COMP229: Introduction to Data Science
Lecture 12: Kuratowski's criterion of planarity

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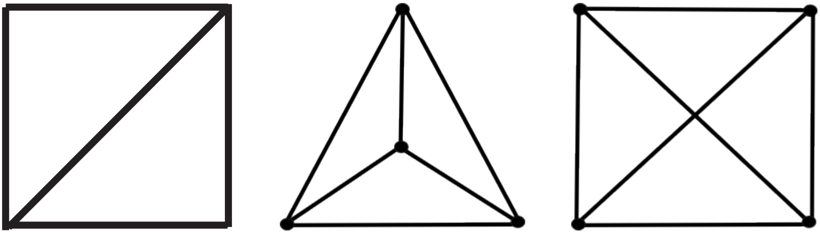
Planar graphs

Definition 12.1. An abstract graph that can be embedded (drawn without intersections) in the plane $\mathbb{R}^2$ (or, equivalently S^2) is called *planar*.



Each of the graphs in the picture has 4 vertices.

Which of them are planar by the definition above?

Criterion for planar graphs

Claim 12.2. [Kuratowski's criterion of planarity] A graph G is planar if and only if G has no (subdivisions of) subgraphs K_5 and $K_{3,3}$, which are called Kuratowski's *forbidden* subgraphs.

We'll prove the easier "only if" part: if a graph G contains K_5 or $K_{3,3}$, then G isn't planar. It's enough to prove that K_5 or $K_{3,3}$ aren't planar.

A proof by contradiction

When we need to prove a claim for all objects, e.g. any continuous map $K_5 \to \mathbb{R}^2$ has intersections in the image, a typical way is by contradiction.

Assume that the graph $K_5 \subset \mathbb{R}^2$ is embedded.

This embedded graph K_5 has $V = 5$ vertices and $E = 10$ edges. Use Euler's formula to find the number F of faces (connected regions of $\mathbb{R}^2 - K_5$).

Counting edges twice

Euler's formula $V - E + F = 2$ for the connected graph $K_5 \subset \mathbb{R}^2$ with $V = 5$, $E = 10$ gives $F = 7$.

Claim 12.3. Each of 7 (hypothetical) faces of a graph $K_5 \subset \mathbb{R}^2$ has at least 3 boundary edges.

Proof. K_5 has no loops (edges connecting a vertex to itself) and no double edges (two different edges between the same pair of vertices). $\square$

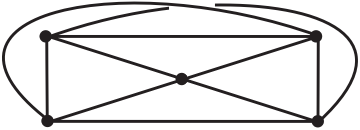
Do you already see a contradiction?

Why is K_5 not planar?

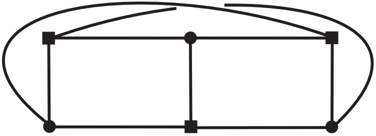
If each of 7 faces of $K_5 \subset \mathbb{R}^2$ has at least 3 boundary edges, the total number of edges is at least $7 \times 3 = 21$. We counted each edge twice, because every edge belongs to the boundaries of 2 faces. The double number of edges is $2E \geq 21$.

However, K_5 has 10 edges, which contradicts $2E \geq 21$ and proves that K_5 isn't planar. $\square$

K_5



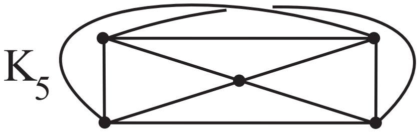
$K_{3,3}$

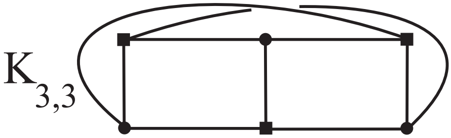


Why is $K_{3,3}$ not planar?

Try the similar approach by contradiction.

Assume that there is an embedding $K_{3,3} \subset \mathbb{R}^2$.



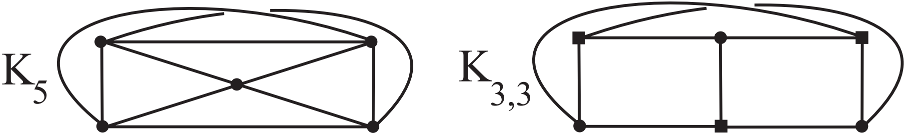


The graph has $V = 6$ vertices, $E = 9$ edges and $F = 5$ faces by Euler's formula $V - E + F = 2$.

If each of $F = 5$ faces has at least 3 edges, the double number of edges is $2E \geq 5 \times 3 = 15$.

$K_{3,3}$ is indeed not planar

$K_{3,3}$ is bipartite, hence two types of vertices (square/round dots) alternate and any cycle has an even number (minimum 4, not 3) of edges.



The double number of edges counted over 5 faces is $2E \geq 5 \times 4 = 20$, which contradicts $E = 9$. $\square$

What graphs can we embed in 3 dimensions?

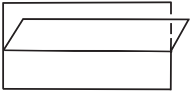
3 dimensions are enough

Claim 12.4. Any graph can be embedded in $\mathbb{R}^3$.

Proof. When we draw an intersection with one edge going over another edge, we implicitly show (a plane projection) of an embedding $G \subset \mathbb{R}^3$

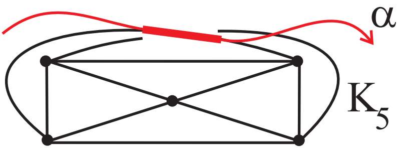
Draw any graph with only double intersections between edges. Such a drawing (in a general position) can be obtained by a slight perturbation. At every crossing, push one arc into the 3rd dimension, keep another arc in the plane. $\square$

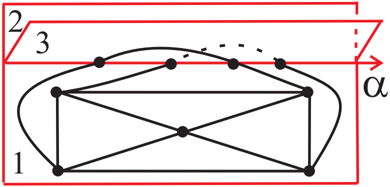
3 half-planes are enough



The *3-page book* B is a union of 3 half-planes glued on the left.

Claim 12.5. Any graph G can be embedded in B.





Proof. Draw a curve α without self-intersections that goes once through all overcrossing arcs. Deform $\mathbb{R}^2$ to make α straight. Push overcrossing arcs to the 3rd half-plane glued along α. $\square$

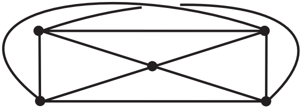
Your questions and the quiz

To benefit from the lecture, now you could

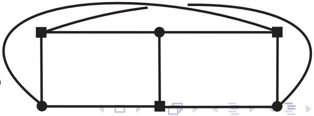
- ask or submit your anonymous questions to the COMP229 folder after the lecture;
- write down your summary in 2-3 phrases, e.g. list key concepts you have learned;
- talk to your classmates to revise the lecture.

Question. Are the graphs K_5 and $K_{3,3}$ not planar, because their drawings below have intersections?

K_5



$K_{3,3}$



Answer to the quiz and summary

Answer. No, these drawings prove nothing.
K_5 and $K_{3,3}$ are not planar, because they have no possible drawings without self-intersections (by Kuratowski's criterion of planarity).

- A *planar* graph: can be embedded (drawn without intersections) in the plane $\mathbb{R}^2$.
- A graph G is planar if and only if G contains no (subdivisions of) the graphs K_5 and $K_{3,3}$. The "only if" part follows from Euler's formula.
- Any graph can be drawn in $\mathbb{R}^3$, in 3 pages.