# Low Default Portfolios - Conservative Estimation of Default Probabilities

## Benjamin-Cathcart-Ryan Approach

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# Low Default Portfolio

- Qualitative descriptions of an Low Default Portfolio (LDP) leave ample room for interpretation, as different individuals may have varying opinions on whether a given portfolio qualifies as an LDP.
- The low number of defaults within a LDP undermines estimates' reliability and statistical validity for quantitative risk parameters based on historical default experience.
- LDPs are not necessarily low-data portfolios. The scarcity of defaults needs to be considered in relation to the size of the portfolio producing them.
- Regulators may be concerned that Probability of Default (PD) estimates based solely on simple historical averages or judgmental considerations may underestimate the bank's capital requirements due to default scarcity.
- The Benjamin-Cathcart-Ryan (BCR) approach is one way to address the estimation of PD for the LDP.
- Alternative methods are also available and warrant examination and comparison with the BCR approach.

# BCR Proposal for the PDs Adjustment

1. Identify the LDP.
2. Identify the historical sample for the LDP (multi-year period) using internal data. The sample should specify each obligor's grade at the start of each year of the historical period.
3. Determine the PD for each grade, derived subjectively, using an analytical method or combining these two methods.
4. Aggregate multi-year data into a single pool of obligors.
5. Calculate the portfolio PD as a weighted average of the grade-level PDs.
6. Compare the portfolio PD with the upper bound of the portfolio PD obtained from the BCR method (explained in the following slides).
7. If the upper bound portfolio PD is higher than the portfolio PD, rescale the grade-level PD to match the upper bound.

The BCR approach adapts the Pluto and Tasche method by consolidating multi-year data into a single pool of obligors. More information on this modification is available here.

# BCR Method - Upper Bound of the Portfolio PD

1. The change in the value of obligor $i$'s assets over a year $t$ is given by:

$$y_{i,t} = \sqrt{\rho} z_t + \sqrt{1-\rho}\epsilon_{i,t}$$

where $\rho$ is asset correlation, $z$ and $\epsilon_i$ systemic and idiosyncratic factor, respectively.
The obligor defaults if the asset value is lower than the value of $c$ for the specific year. Given the above assumptions, we are interested in the probability ($PD$):

$$P(y_{i,t} \leq c) = PD_{i,t} = PD$$

which describes the long-term average 1-year probability of default among the obligors that have not defaulted before and may be considered a through-the-cycle PD.

2. Given the number of years ($T$) and year-to-year correlation ($\theta$) we simulate the systemic factor for each year as follows:

$$z_{t,t\leq T} = \theta z_{t-1} + \sqrt{1-\theta^2}\epsilon_t$$

where $z_{t,t=1}$ and $\epsilon_t$ are drawn from the standard normal distribution (`N[0, 1]`).

3. Using the $z_{t,t\leq T}$ we define the conditional PD ($PD_{c_t}$) for each year $t, t \leq T$ as follows:

$$PD_{c_t} = N\left[\frac{N^{-1}(PD) - z_t\sqrt{\rho}}{\sqrt{1-\rho}}\right]$$

where $N$ and $N^{-1}$ represent the distribution and quantile function of the normal standard distribution, respectively.

# BCR Method - Upper Bound of the Portfolio PD cont.

4. For the multi-year period, we calculate the cumulative $PD$ as follows:

$$PD = 1 - \prod_{t=1}^{T}(1 - PD_{c_t})$$

5. Given the average number of obligors per year $n$, the total number of defaults $k$, and the cumulative probability of default $PD$, we can calculate the likelihood of observing no more than $k$ defaults out of $n$ obligors as:

$$1 - \gamma = \sum_{i=0}^{k} \binom{n}{i} PD^{i}(1 - PD)^{n-i}$$

where $\gamma$ denotes the selected confidence level.

6. Finally, we determine the upper bound of the portfolio PD through numerical optimization for $PD$ using Monte Carlo simulations to minimize the difference between the average simulated and the chosen confidence level ($\gamma$).

# Simulation Datasets

Number of obligors in each grade per year:

```
##   GRADE Y1 Y2 Y3 Y4 Y5
## 1     A  8  7  6  4  1
## 2     B 24 25 25 24 24
## 3     C 37 37 36 37 35
## 4     D 23 24 25 26 25
## 5     E  5  6  4  4  5
## 6     F  1  3  3  2  5
## 7     G  1  1  2  2  3
```

Number of defaults in each grade during the year:

```
##   GRADE Y1 Y2 Y3 Y4 Y5
## 1     A  0  0  0  0  0
## 2     B  0  0  0  0  0
## 3     C  0  0  0  0  0
## 4     D  0  0  0  0  0
## 5     E  1  0  0  0  0
## 6     F  0  1  0  0  0
## 7     G  0  0  0  1  1
```

Aggregated dataset with the PD values:

```
##   GRADE     PD OBLIGORS    PD_SCALED
## 1     A 0.0003       26 0.0003740935
## 2     B 0.0010      122 0.0012469782
## 3     C 0.0030      182 0.0037409347
## 4     D 0.0100      123 0.0124697824
## 5     E 0.0300       24 0.0374093472
## 6     F 0.1000       14 0.1246978241
## 7     G 0.3000        9 0.3740934723
```

Summary:

Number of obligor years: 500

Average number of obligors per year: 100

Number of defaults (`Y1-Y5`): 4

Portfolio PD: 1.35%

# R Code

```r
#source r script (data & functions)
source("https://raw.githubusercontent.com/andrija-djurovic/adsfcr/main/ldp/bcr.R")
#inputs
n <- 100        #number of obligors
k <- 4          #number of defaults
theta <- 0.30   #year-to-year correlation
rho <- 0.12     #asset correlation
T <- 5          #number of years
cl <- 0.75      #confidence level
N <- 1e4        #number of simulations
#random seed
set.seed(1)
#optimize the function (opt.f)
res <- uniroot(f = opt.f,
               interval = c(0,1),
               n = n,
               k = k,
               theta = theta,
               rho = rho,
               T = T,
               cl = cl,
               N = N)
#upper bound of the pd
pd.ub <- res[[1]]
pd.ub
```

```
## [1] 0.01677385
```

# R Code cont.

```
#portfolio pd
pd.wa <- weighted.mean(x = tbl.3$PD,
                       w = tbl.3$OBLIGORS)
pd.wa
```

```
## [1] 0.0134516
```

```
#scaling factor
sf <- pd.ub / pd.wa
sf
```

```
## [1] 1.246978
```

```
#scaled pds
tbl.3$PD_SCALED <- sf * tbl.3$PD
tbl.3
```

```
##   GRADE     PD OBLIGORS    PD_SCALED
## 1     A 0.0003       26 0.0003740935
## 2     B 0.0010      122 0.0012469782
## 3     C 0.0030      182 0.0037409347
## 4     D 0.0100      123 0.0124697824
## 5     E 0.0300       24 0.0374093472
## 6     F 0.1000       14 0.1246978241
## 7     G 0.3000        9 0.3740934723
```

# Python Code

```
#source python script (data & functions)
import requests
url = "https://raw.githubusercontent.com/andrija-djurovic/adsfcr/main/ldp/bcr.py"
r = requests.get(url)
exec(r.text)
#inputs
n = 100        #number of obligors
k = 4          #number of defaults
theta = 0.30   #year-to-year correlation
rho = 0.12     #asset correlation
T = 5          #number of years
cl = 0.75      #confidence level
N = int(1e4)   #number of simulations
#random seed
np.random.seed(321)
#optimize the function (opt_f)
res = root_scalar(f = opt_f,
                  args = (n, k, theta, rho, T, cl, N),
                  bracket = [0, 1])
#upper bound of the pd
pd_ub = res.root
pd_ub
```

```
## 0.016866226993434416
```

# Python Code cont.

```
#portfolio pd
pd_wa = np.average(a = tbl_3["PD"],
                   weights = tbl_3["OBLIGORS"])
pd_wa
```

```
## 0.0134516
```

```
#scaling factor
sf = pd_ub / pd_wa
sf
```

```
## 1.2538454156705832
```

```
#scaled pds
tbl_3["PD_SCALED"] = sf * tbl_3["PD"]
tbl_3
```

```
##   GRADE      PD  OBLIGORS  PD_SCALED
## 0     A  0.0003        26   0.000376
## 1     B  0.0010       122   0.001254
## 2     C  0.0030       182   0.003762
## 3     D  0.0100       123   0.012538
## 4     E  0.0300        24   0.037615
## 5     F  0.1000        14   0.125385
## 6     G  0.3000         9   0.376154
```