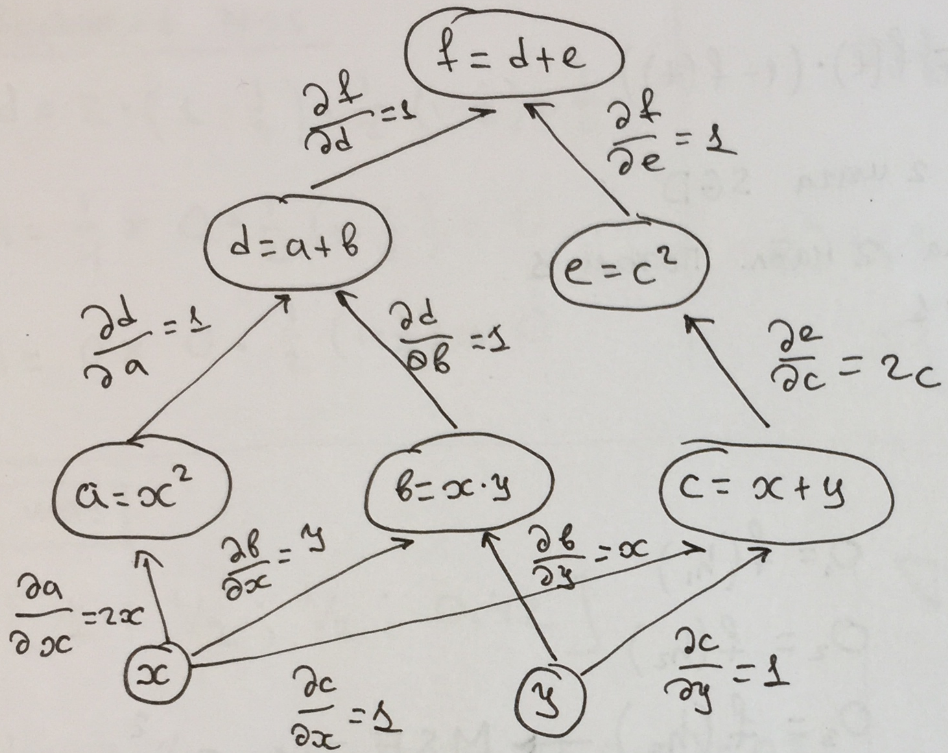
Backpropagation

Упражнение 1

$$f(x,y) = x^2 + x \cdot y + (x+y)^2$$



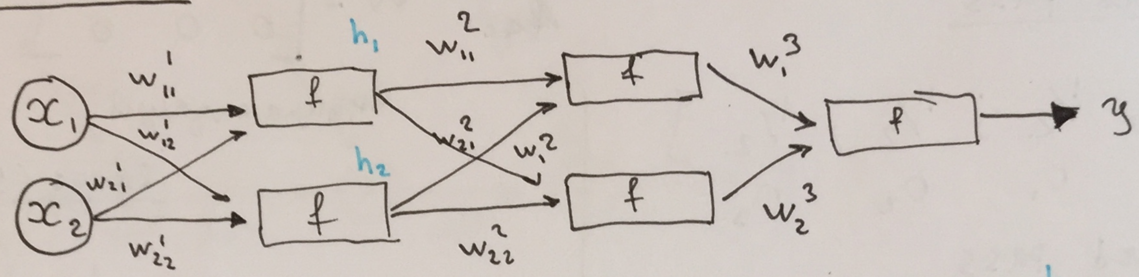
$$\frac{\partial f}{\partial x} = \underset{1}{\frac{\partial f}{\partial d}} \cdot \left(\underset{1}{\frac{\partial d}{\partial a}} \cdot \underset{2x}{\frac{\partial a}{\partial x}} + \underset{1}{\frac{\partial d}{\partial b}} \cdot \underset{y}{\frac{\partial b}{\partial x}} \right) + \underset{1}{\frac{\partial f}{\partial e}} \cdot \underbrace{\underset{2c}{\frac{\partial e}{\partial c}}}_{2(x+y)} \cdot \underset{1}{\frac{\partial c}{\partial x}}$$

$$\frac{\partial f}{\partial x} = 2x + y + 2 \cdot (x+y)$$

$$\frac{\partial f}{\partial y} = x + 2(x+y)$$

Упражнение 3

а)



$$y = f\Big(w_1^3 \cdot f\big(w_{11}^2 \cdot f(\overbrace{w_{11}^1 x_1 + w_{21}^1 x_2}^{h_1}) + w_{21}^2 \cdot f(\overbrace{w_{12}^1 \cdot x_1 + w_{22}^1 \cdot x_2}^{h_2})\big) +$$
$$+ w_2^3 \cdot f\big(w_{12}^2 \cdot f(\underbrace{w_{12}^1 x_1 + w_{21}^1 x_2}_{h_1}) + w_{22}^2 \cdot f(\underbrace{w_{12}^1 \cdot x_1 + w_{22}^1 \cdot x_2}_{h_2})\big)\Big)$$

б) $X = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ \vdots & \vdots \\ x_{1n} & x_{2n} \end{pmatrix}_{n\times 2}$ $\qquad h_1 = X \cdot W_1 = \begin{pmatrix} h_{11} & h_{21} \\ \vdots & \vdots \\ h_{1n} & h_{2n} \end{pmatrix}_{n\times 2}$

$W_1 = \begin{pmatrix} w_{11}^1 & w_{12}^1 \\ w_{21}^1 & w_{22}^2 \end{pmatrix}_{2\times 2}$

$(x_1 \; x_2) \cdot \begin{pmatrix} w_{11}^1 & w_{12}^1 \\ w_{21}^1 & w_{22}^1 \end{pmatrix} = (\cancel{x_1}\underbrace{w_{11}^1 x_1 + w_{21}^1 x_2}_{h_1} ; \underbrace{w_{12}^1 x_1 + w_{22}^1 x_2}_{h_2})$

отдельное наблюдение

Первый слой: $f(Xw_1) \rightarrow h_1$

$\cancel{y} =$ $\quad f(h_1 w_2) \rightarrow h_2$

$f(h_2 w_3) \rightarrow y$

$$y = f(f(f(xw_1)w_2)w_3)$$

в) $L(w_1, w_2, w_3) = \frac{1}{2} \cdot (y - \underbrace{f(f(f(xw_1)w_2)w_3))}_{\hat{y}})^2$

горитм.

I forward $\longrightarrow$

$[\; X \quad O_1 \quad O_2 \;]$ Запоминаем все выходы

II backward $\longleftarrow$

$d = \frac{\partial MSE}{\partial h_3}$ В d копим производную

$\boxed{\frac{\partial MSE}{\partial w_3}}$

$d = d \cdot \frac{\partial h_3}{\partial h_2}$

$\boxed{\frac{\partial MSE}{\partial w_2}}$

$d = d \cdot \frac{\partial h_2}{\partial h_1}$

$\boxed{\frac{\partial MSE}{\partial w_1}}$

Нашли за линейную сложность $\Rightarrow$ можем делать градиентный спуск.

$$\frac{\partial L}{\partial w_3} = -(y-\hat{y}) \cdot f'_{w_3}(\dots) \cdot f(\dots)$$

$$\frac{\partial L}{\partial w_2} = -(y-\hat{y}) \cdot f'_{w_2}(\dots) \cdot f'_{w_2}(\dots) \cdot w_3 \cdot f(\dots)$$

$$\frac{\partial L}{\partial w_1} = -(y-\hat{y}) \cdot f'_{w_1}(\dots) \cdot f'_{w_1}(\dots) \cdot w_3 \cdot f'_{w_1}(X w_1) \cdot X$$

2)

$$X \xrightarrow{w_1} h_1 \xrightarrow{f} O_1 \xrightarrow{w_2} h_2 \xrightarrow{f} O_2 \xrightarrow{w_3} h_3 \xrightarrow{f} O_3 = y \rightarrow MSE$$

$\times X_1$ $\quad \frac{\partial h_2}{\partial h_1} \cdot$ $\quad \times O_1$ $\quad \frac{\partial h_3}{\partial h_2} \cdot$ $\quad \times O_2$ $\quad \frac{\partial MSE}{\partial h_3}$

$h_3 = O_2 \cdot w_3$

$$\frac{\partial MSE}{\partial w_3} = \boxed{\frac{\partial MSE}{\partial O_3} \cdot \frac{\partial O_3}{\partial h_3}} \cdot \frac{\partial h_3}{\partial w_3} \quad O_2$$

$$\frac{\partial MSE}{\partial w_2} = \boxed{\frac{\partial MSE}{\partial O_3} \cdot \frac{\partial O_3}{\partial h_3}} \cdot \boxed{\frac{\partial h_3}{\partial O_2} \cdot \frac{\partial O_2}{\partial h_2}} \cdot \frac{\partial h_2}{\partial w_2} \quad O_1$$

$$\frac{\partial MSE}{\partial w_1} = \boxed{\frac{\partial MSE}{\partial O_3} \cdot \frac{\partial O_3}{\partial h_3}} \cdot \boxed{\frac{\partial h_3}{\partial O_2} \cdot \frac{\partial O_2}{\partial h_2}} \cdot \boxed{\frac{\partial h_2}{\partial O_1} \cdot \frac{\partial O_1}{\partial h_1}} \cdot \frac{\partial h_1}{\partial w_1} \quad X$$

~~Алгоритм:~~

~~I forward ⟼ запоминаем все выходы слоёв~~

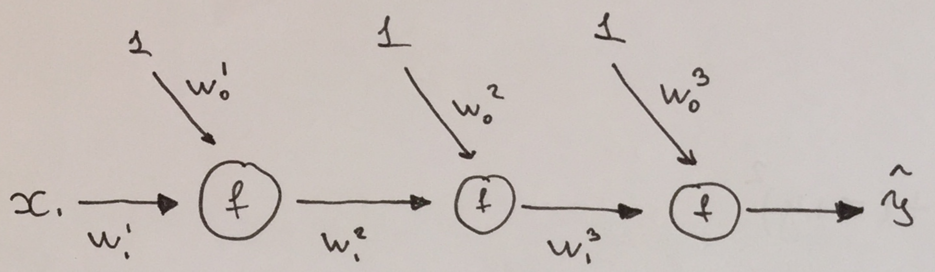
~~$[X; O_1; O_2]$~~

~~II backward ⟻~~

~~$d = \frac{\partial MSE}{\partial h_3}$~~

выходы с каждого слоя – всегда в конце.

$\boxed{№2}$



$$f(t) = \frac{e^t}{1+e^t} \qquad f'(t) = f(t)\cdot(1-f(t))$$

x	y
5	0
1	1

сделать 2 шага SGD

[сначала 2 набл. потом 1.
$\gamma = 1$

~~1 шаг~~ $\boxed{\text{готовимся}}$

$$h_1 = w_1^1 \cdot x_1 + w_0^1 \qquad O_1 = f(h_1)$$
$$h_2 = w_1^2 \cdot O_1 + w_0^2 \qquad O_2 = f(h_2)$$
$$h_3 = w_1^3 \cdot O_2 + w_0^3 \qquad O_3 = f(h_3) \longrightarrow MSE = (y - O_3)^2$$

$$[\, x \,;\, O_1 \,;\, O_2 \,]$$
$$[\, d\cdot\ldots \,;\, d\cdot\ldots \,;\, d \,]$$

$$\cancel{\frac{\partial MSE}{\partial h_3}} =$$

$$\frac{\partial MSE}{\partial h_3} = 2\cdot(y - O_3)\cdot O_3(1 - O_3)$$
$$\frac{\partial h_3}{\partial h_2} = w_1^3 \cdot O_2 \cdot (1 - O_2)$$
$$\frac{\partial h_2}{\partial h_1} = w_1^2 \cdot O_1 (1 - O_1)$$

[...]шаг

[...]orward pass

$$[1;\ 1/2;\ 1/2;\ 1/2]$$
$$x \quad o_1 \quad o_2 \quad o_3$$

backward pass

$$d = 2\cdot\left(1-\frac{1}{2}\right)\cdot\frac{1}{2}\cdot\left(1-\frac{1}{2}\right)=\frac{1}{4}$$

$$d = \frac{1}{4}\times 0\cdot\frac{1}{2}\left(1-\frac{1}{2}\right)=0$$

$$d = 0\times 0\cdot\frac{1}{2}\left(1-\frac{1}{2}\right)=0$$

$$W^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

инициализация

вход: $(1, x)$

$$\nabla \underline{L}(w) = \begin{bmatrix} 0.1 & 0.1 & \frac{1}{4}\cdot 1 \\ 0.1 & 0.\frac{1}{2} & \frac{1}{4}\cdot\frac{1}{2} \end{bmatrix}$$

$$W^1 = W^0 - \gamma\cdot\nabla L =$$
$$= \begin{bmatrix} 0 & 0 & -1/4 \\ 0 & 0 & -1/8 \end{bmatrix}$$

2 шаг

$$[1;\ 1/2;\ 1/2;\ 0.42]$$

$$d = 2\cdot(0-0.42)\cdot 0.42\cdot(1-0.42)=0.21$$

$$d = 0.21\times -\frac{1}{8}\cdot\frac{1}{2}\left(1-\frac{1}{2}\right)=-0.013$$

$$d = -0.013\times 0\cdot\frac{1}{2}\left(1-\frac{1}{2}\right)=0$$

$$\nabla L = \begin{bmatrix} 0.1 & -0.013\cdot 1 & 0.21\cdot 1 \\ 0.1 & -0.013\cdot\frac{1}{2} & 0.21\cdot\frac{1}{2} \end{bmatrix}$$

$$W^2 = W^1 - \gamma\cdot\nabla L =$$
$$= \begin{bmatrix} 0 & 0.013 & -0.46 \\ 0 & 0.007 & -0.275 \end{bmatrix}$$