Contents

1 Introduction

Over the last years, there has been a great interest in alpha-stable distributions. This is a rich class of distributions that include the Gaussian and Cauchy distributions in a family that allows skewness and heavy tails.

Stable distributions are frequently found in analysis of critical behavior (like market crashes or natural catastrophes) and financial time series. They are very important in both theory and practice for the generalization of the Central Limit Theorem to random variables without second or even first order moments and the accompanying self-similarity of the stable family. For example, the shape of the distribution for yearly asset price changes is similar to daily or monthly price changes.

In order to use stable distributions for real data modeling, one needs to estimate its parameters. Well known maximum likelihood (ML) estimation leads to the best asymptotically normal estimators under certain conditions [4]. But building ML estimators is quite complicated for alpha-stable processes due to the lack of an analytical expressions for the probability density functions.

In this course work we consider four-parameter estimation methods such as the quantiles [3] and logarithmic moments method [2]. The main advantage of these methods comparing to ML method is that they have closed-form solutions that require substantially less computation.

The course work is organized as follows. In Section 2 we define a stable random variable and consider the geometric interpretation of parameters of a stable distribution. Then we consider the most commonly used parametrizations and special cases of stable random variables. In Section 3 we consider quantiles and logarithmic moments methods of parameter estimation, its realization in Python and possible calculation issues.

2 Stable distributions

2.1 Definitions and construction

There are plenty of equivalent definitions of stable random variable. We will use the one that is described in [7].

Definition 2.1.1. A random variable X is stable if there exists a family of independent identically distributed random variables $\{X_i\}_{i=1}^{\infty}$ and number sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$, $b_n > 0\ \forall n \in \mathbb{N}$ such that

$$\frac{1}{b_n}\sum_{i=1}^{n} X_i - a_n \xrightarrow{d} X,\ n \to \infty.$$

Definition 2.1.2. A stable distribution can be represented as a four-parameter distribution denoted by $S(\alpha, \beta, \sigma, \mu)$, where

- $\alpha \in (0, 2]$ is the stability index, the characteristics exponent that defines the shape of the distribution;
- $\beta \in [-1, 1]$ is the coefficient of skewness;
- $\sigma > 0$ is the scale parameter, it narrows or extends the distribution around μ;
- $\mu \in \mathbb{R}$ is the location (or shift) parameter.

When $\alpha < 2$, the variance is infinite. When $\alpha > 1$, the mean of the distribution exists and is equal to μ. However, when $\alpha \leq 1$, the tail(s) are so heavy that even the mean does not exist.

2.2 Parametrizations

The characteristic function of a univariate stable distribution takes different parameterizations. Let us consider the forms, that are better for modeling purposes, in the same way as they were described in [5].

We take as our basic quantity a standardized stable distribution, which parametrization is defined in [8] with $\sigma = 1$ and $\mu = 0$. Let α, β be fixed, Z be a random variable such that its characteristic function (chf) is

$$\phi_Z(t) = \begin{cases} \exp\Big\{ -|t|^{\alpha}\big[1 + i\beta\,\mathrm{sign}(t)\tan(\frac{\pi\alpha}{2})[(|t|)^{1-\alpha} - 1]\big]\Big\}, & \text{if } \alpha \neq 1, \\ \exp\Big\{ -|t|\big[1 + i\beta\,\mathrm{sign}(t)\frac{2}{\pi}\log|t|\big]\Big\}, & \text{if } \alpha = 1. \end{cases}$$

The value of this representation is that the chf is continuous that makes calculations of probability density function (pdf) and distribution function of Z much easier.

We now define random variable $X_0 \stackrel{d}{=} \sigma Z + \mu_0$ and say that $X_0 \sim S_0(\alpha, \beta, \sigma, \mu_0)$.

Definition 2.2.1. If random variable X_0 follows a stable distribution $S_0(\alpha, \beta, \sigma, \mu_0)$, then the chf of X_0 in S_0 parameterization is given by

$$\phi_{X_0}(t) = \begin{cases} \exp\Big\{ -|\sigma t|^\alpha \big[1 + i\beta \operatorname{sign}(t) \tan(\frac{\pi\alpha}{2})[(\sigma|t|)^{1-\alpha} - 1]\big] + it\mu_0 \Big\}, & \text{if } \alpha \neq 1, \\ \exp\Big\{ -|\sigma t| \big[1 + i\beta \operatorname{sign}(t) \frac{2}{\pi} \log|t\sigma|\big] + it\mu_0 \Big\}, & \text{if } \alpha = 1. \end{cases}$$

Definition 2.2.2. If random variable X_1 follows a stable distribution $S_1(\alpha, \beta, \sigma, \mu_1)$, then the chf of X_1 in S_1 parameterization is given by

$$\phi_{X_1}(t) = \begin{cases} \exp\Big\{ -|\sigma t|^\alpha \big[1 - i\beta \operatorname{sign}(t) \tan(\frac{\pi\alpha}{2})\big] + it\mu_1 \Big\}, & \text{if } \alpha \neq 1, \\ \exp\Big\{ -|\sigma t| \big[1 + i\beta \operatorname{sign}(t) \frac{2}{\pi} \log|t|\big] + it\mu_1 \Big\}, & \text{if } \alpha = 1. \end{cases}$$

The chf in S_0 parameterization are continuous functions of all four parameters while the chf and pdf in S_1 parameterization both have a discontinuity at $\alpha = 1$. The parameters α, β and σ have the same meaning for the S_0 and the S_1 parameterizations, while the location parameters of the two representations are related by

$$\mu_1 = \begin{cases} \mu_0 - \beta\sigma \tan\frac{\pi\alpha}{2}, & \text{if } \alpha \neq 1, \\ \mu_0 - \beta\sigma \frac{2}{\pi} \log\sigma, & \text{if } \alpha = 1. \end{cases}$$

Moreover, definitions 2.1.1 and 2.2.2 are equivalent. A proof can be found in ([7], Theorem 2.1.1).

2.3 Special cases

In the only three important cases, pdf of stable random variable can be represented as a closed-form expression.

(1) Gaussian distribution $N(\mu, \sigma^2)$: $S\Big(2, 0, \frac{\sigma}{\sqrt{2}}, \mu\Big)$.

$$f^G(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big\{ -\frac{(x-\mu)^2}{2\sigma^2} \Big\}; \ -\infty < x < \infty.$$

(2) Cauchy distribution: $S(1, 0, \sigma, \mu)$.

$$f^C(x) = \frac{\sigma}{\pi[(x-\mu)^2 + \sigma^2]}; \ -\infty < x < \infty.$$

(3) Levy distribution (Inverse-Gaussian or Pearson): $S\Big(\frac{1}{2}, 1, \sigma, \mu\Big)$.

$$f^L(x) = \sqrt{\frac{\sigma}{2\pi}} (x-\mu)^{-\frac{3}{2}} \exp\Big\{ -\frac{\sigma}{2(x-\mu)} \Big\}; \ \mu < x < \infty.$$

3 Parameter estimation of stable distributions

We consider two different methods of parameter estimation of a stable distribution. Using Python method **scipy.stats.levy_stable.rvs(α, β, scale=σ, loc=μ , size=n)** we generate n random variables that follow a stable distribution $S_1(\alpha, \beta, \sigma, \mu)$. This allows us to compare the obtained estimates with theoretical values. Data link: **Stable1.csv**.

```
from scipy.stats import levy_stable
import seaborn as sns

n = 20000
X = levy_stable.rvs(1.8, 0.5, scale=60, loc=520, size=n)
sns.distplot(X, bins=60)
```

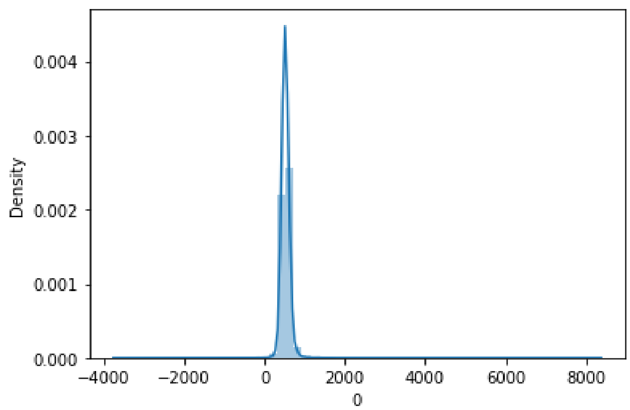

Figure 1: Density estimate of data X and its histogram

3.1 The quantiles method

The quantiles method was pioneered by Fama and Roll (1971) [1]. Then it was improved by McCulloch (1986) [3]. He extended the method to include asymmetric distributions and for cases, where $\alpha \in [0.6, 2]$.

Suppose we have n independent drawings $\hat{X}$ from the stable distribution $S(\alpha, \beta, \sigma, \mu)$, whose parameters are to be estimated. To find the estimates of α and β we need to define $\hat{\nu}_\alpha$ and $\hat{\nu}_\beta$ – the ratios of interquantile ranges of $\hat{X}$ such that $\hat{\alpha} = \psi_1(\hat{\nu}_\alpha, \hat{\nu}_\beta)$ and $\hat{\beta} = \psi_2(\hat{\nu}_\alpha, \hat{\nu}_\beta)$:

$$\hat{\nu}_\alpha = \frac{\hat{X}_{0.95} - \hat{X}_{0.05}}{\hat{X}_{0.75} - \hat{X}_{0.25}}, \quad \hat{\nu}_\beta = \frac{\hat{X}_{0.95} + \hat{X}_{0.05} - 2\hat{X}_{0.5}}{\hat{X}_{0.95} - \hat{X}_{0.05}}.$$

The notation $\hat{X}_q$ represents the q-th quantile of sample $\hat{X}$, suitably corrected for continuity, then by the theorem on the strict consistency of sample quantiles

([9], p. 349) we have that $\hat{X}_q$ is a consistent estimate of theoretical quantile X_q. The functions $\psi_1(\hat{\nu}_\alpha, \hat{\nu}_\beta)$ and $\psi_2(\hat{\nu}_\alpha, \hat{\nu}_\beta)$ can be found through bilinear interpolation in the Tables III and IV in [3].

The scale parameter is given by

$$\hat{\sigma} = \frac{\hat{X}_{0.75} - \hat{X}_{0.25}}{\phi_3(\hat{\alpha}, \hat{\beta})}$$

where $\phi_3(\hat{\alpha}, \hat{\beta})$ is given by Table V in [3].

The location parameter may be estimated by

$$\hat{\mu} = \begin{cases} \hat{\zeta} - \hat{\beta}\hat{\sigma}\tan\frac{\pi\alpha}{2}, & \hat{\alpha} \neq 1, \\ \hat{\zeta}, & \hat{\alpha} = 1, \end{cases} \qquad \hat{\zeta} = \hat{X}_{0.5} + \hat{\sigma}\phi_5(\hat{\alpha}, \hat{\beta}),$$

where $\phi_5(\hat{\alpha}, \hat{\beta})$ is given by Table VII in [3].

Calculations were performed using **numpy.quantile()** function:

```
import numpy as np

nu_a = (np.quantile(X, 0.95) - np.quantile(X, 0.05)) /
    (np.quantile(X, 0.75) - np.quantile(X, 0.25))

nu_b = (np.quantile(X, 0.95) + np.quantile(X, 0.05) -
    2 * np.quantile(X, 0.5)) / (np.quantile(X, 0.95) -
    np.quantile(X, 0.05))

# interpolated table values
a = 1.825
b = 0.72
phi_3 = 1.93
phi_5 = -0.078

sig = (np.quantile(X, 0.75) - np.quantile(X, 0.25)) / phi_3

zet = np.quantile(X, 0.5) + sig * phi_5

mu = zet - b * sig * np.tan(np.pi * a / 2)
```

Bilinear interpolation of table values is calulated by following function

```
def interp(x, y, x1, x2, y1, y2, a11, a21, a12, a22):
  a1 = (y-y1)*(a12-a11)/(y2-y1)+a11
  a2 = (y-y1)*(a22-a21)/(y2-y1)+a21
  a = (x-x1)*(a2-a1)/(x2-x1)+a1
  return a
```

As a result, $\hat{\nu}_\alpha = 2.588$, $\hat{\nu}_\beta = 0.084$, $\phi_3(\hat{\alpha},\hat{\beta}) = 1.93$, $\phi_5(\hat{\alpha},\hat{\beta}) = -0.078$, $\hat{\zeta} = 519$ parameters estimates: $\hat{\alpha} = 1.825, \hat{\beta} = 0.72, \hat{\sigma} = 60.98, \hat{\mu} = 522$ for $S_1(1.8, 0.5, 60, 520)$.

3.2 Logarithmic moments method

3.2.1 Weighted sums of independent stable random variables

Let $\{X_k\}_{k=1}^n$ be independent stable random variables such that

$$\forall k : X_k \sim S(\alpha, \beta, \sigma, \mu).$$

The distribution of a weighted sum of these variables with weights a_k can be obtained from the properties of stable distribution described in ([6], pp. 10-11)

$$Y = \sum_{k=1}^n a_k X_k \sim S\left(\alpha, \frac{\sum_{k=1}^n a_k^{\langle\alpha\rangle}}{\sum_{k=1}^n |a_k|^\alpha}\beta, \sum_{k=1}^n |a_k|^\alpha \sigma, \sum_{k=1}^n a_k \mu\right),$$

where $a_k^{\langle\alpha\rangle} = \mathrm{sign}(a_k)|a_k|^\alpha$. As a result, we can obtain sequences of independent stable random variables with $\mu = 0$ or $\beta = 0$ as well as both $\beta = 0$ and $\mu = 0$ for $\alpha \neq 1$. They are called the centered, deskewed, and symmetrized sequences, respectively, in [2]:

$$X_k^C = X_{3k} + X_{3k-1} - 2X_{3k-2} \sim S\left(\alpha, \frac{2-2^\alpha}{2+2^\alpha}\beta, (2+2^\alpha)\sigma, 0\right),$$

$$X_k^D = X_{3k} + X_{3k-1} - 2^{1/\alpha}X_{3k-2} \sim S\left(\alpha, 0, 4\sigma, (2-2^{1/\alpha})\mu\right),$$

$$X_k^S = X_{2k} - X_{2k-1} \sim S\left(\alpha, 0, 2\sigma, 0\right).$$

Using such sequences allows us to apply methods for symmetric distribution to skewed distributions and we may apply skew-estimation methods for centered sequences to non-centered sequences, but during such estimation we lose a half or a bigger part of sample size.

3.2.2 Logarithmic moments and parameter estimates

Proposition 3.2.1. Let $X \sim S(\alpha, \beta, \sigma, 0)$. Then

$$L_1 = \mathbb{E}[\log|X|] = \psi_0\left(1 - \frac{1}{\alpha}\right) + \frac{1}{\alpha}\log\left|\frac{\sigma}{\cos\theta}\right|,$$

$$L_2 = \mathbb{E}[(\log|X| - L_1)^2] = \psi_1\left(\frac{1}{2} + \frac{1}{\alpha^2}\right) - \frac{\theta^2}{\alpha^2},$$

where $\psi_{k-1} = \frac{d^k}{dx^k} log\Gamma(x)\Big|_{x=1}$ is the polygamma function and $\theta = \arctan(\beta\tan(\alpha\pi/2))$.

The proof of this proposition is given in ([2], Appendix C).

Let us denote $\mathbb{E}[\log|X|] = L_1 = f_1(\alpha,\beta,\sigma,\mu)$ and $\mathbb{E}[(\log|X|)^2] = L_2 + L_1^2 = f_2(\alpha,\beta,\sigma,\mu)$. Since f_1, f_2 are continuous functions then by theorem on the strict consistency of estimates of the method of moments ([9], p. 357) we have that $\hat{\alpha},\hat{\beta},\hat{\sigma},\hat{\mu}$ are consistent estimates of α,β,σ,μ, respectively.

Applying symmetrization to the observed data X to obtain transformed data X^S allows us to evaluate $\hat{\alpha}$, since $\beta = 0$ for X^S, and consequently $\theta = 0$

$$\hat{\alpha} = \left(\frac{\hat{L}_2}{\psi_1} - \frac{1}{2}\right)^{-1/2}.$$

Then we consider centered data X^C and estimate $\hat{L}_2$ for this data to obtain $|\hat{\theta}|$

$$|\hat{\theta}| = \left(\left(\frac{\psi_1}{2} - \hat{L}_2\right)\hat{\alpha}^2 + \psi_1\right)^{1/2} \tag{1}$$

Given the estimate of $|\theta|$, we can obtain the estimate for $|\beta_0|$

$$|\hat{\beta}_0| = \left|\frac{\tan|\hat{\theta}|}{\tan\left(\frac{\alpha\pi}{2}\right)}\right|.$$

Next we need to transform the resulting $|\hat{\beta}_0|$ by multiplying by $|(2+2^{\hat{\alpha}})/(2-2^{\hat{\alpha}})|$ and $\mathrm{sign}(\hat{\beta})$

$$\hat{\beta} = \mathrm{sign}(\hat{\beta})|\hat{\beta}| = \mathrm{sign}(|X_{max} - X_{med}| - |X_{min} - X_{med}|)\left|\frac{2+2^{\hat{\alpha}}}{2-2^{\hat{\alpha}}}\hat{\beta}_0\right|,$$

where $X_{max}, X_{med}, X_{min}$ is the maximum, median and minimum of the original data X.

Then we estimate $\hat{L}_1$ for centered data X^C, and hence

$$\hat{\sigma}_0 = \cos|\hat{\theta}|\exp\{(\hat{L}_1 - \psi_0)\alpha + \psi_0\}.$$

And finally,

$$\hat{\sigma} = \frac{\hat{\sigma}_0}{2+2^{\hat{\alpha}}}.$$

If $\alpha > 1$, then the mean of the distribution exists and is equal to μ, so we can obtain $\hat{\mu}$.

3.2.3 Calculation of parameter estimates

We will calculate polygamma function using Pyhton function **scipy.special.polygamma(k-1, x)**.

```
from scipy import special
import numpy as np

#symmetrized data
X_s = []
for k in range(int(n/2)):
  X_s_k = X[2*k+1] - X[2*k]
  X_s.append(X_s_k)

phi_0 = special.polygamma(0, 1)
phi_1 = special.polygamma(1, 1)

L_1_Xs = np.mean(np.log(np.abs(X_s)))
L_2_Xs = np.mean((np.log(np.abs(X_s)) - L_1_Xs)**2)

alpha = (L_2_Xs / phi_1 - 0.5)**(-0.5)

#centered data
X_c = []
for k in range(int(n/3)):
  X_c_k = X[3*k+2] + X[3*k+1] - 2 * X[3*k]
  X_c.append(X_c_k)

L_1_Xc = np.mean(np.log(np.abs(X_c)))
L_2_Xc = np.mean((np.log(np.abs(X_c)) - L_1_Xc)**2)

abs_theta = ((phi_1 / 2 - L_2_Xc) * alpha**2 + phi_1)**(0.5)
abs_beta_0 = np.abs(np.tan(abs_theta) / np.tan(alpha * np.pi / 2))
abs_beta = np.abs(abs_beta_0 * (2 + 2**alpha) / (2 - 2**alpha))

K = np.sign(np.abs(np.max(X) - np.median(X)) -
            np.abs(np.min(X) - np.median(X)))

beta = K * abs_beta

sigma_0 = np.abs(np.cos(abs_theta)) * np.exp((L_1_Xc - phi_0) * alpha + phi_0)
sigma = sigma_0 / (2 + 2**alpha)
mu = np.mean(X)
```

Finally, $\hat{\alpha} = 1.85, \hat{\beta} = 7.66, \hat{\sigma} = 1984.49, \hat{\mu} = 520.54$ for $S_1(1.8, 0.5, 60, 520)$. As we can see, estimates of β and σ are far from real values while estimates of α and μ are quite good.

3.3 Estimation issues

Let us consider another n random variables that follow the same stable distribution $S_1(1.8, 0.5, 60, 520)$. Data link: **Stable2.csv**.

```
n = 20000
X = levy_stable.rvs(1.8, 0.5, scale=60, loc=520, size=n)
```

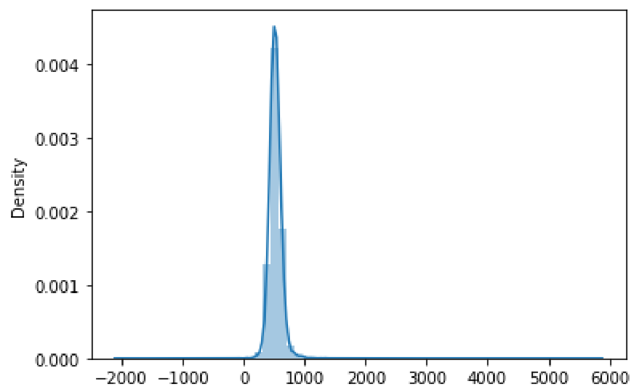

Figure 2: Density estimate of data X and its histogram

The quantiles method gives us the following results

$$\hat{\alpha} = 1.843, \hat{\beta} = 0.69, \hat{\sigma} = 60.589, \hat{\mu} = 528.6.$$

But if we apply logarithmic moments method, we receive

$$\hat{\alpha} = 1.805, \hat{\beta} = nan, \hat{\sigma} = nan, \hat{\mu} = 519.14,$$

where nan means not a (real) number. This result has arisen because $\hat{\theta}^2 = -0.047$, that should not be possible (we supposed, that $\theta \in [-\pi/2, \pi/2]$, since $\theta = \arctan(\beta \tan(\alpha\pi/2))$). In formula (1) $\psi_1 = 1.64493407$ as in [2], $\hat{\alpha} \approx \alpha$. So, probably, the problem is in calculating $\hat{L}_2$ for X^C.

We tried to increase the number of random variables in X or change parameters. But there always exist a sample X for which $\hat{\theta}^2 < 0$. Thus, the method and its implementation in Python should be studied in more detail to explain the cause of such error.

4 Conclusions

In this course work we briefly acquainted with a wide class of stable distributions. There are a lot of little typos in formulas of some articles on a similar topic, so each of them should be read carefully.

Also there are a lot of different parametrizations of chf of stable random variables and it is important to mind the parametrization when using the research results. We mainly used S_1 parametrization that is the most common for parameter estimation.

Either the quantile method or logarithmic moments method has little calculation and high estimation accuracy for α. Our try to estimate β can not be called successful. Probably, more complicated methods are needed. Close to real value $\hat{\sigma}$ and $\hat{\mu}$ can be easily calculated by quantiles method.

5 Appendices

ν_α	ν_β 0.0	0.1	0.2	0.3	0.5	0.7	1.0
2.439	2.0	2.0	2.0	2.0	2.0	2.0	2.0
2.5	1.916	1.924	1.924	1.924	1.924	1.924	1.924
2.6	1.808	1.813	1.829	1.829	1.829	1.829	1.829
2.7	1.729	1.730	1.737	1.745	1.745	1.745	1.745
2.8	1.664	1.663	1.663	1.668	1.676	1.676	1.676
3.0	1.563	1.560	1.553	1.548	1.547	1.547	1.547
3.2	1.484	1.480	1.471	1.460	1.448	1.438	1.438
3.5	1.391	1.386	1.378	1.364	1.337	1.318	1.318
4.0	1.279	1.273	1.266	1.250	1.210	1.184	1.150
5.0	1.128	1.121	1.114	1.101	1.067	1.027	0.973
6.0	1.029	1.021	1.014	1.004	0.974	0.935	0.874
8.0	0.896	0.892	0.887	0.883	0.855	0.823	0.769
10.0	0.818	0.812	0.806	0.801	0.780	0.756	0.691
15.0	0.698	0.695	0.692	0.689	0.676	0.656	0.595
25.0	0.593	0.590	0.588	0.586	0.579	0.563	0.513

Note that $\psi_1(\nu_\alpha, -\nu_\beta) = \psi_1(\nu_\alpha, \nu_\beta)$.

Figure 3: Table III of values $\alpha = \psi_1(\nu_\alpha, \nu_\beta)$ [3]

ν_α	ν_β 0.0	0.1	0.2	0.3	0.5	0.7	1.0
2.439	0.0	2.160	1.0	1.0	1.0	1.0	1.0
2.5	0.0	1.592	3.390	1.0	1.0	1.0	1.0
2.6	0.0	0.759	1.800	1.0	1.0	1.0	1.0
2.7	0.0	0.482	1.048	1.694	1.0	1.0	1.0
2.8	0.0	0.360	0.760	1.232	2.229	1.0	1.0
3.0	0.0	0.253	0.518	0.823	1.575	1.0	1.0
3.2	0.0	0.203	0.410	0.632	1.244	1.906	1.0
3.5	0.0	0.165	0.332	0.499	0.943	1.560	1.0
4.0	0.0	0.136	0.271	0.404	0.689	1.230	2.195
5.0	0.0	0.109	0.216	0.323	0.539	0.827	1.917
6.0	0.0	0.096	0.190	0.284	0.472	0.693	1.759
8.0	0.0	0.082	0.163	0.243	0.412	0.601	1.596
10.0	0.0	0.074	0.147	0.220	0.377	0.546	1.482
15.0	0.0	0.064	0.128	0.191	0.330	0.478	1.362
25.0	0.0	0.056	0.112	0.167	0.285	0.428	1.274

Note that $\psi_2(\nu_\alpha, -\nu_\beta) = -\psi_2(\nu_\alpha, \nu_\beta)$. Entries in this table greater than 1.0 are required in order to permit accurate bivariate linear interpolation as β approaches 1.0 from below. As a result, sampling error in finite samples may yield an interpolated estimate of β greater than 1.0. In this case, the estimate should be truncated back to 1.0.

Figure 4: Table IV of values $\beta = \psi_2(\nu_\alpha, \nu_\beta)$ [3]

α	β 0.0	0.25	0.50	0.75	1.00
2.00	1.908	1.908	1.908	1.908	1.908
1.90	1.914	1.915	1.916	1.918	1.921
1.80	1.921	1.922	1.927	1.936	1.947
1.70	1.927	1.930	1.943	1.961	1.987
1.60	1.933	1.940	1.962	1.997	2.043
1.50	1.939	1.952	1.988	2.045	2.116
1.40	1.946	1.967	2.022	2.106	2.211
1.30	1.955	1.984	2.067	2.188	2.333
1.20	1.965	2.007	2.125	2.294	2.491
1.10	1.980	2.040	2.205	2.435	2.696
1.00	2.000	2.085	2.311	2.624	2.973
0.90	2.040	2.149	2.461	2.886	3.356
0.80	2.098	2.244	2.676	3.265	3.912
0.70	2.189	2.392	3.004	3.844	4.775
0.60	2.337	2.635	3.542	4.808	6.247
0.50	2.588	3.073	4.534	6.636	9.144

Note that $\phi_3(\alpha, -\beta) = \phi_3(\alpha, \beta)$.

Figure 5: Table V of values $\phi_3(\alpha, \beta)$ [3]

α	β 0.0	0.25	0.50	0.75	1.00
2.00	0.0	0.0	0.0	0.0	0.0
1.90	0.0	-0.017	-0.032	-0.049	-0.064
1.80	0.0	-0.030	-0.061	-0.092	-0.123
1.70	0.0	-0.043	-0.088	-0.132	-0.179
1.60	0.0	-0.056	-0.111	-0.170	-0.232
1.50	0.0	-0.066	-0.134	-0.206	-0.283
1.40	0.0	-0.075	-0.154	-0.241	-0.335
1.30	0.0	-0.084	-0.173	-0.276	-0.390
1.20	0.0	-0.090	-0.192	-0.310	-0.447
1.10	0.0	-0.095	-0.208	-0.346	-0.508
1.00	0.0	-0.098	-0.223	-0.383	-0.576
0.90	0.0	-0.099	-0.237	-0.424	-0.652
0.80	0.0	-0.096	-0.250	-0.469	-0.742
0.70	0.0	-0.089	-0.262	-0.520	-0.853
0.60	0.0	-0.078	-0.272	-0.581	-0.997
0.50	0.0	-0.061	-0.279	-0.659	-1.198

Note that $\phi_5(\alpha, -\beta) = -\phi_5(\alpha, \beta)$.

Figure 6: Table VII of values $\phi_5(\alpha, \beta)$ [3]

References

[1] E. F. Fama and R. Roll. *Parameter estimates for symmetric stable distributions. In: Journal of the American Statistical Association 66*. 1971, pp. 331–338.

[2] E. E. Kuruoglu. *Density Parameter Estimation of Skewed Alpha-Stable Distributions. In: IEEE Transactions on signal processing 49.10*. Taylor & Francis, 2001, pp. 2192–2201.

[3] J. H. McCulloch. *Simple consistent estimators of stable distribution parameters. In: Communications in Statistics - Simulation and Computation 15.4*. Taylor & Francis, 1986, pp. 1109–1136.

[4] J. P. Nolan. *Maximum Likelihood Estimation and Diagnostics for Stable Distributions. In: Barndorff-Nielsen, Lévy Processes. Birkhäuser, Boston, MA*. 2001, pp. 379–400.

[5] J. P. Nolan. *Parameterizations and modes of stable distributions. In: Statistics & Probability Letters 38.2*. 1998, pp. 187–195.

[6] G. Samorodnitsky and M. S. Taqqu. *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman & Hall, New York, 1994.

[7] E. Spodarev. *Stable Distributions*. Lecture Notes. Ulm, 2016.

[8] V. M. Zolotarev. *One-dimensional Stable Distributions*. Vol. 65. Translations of Mathematical Monographs. American Mathematical Society, 1986.

[9] М. В. Карташов. *Імовірність, процеси, статистика. Посібник. – К.: Видавничо-поліграфічний центр 'Київський університет'*. 2008, p. 494.