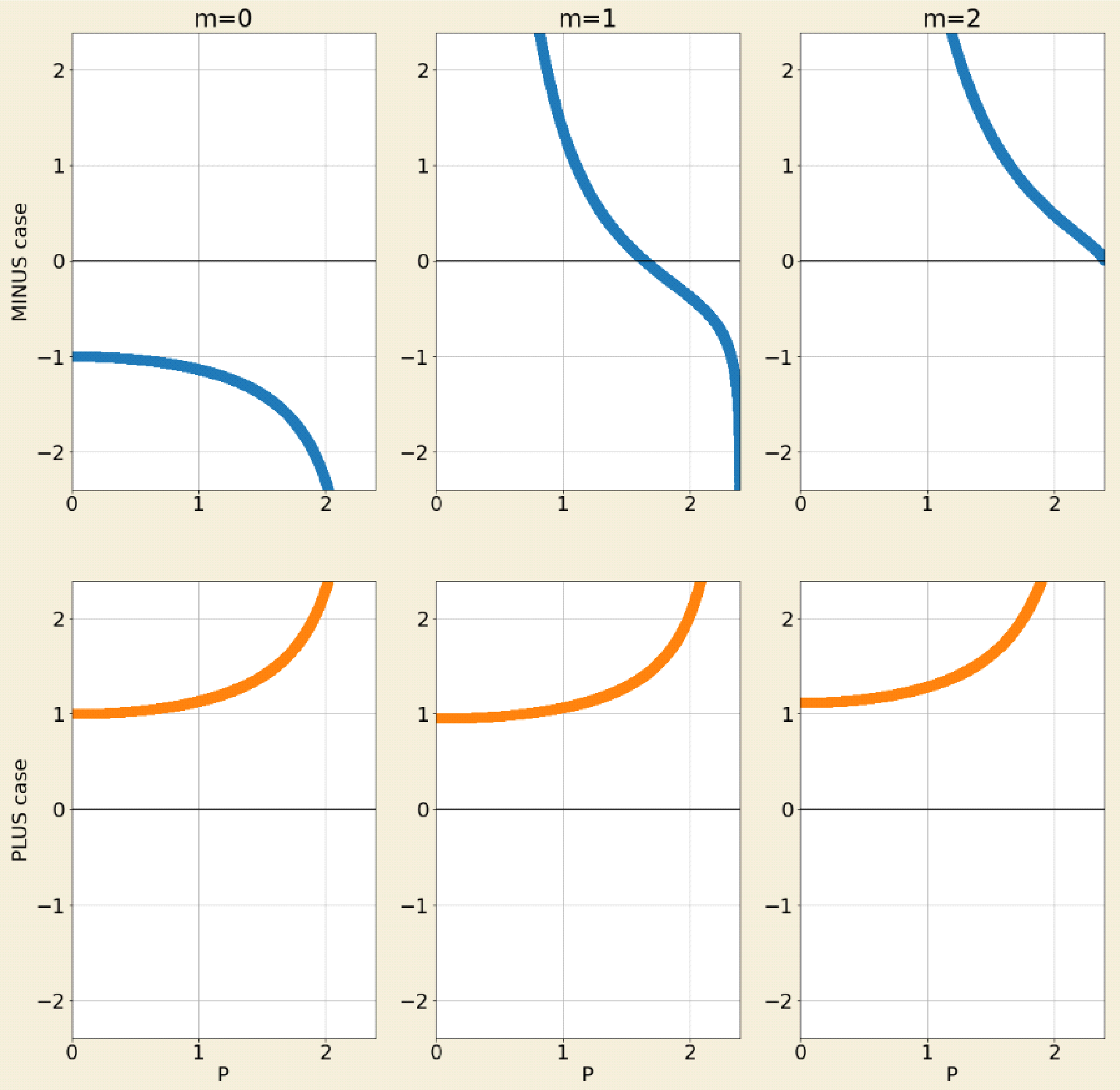


When do we only have 1 mode?

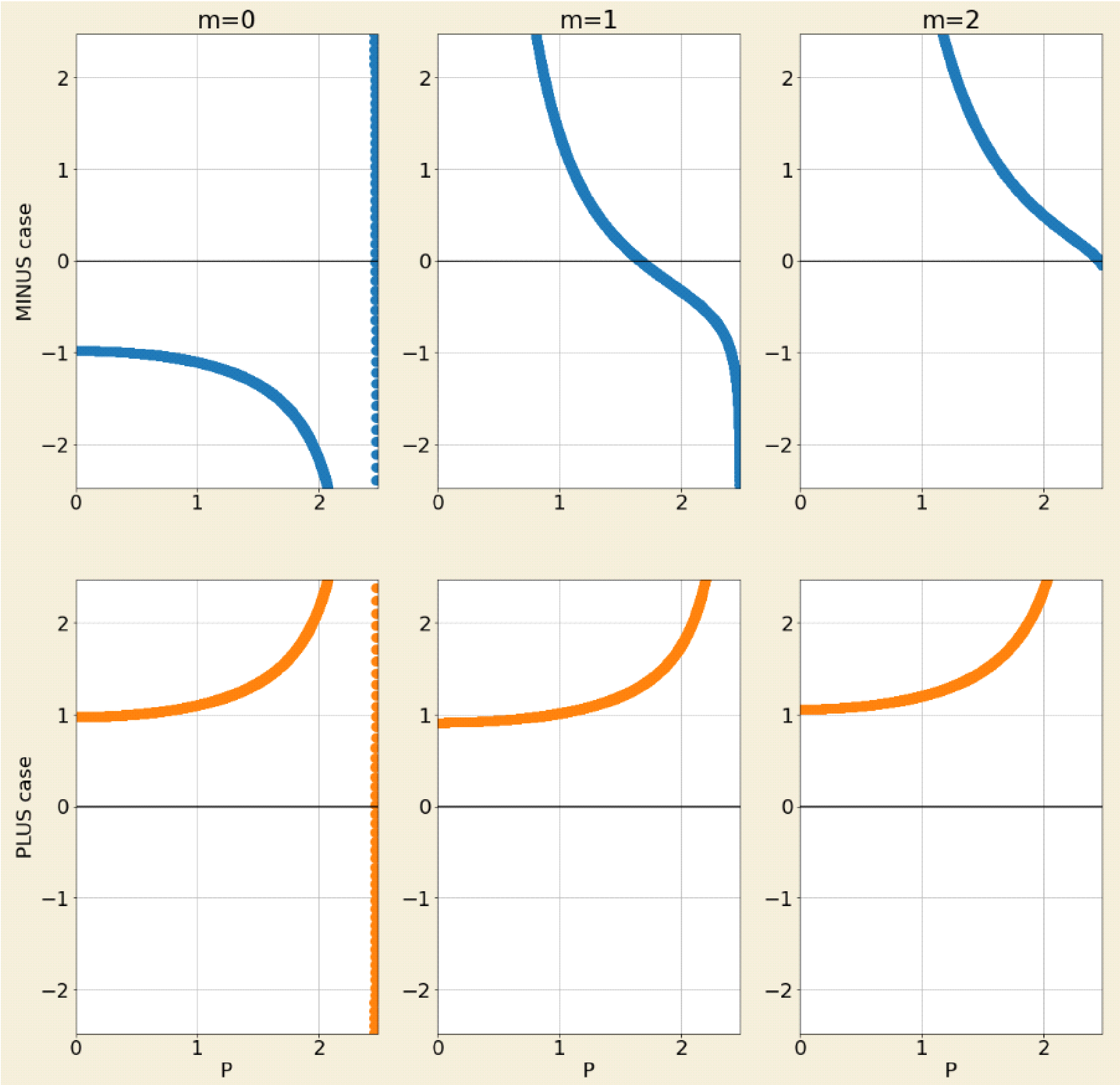
If we generate the plots for V=2.1, we see that only a single mode is available:



At V=2.39, we almost have a 2nd one for the m=2 MINUS case:



At V=2.48, we not only get the m=2 MINUS mode, but also the m=0 MINUS mode and the m=0 PLUS mode:



In short, the PLUS and MINUS equations allow for more than 1 solution in total when V>2.4, which is the value at which the zeroth order Bessel function is zero, $J_0(2.4\ldots) = 0$.

In other words, a fiber, which is designed so that

$$V = \frac{2\pi}{\lambda_0} R \sqrt{{n_{core}}^2 - {n_{clad}}^2} < 2.4,$$

will only support a single mode; i.e. it will be a "single mode fiber" (SMF). Note that describing a fiber as "single mode" as if this is an inherent property is a bit misleading, since any fiber is multimode if the incident light has a short enough wavelength. For example, a fiber with $n_{core} = 1.454, n_{clad} = 1.450$ and a radius of $\mathrm{R} = 4\mu m$ will be multimode for wavelengths below:

$$2.4 < \frac{2\pi}{\lambda_{multi}} R \sqrt{n_{core}^2 - n_{clad}^2}$$

$$\lambda_{multi} < \frac{2\pi}{2.4} R \sqrt{n_{core}^2 - n_{clad}^2} = \frac{2\pi}{2.4} 4\mu m \cdot \sqrt{1.454^2 - 1.450^2} = 1.13\mu m$$

Anyways, let's assume that V = 2.39, which is just below the value of 2.4 that causes additional modes to appear. We want to investigate the funamdental m=1 MINUS mode. What electric field distribution does this correspond to? Let $P_{11} = 1.6428$ be the value of P where the blue line crosses zero when V=2.39. For smaller values of V, P_{11} will also be smaller. We will concentrate on the field inside the core, but a similar analysis can be carried out for the field outside the core.

For the m=1 case

$$0 = \frac{J_0(P_{11})}{P_{11}J_1(P_{11})} - \frac{K_0(Q_{11})}{Q_{11}K_1(Q_{11})}$$

$$R^2 p_{11}^2 = P_{11}^2 = R^2\left(\left(n_{core}k_0\right)^2 - \beta_{11}^2\right)$$

$$p_{11} = \frac{P_{11}}{R}$$

$$\beta_{11}^2 = \left(\left(n_{core}k_0\right)^2 - \frac{P_{11}^2}{R^2}\right)$$

$$\beta_{11} = \sqrt{\left(n_{core}k_0\right)^2 - \frac{P_{11}^2}{R^2}}$$

Let's consider the electric field inside:

<u>Inside:</u>

$$E_\rho = \frac{i}{p_{11}^2}\left(D_{core}\cdot\omega\mu_0\frac{i}{\rho}J_1(\rho p_{11}) + C_{core}\cdot\beta_{11}p_{11}J_1{}'(\rho p_{11})\right)e^{i\phi}e^{i\beta_{11}z}$$

$$E_\phi = \frac{i}{p_{11}^2}\left[C_{core}\cdot\beta_{11}\frac{i}{\rho}J_1(\rho p_{11}) - D_{core}\cdot\mu_0 c k_0 p_{11}J'_1(\rho p_{11})\right]e^{i\phi}e^{i\beta_{11}z}$$

$$E_z = C_{core}\cdot J_1(\rho p_{11})\,e^{i\phi}e^{i\beta_{11}z}$$

Having D_{core}, which represents the magnitude of the magnetic field in the core, is a bit annoying, but fortunately the "Matrix equation" presented earlier can be rearranged to show that for m=1:

$$\frac{C_{core}^2}{D_{core}^2} = -\frac{\mu_0^2c^2}{n_{clad}^2}$$

$$-i\frac{n_{clad}}{\mu_0 c}C_{core} = D_{core}$$

<u>Inside:</u>

$$E_\rho = \frac{i}{p_{11}^2}\left(-i\frac{n_{clad}}{\mu_0 c}C_{core}\cdot\omega\mu_0\frac{i}{\rho}J_1(\rho p_{11}) + C_{core}\cdot\beta_{11}p_{11}J_1{}'(\rho p_{11})\right)e^{i\phi}e^{i\beta_{11}z}$$

$$E_\phi = \frac{i}{p_{11}^2}\left[C_{core}\cdot\beta_{11}\frac{i}{\rho}J_1(\rho p_{11}) + i\frac{n_{clad}}{\mu_0 c}C_{core}\cdot\mu_0 c k_0 p_{11}J'_1(\rho p_{11})\right]e^{i\phi}e^{i\beta_{11}z}$$

$$E_z = C_{core}\cdot J_1(\rho p_{11})\,e^{i\phi}e^{i\beta_{11}z}$$

Rearranging and simplifying:

<u>Inside:</u>

$$E_\rho = C_{core}\frac{i}{p_{11}^2}\left(n_{clad}\frac{k_0}{\rho}J_1(\rho p_{11}) + \beta_{11}p_{11}J_1{}'(\rho p_{11})\right)e^{i\phi}e^{i\beta_{11}z}$$

$$E_\phi = C_{core}\frac{-1}{p_{11}^2}\left[\frac{\beta_{11}}{\rho}J_1(\rho p_{11}) + n_{clad}\,k_0 p_{11}J'_1(\rho p_{11})\right]e^{i\phi}e^{i\beta_{11}z}$$
$$E_z = C_{core}\cdot J_1(\rho p_{11})\,e^{i\phi}e^{i\beta_{11}z}$$

Exploit that $n_{clad} \approx n_{core} \approx \beta_{11}/k_0$

<u>Inside:</u>

$$E_\rho = C_{core}\frac{i}{p_{11}^2}\,n_{clad}k_0\left(\frac{1}{\rho}J_1(\rho p_{11}) + p_{11}J_1'(\rho p_{11})\right)e^{i\phi}e^{i\beta_{11}z}$$
$$E_\phi = C_{core}\frac{-1}{p_{11}^2}n_{clad}\,k_0\left[\frac{1}{\rho}J_1(\rho p_{11}) + p_{11}J'_1(\rho p_{11})\right]e^{i\phi}e^{i\beta_{11}z}$$
$$E_z = C_{core}\cdot J_1(\rho p_{11})\,e^{i\phi}e^{i\beta_{11}z}$$

<u>Inside:</u>

$$E_\rho = C_{core}\frac{i}{p_{11}^2}\,n_{clad}k_0\left(\frac{1}{\rho}J_1(\rho p_{11}) + p_{11}J_1'(\rho p_{11})\right)e^{i\phi}e^{i\beta_{11}z}$$
$$E_\phi = iE_\rho$$
$$E_z = C_{core}\cdot J_1(\rho p_{11})\,e^{i\phi}e^{i\beta_{11}z}$$

Exploit the following property for the derivatives of the Bessel function:

- $$\frac{\partial J_\nu(z)}{\partial z} = J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z)$$

$$J_1'(\rho p_{11}) = J_0(\rho p_{11}) - \frac{1}{\rho p_{11}}J_1(\rho p_{11})$$

<u>Inside:</u>

$$E_\rho = C_{core}\frac{i}{p_{11}^2}\,n_{clad}k_0\left(\frac{1}{\rho}J_1(\rho p_{11}) + p_{11}\left[J_0(\rho p_{11}) - \frac{1}{\rho p_{11}}J_1(\rho p_{11})\right]\right)e^{i\phi}e^{i\beta_{11}z}$$
$$E_\phi = iE_\rho$$
$$E_z = C_{core}\cdot J_1(\rho p_{11})\,e^{i\phi}e^{i\beta_{11}z}$$

<u>Inside:</u>

$$E_\rho = C_{core}\frac{i}{p_{11}^2}\,n_{clad}k_0\left(\frac{1}{\rho}J_1(\rho p_{11}) + p_{11}J_0(\rho p_{11}) - \frac{1}{\rho}J_1(\rho p_{11})\right)e^{i\phi}e^{i\beta_{11}z}$$
$$E_\phi = iE_\rho$$
$$E_z = C_{core}\cdot J_1(\rho p_{11})\,e^{i\phi}e^{i\beta_{11}z}$$

<u>Inside:</u>

$$E_\rho = iC_{core}\frac{n_{clad}k_0}{p_{11}}\,J_0(\rho p_{11})e^{i\phi}e^{i\beta_{11}z}$$
$$E_\phi = iE_\rho$$
$$E_z = C_{core}\cdot J_1(\rho p_{11})\,e^{i\phi}e^{i\beta_{11}z}$$

Now we consider the unit vectors

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi$$
$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi$$

Compute the real electric field:

$$\overline{\boldsymbol{E}}_{in} = \boldsymbol{Re}\left(E_\rho\right)(\hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi) + \boldsymbol{Re}\left(E_\phi\right)(-\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi) + \boldsymbol{Re}(E_z)\hat{\boldsymbol{z}}$$

$$\overline{\boldsymbol{E}}_{in} = \hat{\boldsymbol{x}}\left(\boldsymbol{Re}(E_\rho)\cos\phi - \boldsymbol{Re}(E_\phi)\sin\phi\right) + \hat{\boldsymbol{y}}\left(\boldsymbol{Re}\left(E_\rho\right)\sin\phi + \boldsymbol{Re}\left(E_\phi\right)\cos\phi\right) + \boldsymbol{Re}(E_z)\hat{\boldsymbol{z}}$$

$$\overline{\boldsymbol{E}}_{in} = \hat{\boldsymbol{x}}\left(\boldsymbol{Re}(E_\rho)\cos\phi - \boldsymbol{Re}(iE_\rho)\sin\phi\right) + \hat{\boldsymbol{y}}\left(\boldsymbol{Re}\left(E_\rho\right)\sin\phi + \boldsymbol{Re}\left(iE_\rho\right)\cos\phi\right) + \boldsymbol{Re}(E_z)\hat{\boldsymbol{z}}$$

$$\boldsymbol{Re}\left(E_\rho\right) = C_{core}\frac{n_{clad}k_0}{p_{11}}\,J_0(\rho p_{11})\,Re(i\,(\cos(\beta_{11}z+\phi) + i\sin(\beta_{11}z+\phi)\,))$$
$$\boldsymbol{Re}\left(E_\rho\right) = -C_{core}\frac{n_{clad}k_0}{p_{11}}\,J_0(\rho p_{11})\,\sin(\beta_{11}z+\phi)$$
$$\boldsymbol{Re}\left(iE_\rho\right) = C_{core}\frac{n_{clad}k_0}{p_{11}}\,J_0(\rho p_{11})\,Re(-\,(\cos(\beta_{11}z+\phi) + i\sin(\beta_{11}z+\phi)\,))$$
$$\boldsymbol{Re}\left(iE_\rho\right) = -C_{core}\frac{n_{clad}k_0}{p_{11}}\,J_0(\rho p_{11})\,\cos(\beta_{11}z+\phi)$$

$$\begin{aligned}\overline{\boldsymbol{E}}_{in} = {} & \\ & \hat{\boldsymbol{x}}\left(-C_{core}\frac{n_{clad}k_0}{p_{11}}\,J_0(\rho p_{11})\right)[\sin(\beta_{11}z+\phi)\cos\phi - \cos(\beta_{11}z+\phi)\sin\phi] \\ & +\hat{\boldsymbol{y}}\left(-C_{core}\frac{n_{clad}k_0}{p_{11}}\,J_0(\rho p_{11})\right)[\sin(\beta_{11}z+\phi)\sin\phi + \cos(\beta_{11}z+\phi)\cos\phi] \\ & +\hat{\boldsymbol{z}}\,C_{core}\cdot J_1(\rho p_{11})\cos(\beta_{11}z+\phi)\end{aligned}$$

Rewrite the trig functions in the square brackets:

Product-to-sum identities [edit]

$$\cos\theta\,\cos\varphi = \frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2}$$
$$\sin\theta\,\sin\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2}$$
$$\sin\theta\,\cos\varphi = \frac{\sin(\theta+\varphi)+\sin(\theta-\varphi)}{2}$$
$$\cos\theta\,\sin\varphi = \frac{\sin(\theta+\varphi)-\sin(\theta-\varphi)}{2}$$

$$\sin(\phi+\beta_{11}z)\cos\phi - \cos(\phi+\beta_{11}z)\sin\phi = \frac{\sin(2\phi+\beta_{11}z)+\sin(\beta_{11}z)}{2} - \frac{\sin(2\phi+\beta_{11}z)-\sin(\beta_{11}z)}{2}$$
$$\boldsymbol{\sin(\phi+\beta_{11}z)\cos\phi - \cos(\phi+\beta_{11}z)\sin\phi = \sin(\beta_{11}z)}$$

$$\sin(\phi+\beta_{11}z)\sin\phi + \cos(\phi+\beta_{11}z)\cos\phi = \frac{\cos(\beta_{11}z)-\cos(2\phi+\beta_{11}z)}{2} + \frac{\cos(\beta_{11}z)+\cos(2\phi+\beta_{11}z)}{2}$$
$$\boldsymbol{\sin(\phi+\beta_{11}z)\sin\phi + \cos(\phi+\beta_{11}z)\cos\phi = \cos(\beta_{11}z)}$$

So the electric field in the core is:

$$\overline{\boldsymbol{E}}_{in} = C_{core}\begin{pmatrix} -\frac{n_{clad}k_0}{p_{11}}\ J_0(\rho p_{11})\sin(\beta_{11}z) \\ -\frac{n_{clad}k_0}{p_{11}}\ J_0(\rho p_{11})\cos(\beta_{11}z) \\ J_1(\rho p_{11})\cos(\beta_{11}z+\phi) \end{pmatrix}$$

It would appear that the fundamental mode has components of equal, angle independent magnitude along the x- and y-directions, but a different, angle dependent component along the z-direction.

Let's investigate their relative magnitudes!

$$field\ ratio = \frac{\frac{n_{clad}k_0}{p_{11}}J_0(\rho p_{11})}{J_1(\rho p_{11})}$$

$$p_{11} = \frac{1.6428}{R}$$

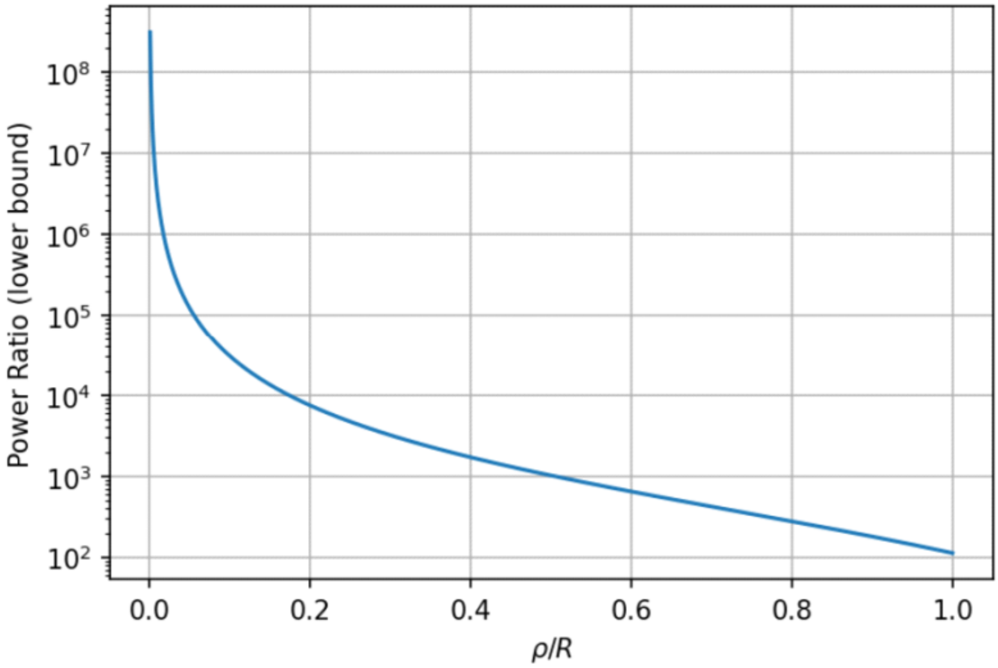
$$field\ ratio = \frac{\frac{n_{clad}k_0}{p_{11}}J_0(\rho p_{11})}{J_1(\rho p_{11})} = \frac{n_{clad}k_0 R J_0\left(\frac{\rho}{R}1.6428\right)}{1.6428 J_1\left(\frac{\rho}{R}1.6428\right)} \approx \frac{1.45\frac{2\pi}{1.5\mu m}\ 4\mu m\ J_0\left(\frac{\rho}{R}1.6428\right)}{1.6428 J_1\left(\frac{\rho}{R}1.6428\right)} = \frac{1.45\frac{2\pi}{1.5\mu m}\ 4\mu m\ J_0\left(\frac{\rho}{R}1.6428\right)}{1.6428 J_1\left(\frac{\rho}{R}1.6428\right)} \approx 14.8\ \frac{J_0\left(\frac{\rho}{R}1.6428\right)}{J_1\left(\frac{\rho}{R}1.6428\right)}$$

$$field\ ratio \approx\ 14.8\ \frac{J_0\left(\frac{\rho}{R}P_{11}\right)}{J_1\left(\frac{\rho}{R}P_{11}\right)}$$

So ratio of the field along the transverse directions compared to the field along the z-direction is at least 10 times the ratio of the zeroth and first order Bessel function. Thus, the power ratio must be

$$Power\ ratio = (field\ ratio)^2 \approx 220\ \left(\frac{J_0\left(\frac{\rho}{R}P_{11}\right)}{J_1\left(\frac{\rho}{R}P_{11}\right)}\right)^2$$

Graphing this expression, we see that the power along the transverse direction is are least 100 times greater than the power along the z-direction:



Note that using a smaller value of V will make P_{11} smaller than 1.6248, which will make the power ratio even higher, implying that the z-component is even more negligible!

Therefore, it is reasonable to neglect the z-component of the electric field of the fundamental mode of an optical fiber and write:

$$\overline{\boldsymbol{E}}_{in} = \frac{-n_{clad}k_0}{p_{11}} J_0(\rho p_{11}) C_{core} \begin{pmatrix} \sin(\beta_{11} z) \\ \cos(\beta_{11} z) \\ 0 \end{pmatrix}$$

This corresponds to <u>clockwise spatial rotation</u> of the electric field at a given instant if we move down the length of the fiber. In turn, this corresponds to <u>counterclockwise temporal rotation</u> of the field if we observe a particular fiber segment over time. Counterclockwise temporal rotation is also called "Right hand circular polarization". However, since the fiber is cylindrically symmetric, "Left hand circular polarization" must also be valid. Furthermore, since all linear and elliptical states of polarizations are just combinations of these two cases with different phase offsets, a fundamental mode with <u>linear polarization</u> must also be valid!

Thus, the (MINUS,m=1) mode is sometimes referred to as "LP_{01}", where LP stands for "Linear Polarization".

Indeed, the prediction that a single mode fiber should be able to support any state of polarization can easily be confirmed experimentally by launching light with a uniform polarization in its cross section into a SMF and observing that the recorded power is the same when the SOP is varied.

Note that the direction of the electric field for the fundamental mode is independent of the radial and angular coordinates; if there is $\hat{\boldsymbol{x}}$ at the center of the fundamental mode, there will be $\hat{\boldsymbol{x}}$ polarization everywhere! For a MMF, it is possible to have modes, where the direction of the electric field depends on the radial and angular coordinates. For example, so-called "Transverse Electric" modes, where the electric field always points radially away from the center are possible!

<u>Conclusion</u>

For cylindrical optical fibers with small refractive index differences, it has been shown how to derive characteristic equations for the propagation constants, β, corresponding to "modes", whose envelopes do not change with propagation along the z-direction. In addition, it was proven that single mode propagation is possible when the V-parameter is below 2.4, and that the fundamental mode supports any state of polarization as long as it is independent of the radial and angular coordinates.

Addendum: Electric field of the fundamental mode outside the core

For the m=1, MINUS case, we have

Outside:

$$E_\rho = \frac{-i}{q_{11}^2}\left(D_{clad}\cdot\omega\mu_0\frac{i}{\rho}K_1(\rho q_{11}) + C_{clad}\cdot\beta_{11}q_{11}K_1{}'(\rho q_{11})\right)e^{i\phi}e^{i\beta_{11}z}$$

$$E_\phi = \frac{-i}{q_{11}^2}\left[C_{clad}\cdot\beta_{11}\frac{i}{\rho}K_1(\rho q_{11}) - D_{clad}\cdot\mu_0 ck_0 q_{11}K_1'(\rho q_{11})\right]e^{i\phi}e^{i\beta_{11}z}$$

$$E_z = C_{clad}\cdot K_1(\rho q_{11})\,e^{i\phi}e^{i\beta_{11}z}$$

Recall that

$$C_{core}\frac{J_m(P)}{K_m(Q)} = C_{clad}$$

$$D_{core}\cdot\frac{J_m(P)}{K_m(Q)} = D_{clad}$$

Outside:

$$E_\rho = \frac{-i}{q_{11}^2}\left(D_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot\omega\mu_0\frac{i}{\rho}K_1(\rho q_{11}) + C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot\beta_{11}q_{11}K_1{}'(\rho q_{11})\right)e^{i\phi}e^{i\beta_{11}z}$$

$$E_\phi = \frac{-i}{q_{11}^2}\left[C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot\beta_{11}\frac{i}{\rho}K_1(\rho q_{11}) - D_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot\mu_0 ck_0 q_{11}K_1'(\rho q_{11})\right]e^{i\phi}e^{i\beta_{11}z}$$

$$E_z = C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot K_1(\rho q_{11})\,e^{i\phi}e^{i\beta_{11}z}$$

As mentioned in the derivation of the field inside the core:

$$-i\frac{n_{clad}}{\mu_0 c}C_{core} = D_{core}$$

Outside:

$$E_\rho = \frac{-i}{q_{11}^2}\left(-i\frac{n_{clad}}{\mu_0 c}C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot\omega\mu_0\frac{i}{\rho}K_1(\rho q_{11}) + C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot\beta_{11}q_{11}K_1{}'(\rho q_{11})\right)e^{i\phi}e^{i\beta_{11}z}$$

$$E_\phi = \frac{-i}{q_{11}^2}\left[C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot\beta_{11}\frac{i}{\rho}K_1(\rho q_{11}) + i\frac{n_{clad}}{\mu_0 c}C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot\mu_0 ck_0 q_{11}K_1'(\rho q_{11})\right]e^{i\phi}e^{i\beta_{11}z}$$

$$E_z = C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot K_1(\rho q_{11})\,e^{i\phi}e^{i\beta_{11}z}$$

Simplify:

Outside:

$$E_\rho = \frac{-i}{q_{11}^2} \frac{J_1(P_{11})}{K_1(Q_{11})} C_{core} \left(\frac{k_0 n_{clad}}{\rho} K_1(\rho q_{11}) + \beta_{11} q_{11} K_1'(\rho q_{11}) \right) e^{i\phi} e^{i\beta_{11} z}$$

$$E_\phi = \frac{1}{q_{11}^2} \frac{J_1(P_{11})}{K_1(Q_{11})} C_{core} \left[\frac{\beta_{11}}{\rho} K_1(\rho q_{11}) + n_{clad} k_0 q_{11} K_1'(\rho q_{11}) \right] e^{i\phi} e^{i\beta_{11} z}$$

$$E_z = C_{core} \frac{J_1(P_{11})}{K_1(Q_{11})} \cdot K_1(\rho q_{11}) e^{i\phi} e^{i\beta_{11} z}$$

Exploit that $n_{clad} \approx n_{core} \approx \beta_{11}/k_0$

Outside:

$$E_\rho = \frac{-i}{q_{11}^2} \frac{J_1(P_{11})}{K_1(Q_{11})} k_0 n_{clad} C_{core} \left(\frac{1}{\rho} K_1(\rho q_{11}) + q_{11} K_1'(\rho q_{11}) \right) e^{i\phi} e^{i\beta_{11} z}$$

$$E_\phi = \frac{1}{q_{11}^2} \frac{J_1(P_{11})}{K_1(Q_{11})} k_0 n_{clad} C_{core} \left[\frac{1}{\rho} K_1(\rho q_{11}) + q_{11} K_1'(\rho q_{11}) \right] e^{i\phi} e^{i\beta_{11} z}$$

$$E_z = C_{core} \frac{J_1(P_{11})}{K_1(Q_{11})} \cdot K_1(\rho q_{11}) e^{i\phi} e^{i\beta_{11} z}$$

Now make use of the fact that

$$\boldsymbol{K_\nu'(x) \;=\; -K_{\nu-1}(x) - \frac{\nu}{x} K_\nu(x)}$$

$$K_1'(\rho q_{11}) = -K_0(\rho q_{11}) - \frac{1}{\rho q_{11}} K_1(\rho q_{11})$$

Outside:

$$E_\rho = \frac{-i}{q_{11}^2} \frac{J_1(P_{11})}{K_1(Q_{11})} k_0 n_{clad} C_{core} \left(\frac{1}{\rho} K_1(\rho q_{11}) + q_{11} \left[-K_0(\rho q_{11}) - \frac{1}{\rho q_{11}} K_1(\rho q_{11}) \right] \right) e^{i\phi} e^{i\beta_{11} z}$$

$$E_\phi = \frac{1}{q_{11}^2} \frac{J_1(P_{11})}{K_1(Q_{11})} k_0 n_{clad} C_{core} \left[\frac{1}{\rho} K_1(\rho q_{11}) + q_{11} \left[-K_0(\rho q_{11}) - \frac{1}{\rho q_{11}} K_1(\rho q_{11}) \right] \right] e^{i\phi} e^{i\beta_{11} z}$$

$$E_z = C_{core} \frac{J_1(P_{11})}{K_1(Q_{11})} \cdot K_1(\rho q_{11}) e^{i\phi} e^{i\beta_{11} z}$$

Outside:

$$E_\rho = \frac{-i}{q_{11}^2} \frac{J_1(P_{11})}{K_1(Q_{11})} k_0 n_{clad} C_{core} \left(\frac{1}{\rho} K_1(\rho q_{11}) - q_{11} K_0(\rho q_{11}) - \frac{1}{\rho} K_1(\rho q_{11}) \right) e^{i\phi} e^{i\beta_{11} z}$$

$$E_\phi = \frac{1}{q_{11}^2} \frac{J_1(P_{11})}{K_1(Q_{11})} k_0 n_{clad} C_{core} \left[\frac{1}{\rho} K_1(\rho q_{11}) - q_{11} K_0(\rho q_{11}) - \frac{1}{\rho} K_1(\rho q_{11}) \right] e^{i\phi} e^{i\beta_{11} z}$$

$$E_z = C_{core} \frac{J_1(P_{11})}{K_1(Q_{11})} \cdot K_1(\rho q_{11}) e^{i\phi} e^{i\beta_{11} z}$$

Outside:

$$E_\rho = \frac{i}{q_{11}^2}\frac{J_1(P_{11})}{K_1(Q_{11})}\; k_0 n_{clad} C_{core}\; q_{11} K_0(\rho q_{11})\; e^{i\phi} e^{i\beta_{11} z}$$
$$E_\phi = \frac{-1}{q_{11}^2}\frac{J_1(P_{11})}{K_1(Q_{11})} k_0 n_{clad} C_{core} q_{11} K_0(\rho q_{11}) e^{i\phi} e^{i\beta_{11} z}$$
$$E_z = C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot K_1(\rho q_{11})\; e^{i\phi} e^{i\beta_{11} z}$$

Outside:

$$E_\rho = iC_{core}\frac{k_0 n_{clad}}{q_{11}}\frac{J_1(P_{11})}{K_1(Q_{11})}\; K_0(\rho q_{11})\; e^{i\phi} e^{i\beta_{11} z}$$
$$E_\phi = -C_{core}\frac{k_0 n_{clad}}{q_{11}}\frac{J_1(P_{11})}{K_1(Q_{11})} K_0(\rho q_{11}) e^{i\phi} e^{i\beta_{11} z}$$
$$E_z = C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot K_1(\rho q_{11})\; e^{i\phi} e^{i\beta_{11} z}$$

Outside:

$$E_\rho = iC_{core}\frac{k_0 n_{clad}}{q_{11}}\frac{J_1(P_{11})}{K_1(Q_{11})}\; K_0(\rho q_{11})\; e^{i\phi} e^{i\beta_{11} z}$$
$$E_\phi = iE_\rho$$
$$E_z = C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\cdot K_1(\rho q_{11})\; e^{i\phi} e^{i\beta_{11} z}$$

Now we consider the unit vectors

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi$$
$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi$$

$$\overline{\boldsymbol{E}}_{out} = \boldsymbol{Re}\left(E_\rho\right)(\hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi) + \boldsymbol{Re}\left(E_\phi\right)(-\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi) + \boldsymbol{Re}(E_z)\hat{\boldsymbol{z}}$$

$$\overline{\boldsymbol{E}}_{out} = \hat{\boldsymbol{x}}\left(\boldsymbol{Re}\left(E_\rho\right)\cos\phi - \boldsymbol{Re}\left(E_\phi\right)\sin\phi\right) + \hat{\boldsymbol{y}}\left(\boldsymbol{Re}\left(E_\rho\right)\sin\phi + \boldsymbol{Re}\left(E_\phi\right)\cos\phi\right) + \boldsymbol{Re}(E_z)\hat{\boldsymbol{z}}$$

$$\boldsymbol{Re}\left(E_\rho\right) = C_{core}\frac{k_0 n_{clad}}{q_{11}}\frac{J_1(P_{11})}{K_1(Q_{11})}\; K_0(\rho q_{11})\; \boldsymbol{Re}\left(i[\cos(\beta_{11} z + \phi) + i\sin(\beta_{11} z + \phi)]\;\right)$$

$$\boldsymbol{Re}\left(E_\rho\right) = -C_{core}\frac{k_0 n_{clad}}{q_{11}}\frac{J_1(P_{11})}{K_1(Q_{11})}\; K_0(\rho q_{11})\; \sin(\beta_{11} z + \phi)$$

$$\boldsymbol{Re}\left(E_\phi\right) = \boldsymbol{Re}\left(iE_\rho\right) = -C_{core}\frac{k_0 n_{clad}}{q_{11}}\frac{J_1(P_{11})}{K_1(Q_{11})}\; K_0(\rho q_{11})\cos(\beta_{11} z + \phi)$$

$$\boldsymbol{Re}(E_z) = C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})} \cdot K_1(\rho q_{11}) \cos(\beta_{11} z + \phi)$$

$$\begin{aligned}
\overline{\boldsymbol{E}}_{out} &= \\
&-\hat{\boldsymbol{x}} C_{core}\frac{k_0 n_{clad}}{q_{11}}\frac{J_1(P_{11})}{K_1(Q_{11})} K_0(\rho q_{11}) \left(\sin(\beta_{11} z + \phi)\cos\phi - \cos(\beta_{11} z + \phi)\sin\phi\right) \\
&- \hat{\boldsymbol{y}} C_{core}\frac{k_0 n_{clad}}{q_{11}}\frac{J_1(P_{11})}{K_1(Q_{11})} K_0(\rho q_{11}) \left(\sin(\beta_{11} z + \phi)\sin\phi + \cos(\beta_{11} z + \phi)\cos\phi\right) \\
&+ \hat{\boldsymbol{z}} C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})} \cdot K_1(\rho q_{11}) \cos(\beta_{11} z + \phi)
\end{aligned}$$

Use the same trig relations as for the inside case:

$$\begin{aligned}
\overline{\boldsymbol{E}}_{out} &= \\
&-\hat{\boldsymbol{x}} C_{core}\frac{k_0 n_{clad}}{q_{11}}\frac{J_1(P_{11})}{K_1(Q_{11})} K_0(\rho q_{11})\ \mathbf{sin}(\boldsymbol{\beta_{11} z}) \\
&-\hat{\boldsymbol{y}} C_{core}\frac{k_0 n_{clad}}{q_{11}}\frac{J_1(P_{11})}{K_1(Q_{11})} K_0(\rho q_{11})\ \mathbf{cos}(\boldsymbol{\beta}_{\mathbf{11}}\mathbf{z}) \\
&+ \hat{\boldsymbol{z}} C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})} \cdot K_1(\rho q_{11}) \cos(\beta_{11} z + \phi)
\end{aligned}$$

$$\begin{aligned}
\overline{\boldsymbol{E}}_{out} &= \\
&-\hat{\boldsymbol{x}} C_{core}\frac{k_0 n_{clad}}{q_{11}}\frac{J_1(P_{11})}{K_1(Q_{11})} K_0(\rho q_{11})\ \mathbf{sin}(\boldsymbol{\beta_{11} z}) \\
&-\hat{\boldsymbol{y}} C_{core}\frac{k_0 n_{clad}}{q_{11}}\frac{J_1(P_{11})}{K_1(Q_{11})} K_0(\rho q_{11})\ \mathbf{cos}(\boldsymbol{\beta}_{\mathbf{11}}\mathbf{z}) \\
&+ \hat{\boldsymbol{z}} C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})} \cdot K_1(\rho q_{11}) \cos(\beta_{11} z + \phi)
\end{aligned}$$

$$\overline{\boldsymbol{E}}_{out} = C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\begin{pmatrix} -\frac{k_0 n_{clad}}{q_{11}} K_0(\rho q_{11}) \sin(\beta_{11} z) \\ -\frac{k_0 n_{clad}}{q_{11}} K_0(\rho q_{11}) \cos(\beta_{11} z) \\ K_1(\rho q_{11}) \cos(\beta_{11} z + \phi) \end{pmatrix}$$

Let's also compare the magnitude of the transverse and z-components

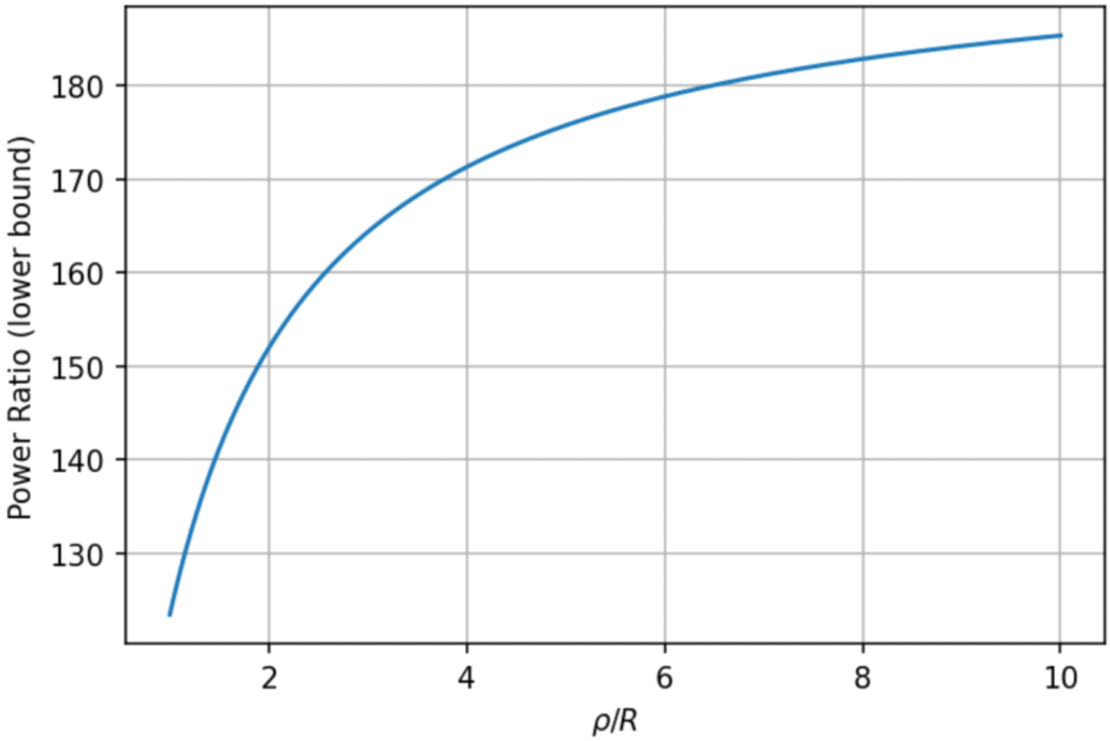
$$q_{11} = \frac{Q_{11}}{R} = \frac{\sqrt{V^2 - P_{11}^2}}{R}$$

$$field\ ratio = \frac{\frac{k_0 n_{clad}}{q_{11}} K_0(\rho q_{11})}{K_1(\rho q_{11})} = \frac{k_0 n_{clad} R}{\sqrt{V^2 - P_{11}^2}}\frac{K_0(\rho q_{11})}{K_1(\rho q_{11})}$$

We assume that V=2.39, so $P_{11} = 1.6428$ and $Q_{11} = \sqrt{2.39^2 - 1.6428^2} = 1.7359$

$$field\ ratio \approx \frac{1.45\frac{2\pi}{1.5\mu m}4\mu m}{1.7359}\frac{K_0(\rho q_{11})}{K_1(\rho q_{11})} \approx 14\ \frac{K_0(\rho q_{11})}{K_1(\rho q_{11})}$$

$$Power\ ratio = (field\ ratio)^2 \approx 196\left(\frac{K_0\left(\frac{\rho}{R}1.7359\right)}{K_1\left(\frac{\rho}{R}1.7359\right)}\right)^2$$



Once again, the z-component has much less power than the transverse components, so it can be neglected and

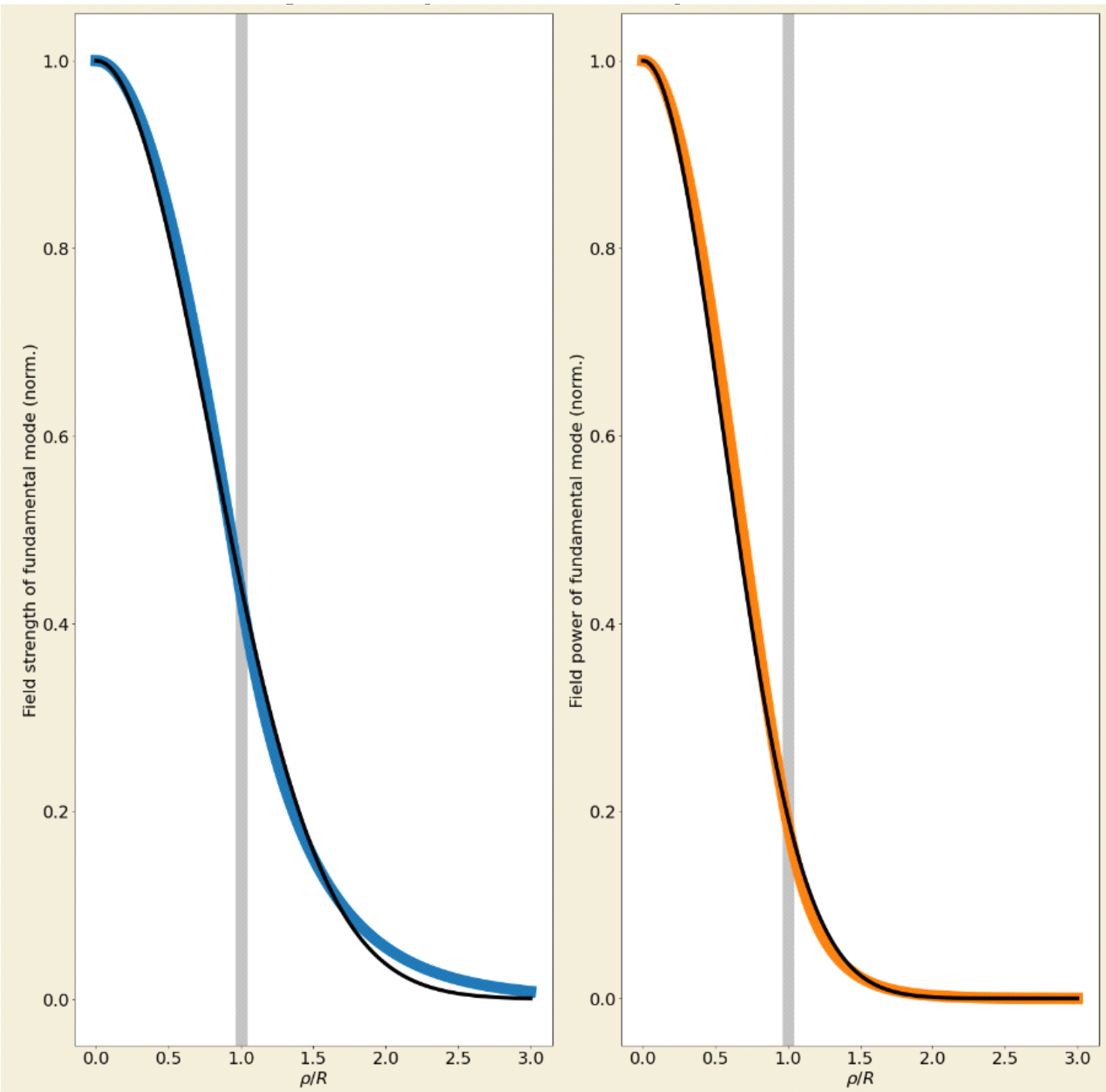
$$\bar{\boldsymbol{E}}_{out} = -C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\frac{k_0 n_{clad}}{q_{11}}K_0(\rho q_{11})\begin{pmatrix}\sin(\beta_{11}z)\\ \cos(\beta_{11}z)\\ 0\end{pmatrix}$$

In summary, the electric field of the fundamental mode of an optical fiber can be described by:

$$\bar{\boldsymbol{E}}_{in} = -C_{core}\qquad\qquad\frac{n_{clad}k_0 R}{P_{11}}J_0\left(\frac{\rho}{R}P_{11}\right)\begin{pmatrix}\sin(\beta_{11}z)\\ \cos(\beta_{11}z)\\ 0\end{pmatrix}$$

$$\bar{\boldsymbol{E}}_{out} = -C_{core}\frac{J_1(P_{11})}{K_1(Q_{11})}\frac{k_0 n_{clad}R}{Q_{11}}K_0\left(\frac{\rho}{R}Q_{11}\right)\begin{pmatrix}\sin(\beta_{11}z)\\ \cos(\beta_{11}z)\\ 0\end{pmatrix}$$

Normalizing away all the constants that the two expressions have in common, the distributions of field strength and field power relative to their peak values are shown below:



The black lines are Gaussian approximations, with

$$E = \exp\left(-\frac{(\rho/R)^2}{w^2}\right)$$

and

$$w \approx \ 0.65 + 1.619V^{-\frac{3}{2}} + 2.879V^{-6}$$

Note that to get the total power, we must integrate using cylindrical coordinates!