COMP229: Introduction to Data Science
Lecture 27: Voronoi diagrams

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Shapes of point clouds

Clustering can be considered as a "0-dimensional approximation", because a data cloud is only represented by (isolated) centres of clusters.

These clouds look like one cluster and require approximation methods beyond clustering.

The Voronoi cell of a point in a cloud

Definition 27.1. For a cloud $C = \{p_1, \dots, p_n\}$ in $\mathbb{R}^d$ and a point $p_i \in C$, the *Voronoi* cell is

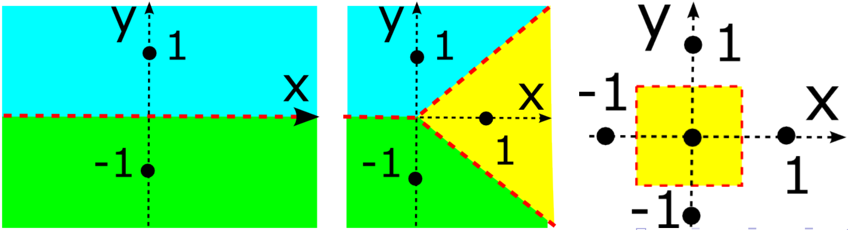
$$V(p_i) = \{q \in \mathbb{R}^d \,:\, d(q, p_i) \le d(q, p_j) \text{ for } j \ne i\}.$$

$V(p_i)$ is the neighbourhood of all points $q \in \mathbb{R}^d$ that are closer to p_i than to other points $p_j \in C$.

The Voronoi cell $V(p_i)$ is similar to a cluster of points (around a centre p_i) from k-means, but $V(p_i)$ is (on one hand) smaller (doesn't include other points of C) and (on another hand) larger as an infinite subset (possibly unbounded) in $\mathbb{R}^d$.

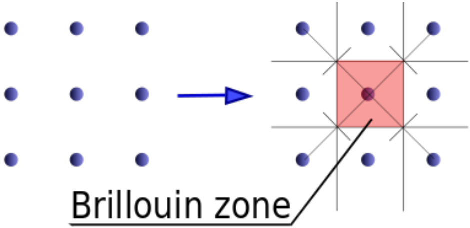
How to draw Voronoi cells

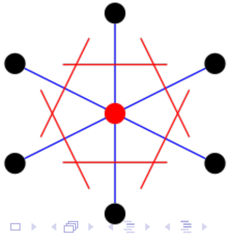
For a cloud C of 2 points $(0, \pm 1) \in \mathbb{R}^2$, the Voronoi cell $V(0,1)$ is the upper half-plane $\{y \geq 0\}$ including the boundary x-axis, $V(0,-1)$ is the lower half-plane $\{y \leq 0\}$. These cells are separated by the *bisector* line of points that have the same Euclidean L_2-distance to both $(0, \pm 1)$.



The Wigner-Seitz cell of a lattice

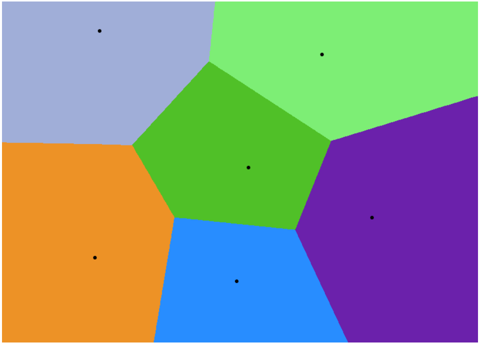
Definition 27.2. A *lattice* in $\mathbb{R}^d$ is the infinite collection of points $L = \{\sum_{i=1}^{d} a_i \vec{v}_i\}$, where the vectors $\vec{v}_1, \dots, \vec{v}_d$ form a basis of $\mathbb{R}^d$ and a_i take all integer values. The *Wigner-Seitz* cell (or the *Brilluoin zone*) of a lattice L is the Voronoi cell of the origin within the infinite periodic cloud L.

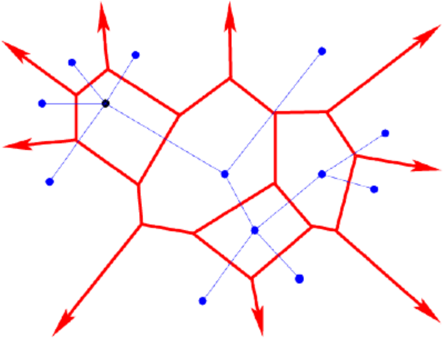




The Voronoi diagram of a cloud C

Definition 27.3. For a cloud $C = \{p_1, \dots, p_n\}$ in $\mathbb{R}^d$, the *Voronoi* diagram $V(C)$ is the splitting of $\mathbb{R}^d$ into the Voronoi cells $V(p_i)$, $i = 1, \dots, n$.





Invariants of a point cloud C

Lecture 5 discussed that not all point clouds C can be distinguished (up to an isometry of $\mathbb{R}^d$) by the pairwise distances between points.

The combinatorial structure, e.g. the 1-dimensional graph, of the Voronoi diagram is the invariant of C, i.e. remains the same under translations, rotations or reflections of C in $\mathbb{R}^d$.

What happens with the Voronoi diagram if we perturb C, i.e. shift points by short vectors?

Continuous invariants

The combinatorial structure of the Voronoi diagram can change: the Wigner-Seitz cell of the integer lattice in $\mathbb{R}^2$ is a square, which becomes a hexagon under any slight perturbation of C.

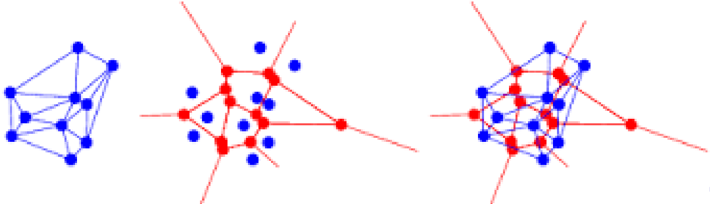
However, geometric invariants such as the areas of (bounded) Voronoi cells change continuously.

Exercise (for your contributions to the wiki page on VITAL): write an equation of the bisector line between any points $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$. Its coefficients will continuously depend on x_i, y_i.

The dual graph of a Voronoi diagram

Connect points $p, q \in C$ by an edge if Voronoi cells intersect: $V(p) \cap V(q) \neq \emptyset$. The resulting graph with the vertex set C and edges above is the dual graph of the Voronoi diagram $V(C)$.

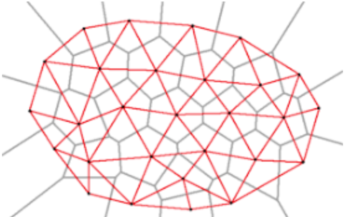
In a general position, three Voronoi cells meet at a common vertex. The corresponding 3 points $p, q, r \in C$ span a triangle as shown below.

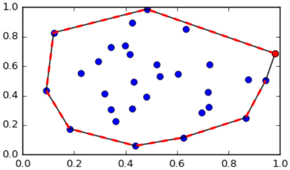


The convex hull of a point cloud

Definition 27.4. A set $S \subset \mathbb{R}^d$ is *convex* if for any points $p, q \in S$, the line segment $[p, q]$ is within S. The *convex hull* of a cloud $C \subset \mathbb{R}^d$ is the intersection of all half-spaces that contain C.

Claim 27.5. Any half-space H is convex. *Outline.* H can be $\{x \geq 0\}$ for suitable coordinates. If p, q have $x \geq 0$, then so does any point in $[p, q]$. $\square$





Your questions and the quiz

To benefit from the lecture, now you could

- ask or submit your anonymous questions to the COMP229 folder after the lecture;
- write down your summary in 2-3 phrases, e.g. list key concepts you have learned;
- talk to your classmates to revise the lecture.

Question. Is the convex hull convex? If yes, why?

Answer to the quiz and summary

Answer follows from Def 27.4 and Claim 27.5, because any intersection of convex sets is convex.

- The *Voronoi cell* of a point p in a cloud $C \subset \mathbb{R}^d$ is the neighbourhood of all points that are closer to p than to other points of C.
- For a cloud $C = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^d$, the *Voronoi* diagram $V(C)$ is the splitting of $\mathbb{R}^d$ into the Voronoi cells $V(p_i)$, $i = 1, \ldots, n$.
- The *convex hull* of a cloud $C \subset \mathbb{R}^d$ is the intersection of all half-spaces that contain C.