PHYS2113 Classical Mechanics

Joey Liang

May 2021

Contents

1 Introduction to Lagrangian Mechanics

1.1 Action

Definition 1.1 (Action) *Action, termed A, is defined as*

$$A = \int_{t_0}^{t_1} L \, \mathrm{d}t. \tag{1.1.1}$$

Where $L(q, \dot{q}) = T - V = \frac{1}{2}m\dot{q}^2 - V(q)$ is what we call the Lagrangian.

Note that action represents the integral over time of the Lagrangian which can be thought as the motion of the object at some point of time.[1]

1.2 The Euler-Lagrange Equation

Definition 1.2 (The Euler-Lagrange Equation) *The Euler-Lagrange equation for a system with a single degree of freedom is*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{1.2.1}$$

1.2.1 Derivation

We want to find a generalised solution for the path that minimises the variational problem integral

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x] \, \mathrm{d}x. \tag{1.2.2}$$

First, let's start off by defining that the 'right' path (path with the least action/that minimise the variational problem) to be $y = y(x)$, and the wrong path is just a variation of the right path known as $Y(y(x), \alpha, \eta(x)) = y(x) + \alpha\eta(x)$.

Since the end points of the right path and the wrong path are the same we get

$$\eta(x_1) = \eta(x_2) = 0. \tag{1.2.3}$$

Now, the variational problem interval in terms of the wrong path S_0 would be

$$\begin{aligned} S_0 &= \int_{x_1}^{x_2} f(Y, Y', x) \, \mathrm{d}x \\ &= \int_{x_1}^{x_2} f(y + \alpha\eta, y' + \alpha\eta', x) dx. \end{aligned} \tag{1.2.4}$$

Note that the only difference between integral S and S_0 is the dependence on α. So, ideally to minimise this integral, we would find the stationary point of S_0 in terms of α, this can be expressed as

$$\begin{aligned}\frac{\mathrm{d}S_0}{\mathrm{d}\alpha} &= 0 \\ &= \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha}\,\mathrm{d}x \\ &= \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}\right)\mathrm{d}x. \end{aligned} \tag{1.2.5}$$

And now, using (1.2.3) and integration by parts, we obtain the following

$$\int_{x_1}^{x_2} \eta(x) \left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right) = 0. \tag{1.2.6}$$

For non-trivial solution, we must have

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0,$$

which is the Euler-Lagrange Equation.

Remark 1.2.1.1 *Note that for* (1.2.5)*, the following needs to be true for* $Y = y + \alpha\eta$*;*

$$\frac{\partial}{\partial Y} f(Y, Y', x) = \frac{\partial}{\partial y} f(y + \alpha\eta, y' + \alpha\eta', x).$$

The proof is simple,

$$\begin{aligned}\frac{\partial}{\partial y} f(y + \alpha\eta, y' + \alpha\eta', x) &= f'(y + \alpha\eta, y' + \alpha\eta', x) \\ &= f'(Y, Y', x).\end{aligned}$$

1.2.2 Example

Refer to Taylor's[1] **Example 6.2** on page 222.

1.3 Proof of Lagrange's Equations with Constraints

Refer to **Section 7.4** (pg 250) in Classical Mechanics by JR Taylor.

One important thing: the right path of any action must follows Newton's second law.

1.4 Lagrangian Multipliers

Suppose we have a system with two variables x, y. Which are linked together with a constraint, say $F(x, y) = C \in \mathbb{R}$.

Now, let us introduce this function $\lambda(t)$ (aka Lagrange multiplier), and now the E-L Equation for x becomes

$$\frac{\partial L}{\partial x} + \lambda(t)\frac{\partial F}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}, \tag{1.4.1}$$

[1] The book 'Classical Mechanics'[2] By John R. Taylor

similarly, the E-L equation for y is

$$\frac{\partial L}{\partial y} + \lambda(t)\frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}}. \tag{1.4.2}$$

Now, fingers cross that we can solve the above two equations. Then we can rewrite one of the variable in terms of the other and obtain their time derivatives, e.g.

$$\begin{aligned} x &= g(y), && (1.4.3)\\ \implies \dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t}g(y), && (1.4.4)\\ \implies \ddot{x} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}g(y). && (1.4.5) \end{aligned}$$

And then we can substitute them into our results from Eq. (1.4.1) & (1.4.2). If we don't drink and do maths, by now, we should just be able to obtain $\lambda(t)$ with a little more algebra.

1.5 Conservation of Energy and the Hamiltonian

Refer to section **Section 7.8** (pg 269) in Taylor's for derivation.

1.6 Legendre Transform

Let's have and input system that consists of

$$\begin{aligned} &F(u_1, \dots u_n, w_1, \dots w_n);\\ &v_i = \frac{\partial F}{\partial u_i};\\ &G = \sum_{i=1}^{n} u_i v_i - F. \end{aligned}$$

By performing a Legendre Transform, we will get the output system that consists of

$$\begin{aligned} &G(v_1, \dots v_n, w_1, \dots w_n);\\ &u_i = \frac{\partial G}{\partial v_i};\\ &F = \sum_{i=1}^{n} u_i v_i - G. \end{aligned}$$

---

2 Very Short Session on Hamiltonian Mechanics

2.1 From Lagrangian to Hamiltonian

Let us record that in the Lagrangian, we have a function of three dependents,

$$L = L(q_1, \dots q_n, \dot{q}_1, \dots \dot{q}_n, t) = T - U, \tag{2.1.1}$$

and obviously, the Euler-Lagrange equation would be

$$\frac{\partial L}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}. \tag{2.1.2}$$

Also remember that the **generalised momentum** (a.k.a canonical momentum/momentum conjugate to q_i) is given by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{2.1.3}$$

The Hamiltonian is defined as

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L. \tag{2.1.4}$$

If we apply the Legendre Transform to the E-L equation, we will find the Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = \frac{\partial H}{\partial q_i}. \tag{2.1.5}$$

References

[1] *Action (physics)*. en. Page Version ID: 1020785959. May 2021. URL: https://en.wikipedia.org/w/index.php?title=Action_(physics)&oldid=1020785959 (visited on 05/31/2021).

[2] John R. (John Robert) Taylor. *Classical mechanics*. eng. Sausalito, Calif. : [Basingstoke: University Science Books ; Palgrave, distributor], 2005. ISBN: 189138922X.