PHYS2114 Cheat Sheet

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Contents

1 Mathematics Preliminary

1.1 IMPORTANT NOTE

The following notations are taken from Griffths' 'Introduction to Electromechanics'.[1]

Where as the line, area and volume integral elements are the following;

- Line integral element: $\mathrm{d}\mathbf{l}$.
- Area integral element: $\mathrm{d}\mathbf{a}$.
- Volume integral element: $\mathrm{d}\tau$.

1.2 Cartesian Coordinates

The line and volume integral elements are

$$\mathrm{d}\mathbf{l} = \hat{\mathbf{x}}\,\mathrm{d}x + \hat{\mathbf{y}}\,\mathrm{d}y + \hat{\mathbf{z}}\,\mathrm{d}z, \qquad \mathrm{d}\tau = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

And the following are some common operators

Gradient: $$\boldsymbol{\nabla} t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}},$$

Divergence: $$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z},$$

Curl: $$\boldsymbol{\nabla} \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}},$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}.$$

1.3 Cylindrical Coordinates

The conversion of Cartesian to Cylindrical coordinates[2] are

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z,$$

note that $\phi \in [0, 2\pi]$.

And the surface integral element with radius r constant is

$$\mathrm{d}\mathbf{S} = r\,\mathrm{d}\phi\,\mathrm{d}z$$

The line and the volume integral elements are

$$\mathrm{d}\mathbf{l} = \hat{\mathbf{s}}\,\mathrm{d}s + s\hat{\phi}\,\mathrm{d}\phi + \hat{\mathbf{z}}\,\mathrm{d}z, \qquad \mathrm{d}\tau = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z.$$

The following are some common vector operators

Gradient: $$\boldsymbol{\nabla} t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{\mathbf{z}},$$

Divergence: $$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z},$$

Curl: $$\boldsymbol{\nabla} \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}},$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}.$$

1.4 Spherical Coordinates

Note that $\theta \in [0, \pi]$ denotes the angle between z-axis and the vector of interest, and that $\phi \in [0, 2\pi]$ denotes the angle between x-axis and the projection of the vector of interest on to the xy-plane.[2]

The conversion of Cartesian to Spherical coordinates are shown as the follows

$$x = r\sin(\theta)\cos(\phi), \qquad y = r\sin(\theta)\sin(\phi), \qquad z = r\cos(\theta).$$

And the line and volume integral elements are

$$\mathrm{d}\mathbf{l} = \hat{\mathbf{r}}\,\mathrm{d}r + r\hat{\theta}\,\mathrm{d}\theta + r\sin(\theta)\hat{\phi}\,\mathrm{d}\phi, \qquad \mathrm{d}\tau = r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

with the surface integral with the radius r constant is

$$\mathrm{d}\mathbf{S} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

The following are the common operators

Gradient: $$\boldsymbol{\nabla} t = \hat{\mathbf{r}}\frac{\partial t}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial t}{\partial \theta} + \frac{\hat{\phi}}{r\sin(\theta)}\frac{\partial t}{\partial \phi},$$

Divergence: $$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi},$$

Curl: $$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{v} = {} & \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}\sin(\theta)v_\phi - \frac{\partial v_\theta}{\partial \phi}\right)\hat{\mathbf{r}} \\ & + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}r v_\phi\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}r v_\theta - \frac{\partial v_r}{\partial \theta}\right)\hat{\phi},\end{aligned}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}.$$

1.5 Gauss' Divergence Theorem

Suppose V is a subset of $\mathbb{R}^n$ (in the case of $n = 3$, V represents a volume in three-dimensional space) which is compact and has a piecewise smooth boundary S (also indicated with $\partial V = S$). If $\mathbf{F}$ is a continuously differentiable vector filed defined on a neighbourhood of V, then:

$$\iiint_V (\boldsymbol{\nabla} \cdot \mathbf{F})\ \mathrm{d}V = \oiint_S (\mathbf{F} \cdot \hat{\mathbf{n}})\ \mathrm{d}S.$$

The left side is a volume integral over the volume V, the right side is the surface integral over the boundary of the volume V. The closed manifold ∂V is oriented by outward-pointing normal, and $\mathbf{n}$ is the outward pointing normal at each point on the boundary ∂V. ($\mathrm{d}\mathbf{S}$ may be used as a shorthand for $\mathbf{n}\,\mathrm{d}S$.) In terms of the intuitive description above, the left-hand side of the equation represents the total of the sources in the volume V, and the right-hand side represents the total flow across the boundary S.[3]

1.6 Stokes' Theorem

Suppose we have a boundary $\partial\Sigma = S$ that bounds the surface Σ with $\mathbf{F}$ defined in Σ, then

$$\iint_\Sigma (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S = \oint_S \mathbf{F} \cdot\ \mathrm{d}\mathbf{l}.$$

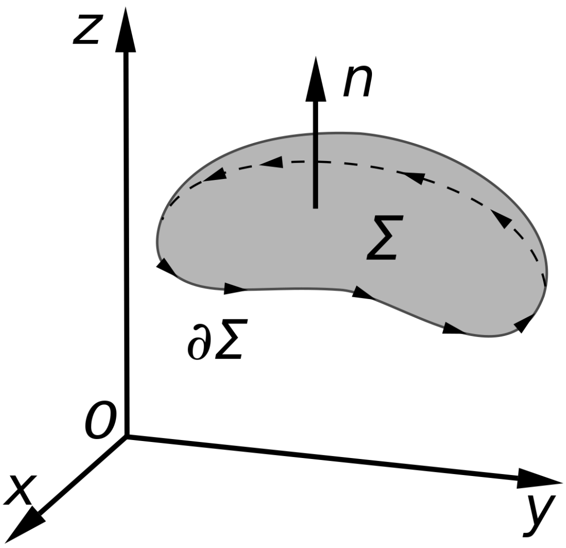

Figure 1: A visual representation for the Stokes' theorem[4]

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2 Electrostatics

2.1 The Holy Trinity

$$V = \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{V}} \frac{\rho}{\mathscr{r}} \, \mathrm{d}\tau \tag{2.1.1}$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \tag{2.1.2}$$

$$E = -\boldsymbol{\nabla} V \tag{2.1.3}$$

$$V = -\int \mathbf{E} \cdot \mathrm{d}\mathbf{l} \tag{2.1.4}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{V}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho \, \mathrm{d}\tau \tag{2.1.5}$$

$$\boldsymbol{\nabla} \cdot E = \frac{\rho}{\varepsilon_0}; \quad \boldsymbol{\nabla} \times E = 0 \tag{2.1.6}$$

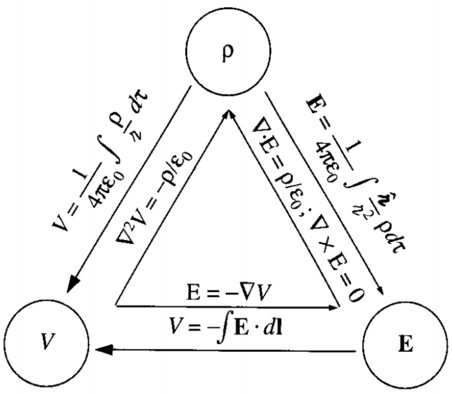


Figure 2: The Griffith's holy trinity of electrostatics[1]

2.2 Electrostatic Boundary Conditions

$$\hat{\mathbf{n}}_2 \cdot [\mathbf{E}_1 - \mathbf{E}_2] = \frac{\sigma}{\varepsilon_0} \tag{2.2.1}$$

$$\hat{\mathbf{n}}_2 \times [\mathbf{E}_1 - \mathbf{E}_2] = 0 \tag{2.2.2}$$

$$V_1 - V_2 = 0 \tag{2.2.3}$$

Note that here the “1” and “2” just refer to the different sides of the interface.

2.3 Work and Energy in Electrostatics

Three important equations for Energy in work and energy in electro statics:
For discrete charges that are not very closed together,

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i). \tag{2.3.1}$$

For volume charge density ρ,

$$W = \frac{1}{2}\iiint_{\mathcal{V}} \rho V \,\mathrm{d}\tau. \tag{2.3.2}$$

And even more simply, we can have,

$$W = \frac{\varepsilon_0}{2}\iiint_{\mathbb{R}^3} E^2 \,\mathrm{d}\tau. \tag{2.3.3}$$

2.4 Conductors

There are **five** fundamental properties of a conductors,

1. $E = 0$ inside a conductor.
2. $\rho = 0$ inside a conductor.
3. Any net charge resides on the surface.
4. A conductor is an equipotential.
5. E is perpendicular to the surface, just outside a conductor.

Now, let's consider the force per unit area $\mathbf{f}$ on any charges that rests on the surface of the conductor, with references to page 102 in the book,[1] we know that

$$\mathbf{f} = \sigma\mathbf{E}_{\text{average}} = \frac{1}{2}\sigma(\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}). \tag{2.4.1}$$

In particular, that for conductors, where we know $\mathbf{E}_{\text{below}} = 0$ (equipotential inside the conductor), we obtain

$$\mathbf{f} = \frac{1}{2\varepsilon_0}\sigma^2\hat{\mathbf{n}} \tag{2.4.2}$$

2.5 Capacitors

Note that the one thing we need to know about capacitance is that it is nothing but a "proportionality" between the stored charge and the potential difference;

$$Q = CV.$$

[1]'The book' is exclusively used in this note to 'Introduction to Electromagnetism' by David J. Griffths

3 Special Techniques

3.1 Uniqueness Theorems

First uniqueness theorem: The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface S.

Second uniqueness theorem: In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the *total charge* on each conductor is given. (The region as a whole can be bounded by another conductor, or else unbounded.)[1]

3.2 Separation of Variables

3.2.1 Cylindrical Coordinate

You can find the full derivation **here.** For those who just care about the result (of potential V) without $\hat{\mathbf{z}}$ dependence

$$\begin{aligned} V(s,\phi) &= a_0 + b_0 \ln s \\ &\quad + \sum_{k=1}^{\infty} \left[s^k (a_k \cos k\phi + b_k \sin k\phi) + s^{-k} (c_k \cos k\phi + d_k \sin k\phi) \right]. \end{aligned} \tag{3.2.1}$$

3.2.2 Spherical Coordinate

Once again, you can find the full derivation **here.** And the general result for the potential $V(s,\theta,\phi)$ is

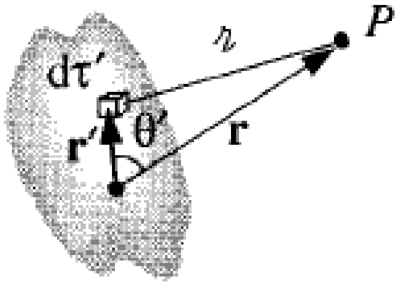
$$V(s,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{l}^{m=-l} \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right) P_l^m(\cos\theta) \exp(im\phi) \tag{3.2.2}$$

3.3 Multipole Expansion

3.3.1 The Multipole Expansion

Note that P_n represents the n^{th} Legendre Polynomial,

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta') \rho(\mathbf{r}') \, \mathrm{d}\tau'. \tag{3.3.1}$$



3.3.2 The Dipole Term

Since $P_0 = 1$, the dipole moment is then

$$\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}')\, \mathrm{d}\tau'. \tag{3.3.2}$$

And the dipole (DIPOLE ONLY) contribution to the potential is

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}. \tag{3.3.3}$$

3.3.3 The Electric Filed of a Dipole

The electric field is simply

$$\mathbf{E}_{\text{dip}}(r, \theta) = \frac{p}{4\pi\varepsilon_0 r^3} (2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}). \tag{3.3.4}$$

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4 Electric Field in Matter

4.1 Polarisation

The torque a dipole experiences due to a field $\mathbf{E}$ is

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}. \tag{4.1.1}$$

And the force the dipole experiences is

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E} \tag{4.1.2}$$

4.2 Induced Dipoles

Suppose we have an uniform electric field $\mathbf{E}$, and an atom (dipole) is in this field. This atom will now posses some dipole moment. The dipole moment is **most of the times** approximately proportional to the field.

$$\mathbf{p} = \alpha \mathbf{E}. \tag{4.2.1}$$

4.3 Bound Charges

Let's suppose we have a blob of polarised material, with dipole moment $\mathbf{p} = \mathbf{P}(\mathbf{r}')\,\mathrm{d}\tau'$ in each volume element. The total potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{V}} \frac{\hat{\mathscr{r}} \cdot \mathbf{P}(\mathbf{r}')}{\mathscr{r}^2}\,\mathrm{d}\tau'.$$

And after some maths, we then obtain

$$V = \frac{1}{4\pi\varepsilon_0} \oiint_{\mathcal{S}} \frac{1}{\mathscr{r}} \mathbf{P} \cdot \mathrm{d}\mathbf{a}' - \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{V}} \frac{1}{\mathscr{r}} (\boldsymbol{\nabla} \cdot \mathbf{P})\,\mathrm{d}\tau'. \tag{4.3.1}$$

Notice that

$$\underbrace{\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}}_{\text{Surface bound charge}}, \qquad \text{and} \qquad \underbrace{\rho_b = -\boldsymbol{\nabla} \cdot \mathbf{P}}_{\text{Volume bound charge}}.$$

Those are called bound charges (surface bound charges & volume bound charge), we can just find them to get the voltage of the polarised dialectic blob instead of calculating the massive integral.

4.4 The Electric Displacement

4.4.1 Gauss's Law in the Presence of DIelectrics

Blah blah blah there is something called the "free charge" because we don't live in a perfect world. So the total charge density is now

$$\rho = \rho_b + \rho_f.$$

And Gauss' law reads

$$\begin{aligned}
& \varepsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\boldsymbol{\nabla} \cdot \mathbf{P} + \rho_f; \\
\implies \quad & \rho_f = \boldsymbol{\nabla} \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}); \\
\implies \quad & \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}; && (4.4.1) \\
\implies \quad & \boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f; && (4.4.2) \\
\implies \quad & \oiint_{\mathcal{S}} \mathbf{D} \cdot \mathrm{d}\mathbf{a} = Q_{f_{\text{enc}}}. && (4.4.3)
\end{aligned}$$

Trick: In most of cases, $\mathbf{D}$ is determined exclusively by the free charge if we are dealing with symmetrical systems.

4.4.2 Boundary Conditions

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f; \tag{4.4.4}$$

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{above}}; \tag{4.4.5}$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\varepsilon_0}; \tag{4.4.6}$$

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = 0. \tag{4.4.7}$$

4.5 Linear Dielectrics

Iff a space that is entirely filled with a homogeneous linear dielectric, then

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f \qquad \text{and} \qquad \boldsymbol{\nabla} \times \mathbf{D} = 0.$$

And, also

$$\mathbf{D} = \varepsilon_0 \mathbf{E}_{\text{vac}}$$

where $\mathbf{E}_{\text{vac}}$ is the filed the same free charge would produce in the absence of dielectric.

So, we can conclude that

$$\mathbf{E} = \frac{1}{\varepsilon}\mathbf{D} = \frac{1}{\varepsilon_r}\mathbf{E}_{\text{vac}}. \tag{4.5.1}$$

Now, just for the sake of a very simple example, if we have a free charge q that is embedded in a large dielectric, the field it produces is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{\mathbf{r}}. \tag{4.5.2}$$

References

[1] David J Griffiths. *Introduction to electrodynamics; 3rd ed.* Upper Saddle River, NJ: Prentice-Hall, 1999. URL: https://cds.cern.ch/record/611579.

[2] *Del in cylindrical and spherical coordinates*. en. Page Version ID: 1016691595. Apr. 2021. URL: https://en.wikipedia.org/w/index.php?title=Del_in_cylindrical_and_spherical_coordinates&oldid=1016691595 (visited on 06/03/2021).

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