COMP229: Introduction to Data Science
Lecture 4: geometric invariants

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Use your card with the scanner for registration

More complicated data objects

A data object can be more complicated than a single point in $\mathbb{R}^n$ (a sequence of ordered coordinates), e.g. a configuration of robots (locations in $\mathbb{R}^2$) or a molecule (atoms in $\mathbb{R}^3$).

Some objects should be considered equivalent, e.g. a molecule remains the same under rotations or translations, though atom coordinates change.

Two cameras can have two images of one object.

The problem of distinguishing objects is a classification with respect to an equivalence.

An equivalence relation

Definition 4.1. If an object A is related to B, we write an ordered pair (A, B). This relation is called an *equivalence* if the following axioms hold

(1) *reflexivity*: any A is equivalent to A, so (A, A);

(2) *symmetry*: if A is equivalent to B, then B is equivalent to A, i.e. $(A, B) \Rightarrow (B, A)$;

(3) *transitivity*: if A is equivalent to B and B is equivalent to C, then A is equivalent to C, i.e. (A, B) and $(B, C) \Rightarrow (A, C)$.

Example relations

Claim 4.2. For any equivalence relation, all objects can be classified (split) into well-defined *classes* consisting of all objects that are pairwisely equivalent to each other.

Outline. The class of $A = \{\text{all } B \text{ equivalent to } A\}$. If two classes overlap, they should coincide. $\square$

Vectors with fixed endpoints are equivalent (equal: $\overrightarrow{AB} = \overrightarrow{CD}$) if $B_i - A_i = D_i - C_i$ for each i-th coordinate. Any class of equal *fixed* vectors is called a *free* vector and has n coordinates in $\mathbb{R}^n$.

Rigid motions and isometries

Definition 4.3. A *rigid motion* of $\mathbb{R}^n$ is any composition of rotations and translations.

An *isometry* of $\mathbb{R}^n$ is a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ that preserves the L_2-metric, i.e. $L_2(p, q) = L_2(f(p), f(q))$ for any $p, q \in \mathbb{R}^n$.

Claim 4.4. Any isometry of $\mathbb{R}^n$ is a composition of rigid motions and reflections (no proof is needed).

Draw the line of reflection for each of these symmetries in $\mathbb{R}^2$: $(x, y) \mapsto (-x, y), (x, -y), (y, x)$.

How to classify up to isometry

An isometry of $\mathbb{R}^2$ applied to triangles is called a *congruence* (in the UK). How can we distinguish and classify triangles (3-point clouds) in $\mathbb{R}^2$?

Are the triangles on these points isometric or not? First: $(0,0),(4,0),(0,3)$; second: $(4,3),(4,0),(0,3)$.

Can they be different, because their vertices are at different points? No, because an isometry can move vertices. Are these triangles isometric, because we can find an isometry between them?

Easy and hard parts of the problem

Yes, the triangles are obtained from each other by the central symmetry (rotation through the angle π or 180°) with respect to the point $(2, 1.5)$.

Usually, the easy part of a classification is to show that two objects are equivalent by specifying a relation between them.

The harder part is to show that two objects aren't equivalent: we need to prove that no equivalence from an infinite variety of possibilities will work.

Invariants help distinguish objects

Definition 4.5. An *invariant* of objects considered up to an equivalence relation is a function f that takes the **same value** on all equivalent objects.

$$f : \frac{\text{objects}}{\text{equivalence}} = \left(\begin{array}{c} \text{classes of} \\ \text{equivalence} \end{array} \right) \to \begin{array}{c} \text{simple} \\ \text{values} \end{array}$$

Claim 4.6. If an invariant takes different values on two objects, then these objects are different. $\square$

A typical mistake is to classify objects by using non-invariants, e.g. people in photos by clothes.

Example invariants

For triangles in $\mathbb{R}^2$, non-invariants (under all isometries) are positions of vertices, centre of mass; invariants are edge-lengths, angles, area.

Invariant f: A is equivalent to $B \Rightarrow f(A) = f(B)$.

If an invariant takes the same value on two objects, what can we conclude? Nothing!

The height of a person is the invariant, but there are millions of people who have the same height.

Can you suggest a better invariant of people?

Invariants vs complete invariants

Definition 4.7. An invariant f is *complete* if f takes the same value only on equivalent objects.

Complete f: $f(A) = f(B) \Rightarrow A$ is equivalent to B.

Complete invariants of people: fingerprints, DNA.

Claim 4.8. For triangles (3-point clouds) in $\mathbb{R}^n$, a complete invariant has 3 edge-lengths (unordered for isometry, cyclically ordered for rigid motion).

Outline. Match longest edges. The 3rd vertex has a unique position in $\mathbb{R}^2$ (up to a reflection). $\square$

Your questions and the quiz

To benefit from the lecture, now you could

- ask or submit your anonymous questions to the COMP229 folder after the lecture;
- write down your summary in 2–3 phrases, e.g. list key concepts you have learned;
- talk to your classmates to revise the lecture.

Question. Are these triangle isometric? First: $(0,0),(4,0),(0,3)$. Second $(0,0),(5,0),(1.8,2.4)$.

Answer to the quiz and summary

Answer. The edge-lengths counter-clockwisely are (4,5,3) and (5,4,3). The given triangles are isometric in $\mathbb{R}^2$, but not up to a rigid motion.

- An isometry preserves the distance (metric).
- An invariant should take the same value on all equivalent objects, hence can distinguish objects in principle (possibly not all objects).
- A complete invariant classifies all objects with respect to a given equivalence.