PHYS2113 Classical Mechanics

Joey Liang

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Contents

1 Introduction to Lagrangian Mechanics

1.1 Action

Definition 1.1 (Action) *Action, termed A, is defined as*

$$A = \int_{t_0}^{t_1} L \, \mathrm{d}t. \tag{1.1.1}$$

Where $L(q, \dot{q}) = T - V = \frac{1}{2}m\dot{q}^2 - V(q)$ is what we call the Lagrangian.

Note that action represents the integral over time of the Lagrangian which can be thought as the motion of the object at some point of time.[1]

1.2 The Euler-Lagrange Equation

Definition 1.2 (The Euler-Lagrange Equation) *The Euler-Lagrange equation for a system with a single degree of freedom is*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{1.2.1}$$

1.2.1 Derivation

We want to find a generalised solution for the path that minimises the variational problem integral

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x] \, \mathrm{d}x. \tag{1.2.2}$$

First, let's start off by defining that the 'right' path (path with the least action/that minimise the variational problem) to be $y = y(x)$, and the wrong path is just a variation of the right path known as $Y(y(x), \alpha, \eta(x)) = y(x) + \alpha\eta(x)$.

Since the end points of the right path and the wrong path are the same we get

$$\eta(x_1) = \eta(x_2) = 0. \tag{1.2.3}$$

Now, the vairational problem interval in terms of the wrong path S_0 would be

$$\begin{aligned} S_0 &= \int_{x_1}^{x_2} f(Y, Y', x) \, \mathrm{d}x \\ &= \int_{x_1}^{x_2} f(y + \alpha\eta, y' + \alpha\eta', x) dx. \end{aligned} \tag{1.2.4}$$

Note that the only difference between integral S and S_0 is the dependence on α. So, idealy to minimise this integral, we would find the stationary point of S_0 in terms of α, this can be expressed as

$$\begin{aligned}\frac{\mathrm{d}S_0}{\mathrm{d}\alpha} &= 0 \\ &= \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha}\,\mathrm{d}x \\ &= \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}\right) \mathrm{d}x. \end{aligned} \tag{1.2.5}$$

And now, using (1.2.3) and integration by parts, we obtain the following

$$\int_{x_1}^{x_2} \eta(x) \left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right) = 0. \tag{1.2.6}$$

For non-trivial solution, we must have

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0,$$

which is the Euler-Lagrange Equation.

Remark 1.2.1.1 *Note that for* (1.2.5)*, the following needs to be true for* $Y = y + \alpha\eta$*;*

$$\frac{\partial}{\partial Y} f(Y, Y', x) = \frac{\partial}{\partial y} f(y + \alpha\eta, y' + \alpha\eta', x).$$

The proof is simple,

$$\begin{aligned}\frac{\partial}{\partial y} f(y + \alpha\eta, y' + \alpha\eta', x) &= f'(y + \alpha\eta, y' + \alpha\eta', x) \\ &= f'(Y, Y', x).\end{aligned}$$

1.2.2 Example

Refer to Taylor's[1] **Example 6.2** on page 222.

1.3 Proof of Lagrange's Equations with Constraints

Refer to **Section 7.4** (pg 250) in Classical Mechnics by JR Taylor.

One important thing: the right path of any action must follows Newton's second law.

[1]The book 'Classical Mechanics'[2] By John R. Taylor

References

[1] *Action (physics)*. en. Page Version ID: 1020785959. May 2021. URL: https://en.wikipedia.org/w/index.php?title=Action_(physics)&oldid=1020785959 (visited on 05/31/2021).

[2] John R. (John Robert) Taylor. *Classical mechanics*. eng. Sausalito, Calif. : [Basingstoke: University Science Books ; Palgrave, distributor], 2005. ISBN: 189138922X.