COMP229: Introduction to Data Science
Lecture 21: eigenvalues and eigenvectors

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The conjugation of matrices

Definition 21.1. Matrices A, B are *conjugated* if there is a matrix D such that $B = D^{-1}AD$.

Claim 21.2. The conjugation (sometimes called *similarity*) is an equivalence relation on matrices.

Proof. Any A is equivalent to itself: $A = I^{-1}AI$.

If $B = D^{-1}AD$, then $A = DBD^{-1} = (D^{-1})^{-1}BD^{-1}$.

If B is conjugated to C, i.e. $C = E^{-1}BE$, then $C = E^{-1}D^{-1}ADE = (DE)^{-1}A(DE)$, i.e. C is conjugated to A, because $(DE)^{-1} = E^{-1}D^{-1}$. $\square$

Invariants of linear operators

Claim 21.3. The trace and determinant are invariants of a linear operator $\vec{v} \mapsto A\vec{v}$.

Proof. If we change the basis by a transition matrix C, by Claim 19.5 the new matrix $C^{-1}AC$ has the determinant $\det(C^{-1}) \det A \det C = \det(A)$, because $\det(C^{-1}) = 1/\det C$.

By Claim 19.9 we can swap two matrices under the trace: $\operatorname{tr}(C^{-1}AC) = \operatorname{tr}(CC^{-1}A) = \operatorname{tr} A$. $\square$

The two numbers $\det A, \operatorname{tr} A$ cannot be complete invariants of matrices under conjugation.

Eigenvalues and eigenvectors

Definition 21.4. If $\vec{v}$ satisfies $A\vec{v} = \lambda\vec{v}$ for $\lambda \in \mathbb{R}$, then λ is an *eigenvalue*, $\vec{v}$ is an *eigenvector* of A.

$A\vec{v} = \lambda\vec{v}$ visually means that the operator scales all vectors in the direction of $\vec{v}$ by the factor λ.

If all basis vectors $\vec{e}_i$ are eigenvectors, then the operator has a diagonal matrix with the corresponding eigenvalues on the diagonal.

Indeed, if $A\vec{e}_i = \lambda_i\vec{e}_i$, then $A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$.

How to find eigenvalues

Claim 21.5. All eigenvalues λ of A are solutions of the *characteristic* equation $\det(A - \lambda I) = 0$.

Outline. $A\vec{v} = \lambda\vec{v}$ is equivalent to $(A - \lambda I)\vec{v} = \vec{0}$, where I is the identity $n \times n$ matrix, $\vec{0} \in \mathbb{R}^n$ is the zero vector. $(A - \lambda I)\vec{v} = \vec{0}$ means that a linear combination of the columns of $A - \lambda I$ (with the coefficients equal to the coordinates of $\vec{v}$) is $\vec{0}$.

The last fact is equivalent to $\det(A - \lambda I) = 0$ since the determinant is preserved when we add a linear combination of columns to any column. $\square$

How to find eigenvectors

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has the characteristic equation

$0 = \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} =$

$(2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$, $\lambda = 1$ and $\lambda = 3$.

Now we find solutions $\vec{v}$ of $(A - \lambda I)\vec{v} = \vec{0}$, e.g.

$(A - I)\vec{v} = \vec{0}$ is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and has infinitely many solutions parallel to $\vec{v}_1 = (1, -1)$.

$(A - 3I)\vec{v} = \vec{0}$ has solutions parallel to $\vec{v}_2 = (1, 1)$.

Diagonalise the operator

In the new basis $\vec{v}_1 = (1, -1)$, $\vec{v}_2 = (1, 1)$ with the transition matrix $C = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, the operator with $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has the new diagonal matrix

$$C^{-1}AC = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}. \text{ Great!}$$

Eigenvalues are invariants

Claim 21.6. A matrix is diagonalisable if and only if there is a basis consisting of eigenvectors. $\square$

Claim 21.7. Eigenvalues of matrices are invariant under conjugation and are invariants of operators.

Proof. Any conjugated matrix $B = C^{-1}AC$ has the characteristic equation $0 = \det(C^{-1}AC - \lambda I) = \det(C^{-1}(A - \lambda I)C) = \det(C^{-1}) \det(A - \lambda I) \det C$, which is equivalent to the characteristic equation $\det(A - \lambda I) = 0$, because $\det(C^{-1}) = 1/\det C$. $\square$

Symmetric positive-definite matrices

Definition 21.8. A symmetric matrix (operator) A is *positive-definite* if $\vec{v}^T A\vec{v} > 0$ for any $\vec{v} \neq \vec{0}$.

For $n = 2$, a symmetric matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive-definite if and only if the polynomial

$$f = (x, y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2 > 0$$

for all $(x, y) \neq (0, 0)$, i.e. when $a > 0$ (or $c > 0$) and $\det A = ac - b^2 > 0$ (the discriminant of f).

A basis of orthogonal eigenvectors

Claim 21.9. [no proof needed] Any symmetric positive-definite matrix (operator) A has an orthogonal basis of eigenvectors, hence can be diagonalised, i.e. there is a transition matrix C such that the matrix $C^{-1}AC$ is a diagonal matrix.

Consequently, the eigenvalues are complete invariants for any symmetric positive-definite matrix, which is conjugated to the diagonal matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$ on the diagonal.

Your questions and the quiz

To benefit from the lecture, now you could

- ask or submit your anonymous questions to the COMP229 folder after the lecture;
- write down your summary in 2–3 phrases, e.g. list key concepts you have learned;
- talk to your classmates to revise the lecture.

Question. Find a basis such that the operator with the matrix $A = \begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$ becomes diagonal.

Answer to the quiz and summary

Answer. $0 = \det(A - \lambda I) = (4-\lambda)(3-\lambda) - 6 = = \lambda^2 - 7\lambda + 6$, $\lambda_1 = 1$ and $\lambda_2 = 6$. For $\lambda_1 = 1$, $(A - I)\vec{v} = \vec{0}$ has a solution $\vec{v}_1 = (1, -3)$. For $\lambda_1 = 6$, $(A - 6I)\vec{v} = \vec{0}$ has a solution $\vec{v}_2 = (1, 2)$. In the basis $\vec{v}_1, \vec{v}_2$, the same operator has $\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$.

- *Eigenvalues* λ and *eigenvectors* $\vec{v}$ of A are solutions of $A\vec{v} = \lambda\vec{v}$, hence $\det(A - \lambda I) = 0$.
- Any symmetric positive-definite operator A has an orthogonal basis of eigenvectors.