COMP229: Introduction to Data Science
Lecture 6: areas and volumes

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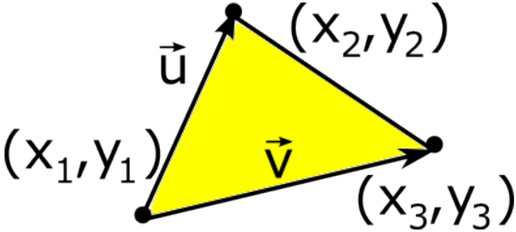
More invariants of shapes

In addition to distances, which aren't always complete, other useful geometric invariants (preserved under isometries) are areas, volumes.

Given 3 points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ in $\mathbb{R}^2$, write down the area of the triangle on these points.



Determinant is the signed area

Claim 6.1. The signed area A of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ in $\mathbb{R}^2$ ordered clockwisely equals $\frac{1}{2} \det \begin{pmatrix} x_3 - x_1 & x_2 - x_1 \\ y_3 - y_1 & y_2 - y_1 \end{pmatrix}$.

Let $\vec{u} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix}$ be the vectors along two sides of the triangle. Claim 6.1 says that the area is $A = \frac{1}{2} \det \begin{pmatrix} v_x & u_x \\ v_y & u_y \end{pmatrix}$, where $\vec{v} = (v_x, v_y)$ is rotated to $\vec{u} = (u_x, u_y)$ through A.

Basic definition of an area

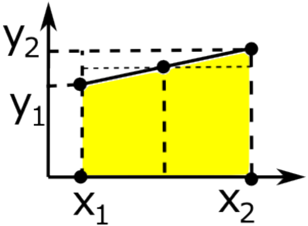
Definition 6.2. The *length* of a segment $[a, b] \subset \mathbb{R}$ is $|a - b|$. The *area* of a rectangle $[a, b] \times [c, d]$ in the plane $\mathbb{R}^2$ is the product $|a - b| \cdot |c - d|$.

The areas of all other shapes in $\mathbb{R}^2$ can be deduced from Definition 6.2, e.g. approximating by small rectangles, which is often non-trivial.

Claim 6.3. The area A of a triangle is $\frac{bh}{2}$, where b is one side (base), h is the height to this base.

Outline. A is a half of the rectangle area $b \cdot h$. $\square$

The area under a line segment

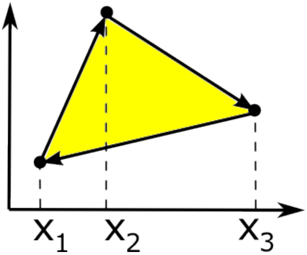


Claim 6.4. The area between the vector segment connecting points $(x_1, y_1), (x_2, y_2)$ and the x-axis is $\dfrac{y_1 + y_2}{2}(x_2 - x_1)$.

Proof. The area of the trapezium on the vertices $(x_1, 0), (x_2, 0), (x_2, y_2), (x_1, y_1)$ is the height $x_2 - x_1$ (assuming $x_1 < x_2$) times the average of the parallel sides $\dfrac{y_1 + y_2}{2}$ (assuming $y_1, y_2 > 0$).
The formula makes sense if $x_1 > x_2$ or $y_1, y_2 < 0$ (negative areas under left-oriented vectors). $\square$

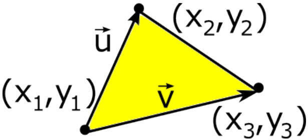
A triangle is a sum of three trapezia

Claim 6.5. The area of the triangle on the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ ordered clockwisely is the sum of the signed areas between the x–axis and $\overrightarrow{(x_1, y_1), (x_2, y_2)}$, $\overrightarrow{(x_2, y_2), (x_3, y_3)}$, $\overrightarrow{(x_3, y_3), (x_1, y_1)}$.



Outline. In the picture the first two areas are positive, the last is negative, triangle $=$ two trapezia minus one trapezium under $\overrightarrow{(x_3, y_3), (x_1, y_1)}$. All other cases are very similar. $\square$

Proof of the triangle area

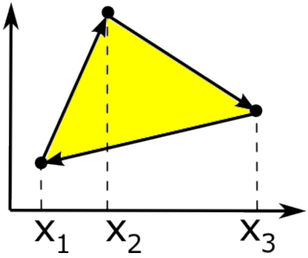


Apply Claims 6.4 and 6.5: the doubled sum of the areas of the trapezia is (please expand all brackets, collect similar terms)

$$2A = (y_1 + y_2)(x_2 - x_1) + (y_2 + y_3)(x_3 - x_2) + (y_3 + y_1)(x_1 - x_3) = y_1x_2 - y_2x_1 + y_2x_3 - y_3x_2 + y_3x_1 - y_1x_3$$
$$= (x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1) = \det\begin{pmatrix} x_3 - x_1 & x_2 - x_1 \\ y_3 - y_1 & y_2 - y_1 \end{pmatrix} = \det\begin{pmatrix} v_x & u_x \\ v_y & u_y \end{pmatrix}. \qquad \square$$

Gauss formula for any polygon

Claim 6.6. The area of the polygon on the points $(x_1, y_1), \ldots, (x_n, y_n)$ ordered clockwisely equals $\sum_{i=1}^{n} \frac{y_i + y_{i+1}}{2}(x_{i+1} - x_i)$, where $(x_{n+1}, y_{n+1}) = (x_1, y_1)$.



Outline. Similarly to Claim 6.5 the area is obtained by adding the signed areas of the trapezia between the x-axis and vectors $\overrightarrow{(x_i, y_i), (x_{i+1}, y_{i+1})}$, $i = 1, \ldots, n$. □

The volume of shapes in $\mathbb{R}^m$

Definition 6.7. The *volume* of a rectangular box $\prod_{i=1}^{m}[a_i, b_i] \subset \mathbb{R}^m$ is the product $\prod_{i=1}^{m}(b_i - a_i)$.

Claim 6.8. In $\mathbb{R}^3$ the signed volume of any parallelepiped (non-rectangular box) spanned by 3 vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$ is $\det \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix}$, where u_x, u_y, u_z are the x, y, z-coordinates of $\vec{u}$.

Can you now state Pythagoras' theorem?

Necessity and sufficiency

Pythagoras' theorem illustrates the key logical concepts of *necessary* and *sufficient* conditions.

Direct claim: any right-angled triangle with sides $a \leq b < c$ satisfies $a^2 + b^2 = c^2$ (it's *necessary*).

Converse claim: any triangle whose sides satisfy $a^2 + b^2 = c^2$ (it's *sufficient*) is right-angled.

Claim 6.9. A triangle with sides $a \leq b < c$ is right-angled *if and only if* $a^2 + b^2 = c^2$.

Hint. Prove the converse using the direct one. $\square$

Your questions and the quiz

To benefit from the lecture, now you could

- ask or submit your anonymous questions to the COMP229 folder after the lecture;
- write down your summary in 2-3 phrases, e.g. list key concepts you have learned;
- talk to your classmates to revise the lecture.

Question. Find the area of the triangle on the vertices $(1,1),(3,2),(2,3)$ in $\mathbb{R}^2$.

Answer to the quiz and summary

Answer. By the determinant formula the area is

$$\frac{1}{2}\det\begin{pmatrix} 3-1 & 2-1 \\ 2-1 & 3-1 \end{pmatrix} = \frac{1}{2}\det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{3}{2}.$$

- The area (take the modulus if you struggle with signs) of the parallelogram spanned by vectors $\vec{u}, \vec{v}$ (clockwisely) is $\det\begin{pmatrix} v_x & u_x \\ v_y & u_y \end{pmatrix}$.
- The signed area of a polygon on vertices (x_i, y_i), $i = 1, \ldots, n$ is $\sum_{i=1}^{n} \frac{y_i + y_{i+1}}{2}(x_{i+1} - x_i)$, where $(x_{n+1}, y_{n+1}) = (x_1, y_1)$.