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Chapter 5
Estimating Probabilities of Default for Low Default Portfolios

Katja Pluto and Dirk Tasche

5.1 Introduction

A core input to modern credit risk modelling and managing techniques is probabilities of default (PD) per borrower. As such, the accuracy of the PD estimations will determine the quality of the results of credit risk models.

One of the obstacles connected with PD estimations can be the low number of defaults, especially in the higher rating grades. These good rating grades might enjoy many years without any defaults. Even if some defaults occur in a given year, the observed default rates might exhibit a high degree of volatility due to the relatively low number of borrowers in that grade. Even entire portfolios with low or zero defaults are not uncommon. Examples include portfolios with an overall good quality of borrowers (e.g. sovereign or bank portfolios) as well as high-exposure low-number portfolios (e.g. specialized lending).

Usual banking practices for deriving PD values in such exposures often focus on qualitative mapping mechanisms to bank-wide master scales or external ratings. These practices, while widespread in the industry, do not entirely satisfy the desire for a statistical foundation of the assumed PD values. One might "believe" that the PDs per rating grade appear correct, as well as thinking that the ordinal ranking and the relative spread between the PDs of two grades is right, but find that there is insufficient information about the absolute PD figures. Lastly, it could be questioned whether these rather qualitative methods of PD calibration fulfil the minimum requirements set out in BCBS (2004a).

The opinions expressed in this chapter are those of the authors and do not necessarily reflect views of their respective employers.

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B. Engelmann and R. Rauhmeier (eds.), *The Basel II Risk Parameters*,
DOI 10.1007/978-3-642-16114-8_5, © Springer-Verlag Berlin Heidelberg 2011

This issue, amongst others, has recently been raised in BBA (2004). In that paper, applications of causal default models and of exogenous distribution assumptions on the PDs across the grades have been proposed as solutions. Schuermann and Hanson (2004) present the "duration method" of estimating PDs by means of migration matrices (see also Jafry and Schuermann 2004). This way, nonzero PDs for high-quality rating grades can be estimated more precisely by both counting the borrower migrations through the lower grades to eventual default and using Markov chain properties.

We present a methodology to estimate PDs for portfolios without any defaults, or a very low number of defaults in the overall portfolio. The proposal by Schuermann and Hanson (2004) does not provide a solution for such cases, because the duration method requires a certain number of defaults in at least some (usually the low-quality) rating grades.

For estimating PDs, we use all available quantitative information of the rating system and its grades. Moreover, we assume that the ordinal borrower ranking is correct. We do not use any additional assumptions or information.[1] Our methodology delivers confidence intervals for the PDs of each rating grade. The PD range can be adjusted by the choice of an appropriate confidence level. Moreover, by the *most prudent estimation principle* our methodology yields monotonic PD estimates. We look both at the cases of uncorrelated and correlated default events, in the latter case under assumptions consistent with the Basel risk weight model.

Moreover, we extend the *most prudent estimation* by two application variants: First we scale our results to overall portfolio central tendencies. Second, we apply our methodology to multi-period data and extend our model by time dependencies of the Basel systematic factor. Both variants should help to align our principle to realistic data sets and to a range of assumptions that can be set according to the specific issues in question when applying our methodology.

The paper is structured as follows: The two main concepts underlying the methodology – estimating PDs as upper confidence bounds and guaranteeing their monotony by the *most prudent estimation principle* – are introduced by two examples that assume independence of the default events. The first example deals with a portfolio without any observed defaults. For the second example, we modify the first example by assuming that a few defaults have been observed. In a further section, we show how the methodology can be modified in order to take into account non-zero correlation of default events. This is followed by two sections discussing extensions of our methodology, in particular the scaling to the overall portfolio central tendency and an extension of our model to the multi-period case. The last two sections are devoted to discussions of the scope of application and of

[1] An important example of additional assumptions is provided by a-priori distributions of the PD parameters which lead to a Bayesian approach as described by Kiefer (2009). Interestingly enough, Dwyer (2006) shows that the confidence bound approach as described in this paper can be interpreted in a Bayesian manner. Another example of an additional assumption is presented in Tasche (2009). In that paper the monotonicity assumption on the PDs is replaced by a stronger assumption on the shape of the PD curve.

open questions. We conclude with a summary of our proposal. In Appendix A, we provide information on the numerics that is needed to implement the estimation approach we suggest. Appendix B provides additional numerical results to Sect. 5.5.

We perceive that our "most prudent estimation principle" has been applied in a wide range of banks since the first edition of this book. However, application has not been limited to PD *estimation*, as intended by us. Rather, risk modellers seem to have made generous use of the methodology to *validate* their rating systems. We have therefore added another short section at the end of this paper that explains the sense and non-sense of using our principle for validation purposes, and clarify what the methodology can and cannot do.

5.2 Example: No Defaults, Assumption of Independence

The obligors are distributed to rating grades A, B, and C, with frequencies n_A, n_B, and n_C. The grade with the highest credit-worthiness is denoted by A, the grade with the lowest credit-worthiness is denoted by C. No defaults occurred in A, B or C during the last observation period.

We assume that the – still to be estimated – PDs p_A of grade A, p_B of grade B, and p_C of grade C reflect the decreasing credit-worthiness of the grades, in the sense of the following inequality:

$$p_A \leq p_B \leq p_C \tag{5.1}$$

The inequality implies that we assume the ordinal borrower ranking to be correct. According to (5.1), the PD p_A of grade A cannot be greater than the PD p_C of grade C. As a consequence, the *most prudent estimate* of the value p_A is obtained under the assumption that the probabilities p_A and p_C are equal. Then, from (5.1) even follows $p_A = p_B = p_C$. Assuming this relation, we now proceed in determining a confidence region for p_A at confidence level γ. This confidence region[2] can be described as the set of all admissible values of p_A with the property that the probability of not observing any default during the observation period is not less than $1-\gamma$ (for instance for $\gamma = 90\%$).

If we have got $p_A = p_B = p_C$, then the three rating grades A, B, and C do not differ in their respective riskiness. Hence we have to deal with a homogeneous sample of size $n_A + n_B + n_C$ without any default during the observation period. Assuming unconditional independence of the default events, the probability of

[2]For any value of p_A not belonging to this region, the hypothesis that the true PD takes on this value would have to be rejected at a type I error level of 1-γ (see Casella and Berger 2002, Theorem 9.2.2 on the duality of hypothesis testing and confidence intervals).

observing no defaults turns out to be $(1 - p_A)^{n_A+n_B+n_C}$. Consequently, we have to solve the inequality

$$1 - \gamma \leq (1 - p_A)^{n_A+n_B+n_C} \tag{5.2}$$

for p_A in order to obtain the confidence region at level γ for p_A as the set of all the values of p_A such that

$$p_A \leq 1 - (1 - \gamma)^{1/(n_A+n_B+n_C)} \tag{5.3}$$

If we choose for the sake of illustration

$$n_A = 100, \quad n_B = 400, \quad n_C = 300, \tag{5.4}$$

Table 5.1 exhibits some values of confidence levels γ with the corresponding maximum values (upper confidence bounds) $\hat{p}_A$ of p_A such that (5.2) is still satisfied.

According to Table 5.1, there is a strong dependence of the upper confidence bound $\hat{p}_A$ on the confidence level γ. Intuitively, values of γ smaller than 95% seem more appropriate for estimating the PD by $\hat{p}_A$.

By inequality (5.1), the PD p_B of grade B cannot be greater than the PD p_C of grade C either. Consequently, the *most prudent estimate* of p_B is obtained by assuming $p_B = p_C$. Assuming additional equality with the PD p_A of the best grade A would violate the *most prudent estimation principle*, because p_A is a lower bound of p_B. If we have got $p_B = p_C$, then B and C do not differ in their respective riskiness and may be considered a homogeneous sample of size $n_B + n_C$. Therefore, the confidence region at level γ for p_B is obtained from the inequality

$$1 - \gamma \leq (1 - p_C)^{n_B+n_C} \tag{5.5}$$

(5.5) implies that the confidence region for p_B consists of all the values of p_B that satisfy

$$p_B \leq 1 - (1 - \gamma)^{1/(n_B+n_C)} \tag{5.6}$$

If we again take up the example described by (5.4), Table 5.2 exhibits some values of confidence levels γ with the corresponding maximum values (upper confidence bounds) $\hat{p}_B$ of p_B such that (5.6) is still fulfilled.

Table 5.1 Upper confidence bound $\hat{p}_A$ of p_A as a function of the confidence level γ. No defaults observed, frequencies of obligors in grades given in (5.4)

γ	50%	75%	90%	95%	99%	99.9%
$\hat{p}_A$	0.09%	0.17%	0.29%	0.37%	0.57%	0.86%

Table 5.2 Upper confidence bound $\hat{p}_B$ of p_B as a function of the confidence level γ. No defaults observed, frequencies of obligors in grades given in (5.4)

γ	50%	75%	90%	95%	99%	99.9%
$\hat{p}_B$	0.10%	0.20%	0.33%	0.43%	0.66%	0.98%

Table 5.3 Upper confidence bound $\hat{p}_C$ of p_C as a function of the confidence level γ. No defaults observed, frequencies of obligors in grades given in (5.4)

γ	50%	75%	90%	95%	99%	99.9%
$\hat{p}_C$	0.23%	0.46%	0.76%	0.99%	1.52%	2.28%

For determining the confidence region at level γ for p_C we only make use of the observations in grade C because by (5.1) there is no obvious upper bound for p_C. Hence the confidence region at level γ for p_C consists of those values of p_C that satisfy the inequality

$$1 - \gamma \leq (1 - p_C)^{n_C} \tag{5.7}$$

Equivalently, the confidence region for p_C can be described by

$$p_C \leq 1 - (1 - \gamma)^{1/n_C} \tag{5.8}$$

Coming back to our example (5.4), Table 5.3 lists some values of confidence levels γ with the corresponding maximum values (upper confidence bounds) $\hat{p}_C$ of p_C such that (5.8) is still fulfilled.

Comparison of Tables 5.1–5.3 shows that – besides the confidence level γ – the applicable sample size is a main driver of the upper confidence bound. The smaller the sample size, the greater will be the upper confidence bound. This is not an undesirable effect, because intuitively the credit-worthiness ought to be the better, the greater the number of obligors in a portfolio without any default observation.

As the results presented so far seem plausible, we suggest using upper confidence bounds as described by (5.3), (5.6) and (5.8) as estimates for the PDs in portfolios without observed defaults. The case of three rating grades we have considered in this section can readily be generalized to an arbitrary number of grades. We do not present the details here.

However, the larger the number of obligors in the entire portfolio, the more often some defaults will occur in some grades at least, even if the general quality of the portfolio is very high. This case is not covered by (5.3), (5.6) and (5.8). In the following section, we will show – still keeping the assumption of independence of the default events – how the *most prudent estimation* methodology can be adapted to the case of a non-zero but still low number of defaults.

5.3 Example: Few Defaults, Assumption of Independence

We consider again the portfolio from Sect. 5.2 with the frequencies n_A, n_B, and n_C. In contrast to Sect. 5.2, this time we assume that during the last period no default was observed in grade A, two defaults were observed in grade B, and one default was observed in grade C.

As in Sect. 5.2, we determine a *most prudent confidence region* for the PD p_A of A. Again, we do so by assuming that the PDs of the three grades are equal. This allows us to treat the entire portfolio as a homogeneous sample of size $n_A + n_B + n_C$. Then the probability of observing not more than three defaults is given by the expression

$$\sum_{i=0}^{3} \binom{n_A + n_B + n_C}{i} p_A^i (1 - p_A)^{n_A+n_B+n_C-i} \tag{5.9}$$

Expression (5.9) follows from the fact that the number of defaults in the portfolio is binomially distributed as long as the default events are independent. As a consequence of (5.9), the confidence region[3] at level γ for p_A is given as the set of all the values of p_A that satisfy the inequality

$$1 - \gamma \leq \sum_{i=0}^{3} \binom{n_A + n_B + n_C}{i} p_A^i (1 - p_A)^{n_A+n_B+n_C-i} \tag{5.10}$$

The tail distribution of a binomial distribution can be expressed in terms of an appropriate beta distribution function. Thus, inequality (5.10) can be solved analytically[4] for p_A. For details, see Appendix A. If we assume again that the obligors' numbers per grade are as in (5.4), Table 5.4 shows maximum solutions $\hat{p}_A$ of (5.10) for different confidence levels γ.

Although in grade A no defaults were observed, the three defaults that occurred during the observation period enter the calculation. They affect the upper confidence bounds, which are much higher than those in Table 5.1. This is a consequence of the precautionary assumption $p_A = p_B = p_C$. However, if we alternatively considered grade A alone (by re-evaluating (5.8) with $n_A = 100$ instead of n_C), we would obtain an upper confidence bound of 1.38% at level $\gamma = 75\%$. This value is still much higher than the one that has been calculated under the precautionary assumption $p_A = p_B = p_C$ – a consequence of the low frequency of obligors in grade A in this example. Nevertheless, we see that the methodology described by (5.10) yields fairly reasonable results.

---

[3]We calculate the simple and intuitive exact Clopper-Pearson interval. For an overview of this approach, as well as potential alternatives, see Brown et al. (2001).

[4]Alternatively, solving directly (5.10) for p_A by means of numerical tools is not too difficult either (see Appendix A, Proposition A.1, for additional information).

Table 5.4 Upper confidence bound $\hat{p}_A$ of p_A as a function of the confidence level γ. No default observed in grade A, two defaults observed in grade B, one default observed in grade C, frequencies of obligors in grades given in (5.4)

γ	50%	75%	90%	95%	99%	99.9%
$\hat{p}_A$	0.46%	0.65%	0.83%	0.97%	1.25%	1.62%

Table 5.5 Upper confidence bound $\hat{p}_B$ of p_B as a function of the confidence level γ. No default observed in grade A, two defaults observed in grade B, one default observed in grade C, frequencies of obligors in grades given in (5.4)

γ	50%	75%	90%	95%	99%	99.9%
$\hat{p}_B$	0.52%	0.73%	0.95%	1.10%	1.43%	1.85%

In order to determine the confidence region at level γ for p_B, as in Sect. 5.2, we assume that p_B takes its greatest possible value according to (5.1), i.e. that we have $p_B = p_C$. In this situation, we have a homogeneous portfolio with $n_B + n_C$ obligors, PD p_B, and three observed defaults. Analogous to (5.9), the probability of observing no more than three defaults in one period then can be written as:

$$\sum_{i=0}^{3} \binom{n_B + n_C}{i} p_B^i (1 - p_B)^{n_B + n_C - i} \tag{5.11}$$

Hence, the confidence region at level γ for p_B turns out to be the set of all the admissible values of p_B which satisfy the inequality

$$1 - \gamma \leq \sum_{i=0}^{3} \binom{n_B + n_C}{i} p_B^i (1 - p_B)^{n_B + n_C - i} \tag{5.12}$$

By analytically or numerically solving (5.12) for p_B – with frequencies of obligors in the grades as in (5.4) – we obtain Table 5.5 with some maximum solutions $\hat{p}_B$ of (5.12) for different confidence levels γ.

From the given numbers of defaults in the different grades, it becomes clear that a stand-alone treatment of grade B would yield still much higher values[5] for the upper confidence bounds. The upper confidence bound 0.52% of the confidence region at level 50% is almost identical with the naïve frequency based PD estimate 2/400 = 0.5% that could alternatively have been calculated for grade B in this example.

For determining the confidence region at level γ for the PD p_C, by the same rationale as in Sect. 5.2, the grade C must be considered a stand-alone portfolio. According to the assumption made in the beginning of this section, one default

[5]At level 99.9%, e.g., 2.78% would be the value of the upper confidence bound.

Table 5.6 Upper confidence bound $\hat{p}_C$ of p_C as a function of the confidence level γ. No default observed in grade A, two defaults observed in grade B, one default observed in grade C, frequencies of obligors in grades given in (5.4)

γ	50%	75%	90%	95%	99%	99.9%
$\hat{p}_C$	0.56%	0.90%	1.29%	1.57%	2.19%	3.04%

occurred among the n_C obligors in C. Hence we see that the confidence region for p_C is the set of all admissible values of p_C that satisfy the inequality

$$1 - \gamma \leq \sum_{i=0}^{1} \binom{n_C}{i} p_C^i (1 - p_C)^{n_C - i} = (1 - p_C)^{n_C} + n_C\, p_C (1 - p_C)^{n_C - 1} \qquad (5.13)$$

For obligor frequencies as assumed in example (5.4), Table 5.6 exhibits some maximum solutions[6] $\hat{p}_C$ of (5.13) for different confidence levels γ.

So far, we have described how to generalize the methodology from Sect. 5.2 to the case where non-zero default frequencies have been recorded. In the following section we investigate the impact of non-zero default correlation on the PD estimates that are effected by applying the *most prudent estimation* methodology.

5.4 Example: Correlated Default Events

In this section, we describe the dependence of the default events with the one-factor probit model[7] that was the starting point for developing the risk weight functions given in BCBS (2004a)[8]. First, we use the example from Sect. 5.2 and assume that no default at all was observed in the whole portfolio during the last period. In order to illustrate the effects of correlation, we apply the minimum value of the asset correlation that appears in the Basel II corporate risk weight function. This minimum value is 12% (see BCBS 2004a, § 272). Our model, however, works with any other correlation assumption as well. Likewise, the *most prudent estimation principle* could potentially be applied to other models than the Basel II type credit risk model as long as the inequalities can be solved for p_A, p_B and p_C, respectively.

[6] If we had assumed that two defaults occurred in grade B but no default was observed in grade C, then we would have obtained smaller upper bounds for p_C than for p_B. As this is not a desirable effect, a possible – conservative – work-around could be to increment the number of defaults in grade C up to the point where p_C would take on a greater value than p_B. Nevertheless, in this case one would have to make sure that the applied rating system yields indeed a correct ranking of the obligors.

[7] According to De Finetti's theorem (see, e.g., Durrett (1996), Theorem 6.8), assuming one systematic factor only is not very restrictive.

[8] See Gordy (2003) and BCBS (2004b) for the background of the risk weight functions. In the case of non-zero realized default rates Balthazar (2004) uses the one-factor model for deriving confidence intervals of the PDs.

Table 5.7 Upper confidence bounds $\hat{p}_A$ of p_A, $\hat{p}_B$ of p_B and $\hat{p}_C$ of p_C as a function of the confidence level γ. No defaults observed, frequencies of obligors in grades given in (5.4). Correlated default events

γ	50%	75%	90%	95%	99%	99.9%
$\hat{p}_A$	0.15%	0.40%	0.86%	1.31%	2.65%	5.29%
$\hat{p}_B$	0.17%	0.45%	0.96%	1.45%	2.92%	5.77%
$\hat{p}_C$	0.37%	0.92%	1.89%	2.78%	5.30%	9.84%

Under the assumptions of this section, the confidence region at level γ for p_A is represented as the set of all admissible values of p_A that satisfy the inequality (cf. Bluhm et al. 2003, Sects. 2.1.2 and 2.5.1 for the derivation)

$$1 - \gamma \leq \int_{-\infty}^{\infty} \varphi(y) \left(1 - \Phi\left(\frac{\Phi^{-1}(p_A) - \sqrt{\rho} y}{\sqrt{1-\rho}}\right)\right)^{n_A+n_B+n_C} dy, \tag{5.14}$$

where φ and Φ stand for the standard normal density and standard normal distribution function, respectively. Φ^{-1} denotes the inverse function of Φ and ρ is the *asset correlation* (here ρ is chosen as $\rho = 12\%$). Similarly to (5.2), the right-hand side of inequality (5.14) tells us the one-period probability of not observing any default among $n_A + n_A + n_A$ obligors with average PD p_A.

Solving[9] (5.14) numerically[10] for the frequencies as given in (5.4) leads to Table 5.7 with maximum solutions $\hat{p}_A$ of (5.14) for different confidence levels γ.

Comparing the values from the first line of Table 5.7 with Table 5.1 shows that the impact of taking care of correlations is moderate for the low confidence levels 50% and 75%. The impact is much higher for the levels higher than 90% (for the confidence level 99.9% the bound is even six times larger). This observation reflects the general fact that introducing unidirectional stochastic dependence in a sum of random variables entails a redistribution of probability mass from the centre of the distribution towards its lower and upper limits.

The formulae for the estimations of upper confidence bounds for p_B and p_C can be derived analogously to (5.14) [in combination with (5.5) and (5.7)]. This yields the inequalities

$$1 - \gamma \leq \int_{-\infty}^{\infty} \varphi(y) \left(1 - \Phi\left(\frac{\Phi^{-1}(p_B) - \sqrt{\rho} y}{\sqrt{1-\rho}}\right)\right)^{n_B+n_C} dy \tag{5.15}$$

[9]See Appendix A, Proposition A.2, for additional information. Taking into account correlations entails an increase in numerical complexity. Therefore, it might seem to be more efficient to deal with the correlation problem by choosing an appropriately enlarged confidence level in the independent default events approach as described in Sects. 5.2 and 5.3. However, it remains open how a confidence level for the uncorrelated case, that "appropriately" adjusts for the correlations, can be derived.

[10]The more intricate calculations for this paper were conducted by means of the software package R (cf. R Development Core Team 2003).

Table 5.8 Upper confidence bounds $\hat{p}_A$ of p_A, $\hat{p}_B$ of p_B and $\hat{p}_C$ of p_C as a function of the confidence level γ. No default observed in grade A, two defaults observed in grade B, one default observed in grade C, frequencies of obligors in grades given in (5.4). Correlated default events

γ	50%	75%	90%	95%	99%	99.9%
$\hat{p}_A$	0.72%	1.42%	2.50%	3.42%	5.88%	10.08%
$\hat{p}_B$	0.81%	1.59%	2.77%	3.77%	6.43%	10.92%
$\hat{p}_C$	0.84%	1.76%	3.19%	4.41%	7.68%	13.14%

and

$$1-\gamma \le \int_{-\infty}^{\infty} \varphi(y)\left(1-\Phi\left(\frac{\Phi^{-1}(p_C)-\sqrt{\rho}y}{\sqrt{1-\rho}}\right)\right)^{n_C} dy, \tag{5.16}$$

to be solved for p_B and p_C respectively. The numerical calculations with (5.15) and (5.16) do not deliver additional qualitative insights. For the sake of completeness, however, the maximum solutions $\hat{p}_B$ of (5.15) and $\hat{p}_C$ of (5.16) for different confidence levels γ are listed in rows 3 and 4 of Table 5.7, respectively.

Secondly, we apply our correlated model to the example from Sect. 5.3 and assume that three defaults were observed during the last period. Analogous to (5.9), (5.10) and (5.14), the confidence region at level γ for p_A is represented as the set of all values of p_A that satisfy the inequality

$$\begin{aligned} 1-\gamma &\le \int_{-\infty}^{\infty} \varphi(y) z(y) dy, \\ z(y) &= \sum_{i=0}^{3} \binom{n_A+n_B+n_C}{i} G(p_A,\rho,y)^i (1-G(p_A,\rho,y))^{n_A+n_B+n_C-i}, \end{aligned} \tag{5.17}$$

where the function G is defined by

$$G(p,\rho,y) = \Phi\left(\frac{\Phi^{-1}(p)-\sqrt{\rho}y}{\sqrt{1-\rho}}\right). \tag{5.18}$$

Solving (5.17) for $\hat{p}_A$ with obligor frequencies as given in (5.4), and the respective modified equations for $\hat{p}_B$ and $\hat{p}_C$ yields the results presented in Table 5.8.

Not surprisingly, as shown in Table 5.8 the maximum solutions for $\hat{p}_A$, $\hat{p}_B$ and $\hat{p}_C$ increase if we introduce defaults in our example. Other than that, the results do not deliver essential additional insights.

5.5 Extension: Calibration by Scaling Factors

One of the drawbacks of the *most prudent estimation principle* is that in the case of few defaults, the upper confidence bound PD estimates for all grades are higher than the average default rate of the overall portfolio. This phenomenon is not surprising,

Table 5.9 Upper confidence bound $\hat{p}_{A,\text{scaled}}$ of p_A, $\hat{p}_{B,\text{scaled}}$ of p_B and $\hat{p}_{C,\text{scaled}}$ of p_C as a function of the confidence level γ after scaling to the central tendency. No default observed in grade A, two defaults observed in grade B, one default observed in grade C, frequencies of obligors in grades given in (5.4). Uncorrelated default events

γ	50%	75%	90%	95%	99%	99.9%
Central Tendency	0.375%	0.375%	0.375%	0.375%	0.375%	0.375%
K	0.71	0.48	0.35	0.30	0.22	0.17
$\hat{p}_A$	0.33%	0.31%	0.29%	0.29%	0.28%	0.27%
$\hat{p}_B$	0.37%	0.35%	0.34%	0.33%	0.32%	0.31%
$\hat{p}_C$	0.40%	0.43%	0.46%	0.47%	0.49%	0.50%

Table 5.10 Upper confidence bound $\hat{p}_{A,\text{scaled}}$ of p_A, $\hat{p}_{B,\text{scaled}}$ of p_B and $\hat{p}_{C,\text{scaled}}$ of p_C as a function of the confidence level γ after scaling to the central tendency. No default observed in grade A, two defaults observed in grade B, one default observed in grade C, frequencies of obligors in grades given in (5.4). Correlated default events

γ	50%	75%	90%	95%	99%	99.9%
Central Tendency	0.375%	0.375%	0.375%	0.375%	0.375%	0.375%
K	0.46	0.23	0.13	0.09	0.05	0.03
$\hat{p}_A$	0.33%	0.33%	0.32%	0.32%	0.32%	0.32%
$\hat{p}_B$	0.38%	0.37%	0.36%	0.36%	0.35%	0.35%
$\hat{p}_C$	0.39%	0.40%	0.41%	0.42%	0.42%	0.42%

given that we include all defaults of the overall portfolio in the upper confidence bound estimation even for the highest rating grade. However, these estimates might be regarded as too conservative by some practitioners.

A remedy would be a scaling[11] of all of our estimates towards the central tendency (the average portfolio default rate). We introduce a scaling factor K to our estimates such that the overall portfolio default rate is exactly met, i.e.

$$\frac{\hat{p}_A n_A + \hat{p}_B n_B + \hat{p}_C n_C}{n_A + n_B + n_C} K = PD_{\text{Portfolio}}. \tag{5.19}$$

The new, scaled PD estimates will then be

$$\hat{p}_{X,\text{scaled}} = K \hat{p}_X, \quad X = A, B, C. \tag{5.20}$$

The results of the application of such a scaling factor to our "few defaults" examples of Sects. 5.3 and 5.4 are shown in Tables 5.9 and 5.10, respectively.

The average estimated portfolio PD will now fit exactly the overall portfolio central tendency. Thus, we remove all conservatism from our estimations. Given the poor default data base in typical applications of our methodology, this might be seen as a disadvantage rather than an advantage. By using the *most prudent estimation*

[11] A similar scaling procedure was suggested by Benjamin et al. (2006). However, the straightforward linear approach as described in (5.19) and (5.20) has the drawback that, in principle, the resulting PDs can exceed 100%. See Tasche (2009, Appendix A) for a non-linear scaling approach based on Bayes' formula that avoids this issue.

principle to derive "relative" PDs before scaling them down to the final results, we preserve the sole dependence of the PD estimates upon the borrower frequencies in the respective rating grades, as well as the monotony of the PDs.

The question of the appropriate confidence level for the above calculations remains. Although the average estimated portfolio PD now always fits the overall portfolio default rate, the confidence level determines the "distribution" of that rate over the rating grades. In the above example, though, the differences in distribution appear small, especially in the correlated case, such that we would not explore this issue further. The confidence level could, in practice, be used to control the spread of PD estimates over the rating grades – the higher the confidence level, the higher the spread.

However, the above scaling works only if there is a nonzero number of defaults in the overall portfolio. Zero default portfolios would indeed be treated more severely if we continued to apply our original proposal to them, compared to using scaled PDs for low default portfolios.

A variant of the above scaling proposal that takes care of both issues is the use of an upper confidence bound for the overall portfolio PD in lieu of the actual default rate. This upper confidence bound for the overall portfolio PD, incidentally, equals the *most prudent estimate* for the highest rating grade. Then, the same scaling methodology as described above can be applied. The results of its application to the few defaults examples as in Tables 5.9 and 5.10 are presented in Tables 5.11 and 5.12.

Table 5.11 Upper confidence bound $\hat{p}_{A,\text{scaled}}$ of p_A, $\hat{p}_{B,\text{scaled}}$ of p_B and $\hat{p}_{C,\text{scaled}}$ of p_C as a function of the confidence level γ after scaling to the upper confidence bound of the overall portfolio PD. No default observed in grade A, two defaults observed in grade B, one default observed in grade C, frequencies of obligors in grades given in (5.4). Uncorrelated default events

γ	50%	75%	90%	95%	99%	99.9%
Upper bound for portfolio PD	0.46%	0.65%	0.83%	0.97%	1.25%	1.62%
K	0.87	0.83	0.78	0.77	0.74	0.71
$\hat{p}_A$	0.40%	0.54%	0.65%	0.74%	0.92%	1.16%
$\hat{p}_B$	0.45%	0.61%	0.74%	0.84%	1.06%	1.32%
$\hat{p}_C$	0.49%	0.75%	1.01%	1.22%	1.62%	2.17%

Table 5.12 Upper confidence bound $\hat{p}_{A,\text{scaled}}$ of p_A, $\hat{p}_{B,\text{scaled}}$ of p_B and $\hat{p}_{C,\text{scaled}}$ of p_C as a function of the confidence level γ after scaling to the upper confidence bound of the overall portfolio PD. No default observed in grade A, two defaults observed in grade B, one default observed in grade C, frequencies of obligors in grades given in (5.4). Correlated default events

γ	50%	75%	90%	95%	99%	99.9%
Upper bound for portfolio PD	0.71%	1.42%	2.50%	3.42%	5.88%	10.08%
K	0.89	0.87	0.86	0.86	0.86	0.87
$\hat{p}_A$	0.64%	1.24%	2.16%	2.95%	5.06%	8.72%
$\hat{p}_B$	0.72%	1.38%	2.39%	3.25%	5.54%	9.54%
$\hat{p}_C$	0.75%	1.53%	2.76%	3.80%	6.61%	11.37%

In contrast to the situation of Tables 5.9 and 5.10, in Tables 5.11 and 5.12 the overall default rate in the portfolio depends on the confidence level, and we observe scaled PD estimates for the grades that increase with growing levels. Nevertheless, the scaled PD estimates for the better grades are still considerably lower than the corresponding unscaled estimates from Sects. 5.3 and 5.4, respectively. For the sake of comparison, we provide in Appendix B the analogous numerical results for the no default case.

The advantage of this latter variant of the scaling approach is that the degree of conservatism is actively manageable by the appropriate choice of the confidence level for the estimation of the upper confidence bound of the portfolio PD. Moreover, it works in the case of zero defaults and few defaults, and thus does not produce a structural break between both scenarios. Lastly, the results are less conservative than those of our original methodology.

5.6 Extension: The Multi-period Case

So far, we have only considered the situation where estimations are carried out on a 1 year (or one observation period) data sample. In case of a time series with data from several years, the PDs (per rating grade) for the single years could be estimated and could then be used for calculating weighted averages of the PDs in order to make more efficient use of the data. By doing so, however, the interpretation of the estimates as upper confidence bounds at some pre-defined level would be lost.

Alternatively, the data of all years could be pooled and tackled as in the 1-year case. When assuming cross-sectional and inter-temporal independence of the default events, the methodology as presented in Sects. 5.2 and 5.3 can be applied to the data pool by replacing the 1-year frequency of a grade with the sum of the frequencies of this grade over the years (analogous for the numbers of defaulted obligors). This way, the interpretation of the results as upper confidence bounds as well as the frequency-dependent degree of conservatism of the estimates will be preserved.

However, when turning to the case of default events which are cross-sectionally and inter-temporally correlated, pooling does not allow for an adequate modelling. An example would be a portfolio of long-term loans, where in the inter-temporal pool every obligor would appear several times. As a consequence, the dependence structure of the pool would have to be specified very carefully, as the structure of correlation over time and of cross-sectional correlation are likely to differ.

In this section, we present two multi-period extensions of the cross-sectional one-factor correlation model that has been introduced in Sect. 5.4. In the first part of the section, we take the perspective of an observer of a cohort of obligors over a fixed interval of time. The advantage of such a view arises from the conceptual separation of time and cross-section effects. Again, we do not present the methodology in full generality but rather introduce it by way of an example.

As in Sect. 5.4, we assume that, at the beginning of the observation period, we have got n_A obligors in grade A, n_B obligors in grade B, and n_C obligors in grade C. In contrast to Sect. 5.4, the length of the observation period this time is $T > 1$. We consider only the obligors that were present at the beginning of the observation period. Any obligors entering the portfolio afterwards are neglected for the purpose of our estimation exercise. Nevertheless, the number of observed obligors may vary from year to year as soon as any defaults occur.

As in the previous sections, we first consider the estimation of the PD p_A for grade A. PD in this section denotes a long-term average 1-year probability of default. Working again with the *most prudent estimation principle*, we assume that the PDs p_A, p_B, and p_C are equal, i.e. $p_A = p_B = p_C = p$. We assume, similar to Gordy (2003), that a default of obligor $i = 1, \ldots, N = n_A + n_B + n_C$ in year $t = 1, \ldots, T$ is triggered if the change in value of their assets results in a value lower than some default threshold c as described below by (5.22). Specifically, if $V_{i,t}$ denotes the change in value of obligor i's assets, $V_{i,t}$ is given by

$$V_{i,t} = \sqrt{\rho}\, S_t + \sqrt{1-\rho}\, \xi_{i,t}, \tag{5.21}$$

where ρ stands for the *asset correlation* as introduced in Sect. 5.4, S_t is the realisation of the *systematic factor* in year t, and $\xi_{i,t}$ denotes the *idiosyncratic* component of the change in value. The cross-sectional dependence of the default events stems from the presence of the systematic factor S_t in all the obligors' change in value variables. Obligor i's default occurs in year t if

$$V_{i,1} > c, \ldots, V_{i,t-1} > c, V_{i,t} \le c. \tag{5.22}$$

The probability

$$\mathrm{P}[\mathrm{V}_{\mathrm{i,t}} \le c] = p_{i,t} = p \tag{5.23}$$

is the parameter we are interested to estimate: It describes the long-term average 1-year probability of default among the obligors that have not defaulted before. The indices i and t at $p_{i,t}$ can be dropped because by the assumptions we are going to specify below $p_{i,t}$ will neither depend on i nor on t. To some extent, therefore, p may be considered a *through-the-cycle* PD.

For the sake of computational feasibility, and in order to keep as close as possible to the Basel II risk weight model, we specify the factor variables S_t, $t = 1,\ldots,T$, and $\xi_{i,t}$, $i = 1,\ldots,N$, $t = 1,\ldots,T$ as standard normally distributed (cf. Bluhm et al. 2003). Moreover, we assume that the random vector $(S_1,\ldots,S_T)$ and the random variables $\xi_{i,t}$, $i = 1,\ldots,N$, $\mathrm{t} = 1,\ldots,T$ are independent. As a consequence, from (5.21) it follows that the change in value variables $V_{i,t}$ are all standard-normally distributed. Therefore, (5.23) implies that the default threshold[12] c is determined by

[12] At first sight, the fact that in our model the default threshold is constant over time seems to imply that the model does not reflect the possibility of rating migrations. However, by construction of the

$$c = \Phi^{-1}(p), \tag{5.24}$$

with Φ denoting the standard normal distribution function.

While the single components S_t of the vector of systematic factors, generate the cross-sectional correlation of the default events at time t, their inter-temporal correlation is affected by the dependence structure of the factors $S_1,\ldots,S_T$. We further assume that not only the components but also the vector as a whole is normally distributed. Since the components of the vector are standardized, its joint distribution is completely determined by the correlation matrix

$$\begin{pmatrix} 1 & r_{1,2} & r_{1,3} & \cdots & r_{1,T} \\ r_{2,1} & 1 & r_{2,3} & \cdots & r_{2,T} \\ \vdots & & \ddots & & \vdots \\ & & & & \\ r_{T,1} & & & r_{T,T-1} & 1 \end{pmatrix}. \tag{5.25}$$

Whereas the cross-sectional correlation within 1 year is constant for any pair of obligors, empirical observation indicates that the effect of inter-temporal correlation becomes weaker with increasing distance in time. We express this distance-dependent behaviour[13] of correlations by setting in (5.25)

$$r_{s,t} = \vartheta^{|s-t|}, \quad s,t = 1,\cdots,T,\ s \neq t, \tag{5.26}$$

for some appropriate $0 < \vartheta < 1$ to be specified below.

Let us assume that within the T years observation period k_A defaults were observed among the obligors that were initially graded A, k_B defaults among the initially graded B obligors and k_C defaults among the initially graded C obligors. For the estimation of p_A according to the most prudent estimation principle, therefore we have to take into account $k = k_A + k_B + k_C$ defaults among N obligors over T years. For any given confidence level γ, we have to determine the maximum value $\hat{p}$ of all the parameters p such that the inequality

$$1 - \gamma \leq \mathrm{P}[\text{No more than } k \text{ defaults observed}] \tag{5.27}$$

is satisfied – note that the right-hand side of (5.27) depends on the one-period probability of default p. In order to derive a formulation that is accessible to numerical calculation, we have to rewrite the right-hand side of (5.27).

model, the *conditional* default threshold at time t given the value $V_{i,t-1}$ will in general differ from c. As we make use of the joint distribution of the $V_{i,t}$, therefore rating migrations are implicitly taken into account.

[13]Blochwitz et al. (2004) proposed the specification of the inter-temporal dependence structure according to (5.26) for the purpose of default probability estimation.

The first step is to develop an expression for obligor i's conditional probability to default during the observation period, given a realization of the systematic factors $S_1,\ldots,S_T$. From (5.21), (5.22), (5.24) and by using the conditional independence of the $V_{i,1}, \ldots, V_{i,T}$ given the systematic factors, we obtain

$$\begin{aligned} &P[\text{Obligor } i \text{ defaults } |S_1, \cdots, \mathrm{S_T}] \\ &\quad = \mathrm{P}\left[\min_{t=1,\ldots,T} V_{i,t} \le \Phi^{-1}(p)|S_1,\ldots,S_T\right] \\ &\quad = 1 - \mathrm{P}\left[\xi_{i,1} > \frac{\Phi^{-1}(p)-\sqrt{\rho}\,S_1}{\sqrt{1-\rho}},\ldots,\xi_{i,T} > \frac{\Phi^{-1}(p)-\sqrt{\rho}\,S_T}{\sqrt{1-\rho}}|S_1,\ldots,S_T\right] \\ &\quad = 1 - \prod_{t=1}^{T}(1-G(p,\rho,S_t)), \end{aligned} \tag{5.28}$$

where the function G is defined as in (5.18). By construction, in the model all the probabilities P[Obligor i defaults $|S_1$, ..., S_T] are equal, so that, for any of the i, we can define

$$\begin{aligned} \pi(S_1,\ldots,S_T) &= \mathrm{P}[\text{Obligor } i \text{ defaults} \mid S_1,\ldots,S_T] \\ &= 1 - \prod_{t=1}^{T}(1-G(p,\rho,S_t)) \end{aligned} \tag{5.29}$$

Using this abbreviation, we can write the right-hand side of (5.27) as

$$\begin{aligned} &\mathrm{P}[\text{No more than } k \text{ defaults observed}] \\ &\quad = \sum_{l=0}^{k} \mathrm{E}[\mathrm{P}[\text{Exactly } l \text{ obligors default} \mid S_1,\ldots,S_T]] \\ &\quad = \sum_{l=0}^{k} \binom{N}{l} \mathrm{E}[\pi(S_1,\ldots,S_T)^l(1-\pi(S_1,\ldots,S_T))^{N-l}]. \end{aligned} \tag{5.30}$$

The expectations in (5.30) are expectations with respect to the random vector $(S_1, \ldots, S_T)$ and have to be calculated as T-dimensional integrals involving the density of the T-variate standard normal distribution with correlation matrix given by (5.25) and (5.26). When solving (5.27) for $\hat{p}$, we calculated the values of these T-dimensional integrals by means of Monte-Carlo simulation, taking advantage of the fact that the term

$$\sum_{l=0}^{k} \binom{N}{l} \mathrm{E}[\pi(S_1,\ldots,S_T)^l(1-\pi(S_1,\ldots,S_T))^{N-l}] \tag{5.31}$$

can be efficiently evaluated by making use of (5.35) of Appendix A.

In order to present some numerical results for an illustration of how the model works, we have to fix a time horizon T and values for the cross-sectional correlation ρ and the inter-temporal correlation parameter ϑ. We choose $T = 5$ as BCBS (2004a) requires the credit institutions to base their PD estimates on a time series with minimum length 5 years. For ρ, we chose $\rho = 0.12$ as in Sect. 5.4, i.e. again a value suggested by BCBS (2004a). Our feeling is that default events with a 5 years time distance can be regarded as being nearly independent. Statistically, this statement might be interpreted as something like "the correlation of S_1 and S_5 is less than 1%". Setting $\vartheta = 0.3$, we obtain corr$[S_1,\ldots,S_T] = \vartheta^4 = 0.81\%$. Thus, the choice $\vartheta = 0.3$ seems reasonable. Note that our choices of the parameters are purely exemplary, as to some extent choosing the values of the parameters is rather a matter of taste or judgement or of decisions depending on the available data or the purpose of the estimations.[14]

Table 5.13 shows the results of the calculations for the case where no defaults were observed during 5 years in the whole portfolio. The results for all the three grades are summarized in one table. To arrive at these results, (5.27) was first evaluated with $N = n_A + n_B + n_C$, then with $N = n_B + n_C$, and finally with $N = n_C$. In all three cases we set $k = 0$ in (5.30) in order to express that no defaults were observed. Not surprisingly, the calculated confidence bounds are much lower than those presented as in Table 5.7, thus demonstrating the potentially dramatic effect of exploiting longer observation periods.

For Table 5.14 we conducted essentially the same computations as for Table 5.13, the difference being that we assumed that over 5 years $k_A = 0$, defaults were observed in grade A, $k_B = 2$ defaults were observed in grade B, and $k_C = 1$

Table 5.13 Upper confidence bounds $\hat{p}_A$ of p_A, $\hat{p}_B$ of p_B and $\hat{p}_C$ of p_C as a function of the confidence level γ. No defaults during 5 years observed, frequencies of obligors in grades given in (5.4). Cross-sectionally and inter-temporally correlated default events

γ	50%	75%	90%	95%	99%	99.9%
$\hat{p}_A$	0.03%	0.06%	0.11%	0.16%	0.30%	0.55%
$\hat{p}_B$	0.03%	0.07%	0.13%	0.18%	0.33%	0.62%
$\hat{p}_C$	0.07%	0.14%	0.26%	0.37%	0.67%	1.23%

Table 5.14 Upper confidence bounds $\hat{p}_A$ of p_A, $\hat{p}_B$ of p_B and $\hat{p}_C$ of p_C as a function of the confidence level γ. During 5 years, no default observed in grade A, two defaults observed in grade B, one default observed in grade C, frequencies of obligors in grades given in (5.4). Cross-sectionally and inter-temporally correlated default events

γ	50%	75%	90%	95%	99%	99.9%
$\hat{p}_A$	0.12%	0.21%	0.33%	0.43%	0.70%	1.17%
$\hat{p}_B$	0.14%	0.24%	0.38%	0.49%	0.77%	1.29%
$\hat{p}_C$	0.15%	0.27%	0.46%	0.61%	1.01%	1.70%

[14] Benjamin et al. (2006) propose a similar methodology that pools multi-year data into one large pool of customers. Effectively, they implicitly assume identical cross-borrower and intra-temporal correlations and disregard borrower duplication within the observation period.

defaults were observed in grade C (as in Sects. 5.3 and 5.4 during 1 year). Consequently, we set $k = 3$ in (5.30) for calculating the upper confidence bounds for p_A and p_B, as well as $k = 1$ for the upper confidence bounds of p_C. Compared with the results presented in Table 5.8, we observe again the very strong effect of taking into account a longer time series.

The methodology described above could be christened "*cohort approach*" – as cohorts of borrowers are observed over multiple years. It does not take into account any changes in portfolio size due to new lending or repayment of loans. Moreover, the approach ignores the information provided by time clusters of defaults (if there are any). Intuitively, time-clustering of defaults should be the kind of information needed to estimate the cross-sectional and time-related correlation parameters ρ and ϑ respectively[15].

A slightly different multi-period approach (called "*multiple binomial*" in the following) allows for variation of portfolio size by new lending and amortization and makes it possible, in principle, to estimate the correlation parameters. In particular this approach ignores the fact that most of the time the portfolio composition this year and next year is almost identical. However, it will turn out that as a consequence of the conditional independence assumptions we have adopted the impact of ignoring the almost constant portfolio composition is reasonably weak.

Assume that the portfolio size in year t was N_t for $t = 1, \ldots, T$, and that d_t defaults were observed in year t. Given realisations $S_1, \ldots, S_T$ of the systematic factors, we then assume that the distribution of the number of defaults in year t conditional on $S_1, \ldots, S_T$ is binomial as in (5.17) and (5.18), i.e.

$$\begin{aligned} &\mathrm{P}[d_t \text{ defaults in year } t | S_1, \ldots, S_T] \\ &= \binom{N_t}{d_t} G(p, \rho, S_t)^{d_t} (1 - G(p, \rho, S_t))^{N_t - d_t} \end{aligned} \tag{5.32}$$

Under the additional assumption of conditional independence of default events at different moments in time conditional on a realisation of the systematic factors, (5.32) implies that the unconditional probability to observe d_1 defaults in year $1, \ldots, d_T$ defaults in year T is given by

$$\begin{aligned} &\mathrm{P}[d_t \text{ defaults in year } t,\ t = 1, \ldots, T] \\ &= \mathrm{E}[\mathrm{P}[d_t \text{ defaults in year } t,\ t = 1, \ldots, T | S_1, \ldots, S_T]] \\ &= \mathrm{E}\left[\prod_{t=1}^{T} \binom{N_t}{d_t} G(p, \rho, S_t)^{d_t} (1 - G(p, \rho, S_t))^{N_t - d_t}\right] \end{aligned} \tag{5.33}$$

[15]Indeed, it is possible to modify the cohort approach in such a way as to take account of portfolio size varying due to other causes than default and of time-clusters of default. This modification, however, comes at a high price because it requires a much more complicated input data structure that causes much longer calculation time.

As (5.33) involves a binomial distribution for each point in time t we call the approach the "multiple binomial" approach. If we assume that the latent systematic factors follow a T-dimensional normal distribution with standard normal marginals as specified by (5.25) and (5.26), then calculation of the right-hand side of (5.33) involves the evaluation of a T-dimensional integral. This can be done by Monte-Carlo simulation as in the case of (5.31).

By means of an appropriate optimisation method[16], the right-hand side of (5.33) can be used as the likelihood function for the determination of joint maximum likelihood estimates of the correlation parameters ρ and ϑ and of the long-run PD parameter p. It however requires at least one of the annual default number observations d_t to be positive. Otherwise the likelihood (5.33) is constant equal to 100% for $p = 0$ and it is not possible to identify unique parameters ρ and ϑ that maximise the likelihood. In the context of Table 5.14, if we assume that the three defaults occurred in the first year and consider the entire portfolio, the maximum likelihood estimates of ρ, ϑ and p are 34.3%, 0%, and 7.5 bps respectively.

In the case where values of the correlation parameters are known or assumed to be known, it is also possible to use the multiple binomial approach to compute confidence bound type estimates of the long-run grade-wise PD estimates as was done for Table 5.14. To be able to do this calculation with the multiple binomial approach, we need to calculate the unconditional probability that the total number of defaults in years 1 to T does not exceed $d = d_1 + \cdots + d_T$. As the sum of binomially distributed random variables with different success probabilities in general is not binomially distributed, we calculate an approximate value of the required unconditional probability based on Poisson approximation:

$$\begin{aligned} &\mathrm{P}[\text{No more than } d \text{ defaults in years 1 to } T] \\ &\quad \approx \mathrm{E}\left[\exp\left(-I_{\rho,p}(S_1, \ldots, S_T)\right) \sum_{k=0}^{d} \frac{I_{\rho,p}(S_1, \ldots, S_T)^k}{k!}\right], \\ &I_{\rho,p}(s_1, \ldots, s_T) = \sum_{t=1}^{T} N_t\, G(p, \rho, S_t). \end{aligned} \tag{5.34}$$

The expected value in (5.34) again has to be calculated by Monte-Carlo simulation. Table 5.15 shows the results of such a calculation in the context of Table 5.13 [i.e. Table 5.13 is recalculated based on (5.34) instead of (5.31)].

Similarly, Table 5.16 displays the recalculated Table 5.14 [i.e. Table 5.14 is recalculated based on (5.34) instead of (5.31)]. Both in Table 5.15 and Table 5.16 results seem hardly different to the results in Table 5.13 and Table 5.14 respectively. Hence the use of (5.34) instead of (5.31) in order to allow for different portfolio sizes due to new lending and amortisation appears to be justified.

[16]For the numerical examples in this paper, the authors made use of the R-procedure nlminb.

Table 5.15 Upper confidence bounds $\hat{p}_A$ of p_A, $\hat{p}_B$ of p_B and $\hat{p}_C$ of p_C as a function of the confidence level γ. No defaults during 5 years observed, frequencies of obligors in grades given in (5.4). Cross-sectionally and inter-temporally correlated default events. Calculation based on (5.34)

Γ	50%	75%	90%	95%	99%	99.9%
$\hat{p}_A$	0.02%	0.05%	0.10%	0.15%	0.29%	0.53%
$\hat{p}_B$	0.03%	0.06%	0.12%	0.17%	0.32%	0.60%
$\hat{p}_C$	0.06%	0.13%	0.26%	0.36%	0.66%	1.19%

Table 5.16 Upper confidence bounds $\hat{p}_A$ of p_A, $\hat{p}_B$ of p_B and $\hat{p}_C$ of p_C as a function of the confidence level γ. During 5 years, no default observed in grade A, two defaults observed in grade B, one default observed in grade C, frequencies of obligors in grades given in (5.4). Cross-sectionally and inter-temporally correlated default events. Calculation based on (5.34)

γ	50%	75%	90%	95%	99%	99.9%
$\hat{p}_A$	0.12%	0.21%	0.33%	0.42%	0.68%	1.12%
$\hat{p}_B$	0.13%	0.23%	0.37%	0.47%	0.76%	1.24%
$\hat{p}_C$	0.14%	0.26%	0.44%	0.59%	0.99%	1.66%

5.7 Applications

The *most prudent estimation* methodology described in the previous sections can be used for a range of applications, both within a bank and in a Basel II context. In the latter case, it might be specifically useful for portfolios where neither internal nor external default data are sufficient to meet the Basel requirements. A good example might be Specialized Lending. In these high-volume, low-number and low-default portfolios, internal data is often insufficient for PD estimations per rating category, and might indeed even be insufficient for central tendency estimations for the entire portfolio (across all rating grades). Moreover, mapping to external ratings – although explicitly allowed in the Basel context and widely used in bank internal applications – might be impossible due to the low number of externally rated exposures.

The (conservative) principle of the *most prudent estimation* could serve as an alternative to the Basel slotting approach, subject to supervisory approval. In this context, the proposed methodology might be interpreted as a specific form of the Basel requirement of conservative estimations if data is scarce.

In a wider context, within the bank, the methodology might be used for all sorts of low default portfolios. In particular, it could complement other estimation methods, whether this be mapping to external ratings, the proposals by Schuermann and Hanson (2004) or others. As such, we see our proposed methodology as an additional source for PD calibrations. This should neither invalidate nor prejudge a bank's internal choice of calibration methodologies.

However, we tend to believe that our proposed methodology should only be applied to whole rating systems and portfolios. One might think of calibrating PDs of individual low default rating grades within an otherwise rich data structure.

Doing so almost unavoidably leads to a structural break between average PDs (data rich rating grades) and upper PD bounds (low default rating grades) which makes the procedure appear infeasible. Similarly, we believe that the application of the methodology for backtesting or similar validation tools would not add much additional information. For instance, purely expert-based average PDs per rating grade would normally be well below our proposed quantitative upper bounds.

5.8 Open Issues

For applications, a number of important issues need to be addressed:

- Which confidence levels are appropriate? The proposed most prudent estimate could serve as a conservative proxy for average PDs. In determining the confidence level, the impact of a potential underestimation of these average PDs should be taken into account. One might think that the transformation of average PDs into some kind of "stress" PDs, as done in the Basel II and many other credit risk models, could justify rather low confidence levels for the PD estimation in the first place (i.e. using the models as providers of additional buffers against uncertainty). However, this conclusion would be misleading, as it mixes two different types of "stresses": the Basel II model "stress" of the single systematic factor over time, and the estimation uncertainty "stress" of the PD estimations. Indeed, we would argue for moderate confidence levels when applying the *most prudent estimation* principle, but for other reasons. The most common alternative to our methodology, namely deriving PDs from averages of historical default rates per rating grade, yields a comparable probability that the true PD will be underestimated. Therefore, high confidence levels in our methodology would be hard to justify.
- At which number of defaults should users deviate from our methodology and use "normal" average PD estimation methods, at least for the overall portfolio central tendency? Can this critical number be analytically determined?
- If the relative number of defaults in one of the better ratings grades is significantly higher than those in lower rating grades (and within low default portfolios, this might happen with only one or two additional defaults), then our PD estimates may turn out to be non-monotonic. In which cases should this be taken as an indication of an incorrect ordinal ranking? Certainly, monotony or non-monotony of our upper PD bounds does not immediately imply that the average PDs are monotonic or non-monotonic. Under which conditions would there be statistical evidence of a violation of the monotony requirement for the PDs?

Currently, we do not have definite solutions to above issues. We believe, though, that some of them will involve a certain amount of expert judgment rather than analytical solutions. In particular, that might be the case with the first item. If our proposed approach were used in a supervisory – say Basel II – context, supervisors might want to think about suitable confidence levels that should be consistently applied.

5.9 Estimation Versus Validation

We have been somewhat surprised to see the methodology described in this chapter being often applied for PD validation rather than PD estimation. This new section for the second edition of the book sets out principles as to when and when not apply the methodology for PD estimation, as well as examples where application might be useful in practice.

First, the low default estimation methodology based on upper confidence bounds has a high degree of inbuilt conservatism. Comparing default rates or PDs estimated by other methodologies against confidence-bound-based PDs must take this estimation bias into account – having observed default rates not breaching our upper confidence bounds should not be regarded as a particular achievement, and observing default rates above the confidence bounds may indicate a serious PD underestimation indeed.

Second, spreading the central tendency of a portfolio across rating grades via the most prudent estimation principle has the grade PDs, in effect, solely driven by grade population and the confidence level. There are limits as to how wide the central tendency can be statistically spread, implying that the slope of the most prudent PDs over rating grades tends to be much flatter than PDs curves derived by alternative methods (e.g. benchmarking to external ratings).

So which benefits can be derived from validation via benchmarking against low default estimates based on upper confidence bounds? As the low default methodology delivers conservative PD estimates, it can offer some insight into the degree of conservatism for PDs calibrated by another method.

For a given PD estimate (derived, for example, by benchmarking to external ratings) and an observed number of defaults, an intermediate step of the calculation of upper confidence bounds gives an *implied confidence level* that would have delivered the same PD from the default rate via the confidence bound calculation. Indeed, using (5.27) with the given PD estimate to determine γ generates an implied confidence level as desired.

While there is no test as to which confidence level is "too conservative" in this context, the approach offers an opportunity for the quantification of conservatism that might be helpful in bank internal and regulatory discussions. The approach is most useful for central tendency comparisons – application at grade level may result in very different confidence levels across the rating scale due to the low number of defaults. The interpretation of such fluctuating levels then becomes somewhat of a challenge. The approach might yield useful results over time, however, as the implicit confidence level changes. The volatility can give some qualitative indication as to how much "point in time" or "through the cycle" a rating system is – the latter should result in higher volatility as observed default rates are always point in time.

5.10 Conclusions

In this article, we have introduced a methodology for estimating probabilities of default in low or no default portfolios. The methodology is based on upper confidence intervals by use of the *most prudent estimation*. Our methodology uses all available quantitative information. In the extreme case of no defaults in the entire portfolio, this information consists solely of the absolute numbers of counter-parties per rating grade.

The lack of defaults in the entire portfolio prevents *reliable* quantitative statements on both the absolute level of *average* PDs per rating grade as well as on the relative risk increase from rating grade to rating grade. Within the *most prudent estimation* methodology, we do not use such information. The only additional assumption used is the *ordinal* ranking of the borrowers, which is assumed to be correct.

Our PD estimates might seem rather high at first sight. However, given the amount of information that is actually available, the results do not appear out of range. We believe that the choice of moderate confidence levels is appropriate within most applications. The results can be scaled to any appropriate central tendency. Additionally, the multi-year context as described in Sect. 5.6 might provide further insight.

Appendix A

This appendix provides additional information on the analytical and numerical solutions of (5.10) and (5.14).

Analytical solution of (5.10). If X is a binomially distributed random variable with size parameter n and success probability p, then for any integer $0 \le k \le n$, we have

$$\sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} = \mathrm{P}[X \le k] = 1 - \mathrm{P}[Y \le p] = \frac{\int_p^1 t^k (1-t)^{n-k-1} dt}{\int_0^1 t^k (1-t)^{n-k-1} dt} \tag{5.35}$$

with Y denoting a beta distributed random variable with parameters $\alpha = k + 1$ and $\beta = n - k$ (see, e.g., Hinderer (1980), Lemma 11.2). The beta distribution function and its inverse function are available in standard numerical tools, e.g. in Excel.

Direct numerical solution of Equation (5.10). The following proposition shows the existence and uniqueness of the solution of (5.10), and, at the same time, provides initial values for the numerical root-finding [see (5.38)].

Proposition A.1. *Let* $0 \leq k < n$ *be integers, and define the function* $f_{n,k}$: $(0, 1) \to \mathrm{R}$ *by*

$$f_{n,k}(p) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i},\ p \in (0,1) \tag{5.36}$$

Fix some $0 < v < 1$. *Then the equation*

$$f_{n,k}(p) = v \tag{5.37}$$

has exactly one solution $0 < p = p(v) < 1$. *Moreover, this solution p(v) satisfies the inequalitie*s

$$1 - \sqrt[n]{v} \leq p(v) \leq \sqrt[n]{1-v} \tag{5.38}$$

Proof. A straight-forward calculation yields

$$\frac{df_{n,k}(p)}{dp} = -(n-k)\binom{n}{k} p^k (1-p)^{n-k-1}. \tag{5.39}$$

Hence $f_{n,k}$ is strictly decreasing. This implies uniqueness of the solution of (5.37). The inequalities

$$f_{n,0}(p) \leq f_{n,k}(p) \leq f_{n,n-1}(p) \tag{5.40}$$

imply the existence of a solution of (5.37) and the inequalities (5.38).

Numerical solution of (5.14). For (5.14) we can derive a result similar to Proposition A.1. However, there is no obvious upper bound to the solution *p(v)* of (5.42) as in (5.38).

Proposition A.2. *For any probability* $0 < p < 1$, *any correlation* $0 < \rho < 1$ *and any real number y define*

$$F_\rho(p, y) = \Phi\left(\frac{\Phi^{-1}(p) + \sqrt{\rho}\, y}{\sqrt{1-\rho}}\right), \tag{5.41}$$

where we make use of the same notations as for (5.14). *Fix a value* $0 < v < 1$ *and a positive integer n. Then the equation*

$$v = \int_{-\infty}^{\infty} \varphi(y)(1 - F_\rho(p, y))^n dy, \tag{5.42}$$

with φ *denoting the standard normal density, has exactly one solution* $0 < p = p(v) < 1$. *This solution p(v) satisfies the inequality*

$$p(v) \geq 1 - \sqrt[n]{v}. \tag{5.43}$$

Proof of Proposition (**A.2**) Note that – for fixed ρ and y – the function $F_\rho(p, y)$ is strictly increasing and continuous in p. Moreover, we have

$$0 = \lim_{p \to 0} F_\rho(p, y) \text{ and } 1 = \lim_{p \to 1} F_\rho(p, y) \tag{5.44}$$

Equation (5.44) implies existence and uniqueness of the solution of (5.42).
Define the random variable Z by

$$Z = F_\rho(p, Y), \tag{5.45}$$

where Y denotes a standard normally distributed random variable. Then Z has the well-known *Vasicek distribution* (cf. Vasicek 1997), and in particular we have

$$\mathrm{E}[Z] = p. \tag{5.46}$$

Using (5.45), (5.42) can be rewritten as

$$v = \mathrm{E}[(1 - Z)^n]. \tag{5.47}$$

Since $y \to (1- \ y)^n$ is convex for $0 < y < 1$, by (5.46) Jensen's inequality implies

$$v = \mathrm{E}[(1 - Z)^n] \geq (1 - p)^n. \tag{5.48}$$

As the right-hand side of (5.42) is decreasing in p, (5.43) now follows from (5.48).

Appendix B

This appendix provides additional numerical results for the "scaling" extension of the *most prudent estimation* principle according to Sect. 5.5 in the case of no default portfolios. In the examples presented in Tables 5.17 and 5.18, the confidence level for deriving the upper confidence bound for the overall portfolio PD, and the confidence levels for the *most prudent estimates* of PDs per rating grade have

Table 5.17 Upper confidence bound $\hat{p}_{A,\text{scaled}}$ of p_A, $\hat{p}_{B,\text{scaled}}$ of p_B and $\hat{p}_{C,\text{scaled}}$ of p_C as a function of the confidence level γ after scaling to the upper confidence bound of the overall portfolio PD. No default observed, frequencies of obligors in grades given in (5.4). Uncorrelated default events

γ	50%	75%	90%	95%	99%	99.9%
Central tendency	0.09%	0.17%	0.29%	0.37%	0.57%	0.86%
K	0.61	0.66	0.60	0.58	0.59	0.59
$\hat{p}_A$	0.05%	0.11%	0.17%	0.22%	0.33%	0.51%
$\hat{p}_B$	0.06%	0.13%	0.20%	0.25%	0.39%	0.58%
$\hat{p}_C$	0.14%	0.24%	0.45%	0.58%	0.89%	1.35%

Table 5.18 Upper confidence bound $\hat{p}_{A,\text{scaled}}$ of p_A, $\hat{p}_{B,\text{scaled}}$ of p_B and $\hat{p}_{C,\text{scaled}}$ of p_C as a function of the confidence level γ after scaling to the upper confidence bound of the overall portfolio PD. No default observed, frequencies of obligors in grades given in (5.4). Correlated default events

γ	50%	75%	90%	95%	99%	99.9%
Central tendency	0.15%	0.40%	0.86%	1.31%	2.65%	5.29%
K	0.62	0.65	0.66	0.68	0.70	0.73
$\hat{p}_A$	0.09%	0.26%	0.57%	0.89%	1.86%	3.87%
$\hat{p}_B$	0.11%	0.29%	0.64%	0.98%	2.05%	4.22%
$\hat{p}_C$	0.23%	0.59%	1.25%	1.89%	3.72%	7.19%

always been set equal. Moreover, our methodology always provides equality between the upper bound of the overall portfolio PD and the *most prudent estimate* for p_A according to the respective examples of Sects. 5.2 and 5.4.

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