PHYS2114 Cheat Sheet

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Contents

1 Mathematics Preliminary

1.1 IMPORTANT NOTE

The following notations are taken from Griffths' 'Introduction to Electromechanics'.[1]

Where as the line, area and volume integral elements are the following;

- Line integral element: $\mathrm{d}\vec{l}$.
- Area integral element: $\mathrm{d}\vec{a}$.
- Volume integral element: $\mathrm{d}\tau$.

1.2 Cartesian Coordinates

The line and volume integral elements are

$$\mathrm{d}\vec{l} = \hat{x}\,\mathrm{d}x + \hat{y}\,\mathrm{d}y + \hat{z}\,\mathrm{d}z, \qquad \mathrm{d}\tau = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

And the following are some common operators

$$\text{Gradient:} \quad \nabla t = \frac{\partial t}{\partial x}\hat{x} + \frac{\partial t}{\partial y}\hat{y} + \frac{\partial t}{\partial z}\hat{z},$$

$$\text{Divergence:} \quad \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z},$$

$$\text{Curl:} \quad \nabla \times \vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z},$$

$$\text{Laplacian:} \quad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}.$$

1.3 Cylindrical Coordinates

The conversion of Cartesian to Cylindrical coordinates[2] are

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z,$$

note that $\phi \in [0, 2\pi]$.

And the surface integral element with radius r constant is

$$\mathrm{d}\vec{S} = r\,\mathrm{d}\phi\,\mathrm{d}z$$

The line and the volume integral elements are

$$\mathrm{d}\vec{l} = \hat{s}\,\mathrm{d}s + s\hat{\phi}\,\mathrm{d}\phi + \hat{z}\,\mathrm{d}z, \qquad \mathrm{d}\tau = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z.$$

The following are some common vector operators

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z},$$

Divergence: $$\nabla \cdot \vec{v} = \frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z},$$

Curl: $$\nabla \times \vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z},$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}.$$

1.4 Spherical Coordinates

Note that $\theta \in [0, \pi]$ denotes the angle between z-axis and the vector of interest, and that $\phi \in [0, 2\pi]$ denotes the angle between x-axis and the projection of the vector of interest on to the xy-plane.[2]

The conversion of Cartesian to Spherical coordinates are shown as the follows

$$x = r\sin(\theta)\cos(\phi), \qquad y = r\sin(\theta)\sin(\phi), \qquad z = r\cos(\theta).$$

And the line and volume integral elements are

$$\mathrm{d}\vec{l} = \hat{r}\,\mathrm{d}r + r\hat{\theta}\,\mathrm{d}\theta + r\sin(\theta)\hat{\phi}\,\mathrm{d}\phi, \qquad \mathrm{d}\tau = r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

with the surface integral with the radius r constant is

$$\mathrm{d}\vec{S} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

The following are the common operators

Gradient: $$\nabla t = \hat{r}\frac{\partial t}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial t}{\partial \theta} + \frac{\hat{\phi}}{r\sin(\theta)}\frac{\partial t}{\partial \phi},$$

Divergence: $$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi},$$

Curl: $$\begin{aligned}\nabla \times \vec{v} = {} & \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}\sin(\theta)v_\phi - \frac{\partial v_\theta}{\partial \phi}\right)\hat{r} \\ & + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}rv_\phi\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}rv_\theta - \frac{\partial v_r}{\partial \theta}\right)\hat{\phi},\end{aligned}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}.$$

1.5 Gauss' Divergence Theorem

Suppose V is a subset of $\mathbb{R}^n$ (in the case of $n = 3$, V represents a volume in three-dimensional space) which is compact and has a piecewise smooth boundary S (also indicated with $\partial V = S$). If $\vec{F}$ is a continuously differentiable vector filed defined on a neighbourhood of V, then:

$$\iiint_V \left(\nabla \cdot \vec{F}\right) \mathrm{d}V = \oiint_S \left(\vec{F} \cdot \hat{n}\right) \mathrm{d}S.$$

The left side is a volume integral over the volume V, the right side is the surface integral over the boundary of the volume V. The closed manifold ∂V is oriented by outward-pointing normal, and $\vec{n}$ is the outward pointing normal at each point on the boundary ∂V. ($\mathrm{d}\vec{S}$ may be used as a shorthand for $\vec{n}\,\mathrm{d}S$.) In terms of the intuitive description above, the left-hand side of the equation represents the total of the sources in the volume V, and the right-hand side represents the total flow across the boundary S.[3]

1.6 Stokes' Theorem

Suppose we have a boundary $\partial\Sigma = S$ that bounds the surface Σ with $\vec{F}$ defined in Σ, then

$$\iint_\Sigma (\nabla \times \vec{F}) \cdot \vec{n}\,\mathrm{d}S = \oint_S \vec{F} \cdot \mathrm{d}\vec{l}.$$

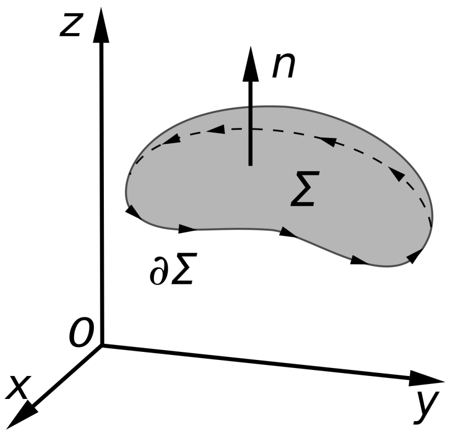

Figure 1: A visual representation for the Stokes' theorem[4]

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2 Electrostatics

2.1 The Holy Trinity

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{\mathscr{r}}\,\mathrm{d}\tau \tag{2.1.1}$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \tag{2.1.2}$$

$$E = -\nabla V \tag{2.1.3}$$

$$V = -\int \vec{E}\cdot \mathrm{d}\vec{l} \tag{2.1.4}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho\,\mathrm{d}\tau \tag{2.1.5}$$

$$\nabla\cdot E = \frac{\rho}{\varepsilon_0};\quad \nabla\times E = 0 \tag{2.1.6}$$

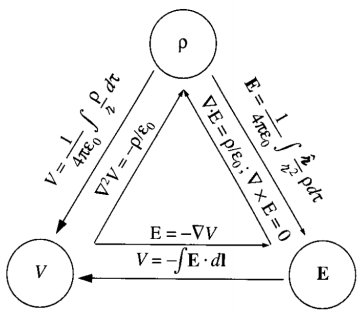

Figure 2: The Griffith's holy trinity of electrostatics[1]

2.2 Electrostatic Boundary Conditions

$$\hat{n}_2\cdot[\vec{E}_1 - \vec{E}_2] = \frac{\sigma}{\varepsilon_0} \tag{2.2.1}$$

$$\hat{n}_2\times[\vec{E}_1 - \vec{E}_2] = 0 \tag{2.2.2}$$

$$V_1 - V_2 = 0 \tag{2.2.3}$$

Note that here the "1" and "2" just refer to the different sides of the interface.

2.3 Work and Energy in Electrostatics

Three important equations for Energy in work and energy in electro statics: For discrete charges that are not very closed together,

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i). \tag{2.3.1}$$

For volume charge density ρ,

$$W = \frac{1}{2}\int \rho V \,\mathrm{d}\tau. \tag{2.3.2}$$

And even more simply, we can have,

$$W = \frac{\varepsilon_0}{2}\int_{\mathbb{R}^3} E^2 \,\mathrm{d}\tau. \tag{2.3.3}$$

2.4 Conductors

There are **five** fundamental properties of a conductors,

1. $E = 0$ inside a conductor.
2. $\rho = 0$ inside a conductor.
3. Any net charge resides on the surface.
4. A conductor is an equipotential.
5. E is perpendicular to the surface, just outside a conductor.

Now, let's consider the force per unit area $\vec{f}$ on any charges that rests on the surface of the conductor, with references to page 102 in the book,[1] we know that

$$\vec{f} = \sigma\vec{E}_{\text{average}} = \frac{1}{2}\sigma(\vec{E}_{\text{above}} - \vec{E}_{\text{below}}). \tag{2.4.1}$$

In particular, that for conductors, where we know $\vec{E}_{\text{below}} = 0$ (equipotential inside the conductor), we obtain

$$\vec{f} = \frac{1}{2\varepsilon_0}\sigma^2\hat{n} \tag{2.4.2}$$

2.5 Capacitors

[1] 'The book' is exclusively used in this note to 'Introduction to Electromagnetism' by David J. Griffths

References

[1] David J Griffiths. *Introduction to electrodynamics; 3rd ed.* Upper Saddle River, NJ: Prentice-Hall, 1999. URL: https://cds.cern.ch/record/611579.

[2] *Del in cylindrical and spherical coordinates.* en. Page Version ID: 1016691595. Apr. 2021. URL: https://en.wikipedia.org/w/index.php?title=Del_in_cylindrical_and_spherical_coordinates&oldid=1016691595 (visited on 06/03/2021).

[3] *Divergence theorem.* en. Page Version ID: 1025269448. May 2021. URL: https://en.wikipedia.org/w/index.php?title=Divergence_theorem&oldid=1025269448 (visited on 06/05/2021).

[4] *Stokes' theorem.* en. Page Version ID: 1014735398. Mar. 2021. URL: https://en.wikipedia.org/w/index.php?title=Stokes%27_theorem&oldid=1014735398 (visited on 06/05/2021).