Time Series Analysis
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HW 4 due prior to the start of class in week 5

1. Consider the GARCH(1,1) model applied to continuously compounded daily returns. Recall that for continuously compounded returns the k-period return can be obtained by summing each of the k daily returns. Here we set the conditional mean equal to zero so that $r_t=\varepsilon_t$

    a. Write the k-step ahead forecast $h_t^k = E\left(r_{t+k}^2 \mid F_t\right)$ as a function of the GARCH parameters and k.
    b. The cumulative continuously compounded return over k periods can be written as $R_t^k = \sum_{j=1}^{k} r_{t+j}$. Assuming the returns are uncorrelated, find the variance of the k-period return given information at time t. Write the forecast as a function of the GARCH parameters and k.
    c. Annualize the cumulative return variance as a function of the GARCH parameters and k. Assuming the GARCH model is stationary, what does it converge to for large k?

2. For this problem, use the file qqq.xls which contains price data for NASDAQ ETF. Create the continuously compounded returns by differencing the log prices.

    a. Report the skewness and the kurtosis. What do these numbers tell you about the distribution of the returns? Report the autocorrelations through lag 20 for the *squared* returns. What does this tell you about the time varying volatility in each series?

    b. Fit an ARMA model for the mean and a GARCH model for the variance to the index and report the results. Use residual diagnostics to select the model for the mean and squared residual diagnostics to find a good model for the volatility.

    c. From the last day in the sample, build the sequence of daily forecasts from one-month ahead through 12 months. Plot the forecasts.

    d. Cumulate the forecasts to get the forecasted variance for the next year for both the index and the stock.

3. Show that a GARCH(p,q) model for the errors implies that the squared errors follow an ARMA(max(p,q),p) model. This generalized the results derived in class.

4. Consider the conditionally Normal GARCH(p,q) model where $f\left(r_t \mid F_{t-1}\right) = N\left(\mu_t, h_t\right)$.
   a. Write down the log likelihood for the model when p=0 and q=1, and μ_t=0 *conditional* on r_1 and h_1. In other words, you can do conditional (on the first observation) maximum likelihood estimation.
   b. Find the first order conditions for ω and α.
   c. Now, consider the model where p=0, q=1, and $\mu_t = \beta_0 + \beta_1 r_{t-1}$. Find the first order conditions for all the parameters. Recall how we saw that (conditional) MLE is the same as OLS for the parameters of the AR model. What is the difference between the first order conditions for the mean parameters when q=1 vs when there is no GARCH so that p=q=0. They are both least squares estimators for the mean parameters but one is ….
   d. Write down the log likelihood as a function for the model when p=1 and q=1, and μ_t=0 conditional on r_1 and h_1.
   e. Find the first order conditions. They will take a recursive form.
   f. Show that the information matrix is block diagonal in the parameters for the mean and the parameters for the variance for the model in part c.
   g. Now, consider the model where p=0, q=1, and $\mu_t = \beta_0 + \beta_1 r_{t-1} + \gamma r_{t-1}^2$. What is the information matrix for this model? Is the information matrix block diagonal in the parameters for the mean and parameters for the variance in this model?