Exponential Object in Geb

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Theorem. Geb category is equivalent to FinSet via a functor $GF: Geb \rightarrow FinSet$ defined on objects by induction:

$$\begin{split} GF(T) &:= \mathbf{1} \\ GF(I) &:= \mathbf{0} \\ GF(X \oplus Y) &:= GF(X) + GF(Y) \\ GF(X \otimes Y) &:= GF(X) \times GF(Y) \end{split}$$

and preserving all categorical structure on the nose, as well as the distribution morphism.

Definition. For $X, Y \in Obj(\texttt{Geb})$ define $X \Rightarrow Y \in Obj(\texttt{Geb})$ by induction:

$$T \Rightarrow Y := Y$$

$$I \Rightarrow Y := T$$

$$(A \oplus B) \Rightarrow Y := (A \Rightarrow Y) \otimes (B \Rightarrow Y)$$

$$(A \otimes B) \Rightarrow Y := (A \Rightarrow (B \Rightarrow Y))$$

Proposition. The above object is an exponential, i.e. $GF(A \Rightarrow B) \cong GF(A)^{GF(B)}$

Proof. By induction:

$$GF(T \Rightarrow B) := GF(B) \cong GF(B)^{1} := GF(B)^{GF(T)}$$

$$GF(I \Rightarrow B) := GF(T) := \mathbf{1} \cong GF(B)^{0} := GF(B)^{GF(I)}$$

$$GF(A \oplus B \Rightarrow Y) := GF((A \Rightarrow Y) \otimes (B \Rightarrow Y))$$

$$\cong GF(A \Rightarrow Y) \times GF(B \Rightarrow Y)$$

$$\cong GF(Y)^{GF(A)} \times GF(Y)^{GF(B)} \quad \text{By induction}$$

$$\cong GF(Y)^{GF(A) \times GF(B)}$$

$$GF(A \otimes B \Rightarrow Y) := GF(A \Rightarrow (B \Rightarrow Y))$$

$$\cong GF(B \Rightarrow Y)^{GF(A)}$$

$$\cong (GF(Y)^{GF(B)})^{GF(A)}$$

$$\cong (GF(Y)^{GF(B) \times GF(A)}$$

$$\cong GF(Y)^{GF(B) \times GF(A)}$$

$$\coloneqq GF(Y)^{GF(B) \times GF(A)}$$

$$\coloneqq GF(Y)^{GF(B) \times GF(A)}$$

For computational details, look in geb.agda

Definition. Define $evalG: (Y \Rightarrow X) \otimes Y \rightarrow X$ by induction:

$$(T \Rightarrow X) \otimes T \xrightarrow{evalG} X$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad X \times T$$

that is, $eval(G) := \lambda(x, *).x$, arguing in **FinSet**

$$X^I \otimes I \xrightarrow{evalG} X$$

that is, evalG is the unique morphism from the initial object: evalG(x, i) := !(i)

$$(A\Rightarrow X)\otimes (B\Rightarrow X)\otimes (A\oplus B) \xrightarrow{evalG} X$$

$$[((A\Rightarrow X)\otimes (B\Rightarrow X))\otimes A]\oplus [((A\Rightarrow X)\otimes (B\Rightarrow X))\otimes B]$$

where D is the distributivity iso, in **FinSet** presented as:

$$\lambda(f, g, inl(a)) = inl(f, g, a)$$

$$\lambda(f, g, inr(b)) = inr(f, g, b)$$

and

$$((A \Rightarrow X) \otimes (B \Rightarrow X)) \otimes A \xrightarrow{fM} (A \Rightarrow X) \otimes A \xrightarrow{evalG} X$$

 $\lambda(f,g,a).GF(evalG(f,a))$ in ${\it FinSet}$

$$((A \Rightarrow X) \otimes (B \Rightarrow X)) \otimes A \xrightarrow{fL} (B \Rightarrow X) \otimes B \xrightarrow{evalG} X$$

 $\lambda(f,g,b).(eval(f,b))$ in **FinSet**

In other words, GF(evalG) here is equivalently

$$\lambda(f, g, (inla)) = eval(f, a)$$

$$\lambda(f,g,(inrb)) = eval(g,b)$$

$$((A\Rightarrow (B\Rightarrow X))\otimes (\underbrace{A\otimes B}) \xrightarrow{evalG} X$$

$$(B\Rightarrow X)\otimes B$$

where

$$((A \Rightarrow (B \Rightarrow X)) \otimes (A \underbrace{\otimes B}) \xrightarrow{\lambda(f,a,b).(f,a)} (A \Rightarrow (B \Rightarrow X)) \otimes A \xrightarrow{evalG} (B \Rightarrow X)$$

$$((A \Rightarrow (B \Rightarrow X)) \otimes \underbrace{(A \otimes B) \xrightarrow{\lambda(f,a,b).b} B \xrightarrow{evalG} (B \Rightarrow X)}_{p1}$$

working in FinSet (in Geb this is of course replaced by projections)

so in **FinSet** this is $\lambda(f, a, b).evalG((eval(f, a)), b)$

Proposition. GF(evalG) is isomorphic to eval, i.e.

$$GF((A\Rightarrow B)\otimes A) \xrightarrow{\cong} GF(A)^{GF(B)} \times GF(A)$$

$$GF(evalG) \xrightarrow{eval} GF(B)$$

Proof. By induction. See exp.agda

Definition. We define $curryG : Hom(X \otimes Y, Z) \to Hom(X, (Y \Rightarrow Z))$ by induction

Let $f \in Hom(X \otimes Y, Z)$

for Y := T, $curryG : Hom(X \otimes T, Z) \rightarrow Hom(X, Z)$ is defined as

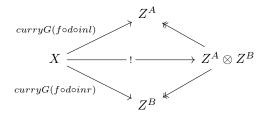
$$curryG(f) = f \circ \langle 1_X, !_X \rangle$$

 $\lambda(x).f(x,*)$ in **FinSet**

for Y := I, $curryG : Hom(X \otimes I, Z) \to Hom(X, T)$ is defined by

$$curryG(f) := !_X$$

for $Y := A \oplus B$ then $curryG : Hom(X \otimes (A \oplus B), Z) \to Hom(X, Z^A \otimes Z^B)$ is defined as



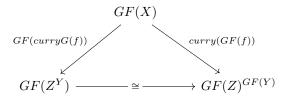
where d is the distributivity morphism $d:(X \otimes Y) \oplus (X \otimes Z) \to X \otimes (Y \oplus Z)$. In **FinSet** it becomes the usual:

$$\lambda(inl(x,y)).(x,inl(y))$$

 $\lambda(inr(x,z)).(x,inr(z))$

for $Y := A \otimes B \ curryG : Hom(X \otimes (A \otimes B)) \to Hom(X, Z^{B^A})$ is defined as

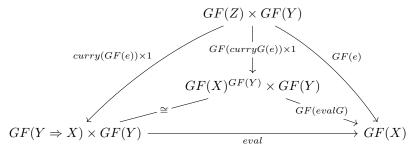
Proposition. GF(curryG(f)) for each f is ismorphic to curry(f):



Proof. By induction. See exp.agda

Since equivalences reflect equalities and are fully faithful, and GF preserves product structure on the nose, it suffices to show that GF(curryG) has the universal property (in terms of making the needed triangle commute) of currying in order to confirm that it is the correct currying function for Geb

Yet this follows as the statement corresponds to the right wall of the following tetrahedron:



as we know the commutativity of all the other sides of the tetrahedron, we are done