Asset Correlation Estimation in the Vasicek Model

The Bias Quantification Process

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The Functional Form and Parameters of the Vasicek-Distributed Variable

The Vasicek distribution is a two-parameter ($0 < p < 1$ and $0 < \rho < 1$) continuous distribution on the range 0 to 1. If a variable x has a Vasicek distribution, then x can be represented as:

$$x = \phi\left(\frac{\phi^{-1}(p) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)$$

where:

- p and ρ are the parameters of the distribution, commonly referred to as the average default rate and asset correlation, respectively;
- z represents the systemic factor drawn from the standard normal distribution; and
- ϕ and ϕ^{-1} denote the distribution and quantile function of the standard normal distribution, respectively.

The Parameters Estimation Methods

The parameters of the Vasicek distribution can be estimated using one of the following methods:

1. Direct Moment Matching
2. Indirect Moment Matching
3. Maximizing the Log-Likelihood of the Vasicek Probability Density Function
4. Quantile-Based Estimation

While each method produces nearly unbiased estimators for parameter p, this is not the case with parameter ρ.

The following slides detail the process of bias quantification in estimating ρ (asset correlation) using the Indirect Moment Matching (IMM) method. First, we introduce the IMM estimation method, then move on to the steps involved in bias quantification, and conclude with a simulation for a hypothetical portfolios.

Note that although only one estimation method will be presented, the same process applies to others as well.

Indirect Moment Matching

$$\hat{p} = \phi\left(\frac{\hat{\mu}_x}{\sqrt{1+\hat{\sigma}_x^2}}\right)$$

$$\hat{\rho} = \frac{\hat{\sigma}_x^2}{1+\hat{\sigma}_x^2}$$

where:

- $\hat{\mu}_x$ is defined as $\hat{\mu}_x = \frac{\sum_{i=1}^{T} \phi^{-1}(x_i)}{T}$ and ϕ^{-1} denotes the quantile function of the standard normal distribution; and
- $\hat{\sigma}_x^2$ is defined as $\hat{\sigma}_x^2 = \frac{\sum_{i=1}^{T} (\phi^{-1}(x_i) - \hat{\mu}_x)^2}{T-1}$ with ϕ^{-1} being the quantile function of the standard normal distribution.

Bias Quantification Process

The following steps outline the bias quantification process using Monte Carlo simulations:

1. Collect default rate data.
2. Select the parameters estimation method for the Vasicek model.
3. Based on the collected default rates and the selected method, estimate the parameters $p_{observed}$ and $\rho_{observed}$.
4. Given the number of observations (years) of the default rate (T) and the estimated parameter $p_{observed}$, define the data-generating process of the Vasicek model with the true value of the asset correlation parameter ρ_{true}.
5. Based on the N Monte Carlo simulations, optimize the data-generating process for ρ_{true} by minimizing the difference between the average value of the ρ_{true} distribution and the $\rho_{observed}$.

Note that the presented process assumes that the parameters of the Vasicek model estimated based on the observed data represent the average value of the true parameters' distribution.

Simulation Setup and Results

The following table presents the simulation inputs of the hypothetical portfolio:

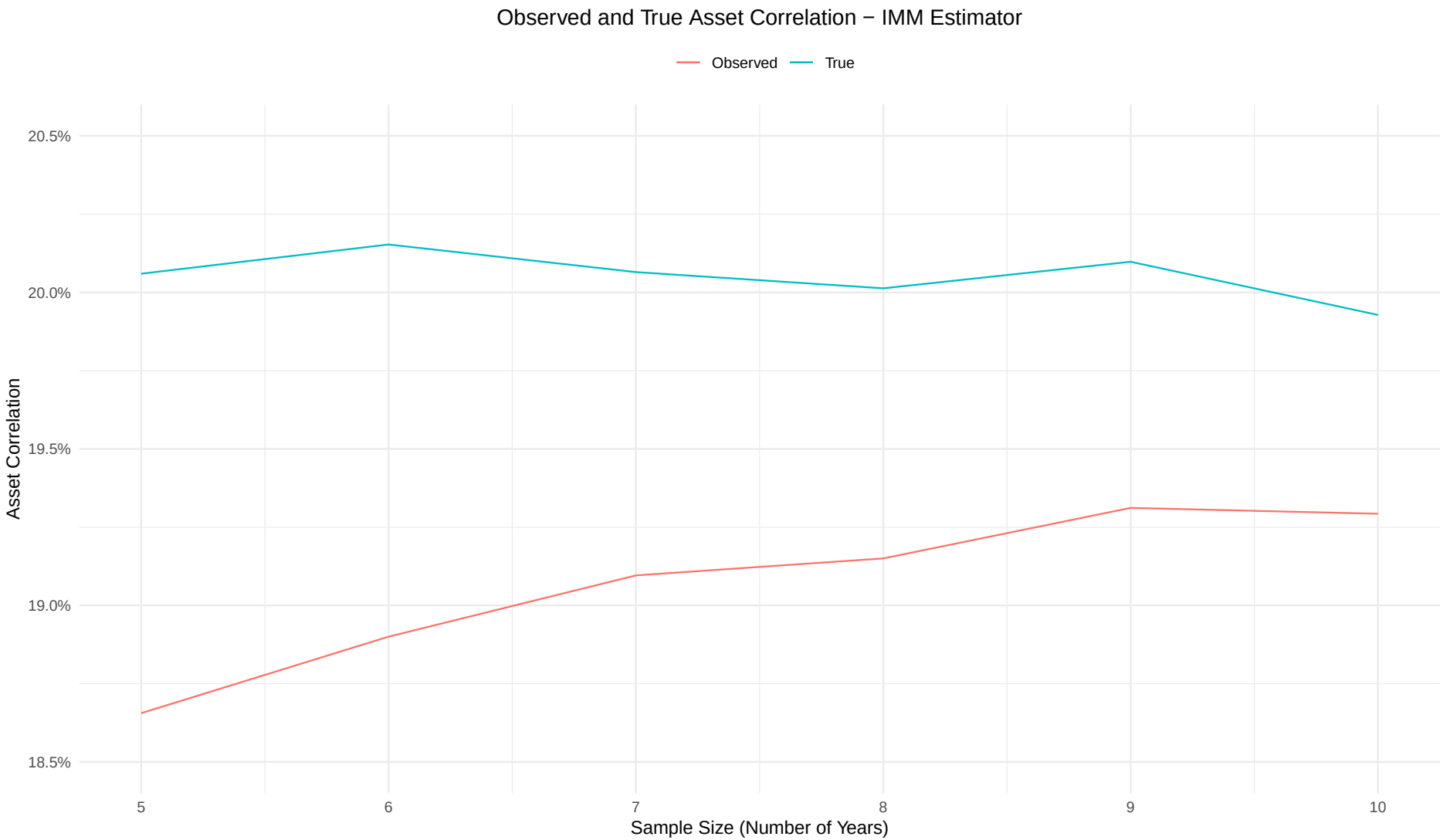
```
##    T p_observed rho_observed
##    5       0.05    0.1865590
##    6       0.05    0.1890025
##    7       0.05    0.1909595
##    8       0.05    0.1915025
##    9       0.05    0.1931158
##   10       0.05    0.1929290
```

Note that the above values are directly obtained from the Vasicek model data-generating process with the value of ρ_{true} being 0.20. Therefore, the expected bias quantification should report values close to the difference between 0.20 and the $\rho_{observed}$ values.

After running the ($N = 10,000$) Monte Carlo simulations for bias quantification of the asset correlation parameter, the following additional values have been obtained:

```
##    T p_observed rho_observed         bias  rho_true
##    5       0.05    0.1865590 -0.014037978 0.2005970
##    6       0.05    0.1890025 -0.012527848 0.2015303
##    7       0.05    0.1909595 -0.009688797 0.2006483
##    8       0.05    0.1915025 -0.008628467 0.2001309
##    9       0.05    0.1931158 -0.007862726 0.2009786
##   10       0.05    0.1929290 -0.006351165 0.1992802
```

Simulation Setup and Results cont.



Simulation Setup and Results cont.