# COMP229: Introduction to Data Science
# Lecture 7: data represented by graphs

Vitaliy Kurlin, vitaliy.kurlin@liverpool.ac.uk
Autumn 2018, Computer Science department
University of Liverpool, United Kingdom

# Data from networks or graphs

The past lectures discussed data (and invariants) given geometrically as a cloud of points. This data often requires a "hard-to-find" metric (distance).

Facebook (or other social nets) may represent their users not as points with real coordinates, but as a network, say with friendship links.

In the next 6 lectures we will study how to classify and visualise these networks: what objects are "the same" and can we draw them?

# Graphs with vertices and edges

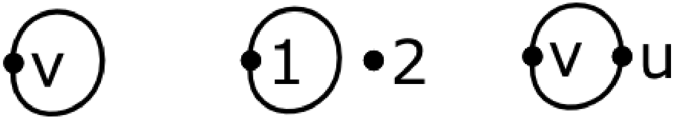
**Definition 7.1.** A (unoriented) *graph* is a pair $(V, E)$, where $V$ is a finite set of $|V|$ *vertices*, $E$ is a set of $|E|$ *edges* (unordered pairs of vertices).

$|V|$ = number of vertices, $|E|$ = number of edges. Other names: graph = network, vertex = node, edge = link (or connection), oriented = directed.

**Example.** The graph with a single edge connecting two vertices (labelled by 1,2) can be described by the pair $(1, 2)$ or $(2, 1)$. The list $(1, 2), (2, 3), (3, 1)$ represents a triangular cycle.

# More conventions and examples

For any vertex $v$, the pair $(v, v)$ represents a *loop* at $v$ (one edge connecting a vertex to itself).




For $|V| = 2$, the list of $(1, 1)$ denotes the graph consisting of one loop and the isolated vertex 2.

For vertices $u, v$, the repeated pair $(u, v), (u, v)$ in a list represents a double edge between $u, v$.

# Questions: graph drawings and lists

Draw the graphs represented by these lists:

1) $V = \{1\}$, $E = \{(1,1),(1,1)\}$;

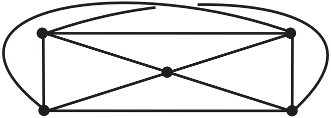
2) $V = \{1,2\}$, $E = \{(1,2),(1,2),(2,1)\}$;

3) $V = \{1,2,3,4\}$, $E = \{(1,3),(2,3),(2,1)\}$.

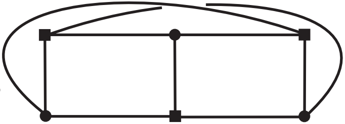
4) $V = \{1,2,3,4\}$, $E = \{(1,3),(2,3),(4,2),(4,1)\}$.

Write down representations of these graphs:

$K_5$

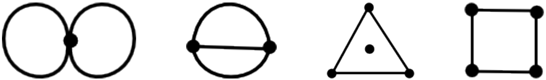


$K_{3,3}$



# Answers: graph drawings and lists

The lists in the last slide represent these graphs:



Graph $K_5$ has 5 vertices and 10 edges $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 4)$,$(3, 5)$, $(4, 5)$.

$K_5$ is a complete graph when every vertex is connected by one edge with every other vertex.

Graph $K_{3,3}$ has 6 vertices and 9 edges $(1, 2)$, $(1, 4)$, $(1, 6)$, $(3, 2)$, $(3, 4)$, $(3, 6)$, $(5, 2)$, $(5, 4)$, $(5, 6)$. Each odd vertex is connected with each even vertex.

# Combinatorial equivalence of graphs

A representation by a list of edges with labelled vertices isn't unique. As usual, when a new object is introduced, the next question is to decide how two objects can be different or the same.

**Definition 7.2.** Graphs $G, H$ are *combinatorially equivalent* if there is a one-to-one map of vertices $f : V(G) \to V(H)$ that *respects* the edges:

$(u, v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(H)$, i.e.

any vertices $u, v$ are connected in $G$ by the same number of edges as the vertices $f(u), f(v)$ in $H$.

# Invariants of graphs

How can we distinguish graphs combinatorially?

**Definition 7.3.** A *combinatorial invariant* of graphs is a function $f$ that takes the same value on all graphs combinatorially equivalent to each other: if $G, H$ are equivalent, then $f(G) = f(H)$. So if $f(G) \neq f(H)$, then $G, H$ aren't equivalent.

$f(G) = f(H)$ may not imply $G, H$ are equivalent.

**Claim 7.4.** The numbers of vertices and edges are combinatorial invariants of graphs. *Proof.* A 1–1 map on vertices induces a 1–1 map on edges. □

# Topological equivalence of graphs

**Definition 7.5.** Graphs are called *topologically equivalent* if they can become combinatorially equivalent after mergers or subdivisions of edges:

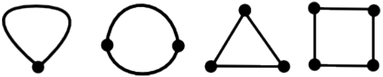

any edge can be subdivided into two edges;

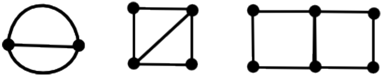
if a vertex has 2 edges, they can merge into one.

The numbers of vertices and edges can change, so they are not topological invariants of graphs.

# Combinatorics vs topology

The graphs in the groups below are topologically equivalent, but not combinatorially, e.g. compare the numbers of vertices and edges in each group.





To prove a topological equivalence, show how edges are merged step-by-step. We draw edges as continuous arcs, not necessarily straight.

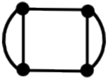
How can we distinguish graphs topologically? We'll introduce topological invariants a bit later.

# Your questions and the quiz

To benefit from the lecture, now you could

- ask or submit your anonymous questions to the COMP229 folder after the lecture;
- write down your summary in 2–3 phrases, e.g. list key concepts you have learned;
- talk to your classmates to revise the lecture.

**Question**. Represent this graph by a list of edges.



# Answer to the quiz and summary

**Answer.** 4 vertices 1,2,3,4. One potential list of edges is $(1,2),(3,4),(2,3),(2,3),(1,4)(1,4)$.

- A graph is a set of vertices and edges.
- A combinatorial equivalence of graphs is a bijection between vertices respecting edges.
- A topological equivalence = combinatorial equivalence plus mergers or subdivisions.
- To prove that graphs are equivalent, it's enough to give an example of equivalence.