PHYS2113 Classical Mechanics

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Contents

1 Introduction to Lagrangian Mechanics

1.1 Action

Definition 1.1 (Action) *Action, termed A, is defined as*

$$A = \int_{t_0}^{t_1} L \, \mathrm{d}t. \tag{1.1.1}$$

Where $L(q, \dot{q}) = T - V = \frac{1}{2}m\dot{q}^2 - V(q)$ is what we call the Lagrangian.

Note that action represents the integral over time of the Lagrangian which can be thought as the motion of the object at some point of time.[1]

1.2 The Euler-Lagrange Equation

Definition 1.2 (The Euler-Lagrange Equation) *The Euler-Lagrange equation for a system with a single degree of freedom is*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{1.2.1}$$

1.2.1 Derivation

We want to find a generalised solution for the path that minimises the variational problem integral

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x] \, \mathrm{d}x. \tag{1.2.2}$$

First, let's start off by defining that the 'right' path (path with the least action/that minimise the variational problem) to be $y = y(x)$, and the wrong path is just a variation of the right path known as $Y(y(x), \alpha, \eta(x)) = y(x) + \alpha\eta(x)$.

Since the end points of the right path and the wrong path are the same we get

$$\eta(x_1) = \eta(x_2) = 0. \tag{1.2.3}$$

Now, the variational problem interval in terms of the wrong path S_0 would be

$$\begin{aligned} S_0 &= \int_{x_1}^{x_2} f(Y, Y', x) \, \mathrm{d}x \\ &= \int_{x_1}^{x_2} f(y + \alpha\eta, y' + \alpha\eta', x) dx. \end{aligned} \tag{1.2.4}$$

Note that the only difference between integral S and S_0 is the dependence on α. So, ideally to minimise this integral, we would find the stationary point of S_0 in terms of α, this can be expressed as

$$\begin{aligned} \frac{\mathrm{d}S_0}{\mathrm{d}\alpha} &= 0 \\ &= \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha} \, \mathrm{d}x \\ &= \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right) \mathrm{d}x. \end{aligned} \tag{1.2.5}$$

And now, using (1.2.3) and integration by parts, we obtain the following

$$\int_{x_1}^{x_2} \eta(x) \left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} \right) = 0. \tag{1.2.6}$$

For non-trivial solution, we must have

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} = 0,$$

which is the Euler-Lagrange Equation.

Remark 1.2.1.1 *Note that for* (1.2.5)*, the following needs to be true for* $Y = y + \alpha\eta$*;*

$$\frac{\partial}{\partial Y} f(Y, Y', x) = \frac{\partial}{\partial y} f(y + \alpha\eta, y' + \alpha\eta', x).$$

The proof is simple,

$$\begin{aligned} \frac{\partial}{\partial y} f(y + \alpha\eta, y' + \alpha\eta', x) &= f'(y + \alpha\eta, y' + \alpha\eta', x) \\ &= f'(Y, Y', x). \end{aligned}$$

1.2.2 Example

Refer to Taylor's[1] **Example 6.2** on page 222.

1.3 Proof of Lagrange's Equations with Constraints

Refer to **Section 7.4** (pg 250) in Classical Mechanics by JR Taylor.

One important thing: the right path of any action must follows Newton's second law.

1.4 Lagrangian Multipliers

Suppose we have a system with two variables x, y. Which are linked together with a constraint, say $F(x, y) = C \in \mathbb{R}$.

Now, let us introduce this function $\lambda(t)$ (aka Lagrange multiplier), and now the E-L Equation for x becomes

$$\frac{\partial L}{\partial x} + \lambda(t) \frac{\partial F}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}}, \tag{1.4.1}$$

similarly, the E-L equation for y is

$$\frac{\partial L}{\partial y} + \lambda(t) \frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{y}}. \tag{1.4.2}$$

[1] The book 'Classical Mechanics'[2] By John R. Taylor

Now, fingers cross that we can solve the above two equations. Then we can rewrite one of the variable in terms of the other and obtain their time derivatives, e.g.

$$x = g(y), \tag{1.4.3}$$
$$\implies \dot{x} = \frac{\mathrm{d}}{\mathrm{d}t} g(y), \tag{1.4.4}$$
$$\implies \ddot{x} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} g(y). \tag{1.4.5}$$

And then we can substitute them into our results from Eq. (1.4.1) & (1.4.2). If we don't drink and do maths, by now, we should just be able to obtain $\lambda(t)$ with a little more algebra.

1.5 Conservation of Energy and the Hamiltonian

Refer to section **Section 7.8** (pg 269) in Taylor's for derivation.

1.6 Legendre Transform

Let's have and input system that consists of

$$F(u_1, \dots u_n, w_1, \dots w_n);$$
$$v_i = \frac{\partial F}{\partial u_i};$$
$$G = \sum_{i=1}^{n} u_i v_i - F.$$

By performing a Legendre Transform, we will get the output system that consists of

$$G(v_1, \dots v_n, w_1, \dots w_n);$$
$$u_i = \frac{\partial G}{\partial v_i};$$
$$F = \sum_{i=1}^{n} u_i v_i - G.$$

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2 Very Short Session on Hamiltonian Mechanics

2.1 From Lagrangian to Hamiltonian

Let us record that in the Lagrangian, we have a function of three dependents,

$$L = L(q_1, \dots q_n, \dot{q}_1, \dots \dot{q}_n, t) = T - U, \tag{2.1.1}$$

and obviously, the Euler-Lagrange equation would be

$$\frac{\partial L}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i}. \tag{2.1.2}$$

Also remember that the **generalised momentum** (a.k.a canonical momentum/momentum conjugate to q_i) is given by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{2.1.3}$$

The Hamiltonian is defined as

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L. \tag{2.1.4}$$

If we apply the Legendre Transform to the E-L equation, we will find the Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{2.1.5}$$

2.2 Poisson Brackets

Definition of Poisson Brackets in **canonical coordinates** (q_i, p_i) on the phase space, given two functions $f(p_i, q_i, t)$ and $g(p_i, q_i, t)$, the Poisson bracket take the form [3]

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right). \tag{2.2.1}$$

The Poisson brackets of the canonical coordinates are

$$\begin{aligned} \{q_i, q_j\} &= 0 \\ \{p_i, p_j\} &= 0 \\ \{q_i, p_j\} &= \delta_{ij}. \end{aligned}$$

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3 Harmonic Oscillator

3.1 Damped Harmonic Oscillators

Suppose we have a system that can be described by the following ODE

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{3.1.1}$$

Where m denotes the mass, b denotes resistance and k denotes the spring constant. Now let's define the damping constant to be

$$\beta = \frac{b}{2m}.$$

And, the system will have a **natural frequency** (the frequency at which it would oscillate if there were no resistive force present)

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

The general solution of this system is

$$x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}t}\right) \tag{3.1.2}$$

3.2 Undamped Oscillators

I hope it is clear that for an undamped oscillator, we just need to take the damping constant β to be zero, so the EOM for the system would be

$$x(t) = C_1 e^{\omega_0 t} + C_2 e^{-\omega_0 t}. \tag{3.2.1}$$

3.3 Weak Damping

In the case of weakly damped oscillators, damping constant β is said to be

$$\beta < \omega_0.$$

Now the EOM becomes

$$x(t) = e^{-\beta t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right), \tag{3.3.1}$$

where

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}.$$

Note, in the case of underdamping. β is said to be the decay parameter, since $1/\beta$ the time taken for the oscillator to fall to $1/e$ of its original amplitude.

$$(\text{decay parameter}) = \beta.$$

3.4 Critical Damping

Critical damping is said to be happening when

$$\beta = \omega_0.$$

The general solution for critical damping is

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}, \tag{3.4.1}$$

with decay parameter

$$(\text{decay parameter}) = \beta = \omega_0.$$

3.5 Strong Damping

Over damping is said to be happening when

$$\beta > \omega_0.$$

The general EOM is

$$x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)t} + C_2 e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t}, \tag{3.5.1}$$

with decay parameter

$$(\text{decay parameter}) = \beta - \sqrt{\beta^2 - \omega_0^2}.$$

3.6 Forced Damped Oscillators

References

[1] *Action (physics)*. en. Page Version ID: 1020785959. May 2021. URL: https://en.wikipedia.org/w/index.php?title=Action_(physics)&oldid=1020785959 (visited on 05/31/2021).

[2] John R. (John Robert) Taylor. *Classical mechanics*. eng. Sausalito, Calif. : [Basingstoke: University Science Books ; Palgrave, distributor], 2005. ISBN: 189138922X.

[3] *Action (physics)*. en. Page Version ID: 1014893060. 2021. URL: https://en.wikipedia.org/w/index.php?title=Poisson_bracket&oldid=1014893060 (visited on 07/02/2021).