COMP229: Introduction to Data Science
Lecture 29: alpha-complexes of clouds

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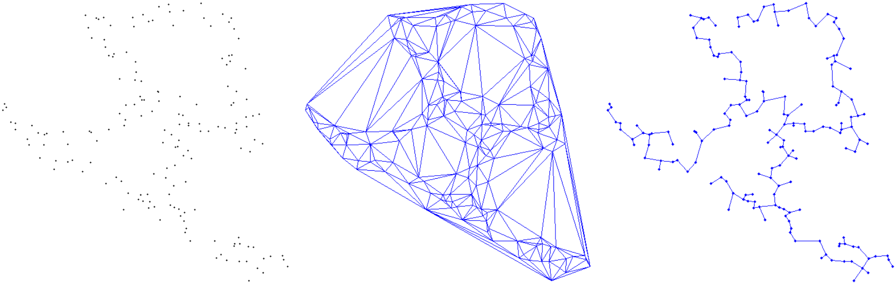
Shapes of point clouds again

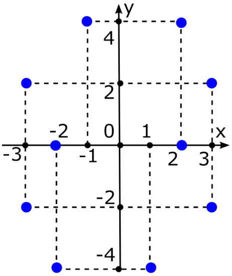
After Lectures 27–28, how can we describe (or approximate) the shape of a data cloud?



Our best option is the convex hull, which will be improved today: "don't look like a convex hull, put yourself into an α-shape" (Prof H. Edelsbrunner).

Life story of a cloud: scale $\alpha = 0$

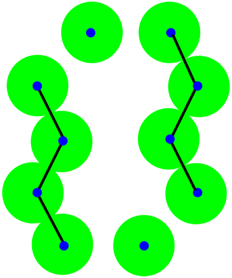
Let's look at a data cloud through a lens of a variable magnitude. At the initial zero scale, the data are isolated points. What happens if increase the scale (decrease the magnitude of the lens)?



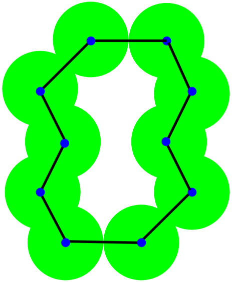
Let's grow disks centred at data points so that the radius of all disks is the increasing scale α.

At what scale α does the union of disks become connected?

Story of a cloud: $\alpha = \sqrt{5}/2$, $\alpha = 1.5$



Growing disks start touching each other at the scale $\alpha = \dfrac{\sqrt{5}}{2}$.

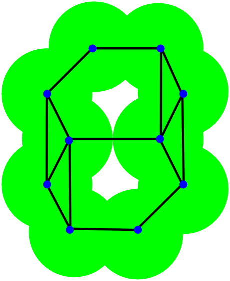


The union of disks becomes connected at $\alpha = 1.5$

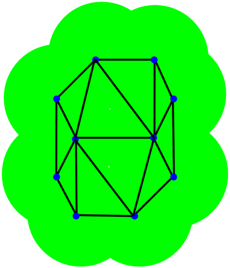
At what scale α will the hole enclosed by the green disks (at $\alpha = 1.5$) be completely filled?

Story of a cloud: $\alpha = 2$ and $\alpha \approx 2.6$

What are the triangles on top of green disks?



The big hole splits at $\alpha = 2$. Both holes are filled if α is the largest circumradius of a Delaunay triangle.

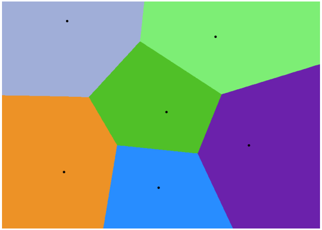


The union of disks is complicated to keep in the computer memory. It would be easier to represent a cloud shape by a graph or a union of triangles.

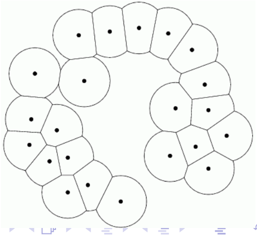
Reduced Voronoi cells

Revision of Definition 27.1: the *Voronoi cell* of a point p in a cloud $C \subset \mathbb{R}^d$ is the set of all points that are closer to p than to other points of C.

Definition 29.1. The *reduced Voronoi* cell of $p \in C$ is $V(p;\alpha) = V(p) \cap B(p;\alpha)$, where $B(p;\alpha)$ is the closed ball with the centre p and radius α.



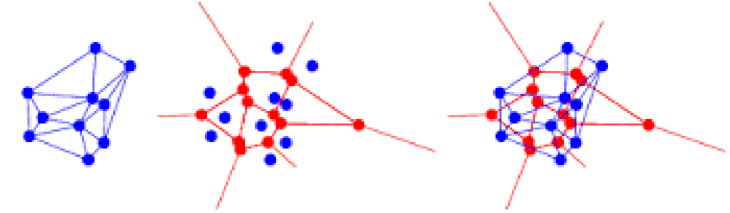
Left: unbounded Voronoi cells. Right: all reduced Voronoi cells are bounded.



From Voronoi to Delaunay

Revision of Definition 28.1: for a cloud $C \subset \mathbb{R}^d$, points $p_1, \ldots, p_k \in C$ span a *Delaunay simplex* if their Voronoi cells intersect: $\bigcap_{i=1}^{k} V(p_i) \neq \emptyset$.

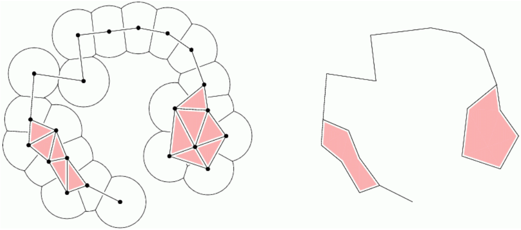
We'll restrict the construction to a fixed scale α.



Delaunay triangulation *Voronoi diagram* *Delaunay and Voronoi*

From Delaunay to α-complexes

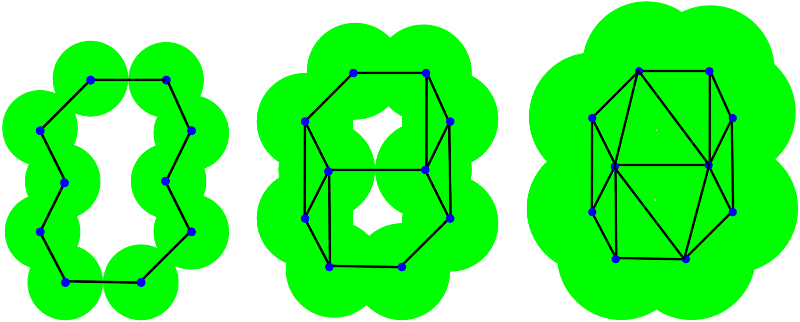
Definition 29.2. For a cloud $C \subset \mathbb{R}^d$ and $\alpha > 0$, the *α-complex* $\mathrm{Del}(C;\alpha)$ consists of all simplices spanned by points $p_1, \ldots, p_k \in C$ such that their reduced Voronoi cells intersect: $\bigcap_{i=1}^{k} V(p_i;\alpha) \neq \emptyset$.



Points p, q are joined by an edge in $\mathrm{Del}(C;\alpha)$ if their Voronoi cells share a side and $d(p,q) \leq 2\alpha$.

The shape of a cloud is preserved

Claim 29.3. For a cloud $C \subset \mathbb{R}^d$ and $\alpha > 0$, the union of balls $\bigcup\limits_{p \in C} B(p; \alpha)$ continuously deforms to $\mathrm{Del}(C; \alpha)$, i.e. the shape of C at any scale α is correctly represented by the α-complex. $\square$



A filtration of α-complexes

Claim 29.4. All α-complexes form a *filtration* (a nested sequence): for any $\alpha < \alpha'$ we have inclusions $C \subset \cdots \subset \mathrm{Del}(C; \alpha) \subset \mathrm{Del}(C; \alpha') \ldots$

Proof follows from Def 29.1: since any reduced cell $V(p; \alpha')$ contains $V(p; \alpha)$ for $\alpha < \alpha'$, so does any intersection $\bigcap\limits_{i=1}^{k} V(p_i; \alpha') \supset \bigcap\limits_{i=1}^{k} V(p_i; \alpha) \neq \emptyset$.

Hence any simplex from the α-complex $\mathrm{Del}(C; \alpha)$ is included into $\mathrm{Del}(C; \alpha')$ for a larger $\alpha' > \alpha$. $\square$

Your questions and the quiz

To benefit from the lecture, now you could

- ask or submit your anonymous questions to the COMP229 folder after the lecture;
- write down your summary in 2–3 phrases, e.g. list key concepts you have learned;
- talk to your classmates to revise the lecture.

Question. What is the α-complex $\mathrm{Del}(C;\alpha)$ of a point cloud $C \subset \mathbb{R}^d$ for a very large scale α?

Answer to the quiz and summary

Answer. When $\alpha \to +\infty$, the reduced Voronoi cell $V(p;\alpha) = V(p) \cap B(p;\alpha)$ becomes the full Voronoi cell $V(p)$, hence the α-complex $\mathrm{Del}(C;\alpha)$ becomes the full Delaunay triangulation $\mathrm{Del}(C)$.

- The *reduced Voronoi* cell of a point $p \in C$ is $V(p;\alpha) = V(p) \cap B(p;\alpha)$, where $B(p;\alpha)$ is the closed ball with centre p and radius α.
- the *α-complex* $\mathrm{Del}(C;\alpha)$ of a cloud C consists of all simplices spanned by points $p_1, \ldots, p_k \in C$ such that $\bigcap_{i=1}^{k} V(p_i;\alpha) \neq \emptyset$.