Metamathema

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CHAPTER 1

Propositional logic

A proposition P signifies a distinct fact P^* that is either the case or not. The set of all propositions is $\mathcal{W}$.

1. Syntax

Definition 1.1. A logical operator o of order $n \in \mathbb{N}$ may act on any n-tuple of propositions $P^n \in \mathcal{W}^n$ to compose a proposition $o(P^n) = P \in \mathcal{W}$.

Each $P_i \in P^n$ is said to be the ith direct descendant of P. A proposition Q is said to be a descendant of P if there is a succession of direct descendants that leads from P to Q.

When a proposition P is atomic it is not composed $P \neq o(P^n)$ of any n-tuple $P^n \in \mathcal{W}^n$ by any operator o of any order n.

Operator 1.1 (Implication). For any $P, Q \in \mathcal{W}$:

$$P \to Q \in \mathcal{W}.$$

If $(P \to Q)^*$ is the case, then if P^* (the implicans) is the case, then so is Q^* (the implicandum). The implicans is said to be a sufficient condition for the implicandum.

Operator 1.2 (Negation). For any $P \in \mathcal{W}$:

$$\neg P \in \mathcal{W}.$$

If $(\neg P)^*$ is the case then P^* is not the case.

Definition 1.2. Given an alphabet α consisting of a finite $|\alpha| \in \mathbb{N}$ set of symbols, we consider any finite $n \in \mathbb{N}$ string of symbols from the alphabet $e^n \in \alpha^n$ an expression. An instance $e_i \in e^n$ of a symbol $e_i = \alpha_j \in \alpha$ at position i is known as a character.

The set of all expressions from the alphabet is $\alpha^\omega = \bigcup_{n \in \mathbb{N}} \alpha^n$.

The substring $d^m = [e^n]_{k+1}^{k+m}$ of an expression $e^n \in \alpha^n$ at position $k+1$ of length m is its restriction to the characters $e_i \in e^n$ at positions $k < i \leq k+m$. The instance $[e^n]_{k+1}^{k+m}$ of the expression d^m is said to be in the scope of the expression e^n at position $k+1$.

Uniform substitution $e(d \setminus c)$ of an expression $c \in \alpha^l$ with an expression $d \in \alpha^m$ in an expression $e \in \alpha^n$ substitutes an instance of d for every instance of c in the scope of e.

Theorem 1.1. The alphabet $\beta = \{\mathrm{a}, \mathrm{o}, ', (,), ,\}$ expresses all propositions.

$$\mathcal{W} \subset \beta^\omega$$

PROOF. We use the atom symbol a and the succession symbol $'$ to denote an atomic proposition for every $n \in \mathbb{N}$ such that $a_0 = \text{a}$ and $a_{n+1} = a_n'$.

We may use the operator symbol o and the succession symbol $'$ to denote any operator. Then parentheses (,) and a comma , allow us to enumerate components in a composite proposition. $\square$

COROLLARY 1.1. The set of all propositions $\mathcal{W}$ may be enumerated though without bound: $|\mathcal{W}| = |\mathbb{N}|$.

PROOF. The alphabet β allows any proposition to be encoded in base 6. $\square$

COROLLARY 1.2. A proposition is in the scope of another only if it descends from the other.

DEFINITION 1.3. A propositional variable p is a distinct reference to any proposition P. An extension $\phi = \beta \cup \{\text{p}\}$ of the alphabet β expresses all such variables.

A formula expresses the logical structure of a proposition. We define the set Φ of all formulas recursively.

If $\varphi \in \Phi$ is a formula then so is the result of uniform substitution $\varphi(p \setminus P) \in \Phi$ of any proposition P in the scope of φ with a propositional variable p. Furthermore, $\mathcal{W} \subset \Phi$. Note that $\Phi \subset \phi^\omega$. A formula $\varphi \in \Phi$ is said to be well formed only if it is a proposition $\varphi \in \mathcal{W}$.

A proposition P is said to be an instance of a formula φ only if it follows from uniform substitution $P = \varphi(P_i \setminus p_i)^n$ of all propositional variables $p_{i \leq n}$ in the scope of φ with some proposition P_i.

DEFINITION 1.4. A set of propositions $\mathcal{E} \subseteq \mathcal{W}$ may allow us to infer $\mathcal{E} \vdash P$ a proposition $P \in \mathcal{W}$. The propositions of $\mathcal{E}$ are known as premises whereas P is said to be a syntactic consequence of $\mathcal{E}$.

The deductive closure $\partial\mathcal{E}$ of a set $\mathcal{E}$ contains all syntactic consequences of the set such that $\mathcal{E} \vdash P$ only if $P \in \partial\mathcal{E}$.

A set of propositions $\mathcal{E}$ may have the following properties with respect to its deductive closure $\partial\mathcal{E}$.

(deductively closed). $\mathcal{E} = \partial\mathcal{E}$
(consistent). if $P \in \partial\mathcal{E}$ then $\neg P \notin \partial\mathcal{E}$
(complete). if $P \notin \partial\mathcal{E}$ then $\neg P \in \partial\mathcal{E}$
(trivial). $\partial\mathcal{E} = \mathcal{W}$

A system that determines all valid inferences is known as an inference system. Given a preferred set of propositions considered axioms, an inference system suffices to produce their deductive closure, known as a theory, where each syntactic consequence is known as a theorem.

Any proposition T that we may infer from the empty set $\vdash T$ is known as a logical theorem.

DEFINITION 1.5. The alphabet $\beta^\vdash = \beta \cup \{\vdash\}$ allows the inference symbol $\vdash$ to act on any finite set of propositions to compose an inferential expression $I \in \mathcal{W}^\vdash$ denoting a potential inference.

A propositional variable may substitute uniformly $\eta = \iota(p \setminus P)$ for a proposition in an inferential formula $\iota, \eta \in \Phi^\vdash$ where $\mathcal{W}^\vdash \subset \Phi^\vdash$. Note that $\Phi^\vdash$ is expressible from $\phi^\vdash = \phi \cup \{\vdash\} = \beta^\vdash \cup \{\text{p}\}$.

A pure inferential formula has no atomic propositions in its scope.

An inferential expression is an instance of an inferential formula if it follows from uniform substitution of all propositional variables in its scope.

Definition 1.6. A rule of inference is a pure inferential formula of which every instance expresses a valid inference.

Note that the premises in any rule of inference are finite.

Note that not every inferential expression is a valid inference, nor every pure inferential formula a rule of inference. However, postulating certain primitive rules and properties of inference allows us to determine all valid inferences.

Rule 1.1 (Modus ponens).

$$\begin{matrix} p \to q, \\ p \end{matrix} \vdash q$$

Property 1.1 (Finite foundation). A set $\mathcal{E}$ infers a proposition $\mathcal{E} \vdash P$ only if some finite subset $\mathcal{D} \subseteq \mathcal{E}$ infers the proposition $\mathcal{D} \vdash P$ such that all finite subsets $\mathcal{G} \subseteq \mathcal{E}$ containing the subset $\mathcal{D} \subseteq \mathcal{G}$ also infer the proposition $\mathcal{G} \vdash P$.

Note that this allows us to infer from infinite sets.

The following property does not hold in adaptive inference systems.

Property 1.2 (Monotony). If $\mathcal{E} \vdash P$ then $\mathcal{E} \cup \{Q\} \vdash P$ for any Q.

This means that a finite subset inferring the proposition suffices to infer a proposition from the total set.

Note that every meaningful application of a rule of inference is marked by the extension of a set of inferred propositions. The following property allows us to apply rules of inference to propositions already inferred.

Property 1.3 (Transitivity). If $\mathcal{E} \vdash P$ for all $P \in \mathcal{D}$ and $\mathcal{D} \vdash Q$ then $\mathcal{E} \vdash Q$.

Theorem 1.2. The deductive closure of a set is deductively closed $\partial\partial\mathcal{E} = \partial\mathcal{E}$.

This allows us to derive rules of inference by showing that the inference is valid for every instance of the rule. The following property ensures that every rule of inference is derived from a finite succession of primitive rules.

Property 1.4 (Finite transition). A set infers a proposition only by a finite succession of primitive rules of inference, taking the set as premises and extending with any proposition inferred.

It also ensures that every inference is an instance of some rule of inference.

1.1. Inferential approach. The following property allows us to infer over the possibility of inference itself. Consider the extension $\mathcal{E} \cup \{P\}$ of some premises $\mathcal{E}$ with a proposition P known as the hypothesis.

Property 1.5 (Deduction). If $\mathcal{E} \cup \{P\} \vdash Q$ then

$$\mathcal{E} \vdash P \to Q.$$

for all $\mathcal{E} \subset \mathcal{W}$.

Hence the subject of hypothesis is sufficient condition for what can be inferred from the extension.

Furthermore this entails a familiar property.

THEOREM 1.3 (Linearity of inference). If $\mathcal{E} \vdash P$ and $\mathcal{E} \cup \{P\} \vdash Q$ then $\mathcal{E} \vdash Q$.

PROOF. From $\mathcal{E} \vdash P \to Q$ and transitivity of inference. $\square$

Some expected properties of the implication can now follow.

THEOREM 1.4 (Transitivity of implication).

$$\begin{matrix} p \to q, \\ q \to r \end{matrix} \vdash p \to r$$

PROOF. From $p \to q, q \to r, p \vdash r$. $\square$

THEOREM 1.5 (Distributivity of implication).

$$p \to (q \to r) \vdash (p \to q) \to (p \to r)$$

The following rule seems all too evident.

RULE 1.2 (Reflection).

$$p \vdash p$$

It allows us to prove the following theorems.

THEOREM 1.6. $\mathcal{E} \subseteq \partial\mathcal{E}$

THEOREM 1.7 (Reflexivity of implication).

$$\vdash p \to p$$

THEOREM 1.8 (Ex quod libet datum).

$$p \vdash q \to p$$

PROOF. $p, q \vdash p$ by monotony. $\square$

The implication thus defined is said to be material.

The preceding rules and theorems have applied exclusively to the implication. The following rule introduces the syntactic role of the negation.

RULE 1.3 (Contraposition).

$$p \to q \vdash \neg q \to \neg p$$

This shows a property of inference.

THEOREM 1.9 (Contrapositivity). If $p \vdash q$ then $\neg q \vdash \neg p$.

THEOREM 1.10 (Ex absurdum negatio quod libet).

$$p, \neg p \vdash \neg q$$

PROOF. $p \vdash q \to p$. $\square$

COROLLARY 1.3 (Ex absurdo falsum). For any formula τ such that $\vdash \tau$:

$$p, \neg p \vdash \neg\tau$$

THEOREM 1.11 (Ex falso negatio quod libet). For any formula τ such that $\vdash \tau$:

$$\vdash \neg\tau \to \neg q$$

PROOF. $\vdash q \to \tau$, also known as ex quod libet verum. $\square$

RULE 1.4 (Introduction of double negation).

$$p \vdash \neg\neg p$$

LEMMA 1.1 (Contraposition with left elimination).

$$p \to \neg q \vdash q \to \neg p$$

PROOF. $q \to \neg\neg q$ is transitive with $\neg\neg q \to \neg p$. □

THEOREM 1.12 (Negatio improbis). For any formula τ such that $\vdash \tau$:

$$p \to \neg\tau \vdash \neg p.$$

THEOREM 1.13 (Introduction of implicative conjuction).

$$p, q \vdash \neg(p \to \neg q)$$

PROOF. We can deduce $p \vdash (p \to \neg q) \to \neg q$ from modus ponens. □

This has a special case with an interesting consequence.

LEMMA 1.2 (Negatio reprobis).

$$p \to \neg p \vdash \neg p$$

PROOF. $p \vdash \neg(p \to \neg p)$. Alternatively by ex contradictio falsum. □

THEOREM 1.14 (Reductio ad absurdum).

$$\begin{array}{r} p \to q, \\ p \to \neg q \end{array} \vdash \neg p$$

By contrapositivity we also have the following.

COROLLARY 1.4 (Commutativity of implicative conjunction).

$$\neg(p \to \neg q) \vdash \neg(q \to \neg p)$$

The final rule completes the law of double negation.

RULE 1.5 (Elimination of double negation).

$$\neg\neg p \vdash p$$

This rule has been controversial since it allows the following.

THEOREM 1.15 (Proof by contradiction).

$$\neg p \to q, \neg p \to \neg q \vdash p$$

This result is rejected by intuitionists, hence they do not allow double negation to be eliminated.

The following completes the implicative variety of the law of junctive commutativity.

LEMMA 1.3 (Commutativity of implicative disjunction).

$$\neg p \to q \vdash \neg q \to p$$

This leads to double negation elimination considering $\vdash \neg p \to \neg\neg\neg p$.

Nonetheless, the following results may be preserved by intuitionists as independent rules. We complete the law of implicative conjuction and show an interesting lemma.

COROLLARY 1.5 (Elimination of implicative conjunction).

$$\neg(p \to \neg q) \vdash p, q$$

PROOF. $\neg p \vdash p \to \neg q$. □

LEMMA 1.4 (Introduction of implicative disjunction).

$$p \vdash \neg p \to q$$

PROOF. $p \vdash \neg q \to p$. □

THEOREM 1.16 (Ex absurdo quod libet).

$$\begin{matrix} p, \\ \neg p \end{matrix} \vdash q$$

A consistent set is clearly not trivial. Now we can also show the following.

COROLLARY 1.6. An inconsistent set is trivial.

Hence a maximally consistent set is maximally not trivial. This is avoided in paraconsistent inferential systems.

For any formula τ such that $\vdash \tau$:

THEOREM 1.17 (Ex falso quod libet).

$$\vdash \neg\tau \to q$$

COROLLARY 1.7 (Ex falso absurdum).

$$\neg\tau \vdash p, \neg p$$

Furthermore the completion of the law of contraposition allows us to conclude with a remarkable result.

LEMMA 1.5 (Converse contraposition).

$$\neg p \to \neg q \vdash q \to p$$

THEOREM 1.18 (Peirce's law).

$$(p \to q) \to p \vdash p$$

PROOF. We know that $\vdash \neg p \to (\neg q \to \neg p)$ so $\vdash \neg p \to (p \to q)$. However, in that case $\neg p \to \neg(p \to q) \vdash p$. □

COROLLARY 1.8 (Consequentia mirabilis).

$$\neg p \to p \vdash p$$

PROOF. $\vdash (p \to \neg p) \to \neg p$ is transitive with $\neg p \to p$. □

By ex absurdum quod libet we can show $\neg\neg p, \neg p \vdash p$. Evidently Peirce's law is unacceptable to intuitionists.

The following rule is rejected in its implicative variety for similar reasons.

THEOREM 1.19 (Implicative dilemma).

$$\neg p \to q, \begin{matrix} p \to r \\ q \to r \end{matrix} \vdash r$$

PROOF. $p \to q, \neg p \to q, \neg q \vdash q$. □

Since $\vdash p \to p$ this is enough to invite consequentia mirabilis.

1.2. Alternative operators. We may disregard the law of double negation and introduce some new operators.

OPERATOR 1.3 (Conjuction). If $(P \,\&\, Q)^*$ is the case, then P^* is the case and Q^* is the case.

RULE 1.6 (Law of conjunction). $p, q \vdash p \,\&\, q$ and $p \,\&\, q \vdash p, q$.

THEOREM 1.20 (Conjunctive syllogism). $\neg(p \,\&\, q), p \vdash \neg q$.

PROOF. $p \vdash q \to (p \,\&\, q)$. □

RULE 1.7 (Law against contradiction).

$$\vdash \neg(p \,\&\, \neg p)$$

LEMMA 1.6. $\vdash p \to \neg\neg p$.

LEMMA 1.7. $p \,\&\, q \vdash \neg(p \to \neg q)$

LEMMA 1.8. $p \to q \vdash \neg(p \,\&\, \neg q)$.

OPERATOR 1.4 (Disjunction). If $(P \vee Q)^*$ is the case, then P^* is the case or Q^* is the case.

RULE 1.8 (Disjunctive syllogism).

$$p \vee q, \neg p \vdash q$$

LEMMA 1.9. $\neg p \vee q \vdash p \to q$.

RULE 1.9 (Introduction of disjunction).

$$p \vdash p \vee q$$

RULE 1.10 (Dilemma).

$$p \vee q, \begin{matrix} p \to r, \\ q \to r \end{matrix} \vdash r$$

RULE 1.11 (Law of junctive commutativity). $p \,\&\, q \vdash q \,\&\, p$ and $p \vee q \vdash q \vee p$.

OPERATOR 1.5 (Equivalence). If $(P \leftrightarrow Q)^*$ is the case then P^* is the case only if Q^* is the case.

RULE 1.12 (Law of equivalence).

$$p \to q, q \to p \vdash p \leftrightarrow q$$
$$p \leftrightarrow q \vdash p \to q, q \to p$$

THEOREM 1.21. $\vdash (p \leftrightarrow q) \leftrightarrow ((p \to q) \,\&\, (q \to p))$

THEOREM 1.22 (De Morgan's laws I).

$$\vdash \neg(p \vee q) \leftrightarrow (\neg p \,\&\, \neg q)$$

PROOF. $\vdash p \to (p \vee q), \vdash q \to (p \vee q)$. $\neg p \vdash (p \vee q) \to q$. □

THEOREM 1.23 (De Morgan's laws IIa).

$$\neg p \vee \neg q \vdash \neg(p \,\&\, q)$$

PROOF. $\neg p \vee \neg q, p \,\&\, q \vdash q, \neg q$. □

Rule 1.13 (Law of the excluded middle).

$$\vdash \neg p \vee p$$

Corollary 1.9. $\vdash \neg\neg p \to p$.

This is the first offence against intuitionism in this approach.
A weaker variety of the offending law suffices to complete De Morgan's laws.

Lemma 1.10. $\vdash \neg p \vee \neg\neg p$

Theorem 1.24 (De Morgan's laws IIb).

$$\neg(p \,\&\, q) \vdash \neg p \vee \neg q$$

Proof. $\neg(p \,\&\, q) \vdash \neg\neg p \to (\neg p \vee \neg q)$ from $p \to \neg q \vdash \neg\neg p \to \neg q$. □

Theorem 1.25.

$$\begin{array}{cccc} \vdash & (p \,\&\, q) & \leftrightarrow & \neg(p \to \neg q) \\ \vdash & (p \vee q) & \leftrightarrow & (\neg p \to q) \end{array}$$

Proof. From elimination of implicative conjuction and from $\neg p \to q \vdash \neg p \to (p \vee q)$ by dilemma over exclusion of the middle. □

Corollary 1.10.

$$\begin{array}{cccc} \vdash & (p \to q) & \leftrightarrow & \neg(p \,\&\, \neg q) \\ \vdash & (p \to q) & \leftrightarrow & (\neg p \vee q) \end{array}$$

1.3. Axiomatic approach. We may disregard all properties and rules from the inferential approach. Instead we consider a special set of formulas $\Xi \subset \Phi$ known as axioms. We may also consider the set of all instances of axioms $\mathcal{X} \subset \mathcal{W}$.

Property 1.6. If $\mathcal{E} \subset \{Q, Q \to P\} \subset (\mathcal{E} \cup \mathcal{X})$ then $\mathcal{E} \vdash P$.

This property is independent from standard modus ponens yet extends it. It is also independent from monotony.

Axiom 1.1. $p \to (q \to p)$

Axiom 1.2. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

Lemma 1.11. $\vdash p \to p$

Proof. Taking $\alpha = p \to ((q \to p) \to p)$ and $\beta = p \to (q \to p)$ we have $\alpha \to (\beta \to (p \to p)), \alpha, \beta \in \Xi$. □

Corollary 1.11. $p \vdash p$

Theorem 1.26 (Deduction). If $\Lambda \cup \{p\} \vdash q$ then $\Lambda \vdash p \to q$.

Proof. We have $\Lambda \vdash p \to \alpha$ for any $\alpha \in \Lambda \cup \{p\}$.
Hence for any $\alpha \to \beta \in \Lambda \cup \{p\}$ we have $\Lambda \vdash p \to (\alpha \to \beta)$.
Evidently $\Lambda \vdash p \to \beta$ permits continued deduction until we show $\Lambda \vdash p \to \gamma$ for any $\gamma \in \partial(\Lambda \cup \{p\})$. □

Axiom 1.3. $(\neg p \to \neg q) \to (q \to p)$

Theorem 1.27 (Double negation elimination). $\vdash \neg\neg p \to p$

PROOF. $\neg\neg p \vdash \neg\neg(\neg\neg p) \to \neg\neg p$. □

THEOREM 1.28 (Double negation introduction). $\vdash p \to \neg\neg p$

PROOF. $\vdash \neg\neg(\neg p) \to \neg p$. □

Evidently the axiomatic approach allows precisely the same syntactic derivations as the inferential.

2. Semantics

The expressive range of P is restricted to a binary distinction $P^* \in \{0,1\}$.

We say that P^* is the case only if $P^* = 1$. Only in that case is P true.

DEFINITION 2.1. A model is a valuation map $v : \mathcal{W} \longrightarrow \{0,1\}$ that assigns a truth value $v(P) \in \{0,1\}$ to each proposition $P \in \mathcal{W}$.

A model v supports $(\mathcal{E}, P)$: if no $E \in \mathcal{E}$ assigns to $v(E) = 0$ then $v(P) = 1$.

Furthermore the following holds for any models v, w:

(inferential). If $\mathcal{E} \vdash P$ then v supports $(\mathcal{E}, P)$.

(regular with respect to negation). If $v(P) = w(Q)$ then $v(\neg P) = w(\neg Q)$.

(not trivial). There is at least one proposition P such that $v(P) = 0$.

We may consider the set of all models $\mathbf{M}$.

The positive domain $v^{-1}(1) \subseteq \mathcal{W}$ of a valuation map v contains all propositions P for which $v(P) = 1$. Clearly a model is determined by the positive domain alone. A model v supports $(\mathcal{E}, P)$ only if $P \in v^{-1}(1)$ if $\mathcal{E} \subseteq v^{-1}(1)$. We can see that a valuation map is inferential only if the positive domain is deductively closed.

DEFINITION 2.2. A proposition is a semantic consequence $\mathcal{E} \vDash P$ of a set of premises only if all models $v \in \mathbf{M}$ support $(\mathcal{E}, P)$.

$\mathcal{E}$ is said to validate P.

LEMMA 2.1. If $\mathcal{E} \vdash P$ then $\mathcal{E} \vDash P$.

The inferential system is semantically sound. The semantics are adequate for the inferential system.

DEFINITION 2.3. A truth function $f \in F^n$ of order $n \in \mathbb{N}$ maps n-tuples of truth values to a single truth value $F^n = \left\{ f \subset \{0,1\}^{n+1} \,\middle|\, f : \{0,1\}^n \longrightarrow \{0,1\} \right\}$ such that $F = \bigcup_{n \in \mathbb{N}} F^n$ is the set of all truth functions.

A logical operator o of order n may be characterised semantically $o \equiv f$ by a truth function $f \in F^n$ such that in every model v the valuation of $P = o(P^n)$ is mapped from its components $v(P) = f(v(P^n))$ where $v(P^n)_i = v(P_i)$ for all $P_i \in P^n$.

Note that regularity with respect to negation is equivalent to the characterisation of negation by a truth function.

Furthermore, since the codomain of any truth function is binary, the characterisation of any operator o may be defined by an equivalence that holds for all models v between $v(o(P^n)) = 1$ and certain conditions on $v(P^n)$.

PROPERTY 2.1. The following properties characterise the negation.

(consistency). $v(\neg P) = 0$ if $v(P) = 1$

(completeness). $v(\neg P) = 1$ if $v(P) = 0$

Proof. A model must be consistent since it may not be trivial.

Consider now any theorem $\vdash T$. Clearly $v(T) = 1$ so $v(\neg T) = 0$ by consistency. However, $\vdash \neg\neg T$ from which $v(\neg P) = 1$ if $v(P) = 0$ by negational regularity. $\square$

A consistent map v is not trivial. If it is also complete, it is regular with respect to negation. Hence, an inferential valuation map is regular with respect to negation and not trivial if it is consistent and complete.

Note that we relied on ex falso quod libet to show the consistency of an inferential map which is not trivial. This rule does not hold in paraconsistent inferential systems. Nonetheless, regularity with respect to negation requires that if some model w supports some contradiction $w(Q) = 1 = w(\neg Q)$ then $v(P) = 1$ immediately implies $v(\neg P) = 1$ for any proposition P in all models v.

Therefore neither consistency nor regularity with respect to negation are a defining property of paraconsistent semantics. Instead a model is required to be complete and not trivial.

Property 2.2. The following characterises the implication.

$$v(P \to Q) = 1 \text{ only if } v(P) = 0 \text{ or } v(Q) = 1$$

Proof. Suppose $v(Q) = 1$ then $v(P \to Q) = 1$ given materiality of implication. Should $v(P) = 0$ then $v(\neg P) = 1$ by completeness so $v(P \to Q) = 1$ by ex contradictio quod libet.

Conversely, if $v(P \to Q) = 1 = v(P)$ then $v(Q) = 1$ by modus ponens. $\square$

Note that we relied on ex contradictio quod libet to show the implicational completeness of an inferential map.

Definition 2.4. A set of logical operators O is functionally complete: the closure under function composition $\partial_\circ O_F$ of the truth functions that characterise its operators $O_F = \{f \in F \mid f \equiv o \text{ for some } o \in O\}$ is the set of all truth functions $\partial_\circ O_F = F$.

Lemma 2.2. If $F^2 \subseteq \partial_\circ O_F$ then $\partial_\circ O_F = F$.

Proof. F^2 contains every binary function independent of its second argument, from which we compose every unary function F^1 by passing the same argument twice. Suppose now $F^n \subseteq \partial_\circ O_F$ where $2 \leq n$ and consider any function $f \in F^{n+1}$. For any $b^{n+1} \in \{0,1\}^{n+1}$ restricted to $b^n = [b^{n+1}]_1^n \in \{0,1\}^n$ we may find some function s such that $f(\ldots b^n, a) = s(a)$ depends only on $a \in \{0,1\}$.

(1) $g(b^n) = 1$ only if $f(\ldots b^n, 0) = f(\ldots b^n, 1)$:
  s is constant.
(2) $h(b^n) = 1$ only if $f(\ldots b^n, 1) = 1$:
  s is veracious.
(3) $l(g, a) = 0$ only if $g = a = 0$.
  We transmit 0 only if $a = 0$ and s depends on its argument a.
(4) $k(l, h) = 1$ only if $l = h$
  We return 1 only if the veracity of s is equivalent to the constancy of s or the positivity of $a = 1$.

We compose $k\left(l\left(g(b^n), a\right), h(b^n)\right) = f(\ldots b^n, a)$. $\square$

Theorem 2.1. The set of operators $\{\neg, \to\}$ is functionally complete.

PROOF. Any $f \in F^2$ has a positive domain which permits at most $|f^{-1}(1)| \leq 4$ pairs of binary values.

The function characterising the implication $m(p,q) \equiv p \to q$ permits three such pairs, yet both of its arguments, each of which it depends on, may be inverted independently by means of the negation $n(p) \equiv \neg p$ to compose any fully dependent function $g \in F^2$ of positive order $|g^{-1}(1)| = 3$, of which there are 4.

Any such function may then be inverted to obtain the fully dependent functions of positive order 1, among which the conjunction $c(p,q) = n(m(p,n(q))) \equiv \neg(p \to \neg q)$. Conjugating converses $e = c(m(p,q), m(q,p)) \equiv \neg((p \to q) \to \neg(q \to p))$ we may complete the fully dependent functions of positive order 2 by singular inversion $n(e(p,q))$.

Passing constant functions $t(p) = m(p,p) = 1 \equiv p \to p$ or $l(p) = n(t(p)) = 0 \equiv \neg(p \to p)$ to any argument eliminates the dependency on the argument. Note that the positive order of a partially independent function must be divisible by 2, lest there be some value for which the function depends on the eliminated variable. Passing t to either argument of e gives 2 selection functions, both of which are distinctly invertible, counting the expected total of $16 = |F^2| = |\{0,1\}|^{|\{0,1\}|^2}$. $\square$

DEFINITION 2.5. A set of propositions $\mathcal{E} \subseteq \mathcal{W}$ is saturable only if there is a model v such that $v(P) = 1$ if $P \in \mathcal{E}$.

The set determines the model only if $P \in \mathcal{E}$ if $v(P) = 1$.

A model set is deductively closed and maximally not trivial.

Note that any deductively closed set which is maximally consistent is a model set if our logic permits it.

LEMMA 2.3. A model set determines an inferential map which is complete with respect to negation and implication.

PROOF. Consider a map $v : \mathcal{W} \longrightarrow \{0,1\}$ such that $v(P) = 0$ only if $P \notin \mathcal{E}$ where $\mathcal{E}$ is a model set.

If $v(P) = 0$ then $\mathcal{E} \cup \{P\} \vdash Q, \neg Q$ by maximal triviality so $v(\neg P) = v(P \to Q) = 1$ by deductive closure. $\square$

Note that the preceding also holds for paraconsistent systems.

In any case, a model set determines a model.

LEMMA 2.4. A set that does not prove a proposition $\mathcal{E} \nvdash P$ is a subset of a model set $\mathcal{E} \subseteq \Gamma$ that does not contain the proposition $P \notin \Gamma$.

PROOF. Consider $\Gamma_{i+1} = \Delta_i$ if $P \notin \Delta_i$ and $\Gamma_{i+1} = \Gamma_i$ otherwise, where $\Delta_i = \partial(\Gamma_i \cup \{\mathcal{W}_i\})$ with $\mathcal{W}_i \in \mathcal{W}$ enumerating all well formed formulas and $\Gamma_0 = \partial\mathcal{E}$. Evidently $\Gamma = \bigcup_{i \in \mathbb{N}} \Gamma_i$ is deductively closed, since for any finite $\mathcal{D} \subset \Gamma$ also $\partial\mathcal{D} \subset \Gamma_i$ for some i.

We can see that $P \to Q \in \Gamma$ for any Q since $\mathcal{W}_i = P \to Q$ for some i so lest Peirce's law come to effect $\Gamma_i \cup \{P \to Q\} \nvdash P$. Moreover, if $R \notin \Gamma$ then $\Gamma_i \cup \{R\} \vdash P$ for some i so $\Gamma \cup \{R\}$ is trivial.

Alternatively, if $\Gamma_0 = \mathcal{E} \cup \{\neg P\}$ is inconsistent we have $\mathcal{E} \vdash P$ by contradiction. Hence Γ_0 is consistent. We may take $\Gamma_{i+1} = \Delta_i$ only if Δ_i is consistent. By construction Γ is maximally consistent. This proof only works if inconsistency implies triviality. It also relies on the consistency of $P \notin \Gamma$. $\square$

THEOREM 2.2. $\mathcal{E} \models P$ only if $\mathcal{E} \vdash P$.

The semantics are characteristic for the inferential system. The converse means that the inferential system is semantically complete.

CHAPTER 2

Predicate logic

We may extend our semantics with a notion of distinct objects and distinct relations. A relation either holds for certain objects or it does not. If there can be only one object in the relation, it may be considered a property of the object.

1. Syntax

In predicate logic a well formed formula is known as a sentence.

The set of all sentences is $\mathcal{S}$.

Definition 1.1. An individual constant is a distinct reference to a particular object.

The set of all constants is $\mathbf{c}$.

Definition 1.2. A predicate P of order $n \in \mathbb{N}$ applies to an n-tuple of constants $c_i \in c^n \subseteq \mathbf{c}$ to formulate a sentence $P(c^n) \in \mathcal{S}$.

A constant $c \in \mathbf{c}$ is in the scope of the sentence $P(c^n)$ at position i if $c_i = c$.

The set of all predicates is $\mathbf{P}$. The set of all predicates of order n is $\mathbf{P}^n \subset \mathbf{P}$.

Definition 1.3. An individual variable x is a distinct reference to any object.

The set of all variables is $\mathbf{x}$.

Note that the distinction applies to the reference and not to the object. Since a variable refers to any object, it refers to no object exclusively.

Definition 1.4. A quantifier q may act on a sentence $s \in \mathcal{S}$ by uniform substitution $\varphi(x) = s(x \setminus c)$ of a constant $c \in \mathbf{c}$ in the scope of $s = \varphi(c)$ with a variable $x \in \mathbf{x}$ nowhere in the scope of s to formulate $qx\ \varphi(x) \in \mathcal{S}$.

We call $\varphi(x)$ a formula and x a free variable. A formula $\varphi(x^n)$ involved in $0 < n$ levels of quantification has n distinct free variables $x_i \in x^n$ in its scope. Such a formula is considered open. Only a closed formula is well formed.

A sentence $s \in \mathcal{S}$ may be an instance $s = \varphi(c^n)$ of an open formula $\varphi(x^n)$ if it follows from uniform substitution $s = \varphi(x^n)(c_i \setminus x_i)^n$ of the free variables x^n. The instance is strict if all c_i are distinct and nowhere in the scope of $\varphi(x^n)$.

Note that no such distinct c_i is in the scope of $\varphi(x^n)$ only if $\varphi(x^n) = s(x_i \setminus c_i)^n$ with no x_i anywhere in s.

Definition 1.5. A logical operator o of order n may act on any n-tuple of sentences $s_i \in s^n \subset \mathcal{S}$ to form a new sentence $o(s^n) \in \mathcal{S}$.

This permits all propositional operators to act on sentences.

A sentence $s \in \mathcal{S}$ may substitute uniformly $\pi(s \setminus p)$ for a propositional variable p in a propositional formula π. Hence a propositional formula π defines an equivalence class on a set of sentences $\mathcal{M} \subseteq \mathcal{S}$ such that every $s \in \mathcal{M}$ is an instance of π.

We permit all propositional rules of inference to apply to sentences.

QUANTIFIER 2.1 (Universal). If $[\forall x\ \varphi(x)]$ holds, then $[\varphi]$ holds for all objects.

RULE 2.1 (Universal instantiation).

$$\forall x\ \varphi(x) \vdash \varphi(x)(c \setminus x)$$

We introduce the following quantifier by equivalence.

QUANTIFIER 2.2 (Existential). If $[\exists x\ \varphi(x)]$ holds, then there is an object for which $[\varphi]$ holds.

$$\vdash \exists x\ \varphi(x) \leftrightarrow \neg\forall x\ \neg\varphi(x)$$

THEOREM 1.1 (Existential generalisation).

$$\varphi(x)(c \setminus x) \vdash \exists x\ \varphi(x)$$

PROOF. $\forall x\ \neg\varphi(x) \vdash \neg\varphi(x)(c \setminus x)$. □

The condition to the weaker universal form forbids that we assume anything about c.

RULE 2.2 (Universal generalisation). Given a constant $c \in \mathbf{c}$ nowhere in the scope of a sentence $s \in \mathcal{S}$.

$$\text{If } s \vdash \varphi(c) \text{ then } s \vdash \forall x\ \varphi(c)(x \setminus c).$$

LEMMA 1.1. Given finite premises $\mathcal{M} \subset \mathcal{S}$ such that c is nowhere in $\mathcal{M}$.

$$\text{If } \mathcal{M} \vdash \varphi(c) \text{ then } \mathcal{M} \vdash \forall x\ \varphi(c)(x \setminus c).$$

PROOF. Consider $s = \&_{i=1}^{|\mathcal{M}|} \mathcal{M}_i$. □

THEOREM 1.2 (Existential instantiation). Given c nowhere in s nor in $\mathcal{M}$ assumed finite.

$$\text{If } \mathcal{M} \cup \{\varphi(c)\} \vdash s \text{ then } \mathcal{M} \cup \{\exists x\ \varphi(c)(x \setminus c)\} \vdash s$$

PROOF. Clearly $\mathcal{M} \cup \{\neg s\} \vdash \neg\varphi(c)$. □

This is clearly a weaker form of instantiation which avoids inferring an actual instance. This ensures $\exists x\ \varphi(x) \nvdash \forall x\ \varphi(x)$.

DEFINITION 1.6 (Bound quantifiers).

$$\vdash (\forall\varphi(x)\ \gamma(x)) \leftrightarrow (\forall x\ \varphi(x) \to \gamma(x)) \tag{1}$$

$$\vdash (\exists\varphi(x)\ \gamma(x)) \leftrightarrow (\exists x\ \varphi(x)\ \&\ \gamma(x)) \tag{2}$$

COROLLARY 1.1. $\forall\varphi(x)\ \gamma(x), \exists x\ \varphi(x) \vdash \exists\varphi(x)\ \gamma(x)$

COROLLARY 1.2. $\neg\forall\varphi(x)\ \gamma(x) \vdash \exists\varphi(x)\ \neg\gamma(x)$

THEOREM 1.3 (Universal hypothesis). Given c nowhere in $\mathcal{M}$ finite.

$$\text{If } \mathcal{M} \cup \{\varphi(c)\} \vdash \gamma(c) \text{ then } \mathcal{M} \vdash \forall\varphi(x)\ \gamma(x).$$

The ubiquitous equality predicate is often introduced as a rule of inference inherent to the logic.

RULE 2.3 (Equality).

$$a = b \vdash \varphi(a) \leftrightarrow \varphi(b)$$

RULE 2.4 (Reflexivity of equality). $\vdash c = c$

This allows us to introduce a special form of existence.

DEFINITION 1.7 (Unique existence).

$$\exists! x\ \varphi(x) \equiv \exists\varphi(x)\ \forall\varphi(y)\ y = x$$
$$\exists!\varphi(x)\ \gamma(x) \equiv \exists! x\ \varphi(x)\ \&\ \gamma(x)$$

We may also embed the notion of a function into the logic.

DEFINITION 1.8. A binary relation ρ_f may define a dependent variable y as a function f of an independent variable x.

The domain of f is defined by formula δ, the codomain by formula κ. They consist of all constants c which we may substitute for the variable x, y respectively in the respective formula $\delta(x), \kappa(y)$ to prove the resulting sentence.

$$f : \delta \longrightarrow \kappa \equiv \forall x\ \forall y\ f(x) = y\ \rightarrow\ \forall y'\ f(x) = y'\ \rightarrow\ y' = y$$
$$\varphi(f(a)) \equiv \delta(a) \rightarrow \exists\kappa(y)\ \rho_f(a, y)\ \&\ \varphi(y)$$

A map $f : \delta \longmapsto \kappa \equiv f : \delta \longrightarrow \kappa\ \&\ \forall x\ \exists y\ f(x) = y$ is a function defined over its entire domain. Note that this is equivalent to $f : \delta \longmapsto \kappa \equiv \forall x\ \exists! y\ f(x) = y$.

A function is surjective $\mathrm{sur}(f)$: any variable in the codomain depends on some domain variable $\forall\kappa(y)\ \exists\delta(x)\ y = f(x)$.

A function is injective $f : \delta \longleftrightarrow \kappa \equiv f : \delta \longrightarrow \kappa\ \&\ f^{-1} : \kappa \longrightarrow \delta$ where the inverse $f^{-1}(b) = a \equiv b = f(a)$. Note $\forall x\ \forall x'\ f(x) = f(x') \rightarrow x' = x$ suffices.

Note that an injective function is surjective if the inverse is a map $f^{-1} : \kappa \longmapsto \delta$. An injective map is said to be bijective $\mathrm{bi}(f)$ only if it is surjective.

Reflecting equality $\mathrm{X} \equiv x = x$ lifts any restriction on the domain by unbinding the respective quantifier $\vdash \forall\mathrm{X}(x)\ \varphi(x) \leftrightarrow \forall x\ \varphi(x)$. Note that the identity function $f_=$ defined by the identity relation $y = x$ is a bijection that restricts neither domain nor codomain $\vdash f_= : \mathrm{X} \longleftrightarrow \mathrm{X}$.

DEFINITION 1.9. A multivariate map f of $n \in \mathbb{N}$ variables $x_i \in x^n$, each from a subdomain defined by property $\delta_i \in \delta^n$, is defined by a relation ρ_f of order $n+1$.

$$f : \delta^n \longrightarrow \kappa \equiv \forall_{i=1}^n x_i\ \exists! y\ f(x^n) = y$$
$$\varphi(f(c^n)) \equiv \&_{i=1}^n \delta_i(c_i) \rightarrow \exists\kappa(y)\ \rho_f(\ldots c^n, y)\ \&\ \varphi(y)$$

A binary map where both subdomains are equal to the codomain is known as an algebraic operation $f : \kappa^2 \longmapsto \kappa$.

2. Semantics

DEFINITION 2.1. The universe U consists of all objects. Even without bound we may not be able to enumerate all objects.

DEFINITION 2.2. An assignment consists of a map u and a set r such that there is a map $r^n \in r$ for each $n \in \mathbb{N}$.

(1) $u : \mathbf{c} \longrightarrow \mathrm{U}$. Each constant c is assigned an object $u(c)$.
(2) $r^n : \mathbf{P}^n \longrightarrow \wp(\mathrm{U}^n)$. Each predicate P of order n is assigned a set $r^n(P)$ of n-tuples of objects.

The constant codomain $u(\mathbf{c}) \subseteq \mathrm{U}$ is the set of all objects $\mathrm{o} \in \mathrm{U}$ which were assigned $\mathrm{o} = u(c) \in u(\mathbf{c})$ from some constant $c \in \mathbf{c}$.

Each constant $c_i \in c^n$ in an n-tuple $c^n \in \mathbf{c}^n$ may be assigned $u(c_i) = u(c^n)_i$ to form an n-tuple $u(c^n) \in u(\mathbf{c})^n \subseteq \mathrm{U}^n$ of objects from the constant codomain. We agree that $u(\mathbf{c})^0 = \mathrm{U}^0 = \{\emptyset\}$ such that $u^0 = \emptyset$.

We may suffice with $r(P) = r^n(P)$ to denote the assignment of a predicate $P \in \mathbf{P}^n$ of order n.

Definition 2.3. A structure $v, (u, r)$ consists of a model $v : \mathcal{S} \longrightarrow \{0, 1\}$ and an assignment u, r.

(1) The valuation of predicative sentences $\mathrm{P} \subset \Phi$ is interdeterminate with the assignment.

$$v(P(c^n)) = 1 \text{ only if } u(c^n) \in r(P)$$

We say that the n-tuple of objects $u(c^n)$ satisfies the predicate P.

(2) Quantifying sentences $\forall \subset \Phi$ are evaluated by universal consensus on each instance of the quantified formula between all structures $w, (t, r)$ interdeterminate with the same predicative assignment r.

$$v(\forall x\ \varphi(x)) = 1 \text{ only if } w(\varphi(c)) = 1 \text{ for } \begin{array}{l}\text{all constants } c \\ \text{all structures } w, (t, r)\end{array}$$

Definition 2.4. A sentence is a semantic consequence of a set of premises $\mathcal{M} \vDash s$ only if all structures $v, (u, r)$ have a model v that supports $(\mathcal{M}, s)$.

Definition 2.5. A model v may be interdeterminate with a total assignment u, R such that $R^n \in R$ for each $n \in \mathbb{N}$ where $R^n : \Phi^n \longrightarrow \wp(\mathrm{U}^n)$ assigns a set of n-tuples of objects to each sentence $s \in \Phi^n \subset \mathcal{S}$ with precisely n distinct constants in its scope $s = \varphi(c^n)$ where $c_i \neq c_j$ for all $c_i, c_j \in c^n$.

$$v(\varphi(c^n)) = 1 \text{ only if } u(c^n) \in R^n(\varphi)$$

Note that $R^n(\varphi(a^n)) = R^n(\varphi(b^n))$ for any a^n, $b^n \in u(\mathbf{c})^n$ such that $R^n(\varphi)$ defines a set of n-tuples of objects that satisfy the formula φ. Furthermore, $\bigcup_{n \in \mathbb{N}} \Phi^n = \mathcal{S}$.

(1) $R^n(\neg\varphi) = R^n(\varphi)^C$ where the complement is over U^n.

(2) $R^{m+n-k}(\varphi \to \gamma) = \left\{ u \in \mathrm{U}^{m+n-k} \;\middle|\; \begin{array}{r} [u]_1^m \in R^m(\neg\varphi) \\ \text{or } [u]_{m-k+1}^{m-k+n} \in R^n(\gamma) \end{array} \right\}$

where common constants are arranged from $m - k + 1$ to m if any.

(3) $R^n(\forall x\ \varphi(\dots c^n, x)) = \left\{ u^n \in \mathrm{U}^n \;\middle|\; \begin{array}{l} \text{all } u \in \mathrm{U} : \\ (\dots u^n, u) \in R^{n+1}(\varphi(\dots c^n, c)) \end{array} \right\}$

where $(\dots \emptyset, u) = u$ so $v(\forall x\ \varphi(x)) = 1$ only if $\emptyset = u^0 \in R^0(\forall x\ \varphi(x)) \subseteq \mathrm{U}^0 = \{\emptyset\}$ such that $R^1(\varphi(c)) = \mathrm{U}$.

Lemma 2.1. A model determines a nonempty set of structures of which it is the model.

Proof. Given any v an injective u ensures that any formula in every Φ^n has n distinctly assigned constants. This determines a total assignment R and thus a predicative assignment $r \subset R$. We show that $v, (u, r)$ is a structure by inductively decomposing and reconstructing between R and r where v is ever the interdeterminate model.

(1) $v(\neg\varphi) = 1$ only if $v(\varphi) = 0$ only if $u(c^n) \notin R(\varphi)$
(2) Similarly from implicational completeness.

(3) Evidently all structures $w, (t, r)$ come to universal consensus on a formula $w(\varphi(c)) = 1$ for all constants c if all objects satisfy it $R(\varphi) = \mathrm{U}$.

Conversely, there must be some object $q \in \mathrm{U}$ that does not satisfy the formula $q \notin R(\varphi)$ so there must be some structure $w, (t, r)$ assigning $t(c) = q$ some constant c such that $w(\varphi(c)) = 0$. This does not preclude $u(\mathbf{c}) \subseteq R(\varphi)$ so $v(\varphi(c)) = 1$ nonetheless for any c.

□

THEOREM 2.1 (Gödel). First order logic is semantically complete.

DEFINITION 2.6. A theory $\mathcal{T} \subseteq \mathcal{S}$ may be:
(ω-consistent). If $\exists x\ \varphi(x) \in \partial\mathcal{T}$, then $\varphi(c) \in \partial\mathcal{T}$ for some c.
(ω-complete). If $\varphi(c) \in \partial\mathcal{T}$ for all c, then $\forall x\ \varphi(x) \in \partial\mathcal{T}$.

Note that consistency follows from ω-consistency if it follows from any theory that is not trivial.

LEMMA 2.2. A consistent ω-consistent theory $\mathcal{T}$ which is ω-incomplete: $\varphi(c) \in \partial\mathcal{T}$ for all c yet $\forall x\ \varphi(x) \notin \partial\mathcal{T}$, is incomplete: $\neg\forall x\ \varphi(x) \notin \partial\mathcal{T}$.

PROOF. By consistency $\neg\varphi(c) \notin \partial\mathcal{T}$ for any c, so $\exists x\ \neg\varphi(x) \notin \partial\mathcal{T}$ by ω-consistency. □

COROLLARY 2.1. A model is ω-complete only if it is ω-consistent.

PROOF. Suppose v is ω-inconsistent. Then $v(\exists x\ \varphi(x)) = 1$ and $v(\varphi(c)) = 0$ so by completeness $v(\neg\varphi(c)) = 1$ for all c yet by consistency $v(\neg\exists x\ \varphi(x)) = 0$. □

COROLLARY 2.2. A consistent, ω-consistent yet ω-incomplete theory $\mathcal{T}$ is saturated by a distinct pair of models, one of which is ω-incomplete and thus ω-inconsistent.

PROOF. Any consistent theory may be saturated by some model, yet by completeness and consistency of any model there must be a saturating pair of models v, w such that $v(\forall x\ \varphi(x)) = 0 = w(\neg\forall x\ \varphi(x))$ and $v(\neg\forall x\ \varphi(x)) = 1 = w(\forall x\ \varphi(x))$.

Since $\varphi(c) \in \mathcal{T}$ so $v(\varphi(c)) = 1$ for all c, v is ω-incomplete. We may show its ω-inconsistency independently since $v(\exists x\ \neg\varphi(x)) = 1$ whereas by consistency $v(\neg\varphi(c)) = 0$ for all c. □

APPENDIX A

Arithmetic

Axiom A.1 (Succession). Each number has a single successor.

$$\mathrm{S} : \mathbb{N} \longmapsto \mathbb{N}$$

Axiom A.2 (Origin). 0 has no predecessor.

$$0 \neq \mathrm{S}(n)$$

Axiom A.3 (Predecession). Each number except 0 has a single predecessor.

$$\mathrm{S}^{-1} : n \neq 0 \longmapsto \mathbb{N}$$

Definition 0.1. A number c is standard: $c = \mathrm{S}^n(0)$ for some $n \in \mathbb{N}$.
An operation is correct: $o(\mathrm{S}^m(0), \mathrm{S}^n(0)) = \mathrm{S}^{o(m,n)}(0)$.

Theorem 0.1. None of the numbers preceding the successor of a standard number are its successor.

$$\mathrm{S}^n(0) \neq \mathrm{S}^{i<n}(0)$$

Proof. Should $\mathrm{S}^n(0) = \mathrm{S}^{i<n}(0)$ then $\mathrm{S}^{n-k}(0) = \mathrm{S}^{i-k}(0)$. Yet $\mathrm{S}^{n-i}(0) \neq 0$. □

Axiom A.4 (Addition).

$$+ : \mathbb{N}^2 \longmapsto \mathbb{N}$$

Axiom A.5 (Identity element of addition).

$$n + 0 = n$$

Axiom A.6 (Term succession).

$$m + \mathrm{S}(n) = \mathrm{S}(m + n)$$

Theorem 0.2. Addition is correct.

$$\mathrm{S}^m(0) + \mathrm{S}^n(0) = \mathrm{S}^{m+n}(0)$$

Proof. $\mathrm{S}^m(0) + \mathrm{S}^n(0) = \mathrm{S}^{k \leq n}(\mathrm{S}^m(0) + \mathrm{S}^{n-k}(0))$. □

Axiom A.7 (Multiplication).

$$\times : \mathbb{N}^2 \longmapsto \mathbb{N}$$

Axiom A.8 (Null element of multiplication).

$$n \times 0 = 0$$

Axiom A.9 (Factor succession).

$$m \times \mathrm{S}(n) = m \times n + m$$

Theorem 0.3. Multiplication is correct.

$$\mathrm{S}^m(0) \times \mathrm{S}^n(0) = \mathrm{S}^{m \times n}(0)$$

PROOF. $S^m(0) \times S^n(0) = S^m(0) \times S^{n-k}(0) + S^{\sum_{i=1}^{k} m}(0)$. □

AXIOM A.10 (Order).

$$m \leq n \leftrightarrow m = n \vee S(m) \leq n$$

DEFINITION 0.2. A sentence in the language of arithmetic is $\sum_0$: all quantifiers in the scope of the sentence are bound such that the binding condition can only be proven for a finite set of standard numbers.

DEFINITION 0.3. All numbers n strictly bounded $n < a$ by a number a may be considered symbols of an alphabet with a symbols. We construct the notion of a string of numbers from any such alphabet, taking 0 as the empty string representing any string whose only character is 0.

(strict order). $m < n \equiv m \leq n \;\&\; m \neq n$
(division). $m \mid n \equiv \exists i \leq n\; m \times i = n$
(prime). $\mathrm{prime}(n) \equiv 1 < n \;\&\; \neg\exists m < n\; 1 < m \;\&\; m \mid n$
(prime power). $n \text{ power } p \equiv 1 < p \;\&\; \forall m \leq n\; (1 < m \;\&\; m \mid n) \to p \mid m$
(concatenation). $i *_p j = n \equiv \exists k \leq n\; i \times k + j = n$
$\quad \&\; k \text{ power } p \;\&\; j < k \;\&\; \forall h < k\; h \text{ power } p \to h \leq j$
(endstring). $j \text{ ends}_p\; n \equiv (j = 0 \to (n = 0 \vee p \mid n)) \;\&\; \exists i \leq n\; i *_p j = n$
(startstring). $i \text{ starts}_p\; n \equiv i = n \vee \exists j < n\; i *_p j = n$
(substring). $m \text{ subs}_p\; n \equiv \exists i \leq n\; i \text{ starts}_p\; n \;\&\; m \text{ ends}_p\; i$
(monostring). $n \text{ monos}_p\; i \equiv \forall m \leq n\; m \text{ subs}_p\; n \to i \text{ subs}_p\; m$
(string). $\mathrm{string}_p^a(n) \equiv \mathrm{prime}(p) \;\&\; a \leq p \;\&\; \forall q < p\; \mathrm{prime}(q) \to q < a$
$\quad \&\; \forall i < p\; i \text{ subs}_p\; n \to i < a$

Note that $i * 0 = 0 * i = i$ yet 0 ends $i * 0$ only if 0 ends i and 0 starts $0 * i$ only if $i = 0$. The alphabet with no symbols has no strings, the alphabet with only the 0 symbol has only the empty string.

DEFINITION 0.4. We may designate a certain symbol $\sigma \in \alpha$ from any finite alphabet α as sequence delimiter to construct the notion of sequences nested evenly to any depth $\delta \in \mathbb{N}$.

(sequence). $\mathrm{seq}^{\sigma,\delta}(n) \equiv \neg(\ulcorner\sigma\urcorner^{\delta+1} \text{ subs } n) \;\&\; \exists m \leq n\; \ulcorner\sigma\urcorner^{\delta} * m * \ulcorner\sigma\urcorner^{\delta} = n$
(occurrence). $i \text{ inseq}^{\sigma,\delta}\; n \equiv \neg(\ulcorner\sigma\urcorner^{\delta} \text{ subs } i) \;\&\; \ulcorner\sigma\urcorner^{\delta} * i * \ulcorner\sigma\urcorner^{\delta} \text{ subs } n$
(precedence). $i \text{ preseq}_n\; j \equiv i \text{ inseq } n \;\&\; j \text{ inseq } n \;\&\; \exists m \leq n\; m \text{ starts } n \;\&$
$\quad j \text{ inseq } m \;\&\; \neg(i \text{ inseq } m)$

Clearly the sequence symbol does not occur in any element of a sequence.

DEFINITION 0.5. A formation sequence s shows recursively that a relation holds between a ν-tuple of strings n^ν from an alphabet of $a \leq p$ symbols.

$$\varphi(n^\nu) \equiv \begin{array}{l} \mathrm{final}(\sigma, p, f, n^\nu), \\ \mathrm{base}(\sigma, p, h, n^\nu), \\ \mathrm{condition}(\sigma, p, h, n^\nu, s) \end{array}$$

We ensure that the sequence symbol does not belong to the alphabet.

$$\begin{gathered} [\varphi(n^\nu)]_s^a \equiv \exists p < s\; \mathrm{prime}(p) \;\&\; a < S^{-1}(p) \;\&\; \mathrm{seq}^{S^{-1}(p),2}(s) \\ \&\; \exists f < s\; \mathrm{final}(S^{-1}(p), p, f, n^\nu) \;\&\; f \text{ inseq } s \;\&\; \forall h < s\; h \text{ inseq } s \\ \to \mathrm{base}(S^{-1}(p), p, h, n^\nu) \vee \mathrm{condition}(S^{-1}(p), p, h, n^\nu, s) \end{gathered}$$

The condition may be decomposed into κ cases, each of which $\text{case}_{\iota\leq\kappa}$ may cache Σ_ι values to justify its formation against Π_ι precedents.

$$\text{condition}(\sigma, p, h, n^\nu, s) \equiv \vee_{\iota=1}^{\kappa} \forall_{\lambda=1}^{\Sigma_\iota} h_\lambda < s \to (\text{case}_\iota(\sigma, p, h, n^\nu, h^{\Sigma_\iota}) \\ \to \exists_{\lambda=1}^{\Pi_\iota} g_\lambda < s \;\&\; g_\lambda \text{ preseq}_s\, h \\ \&\; \text{formation}_\iota(\sigma, p, h, n^\nu, h^{\Sigma_\iota}, g^{\Pi_\iota}))$$

Definition 0.6. We may construct uniform substitution $n = m(j \backslash i)$ by means of a formation sequence.

$$\begin{aligned} \text{final}(f, m) &\equiv f = m \\ \text{base}(h, n, i) &\equiv h = n \;\&\; \neg(i \text{ subs } h) \\ \text{case}(h, i, k, l) &\equiv h = k * i * l \\ \text{formation}(j, k, l, t) &\equiv t = k * j * l \end{aligned}$$

Note the order of formation whereby n is base and m final. This permits j to occur in m and ensures the unbroken chain of substitution from m to n.

Definition 0.7. We may represent constants as well as any predicate ρ of order δ.

(zero constant). $\text{zero}(n) \equiv n = \ulcorner \text{c} \urcorner$
(constant). $\text{const}(n) \equiv \text{zero}(n) \vee \exists m < n\; n = \ulcorner \text{c} \urcorner * m \;\&\; m \text{ monos } \ulcorner {}' \urcorner$
(variable). $\text{var}(n) \equiv n = \ulcorner \text{x} \urcorner \vee \exists m < n\; n = \ulcorner \text{x} \urcorner * m \;\&\; m \text{ monos } \ulcorner {}' \urcorner$
(predicative formula). $\text{pred}^{\rho,\delta}(n) \equiv \exists_{\iota=1}^{\delta} i_\iota \leq n \;\&_{\iota=1}^{\delta} (\text{const}(i_\iota) \vee \text{var}(i_\iota))$
$\quad \& \; n = \ulcorner \rho \urcorner * \ulcorner (\urcorner * (*_{\iota=1}^{\delta} i_\iota * (\iota \neq \delta \;\&\& \ulcorner , \urcorner)) * \ulcorner) \urcorner$

Then there is a formation sequence for well formed formulas.

$$\begin{aligned} \text{base}(h) &\equiv \vee_{\iota=1}^{|\pi|} \text{pred}^{\pi_\iota, \text{order}(\pi_\iota)}(h) \\ \text{case}_1(h, i, j) &\equiv h = \ulcorner \to \urcorner * \ulcorner (\urcorner * i * \ulcorner , \urcorner * j * \ulcorner) \urcorner \\ \text{case}_2(h, i) &\equiv h = \ulcorner \neg \urcorner * \ulcorner (\urcorner * i * \ulcorner) \urcorner \\ \text{case}_3(h, m, c, x, n) &\equiv \text{const}(c) \;\&\; c \text{ subs } m \;\&\; \text{var}(x) \;\&\; \neg(x \text{ subs } m) \\ &\quad \& \; n = m(x \setminus c) \;\&\; h = \ulcorner \forall \urcorner * x * \ulcorner (\urcorner * n * \ulcorner) \urcorner \end{aligned}$$

Definition 0.8. We may construct exponentiation by means of a formation sequence.

$$\begin{aligned} \text{final}(\sigma, p) &\equiv m *_p \sigma *_p n \\ \text{base}(\sigma, p, h) &\equiv h = 0 *_p \sigma *_p \mathrm{S}(0) \\ \text{case}(h, i, l) &\equiv h = i *_p \sigma *_p l \\ \text{formation}(i, l, g) &\equiv \exists k < g\; g = \mathrm{S}^{-1}(i) *_p \sigma *_p l \;\&\; l = k \times b \end{aligned}$$