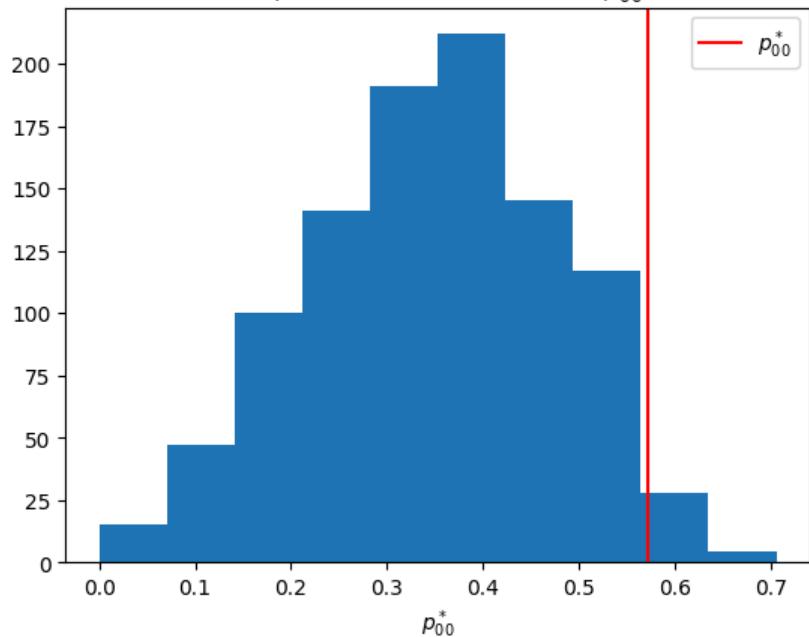


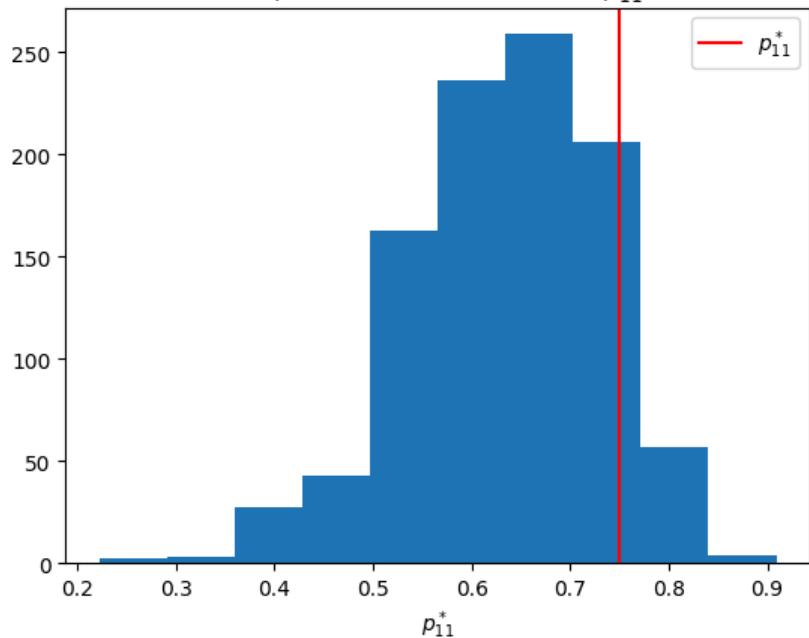
HW1

b)

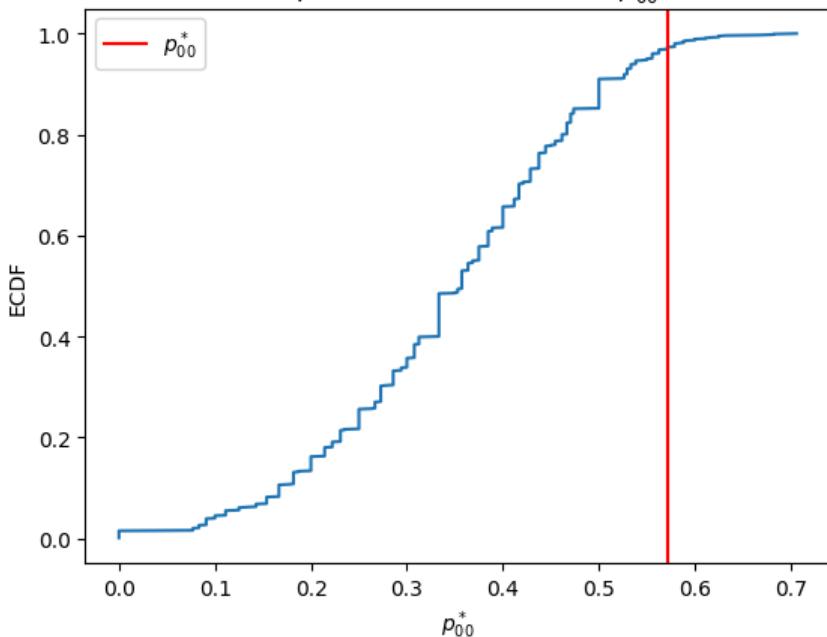
Empirical null distribution for p_{00}^*



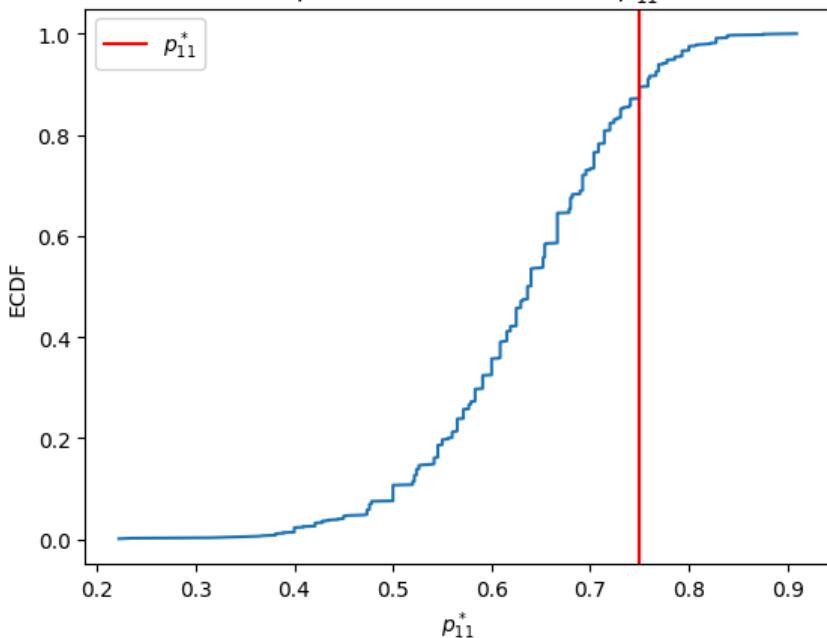
Empirical null distribution for p_{11}^*



Empirical cdf distribution for p_{00}^*



Empirical cdf distribution for p_{11}^*



For p_{00} , the empirical p-value of 0.027 indicates that the observed transition probability is unlikely to arise from an i.i.d. sequence, providing evidence for a first-order Markov model. Assuming an alpha of 0.05 we reject our null of a simple coin flip model.

For p_{11} , the empirical p-value of 0.108 indicates that we do not have statistical significance, assuming an alpha of 0.05, to reject the null hypothesis that the data was generated via a simple coin flip model.

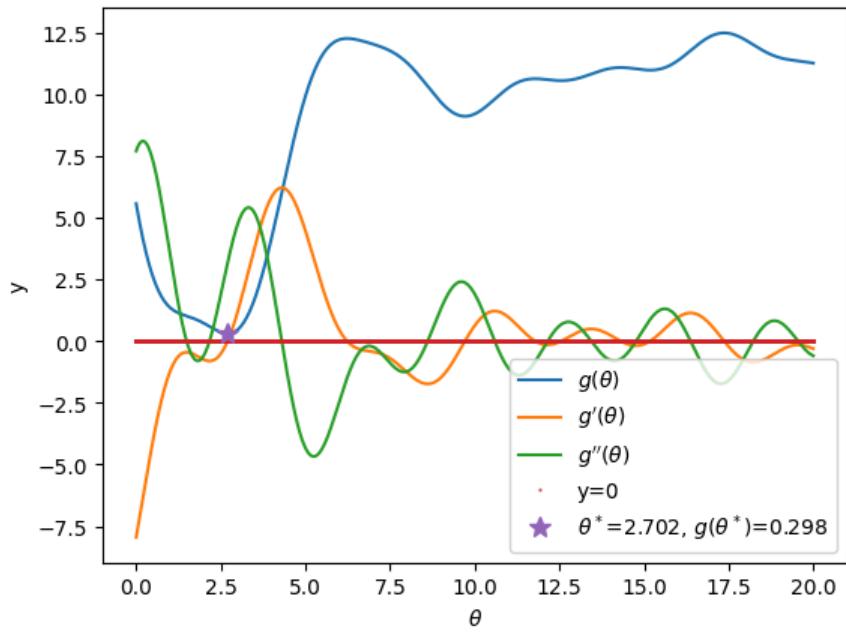
HW2

b)

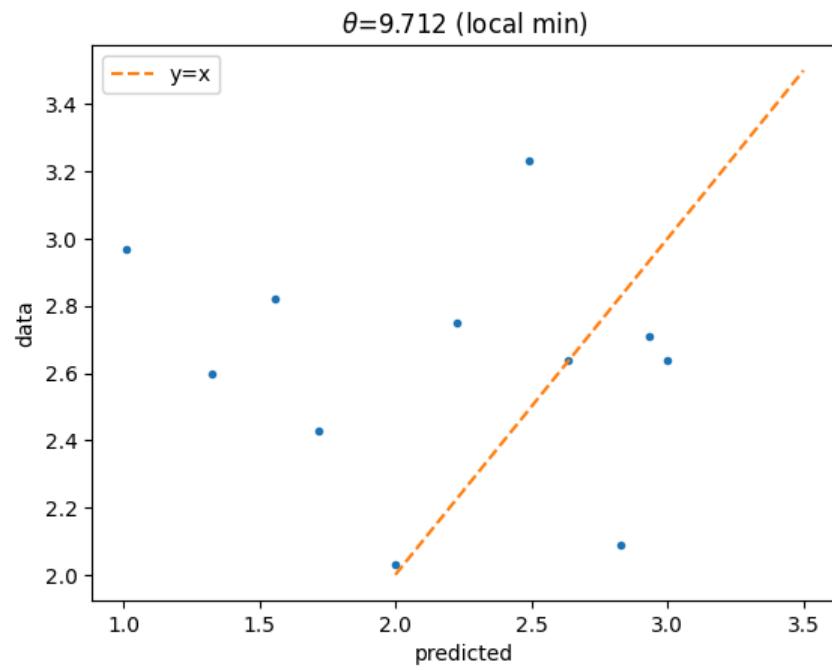
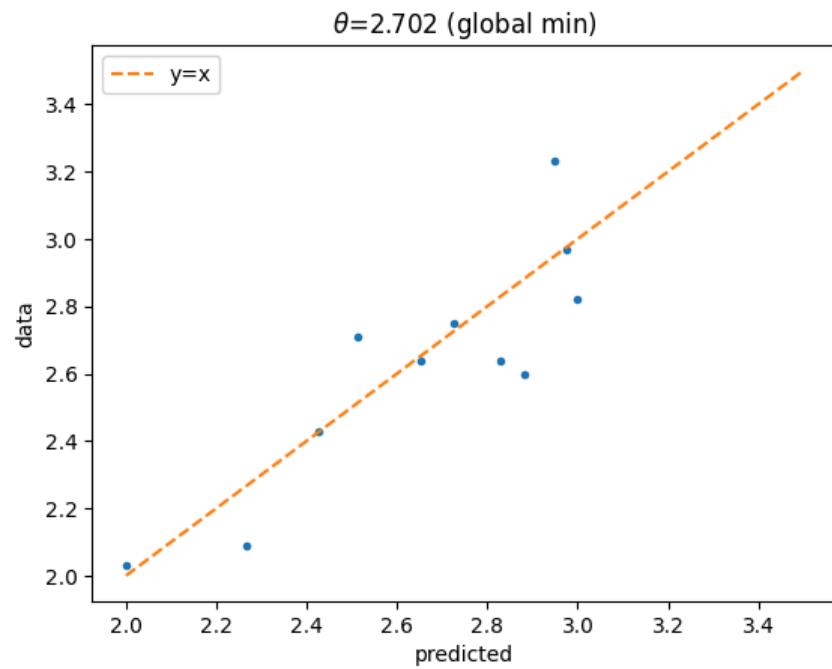
All optima are: [0.298, 9.104, 10.547, 10.971]

Respectively, the theta values corresponding to all optima are: [2.702, 9.712, 12.553, 15.181]

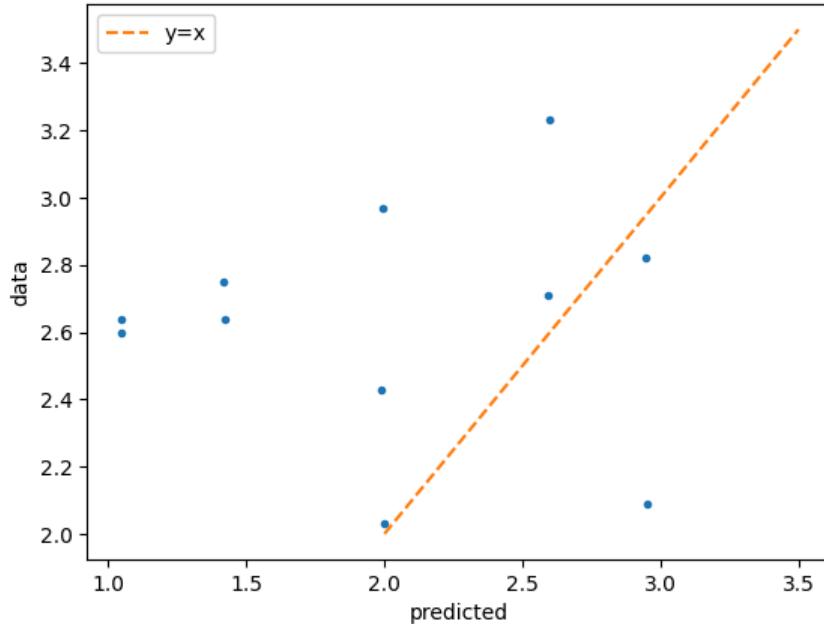
<matplotlib.legend.Legend at 0x113d109a0>



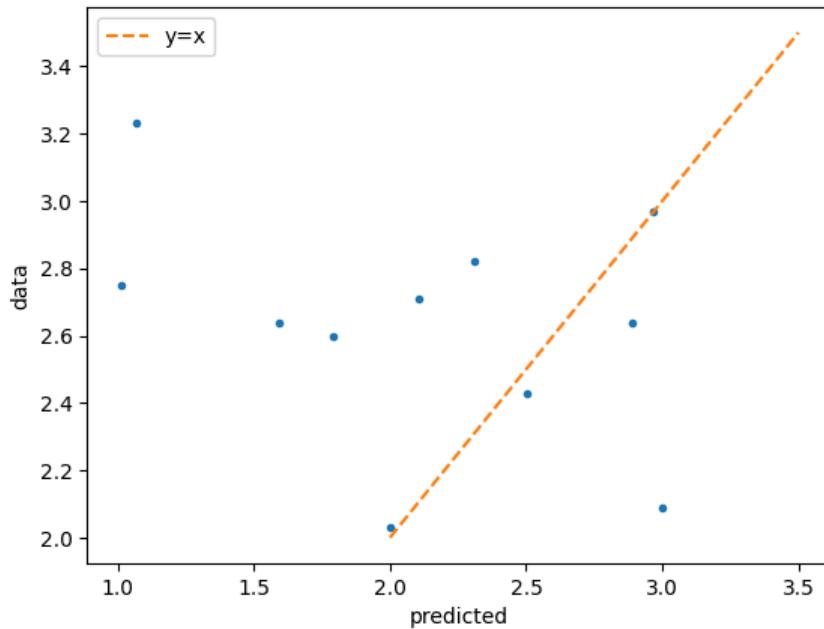
c)



$\theta=12.553$ (local min)



$\theta=15.181$ (local min)



Newton method: $\theta=2.702120$, $g(\theta)=0.298142$

Global minimum value parameter for the objective function from the plot: $\theta=2.702103$, $g(\theta)=0.298142$

'The solution for the objective function θ^* associated with the local minimum after running the Newton method with the starting point θ_0 equals to 10: $\theta=9.711984372034813$, $g(\theta)=9.10391889626$ '

As we see, the Newton method executed for the starting point $\theta_0 = 3$ aligns with the values of the parameter θ associated with the global minimum, and finds the global optimum correctly.

Despite this fact, the Newton method does not guarantee to find the global optimum as it depends on the starting point. It might find the local minimum which does not model our data correctly. For example, for starting $\theta_0 = 10$, it finds the local optimum: $g(\theta^*) \approx 9.1$

HW3

$$y_i = x_{i|\theta} + \eta_i \text{ and } \eta_i \sim \text{Norm}(0, \sigma_i^2)$$

$$L(y_i|\theta) = \prod_{i=1}^n p(y_i|\theta)$$

$$\text{where } p(y_i|\theta) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(y_i - x_{i|\theta})^2}{2\sigma_i^2}\right)$$

$$L(y_i|\theta) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(y_i - x_{i|\theta})^2}{2\sigma_i^2}\right) = \frac{1}{(2\pi)^{\frac{n}{2}} \sigma_1 \dots \sigma_n} \exp\left(-\sum_{i=1}^n \frac{(y_i - x_{i|\theta})^2}{2\sigma_i^2}\right)$$

$$l(y_i|\theta) = \log L(y_i|\theta) = \underbrace{\frac{n}{2} \log(\frac{1}{2\pi})}_{constant} - \sum_{i=1}^n \log(\sigma_i) - \sum_{i=1}^n \frac{(y_i - x_{i|\theta})^2}{2\sigma_i^2}$$

2 is a constant

$$\text{As } \theta^* = \arg \min_{\theta} (-l(y_i|\theta))$$

Therefore

$$\text{As } \theta^* = \arg \min_{\theta} (\sum_{i=1}^n \frac{(y_i - x_{i|\theta})^2}{\sigma_i^2})$$

$$\text{The objective function is } g(\theta) = \sum_{i=1}^n \frac{(y_i - x_{i|\theta})^2}{\sigma_i^2}$$

HW4

$$N = 2$$

$$t = (1, 3)$$

$$y = (6, 7)$$

$$(\theta_1, \theta_2)^\top = (1, 1)$$

$$x_i = \theta_1 e^{\theta_2 t_i}$$

$$g(\theta) = \sum_{i=1}^N (x_{i|\theta} - y_i)^2 = \sum_{i=1}^N (\theta_1 e^{\theta_2 t_i} - y_i)^2$$

$$\frac{dg(\theta)}{d\theta_1} = 2 \sum_{i=1}^N (\theta_1 e^{\theta_2 t_i} - y_i) e^{\theta_2 t_i} = 2 \sum_{i=1}^N (1 e^{1t_i} - y_i) e^{t_i} = 2(e - 6)e + 2(e^3 - 7)e^3$$

$$\frac{dg(\theta)}{d\theta_2} = 2 \sum_{i=1}^N (\theta_1 e^{\theta_2 t_i} - y_i) e^{\theta_2 t_i} \theta_1 t_i = 2 \sum_{i=1}^N (e^{t_i} - y_i) e^{t_i} t_i = 2e(e - 6) + 6e^3(e^3 - 7)$$

$$\frac{d^2g(\theta)}{d\theta_1^2} = 2 \sum_{i=1}^N e^{\theta_2 t_i} = 2e^2 + 2e^{2 \cdot 3} = 2(e^2 + e^6)$$

$$\frac{d^2g(\theta)}{d\theta_1 d\theta_2} = 2 \sum_{i=1}^N (\theta_1 e^{2\theta_2 t_i} 2t_i - y_i e^{\theta_2 t_i} t_i) = 2 \sum_{i=1}^N (1 e^{2t_i} 2t_i - y_i e^{1t_i} t_i) = 2(2e^2 - 6e) + 2(6e^6 - 7 \cdot 3e^3) = 2(2e^2 - 6e) + 2(6e^6 - 21e^3)$$

$$\frac{d^2g(\theta)}{d\theta_2^2} = 2 \sum_{i=1}^N (\theta_1^2 e^{2\theta_2 t_i} 2t_i^2 - y_i \theta_1 e^{\theta_2 t_i} t_i^2) = 2(1e^2 2 - 6e) + 2(e^{2 \cdot 3} 2 \cdot 9 - 7e^3 9) = 2(2e^2 - 6e + 18e^6 - 63e^3)$$

```
d g(theta)/ d theta1: 507.8188003171955
d g(theta)/ d theta2: 1559.1389404388808
```

```
d^2 g(theta)/ d theta1^2: 821.6356991833313
d^2 g(theta)/ d theta1 d theta2: 3994.4898135931526
d^2 g(theta)/ d theta1^2: 11989.595755871029
```

$$H = \begin{bmatrix} \frac{d^2g(\theta)}{d\theta_1^2} & \frac{d^2g(\theta)}{d\theta_1 \theta_2} \\ \frac{d^2g(\theta)}{d\theta_1 \theta_2} & \frac{d^2g(\theta)}{d\theta_2^2} \end{bmatrix} = \begin{bmatrix} 821.6 & 3994.5 \\ 3994.5 & 11989.6 \end{bmatrix}$$

If we consider H as: $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$$H^{-1} = \frac{1}{a \cdot c - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} = \frac{1}{821.6 \cdot 11989.6 - 3994.5^2} \begin{bmatrix} 11989.6 & -3994.5 \\ -3994.5 & 821.6 \end{bmatrix} = \frac{1}{-6105374.8} \begin{bmatrix} 11989.6 & -3994.5 \\ -3994.5 & 821.6 \end{bmatrix} = \begin{bmatrix} -0.001964 & 0.00065426 \\ 0.00065426 & -0.0001346 \end{bmatrix}$$

$$\theta^{s+1} = \theta^s + \Delta\theta = \theta^s - H^{-1} \cdot \nabla g(\theta^s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -0.001964 & 0.00065426 \\ 0.00065426 & -0.0001346 \end{bmatrix} \begin{bmatrix} 507.82 \\ 1559.14 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.0228 \\ 0.1224 \end{bmatrix} = \begin{bmatrix} 0.9772 \\ 0.8776 \end{bmatrix}$$