

Metamathema

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CHAPTER 1

Propositional logic

A proposition P signifies a distinct fact P^* that is either the case or not.
The set of all propositions is \mathcal{W} .

1. Syntax

DEFINITION 1.1. A logical operator o of order $n \in \mathbb{N}$ may act on any n -tuple of propositions $P^n \in \mathcal{W}^n$ to compose a proposition $o(P^n) = P \in \mathcal{W}$.

Each $P_i \in P^n$ is said to be the i th direct descendant of P . A proposition Q is said to be a descendant of P if there is a succession of direct descendants that leads from P to Q .

When a proposition P is atomic it is not composed $P \neq o(P^n)$ of any n -tuple $P^n \in \mathcal{W}^n$ by any operator o of any order n .

OPERATOR 1.1 (Implication). For any $P, Q \in \mathcal{W}$:

$$P \rightarrow Q \in \mathcal{W}.$$

If $(P \rightarrow Q)^*$ is the case, then if P^* (the implicans) is the case, then so is Q^* (the impicandum). The implicans is said to be a sufficient condition for the impicandum.

OPERATOR 1.2 (Negation). For any $P \in \mathcal{W}$:

$$\neg P \in \mathcal{W}.$$

If $(\neg P)^*$ is the case then P^* is not the case.

DEFINITION 1.2. Given an alphabet α consisting of a finite $|\alpha| \in \mathbb{N}$ set of symbols, we consider any finite $n \in \mathbb{N}$ string of symbols from the alphabet $e^n \in \alpha^n$ an expression. An instance $e_i \in e^n$ of a symbol $e_i = \alpha_j \in \alpha$ at position i is known as a character.

The set of all expressions from the alphabet is $\alpha^\omega = \bigcup_{n \in \mathbb{N}} \alpha^n$.

The substring $d^m = [e^n]_{k+1}^{k+m}$ of an expression $e^n \in \alpha^n$ at position $k+1$ of length m is its restriction to the characters $e_i \in e^n$ at positions $k < i \leq k+m$. The instance $[e^n]_{k+1}^{k+m}$ of the expression d^m is said to be in the scope of the expression e^n at position $k+1$.

Uniform substitution $e(d \setminus c)$ of an expression $c \in \alpha^l$ with an expression $d \in \alpha^m$ in an expression $e \in \alpha^n$ substitutes an instance of d for every instance of c in the scope of e .

THEOREM 1.1. The alphabet $\beta = \{a, o, ', (,), , \}$ expresses all propositions.

$$\mathcal{W} \subset \beta^\omega$$

PROOF. We use the atom symbol a and the succession symbol $'$ to denote an atomic proposition for every $n \in \mathbb{N}$ such that $a_0 = a$ and $a_{n+1} = a'_n$.

We may use the operator symbol o and the succession symbol $'$ to denote any operator. Then parentheses $(,)$ and a comma $,$ allow us to denote components in a composite proposition. \square

COROLLARY 1.1. The set of all propositions \mathcal{W} may be enumerated though without bound: $|\mathcal{W}| = |\mathbb{N}|$.

PROOF. The alphabet β allows any proposition to be encoded in base 6. \square

COROLLARY 1.2. A proposition is in the scope of another only if it descends from the other.

DEFINITION 1.3. A propositional variable p is a distinct reference to any proposition P . An extension $\phi = \beta \cup \{p\}$ of the alphabet β expresses all such variables.

A formula expresses the logical structure of a proposition. We define the set Φ of all formulas recursively.

If $\varphi \in \Phi$ is a formula then so is the result of uniform substitution $\varphi(p \setminus P) \in \Phi$ of any proposition P in the scope of φ with a propositional variable p . Furthermore, $\mathcal{W} \subset \Phi$. Note that $\Phi \subset \phi^\omega$. A formula $\varphi \in \Phi$ is said to be well formed only if it is a proposition $\varphi \in \mathcal{W}$.

A proposition P is said to be an instance of a formula φ only if it follows from uniform substitution $P = \varphi(P_i \setminus p_i)^n$ of all propositional variables $p_{i \leq n}$ in the scope of φ with some proposition P_i .

DEFINITION 1.4. A set of propositions $\mathcal{E} \subseteq \mathcal{W}$ may allow us to infer $\mathcal{E} \vdash P$ a proposition $P \in \mathcal{W}$. The propositions of \mathcal{E} are known as premises whereas P is said to be a syntactic consequence of \mathcal{E} .

The deductive closure $\partial\mathcal{E}$ contains all syntactic consequences of the set such that $\mathcal{E} \vdash P$ only if $P \in \partial\mathcal{E}$.

A system that determines all valid inferences is known as an inference system. Given a set of propositions considered axioms, an inference system suffices to produce their deductive closure, known as a theory, where each syntactic consequence is known as a theorem.

A set of propositions \mathcal{E} may have the following properties with respect to its deductive closure $\partial\mathcal{E}$.

- (deductively closed). $\mathcal{E} = \partial\mathcal{E}$
- (consistent). if $P \in \partial\mathcal{E}$ then $\neg P \notin \partial\mathcal{E}$
- (trivial). $\partial\mathcal{E} = \mathcal{W}$

The alphabet $\beta^\vdash = \beta \cup \{\vdash\}$ allows the inference symbol \vdash to act on any finite set of propositions to compose an inferential expression $I \in \mathcal{W}^\vdash$.

A propositional variable may substitute uniformly $\eta = \iota(p \setminus P)$ for a proposition in an inferential formula $\iota, \eta \in \Phi^\vdash$ where $\mathcal{W}^\vdash \subset \Phi^\vdash$. Note that Φ^\vdash is expressible from $\phi^\vdash = \phi \cup \{\vdash\} = \beta^\vdash \cup \{p\}$.

A pure inferential formula has no atomic propositions in its scope.

An inferential expression is an instance of an inferential formula if it follows from uniform substitution of all propositional variables in its scope.

A rule of inference is a pure inferential formula such that each instance of the formula expresses a valid inference.

Note that not every inferential expression is a valid inference, nor every inferential formula a rule of inference. However, postulating or deriving certain properties and rules of inference allows us to determine all valid inferences.

The first such property ensures that inference is structural, so no set of propositions is privileged.

PROPERTY 1.1 (Universality). If $\mathcal{E} \vdash P$ then $\mathcal{E}(Q \setminus a) \vdash P(Q \setminus a)$ where Q is any proposition and a any atomic proposition.

Equivalently, every inference is an instance of some rule of inference, namely by uniform substitution of each atom in the scope of the inference with a variable such that each variable substitutes for only one atom.

PROPERTY 1.2 (Transitivity). If $\mathcal{E} \vdash P$ for all $P \in \mathcal{D}$ and $\mathcal{D} \vdash Q$ then $\mathcal{E} \vdash Q$.

The following property does not hold in adaptive inference systems.

PROPERTY 1.3 (Monotony). If $\mathcal{E} \vdash P$ then $\mathcal{E} \cup \{Q\} \vdash P$ for any Q .

RULE 1.1 (Modus ponens).

$$\frac{p \rightarrow q, p}{q}$$

1.1. Inferential approach. The following property allows us to infer over the possibility of inference itself. Consider the extension $\mathcal{E} \cup \{P\}$ of some premises \mathcal{E} with a proposition P known as the hypothesis.

PROPERTY 1.4 (Deduction). If $\mathcal{E} \cup \{P\} \vdash Q$ then

$$\mathcal{E} \vdash P \rightarrow Q.$$

for all $\mathcal{E} \subset \mathcal{W}$.

Hence the subject of hypothesis is sufficient condition for what can be inferred from the extension.

This holds a fortiori for any finite $A \subset \Phi$ such that $A \cup \{p\} \vdash q$ since it holds for all instances.

Furthermore this entails a familiar property.

THEOREM 1.2 (Linearity of inference). If $A \vdash p$ and $A \cup \{p\} \vdash q$ then $A \vdash q$ for all $A \subset \Phi$.

PROOF. From $A \vdash p \rightarrow q$ and transitivity of inference. □

Some expected properties of the implication can now follow.

THEOREM 1.3 (Transitivity of implication).

$$\frac{p \rightarrow q, q \rightarrow r}{p \rightarrow r}$$

PROOF. From $p \rightarrow q, q \rightarrow r, p \vdash r$. □

THEOREM 1.4 (Distributivity of implication).

$$p \rightarrow (q \rightarrow r) \vdash (p \rightarrow q) \rightarrow (p \rightarrow r)$$

The following rule seems all too evident.

RULE 1.2 (Reflection).

$$p \vdash p$$

It allows us to prove the following theorems.

THEOREM 1.5 (Reflexivity of implication).

$$\vdash p \rightarrow p$$

THEOREM 1.6 (Ex quod libet datum).

$$p \vdash q \rightarrow p$$

PROOF. $p, q \vdash p$ by monotony. □

The implication thus defined is said to be material.

The preceding rules and theorems have applied exclusively to the implication.

The following rule introduces the syntactic role of the negation.

RULE 1.3 (Contraposition).

$$p \rightarrow q \vdash \neg q \rightarrow \neg p$$

This shows a property of inference.

THEOREM 1.7 (Contrapositivity). If $p \vdash q$ then $\neg q \vdash \neg p$.

THEOREM 1.8 (Ex absurdum negatio quod libet).

$$p, \neg p \vdash \neg q$$

PROOF. $p \vdash q \rightarrow p$. □

COROLLARY 1.3 (Ex absurdo falsum). For any formula τ such that $\vdash \tau$:

$$p, \neg p \vdash \neg \tau$$

THEOREM 1.9 (Ex falso negatio quod libet). For any formula τ such that $\vdash \tau$:

$$\vdash \neg \tau \rightarrow \neg q$$

PROOF. $\vdash q \rightarrow \tau$, also known as ex quod libet verum. □

RULE 1.4 (Introduction of double negation).

$$p \vdash \neg \neg p$$

LEMMA 1.1 (Contraposition with left elimination).

$$p \rightarrow \neg q \vdash q \rightarrow \neg p$$

PROOF. $q \rightarrow \neg \neg q$ is transitive with $\neg \neg q \rightarrow \neg p$. □

THEOREM 1.10 (Negatio improbis). For any formula τ such that $\vdash \tau$:

$$p \rightarrow \neg \tau \vdash \neg p.$$

THEOREM 1.11 (Introduction of implicative conjunction).

$$p, q \vdash \neg(p \rightarrow \neg q)$$

PROOF. We can deduce $p \vdash (p \rightarrow \neg q) \rightarrow \neg q$ from modus ponens. □

This has a special case with an interesting consequence.

LEMMA 1.2 (Negatio reprobis).

$$p \rightarrow \neg p \vdash \neg p$$

PROOF. $p \vdash \neg(p \rightarrow \neg p)$. Alternatively by ex contradictio falsum. \square

THEOREM 1.12 (Reductio ad absurdum).

$$\begin{array}{c} p \rightarrow q, \\ p \rightarrow \neg q \end{array} \vdash \neg p$$

By contrapositionality we also have the following.

COROLLARY 1.4 (Commutativity of implicative conjunction).

$$\neg(p \rightarrow \neg q) \vdash \neg(q \rightarrow \neg p)$$

The final rule completes the law of double negation.

RULE 1.5 (Elimination of double negation).

$$\neg\neg p \vdash p$$

This rule has been controversial since it allows the following.

THEOREM 1.13 (Proof by contradiction).

$$\neg p \rightarrow q, \neg p \rightarrow \neg q \vdash p$$

This result is rejected by intuitionists, hence they do not allow double negation to be eliminated.

The following completes the implicative variety of the law of junctive commutativity.

LEMMA 1.3 (Commutativity of implicative disjunction).

$$\neg p \rightarrow q \vdash \neg q \rightarrow p$$

This leads to double negation elimination considering $\vdash \neg p \rightarrow \neg\neg\neg p$.

Nonetheless, the following results may be preserved by intuitionists as independent rules. We complete the law of implicative conjunction and show an interesting lemma.

COROLLARY 1.5 (Elimination of implicative conjunction).

$$\neg(p \rightarrow \neg q) \vdash p, q$$

PROOF. $\neg p \vdash p \rightarrow \neg q$. \square

LEMMA 1.4 (Introduction of implicative disjunction).

$$p \vdash \neg p \rightarrow q$$

PROOF. $p \vdash \neg q \rightarrow p$. \square

THEOREM 1.14 (Ex absurdo quod libet).

$$\begin{array}{c} p, \\ \neg p \end{array} \vdash q$$

A consistent set is clearly not trivial. Now we can also show the following.

COROLLARY 1.6. An inconsistent set is trivial.

Hence a maximally consistent set is maximally not trivial. This is avoided in paraconsistent inferential systems.

For any formula τ such that $\vdash \tau$:

THEOREM 1.15 (Ex falso quod libet).

$$\vdash \neg\tau \rightarrow q$$

COROLLARY 1.7 (Ex falso absurdum).

$$\neg\tau \vdash p, \neg p$$

Furthermore the completion of the law of contraposition allows us to conclude with a remarkable result.

LEMMA 1.5 (Converse contraposition).

$$\neg p \rightarrow \neg q \vdash q \rightarrow p$$

THEOREM 1.16 (Peirce's law).

$$(p \rightarrow q) \rightarrow p \vdash p$$

PROOF. We know that $\vdash \neg p \rightarrow (\neg q \rightarrow \neg p)$ so $\vdash \neg p \rightarrow (p \rightarrow q)$. However, in that case $\neg p \rightarrow \neg(p \rightarrow q) \vdash p$. \square

COROLLARY 1.8 (Consequentia mirabilis).

$$\neg p \rightarrow p \vdash p$$

PROOF. $\vdash (p \rightarrow \neg p) \rightarrow \neg p$ is transitive with $\neg p \rightarrow p$. \square

By ex absurdum quod libet we can show $\neg\neg p, \neg p \vdash p$. Evidently Peirce's law is unacceptable to intuitionists.

The following rule is rejected in its implicative variety for similar reasons.

THEOREM 1.17 (Implicative dilemma).

$$\neg p \rightarrow q, \begin{matrix} p \rightarrow r \\ q \rightarrow r \end{matrix} \vdash r$$

PROOF. $p \rightarrow q, \neg p \rightarrow q, \neg q \vdash q$. \square

Since $\vdash p \rightarrow p$ this is enough to invite consequentia mirabilis.

1.2. Alternative operators. We may disregard the law of double negation and introduce some new operators.

OPERATOR 1.3 (Conjunction). If $(P \& Q)^*$ is the case, then P^* is the case and Q^* is the case.

RULE 1.6 (Law of conjunction). $p, q \vdash p \& q$ and $p \& q \vdash p, q$.

THEOREM 1.18 (Conjunctive syllogism). $\neg(p \& q), p \vdash \neg q$.

PROOF. $p \vdash q \rightarrow (p \& q)$. \square

RULE 1.7 (Law against contradiction).

$$\vdash \neg(p \& \neg p)$$

LEMMA 1.6. $\vdash p \rightarrow \neg\neg p$.

LEMMA 1.7. $p \& q \vdash \neg(p \rightarrow \neg q)$

LEMMA 1.8. $p \rightarrow q \vdash \neg(p \& \neg q)$.

OPERATOR 1.4 (Disjunction). If $(P \vee Q)^*$ is the case, then P^* is the case or Q^* is the case.

RULE 1.8 (Disjunctive syllogism).

$$p \vee q, \neg p \vdash q$$

LEMMA 1.9. $\neg p \vee q \vdash p \rightarrow q$.

RULE 1.9 (Introduction of disjunction).

$$p \vdash p \vee q$$

RULE 1.10 (Dilemma).

$$p \vee q, \begin{array}{l} p \rightarrow r, \\ q \rightarrow r \end{array} \vdash r$$

RULE 1.11 (Law of junctive commutativity). $p \& q \vdash q \& p$ and $p \vee q \vdash q \vee p$.

OPERATOR 1.5 (Equivalence). If $(P \leftrightarrow Q)^*$ is the case then P^* is the case only if Q^* is the case.

RULE 1.12 (Law of equivalence).

$$\begin{array}{l} p \rightarrow q, q \rightarrow p \vdash p \leftrightarrow q \\ p \leftrightarrow q \vdash p \rightarrow q, q \rightarrow p \end{array}$$

THEOREM 1.19. $\vdash (p \leftrightarrow q) \leftrightarrow ((p \rightarrow q) \& (q \rightarrow p))$

THEOREM 1.20 (De Morgan's laws I).

$$\vdash \neg(p \vee q) \leftrightarrow (\neg p \& \neg q)$$

PROOF. $\vdash p \rightarrow (p \vee q), \vdash q \rightarrow (p \vee q), \neg p \vdash (p \vee q) \rightarrow q$. □

THEOREM 1.21 (De Morgan's laws IIa).

$$\neg p \vee \neg q \vdash \neg(p \& q)$$

PROOF. $\neg p \vee \neg q, p \& q \vdash q, \neg q$. □

RULE 1.13 (Law of the excluded middle).

$$\vdash \neg p \vee p$$

COROLLARY 1.9. $\vdash \neg\neg p \rightarrow p$.

This is the first offence against intuitionism in this approach.

A weaker variety of the offending law suffices to complete De Morgan's laws.

LEMMA 1.10. $\vdash \neg p \vee \neg\neg p$

THEOREM 1.22 (De Morgan's laws IIb).

$$\neg(p \& q) \vdash \neg p \vee \neg q$$

PROOF. $\neg(p \& q) \vdash \neg\neg p \rightarrow (\neg p \vee \neg q)$ from $p \rightarrow \neg q \vdash \neg\neg p \rightarrow \neg q$. □

THEOREM 1.23.

$$\begin{array}{llll} \vdash & (p \& q) & \leftrightarrow & \neg(p \rightarrow \neg q) \\ \vdash & (p \vee q) & \leftrightarrow & (\neg p \rightarrow q) \end{array}$$

PROOF. From elimination of implicative conjunction and from $\neg p \rightarrow q \vdash \neg p \rightarrow (p \vee q)$ by dilemma over exclusion of the middle. \square

COROLLARY 1.10.

$$\begin{array}{llll} \vdash & (p \rightarrow q) & \leftrightarrow & \neg(p \& \neg q) \\ \vdash & (p \rightarrow q) & \leftrightarrow & (\neg p \vee q) \end{array}$$

1.3. Axiomatic approach. We may disregard all properties and rules from the inferential approach. Instead we consider a special set of formulas $\Xi \subset \Phi$ known as axioms. We may also consider the set of all instances of axioms $\mathcal{X} \subset \mathcal{W}$.

PROPERTY 1.5. If $\mathcal{E} \subset \{Q, Q \rightarrow P\} \subset (\mathcal{E} \cup \mathcal{X})$ then $\mathcal{E} \vdash P$.

This property is independent from standard modus ponens yet extends it. It is also independent from monotony.

AXIOM 1.1. $p \rightarrow (q \rightarrow p)$

AXIOM 1.2. $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

LEMMA 1.11. $\vdash p \rightarrow p$

PROOF. Taking $\alpha = p \rightarrow ((q \rightarrow p) \rightarrow p)$ and $\beta = p \rightarrow (q \rightarrow p)$ we have $\alpha \rightarrow (\beta \rightarrow (p \rightarrow p)), \alpha, \beta \in \Xi$. \square

COROLLARY 1.11. $p \vdash p$

THEOREM 1.24 (Deduction). If $\Lambda \cup \{p\} \vdash q$ then $\Lambda \vdash p \rightarrow q$.

PROOF. We have $\Lambda \vdash p \rightarrow \alpha$ for any $\alpha \in \Lambda \cup \{p\}$.

Hence for any $\alpha \rightarrow \beta \in \Lambda \cup \{p\}$ we have $\Lambda \vdash p \rightarrow (\alpha \rightarrow \beta)$.

Evidently $\Lambda \vdash p \rightarrow \beta$ permits continued deduction until we show $\Lambda \vdash p \rightarrow \gamma$ for any $\gamma \in \partial(\Lambda \cup \{p\})$. \square

AXIOM 1.3. $(\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p)$

THEOREM 1.25 (Double negation elimination). $\vdash \neg\neg p \rightarrow p$

PROOF. $\neg\neg p \vdash \neg\neg(\neg\neg p) \rightarrow \neg\neg p$. \square

THEOREM 1.26 (Double negation introduction). $\vdash p \rightarrow \neg\neg p$

PROOF. $\vdash \neg\neg(\neg p) \rightarrow \neg p$. \square

Evidently the axiomatic approach allows precisely the same syntactic derivations as the inferential.

2. Semantics

The expressive range of P is restricted to a binary distinction $P^* \in \{0, 1\}$.

We say that P^* is the case only if $P^* = 1$. Only in that case is P true.

DEFINITION 2.1. A model is a valuation map $v : \mathcal{W} \longrightarrow \{0, 1\}$ that assigns a truth value $v(P) \in \{0, 1\}$ to each proposition $P \in \mathcal{W}$.

A model v supports (\mathcal{E}, P) : if no $E \in \mathcal{E}$ assigns to $v(E) = 0$ then $v(P) = 1$.

Furthermore the following holds for any models v, w :

(inferential). If $\mathcal{E} \vdash P$ then v supports (\mathcal{E}, P) .

(regular with respect to negation). If $v(P) = w(Q)$ then $v(\neg P) = w(\neg Q)$.

(not trivial). There is at least one proposition P such that $v(P) = 0$.

We may consider the set of all models \mathbf{M} .

The positive domain $v^{-1}(1) \subseteq \mathcal{W}$ of a valuation map v contains all propositions P for which $v(P) = 1$. Clearly a model is determined by the positive domain alone. A model v supports (\mathcal{E}, P) only if $P \in v^{-1}(1)$ if $\mathcal{E} \subseteq v^{-1}(1)$. We can see that a valuation map is inferential only if the positive domain is deductively closed.

DEFINITION 2.2. A proposition is a semantic consequence $\mathcal{E} \models P$ of a set of premises only if all models $v \in \mathbf{M}$ support (\mathcal{E}, P) .

\mathcal{E} is said to validate P .

LEMMA 2.1. If $\mathcal{E} \vdash P$ then $\mathcal{E} \models P$.

The inferential system is semantically sound. The semantics are adequate for the inferential system.

DEFINITION 2.3. A truth function $f \in F^n$ of order $n \in \mathbb{N}$ maps n -tuples of truth values to a single truth value $F^n = \left\{ f \subset \{0, 1\}^{n+1} \mid f : \{0, 1\}^n \longrightarrow \{0, 1\} \right\}$ such that $F = \bigcup_{n \in \mathbb{N}} F^n$ is the set of all truth functions.

A logical operator o of order n may be characterised semantically $o \equiv f$ by a truth function $f \in F^n$ such that in every model v the valuation of $P = o(P^n)$ is mapped from its components $v(P) = f(v(P^n))$ where $v(P^n)_i = v(P_i)$ for all $P_i \in P^n$.

Note that regularity with respect to negation is equivalent to the characterisation of negation by a truth function.

Furthermore, since the codomain of any truth function is binary, the characterisation of any operator o may be defined by an equivalence that holds for all models v between $v(o(P^n)) = 1$ and certain conditions on $v(P^n)$.

PROPERTY 2.1. The following properties characterise the negation.

(consistency). $v(\neg P) = 0$ if $v(P) = 1$

(completeness). $v(\neg P) = 1$ if $v(P) = 0$

PROOF. A model must be consistent since it may not be trivial.

Consider now any theorem $\vdash T$. Clearly $v(T) = 1$ so $v(\neg T) = 0$ by consistency. However, $\vdash \neg \neg T$ from which $v(\neg P) = 1$ if $v(P) = 0$ by negational regularity. \square

A consistent map v is not trivial. If it is also complete, it is regular with respect to negation. Hence, an inferential valuation map is regular with respect to negation and not trivial if it is consistent and complete.

Note that we relied on *ex falso quod libet* to show the consistency of an inferential map which is not trivial. This rule does not hold in paraconsistent inferential systems. Nonetheless, regularity with respect to negation requires that if some model w supports some contradiction $w(Q) = 1 = w(\neg Q)$ then $v(P) = 1$ immediately implies $v(\neg P) = 1$ for any proposition P in all models v .

Therefore neither consistency nor regularity with respect to negation are a defining property of paraconsistent semantics. Instead a model is required to be complete and not trivial.

PROPERTY 2.2. The following characterises the implication.

$$v(P \rightarrow Q) = 1 \text{ only if } v(P) = 0 \text{ or } v(Q) = 1$$

PROOF. Suppose $v(Q) = 1$ then $v(P \rightarrow Q) = 1$ given materiality of implication. Should $v(P) = 0$ then $v(\neg P) = 1$ by completeness so $v(P \rightarrow Q) = 1$ by *ex contradictio quod libet*.

Conversely, if $v(P \rightarrow Q) = 1 = v(P)$ then $v(Q) = 1$ by *modus ponens*. \square

Note that we relied on *ex contradictio quod libet* to show the implicational completeness of an inferential map.

DEFINITION 2.4. A set of logical operators O is functionally complete: the closure under function composition $\partial_o O_F$ of the truth functions that characterise its operators $O_F = \{f \in F \mid f \equiv o \text{ for some } o \in O\}$ is the set of all truth functions $\partial_o O_F = F$.

LEMMA 2.2. If $F^2 \subseteq \partial_o O_F$ then $\partial_o O_F = F$.

PROOF. F^2 contains every binary function independent of its second argument, from which we compose every unary function F^1 by passing the same argument twice. Suppose now $F^n \subseteq \partial_o O_F$ where $2 \leq n$ and consider any function $f \in F^{n+1}$. For any $b^{n+1} \in \{0, 1\}^{n+1}$ restricted to $b^n = [b^{n+1}]_1^n \in \{0, 1\}^n$ we may find some function s such that $f(\dots b^n, a) = s(a)$ depends only on $a \in \{0, 1\}$.

(1) $g(b^n) = 1$ only if $f(\dots b^n, 0) = f(\dots b^n, 1)$:

s is constant.

(2) $h(b^n) = 1$ only if $f(\dots b^n, 1) = 1$:

s is veracious.

(3) $l(g, a) = 0$ only if $g = a = 0$.

We transmit 0 only if $a = 0$ and s depends on its argument a .

(4) $k(l, h) = 1$ only if $l = h$

We return 1 only if the veracity of s is equivalent to the constancy of s or the positivity of $a = 1$.

We compose $k(l(g(b^n), a), h(b^n)) = f(\dots b^n, a)$. \square

THEOREM 2.1. The set of operators $\{\neg, \rightarrow\}$ is functionally complete.

PROOF. Any $f \in F^2$ has a positive domain which permits at most $|f^{-1}(1)| \leq 4$ pairs of binary values.

The function characterising the implication $m(p, q) \equiv p \rightarrow q$ permits three such pairs, yet both of its arguments, each of which it depends on, may be inverted independently by means of the negation $n(p) \equiv \neg p$ to compose any fully dependent function $g \in F^2$ of positive order $|g^{-1}(1)| = 3$, of which there are 4.

Any such function may then be inverted to obtain the fully dependent functions of positive order 1, among which the conjunction $c(p, q) = n(m(p, n(q))) \equiv \neg(p \rightarrow \neg q)$. Conjugating converses $e = c(m(p, q), m(q, p)) \equiv \neg((p \rightarrow q) \rightarrow \neg(q \rightarrow p))$ we may complete the fully dependent functions of positive order 2 by singular inversion $n(e(p, q))$.

Passing constant functions $t(p) = m(p, p) = 1 \equiv p \rightarrow p$ or $l(p) = n(t(p)) = 0 \equiv \neg(p \rightarrow p)$ to any argument eliminates the dependency on the argument. Note that the positive order of a partially independent function must be divisible by 2, lest there be some value for which the function depends on the eliminated variable. Passing t to either argument of e gives 2 selection functions, both of which are distinctly invertible, counting the expected total of $16 = |F^2| = |\{0, 1\}|^{|\{0, 1\}|^2}$. \square

DEFINITION 2.5. A set of propositions $\mathcal{E} \subseteq \mathcal{W}$ is saturable only if there is a model v such that $v(P) = 1$ if $P \in \mathcal{E}$.

The set determines the model only if $P \in \mathcal{E}$ if $v(P) = 1$.

A model set is deductively closed and maximally not trivial.

Note that any deductively closed set which is maximally consistent is a model set if our logic permits it.

LEMMA 2.3. A model set determines an inferential map which is complete with respect to negation and implication.

PROOF. Consider a map $v : \mathcal{W} \longrightarrow \{0, 1\}$ such that $v(P) = 0$ only if $P \notin \mathcal{E}$ where \mathcal{E} is a model set.

If $v(P) = 0$ then $\mathcal{E} \cup \{P\} \vdash Q, \neg Q$ by maximal triviality so $v(\neg P) = v(P \rightarrow Q) = 1$ by deductive closure. \square

Note that the preceding also holds for paraconsistent systems.

In any case, a model set determines a model.

LEMMA 2.4. A set that does not prove a proposition $\mathcal{E} \not\vdash P$ is a subset of a model set $\mathcal{E} \subseteq \Gamma$ that does not contain the proposition $P \notin \Gamma$.

PROOF. Consider $\Gamma_{i+1} = \Delta_i$ if $P \notin \Delta_i$ and $\Gamma_{i+1} = \Gamma_i$ otherwise, where $\Delta_i = \partial(\Gamma_i \cup \{\mathcal{W}_i\})$ with $\mathcal{W}_i \in \mathcal{W}$ enumerating all well formed formulas and $\Gamma_0 = \partial\mathcal{E}$. Evidently $\Gamma = \bigcup_{i \in \mathbb{N}} \Gamma_i$ is deductively closed, since for any finite $\mathcal{D} \subset \Gamma$ also $\partial\mathcal{D} \subset \Gamma_i$ for some i .

We can see that $P \rightarrow Q \in \Gamma$ for any Q since $\mathcal{W}_i = P \rightarrow Q$ for some i so lest Peirce's law come to effect $\Gamma_i \cup \{P \rightarrow Q\} \not\vdash P$. Moreover, if $R \notin \Gamma$ then $\Gamma_i \cup \{R\} \vdash P$ for some i so $\Gamma \cup \{R\}$ is trivial.

Alternatively, if $\Gamma_0 = \mathcal{E} \cup \{\neg P\}$ is inconsistent we have $\mathcal{E} \vdash P$ by contradiction. Hence Γ_0 is consistent. We may take $\Gamma_{i+1} = \Delta_i$ only if Δ_i is consistent. By construction Γ is maximally consistent. This proof only works if inconsistency implies triviality. It also relies on the consistency of $P \notin \Gamma$. \square

THEOREM 2.2. $\mathcal{E} \models P$ only if $\mathcal{E} \vdash P$.

The semantics are characteristic for the inferential system. The converse means that the inferential system is semantically complete.

CHAPTER 2

Predicate logic

We may extend our semantics with a notion of distinct objects and distinct relations. A relation either holds for certain objects or it does not. If there can be only one object in the relation, it may be considered a property of the object.

1. Syntax

In predicate logic a well formed formula is known as a sentence.

The set of all sentences is \mathcal{S} .

DEFINITION 1.1. An individual constant is a distinct reference to a particular object.

The set of all constants is \mathbf{c} .

DEFINITION 1.2. A predicate P of order $n \in \mathbb{N}$ applies to an n -tuple of constants $c_i \in c^n \subseteq \mathbf{c}$ to formulate a sentence $P(c^n) \in \mathcal{S}$.

A constant $c \in \mathbf{c}$ is in the scope of the sentence $P(c^n)$ at position i if $c_i = c$.

The set of all predicates is \mathbf{P} . The set of all predicates of order n is $\mathbf{P}^n \subset \mathbf{P}$.

DEFINITION 1.3. An individual variable x is a distinct reference to any object.

The set of all variables is \mathbf{x} .

Note that the distinction applies to the reference and not to the object. Since a variable refers to any object, it refers to no object exclusively.

DEFINITION 1.4. A quantifier q may act on a sentence $s \in \mathcal{S}$ by uniform substitution $\varphi(x) = s(x \setminus c)$ of a constant $c \in \mathbf{c}$ in the scope of $s = \varphi(c)$ with a variable $x \in \mathbf{x}$ nowhere in the scope of s to formulate $qx \varphi(x) \in \mathcal{S}$.

We call $\varphi(x)$ a formula and x a free variable. A formula $\varphi(x^n)$ involved in $0 < n$ levels of quantification has n distinct free variables $x_i \in x^n$ in its scope. Such a formula is considered open. Only a closed formula is well formed.

A sentence $s \in \mathcal{S}$ may be an instance $s = \varphi(c^n)$ of an open formula $\varphi(x^n)$ if it follows from uniform substitution $s = \varphi(x^n)(c_i \setminus x_i)^n$ of the free variables x^n . The instance is strict if all c_i are distinct and nowhere in the scope of $\varphi(x^n)$.

Note that no such distinct c_i is in the scope of $\varphi(x^n)$ only if $\varphi(x^n) = s(x_i \setminus c_i)^n$ with no x_i anywhere in s .

DEFINITION 1.5. A logical operator o of order n may act on any n -tuple of sentences $s_i \in s^n \subset \mathcal{S}$ to form a new sentence $o(s^n) \in \mathcal{S}$.

This permits all propositional operators to act on sentences.

A sentence $s \in \mathcal{S}$ may substitute uniformly $\pi(s \setminus p)$ for a propositional variable p in a propositional formula π . Hence a propositional formula π defines an equivalence class on a set of sentences $\mathcal{M} \subseteq \mathcal{S}$ such that every $s \in \mathcal{M}$ is an instance of π .

We permit all propositional rules of inference to apply to sentences.

QUANTIFIER 2.1 (Universal). If $[\forall x \varphi(x)]$ holds, then $[\varphi]$ holds for all objects.

RULE 2.1 (Universal instantiation).

$$\forall x \varphi(x) \vdash \varphi(x)(c \setminus x)$$

We introduce the following quantifier by equivalence.

QUANTIFIER 2.2 (Existential). If $[\exists x \varphi(x)]$ holds, then there is an object for which $[\varphi]$ holds.

$$\vdash \exists x \varphi(x) \leftrightarrow \neg \forall x \neg \varphi(x)$$

THEOREM 1.1 (Existential generalisation).

$$\varphi(x)(c \setminus x) \vdash \exists x \varphi(x)$$

PROOF. $\forall x \neg \varphi(x) \vdash \neg \varphi(x)(c \setminus x)$. □

The condition to the weaker universal form forbids that we assume anything about c .

PROPERTY 1.1 (Universal generalisation). Given a constant $c \in \mathbf{c}$ nowhere in the scope of a sentence $s \in \mathcal{S}$.

$$\text{If } s \vdash \varphi(c) \text{ then } s \vdash \forall x \varphi(c)(x \setminus c).$$

Note that $\mathcal{M} \vdash \varphi(c)$ does not imply $\mathcal{M} \vdash \forall x \varphi(c)(x \setminus c)$. This does not preclude the following.

LEMMA 1.1. Given finite premises $\mathcal{M} \subset \mathcal{S}$ such that c is nowhere in \mathcal{M} .

$$\text{If } \mathcal{M} \vdash \varphi(c) \text{ then } \mathcal{M} \vdash \forall x \varphi(c)(x \setminus c).$$

PROOF. Consider $s = \&_{i=1}^{|\mathcal{M}|} \mathcal{M}_i$. □

Furthermore we may contrapose into the following.

THEOREM 1.2 (Existential instantiation). Given c nowhere in s nor in \mathcal{M} assumed finite.

$$\text{If } \mathcal{M} \cup \{\varphi(c)\} \vdash s \text{ then } \mathcal{M} \cup \{\exists x \varphi(c)(x \setminus c)\} \vdash s$$

PROOF. Clearly $\mathcal{M} \cup \{\neg s\} \vdash \neg \varphi(c)$. □

This is clearly a weaker form of instantiation which avoids inferring an actual instance. This is necessary to avoid $\exists x \varphi(x) \vdash \forall x \varphi(x)$.

2. Semantics

DEFINITION 2.1. The universe U consists of all objects. Even without bound we may not be able to enumerate all objects.

DEFINITION 2.2. An assignment consists of a map u and a set r such that there is a map $r^n \in r$ for each $n \in \mathbb{N}$.

- (1) $u : \mathbf{c} \longrightarrow U$. Each constant c is assigned an object $u(c)$.
- (2) $r^n : \mathbf{P}^n \longrightarrow \wp(U^n)$. Each predicate P of order n is assigned a set $r^n(P)$ of n -tuples of objects.

The constant codomain $u(\mathbf{c}) \subseteq U$ is the set of all objects $o \in U$ which were assigned $o = u(c) \in u(\mathbf{c})$ from some constant $c \in \mathbf{c}$.

Each constant $c_i \in c^n$ in an n -tuple $c^n \in \mathbf{c}^n$ may be assigned $u(c_i) = u(c^n)_i$ to form an n -tuple $u(c^n) \in u(\mathbf{c})^n$ from the constant codomain. We agree that $u(\mathbf{c})^0 = \{\emptyset\}$. Note that $\emptyset \times S = \emptyset$ and $\{\emptyset\} \times S = S$ for any set S .

We may suffice with $r(P) = r^n(P)$ to denote the assignment of a predicate $P \in \mathbf{P}^n$ of order n .

DEFINITION 2.3. A structure $v, (u, r)$ consists of a model $v : \Phi \longrightarrow \{0, 1\}$ and an assignment u, r .

- (1) The valuation of constant predicative formulas $P \subset \Phi$ is interdeterminate with the assignment.

$$v(P(c^n)) = 1 \text{ only if } u(c^n) \in r(P)$$

We say that the n -tuple of objects $u(c^n)$ satisfies the formula P .

- (2) Quantifying formulas $\forall \subset \Phi$ are evaluated by universal consensus on the quantified formula between all structures $w, (t, r)$ interdeterminate with the same predicative assignment r .

$$v(\forall x \varphi(x)) = 1 \text{ only if } w(\varphi(c)) = 1 \text{ for } \begin{array}{l} \text{all constants } c \\ \text{all structures } w, (t, r) \end{array}$$

DEFINITION 2.4. A sentence is a semantic consequence of a set of premises $\mathcal{M} \models s$ only if all structures $v, (u, r)$ have a model v that supports (\mathcal{M}, s) .

DEFINITION 2.5. We denote the set of sentences with instances of precisely n distinct constants as Φ^n . Note that $\mathcal{S} = \bigcup_{n \in \mathbb{N}} \Phi^n$.

A model v may be interdeterminate with a total assignment u, R such that $R^n \in R$ for each $n \in \mathbb{N}$ where $R^n : \Phi^n \longrightarrow \wp(u(\mathbf{c})^n)$.

$$v(\varphi(c^n)) = 1 \text{ only if } u(c^n) \in R(\varphi)$$

- (1) $R^n(\neg\varphi) = R^n(\varphi)^C$ where the complement is over $u(\mathbf{c})^n$.

$$(2) \quad R^{m+n-k}(\varphi \rightarrow \gamma) = \left\{ u \in U^{m+n-k} \mid \begin{array}{l} [u]_1^m \in R^m(\neg\varphi) \\ \text{or } [u]_{m-k+1}^{m-k+n} \in R^n(\gamma) \end{array} \right\}$$

where common constants are arranged from $m - k + 1$ to m if any.

We define $R^0(\forall x \varphi(x)) = \{\emptyset\}$.

$$(3) \quad R^n(\forall x \varphi(\dots c^n, x)) = \left\{ u^n \in u(\mathbf{c})^n \mid \text{all } u \in U : (\dots u^n, u) \in R^{n+1}(\varphi(\dots c^n, c)) \right\}$$

LEMMA 2.1. A model determines a nonempty set of structures of which it is the model.

PROOF. Given any v an injective u ensures that any formula in every Φ^n has n distinctly assigned constants. This determines a total assignment R and thus a predicative assignment $r \subset R$. We show that $v, (u, r)$ is a structure by inductively decomposing and reconstructing between R and r where v is ever the interdeterminate model.

- (1) $v(\neg\varphi) = 1$ only if $v(\varphi) = 0$ only if $u(c^n) \notin R(\varphi)$
- (2) Similarly from implicational completeness.

- (3) Evidently all structures $w, (t, r)$ come to universal consensus on a formula $w(\varphi(c)) = 1$ for all constants c if all objects satisfy it $R(\varphi) = U$.

Conversely, there must be some object $q \in U$ that does not satisfy the formula $q \notin R(\varphi)$ so there must be some structure $w, (t, r)$ assigning $t(c) = q$ some constant c such that $w(\varphi(c)) = 0$. This does not preclude $R(\varphi) = u(\mathbf{c})$ so nonetheless $v(\varphi(c)) = 1$ for any c .

□

THEOREM 2.1 (Gödel). First order logic is semantically complete.

DEFINITION 2.6.