COMP229: Introduction to Data Science
Lecture 28: Delaunay triangulations

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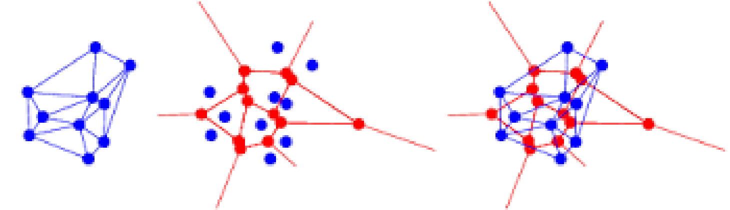
A graph dual to a Voronoi diagram

Lecture 27 has shown a triangulation dual to the Voronoi diagram of a point cloud $C \subset \mathbb{R}^d$.



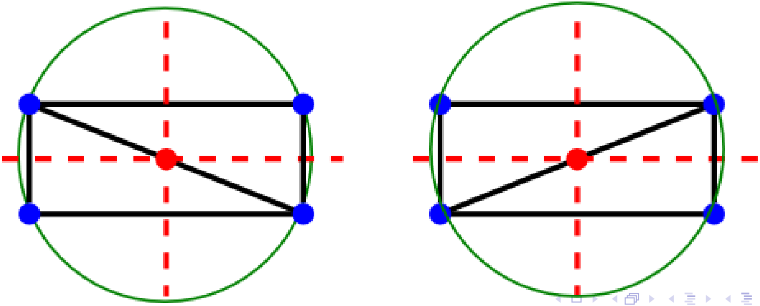
Delaunay triangulation *Voronoi diagram* *Delaunay and Voronoi*

Definition 28.1. For a cloud $C \subset \mathbb{R}^d$ of points, two points $p, q \in C$ are joined by an edge if their (closed) Voronoi cells intersect: $V(p) \cap V(q) \neq \emptyset$.

A Delaunay triangle on 3 points

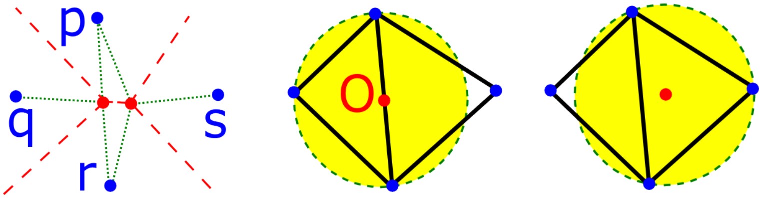
Points $p, q, r \in C$ span a *Delaunay triangle* if the Voronoi cells intersect: $V(p) \cap V(q) \cap V(r) \neq \emptyset$.

When 4 points lie in the same circle centred at a point O, their Voronoi cells meet at O, one of two possible pairs of Delaunay triangles is chosen.



From Voronoi cells to Delaunay $\triangle$

Claim 28.2. The vertex O where 3 Voronoi cells $V(p), V(q), V(r)$ meet is the centre of the circumcircle of the triangle $\triangle pqr$.

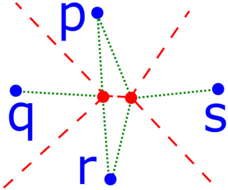


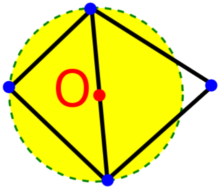
Proof. By Definition 27.1, the vertex O in the boundaries of $V(p),V(q),V(r)$ has equal distances to p, q, r, so O is the circumcentre of $\triangle pqr$. $\square$

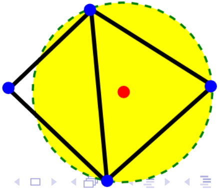
A criterion of Delaunay triangles

Claim 28.3. A triangle on points p, q, r in a given cloud C is Delaunay if and only if the open circumdisk of $\triangle pqr$ contains no points of C.

Proof. For any Delaunay triangle $\triangle pqr$, the centre O of its circumdisk doesn't belong to the cell $V(s)$ of any other $s \in C$, which happens if and only if s is more distant from O than p, q, r. $\square$



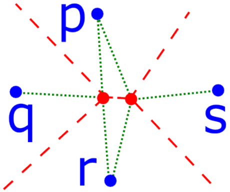


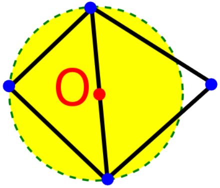


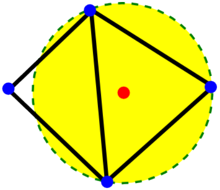
How to choose Delaunay triangles

Claim 28.4. For a convex quadrilateral on points p, q, r, s, the edge $[pr]$ is Delaunay if and only if the sum of opposite angles $\angle q + \angle s \leq 180^\circ$.

Outline. Points p, q, r, s lie in a circle if and only if $\angle q + \angle s = 180^\circ$. The open circumdisk of $\triangle pqr$ contains s if and only if $\angle q + \angle s > 180^\circ$. $\square$

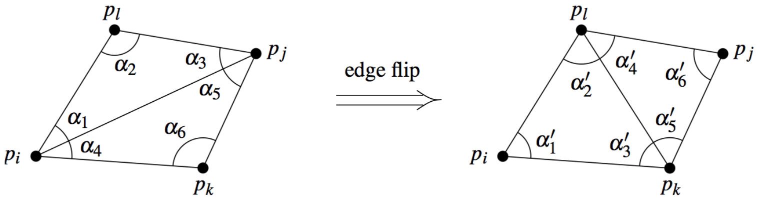






An incremental algorithm in $\mathbb{R}^2$

Idea: add points one by one and maintain a Delaunay triangulation by edge flips below.

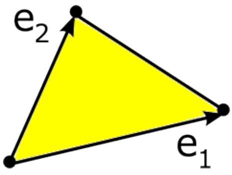


The flip increases the minimum of the 6 angles, hence edge flips will not form a cycle of flips.

Claim 28.5. A Delaunay triangulation for n points in $\mathbb{R}^2$ can be computed in time $O(n \log n)$. $\square$

High-dimensional simplices

Definition 28.6. A *k-dimensional* simplex is the set of points $\{\sum_{i=1}^{k} t_i \vec{e}_i \,:\, t_i \geq 0, \sum_{i=1}^{k} t_i \leq 1\} \subset \mathbb{R}^d$, where $\vec{e}_1, \ldots, \vec{e}_k$ are linearly independent (fixed) vectors that share the same start point in $\mathbb{R}^d$.



The 1-dimensional simplex coincides with the line segment of $\vec{e}_1$. The 2-dimensional simplex is the triangle between $\vec{e}_1, \vec{e}_2$ in $\mathbb{R}^d$.

High-dimensional Delaunay simplices

Definition 28.7. A *Delaunay* triangulation $\mathrm{Del}(C)$ of a cloud $C \subset \mathbb{R}^d$, consists of all d-dimensional simplices on points of C whose circumspheres (of dimension $d-1$) enclose no points of C.

For n points in $\mathbb{R}^d$, a Delaunay triangulation contains $O(n^{\lceil d/2 \rceil})$ simplices and has the running time $O(n^{\lceil d/2 \rceil})$, where $\lceil d/2 \rceil$ is the integer part of $(d+1)/2$. The *DeWall* (Delaunay Wall) is the fastest algorithm for $\mathrm{Del}(C)$ in practice with the same worst-time complexity as above.

A revision of Euler's formula

Claim 28.8. If a cloud $C \subset \mathbb{R}^2$ of n points has b points on the boundary of its convex hull, then a triangulation $\mathrm{Del}(C)$ has $2n - 2 - b$ triangles.

Proof. The 1-dimensional skeleton of $\mathrm{Del}(C)$ is embedded in $\mathbb{R}^2$ with n vertices, E edges, T triangles such that $n - E + T = 1$ (without 1 unbounded face). Each triangle has 3 edges, which are counted twice apart from b boundary edges: $3T = 2(E - b) + b = 2E - b$, $2E = 3T + b$. Then $2 = 2(n - E + T) = 2n - (3T + b) + 2T$. $\square$

Your questions and the quiz

To benefit from the lecture, now you could

- ask or submit your anonymous questions to the COMP229 folder after the lecture;
- write down your summary in 2–3 phrases, e.g. list key concepts you have learned;
- talk to your classmates to revise the lecture.

Question. If a cloud $C \subset \mathbb{R}^2$ of n points has b points on the boundary of its convex hull, how many edges does a triangulation $\mathrm{Del}(C)$ have?

Answer to the quiz and summary

Answer. $2E = 3T + b$ and $T = 2n - 2 - b$ from (the proof of) Claim 28.8 give $E = 3n - 3 - b$.

- A *k-dimensional* simplex is the set of points $\{\sum_{i=1}^{k} t_i \vec{e}_i \ : \ t_i \geq 0, \sum_{i=1}^{k} t_i \leq 1\} \subset \mathbb{R}^d$ for linearly independent $\vec{e}_1, \ldots, \vec{e}_k$ sharing a start point.
- A *Delaunay* triangulation $\mathrm{Del}(C)$ of a point cloud $C \subset \mathbb{R}^d$, consists of all d-dimensional simplices on points of C whose circumspheres (of dimension $d-1$) enclose no points of C.