IBM Ponder This
2019-Nov
Maximal Determainant

If denote rows of the matrix M as
$r_1 = (a_1, b_1, ..., h_1, i_1)$,
$r_2 = (a_2, b_2, ..., h_2, i_2)$,
$\ldots$,
$r_9 = (a_9, b_9, ..., h_9, i_9)$,
then consider the product $W = M \times M^\top$, where $M^\top$ is transposed matrix.

Based on the MSE answer, W has to be very close to the matrix

$$W_0 = \begin{pmatrix} \mathbf{a} & b & b & b & b & b & b & b & b \\ b & \mathbf{a} & b & b & b & b & b & b & b \\ b & b & \mathbf{a} & b & b & b & b & b & b \\ b & b & b & \mathbf{a} & b & b & b & b & b \\ b & b & b & b & \mathbf{a} & b & b & b & b \\ b & b & b & b & b & \mathbf{a} & b & b & b \\ b & b & b & b & b & b & \mathbf{a} & b & b \\ b & b & b & b & b & b & b & \mathbf{a} & b \\ b & b & b & b & b & b & b & b & \mathbf{a} \end{pmatrix},$$

where
$a = \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6 = 285,$
$b = n(n+1)(3n+2)/12 = 217.5$.

So, the strongest condition for $a_k, b_k, \ldots, h_k, i_k$ is:

$$a_k^2 + b_k^2 + \ldots + h_k^2 + i_k^2 = 285; \tag{1}$$

$$a_k \cdot a_m + b_k \cdot b_m + \ldots + h_k \cdot h_m + i_k \cdot i_m = 217 \textit{ or } 218. \quad (k \neq m) \tag{2}$$

There are 83 ordered 9-tuples (candidates, written down as strings - without delimiters) which match equation (1):
111158888, 111166889, 111246899, ..., 123456789, ..., 445555588, 445555669.

We can focus on those ones which have not more than 2 equal digits in the row, so the 'candidate' set could be reduced to 23 candidates: $\{v_1, v_2, ..., v_{23}\} = \{$
112356889, 112446799, 113566788, 114457788, 114466779,

$114556689, 122337889, 122345899, 122557788, 122566779,$
$123367788, 123456789, 124455699, 133445889, 133446699,$
$134566778, 144556679, 223347789, 223355889, 223356699,$
$223445799, 233556788, 234456779\}$.

First row r_1 of the solution could be chosen from candidate list - as is: one of 23. And $r_2, r_3, ..., r_9$ can be built from the set constructed of those candidates, but with permuted their elements inside.

So we have to choose 9 such candidates (with permuted elements) which are linked by condition (2) *each other*.

Talking in GEOMETRY terms:
we have $9D$-space,
and there is hyper-sphere with radius $\sqrt{285}$,
and we need to find 9 'equidistant' points (with integral coordinates) on 'positive chunk' of the hyper-sphere.
Extended version: consider the 'spherical cloud' with radii $\sqrt{284}...\sqrt{286}$, etc.

Talking in GRAPH terms:
if consider candidates (and their internal element permutations) as 'vertices' of a graph,
and consider satisfied condition (2) as 'edges',
then we have to build 9-clique(s) from extended (by permutations) candidate set.
(see Clique problem)
(And finally check each 9-clique for nine '1's, nine '2's, ..., nine '9's).

This problem is quite hard (slowly-solvable), but the subtask with described restrictions:
r_1 coincides with *ordered* candidate;
- condition (1) with 285;
- condition (2) with 217 or 218;
- candidate set reduced to 23 ones with no more 2 equal digits on the row;
is solvable within $4 - 6$ hours.

When I extended the candidate set to entire 83 candidates,
or tried to extend condition (1) to '284 or 285 or 286';
or tried to extend condition (2) to '216 or 217 or 218 or 219';
it took me much more time: more than 1 day (1 week?, 1 month?)...

As current result: the same max value for determinant: $933251220 =$

0x37a04894
and 2 nonequivalent matrices:
955332288
492675318
479218635
637941825
628597143
741829563
763184952
186753492
114466779
and
122557788
916644339
951378642
658132974
674815293
473961825
485293167
387429516
329786451