PHYS2111 Cheat Sheet

Moisty Eyes

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1 Formula Sheet

1.1 Expectation Value

Expectation value of function $f(x)$ subject to $\Psi(x,t)$

$$\mathbb{P}(f(x)) = \int_{-\infty}^{\infty} f(x) \|\Psi(x,t)\|^2 \, \mathrm{d}x.$$

Note that the Hamiltonian operator H is

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t),$$

and the expectation value of the energy $\langle H \rangle$ is

$$\begin{aligned} \langle H \rangle &= \left\langle -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t) \right\rangle \\ &= \int_{-\infty}^{\infty} \Psi^*(x,t) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t) \right) \Psi(x,t) \, \mathrm{d}x. \end{aligned}$$

General case, the expectation value for any operator $Q(x, \hat{p})$ is

$$\int_{-\infty}^{\infty} \Psi^*(x,t) Q(x,\hat{p}) \Psi(x,t) \, \mathrm{d}x.$$

1.2 Poisition and Momentum Operators

The position operator,

$$\hat{x} = x.$$

The momentum operator,

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}.$$

1.3 The Infinite Square Well

Time Dependent Schrodinger's Equation

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right) \exp\left(-iE_n t\right).$$

with constant

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\,\mathrm{d}x.$$

and allowed energy

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}.$$

1.4 The Harmonic Oscillator

A harmonic oscillator has potential energy

$$V(x) = \frac{1}{2}m\omega^2x^2,$$

With ground state $\psi_0(x)$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

The ladder operator

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega x), \qquad (\omega = \sqrt{\tfrac{k}{m}})$$

and to extract the n-th state with the ladder operator

$$\psi_n(x) = A_n(\hat{a}_+)^n\psi_0(x),$$

with

$$E_n = \left(n + \frac{1}{2}\hbar\omega\right).$$

1.5 The Free Particle

The initial condition can be expressed in the Fourier k space,

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0)\exp(-ikx)\,\mathrm{d}x, \qquad (k = \tfrac{\sqrt{2mE}}{\hbar})$$

we can then use this $\phi(k)$ calculated above to determine the time dependent wave equation for the free particle

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\exp\left(i\left(kx - \frac{\hbar k^2}{2m}t\right)\right)\,\mathrm{d}k.$$

Notably, the particle has a phase velocity and a group velocity,

$$v_{\text{phase}} = \frac{\omega}{k}, \qquad v_{\text{group}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \qquad (\omega = \tfrac{\hbar k^2}{2m})$$

Also note that

$$v_{\text{group}} = v_{\text{classical}}.$$

For a free particle, note that the potential energy is zero namely $V(x,t) = 0$. Also note that we can switch from the Fourier k space and the momentum p space ($p = \hbar k$), obtaining the following

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x,0)\exp\left(-\frac{ip}{\hbar}x\right)\,\mathrm{d}x,$$

and

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) \exp\left(\frac{ip}{\hbar}x\right) \exp\left(\frac{-iEt}{\hbar}\right) \mathrm{d}p \qquad (E = \tfrac{\hbar^2 k^2}{2m})$$

1.6 The Delta-Function Potential

The delta function well has **only one** bound state (for $\alpha > 0$) namely

$$\phi(x) = \frac{\sqrt{m\alpha}}{\hbar} \exp\left(\frac{-m\alpha|x|}{\hbar}\right),$$

with **only one** allow energy

$$E = -\frac{m\alpha^2}{2\hbar^2}.$$

The reflection coefficient R and the transmission coefficient T can be expressed as the following

$$R = \frac{1}{1 + \frac{2\hbar^2 E}{m\alpha^2}}, \qquad T = \frac{1}{1 + \frac{m\alpha^2}{2\hbar^2 E}}.$$

1.7 The Finite Squre Well

The finite squre well has the potential function such that (note that this is a flipped tophat function),

$$V(x) = \begin{cases} -V_0 & \text{if } -a \le x \le a, \\ 0 & \text{if } |x| > a. \end{cases}$$

For scattering states, the energy for perfect transmission is given by

$$E_n + V_0 = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}.$$

Note that is also the allowed eneries for the infinite square well.

2 Formalism

2.1 Hermitian Matrices

Properties of Hermitian Matrices:

1. The diagonal elements are real, as they must be their complex conjugate.
2. The off-diagonal symmetric pairs must be complex conjugates $m_{ij} = m_{ji}^*$. If they are real, then they will be equal.
3. Hermitian matrices are **normal**, i.e. $M^\dagger M = MM^\dagger$, and therefore **diagonalizable**, which means they can be transformed such that all off-diagonal elements are zero.
4. The sum of any two Hermitian matrices is also Hermitian.
5. The determinant of a Hermitian matrix is real.

Basically, let M_n be the set of $n \times n$ complex-valued matrices. Let us consider a matrix $A = [a_{ij}] \in M_n$ and denote its complex conjugate by $\bar{A} = [\bar{a}_{ij}]$ and its transpose by $A^T = [a_{ji}]$. We then have the following: A matrix $A = [a_{ij}] \in M_n$ is said to be Hermitian if $A = A^*$, where $A^* = \bar{A}^T = [\bar{a}_{ji}]$.

2.2 Fundamental Theorem of Quantum Mechanics

The Fundamental Theorem

1. If λ_1 and λ_2 are two unequal eigenvalues of a Hermitian operator, then the corresponding eigenvectors are orthogonal.
2. Even if $\lambda_1 = \lambda_2$, the coreresponding eigenvectors can be chosen to be orthogonal. We use the term degeneracy to describe the case where two different eigenvectors have the same eigenvalue. λ_1 and λ_2 are referred to as degenerate.
3. The eigenvectors of a Hermitian operator are a complete set, i.e. any vector the operator can generate can be expanded as a sum of its eigenvectors.

A **distillation** of the above: For any observable, we have an operator, the eigenvectors of that operator will be the basis for the vector space we operate in.

The Principles

1. The observable or measurable quantities of quantum mechanics are represented by linear opertors $\mathbf{L}$, with λ_i, $|\lambda_i\rangle$ as its eigenvalue and eigenvector respectively.
2. The possible results of a measuremeant are the eigenvalues of the operator that represents the observable.
3. unambiguously distinguishable states are representsed by orthogonal vectors.
4. if $|A\rangle$ is the state-vector of a system, and the observable $\mathbf{L}$ is measured, the probability to observe value λ_i is

$$\mathbb{P}(\lambda_i) = \|\langle A|\lambda_i\rangle\|^2 = \langle A|\lambda_i\rangle \langle\lambda_i|A\rangle$$

2.3 Recalling on Statistics

2.3.1 Statitical Correlation

Consider

$$P = \langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle,$$

if $P \neq 0$, then $\langle \sigma_A \rangle$ unrelated to $\langle \sigma_B \rangle$, else they are related.

For event a and b to be independent,

$$\mathbb{P}(a, b) = \mathbb{P}(a)\mathbb{P}(b).$$

2.3.2 The Cauchy-Swarz Inequality

It states that

$$\langle A|A \rangle \langle B|B \rangle \geq |\langle A|B \rangle|^2.$$

2.3.3 Variance/Uncertainty

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2.$$

2.4 Commutators

For some $[L, M] = LM - ML$, if $[L, M] = 0$, then L and M commute and therefore there can exist a zero undertainty (we can know both of the observable precisely).

2.5 Some useful Propeties

2.5.1 Matrices

Let r be a real number and A and B be matrices. Then

1. $(A^T)^T = A$,
2. $(A + B)^T = A^T + B^T$,
3. $(AB)^T = B^T A^T$,
4. $(rA)^T = rA^T$.

2.5.2 Hermitian Conjugate

$$\langle \phi | H | \psi \rangle = \int_{-\infty}^{\infty} \phi^* H \psi \, \mathrm{d}x = \left\langle H^\dagger \phi \middle| \psi \right\rangle.$$