COMP229: Introduction to Data Science
Lecture 9: topological invariants of graphs

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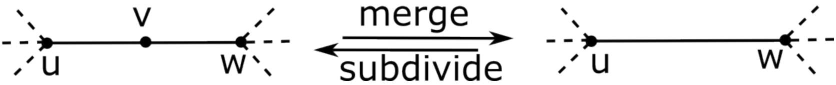
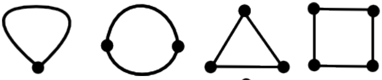
Topological non-invariants

The numbers of vertices and edges aren't preserved under a topological equivalence (introduced through mergers and subdivisions).



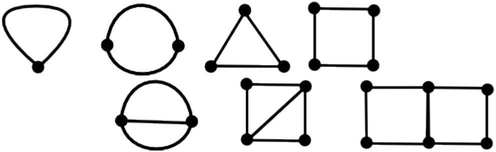
Hence the numbers of vertices and edges aren't topological invariants. How can we distinguish the graphs from the two groups below?



The Euler characteristic

Definition 9.1. The *Euler characteristic* of a graph G is $\chi(G) = |V(G)| - |E(G)|$ = the number of the vertices minus the number of edges in G.

Please compute $\chi(G)$ for the graphs below.



For many graphs $\chi(G) < 0$, because graphs often contain more edges than vertices. Can you make any conclusions from your computations?

$\chi(G)$ is a topological invariant

Claim 9.2. The Euler characteristic is preserved by any topological equivalence of graphs.

Hence, if $\chi(G) \neq \chi(H)$, then the graphs G, H are not topologically equivalent.

Proof. χ is preserved by any combinatorial equivalence (one-to-one maps between vertices respecting edges). When two edges merge, $|V|, |E|$ drop by one, hence $|V| - |E|$ is preserved. When an edge is subdivided, both $|V|, |E|$ increase by one, hence $|V| - |E|$ is preserved. $\square$

$\chi(G)$ is a non-complete invariant

Find connected graphs that have the same Euler characteristic, but aren't topologically equivalent.

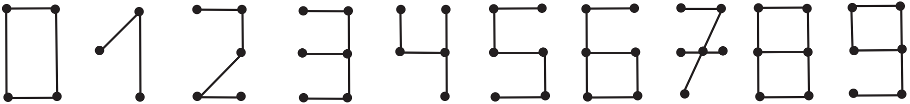
Definition 9.3. The *degree* $\deg v$ of a vertex v in a graph G is the number of edges at v (any loop connecting a vertex to itself is counted twice).

Claim 9.4. The number of vertices of any fixed degree $k \geq 0$ is a combinatorial invariant of graphs. The number of vertices of any fixed degree $k \neq 2$ is a topological invariant of graphs.

A proof for degrees

Proof. A combinatorial equivalence maps any vertex to a vertex of the same degree, because all edges are mapped accordingly. Then under any topological equivalence (mergers and subdivisions of edges), only degree 2 vertices are affected. □

Topologically classify the "digits" graphs below.



Find topologically equivalent graphs, use invariants.

Numbers of vertices of degree k

graphs/invariants	$\chi(G)$	deg 1	deg 3	deg 4
0	0	0	0	0
1	1	2	0	0
2	1	2	0	0
3	1	3	1	0
4	1	3	1	0
5	1	2	0	0
6	0	1	1	0
7	1	4	0	1
8	–1	0	2	0
9	0	1	1	0

Example classification

Digits $1, 2, 5$ are topologically equivalent (as well as $3, 4$) to each other via mergers/subdivisions. Digits $6, 9$ are geometrically symmetric.

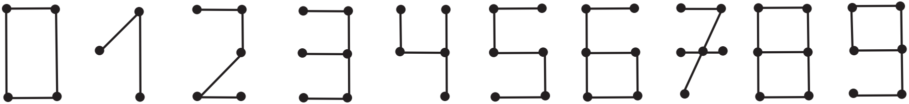
The classes represented by the digits $0, 1, 3, 6, 7, 8$ are topologically different, because they are distinguished by the numbers of degree k vertices for $k = 1, 3, 4$ (considered together) in the table.

Six classes: $\{0\}$, $\{1, 2, 5\}$, $\{3, 4\}$, $\{6, 9\}$, $\{7\}$, $\{8\}$.

Trees and forests

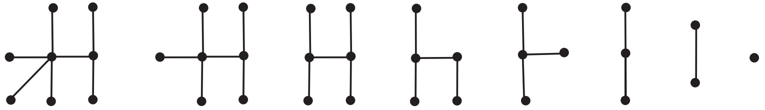
Definition 9.5. A *cycle* in a graph is a path connecting a vertex to itself, e.g. a loop is a cycle of one edge. A graph without cycles is called a *tree* (if connected) or a *forest* (if disconnected).

A picture to remember: no cycles in forests. Can you recognise a tree using topological invariants?



Euler characteristic of forests

Claim 9.6. The number of connected components in any forest G equals $\chi(G)$. In particular, it's enough to prove that any tree T has $\chi(T) = 1$.



Proof. Any tree T with at least 1 edge has a vertex of degree 1. Indeed, we can walk along edges and form a cycle. Then remove a vertex of degree 1 with its edge, which preserves $\chi(T)$. The new graph is a tree, keep removing edges until T is a single vertex and has $\chi(T) = 1 - 0 = 1$.

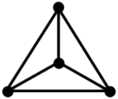
Your questions and the quiz

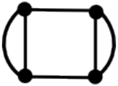
To benefit from the lecture, now you could

- ask or submit your anonymous questions to the COMP229 folder after the lecture;
- write down your summary in 2–3 phrases, e.g. list key concepts you have learned;
- talk to your classmates to revise the lecture.

Question. Find connected graphs that have the same Euler characteristic and the same numbers of vertices for any fixed degree $k \neq 2$, but are not topologically equivalent.

Answer to the quiz and summary





Answer. Both graphs have $\chi = -2$ and only 4 vertices of degree 3. The second graph, but not the first, has two disjoint cycles (a topological invariant), hence they are not topologically equivalent.

- The *Euler* characteristic is $\chi = |V| - |E|$.
- χ and the number of vertices of any fixed degree $k \neq 2$ are topological invariants.
- A *tree* is a connected graph that has no cycles and the Euler characteristic $\chi = 1$.