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ON FORMULAS OF ONE VARIABLE IN INTUITIONISTIC PROPOSITIONAL CALCULUS[1]

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Introduction. McKinsey and Tarski [3] described Gödel's proof that the number of Brouwerian-algebraic functions is infinite. They gave an example of a sequence of infinitely many distinct Brouwerian-algebraic functions of one argument, which means that there are infinitely many non-equivalent formulas of one variable in the intuitionistic propositional calculus LJ of Gentzen [1]. However they did not completely characterize such formulas. In § 1 of this note, we define a sequence of basic formulas $P_\infty(X), P_0(X), P_1(X), \cdots$ and prove the following theorems.

THEOREM I. *Any formula of one variable X is equivalent to one of the basic formulas in LJ.*

THEOREM II. *No two of the basic formulas are equivalent to each other in LJ.*

From Theorem I, we can also characterize all the propositional logics obtained from the intuitionistic propositional calculus LJ of Gentzen by adding an *axiomatic sequent*[2] of the form $\rightarrow P(A)$. In § 2 we consider *inclusion* and *non-inclusion relationships*[3] between these logics.

To every formula $P(X)$ of one variable X corresponds a class CP of propositions A for which $\rightarrow P(A)$ holds in LJ. If CP is closed with respect to the logical connectives $\neg$, $\vee$, $\&$, and $\supset$, then $P(X)$ is called *LJ-closed*. In § 3 we prove the following theorem.

THEOREM III. *$P(X)$ is LJ-closed if and only if $P(X)$ is equivalent to one of $P_\infty(X)$, $P_0(X)$, $P_3(X)$, and $P_5(X)$ in LJ.*

CP_∞ is the class of all propositions, CP_0 is void, CP_3 is the class of all the propositions which satisfy the law of the excluded middle, and CP_5 is the class of all the propositions whose negations satisfy the law of the excluded middle. It is remarkable that only these few formulas are LJ-closed. Moreover it should be noticed that $P_1(X)$, $P_2(X)$, $P_4(X)$, $P_6(X)$, and $P_8(X)$ are not LJ-closed, while each one of the axiomatic sequents $\rightarrow P_1(A)$, $\rightarrow P_2(A)$, and $\rightarrow P_4(A)$ is equivalent to $\rightarrow P_0(A)$, $\rightarrow P_6(A)$ is equivalent to $\rightarrow P_3(A)$, and $\rightarrow P_8(A)$ is equivalent to $\rightarrow P_5(A)$.

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[2] See [6] p. 22. In the present note results of Gentzen [1], Mostowski [4], and Umezawa [5], [6], are used without refering on every occasion.

Concerning terminology and notations we follow [6] mainly.

[3] See [6] p. 22.

§ 1. **Basic formulas.** *Basic formulas* are defined recursively as follows.

DEFINITION. $P_\infty(X) \cdot\equiv\cdot X \supset X$, $P_0(X) \cdot\equiv\cdot X \,\&\, \neg X$, $P_1(X) \equiv X$, $P_2(X) \cdot\equiv\cdot \neg X$, $P_{2n+3}(X) \cdot\equiv\cdot P_{2n+1}(X) \vee P_{2n+2}(X)$, $P_{2n+4}(X) \cdot\equiv\cdot P_{2n+3}(X) \supset P_{2n+1}(X)$, for $n \geqq 0$.

THEOREM I. *Any formula of one variable X is equivalent to one of the basic formulas in LJ.*

Proof. We get $P_k(X) \rightarrow P_{2n+1}(X)$, for $0 \leqq k \leqq 2n+1$, by definition and induction on n. Hence we can prove $P_i(X) \rightarrow P_j(X)$, for $0 \leqq i \leqq j \leqq \infty$, except for the two cases ($i = 2n+1$, and $j = 2n+2$) and ($i = 2n+2$, and $j = 2n+4$). From this, by considering all possible cases, we can prove that each one of $P_i(X) \supset P_j(X)$, $P_i(X) \,\&\, P_j(X)$, $P_i(X) \vee P_j(X)$, and $\neg P_i(X)$ is equivalent to one of the basic formulas. By induction on the number of logical connectives we can complete the proof. (See appendix.)

THEOREM II. *No two of the basic formulas are equivalent to each other in LJ.*

Proof. Let S be a set such that

$S = \{(x, y) | (0 \leqq x < \infty, 0 \leqq y < \infty$, and $x-1 \leqq y \leqq x+2)$ or $x = y = \infty\}$.

Let us introduce in S an order relation such that $(x_1, y_1) \leqq (x_2, y_2)$ if and only if both $x_1 \geqq x_2$ and $y_1 \geqq y_2$. Then S is a *complete Brouwerian lattice*[4] and every provable sequent *fulfills*[5] S. Therefore if $v(A) \neq v(B)$ for any *normal mapping*[6], then the two formulas A and B are not equivalent in LJ. Now if X corresponds to $(0, 1)$, then we have

$$v(P_0(X)) = (0, 0) = I,\ v(P_\infty(X)) = (\infty, \infty) = 0,$$
$$v(P_{4k+1}(X)) = (k, k+1),\ v(P_{4k+2}(X)) = (k+1, k),$$
$$v(P_{4k+3}(X)) = (k+1, k+1),\ v(P_{4k+4}(X)) = (k, k+2), \text{ for } k \geqq 0$$

therefore if $i \neq j$, $P_i(X) \not\equiv P_j(X)$.

Remark: The sequence of Brouwerian-algebraic functions which Gödel gave corresponds to $P_7(X), P_{13}(X), \cdots, P_{6n+1}(X), :\cdots$.

By a suitable interpretation we have

COROLLARY I. *The basic formulas constitute a lattice isomorphic to S* (Fig. 1).

COROLLARY II. *If $i \neq 0$, then there are just $i+1$ basic formulas non-deducible from $P_i(X)$ in LJ.*

COROLLARY III. *Provided that $A_1, \cdots, A_n$, and B are formulas of one variable X, B is deducible from $A_1, \cdots, A_n$ in LJ, if and only if $A_1 \,\&\, \cdots \,\&\, A_n \supset B$ is equivalent to $P_\infty(X)$ in LJ.*

§ 2. Inclusion and non-inclusion relationships of logics.

From Theorem I, in order to characterize all the propositional logics each of which is obtained from LJ by adding an axiomatic sequent of the

[4] See [4] p. 204.
[5] See [6] p. 23.
[6] See [6] p. 23.

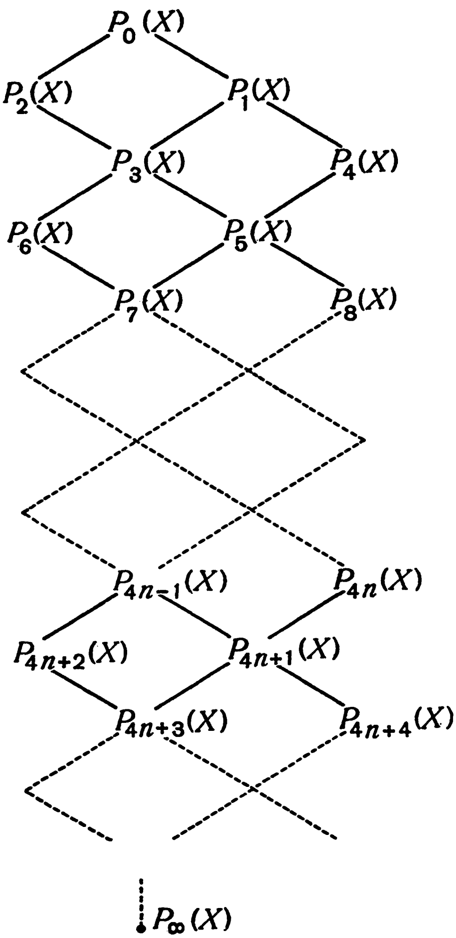

Fig. 1.

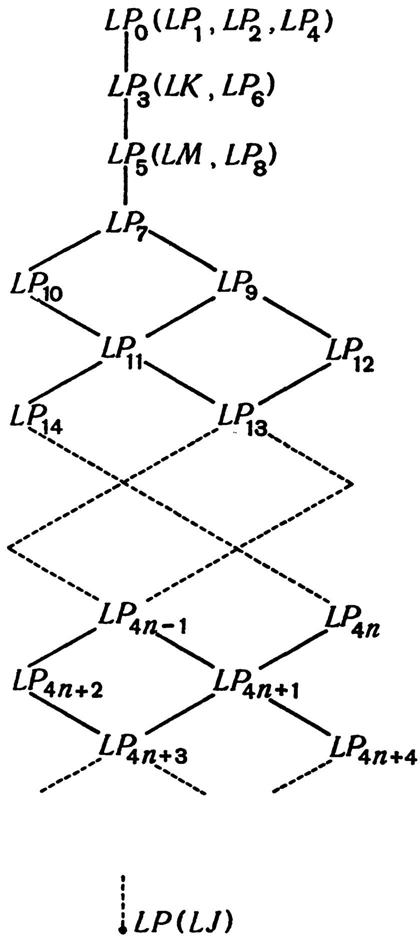

Fig. 2.

form $\rightarrow P(A)$, we have only to consider the cases where $P(X)$'s are the basic formulas. We call these logics LP_i. We can investigate inclusion, non-inclusion and equivalence relationships between these logics by methods of Gödel [2], or Umezawa [5], [6]. As models, we use the lattice S introduced in § 1, and its sublattices $S_{0,1}$, $S_{0,2}$, $S_{n,n+1}$, $S_{n+1,n}$, and $S_{n,n+2}$ for $n \geqq 1$, where

$$S_{p,q} = \{(x, y) | 0 \leqq x \leqq p,\ 0 \leqq y \leqq q, \text{ and } x-1 \leqq y \leqq x+2\}.$$

If we rewrite the inclusion relation $LP_i \supset LP_j$ by an order relation $LP_i > LP_j$, then our results are expressed in Fig. 2.

§ 3. LJ-closed formulas.

THEOREM III. *$P(X)$ is LJ-closed if and only if $P(X)$ is equivalent to one of $P_\infty(X)$, $P_0(X)$, $P_3(X)$, and $P_5(X)$ in LJ.*

Proof. a) To prove that $P(X)$ is LJ-closed, it is sufficient that the following four sequents are provable in LJ.

$$P(X) \to P(\neg X). \ P(X), P(Y) \to P(X \vee Y).$$
$$P(X), P(Y) \to P(X \,\&\, Y). \ P(X), P(Y) \to P(X \supset Y).$$

These sequents can be easily verified to be provable in LJ, when $P(X)$ is equivalent to one of $P_\infty(X)$, $P_0(X)$, $P_3(X)$, and $P_5(X)$ in LJ.

b) Suppose $P(X)$ is LJ-closed. Then $P(X) \to P(F(X))$ and $P(X)$, $P(Y) \to P(G(X, Y))$ must be provable in LJ for any formulas $F(X)$ and $G(X, Y)$. Therefore, from Corollary III of § 1, $P(X) \supset P(F_1(X))$ and $P(F_1(X)) \,\&\, P(F_2(X)) \supset P(G(F_1(X), F_2(X))$ must be both equivalent to $P_\infty(X)$ in LJ for any formulas of one variable $F_1(X)$ and $F_2(X)$. Hence in order to prove that $P(X)$ is not LJ-closed when $P(X)$ is equivalent to none of $P_\infty(X)$, $P_0(X)$, $P_3(X)$, and $P_5(X)$, it is sufficient to prove that the following equivalences hold in LJ.

$$P_1(X) \supset P_1(\neg X) \equiv P_2(X),\ P_2(X) \supset P_2(\neg X) \equiv P_4(X),$$
$$P_4(X) \supset P_4(\neg X) \cdot\!\equiv\!\cdot P_2(X),\ P_6(X) \,\&\, P_6(\neg X) \supset P_6(X \vee \neg X) \equiv P_8(X),$$
$$P_8(X) \supset P_8(\neg X) \equiv P_{10}(X),$$
$$P_k(P_3(X)) \,\&\, P_k(P_4(X)) \supset P_k(P_3(X) \,\&\, P_4(X)) \cdot\!\equiv\!\cdot P_k(X),$$
$$\text{for } k = 7 \text{ and for } 9 \leqq k < \infty.$$

These equivalences can be proved easily from the definition.

Appendix. In order to prove that each one of $P_i(X) \supset P_j(X)$, $P_i(X) \,\&\, P_j(X)$, $P_i(X) \vee P_j(X)$, and $\neg P_i(X)$ is equivalent to one of the basic formulas, it is sufficient to prove the following equivalences (1)$\cdots$(24).

(1) $\neg P_i(X) \equiv P_i(X) \supset P_0(X)$.

(2) $P_{2n+1}(X) \supset P_{2n+2}(X) \equiv P_{2n+2}(X)$.

(3) $P_{2n+2}(X) \supset P_{2n+4}(X) \equiv P_{2n+4}(X)$.

(4) $P_i(X) \supset P_j(X) \equiv P_\infty(X)$, for $i \leqq j$ except the two cases (2) and (3).

(5) $P_1(X) \supset P_0(X) \equiv P_2(X)$.

(6) $P_2(X) \supset P_0(X) \equiv P_4(X)$.

(7) $P_3(X) \supset P_0(X) \equiv P_0(X)$.

(8) $P_4(X) \supset P_2(X) \equiv P_2(X)$.

(9) $P_i(X) \supset P_0(X) \equiv P_0(X)$, for $i \geqq 5$.

(10) $P_{2n+2}(X) \supset P_{2n+1}(X) \equiv P_{2n+4}(X)$.

(11) $P_{2n+3}(X) \supset P_{2n+1}(X) \equiv P_{2n+4}(X)$.

(12) $P_{2n+4}(X) \supset P_{2n+1}(X) \equiv P_{2n+6}(X)$.

(13) $P_{2n+5}(X) \supset P_{2n+1}(X) \equiv P_{2n+1}(X)$.

(14) $P_{2n+6}(X) \supset P_{2n+1}(X) \equiv P_{2n+4}(X)$.

(15) $P_i(X) \supset P_{2n+1}(X) \equiv P_{2n+1}(X)$, for $i \geqq 2n+7$.

(16) $P_i(X) \supset P_{2n+2}(X) \equiv P_{2n+2}(X)$, for $i \geqq 2n+3$.

(17) $P_1(X) \,\&\, P_2(X) \equiv P_0(x)$.

(18) $P_{2n+3}(X) \,\&\, P_{2n+4}(X) \equiv P_{2n+1}(X)$.

(19) $P_2(X) \,\&\, P_4(X) \equiv P_0(X)$.

(20) $P_{2n+4}(X) \,\&\, P_{2n+6}(X) \equiv P_{2n+1}(X)$.

(21) $P_i(X) \,\&\, P_j(X) \equiv P_i(X)$, for $i \leqq j$ except the four cases (20), (17), (18), and (19).

(22) $P_{2n+1}(X) \vee P_{2n+2}(X) \equiv P_{2n+3}(X)$.

(23) $P_{2n+2}(X) \vee P_{2n+4}(X) \equiv P_{2n+5}(X)$.

(24) $P_i(X) \vee P_j(X) \equiv P_j(x)$, for $i \leqq j$ except the two cases (22) and (23).

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