Равномерная непрерывность

Непр-ть $\Rightarrow$ $\forall x_0 \in I \quad \forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in U_\delta(x_0) \hookrightarrow |f(x) - f(x_0)| < \varepsilon$

$\forall \varepsilon > 0 \; \exists \delta > 0 : \; \forall x', x'' \in U_\delta(x_0) \hookrightarrow |f(x'') - f(x')| < \varepsilon$

$f(x)$ наз равномерно непрерывной на I, если

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x', x'' \in I : |x'' - x'| < \delta \hookrightarrow |f(x'') - f(x')| < \varepsilon$

Не Р.Н: $\exists \varepsilon > 0 : \forall \delta > 0 \; \exists x', x'' \in I \; |x'' - x'| < \delta \hookrightarrow |f(x'') - f(x')| \geq \varepsilon$

№1

$f(x) = x \quad , \quad I = \mathbb{R}$

$\forall \varepsilon > 0 \; \exists \delta = \varepsilon : \; \forall x', x'' \in \mathbb{R} \; : |x'' - x'| < \delta \hookrightarrow |f(x'') - f(x')| = |x'' - x'| < \delta = \varepsilon$

№2

$f(x) = \sqrt{x} \qquad I = [0, 1]$

$1 \geq x'' > x' \geq 0 \qquad x'' = x' + \Delta \; , \; \Delta > 0$

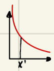
$|\sqrt{x''} - \sqrt{x'}| = |\sqrt{x' + \Delta} - \sqrt{x'}| = \left| \frac{\Delta}{\sqrt{x' + \Delta} + \sqrt{x'}} \right| \leq \frac{|\Delta|}{\sqrt{\Delta}} = \sqrt{\Delta} < \varepsilon$

$\Delta < \delta = \varepsilon^2$

$\forall \varepsilon > 0 \; \exists \delta = \varepsilon^2 : \; \forall x', x'' \in [0, 1] : |x'' - x'| < \delta \hookrightarrow |\sqrt{x''} - \sqrt{x'}| < \varepsilon$

№3

$f(x) = \frac{1}{x}$ на $(0, 1]$



$\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists x', x'' \in I \; , \; |x'' - x'| < \delta \hookrightarrow |f(x') - f(x'')| \geq \varepsilon$

$0 < x'' < x' \leq 1 \qquad x'' = x' - \Delta \; , \; \Delta > 0, \; \Delta < x', \; \Delta < \delta$

$\left| \frac{1}{x'} - \frac{1}{x' - \Delta} \right| = \left| \frac{x' - \Delta - x'}{x'(x' - \Delta)} \right| = \left| \frac{-\Delta}{x' \cdot \underbrace{(x' - \Delta)}_{\to 0}} \right| \geq \frac{\Delta}{x'^2} = \frac{1}{2x'} \geq \varepsilon$

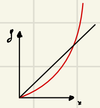
Пусть $\boxed{\Delta = \frac{x'}{2}}$

$x' = x'$

$x'' = \frac{x'}{2}$

$$\begin{cases} \frac{1}{2x'} \geq \varepsilon \to & x' \leq \frac{1}{2\varepsilon} \\ \frac{x'}{2} < \delta \to & 0 < x' < 2\delta \end{cases}$$

$\exists \varepsilon = \frac{1}{2} \quad \forall \delta > 0 \quad \exists x' = \min(\delta; 1); x'' = \frac{x'}{2} : \; |x'' - x'| < \delta \hookrightarrow \left| \frac{1}{x''} - \frac{1}{x'} \right| \geq \frac{1}{2x'} \geq \frac{1}{2}$



№4

$f(x) = x^2$ на $\mathbb{R}$

$\exists \varepsilon = 1 \quad \forall \delta > 0 \quad \exists \, x' = \frac{1}{\delta}, \; x'' = \frac{1}{\delta} + \frac{\delta}{2} : \; |x'' - x'| < \delta \hookrightarrow |x''^2 - x'^2| \geq 1$

$x' > 0$

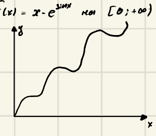
$x'' = x' + \frac{\delta}{2}$

$x' = \frac{1}{\delta}$

$x''^2 - x'^2 = x'\delta + \frac{\delta^2}{4} > x'\delta = 1$

№5

$f(x) = x \cdot e^{\sin x}$ на $[0; +\infty)$



$x_n = 2\pi n \qquad \Big| \quad \lim_{n \to \infty} |x_n - y_n| = 0$

$y_n = 2\pi n + \frac{1}{n} \quad \Big| \quad \forall \delta > 0 \quad \exists n_0 : \; \forall n \geq n_0 \hookrightarrow |x_n - y_n| < \delta$

$f(x_n) = 2\pi n$

$f(y_n) = \left(2\pi n + \frac{1}{n}\right) \cdot e^{\sin \frac{1}{n}}$

$$f(y_n) - f(x_n) = 2\pi n\left(e^{\sin\frac{1}{n}} - 1\right) + \cancel{\frac{1}{n} e^{\sin\left(\frac{1}{n}\right)}} \to 0 = 2\pi n \cdot \left(1 + \frac{1}{n} - 1 + \bar{\bar{o}}\left(\frac{1}{n}\right)\right) = 2\pi + \underset{n\to\infty}{\bar{\bar{o}}}(1)$$

$$e^{\sin\frac{1}{n}} = 1 + \sin\frac{1}{n} + \underset{n\to\infty}{\bar{\bar{o}}}\left(\sin\left(\frac{1}{n}\right)\right) = 1 + \frac{1}{n} + \bar{\bar{o}}\left(\frac{1}{n}\right)$$

$\forall \varepsilon > 0 \quad \exists n_0' : \forall n \geq n_0' \quad |f(y_n) - f(x_n) - 2\pi| < \varepsilon$

Также: $\exists n_0' : \forall n \geq n_0' \quad |f(y_n) - f(x_n)| > \pi$

Обобщим

$\exists \varepsilon = \pi \qquad \forall \delta > 0 \qquad \exists x_{\tilde{n}}, y_{\tilde{n}} \in [0; +\infty) : |y_{\tilde{n}} - x_{\tilde{n}}| < \delta :$

$\tilde{n} : \quad \tilde{n} = \max(n_0; n_0') : |f(y_{\tilde{n}}) - f(x_{\tilde{n}})| \geq \pi$