

Convergence of Standard Gradient Descent

Problem Setup

We consider the minimization problem:

$$\min_{x \in \mathbb{R}^n} f(x)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex and differentiable function. The standard gradient descent update rule is:

$$x_{k+1} = x_k - \alpha \nabla f(x_k) \quad (1)$$

where $\alpha > 0$ is the step size (learning rate), and $\nabla f(x_k)$ is the gradient of f at x_k with respect to the standard Euclidean inner product $\langle u, v \rangle = u^T v$. We use the standard Euclidean norm $\|u\| = \sqrt{\langle u, u \rangle}$.

Assumptions

1. **Convexity of f :** For any $x, y \in \mathbb{R}^n$:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \quad (2)$$

2. **L -smoothness (Lipschitz continuous gradient):** The gradient ∇f is L -Lipschitz continuous with respect to the Euclidean norm. That is, there exists a constant $L > 0$ such that:

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \quad \forall x, y \in \mathbb{R}^n \quad (3)$$

This assumption implies the following inequality (Descent Lemma):

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2}\|y - x\|^2 \quad (4)$$

3. **Existence of a minimizer:** There exists $x^* \in \mathbb{R}^n$ such that $f(x^*) = f^* = \min_{x \in \mathbb{R}^n} f(x)$. For convex f , this implies $\nabla f(x^*) = 0$.

Convergence Proof

Step 1: Bounding the function decrease in one step. We use the Descent Lemma (4) with $x = x_k$ and $y = x_{k+1} = x_k - \alpha \nabla f(x_k)$. Then $y - x = -\alpha \nabla f(x_k)$.

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + \langle \nabla f(x_k), -\alpha \nabla f(x_k) \rangle + \frac{L}{2}\|-\alpha \nabla f(x_k)\|^2 \\ &= f(x_k) - \alpha \langle \nabla f(x_k), \nabla f(x_k) \rangle + \frac{L\alpha^2}{2}\|\nabla f(x_k)\|^2 \\ &= f(x_k) - \alpha \|\nabla f(x_k)\|^2 + \frac{L\alpha^2}{2}\|\nabla f(x_k)\|^2 \\ &= f(x_k) - \alpha \left(1 - \frac{L\alpha}{2}\right) \|\nabla f(x_k)\|^2 \end{aligned}$$

To guarantee descent, we choose the step size α such that $1 - \frac{L\alpha}{2} \geq 0$, i.e., $\alpha \leq \frac{2}{L}$. A common choice is $\alpha = \frac{1}{L}$. With this choice:

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) - \frac{1}{L} \left(1 - \frac{L(1/L)}{2}\right) \|\nabla f(x_k)\|^2 \\ &= f(x_k) - \frac{1}{L} \left(1 - \frac{1}{2}\right) \|\nabla f(x_k)\|^2 \\ f(x_{k+1}) &\leq f(x_k) - \frac{1}{2L} \|\nabla f(x_k)\|^2 \end{aligned} \tag{5}$$

This shows that the function value decreases at each step, provided $\nabla f(x_k) \neq 0$.

Step 2: Analyzing the distance to the optimum. Consider the squared Euclidean distance from x_{k+1} to x^* :

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &= \|x_k - \alpha \nabla f(x_k) - x^*\|^2 \\ &= \|(x_k - x^*) - \alpha \nabla f(x_k)\|^2 \\ &= \|x_k - x^*\|^2 - 2\langle x_k - x^*, \alpha \nabla f(x_k) \rangle + \|\alpha \nabla f(x_k)\|^2 \\ &= \|x_k - x^*\|^2 - 2\alpha \langle \nabla f(x_k), x_k - x^* \rangle + \alpha^2 \|\nabla f(x_k)\|^2 \end{aligned}$$

Step 3: Using convexity. From the convexity inequality (2), substitute $y = x^*$:

$$f(x^*) \geq f(x_k) + \langle \nabla f(x_k), x^* - x_k \rangle$$

Rearranging gives:

$$\langle \nabla f(x_k), x_k - x^* \rangle \geq f(x_k) - f(x^*) = f(x_k) - f^* \tag{6}$$

Step 4: Combining the results. Substitute inequality (6) into the expression for the distance:

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - 2\alpha(f(x_k) - f^*) + \alpha^2 \|\nabla f(x_k)\|^2$$

Now, use the step size $\alpha = 1/L$ and the function decrease inequality (5). From (5), we have $\|\nabla f(x_k)\|^2 \leq 2L(f(x_k) - f(x_{k+1}))$.

$$\begin{aligned} \|x_{k+1} - x^*\|^2 &\leq \|x_k - x^*\|^2 - \frac{2}{L}(f(x_k) - f^*) + \frac{1}{L^2} \|\nabla f(x_k)\|^2 \\ &\leq \|x_k - x^*\|^2 - \frac{2}{L}(f(x_k) - f^*) + \frac{1}{L^2} [2L(f(x_k) - f(x_{k+1}))] \\ &= \|x_k - x^*\|^2 - \frac{2}{L}(f(x_k) - f^*) + \frac{2}{L}(f(x_k) - f(x_{k+1})) \\ &= \|x_k - x^*\|^2 - \frac{2}{L} [(f(x_k) - f^*) - (f(x_k) - f(x_{k+1}))] \\ &= \|x_k - x^*\|^2 - \frac{2}{L} (f(x_{k+1}) - f^*) \end{aligned}$$

Step 5: Telescoping sum. Let $\delta_k = \|x_k - x^*\|^2$ (squared Euclidean distance) and $\varepsilon_k = f(x_k) - f^*$ (function error). The inequality becomes:

$$\delta_{k+1} \leq \delta_k - \frac{2}{L}\varepsilon_{k+1}$$

or

$$\varepsilon_{k+1} \leq \frac{L}{2}(\delta_k - \delta_{k+1})$$

Summing this inequality from $k = 0$ to $K - 1$:

$$\begin{aligned} \sum_{k=0}^{K-1} \varepsilon_{k+1} &\leq \sum_{k=0}^{K-1} \frac{L}{2}(\delta_k - \delta_{k+1}) \\ \sum_{k=1}^K \varepsilon_k &\leq \frac{L}{2} \left(\sum_{k=0}^{K-1} (\delta_k - \delta_{k+1}) \right) \\ &= \frac{L}{2}(\delta_0 - \delta_K) \quad (\text{Telescoping sum}) \end{aligned}$$

Since $\delta_K = \|x_K - x^*\|^2 \geq 0$, we have $\delta_0 - \delta_K \leq \delta_0 = \|x_0 - x^*\|^2$. Thus:

$$\sum_{k=1}^K \varepsilon_k \leq \frac{L}{2} \|x_0 - x^*\|^2 \quad (7)$$

Step 6: Obtaining the convergence rate. From Step 1, we know that $f(x_{k+1}) \leq f(x_k)$, so the sequence $\varepsilon_k = f(x_k) - f^*$ is non-increasing ($\varepsilon_{k+1} \leq \varepsilon_k$). Therefore:

$$K \cdot \varepsilon_K = K(f(x_K) - f^*) \leq \sum_{k=1}^K \varepsilon_k$$

Combining this with inequality (7):

$$K\varepsilon_K \leq \frac{L}{2} \|x_0 - x^*\|^2$$

From this, we obtain the convergence rate:

$$f(x_K) - f^* = \varepsilon_K \leq \frac{L \|x_0 - x^*\|^2}{2K} \quad (8)$$

Conclusion

For a convex and L -smooth function f , the standard gradient descent method with step size $\alpha = 1/L$ converges in function value to the minimum f^* with a rate of $O(1/K)$. That is, $f(x_K) \rightarrow f^*$ as $K \rightarrow \infty$.

Derivation of the Preconditioned Gradient $\nabla_P f(x)$

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function. Let P be a symmetric positive definite matrix defining the P -inner product :

$$\begin{aligned}\langle u, v \rangle_P &= u^T P v = \langle P u, v \rangle \\ \|u\|_P &= \sqrt{\langle u, u \rangle_P} = \sqrt{u^T P u}\end{aligned}$$

The differential of f at x , denoted $df_x(h)$, is a linear functional representing the best linear approximation of the change $f(x+h) - f(x)$ for a small displacement h . By the Riesz Representation Theorem, this linear functional df_x can be represented via an inner product with a unique vector. The specific vector depends on the chosen inner product:

1. Using the **standard inner product** $\langle \cdot, \cdot \rangle$, there exists a unique vector, the **standard gradient** $\nabla f(x)$, such that for all $h \in \mathbb{R}^n$:

$$df_x(h) = \langle \nabla f(x), h \rangle$$

2. Using the **P -inner product** $\langle \cdot, \cdot \rangle_P$, there exists a unique vector, the **pre-conditioned gradient** $\nabla_P f(x)$, such that for all $h \in \mathbb{R}^n$:

$$df_x(h) = \langle \nabla_P f(x), h \rangle_P$$

Since both expressions represent the same differential $df_x(h)$, they must be equal:

$$\langle \nabla f(x), h \rangle = \langle \nabla_P f(x), h \rangle_P$$

Using the definition $\langle u, v \rangle_P = \langle P u, v \rangle$:

$$\langle \nabla f(x), h \rangle = \langle P(\nabla_P f(x)), h \rangle$$

Rearranging the terms:

$$\langle \nabla f(x) - P \nabla_P f(x), h \rangle = 0$$

This must hold for all h , which implies the vector inside the inner product must be zero:

$$\nabla f(x) - P \nabla_P f(x) = 0$$

Solving for $\nabla_P f(x)$ (using the invertibility of P):

$$P \nabla_P f(x) = \nabla f(x)$$

$$\boxed{\nabla_P f(x) = P^{-1} \nabla f(x)} \tag{9}$$

This gives the explicit relationship between the preconditioned gradient and the standard gradient.

Convergence of Preconditioned Gradient Descent

Problem Setup

We consider the minimization problem:

$$\min_{x \in \mathbb{R}^n} f(x)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex and differentiable function. The preconditioned gradient descent update rule is:

$$x_{k+1} = x_k - \alpha \nabla_P f(x_k) = x_k - \alpha P^{-1} \nabla f(x_k) \quad (10)$$

where $\alpha > 0$ is the step size (learning rate).

Assumptions

1. **Convexity of f :** For any $x, y \in \mathbb{R}^n$:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle = f(x) + \langle \nabla_P f(x), y - x \rangle_P \quad (11)$$

2. **L_P -smoothness with respect to the P -norm:** We assume that the preconditioned gradient $\nabla_P f$ is L_P -Lipschitz continuous with respect to the P -norm. That is, there exists a constant $L_P > 0$ such that:

$$\|\nabla_P f(x) - \nabla_P f(y)\|_P \leq L_P \|x - y\|_P \quad \forall x, y \in \mathbb{R}^n \quad (12)$$

This assumption is equivalent to the following inequality (the P -Descent Lemma):

$$f(y) \leq f(x) + \langle \nabla_P f(x), y - x \rangle_P + \frac{L_P}{2} \|y - x\|_P^2 \quad (13)$$

3. **Existence of a minimizer:** There exists $x^* \in \mathbb{R}^n$ such that $f(x^*) = f^* = \min_{x \in \mathbb{R}^n} f(x)$. For convex f , this implies $\nabla f(x^*) = 0$, and consequently $\nabla_P f(x^*) = P^{-1}0 = 0$.

Convergence Proof

Step 1: Bounding the function decrease in one step. We use the P -Descent Lemma (13) with $x = x_k$ and $y = x_{k+1} = x_k - \alpha \nabla_P f(x_k)$. Then $y - x = -\alpha \nabla_P f(x_k)$.

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + \langle \nabla_P f(x_k), -\alpha \nabla_P f(x_k) \rangle_P + \frac{L_P}{2} \| -\alpha \nabla_P f(x_k) \|_P^2 \\ &= f(x_k) - \alpha \langle \nabla_P f(x_k), \nabla_P f(x_k) \rangle_P + \frac{L_P \alpha^2}{2} \|\nabla_P f(x_k)\|_P^2 \\ &= f(x_k) - \alpha \|\nabla_P f(x_k)\|_P^2 + \frac{L_P \alpha^2}{2} \|\nabla_P f(x_k)\|_P^2 \\ &= f(x_k) - \alpha \left(1 - \frac{L_P \alpha}{2} \right) \|\nabla_P f(x_k)\|_P^2 \end{aligned}$$

To guarantee descent, we choose the step size α such that $1 - \frac{L_P \alpha}{2} \geq 0$, i.e., $\alpha \leq \frac{2}{L_P}$. A standard choice is $\alpha = \frac{1}{L_P}$. With this choice:

$$\begin{aligned}
f(x_{k+1}) &\leq f(x_k) - \frac{1}{L_P} \left(1 - \frac{L_P(1/L_P)}{2}\right) \|\nabla_P f(x_k)\|_P^2 \\
&= f(x_k) - \frac{1}{L_P} \left(1 - \frac{1}{2}\right) \|\nabla_P f(x_k)\|_P^2 \\
f(x_{k+1}) &\leq f(x_k) - \frac{1}{2L_P} \|\nabla_P f(x_k)\|_P^2
\end{aligned} \tag{14}$$

This shows that the function value decreases at each step, provided $\nabla_P f(x_k) \neq 0$.

Step 2: Analyzing the distance to the optimum in P -norm. Consider the squared P -norm of the distance from x_{k+1} to x^* :

$$\begin{aligned}
\|x_{k+1} - x^*\|_P^2 &= \|x_k - \alpha \nabla_P f(x_k) - x^*\|_P^2 \\
&= \|(x_k - x^*) - \alpha \nabla_P f(x_k)\|_P^2 \\
&= \|x_k - x^*\|_P^2 - 2\langle x_k - x^*, \alpha \nabla_P f(x_k) \rangle_P + \|\alpha \nabla_P f(x_k)\|_P^2 \\
&= \|x_k - x^*\|_P^2 - 2\alpha \langle \nabla_P f(x_k), x_k - x^* \rangle_P + \alpha^2 \|\nabla_P f(x_k)\|_P^2
\end{aligned}$$

Step 3: Using convexity. From the convexity inequality (11), substitute $y = x^*$:

$$f(x^*) \geq f(x_k) + \langle \nabla_P f(x_k), x^* - x_k \rangle_P$$

Rearranging gives:

$$\langle \nabla_P f(x_k), x_k - x^* \rangle_P \geq f(x_k) - f(x^*) = f(x_k) - f^* \tag{15}$$

Step 4: Combining the results. Substitute inequality (15) into the expression for the distance:

$$\|x_{k+1} - x^*\|_P^2 \leq \|x_k - x^*\|_P^2 - 2\alpha(f(x_k) - f^*) + \alpha^2 \|\nabla_P f(x_k)\|_P^2$$

Now, use the step size $\alpha = 1/L_P$ and the function decrease inequality (14). From (14), we have $\|\nabla_P f(x_k)\|_P^2 \leq 2L_P(f(x_k) - f(x_{k+1}))$.

$$\begin{aligned}
\|x_{k+1} - x^*\|_P^2 &\leq \|x_k - x^*\|_P^2 - \frac{2}{L_P}(f(x_k) - f^*) + \frac{1}{L_P^2} \|\nabla_P f(x_k)\|_P^2 \\
&\leq \|x_k - x^*\|_P^2 - \frac{2}{L_P}(f(x_k) - f^*) + \frac{1}{L_P^2} [2L_P(f(x_k) - f(x_{k+1}))] \\
&= \|x_k - x^*\|_P^2 - \frac{2}{L_P}(f(x_k) - f^*) + \frac{2}{L_P}(f(x_k) - f(x_{k+1})) \\
&= \|x_k - x^*\|_P^2 - \frac{2}{L_P} [(f(x_k) - f^*) - (f(x_k) - f(x_{k+1}))] \\
&= \|x_k - x^*\|_P^2 - \frac{2}{L_P}(f(x_{k+1}) - f^*)
\end{aligned}$$