The Vasicek Distribution

The functional form and parameters estimation methods

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The Functional Form and Parameters

The Vasicek distribution is a two-parameter ($0 < p < 1$ and $0 < \rho < 1$) continuous distribution on the range 0 to 1. If a variable x has a Vasicek distribution, then x can be represented as:

$$x = \phi\left(\frac{\phi^{-1}(p) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)$$

where:

- p and ρ are the parameters of the distribution, commonly referred to as the average default rate and asset correlation, respectively;
- z represents the systemic factor drawn from the standard normal distribution; and
- ϕ and ϕ^{-1} denote the distribution and quantile function of the standard normal distribution, respectively.

Distribution, Density, and Quantile Functions

Cumulative distribution function:

$$F_{p,\rho}(x) = \phi\left(\frac{\sqrt{1-\rho}\ \phi^{-1}(x) - \phi^{-1}(p)}{\sqrt{\rho}}\right)$$

Probability density function:

$$f_{p,\rho}(x) = \sqrt{\frac{1-\rho}{\rho}} e^{\frac{1}{2}\left(\phi^{-1}(x)^2 - \left(\frac{\sqrt{1-\rho}\phi^{-1}(x) - \phi^{-1}(p)}{\sqrt{\rho}}\right)^2\right)}$$

Quantile function:

$$F_{p,\rho}^{-1}(\alpha) = \phi\left(\frac{\phi^{-1}(p) + \sqrt{\rho}\ \phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)$$

The Parameters Estimation Methods

The parameters of the Vasicek distribution can be estimated using one of the following methods:

1. Direct Moment Matching
2. Indirect Moment Matching
3. Maximizing the Log-Likelihood of the Vasicek Probability Density Function
4. Quantile-Based Estimation

Direct Moment Matching

$$\hat{p} = \frac{\sum_{i=1}^{T} x_i}{T}$$

$$\hat{\sigma}_x^2 = \phi_2(y_1 \leq \phi^{-1}(\hat{p}), y_2 \leq \phi^{-1}(\hat{p}), \rho) - \hat{p}^2$$

where:

- T denotes the number of observations;
- x_i represents the observed default rates;
- $\hat{\sigma}_x^2$ is the variance of the observed default rates;
- ϕ_2 denotes the bivariate standard normal cumulative distribution function with $\mu_{y_i} = 0$ and $\sigma_{y_i} = 1$; and
- ϕ^{-1} stands for the quantile function of the standard normal distribution.

The asset correlation parameter ρ is estimated based on a numerical root-finding procedure.

Indirect Moment Matching

$$\hat{p} = \phi\left(\frac{\hat{\mu}_x}{\sqrt{1+\hat{\sigma}_x^2}}\right)$$

$$\hat{\rho} = \frac{\hat{\sigma}_x^2}{1+\hat{\sigma}_x^2}$$

where:

- $\hat{\mu}_x$ is defined as $\hat{\mu}_x = \frac{\sum_{i=1}^{T} \phi^{-1}(x_i)}{T}$ and ϕ^{-1} denotes the quantile function of the standard normal distribution; and
- $\hat{\sigma}_x^2$ is defined as $\hat{\sigma}_x^2 = \frac{\sum_{i=1}^{T} (\phi^{-1}(x_i) - \hat{\mu}_x)^2}{T-1}$ with ϕ^{-1} being the quantile function of the standard normal distribution.

This method is also known as the analytical solution for the Maximum Likelihood Estimator.

Maximizing the Log-Likelihood of the Vasicek Probability Density Function

The Log-Likelihood of the Vasicek Probability Density Function is given by:

$$\sum_{i=1}^{T} ln(f_{p,\rho}(x_i))$$

with

$$f_{p,\rho}(x) = \sqrt{\frac{1-\rho}{\rho}} e^{\frac{1}{2}\left(\phi^{-1}(x)^2 - \left(\frac{\sqrt{1-\rho}\phi^{-1}(x) - \phi^{-1}(p)}{\sqrt{\rho}}\right)^2\right)}$$

where:

- T represents the number of observations;
- $f_{p,\rho}(x)$ denotes the Vasicek Probability Density Function;
- x represents the observed default rates;
- p denotes the average default rate; and
- ϕ^{-1} is the quantile function of the standard normal distribution.

With p calculated as $\frac{\sum_{i=1}^{T} x_i}{T}$, ρ is derived by maximizing the Log-Likelihood function based on the observed default rates.

Quantile-Based Estimation

After selecting the probabilities (α_1 and α_2), a system of two equations ($\hat{\mu}_x$ and $\hat{\sigma}_x$) is solved to obtain estimates for p and ρ using the formulas derived from the Indirect Moment Matching method.

$$\hat{q}(\alpha_1) = \hat{\mu}_x + \hat{\sigma}_x \phi^{-1}(\alpha_1)$$

$$\hat{q}(\alpha_2) = \hat{\mu}_x + \hat{\sigma}_x \phi^{-1}(\alpha_2)$$

where:

- $\hat{q}$ represents the observed quantiles for the selected probabilities α_1 and α_1; and
- ϕ^{-1} is the quantile function of the standard normal distribution.