Asset Correlation Estimation in the Vasicek Model

Maximum Likelihood: Analytical vs Numerical Optimization Approach

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The Functional Form and Parameters of the Vasicek-Distributed Variable

The Vasicek distribution is a two-parameter ($0 < p < 1$ and $0 < \rho < 1$) continuous distribution on the range 0 to 1. If a variable x has a Vasicek distribution, then x can be represented as:

$$x = \phi\left(\frac{\phi^{-1}(p) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)$$

where:

- p and ρ are the parameters of the distribution, commonly referred to as the average default rate and asset correlation, respectively;
- z represents the systemic factor drawn from the standard normal distribution; and
- ϕ and ϕ^{-1} denote the distribution and quantile function of the standard normal distribution, respectively.

The Parameters Estimation Methods

The parameters of the Vasicek distribution can be estimated using one of the following methods:

1. Direct Moment Matching
2. Indirect Moment Matching
3. Maximizing the Log-Likelihood of the Vasicek Probability Density Function
4. Quantile-Based Estimation

While each method produces nearly unbiased estimators for parameter p, this is not the case with parameter ρ.

The most frequently used method for estimating ρ in practice is Indirect Moment Matching. This method is also known as the analytical solution for the Maximum Likelihood Estimator, but how does it compare with the method of maximizing the Log-Likelihood of the Vasicek Probability Density Function?

The following slides provide a brief description of these two approaches along with simulation results for different sample sizes and selected elements of the variable with the Vasicek distribution.

Indirect Moment Matching

$$\hat{p} = \phi\left(\frac{\hat{\mu}_x}{\sqrt{1+\hat{\sigma}_x^2}}\right)$$

$$\hat{\rho} = \frac{\hat{\sigma}_x^2}{1+\hat{\sigma}_x^2}$$

where:

- $\hat{\mu}_x$ is defined as $\hat{\mu}_x = \frac{\sum_{i=1}^{T} \phi^{-1}(x_i)}{T}$ and ϕ^{-1} denotes the quantile function of the standard normal distribution; and
- $\hat{\sigma}_x^2$ is defined as $\hat{\sigma}_x^2 = \frac{\sum_{i=1}^{T} (\phi^{-1}(x_i) - \hat{\mu}_x)^2}{T-1}$ with ϕ^{-1} being the quantile function of the standard normal distribution.

Maximizing the Log-Likelihood of the Vasicek Probability Density Function

The Log-Likelihood of the Vasicek Probability Density Function is given by:

$$\sum_{i=1}^{T} ln(f_{p,\rho}(x_i))$$

with

$$f_{p,\rho}(x) = \sqrt{\frac{1-\rho}{\rho}} e^{\frac{1}{2}\left(\phi^{-1}(x)^2 - \left(\frac{\sqrt{1-\rho}\phi^{-1}(x) - \phi^{-1}(p)}{\sqrt{\rho}}\right)^2\right)}$$

where:

- T represents the number of observations;
- $f_{p,\rho}(x)$ denotes the Vasicek Probability Density Function;
- x represents the observed default rates;
- p denotes the average default rate; and
- ϕ^{-1} is the quantile function of the standard normal distribution.

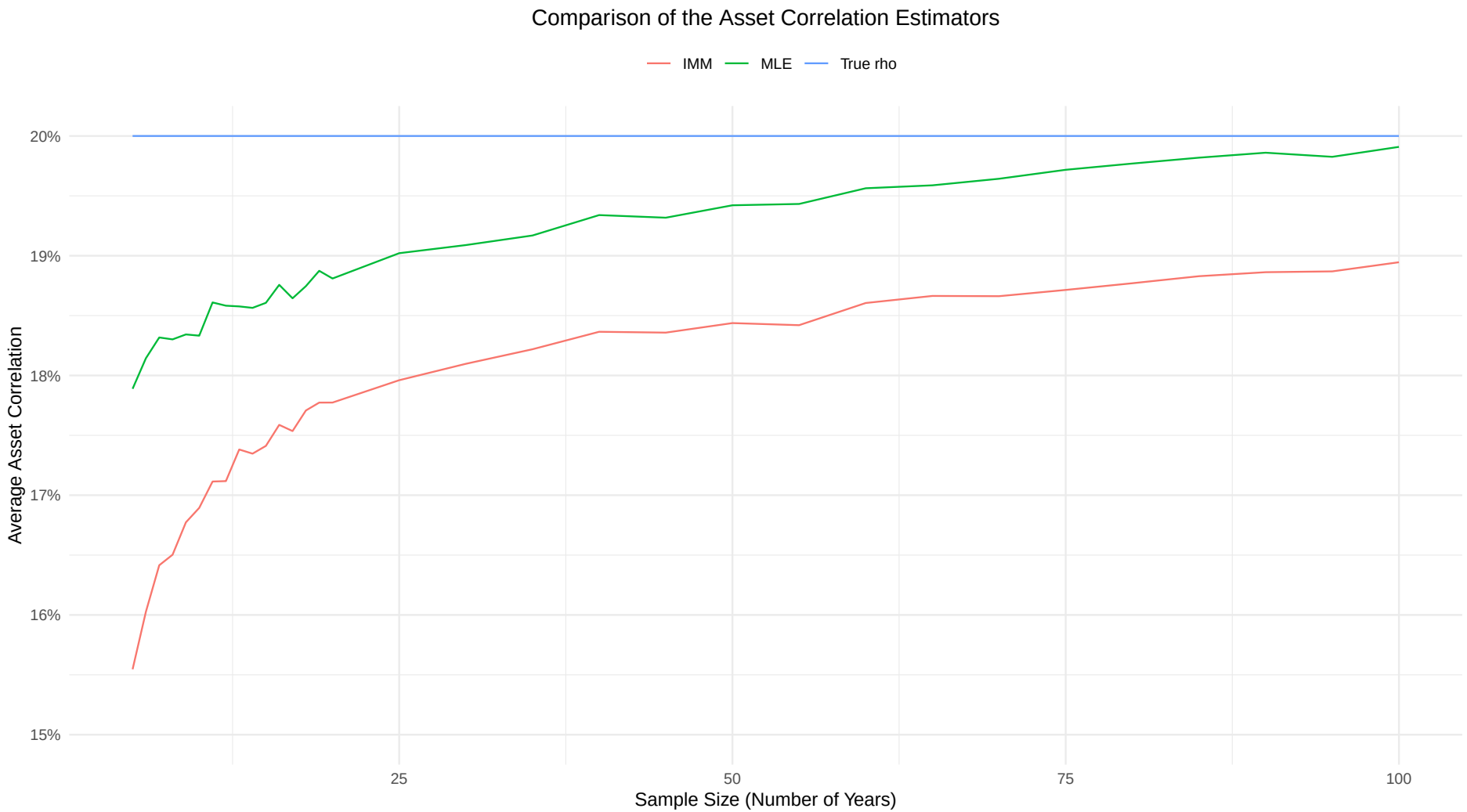
With p calculated as $\frac{\sum_{i=1}^{T} x_i}{T}$, ρ is derived by maximizing the Log-Likelihood function based on the observed default rates.

Simulation Setup

1. Select the inputs for simulating random data from the Vasicek distribution: sample size ($T = 5$), unconditional PD ($pd = 0.05$), asset correlation ($\rho = 0.20$), time-dependent systemic factor (z) from the standard normal distribution with an autoregression coefficient $\theta = 0.30$.
2. Using the simulated data, estimate the asset correlation by employing Indirect Moment Matching and maximizing the Log-Likelihood of the Vasicek Probability Density Function.
3. Store the results and repeat step 2 a total of N times ($N = 10,000$).
4. Calculate the average asset correlation for each estimation method, then compare these averages to the true ρ.
5. Change the sample size T from step 1, and repeat steps 2 through 4.

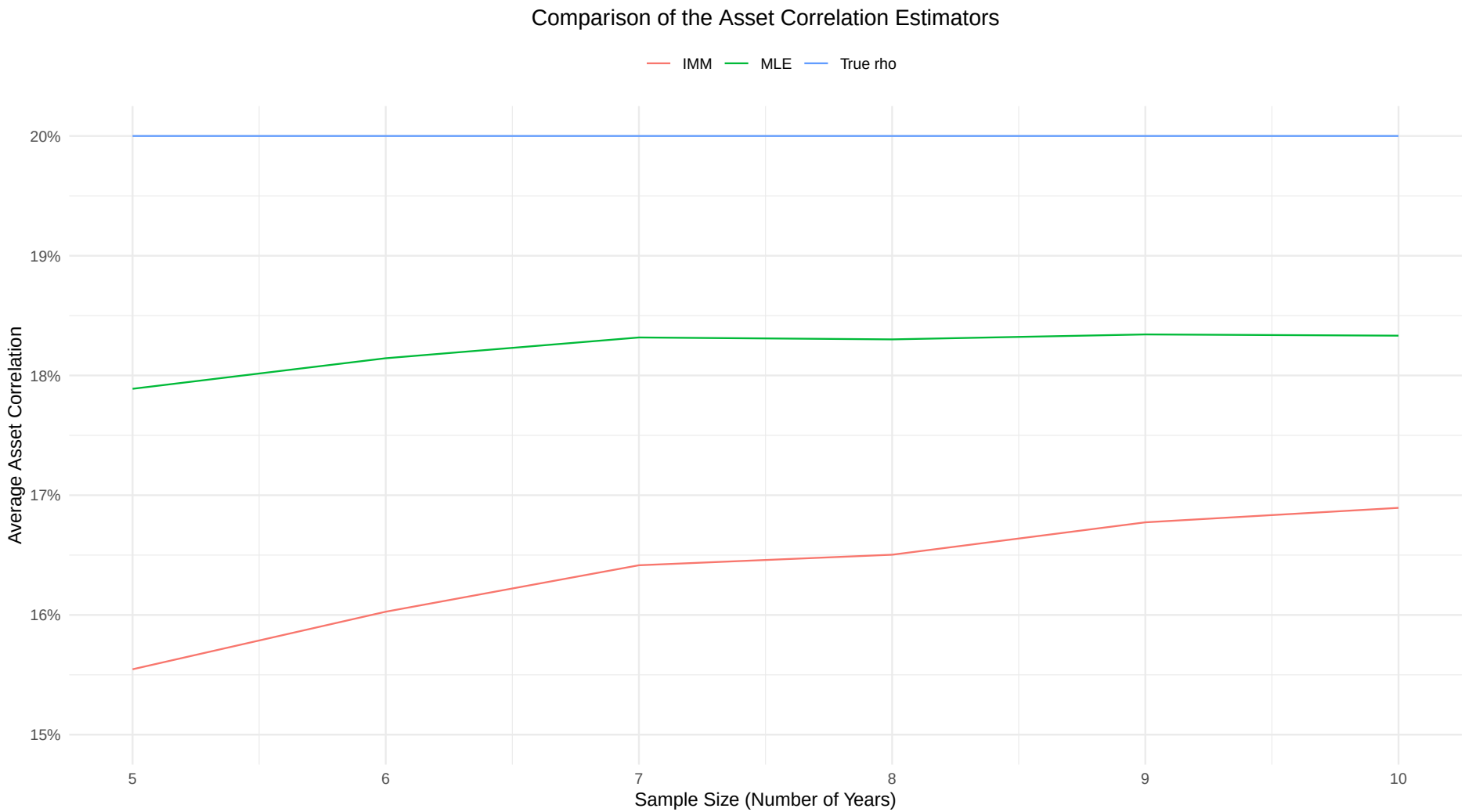
Practitioners are encouraged to test and simulate the bias of the estimators with other parameters of the Vasicek-distributed variable.

Simulation Results



IMM − Indirect Moment Matching
MLE − Maximizing the Log−Likelihood of the Vasicek pdf

Simulation Results cont.



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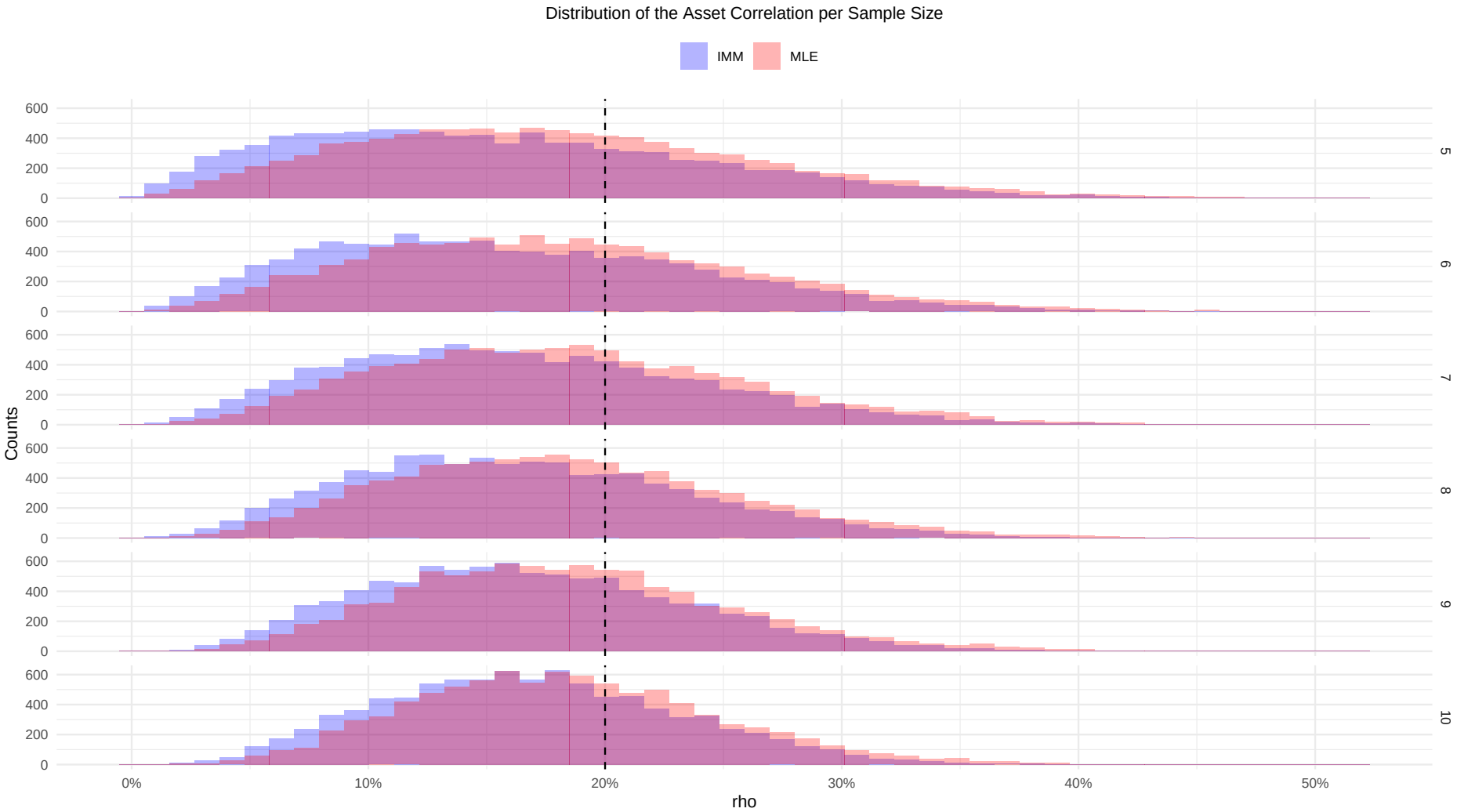
Simulation Results cont.



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