

PHYS2113 Classical Mechanics

Joey Liang

May 2021

Contents

1	Introduction to Lagrangian Mechanics	3
1.1	Action	3
1.2	The Euler-Lagrange Equation	3
1.2.1	Derivation	3
1.2.2	Example	4
1.3	Proof of Lagrange's Equations with Constraints	4
	References	5

1 Introduction to Lagrangian Mechanics

1.1 Action

Definition 1.1 (Action) *Action, termed A , is defined as*

$$A = \int_{t_0}^{t_1} L \, dt. \quad (1.1.1)$$

Where $L(q, \dot{q}) = T - V = \frac{1}{2}m\dot{q}^2 - V(q)$ is what we call the *Lagrangian*.

Note that action represents the integral over time of the Lagrangian which can be thought as the motion of the object at some point of time.[1]

1.2 The Euler-Lagrange Equation

Definition 1.2 (The Euler-Lagrange Equation) *The Euler-Lagrange equation for a system with a single degree of freedom is*

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \quad (1.2.1)$$

1.2.1 Derivation

We want to find a generalised solution for the path that minimises the variational problem integral

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x] \, dx. \quad (1.2.2)$$

First, let's start off by defining that the 'right' path (path with the least action/that minimise the variational problem) to be $y = y(x)$, and the wrong path is just a variation of the right path known as $Y(y(x), \alpha, \eta(x)) = y(x) + \alpha\eta(x)$.

Since the end points of the right path and the wrong path are the same we get

$$\eta(x_1) = \eta(x_2) = 0. \quad (1.2.3)$$

Now, the variational problem interval in terms of the wrong path S_0 would be

$$\begin{aligned} S_0 &= \int_{x_1}^{x_2} f(Y, Y', x) \, dx \\ &= \int_{x_1}^{x_2} f(y + \alpha\eta, y' + \alpha\eta', x) \, dx. \end{aligned} \quad (1.2.4)$$

Note that the only difference between integral S and S_0 is the dependence on α . So, ideally to minimise this integral, we would find the stationary point

of S_0 in terms of α , this can be expressed as

$$\begin{aligned}\frac{dS_0}{d\alpha} &= 0 \\ &= \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha} dx \\ &= \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right) dx.\end{aligned}\tag{1.2.5}$$

And now, using (1.2.3) and integration by parts, we obtain the following

$$\int_{x_1}^{x_2} \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) dx = 0.\tag{1.2.6}$$

For non-trivial solution, we must have

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0,$$

which is the Euler-Lagrange Equation.

Remark 1.2.1.1 *Note that for (1.2.5), the following needs to be true for $Y = y + \alpha\eta$;*

$$\frac{\partial}{\partial Y} f(Y, Y', x) = \frac{\partial}{\partial y} f(y + \alpha\eta, y' + \alpha\eta', x).$$

The proof is simple,

$$\begin{aligned}\frac{\partial}{\partial y} f(y + \alpha\eta, y' + \alpha\eta', x) &= f'(y + \alpha\eta, y' + \alpha\eta', x) \\ &= f'(Y, Y', x).\end{aligned}$$

1.2.2 Example

Refer to Taylor's¹ **Example 6.2** on page 222.

1.3 Proof of Lagrange's Equations with Constraints

Refer to **Section 7.4** (pg 250) in Classical Mechanics by JR Taylor.

One important thing: the right path of any action must follow Newton's second law.

¹The book 'Classical Mechanics'[2] By John R. Taylor

References

- [1] *Action (physics)*. en. Page Version ID: 1020785959. May 2021. URL: [https://en.wikipedia.org/w/index.php?title=Action_\(physics\)&oldid=1020785959](https://en.wikipedia.org/w/index.php?title=Action_(physics)&oldid=1020785959) (visited on 05/31/2021).
- [2] John R. (John Robert) Taylor. *Classical mechanics*. eng. Sausalito, Calif. : [Basingstoke: University Science Books ; Palgrave, distributor], 2005. ISBN: 189138922X.