Homework Assignment 1

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- **Problem 1**

  Prove that a strongly convex function has a unique minimum.

- **Solution 1**

  Since f is continuous on the compact interval $[a, b]$, by the extreme value theorem, f attains its minimum at some point $x_0 \in [a, b]$. That is,

  $$f(x_0) \leq f(x) \quad \forall x \in [a, b].$$

  Suppose, for the sake of contradiction, that there exists another point $x_1 \in [a, b]$, with $x_0 \neq x_1$, such that $f(x_0) = f(x_1)$, i.e., x_0 and x_1 are both global minima.

  Since f is strictly convex, for any $t \in (0, 1)$, the combination $x_t = tx_0 + (1 - t)x_1$ satisfies:

  $$f(x_t) < tf(x_0) + (1 - t)f(x_1).$$

  Substituting $f(x_0) = f(x_1)$, we get:

  $$f(x_t) < tf(x_0) + (1 - t)f(x_0) = f(x_0).$$

  This contradicts the assumption that $f(x_0)$ is a global minimum, as it implies the existence of a point x_t where $f(x_t) < f(x_0)$.

  Thus, our assumption that f has more than one global minimum must be false, meaning that f has a unique global minimum.

- **Problem 2**

  Prove that if the gradient of a convex function is zero at a point, then that point is a minimum. Also, show that this does not necessarily hold for a non-convex function.

- **Solution 2**

  Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. We want to show that if $\nabla f(x^*) = 0$, then x^* is a global minimum.

Step 1: First-Order Condition for Convexity

By the definition of convexity, for any $x, y \in \mathbb{R}^n$,

$$f(y) \geq f(x) + \nabla f(x)^T (y - x).$$

Setting $x = x^*$ and using $\nabla f(x^*) = 0$, we get:

$$f(y) \geq f(x^*) \quad \forall y \in \mathbb{R}^n.$$

Thus, x^* is a global minimum.

Counterexample for a Non-Convex Function

Consider the function:

$$f(x) = x^3.$$

Its derivative is:

$$\nabla f(x) = 3x^2.$$

Setting $\nabla f(0) = 0$, we find that $x = 0$ is a critical point. However, $f(x)$ is **not** convex, and $x = 0$ is neither a minimum nor a maximum, but an inflection point.

Thus, the statement does not hold for non-convex functions.