

# PHYS2111 Cheat Sheet

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## 1 Formula Sheet

### 1.1 Expectation Value

Expectation value of function  $f(x)$  subject to  $\Psi(x, t)$

$$\mathbb{P}(f(x)) = \int_{-\infty}^{\infty} f(x) \|\Psi(x, t)\|^2 dx.$$

Note that the Hamiltonian operator  $H$  is

$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, t),$$

and the expectation value of the energy  $\langle H \rangle$  is

$$\begin{aligned} \langle H \rangle &= \left\langle -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, t) \right\rangle \\ &= \int_{-\infty}^{\infty} \Psi^*(x, t) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, t) \right) \Psi(x, t) dx. \end{aligned}$$

General case, the expectation value for any operator  $Q(x, \hat{p})$  is

$$\int_{-\infty}^{\infty} \Psi^*(x, t) Q(x, \hat{p}) \Psi(x, t) dx.$$

### 1.2 Position and Momentum Operators

The position operator,

$$\hat{x} = x.$$

The momentum operator,

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}.$$

### 1.3 The Infinite Square Well

Time Dependent Schrodinger's Equation

$$\Psi(x, t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \exp(-iE_n t).$$

with constant

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x, 0) dx.$$

and allowed energy

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}.$$

## 1.4 The Harmonic Oscillator

A harmonic oscillator has potential energy

$$V(x) = \frac{1}{2}m\omega^2 x^2,$$

With ground state  $\psi_0(x)$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

The ladder operator

$$a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega x), \quad (\omega = \sqrt{\frac{k}{m}})$$

and to extract the n-th state with the ladder operator

$$\psi_n(x) = A_n(\hat{a}_+)^n \psi_0(x),$$

with

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$$

## 1.5 The Free Particle

The initial condition can be expressed in the Fourier  $k$  space,

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x, 0) \exp(-ikx) dx, \quad (k = \frac{\sqrt{2mE}}{\hbar})$$

we can then use this  $\phi(k)$  calculated above to determine the time dependent wave equation for the free particle

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) \exp\left(i\left(kx - \frac{\hbar k^2}{2m}t\right)\right) dk.$$

Notably, the particle has a phase velocity and a group velocity,

$$v_{\text{phase}} = \frac{\omega}{k}, \quad v_{\text{group}} = \frac{d\omega}{dk}. \quad (\omega = \frac{\hbar k^2}{2m})$$

Also note that

$$v_{\text{group}} = v_{\text{classical}}.$$

For a free particle, note that the potential energy is zero namely  $V(x, t) = 0$ . Also note that we can switch from the Fourier  $k$  space and the momentum  $p$  space ( $p = \hbar k$ ), obtaining the following

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x, 0) \exp\left(-\frac{ip}{\hbar}x\right) dx,$$

and

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) \exp\left(\frac{ip}{\hbar}x\right) \exp\left(\frac{-iEt}{\hbar}\right) dp \quad (E = \frac{\hbar^2 k^2}{2m})$$

## 1.6 The Delta-Function Potential

The delta function well has **only one** bound state (for  $\alpha > 0$ ) namely

$$\phi(x) = \frac{\sqrt{m\alpha}}{\hbar} \exp\left(\frac{-m\alpha|x|}{\hbar}\right),$$

with **only one** allowed energy

$$E = -\frac{m\alpha^2}{2\hbar^2}.$$

The reflection coefficient  $R$  and the transmission coefficient  $T$  can be expressed as the following

$$R = \frac{1}{1 + \frac{2\hbar^2 E}{m\alpha^2}}, \quad T = \frac{1}{1 + \frac{m\alpha^2}{2\hbar^2 E}}.$$

## 1.7 The Finite Square Well

The finite square well has the potential function such that (note that this is a flipped tophat function),

$$V(x) = \begin{cases} -V_0 & \text{if } -a \leq x \leq a, \\ 0 & \text{if } |x| > a. \end{cases}$$

For scattering states, the energy for perfect transmission is given by

$$E_n + V_0 = \frac{n^2 \pi^2 \hbar^2}{2m(2a)^2}.$$

Note that is also the allowed energies for the infinite square well.

## 2 Formalism

### 2.1 Hermitian Matrices

#### Properties of Hermitian Matrices:

1. The diagonal elements are real, as they must be their complex conjugate.
2. The off-diagonal symmetric pairs must be complex conjugates  $m_{ij} = m_{ji}^*$ . If they are real, then they will be equal.
3. Hermitian matrices are **normal**, i.e.  $M^\dagger M = M M^\dagger$ , and therefore **diagonalizable**, which means they can be transformed such that all off-diagonal elements are zero.
4. The sum of any two Hermitian matrices is also Hermitian.
5. The determinant of a Hermitian matrix is real.

### 2.2 Fundamental Theorem of Quantum Mechanics

#### The Fundamental Theorem

1. If  $\lambda_1$  and  $\lambda_2$  are two unequal eigen values of a Hermitian operator, then the corresponding eigenvectors are orthogonal.
2. Even if  $\lambda_1 = \lambda_2$ , the corresponding eigenvectors can be chosen to be orthogonal. We use the term degeneracy to describe the case where two different eigenvectors have the same eigenvalue.  $\lambda_1$  and  $\lambda_2$  are referred to as degenerate.
3. The eigenvectors of a Hermitian operator are a complete set, i.e. any vector the operator can generate can be expanded as a sum of its eigenvectors.

A **distillation** of the above: For any observable, we have an operator, the eigenvectors of that operator will be the basis for the vector space we operate in.

#### The Principles

1. The observable or measurable quantities of quantum mechanics are represented by linear operators  $\mathbf{L}$ , with  $\lambda_i$ ,  $|\lambda_i\rangle$  as its eigenvalue and eigenvector respectively.
2. The possible results of a measurement are the eigenvalues of the operator that represents the observable.
3. unambiguously distinguishable states are represented by orthogonal vectors.
4. if  $|A\rangle$  is the state-vector of a system, and the observable  $\mathbf{L}$  is measured, the probability to observe value  $\lambda_i$  is

$$\mathbb{P}(\lambda_i) = \|\langle A|\lambda_i\rangle\|^2 = \langle A|\lambda_i\rangle \langle \lambda_i|A\rangle$$

## 2.3 Recalling on Statistics

### 2.3.1 Statitital Correlation

Consider

$$P = \langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle,$$

if  $P \neq 0$ , then  $\langle \sigma_A \rangle$  unrelated to  $\langle \sigma_B \rangle$ , else they are related.

For event  $a$  and  $b$  to be independent,

$$\mathbb{P}(a, b) = \mathbb{P}(a)\mathbb{P}(b).$$