COMP229: Introduction to Data Science
Lecture 3: linear transformations

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The scanner for registrations will be from week 2.

Data normalisation

Real data often comes from different sources and requires some normalisation, e.g. scaling real values to [0,1] or shifting so that the average is 0.

Let measurements be in the range $[p, q]$. How should you map the variable $x \in [p, q]$ to $[0, 1]$?

Find a linear function $f(x) = ax + b$ that bijectively (one-to-one) maps $[p, q]$ to $[0, 1]$.

You may assume that $f(p) = 0$ and $f(q) = 1$.

1-variable normalisation

One potential solution: shift $[p, q]$ to $[0, q-p]$ and divide by $q-p$. The function is $f(x) = \dfrac{x-p}{q-p}$.

The coefficients in $f(x) = ax + b$ are $a = \dfrac{1}{q-p}$, $b = -\dfrac{p}{q-p}$. Check that $f(p) = 0$, $f(q) = 1$.

Any point in $\mathbb{R}^n$ can be represented by a vector:

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ has the coordinates $x = 2$, $y = 1$ in $\mathbb{R}^2$.

Operations on vectors and matrices

Vectors are added and multiplied by a scalar *coordinate-wise*:

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} x+s \\ y+t \end{pmatrix}.$$

Matrices are added and multiplied by a scalar "entry-wise" similarly to vectors, but are multiplied together "rows-by-columns":

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}.$$

Possible: $m \times n$ matrix times $n \times k$ matrix.

Translations and scalings

Definition 3.1. The *translation* by a fixed vector $\vec{u} = \begin{pmatrix} s \\ t \end{pmatrix}$ is defined for any $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ by $f(\vec{v}) = \vec{v} + \vec{u} = \begin{pmatrix} x+s \\ y+t \end{pmatrix}$, similarly in $\mathbb{R}^n$.

The (non-uniform) *scaling* is the multiplication by a diagonal matrix $f(\vec{v}) = A\vec{v}$ for a fixed matrix $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where non-diagonal entries $= 0$.

A linear transformation

Definition 3.2. An *affine* transformation of $\mathbb{R}^n$ is a map $\mathbb{R}^n \to \mathbb{R}^n$ given by a formula $f(v) = Av + B$, where A is a fixed $n \times n$ matrix, $B \in \mathbb{R}^n$ is a fixed vector, $v \in \mathbb{R}^n$ is a variable vector. If $B = \vec{0}$, then $f(v) = Av$ is called a *linear* transformation.

If $n = 2$, any affine transformation has the form

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} s \\ t \end{pmatrix} =$$

$$= \begin{pmatrix} ax + by + s \\ cx + dy + t \end{pmatrix}$$ for any coordinates $x, y \in \mathbb{R}$.

Transformation exercises

Write down matrices of two affine transformations:

1) first translate by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then scale the first coordinate by 2 and the second by 3.

2) first scale as above, then translate by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Geometrically describe $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 3x - 3 \\ 2y + 4 \end{pmatrix}$.

Transformations don't commute

1) $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+2 \\ y+1 \end{pmatrix} \mapsto \begin{pmatrix} 2x+4 \\ 3y+3 \end{pmatrix}.$

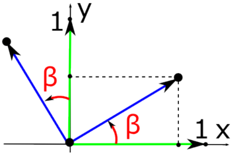
2) $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x \\ 3y \end{pmatrix} \mapsto \begin{pmatrix} 2x+2 \\ 3y+1 \end{pmatrix}.$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 3x \\ 2y \end{pmatrix} \mapsto \begin{pmatrix} 3x-3 \\ 2y+4 \end{pmatrix} = \begin{pmatrix} 3(x-1) \\ 2(y+2) \end{pmatrix},$$

we first scale by $(3,2)$, then translate by $(-3,4)$; or we first translate by $(-1,2)$, scale by $(3,2)$.

Rotations in the plane

Definition 3.3. The rotation in the plane around the origin in the counterclockwise direction through an angle β is the linear transformation $\vec{v} \mapsto A\vec{v}$ with $A = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$.



The basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ map to the columns $\begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}, \begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix}$.

Invertible transformations

Definition 3.4. A linear transformation $\vec{v} \mapsto A\vec{v}$ in $\mathbb{R}^2$ (or a matrix A) is called *invertible* if there is $\vec{v} \mapsto B\vec{v}$ such that $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = BA$.

The *determinant* is $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

Claim 3.5. A is invertible if and only if $\det A \neq 0$.

Outline. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. $\square$

Your questions and the quiz

To benefit from the lecture, now you could

- ask or submit your anonymous questions to the COMP229 folder after the lecture;
- write down your summary in 2-3 phrases, e.g. list key concepts you have learned;
- talk to your classmates to revise the lecture.

Question. Is the linear transformation $\vec{v} \mapsto A\vec{v}$ with $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ invertible? Find $\vec{v} \mapsto A^{-1}\vec{v}$.

Answer to the quiz and summary

Answer. A is invertible (or non-singular), because $\det A = 2 \neq 0$. The inverse is $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

A is a rotation through $\pi/4$ with a scaling by $\sqrt{2}$.

- Any linear transformation of $\mathbb{R}^n$ is defined by $\vec{v} \mapsto A\vec{v}$, where A is any $n \times \mathrm{n}$ matrix.
- The rotation of $\mathbb{R}^2$ around $(0,0)$ through an angle β is defined by $A = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$.
- $\vec{v} \mapsto A\vec{v}$ is invertible if and only if $\det A \neq 0$.