COMP229: Introduction to Data Science
Lecture 20: a change of a linear basis

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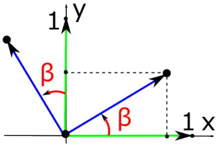
A basis of vectors

Why was the past lecture called "invariants of operators"? The determinant and the trace are invariants under an equivalence between matrices (under a change of a basis discussed today).

Definition 20.1. Vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ form a *basis* if the matrix A with the columns $\vec{v}_1, \ldots, \vec{v}_n$ has $\det A \neq 0$. Then any vector $\vec{v} \in \mathbb{R}^n$ can be written as a unique linear combination $\sum_{i=1}^{n} \lambda_i \vec{v}_i$ with some coefficients $\lambda_i \in \mathbb{R}$.

Images of the basis vectors

The standard basis in $\mathbb{R}^n$: each vector $\vec{e}_i$ has coordinate 1 on the i–th place, 0 anywhere else.



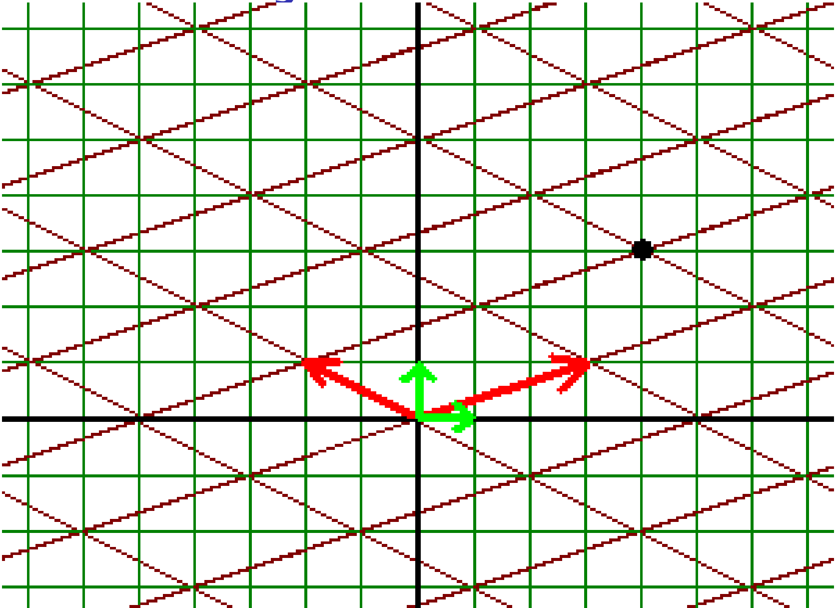
The basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ map to the columns $\begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}, \begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix}$.

For any linear map $\vec{v} \mapsto A\vec{v}$ in $\mathbb{R}^n$, the columns of the matrix A are images of the basis vectors.

What if we change basis vectors: replace $\vec{e}_1, \vec{e}_2$.

Will the matrix change, will the map change?

From the green to the red basis



From the old basis to the new basis

Let a new basis be $\vec{v}_1 = 3\vec{e}_1 + \vec{e}_2$, $\vec{v}_2 = -2\vec{e}_1 + \vec{e}_2$.

This change can be represented by the matrix

$C = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$, because $(\vec{v}_1, \vec{v}_2) = C(\vec{e}_1, \vec{e}_2)$

considered as a simple matrix identity:

$$\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The vector $\vec{v} = (2, 1)_v$ in the new basis $(\vec{v}_1, \vec{v}_2)$ is

$2\vec{v}_1 + \vec{v}_2 = 2(3\vec{e}_1 + \vec{e}_2) + (-2\vec{e}_1 + \vec{e}_2) = 4\vec{e}_1 + 3\vec{e}_2$

has the coordinates $(4, 3)_e$ in the old basis $(\vec{e}_1, \vec{e}_2)$.

The same vector in the two bases

In the matrix form the last identity is $\vec{r}_e = C\vec{r}_v$, where $\vec{r}_e, \vec{r}_v$ represented the same point in the bases $(\vec{e}_1, \vec{e}_2)$, $(\vec{v}_1, \vec{v}_2)$: $\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Invert: $C^{-1} = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$.

The point with coordinates $(4, 3)_e$ in the old basis $(\vec{e}_1, \vec{e}_2)$ has these coordinates in the basis $(\vec{v}_1, \vec{v}_2)$:

$$C^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix}_e = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_v.$$

The same rotation in the new basis

Let A be a matrix of a linear operator in the basis $(\vec{e}_1, \vec{e}_2)$, e.g. the rotation around 0 through $\pi/2$ has $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_e$, any $\vec{r}_e \in \mathbb{R}^2$ maps to $\vec{s}_e = A\vec{r}_e$.

When we change to the new basis $(\vec{v}_1, \vec{v}_2)$, the same vectors change their coordinates according to the transition matrix C: $\vec{r}_e = C\vec{r}_v, \vec{s}_e = C\vec{s}_v$.

Then $C\vec{s}_v = A(C\vec{r}_v)$, $\vec{s}_v = (C^{-1}AC)\vec{r}_v$, which proves

Claim 20.2. In the new basis $(\vec{v}_1, \vec{v}_2) = C(\vec{e}_1, \vec{e}_2)$, the operator $\vec{r}_e \mapsto A\vec{r}_e$ becomes $\vec{r}_v \mapsto (C^{-1}AC)\vec{r}_v$.

An example for the new rotation matrix

$$A \cdot C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 3 & -2 \end{pmatrix}.$$

$$C^{-1}AC = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}.$$

Indeed, we rotate $\vec{v}_1 = (3,1)_e$, $\vec{v}_2 = (-2,1)_e$ as follows: $\vec{v}_1 \mapsto (-1,3)_e = \vec{v}_1 + 2\vec{v}_2 = (1,2)_v$ and $\vec{v}_2 \mapsto (-1,-2)_e = -\vec{v}_1 - \vec{v}_2 = (-1,-1)_v$ as predicted by the matrix $C^{-1}AC$ above. So the operator is the same, but the matrix changes.

An extra check for a point

What's happened with the point $(4,3)_e = (2,1)_v$?

In the basis $(\vec{e}_1, \vec{e}_2)$, the point is rotated to the new point $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix}_e = \begin{pmatrix} -3 \\ 4 \end{pmatrix}_e$.

In the basis $(\vec{v}_1, \vec{v}_2)$, the same point is rotated to $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}_v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}_v$, which is indeed

$$\vec{v}_1 + 3\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}_e + 3 \begin{pmatrix} -2 \\ 1 \end{pmatrix}_e = \begin{pmatrix} -3 \\ 4 \end{pmatrix}_e.$$

Your questions and the quiz

To benefit from the lecture, now you could

- ask or submit your anonymous questions to the COMP229 folder after the lecture;
- write down your summary in 2–3 phrases, e.g. list key concepts you have learned;
- talk to your classmates to revise the lecture.

Question. Find the matrix of the operator with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the basis $\vec{v}_1 = (3, 1)$, $\vec{v}_3 = (1, 2)$.

Another operator in the new basis

The transition matrix $C = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ has columns equal to $\vec{v}_1, \vec{v}_3$ and $C^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$.

The matrix of the operator in the new basis is

$$C^{-1}AC = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} =$$

$$\frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

Question. What happens with $\vec{v} = (4, 3)_e$?

Answer to the quiz and summary

Answer. The point $(4,3)_e = (1,1)_v$ is rotated through $-\pi/2$ to the point $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix}_e =$
$\begin{pmatrix} 3 \\ -4 \end{pmatrix}_e = \begin{pmatrix} 2 \\ -3 \end{pmatrix}_v = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_v$.

- When an old basis is replaced by a new basis, the columns of the transition matrix C are the new basis vectors (in the old basis).
- A linear operator with a matrix A in the old basis has the matrix $C^{-1}AC$ in the new basis.