Exercises Dynamics of Neural Systems

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Exercise Sheet 8 due Feb 2^{nd} 2017

1 Exercise: Linear Neural Field. Credits: 6

Assume a linear neural field that is defined over the infinite spatial domain by the equation:

$$\tau \dot{u}(x,t) = -u(x,t) + \int_{-\infty}^{\infty} w(x-x')u(x',t)\,\mathrm{d}x' + s(x,t) \tag{1}$$

1.1

Assume first that the input signal is constant over time and given by:

$$s(x) = \exp\left(-\frac{x^2}{4d^2}\right)/(2\sqrt{\pi}\,d) \tag{2}$$

Assume further that the interaction kernel is given by the Gabor function:

$$w(x) = a\left(\exp\left(-\frac{x^2}{4b^2}\right)\cos(k_0 x)\right)/(\sqrt{\pi}\,b) \tag{3}$$

Derive an equation for the stationary solution from equation (1). *Hint: Assume the system has a stable solution that does not depend on time.*

1.2

Solve the resulting integral equation for the stationary solution $u^*(x)$ using the Fourier transform. (The solution has a closed form in frequency space.)

Hint: Use the Fourier transformation of the Gaussian function $f(x) = 1/(\sqrt{2\pi}\,\sigma)\exp\left(-x^2/(2\sigma^2)\right)$ *that is given by the equation:* $\tilde{f}(k) = \mathcal{F}[f](k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x = \exp\left(-\sigma^2 k^2/2\right)$.

1.3

What do you predict for the solution in the x-domain if the parameter a is approaching 1 from below?

Hint: Use Euler's formula $e^{ix} = \cos x + i\sin x$ *and the the shift theorem of the Fourier transformation:* $\mathcal{F}[f(x)\exp^{ixk_0}](k) = \tilde{f}(k-k_0)$.

1.4

Try to simulate the equation, using an Euler approximation for the time derivative, and replacing the integral by a Riemann sum (i.e. $\int_A^B f(x)\,\mathrm{d}x \approx \sum_0^M f(x_n)\Delta x$ with $x_n = A + n\Delta x$ and $\Delta x = (B-A)/M$). Simulate the stationary solutions for the following parameters: $A = 10$, $B = -10$, $M \geq 200$, and $\tau = 10$, $a = 1.0$, $b = 0.6$, $d = 2$, $k_0 = 4$. What do you observe? Add a little bit of Gaussian white noise to the stimulus signal (choose a small variance with $\mathrm{Var}(s(x)) \approx 0.01^2$.) What does change and how can this be explained. Choose $a = 0.7$ and report what changes. Then choose $a = 1.5$. Can you explain the result? Finally, choose $k_0 = 8$. How does this influence the form of the stationary solution?

1.5

The stationary solution of the linear neural field equation can be interpreted as the result of the application of a linear integro-differential operator to the input function. This implies that we can define a Green's function for it that describes the response of the field to a delta input signal of the form $s(x,t) = \delta(x)\delta(t)$. If the function $g(x,t)$ is known solutions for arbitrary inputs s can be easily computed according to the formula:

$$u(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-x', t-t')s(x',t')\,\mathrm{d}x'\mathrm{d}t' \tag{4}$$

Compute the (two-dimensional!) Fourier transform of this Green's function $\tilde{g}(k,\omega)$.

Hint: Use the two-dimensional Fourier transform with respect to space and time.

1.6

Assume now another interaction kernel with the form $w(x) = \exp^{-c|x|}\mathrm{sign}(x)$ with $c > 0$ and a time-dependent stimulus with the form:

$$s(x,t) = c\exp\left(-\frac{(x-vt)^2}{4d_1^2}\right)/(2\sqrt{\pi}d_1), \tag{5}$$

where v is the traveling speed of the stimulus peak.

Simulate this solution for the parameters: $c = 1$, $d_1 = 0.5$, and $v = 0.1$. Try then also $v = -0.1$. What is the difference and what does this imply for possible computations that could be done with this field?

1.7 (optional; not required for maximum credit)

Can you compute approximately the optimal input speed that leads to the highest output amplitude of the field in dependence of the parameters? Assume for this computation $|k| \ll c$.

Hint: Analyze the system in space-time frequency space. What is the best delta input of the form $s(x,t) = \delta(x - vt)$? Use the Fourier transformation pair: $\mathcal{F}[\exp(-c|x|)\mathrm{sign}(x)](k) = \frac{2ik}{c^2+k^2}$

2 Exercise: Nonlinear Neural Field. Credits: 6

Assume a nonlinear neural field of Amari type with step threshold that is given by the equation:

$$\tau\dot{u}(x,t) = -u(x,t) + \int_{-\infty}^{\infty} w(x-x')1(u(x',t))\,\mathrm{d}x' + s(x,t) - h \tag{6}$$

Assume for the resting level parameter $h = 1$. To keep the model mathematically simple assume that the input and the recurrent interaction kernel are given by piece-wise linear functions with

$$s(x) = \begin{cases} C(1-|x|/d) & \text{for } |x| \le d \\ 0 & \text{else.} \end{cases} \tag{7}$$

and the interaction kernel

$$w(x) = \begin{cases} A & \text{for } |x| \le a \\ -B & \text{for } a \le |x| \le b \\ 0 & \text{else.} \end{cases} \tag{8}$$

2.1

Simulate this system for the parameters $A = 3$, $B = 2$, $C = 0.6$, $a = 1$, $b = 3$, $d = 4$, and $h = 1$. First start with the initial condition $u(x,0) = -h$. Then use the initial condition $u(x,0) = 3.3\,s(x) - h$. What is changing in terms of the stable stationary solution and what does this mean. Now reduce the amplitude of the interaction kernel by factor ten (i.e. $w(x) \to 0.1w(x)$). Redo the same simulation. How can this result be explained?

Hint: It is recommended to use the same Riemann sum approximation as above to implement the convolution, e.g. using the MATLAB `conv.m`. *If you use FFT you have to be careful because it is defined only for periodic functions. One can use it, but one needs really to know how (zero padding etc.).*

2.2

Compute the exact size of the stable peak solution for the first parameter setting. Is this solution stable and why? Exploit the linearity of the relevant functions.

Hint: Derive a linear equation system for the boundaries x_1 and x_2 of the activated region with $u(x) > 0$. After having computed the stationary solution points x_1^* and x_2^* perturb these points by small amounts, assuming $x_i(t) = x_i^* + \delta x_i(t)$ and analyze the (linear) dynamics of the small perturbations $\delta x_i(t)$. This results in a stability condition that is similar to the one in the lecture.

2.3

Compute also the stable solution for case that the initial condition does not create an activated region with $u(x) > 0$ in the field. Why can in this case no peak arise? What is the threshold amplitude C for the input signal where a peak solution with an activated region will start to emerge?

2.4

Use again the initial condition $u(x,0) = 3.3s(x) - h$ and the initial interaction kernel. Set now the input signal s (but not the initial condition!) to zero. Which stationary solution emerges?

2.5 (optional; not required for maximum credit)

Add or subtract the following asymmetric function to the interaction kernel w, i.e. redo the simulation with the effective interaction kernel $w(x) \pm w_u(x)$, where:

$$w_u(x) = \begin{cases} 0.8\,x & \text{for } |x| \le b \\ 0 \text{ else.} & \end{cases} \tag{9}$$

What do you observe and how can this observation be explained?