# PHYS2114 Cheat Sheet

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# Contents

# 1 Mathematics Preliminary

## 1.1 IMPORTANT NOTE

The following notations are taken from Griffths' 'Introduction to Electromechanics'.[1]

Where as the line, area and volume integral elements are the following;

- Line integral element: $\mathrm{d}\mathbf{l}$.
- Area integral element: $\mathrm{d}\mathbf{a}$.
- Volume integral element: $\mathrm{d}\tau$.

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## 1.2 Cartesian Coordinates

The line and volume integral elements are

$$\mathrm{d}\mathbf{l} = \hat{\mathbf{x}}\,\mathrm{d}x + \hat{\mathbf{y}}\,\mathrm{d}y + \hat{\mathbf{z}}\,\mathrm{d}z, \qquad \mathrm{d}\tau = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

And the following are some common operators

Gradient: $$\boldsymbol{\nabla} t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}},$$

Divergence: $$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z},$$

Curl: $$\boldsymbol{\nabla}\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}},$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}.$$

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## 1.3 Cylindrical Coordinates

The conversion of Cartesian to Cylindrical coordinates[2] are

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z,$$

note that $\phi \in [0, 2\pi]$.

And the surface integral element with radius $r$ constant is

$$\mathrm{d}\mathbf{S} = r\,\mathrm{d}\phi\,\mathrm{d}z$$

The line and the volume integral elements are

$$\mathrm{d}\mathbf{l} = \hat{\mathbf{s}}\,\mathrm{d}s + s\hat{\phi}\,\mathrm{d}\phi + \hat{\mathbf{z}}\,\mathrm{d}z, \qquad \mathrm{d}\tau = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z.$$

The following are some common vector operators

$$\text{Gradient:} \qquad \boldsymbol{\nabla} t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{\mathbf{z}},$$

$$\text{Divergence:} \qquad \boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z},$$

$$\text{Curl:} \qquad \boldsymbol{\nabla} \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}},$$

$$\text{Laplacian:} \qquad \nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}.$$

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## 1.4 Spherical Coordinates

Note that $\theta \in [0, \pi]$ denotes the angle between z-axis and the vector of interest, and that $\phi \in [0, 2\pi]$ denotes the angle between x-axis and the projection of the vector of interest on to the xy-plane.[2]

The conversion of Cartesian to Spherical coordinates are shown as the follows

$$x = r\sin(\theta)\cos(\phi), \qquad y = r\sin(\theta)\sin(\phi), \qquad z = r\cos(\theta).$$

And the line and volume integral elements are

$$\mathrm{d}\mathbf{l} = \hat{\mathbf{r}}\,\mathrm{d}r + r\hat{\theta}\,\mathrm{d}\theta + r\sin(\theta)\hat{\phi}\,\mathrm{d}\phi, \qquad \mathrm{d}\tau = r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

with the surface integral with the radius $r$ constant is

$$\mathrm{d}\mathbf{S} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

The following are the common operators

$$\text{Gradient:} \qquad \boldsymbol{\nabla} t = \hat{\mathbf{r}}\frac{\partial t}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial t}{\partial \theta} + \frac{\hat{\phi}}{r\sin(\theta)}\frac{\partial t}{\partial \phi},$$

$$\text{Divergence:} \qquad \boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi},$$

$$\begin{aligned}\text{Curl:} \qquad \boldsymbol{\nabla} \times \mathbf{v} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}\sin(\theta)v_\phi - \frac{\partial v_\theta}{\partial \phi}\right)\hat{\mathbf{r}} \\ &+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}rv_\phi\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}rv_\theta - \frac{\partial v_r}{\partial \theta}\right)\hat{\phi},\end{aligned}$$

$$\text{Laplacian:} \qquad \nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}.$$

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## 1.5 Gauss' Divergence Theorem

Suppose $V$ is a subset of $\mathbb{R}^n$ (in the case of $n = 3$, $V$ represents a volume in three-dimensional space) which is compact and has a piecewise smooth boundary $S$ (also indicated with $\partial V = S$). If $\mathbf{F}$ is a continuously differentiable vector filed defined on a neighbourhood of $V$, then:

$$\iiint_V (\boldsymbol{\nabla} \cdot \mathbf{F})\ \mathrm{d}V = \oiint_S (\mathbf{F} \cdot \hat{\mathbf{n}})\ \mathrm{d}S.$$

The left side is a volume integral over the volume $V$, the right side is the surface integral over the boundary of the volume $V$. The closed manifold $\partial V$ is oriented by outward-pointing normal, and $\mathbf{n}$ is the outward pointing normal at each point on the boundary $\partial V$. ( $\mathrm{d}\mathbf{S}$ may be used as a shorthand for $\mathbf{n}\,\mathrm{d}S$.) In terms of the intuitive description above, the left-hand side of the equation represents the total of the sources in the volume $V$, and the right-hand side represents the total flow across the boundary $S$.[3]

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## 1.6 Stokes' Theorem

Suppose we have a boundary $\partial\Sigma = S$ that bounds the surface $\Sigma$ with $\mathbf{F}$ defined in $\Sigma$, then

$$\iint_\Sigma (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S = \oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{l}.$$

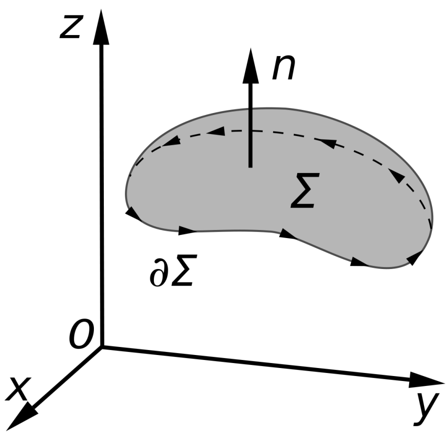


Figure 1: A visual representation for the Stokes' theorem[4]

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# 2 Electrostatics

## 2.1 The Holy Trinity

$$V = \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{V}} \frac{\rho}{\mathscr{r}} \,\mathrm{d}\tau \tag{2.1.1}$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \tag{2.1.2}$$

$$E = -\boldsymbol{\nabla} V \tag{2.1.3}$$

$$V = -\int \mathbf{E} \cdot \mathrm{d}\mathbf{l} \tag{2.1.4}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{V}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho \,\mathrm{d}\tau \tag{2.1.5}$$

$$\boldsymbol{\nabla} \cdot E = \frac{\rho}{\varepsilon_0}; \quad \boldsymbol{\nabla} \times E = 0 \tag{2.1.6}$$

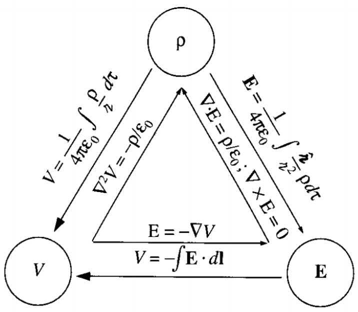


Figure 2: The Griffith's holy trinity of electrostatics[1]

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## 2.2 Electrostatic Boundary Conditions

$$\hat{\mathbf{n}}_2 \cdot [\mathbf{E}_1 - \mathbf{E}_2] = \frac{\sigma}{\varepsilon_0} \tag{2.2.1}$$

$$\hat{\mathbf{n}}_2 \times [\mathbf{E}_1 - \mathbf{E}_2] = 0 \tag{2.2.2}$$

$$V_1 - V_2 = 0 \tag{2.2.3}$$

Note that here the "1" and "2" just refer to the different sides of the interface.

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## 2.3 Work and Energy in Electrostatics

Three important equations for Energy in work and energy in electro statics: For discrete charges that are not very closed together,

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i). \tag{2.3.1}$$

For volume charge density $\rho$,

$$W = \frac{1}{2} \iiint_{\mathcal{V}} \rho V \, \mathrm{d}\tau. \tag{2.3.2}$$

And even more simply, we can have,

$$W = \frac{\varepsilon_0}{2} \iiint_{\mathbb{R}^3} E^2 \, \mathrm{d}\tau. \tag{2.3.3}$$

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## 2.4 Conductors

There are **five** fundamental properties of a conductors,

1. $E = 0$ inside a conductor.
2. $\rho = 0$ inside a conductor.
3. Any net charge resides on the surface.
4. A conductor is an equipotential.
5. $E$ is perpendicular to the surface, just outside a conductor.

Now, let's consider the force per unit area $\mathbf{f}$ on any charges that rests on the surface of the conductor, with references to page 102 in the book,[1] we know that

$$\mathbf{f} = \sigma \mathbf{E}_{\text{average}} = \frac{1}{2}\sigma(\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}). \tag{2.4.1}$$

In particular, that for conductors, where we know $\mathbf{E}_{\text{below}} = 0$ (equipotential inside the conductor), we obtain

$$\mathbf{f} = \frac{1}{2\varepsilon_0} \sigma^2 \hat{\mathbf{n}} \tag{2.4.2}$$

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## 2.5 Capacitors

$$Q = CV.$$

[1]'The book' is exclusively used in this note to 'Introduction to Electromagnetism' by David J. Griffths

# 3 Special Techniques

## 3.1 Uniqueness Theorems

**First uniqueness theorem:** The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if $V$ is specified on the boundary surface $S$.

**Second uniqueness theorem:** In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density $\rho$, the electric field is uniquely determined if the *total charge* on each conductor is given. (The region as a whole can be bounded by another conductor, or else unbounded.)[1]

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# 4 Electric Field in Matter

## 4.1 Induced Dipoles

Suppose we have an uniform electric field $\mathbf{E}$, and an atom (dipole) is in this field. This atom will now posses some dipole moment. The dipole moment is **most of the times** approximately proportional to the field.

$$\mathbf{p} = \alpha \mathbf{E}. \tag{4.1.1}$$

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## 4.2 Bound Charges

Let's suppose we have a blob of polarised material, with dipole moment $\mathbf{p} = \mathbf{P}(\mathbf{r}')\,\mathrm{d}\tau'$ in each volume element. The total potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{V}} \frac{\hat{\mathscr{r}} \cdot \mathbf{P}(\mathbf{r}')}{\mathscr{r}^2}\,\mathrm{d}\tau'.$$

And after some maths, we then obtain

$$V = \frac{1}{4\pi\varepsilon_0} \oiint_{\mathcal{S}} \frac{1}{\mathscr{r}}\mathbf{P} \cdot \mathrm{d}\mathbf{a}' - \frac{1}{4\pi\varepsilon_0} \iiiint_{\mathcal{V}} \frac{1}{\mathscr{r}}(\boldsymbol{\nabla} \cdot \mathbf{P})\,\mathrm{d}\tau'. \tag{4.2.1}$$

Notice that

$$\underbrace{\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}}_{\text{Surface bound charge}}, \qquad \text{and} \qquad \underbrace{\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P}}_{\text{Volume bound charge}}.$$

Those are called bound charges (surface bound charges & volume bound charge), we can just find them to get the voltage of the polarised dialectic blob instead of calculating the massive integral.

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## 4.3 The Electric Displacement

### 4.3.1 Gauss's Law in the Presence of DIelectrics

Blah blah blah there is something called the "free charge" because we don't live in a perfect world. So the total charge density is now

$$\rho = \rho_b + \rho_f.$$

and Gauss' law reads

$$\begin{aligned}
& \varepsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\boldsymbol{\nabla} \cdot \mathbf{P} + \rho_f; \\
\Longrightarrow \quad & \rho_f = \boldsymbol{\nabla} \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}); \\
\Longrightarrow \quad & \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}; \qquad (4.3.1) \\
\Longrightarrow \quad & \boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f; \qquad (4.3.2) \\
\Longrightarrow \quad & \oiint_{\mathcal{S}} \mathbf{D} \cdot \mathrm{d}\mathbf{a} = Q_{f_{\mathrm{enc}}}. \qquad (4.3.3)
\end{aligned}$$

*Trick*: In most of cases, **D** is determined exclusively by the free charge if we are dealing with symmetrical systems.

### 4.3.2 Boundary Conditions

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f; \tag{4.3.4}$$

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{above}}; \tag{4.3.5}$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\varepsilon_0}; \tag{4.3.6}$$

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = 0. \tag{4.3.7}$$

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## 4.4 Linear Dielectrics

Iff a space that is entirely filled with a homogeneous linear dielectric, then

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f \qquad \text{and} \qquad \boldsymbol{\nabla} \times \mathbf{D} = 0.$$

And, also

$$\mathbf{D} = \varepsilon_0 \mathbf{E}_{\text{vac}}$$

where $\mathbf{E}_{\text{vac}}$ is the filed the same free charge would produce in the absence of dielectric.

So, we can conclude that

$$\mathbf{E} = \frac{1}{\varepsilon} \mathbf{D} = \frac{1}{\varepsilon_r} \mathbf{E}_{\text{vac}}. \tag{4.4.1}$$

Now, just for the sake of a very simple example, if we have a free charge $q$ that is embedded in a large dielectric, the field it produces is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \frac{q}{r^2} \hat{\mathbf{r}}. \tag{4.4.2}$$

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## References


[1] David J Griffiths. *Introduction to electrodynamics; 3rd ed.* Upper Saddle River, NJ: Prentice-Hall, 1999. URL: https://cds.cern.ch/record/611579.

[2] *Del in cylindrical and spherical coordinates.* en. Page Version ID: 1016691595. Apr. 2021. URL: https://en.wikipedia.org/w/index.php?title=Del_in_cylindrical_and_spherical_coordinates&oldid=1016691595 (visited on 06/03/2021).

[3] *Divergence theorem.* en. Page Version ID: 1025269448. May 2021. URL: https://en.wikipedia.org/w/index.php?title=Divergence_theorem&oldid=1025269448 (visited on 06/05/2021).

[4] *Stokes' theorem.* en. Page Version ID: 1014735398. Mar. 2021. URL: https://en.wikipedia.org/w/index.php?title=Stokes%27_theorem&oldid=1014735398 (visited on 06/05/2021).