COMP229: Introduction to Data Science
Lecture 11: Euler's theorem for planar graphs

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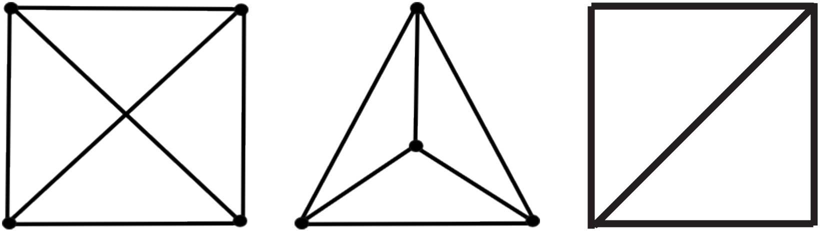
A drawing of an abstract graph

Abstract graphs are given by vertices and edges. When we draw a graph G, say in the plane, we define a function $G \to \mathbb{R}^2$ that map vertices to points in $\mathbb{R}^2$ and edges to continuous arcs.

Definition 11.1. A drawing (continuous function) $f : G \to \mathbb{R}^2$ is called an *embedding* if the (images under f of) edges don't intersect in the plane.

We can draw a graph, say the complete graph K_4 on 4 vertices, in a good way (an embedded graph) or in a bad way with intersected edges, try!

Faces of an embedded graph

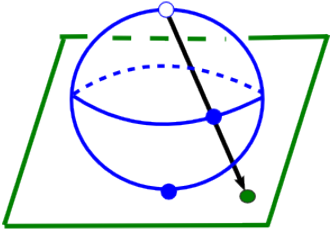


If a graph $G \subset \mathbb{R}^2$ is embedded (drawn in the plane without intersections), one can cut G from $\mathbb{R}^2$ and consider the connected region of $\mathbb{R}^2 - G$.

Definition 11.2. A *face* of an embedded graph $G \subset \mathbb{R}^2$ is a connected region of $\mathbb{R}^2 - G$.

A flat map of a round Earth?

One can stereographically project a sphere (without the north pole N) to a plane: $S^2 - N \to \mathbb{R}^2$.

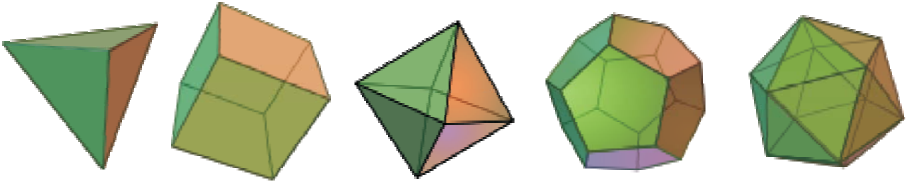


One can map a graph from the plane to the sphere. For any connected graph $G \subset S^2$, every face (region of $S^2 - G$) is a curved polygon.

For embedded graphs, let's now count the numbers of vertices (V), edges (E), faces (F).

Five Platonic polyhedra

have 1-dimensional skeletons (graphs) $G \subset S^2$.



polyhedron	V	E	F	V-E+F
tetrahedron	4	6	4	2
cube	8	12	6	2
octahedron	6	12	8	2
dodecahedron	20	30	12	2
icosahedron	12	30	20	2

Euler's formula for graphs

Claim 11.3. [Euler's theorem] Any embedded connected graph $G \subset S^2$ with V vertices, E edges, F faces satisfies the formula $V - E + F = 2$.

A graph G should be embedded (drawn without self-intersections), otherwise faces are undefined.

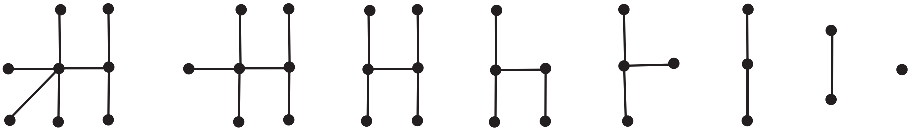
A disconnected graph may not satisfy the formula. Can you guess a corrected formula for a disconnected graph?

We start a proof from the simpler case of trees.

A proof of Euler's formula for trees

Claim 11.4. Any embedded tree $T \subset S^2$ has one face. *Proof.* Assuming that $T \subset S^2$ has two faces, they are separated by a cycle, a contradiction. $\square$

Claim 9.6 says that any tree T has the Euler characteristic $\chi(T) = V - E = 1$, see below.

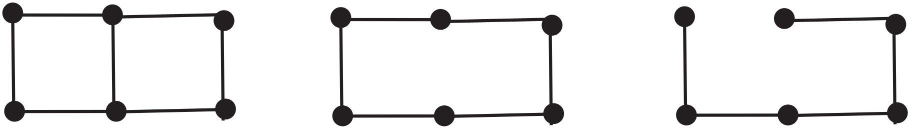


By Claim 11.4 Euler's formula holds for any embedded tree T, i.e. $(V - E) + F = 1 + 1 = 2$.

A proof of $V - E + F = 2$ in general

If a graph $G \subset S^2$ isn't a tree, G contains a cycle.

Removing one edge (without vertices) from this cycle keeps G connected. How about $V - E + F$?

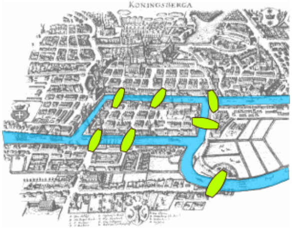


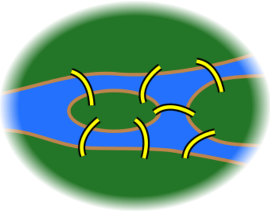
We lost one edge and merged two faces into one. Hence the smaller graph $G' \subset S^2$ has $V' = V$ vertices, $E' = E - 1$ edges, $F' = F - 1$ faces.

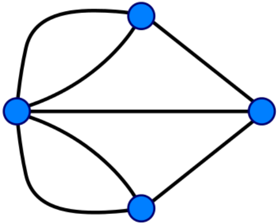
Continue removing edges from cycles until we get a tree with one face, hence $(V - E) + F = 2$. $\square$

Euler's bridges in Königsberg

The graph theory was born when Leonhard Euler tried to solve the real-life problem: find a minimal way to pass through all bridges exactly once.







Claim 11.5. [no proof needed] A path passing via all edges of a graph G exactly once exists if and only if G has 0 or 2 vertices of an odd degree.

Disconnected graphs

Find a disconnected graph when $V - E + F \neq 2$.

Find a formula for a disconnected embedded graph $G \subset S^2$ using the number of components.

You could give a proof (using the connected case) on the wiki page or give a link to a nice proof.

This proof isn't need for the exam, however you have a good chance to make a contribution, also for a proof of Claim 11.5 about Eulerian paths.

Your questions and the quiz

To benefit from the lecture, now you could

- ask or submit your anonymous questions to the COMP229 folder after the lecture;
- write down your summary in 2–3 phrases, e.g. list key concepts you have learned;
- talk to your classmates to revise the lecture.

Question. Suppose that graphs K_5 and $K_{3,3}$ are embedded (drawn without intersections) in $\mathbb{R}^2$. How many faces will we get in each case?

Answer to the quiz and summary

Answer. 7 faces for K_5, because $5 - 10 + 7 = 2$.
5 faces for $K_{3,3}$, because $6 - 9 + 5 = 2$.

- An embedding of a graph is a continuous drawing without intersections of edges.
- Graphs drawn in the plane $\mathbb{R}^2$ can be also drawn on the sphere S^2 and vice versa.
- For a connected graph embedded in $\mathbb{R}^2$ (or, equivalently, S^2) with V vertices, E edges, F faces, Euler's formula is $V - E + F = 2$.