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Multivariate distribution defined with Farlie–Gumbel–Morgenstern copula and mixed Erlang marginals: Aggregation and capital allocation

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HIGHLIGHTS

- We consider a portfolio of dependent risks.
- The joint distribution is defined with the FGM copula.
- Mixed Erlang distributions are assumed for the marginals.
- We show that the aggregate claim amount has a mixed Erlang distribution.
- The contributions of each risk are derived using the TVaR and covariance rules.

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ABSTRACT

In this paper, we investigate risk aggregation and capital allocation problems for a portfolio of possibly dependent risks whose multivariate distribution is defined with the Farlie–Gumbel–Morgenstern copula and mixed Erlang distribution marginals. In such a context, we first show that the aggregate claim amount has a mixed Erlang distribution. Based on a top-down approach, closed-form expressions for the contribution of each risk are derived using the TVaR and covariance rules. These findings are illustrated with numerical examples.

1. Introduction

In light of the new regulation requirements, insurance companies are required to determine their capital allocation according to their risk exposure. In such a context, risk management raises some issues about risk aggregation and capital allocation. A risk capital must be held by the institution for the whole business portfolio to insure a safety financial level and also to be allocated adequately to each risk. Required capitals are commonly determined using an adequate risk measure, which is a mapping from the random variable space into the real numbers that allows risk ordering. Artzner et al. (1999) give an axiomatic definition of a risk measure and introduce the concept of coherent measures of risk. Artzner (1999) examines the implication of using coherent risk measures on capital requirements in an insurance context. As for Wang (2002), he notably discusses coherent methods to determine the aggregate capital requirement for a firm and the capital allocation to individual business units. These methods for enterprise risk management can be used for asset/loss portfolio optimization. Both Artzner (1999) and Wang (2002) suggest using the Tail Value at Risk (TVaR), also called the Expected Shortfall (ES), to replace the usual Value at Risk given that it does not meet the subadditivity criterion. The TVaR is a coherent risk measure and it is equal to the Conditional Tail Expectation (CTE) in the continuous case. In a discrete setting, as explained in Acerbi et al. (2001) and Acerbi and Tasche (2002), the TVaR remains a coherent risk measure while the CTE is no longer coherent. See also McNeil et al. (2005) for details on risk measures

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and their applications in a quantitative risk management context. For a recent discussion and further details on the use and origin of the VaR and TVaR measures of risk, and the families they belong too, the reader is referred to Goovaerts et al. (2010).

To allocate capital to different lines of business, Denault (2001) suggests a set of desirable properties for a fair risk capital allocation principle. More precisely, his axioms define the coherence of risk capital allocation principles, in a similar way as Artzner et al. (1999) in the context of risk measures. The top down allocation method introduced by Tasche (2000) has been used to provide several closed-form formulas and approximations of the TVaR and the TVaR-based allocations for different types of multivariate continuous distributions. For example, Panjer (2002) shows that the TVaR-based allocation principle is identical to the covariance-based principle when multivariate normal distributions are considered. Dhaene et al. (2008) develop a closed-form expression for the TVaR allocation under multivariate elliptical distributions. Bargès et al. (2009) give a closed-form expression for the TVaR-based allocation when lines of business of an insurance portfolio are linked with a Farlie–Gumbel–Morgenstern (FGM) copula and when marginal risks are distributed as mixtures of exponentials. Other applications of the TVaR-based allocation principle are also provided in Cossette et al. (2012). Buch and Dorfleitner (2008) discuss the gradient allocation principle which generalizes well known allocation principles including the TVaR and covariance rules. The risk aggregation problem using VaR and TVaR, and the key differences between these two measures of risk, have been studied thoroughly in Kaas et al. (2009).

In this paper, we address risk aggregation and capital allocation problems for a portfolio of dependent risks whose multivariate distribution is defined with a copula and mixed Erlang distributed marginals. The class of mixed Erlang distributions has many interesting features which are discussed in detail notably in Willmot and Lin (2010) (see also references therein). With several examples, they illustrate the versatility and the usefulness of this class of distributions for modeling claim amounts and the availability of closed-form expressions for various quantities of interest in risk theory. Any positive continuous distribution may be approximated by a member of the mixed Erlang class which includes distributions such as the generalized Erlang distribution and phase-type distributions. Furthermore, the mixed Erlang class provides many possible shapes of probability density functions (pdf) and a variety of skewness in the right tail. These features make this class very useful in actuarial and risk management applications. Lee and Lin (2010) suggest the use of mixed Erlang distributions to model insurance losses. For the dependence structure, we use the FGM copula which is attractive due to its simplicity and hence allows explicit results. This copula is a perturbation of the independence copula. In this paper, we capitalize on the ability to write a joint pdf defined by an FGM copula in terms of the product of marginal pdfs and their corresponding survival functions. We combine the tractability of the FGM copula and the mixed Erlang distribution class to analyze the stochastic behavior of the aggregate claim amount for a portfolio of dependent risks in order to determine the amount of economic capital needed for the whole portfolio. More precisely, we show under these assumptions that the aggregate claim amount follows a mixed Erlang distribution. Then, we find closed-form expressions for the corresponding TVaR risk measure and stop-loss premium. Based on a top-down approach, explicit expressions for the amount of capital to be allocated to each risk based on the TVaR and covariance rules are derived.

The paper is organized as follows. Section 2 fixes some notations, definitions and provides some preliminary results. The expressions for the bivariate case are given in Section 3 together with essential theorems and an illustrative example. Section 4 shows how to extend the results to the multivariate case and provides a numerical example for the trivariate case.

2. Definitions

In this section, we briefly recall the definition and characteristics of the FGM copula. We also give the definitions of the risk measures Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) as well as the allocation rules based on the TVaR and the covariance (see e.g. McNeil et al., 2005). We end this section by presenting the mixed Erlang distribution and its properties.

2.1. Farlie–Gumbel–Morgenstern copula

Let $\underline{X} = (X_1, \ldots, X_n)$ be a vector of n continuous random variables (rvs) with joint cumulative distribution function (cdf) denoted by $F_{\underline{X}}$ and univariate marginals F_{X_i}, $i = 1, \ldots, n$. According to Sklar's theorem, see e.g. Sklar (1959) and Nelsen (2006), $F_{\underline{X}}$ can be written as a function of the univariate marginals F_{X_i}, $i = 1, \ldots, n$, and the copula C describing the dependence structure as follows:

$$F_{\underline{X}}(x_1, \ldots, x_n) = C\left(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\right).$$

The joint probability density function (pdf) of $\underline{X}$ is given by

$$f_{\underline{X}}(x_1, \ldots, x_n) = f_{X_1}(x_1) \ldots f_{X_n}(x_n)\, c\left(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\right), \quad (1)$$

where c is the corresponding pdf of the copula C defined by

$$c(u_1, \ldots, u_n) = \frac{\partial C(u_1, \ldots, u_n)}{\partial u_1 \ldots \partial u_n}.$$

In this paper, we are interested in the FGM copula. The bivariate FGM copula is defined by the joint cdf

$$C(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2), \quad (2)$$

where the scalar θ is the dependence parameter with $\theta \in [-1, 1]$. The independence structure is reached when $\theta = 0$, i.e. $C^I(u_1, u_2) = u_1 u_2$. The pdf of the bivariate FGM copula is given by

$$c(u_1, u_2) = (1 + \theta) - \theta 2\overline{u}_1 - \theta 2\overline{u}_2 + \theta 2\overline{u}_1 2\overline{u}_2, \quad (3)$$

where $\overline{u}_i = 1 - u_i$.

The FGM copula is a perturbation of the product copula and, as mentioned in Nelsen (2006), is a first order approximation to the Ali Mikhail Haq, Frank and Placket copulas. With $\theta \in [-1, 1]$ and association measures such as Kendall's tau and Spearman's rho respectively given by $\tau = \frac{2\theta}{9}$ and $\rho = \frac{\theta}{3}$, moderate positive and negative dependence can be modeled with the FGM copula. This copula is attractive due to its simplicity and its form which allows explicit calculus and exact results. For example, Bargès et al. (2009) investigate aggregation and capital allocation problems for an insurance company with several lines of business with dependence structure based on the FGM copula and with exponentially distributed risks. Prieger (2002) highlights its usefulness in model selection into health insurance plans. The FGM copula was also used to link claim variables in a credibility model in Yeo and Valdez (2006). The FGM copula with exponential margins was proposed by Jang and Fu (2011) to measure tail dependence between collateral losses. In finance, Cherubini et al. (2011) use the FGM copula for the analysis of financial time series and suggest a new technique to construct first order Markov processes using this copula. The FGM copula was also used to describe different correlation relations on the financial markets in Gatfaoui (2005, 2007). In risk theory, Cossette et al. (2008), Zhang and Yang (2011) and Chadjiconstantinidis and Vrontos (2012) consider risk models with a dependence structure between claim sizes and interclaim times based on the FGM copula.

In the multivariate case, for $n \geq 3$, the FGM n-copula, which has $2^n - n - 1$ parameters, is defined as follows:

$$C\left(\underline{u}\right) = u_1 u_2 \ldots u_n \left(1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \cdots < j_k \leq n} \theta_{j_1 j_2 \ldots j_k} \overline{u}_{j_1} \overline{u}_{j_2} \ldots \overline{u}_{j_k}\right),$$

which is equivalent to

$$C\left(\underline{u}\right) = u_1 u_2 \ldots u_n P\left(u_1, u_2, \ldots, u_n\right), \tag{4}$$

where P denotes the polynomial

$$P\left(u_1, u_2, \ldots, u_n\right) = 1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \cdots < j_k \leq n} \theta_{j_1 j_2 \ldots j_k} \overline{u}_{j_1} \overline{u}_{j_2} \ldots \overline{u}_{j_k}. \tag{5}$$

The pdf of the FGM n-copula is given by

$$c\left(\underline{u}\right) = P\left(2u_1, 2u_2, \ldots, 2u_n\right), \tag{6}$$

where the polynomial P is linear in argument u_i, for $i = 1, \ldots, n$.

According to Nelsen (2006), the FGM n-copula exists if the following constraints hold:

$$1 + \sum_{k=2}^{n} \sum_{1 \leq j_1 < \cdots < j_k \leq n} \theta_{j_1 j_2 \ldots j_k} \varepsilon_{j_1} \varepsilon_{j_2} \ldots \varepsilon_{j_k} \geq 0,$$

$$\text{where } \varepsilon_{j_1} \varepsilon_{j_2} \ldots \varepsilon_{j_k} \in \{-1, 1\}. \tag{7}$$

We also mention that, for $k \in \{2, 3, \ldots, n\}$, each k-margin of an FGM n-copula is an FGM k-copula.

2.2. Risk measures and risk allocation

Let us consider the aggregate claim amount rv $S_n = \sum_{i=1}^{n} X_i$ with cdf F_{S_n}. The Value-at-Risk at level κ, $0 \leq \kappa < 1$, of S_n is defined by

$$\mathrm{VaR}_\kappa\left(S_n\right) = \inf\left\{x \in \mathbb{R} : F_{S_n}(x) \geq \kappa\right\},$$

and the Tail-Value-at-Risk at level κ, $0 \leq \kappa < 1$, is defined by

$$\begin{aligned} \mathrm{TVaR}_\kappa\left(S_n\right) &= \frac{1}{1-\kappa} \int_\kappa^1 \mathrm{VaR}_u\left(S_n\right) du \\ &= \frac{E\left[S_n \times 1_{\{S_n > \mathrm{VaR}_\kappa(S_n)\}}\right] + \mathrm{VaR}_\kappa\left(S_n\right)\left(F_{S_n}\left(\mathrm{VaR}_\kappa\left(S_n\right)\right) - \kappa\right)}{1-\kappa}, \end{aligned} \tag{8}$$

where $1_{\{S_n > b\}}$ is the indicator function such that $1_{\{S_n > b\}} = 1$, if $S_n > b$, and $1_{\{S_n > b\}} = 0$, if $S_n \leq b$. Note that the truncated expectation of S_n, denoted by $E\left[S_n \times 1_{\{S_n > b\}}\right]$, can be expressed as $E\left[S_n\right] - E\left[S_n \times 1_{\{S_n \leq b\}}\right]$. See e.g. Acerbi and Tasche (2002) and McNeil et al. (2005) for details on the risk measures VaR and TVaR.

When the rv S_n is continuous, we have $F_{S_n}\left(\mathrm{VaR}_\kappa\left(S_n\right)\right) - \kappa = 0$ and (8) becomes

$$\begin{aligned} \mathrm{TVaR}_\kappa\left(S_n\right) &= \frac{E\left[S_n \times 1_{\{S_n > \mathrm{VaR}_\kappa(S_n)\}}\right]}{1-\kappa} \\ &= E\left[S_n | S_n > \mathrm{VaR}_\kappa\left(S_n\right)\right], \end{aligned} \tag{9}$$

which means that the Tail-Value-at-Risk of a continuous rv is equal to its conditional tail expectation (which is not the case generally). In the present work, we prefer to use the term TVaR which is always coherent rather than the term CTE.

Risk measures allow us to determine the amount of capital needed for the whole portfolio but one also needs to find the part of the capital that is allocated to each risk. For that purpose, we consider two capital allocation rules: the TVaR-based allocation rule and the covariance-based allocation rule.

The aim of capital allocation is to determine the amount of contribution C_i that is allocated to risk i ($i = 1, \ldots, n$) such that

$$\mathrm{TVaR}_\kappa\left(S_n\right) = \sum_{i=1}^{n} C_i.$$

Under the TVaR-based allocation rule, the amount of contribution C_i, denoted by $\mathrm{TVaR}_\kappa\left(X_i, S_n\right)$, is given by

$$\begin{aligned} C_i &= \mathrm{TVaR}_\kappa\left(X_i, S_n\right) \\ &= \frac{E\left[X_i 1_{\{S_n > \mathrm{VaR}_\kappa(S_n)\}}\right] + \beta_{S_n} E\left[X_i 1_{\{S_n = \mathrm{VaR}_\kappa(S_n)\}}\right]}{1-\kappa}, \\ & i = 1, \ldots, n, \end{aligned}$$

where

$$\beta_{S_n} = \begin{cases} \dfrac{\Pr\left(S_n \leq \mathrm{VaR}_\kappa\left(S_n\right)\right) - \kappa}{\Pr\left(S_n = \mathrm{VaR}_\kappa\left(S_n\right)\right)}, & \text{if } \Pr\left(S_n = \mathrm{VaR}_\kappa\left(S_n\right)\right) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

For continuous rvs, the TVaR-based allocations are equal to

$$\mathrm{TVaR}_\kappa\left(X_i, S_n\right) = \frac{E\left[X_i 1_{\{S_n > \mathrm{VaR}_\kappa(S_n)\}}\right]}{1-\kappa}.$$

Under the covariance-based allocation rule, the contribution amount C_i, denoted by $C_\kappa\left(X_i, S_n\right)$, is given by

$$\begin{aligned} C_\kappa\left(X_i, S_n\right) &= E\left[X_i\right] + \frac{\mathrm{Cov}(X_i, S_n)}{\mathrm{Var}(S_n)} \\ &\quad \times \left(\mathrm{TVaR}_\kappa\left(S_n\right) - E\left[S_n\right]\right), \quad i = 1, \ldots, n. \end{aligned} \tag{10}$$

For both allocation rules, the sum of the allocations $\sum_{i=1}^{n} C_i$ is equal to $\mathrm{TVaR}_\kappa\left(S_n\right)$, the amount of capital needed for the entire portfolio. Note also that both allocation rules satisfy Euler's principle. See e.g. Tasche (2000) and McNeil et al. (2005) for details on both allocation rules and Hesselager and Andersson (2002) for further information on the covariance-based allocation rule.

2.3. Mixed Erlang distribution

The mixed Erlang class has many useful analytic properties for actuarial and risk management applications. Closed-form expressions for important quantities in actuarial science are derived using this class of distributions (e.g. stop-loss premium, finite time ruin probabilities among others). Willmot and Lin (2010) and Lee and Lin (2010) give several illustrative examples. Tijms (1994) shows that the mixed Erlang class is dense in the set of positive continuous probability distributions. This is a significant advantage of this class since any positive continuous distribution can be approximated by a member of this class. In this subsection, we define the mixed Erlang distribution and present a list of its useful properties.

Let the pdf and cdf of an Erlang distribution of order $k \in \mathbb{N}^*$ and scale factor $\beta \in \mathbb{R}^+$ be defined by

$$h\left(x; k, \beta\right) = \frac{\beta^k}{\Gamma(k)} x^{k-1} e^{-\beta x}, \quad x > 0,$$

and

$$H\left(x; k, \beta\right) = 1 - e^{-\beta x} \sum_{j=0}^{n-1} \frac{(\beta x)^j}{j!}, \quad x > 0.$$

Let Y be a mixed Erlang rv with scale parameter β. The pdf and cdf of Y are respectively given by

$$f_Y(x) = \sum_{k=1}^{\infty} p_k h(x; k, \beta),$$

$$F_Y(x) = \sum_{k=1}^{\infty} p_k H(x; k, \beta),$$

where p_k is the probability mass associated with the kth Erlang distribution in the mixture and β is the scale parameter. We use the notation $Y \sim \text{MixErl}\left(\underline{p}, \beta\right)$ with $\underline{p} = \{p_1, p_2, \ldots\}$.

One of the main advantages of the mixed Erlang distribution is that, most of the time, risk measures or other risk related quantities, such as the stop-loss premium, have either an exact expression or can be easily computed (see e.g. Willmot and Lin (2010) and references therein). For instance, since the expression for F_Y is analytic, the value of $\text{VaR}_\kappa(Y)$ can easily be obtained with any optimization tool (e.g. `optimize` in R and `solver` in Excel). The expression for $\text{TVaR}_\kappa(Y)$ is given by

$$\text{TVaR}_\kappa(Y) = \frac{1}{1-\kappa}\sum_{k=1}^{\infty} p_k \frac{k}{\beta}\overline{H}\left(\text{VaR}_\kappa(Y); k+1, \beta\right), \tag{11}$$

where $\overline{H}(x; k, \beta) = 1 - H(x; k, \beta)$ is the survival function of an Erlang distribution. For the expression of the stop-loss premium for a given retention $d \geq 0$, defined by $\pi_Y(d) = E\left[\max\left(Y - d; 0\right)\right]$, we have

$$\pi_Y(d) = \sum_{k=1}^{\infty} p_k e^{-\beta d}\frac{(\beta d)^k}{k!}. \tag{12}$$

In the rest of this subsection, we present some useful properties of the mixed Erlang distribution. First, we show that the equilibrium density of Y, defined by $f_Y^{\varepsilon}(x) = \frac{\overline{F}_Y(x)}{E[Y]}$ with $\overline{F}_Y(x)$ the survival function of Y, can be written as the pdf of a mixed Erlang distribution.

Lemma 2.1. *The equilibrium distribution associated with the rv Y is a mixed Erlang distribution with parameters $(\underline{\varepsilon}(\underline{p}) = \{\varepsilon(k, \underline{p}), k \in \mathbb{N}^*\}, \beta)$ and pdf given by*

$$f_Y^{\varepsilon}(x) = \sum_{k=1}^{\infty} \varepsilon(k, \underline{p}) h\left(x; k, \beta\right),$$

where

$$\varepsilon(k, \underline{p}) = \frac{\sum_{j=k}^{\infty} p_j}{\sum_{j=1}^{\infty} j p_j}, \quad \text{for } k = 1, 2, \ldots.$$

Proof. See Section 3 of Willmot and Lin (2010). □

Let Y_1 and Y_2 be two independent rvs with $Y_i \sim \text{MixErl}\left(\underline{p}_i, \beta_i\right)$ and $\beta_i = \beta,\ i = 1, 2$. The next lemma shows that the sum $Y_1 + Y_2$ also has a mixed Erlang distribution.

Lemma 2.2. *Under the condition that $\beta_1 = \beta_2 = \beta$, the rv $T_2 = Y_1 + Y_2$ has a mixed Erlang distribution with parameters $(\underline{\sigma}^{(2)}(\underline{p}_1, \underline{p}_2) = \{\sigma^{(2)}(k, \underline{p}_1, \underline{p}_2), k \in \mathbb{N}^*\}, \beta)$ and pdf given by*

$$f_{T_2}(s) = \sum_{k=1}^{\infty} \sigma^{(2)}(k, \underline{p}_1, \underline{p}_2) h\left(s; k, \beta\right),$$

where

$$\sigma^{(2)}(k, \underline{p}_1, \underline{p}_2) = \begin{cases} 0, & \text{for } k = 1, \\ \sum_{j=1}^{k-1} p_{1,j} p_{2,k-j}, & \text{for } k = 2, 3, \ldots. \end{cases}$$

Proof. This result is a special case of Proposition 5 in Cossette et al. (2012). □

Remark 2.1. The result of Lemma 2.2 can be generalized to a portfolio of n independent risks $Y_1, \ldots, Y_n$, where $Y_i \sim \text{MixErl}\left(\underline{p}_i, \beta\right)$ and $\beta_i = \beta$ for $i = 1, 2, \ldots, n$. More precisely, for the aggregate claim amount $T_n = \sum_{i=1}^{n} Y_i$, we have

$$T_n \sim \text{MixErl}\left(\underline{\sigma}^{(n)}(\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n), \beta\right),$$

where the probabilities $\underline{\sigma}^{(n)}(\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n) = \left(\sigma^{(n)}(k, \underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n), k \in \mathbb{N}^*\right)$ are computed with the following recursive expression:

$$\sigma^{(n+1)}(k, \underline{p}_1, \underline{p}_2, \ldots, \underline{p}_{n+1}) = \begin{cases} 0, \quad \text{for } k = 1, \ldots, n \\ \sum_{j=n}^{k-1} \sigma^{(n)}(j, \underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n) p_{n+1,k-j}, \\ \quad \text{for } k = n+1, n+2, \ldots \end{cases}$$

for $n = 2, 3, \ldots$.

Let Y be a positive rv with $Y \sim \text{MixErl}\left(\underline{p}, \beta\right)$. In the lemma that follows, we state that $2f_Y(x)\overline{F}_Y(x)$ is the pdf of a mixed Erlang distribution.

Lemma 2.3. *Let $f_Y^{\pi}(x) = 2f_Y(x)\overline{F}_Y(x)$ where $Y \sim \text{MixErl}\left(\underline{p}, \beta\right)$. Then, $f_Y^{\pi}(x)$ is the pdf of a mixed Erlang distribution with parameters $(\underline{\pi}(\underline{p}) = \{\pi(j, \underline{p}), j \in \mathbb{N}^*\}, 2\beta)$, i.e.*

$$f_Y^{\pi}(x) = \sum_{j=1}^{\infty} \pi(j, \underline{p}) h\left(x; j, 2\beta\right),$$

where

$$\pi(j, \underline{p}) = \frac{1}{2^{j-1}}\sum_{k=1}^{j}\binom{j-1}{k-1} p_k \sum_{l=j-k+1}^{\infty} p_l, \quad \text{for } j = 1, 2, \ldots. \tag{13}$$

Proof. To identify the expression of $\pi(j, \underline{p})$, we begin by writing $f_Y^{\pi}(x)$ as

$$f_Y^{\pi}(x) = 2E\left[Y\right] f_Y(x) f_Y^{\varepsilon}(x),$$

where

$$\begin{aligned} f_Y(x) f_Y^{\varepsilon}(x) &= \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} p_k \varepsilon(j-k+1, \underline{p}) \beta^{j+1} e^{-2\beta x}\frac{x^{j-1}}{(k-1)!(j-k)!} \\ &= \sum_{j=1}^{\infty}\frac{\beta}{2^j}\sum_{k=1}^{j}\binom{j-1}{k-1} p_k \varepsilon(j-k+1, \underline{p}) h\left(x; j, 2\beta\right). \end{aligned} \tag{14}$$

Using Lemma 2.1, we multiply both sides of (14) by $2E\left[Y\right]$ and we obtain

$$\begin{aligned} f_Y^{\pi}(x) &= \sum_{j=1}^{\infty}\frac{\beta E\left[Y\right]}{2^{j-1}}\sum_{k=1}^{j}\binom{j-1}{k-1} p_k \varepsilon(j-k+1, \underline{p}) h\left(x; j, 2\beta\right) \\ &= \sum_{j=1}^{\infty}\left(\frac{1}{2^{j-1}}\sum_{k=1}^{j}\binom{j-1}{k-1} p_k \sum_{l=j-k+1}^{\infty} p_l\right) h\left(x; j, 2\beta\right) \\ &= \sum_{j=1}^{\infty}\pi(j, \underline{p}) h\left(x; j, 2\beta\right), \end{aligned}$$

where the expression for $\pi(j, \underline{p})$ is as given in (13).

From (13), we observe that $\pi(j, \underline{p}) \geq 0$, for $j = 1, 2, \ldots$ and since $\int_0^{\infty} f_Y^{\pi}(x)dx = 2E\left[1 - F_Y(Y)\right] = 1$, we have $\sum_{j=1}^{\infty}\pi(j, \underline{p}) = 1$. We conclude that f_Y^{π} can be written as the pdf of a mixed Erlang distribution. □

We may write the pdf of a mixed Erlang rv with scale parameter β_1 in terms of the pdf of a mixed Erlang rv with scale parameter β_2. This is shown below.

Lemma 2.4. *Let $Y \sim \mathrm{MixErl}\left(\underline{p}, \beta_1\right)$ and $\beta_1 \le \beta_2$. Then, the pdf of Y can be expressed as the pdf of a mixed Erlang distribution with parameters $(\underline{\omega}(\underline{p}, \beta_1, \beta_2) = \{\omega(k, \underline{p}, \beta_1, \beta_2), k \in \mathbb{N}^*\}, \beta_2)$, i.e.*

$$f_Y(x) = \sum_{k=1}^{\infty} \omega(k, \underline{p}, \beta_1, \beta_2) h(x; k, \beta_2),$$

with

$$\omega(k, \underline{p}, \beta_1, \beta_2) = \sum_{j=1}^{k} p_j \binom{k-1}{k-j} \left(\frac{\beta_1}{\beta_2}\right)^j \left(1 - \frac{\beta_1}{\beta_2}\right)^{k-j},$$

for $k = 1, 2, \ldots$.

Proof. See Proposition 2 in Cossette et al. (2012). □

Remark 2.2. Note that for a given positive rv $Y \sim \mathrm{MixErl}\left(\underline{p}, \beta\right)$, we have $\underline{\omega}(\underline{p}, \beta, \beta) = \underline{p}$.

For a given rv $Y \sim \mathrm{MixErl}\left(\underline{p}, \beta\right)$, we define $f_Y^{\alpha}(x) = \frac{x f_Y(x)}{E[Y]}$. The following lemma shows that f_Y^{α} is the pdf of a mixed Erlang distribution.

Lemma 2.5. *Let $f_Y^{\alpha}(x) = \frac{x f_Y(x)}{E[Y]}$ where $Y \sim \mathrm{MixErl}\left(\underline{p}, \beta\right)$. Then, $f_Y^{\alpha}(x)$ can be expressed as the pdf of a mixed Erlang distribution with parameters $(\underline{\alpha}(\underline{p}) = \{\alpha(k, \underline{p}), k \in \mathbb{N}^*\}, \beta)$, i.e.*

$$f_Y^{\alpha}(x) = \sum_{k=1}^{\infty} \alpha(k, \underline{p}) h(x; k, \beta),$$

with

$$\alpha(k, \underline{p}) = \begin{cases} 0, & \text{if } k = 1, \\ \dfrac{(k-1)p_{k-1}}{\sum_{j=1}^{\infty} j p_j}, & \text{if } k = 2, 3, \ldots. \end{cases}$$

Proof. We have

$$\begin{aligned} x f_Y(x) &= x \sum_{k=1}^{\infty} p_k h(x; k, \beta) \\ &= \sum_{k=1}^{\infty} p_k e^{-\beta x} \beta^k \frac{x^k}{(k-1)!} \\ &= \sum_{k=2}^{\infty} p_{k-1} e^{-\beta x} \beta^{k-1} \frac{x^{k-1}}{(k-2)!} \\ &= \sum_{k=2}^{\infty} \frac{k-1}{\beta} p_{k-1} h(x; k, \beta). \end{aligned}$$

Since $E[Y] = \sum_{j=1}^{\infty} \frac{j}{\beta} p_j$ it follows that f_Y^{α} is a mixed Erlang pdf with the specified parameters. □

3. Bivariate distribution defined with the FGM copula and mixed Erlang marginals

In this section, we assume that the couple (X_1, X_2) has a bivariate distribution defined with the FGM copula given in (2) and $X_i \sim \mathrm{MixErl}\left(\underline{p}_i, \beta_i\right)$, $i = 1, 2$. Without loss of generality, we assume that $\beta_1 \le \beta_2$.

In the following proposition, we provide a closed-form expression for the covariance between X_1 and X_2.

Proposition 3.1. *Let (X_1, X_2) have a bivariate distribution defined with the FGM copula and $X_i \sim \mathrm{MixErl}\left(\underline{p}_i, \beta_i\right)$, $i = 1, 2$ with $\beta_1 \le \beta_2$. We have the following expression for the covariance between X_1 and X_2:*

$$\begin{aligned} \mathrm{Cov}(X_1, X_2) = \frac{\theta}{\beta_1 \beta_2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} ij \left(\pi(i, \underline{p}_1) - p_{1,i}\right) \\ \times \left(\pi(j, \underline{p}_2) - p_{2,j}\right). \end{aligned} \quad (15)$$

Proof. Assuming that the dependence structure for (X_1, X_2) is given by the FGM copula with dependence parameter θ, Armstrong and Galli (2002) obtain the following expression for the covariance $\mathrm{Cov}(X_1, X_2)$:

$$\mathrm{Cov}(X_1, X_2) = \theta \gamma_1 \gamma_2, \quad (16)$$

where

$$\gamma_i = 2E\left[X_i \overline{F}_{X_i}(X_i)\right] - E[X_i],$$

for $i = 1, 2$.

Using Lemma 2.3, we have

$$2E\left[X_i \overline{F}_{X_i}(X_i)\right] = \sum_{j=1}^{\infty} \frac{j}{\beta_i} \pi(j, \underline{p}_i)$$

for $i = 1, 2$, which leads to the expression given in (15). □

Remark 3.1. According to Lemma 2.4, the covariance formula given in (15) can also be written as

$$\begin{aligned} \mathrm{Cov}(X_1, X_2) = \frac{\theta}{\beta_2^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} ij \Big(\pi(i, \underline{\omega}(\underline{p}_1, \beta_1, \beta_2)) \\ - \omega(i, \underline{p}_1, \beta_1, \beta_2)\Big) \left(\pi(j, \underline{p}_2) - p_{2,j}\right). \end{aligned}$$

3.1. Distribution of S_2

In the next proposition, we show that $S_2 = X_1 + X_2$ follows a mixed Erlang distribution.

Proposition 3.2. *Let (X_1, X_2) have a bivariate distribution defined with the FGM copula and $X_i \sim \mathrm{MixErl}\left(\underline{p}_i, \beta_i\right)$, $i = 1, 2$ with $\beta_1 \le \beta_2$. Then, $S_2 = X_1 + X_2 \sim \mathrm{MixErl}\left(\underline{p}^{(2)}, 2\beta_2\right)$, where $\underline{p}^{(2)} = \{p_j^{(2)}, j \in \mathbb{N}^*\}$ with*

$$\begin{aligned} p_j^{(2)} &= (1+\theta)\sigma^{(2)}(j, \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_2), \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2)) \\ &\quad - \theta\sigma^{(2)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_1), 2\beta_1, 2\beta_2), \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2)) \\ &\quad - \theta\sigma^{(2)}(j, \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_2), \underline{\pi}(\underline{p}_2)) \\ &\quad + \theta\sigma^{(2)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_1), 2\beta_1, 2\beta_2), \underline{\pi}(\underline{p}_2)), \end{aligned} \quad (17)$$

for $j = 2, 3, \ldots$ and $p_1^{(2)} = 0$.

Proof. Given (1) and (3), the pdf of S_2 is obtained from the joint pdf of (X_1, X_2) as

$$f_{S_2}(s) = \int_0^s f_{\underline{X}}(x, s-x)\, dx,$$

where

$$\begin{aligned} f_{\underline{X}}(x, s-x) = \Big((1+\theta) - 2\theta\overline{F}_{X_1}(x) - 2\theta\overline{F}_{X_2}(s-x) \\ + 4\theta\overline{F}_{X_1}(x)\overline{F}_{X_2}(s-x)\Big) f_{X_1}(x) f_{X_2}(s-x). \end{aligned}$$

This leads to

$$f_{S_2}(s) = I_1(s) + I_2(s) + I_3(s) + I_4(s),$$

where

$$I_1(s) = (1+\theta)\int_0^s f_{X_1}(x)f_{X_2}(s-x)dx,$$

$$I_2(s) = -2\theta\int_0^s \overline{F}_{X_1}(x)f_{X_1}(x)f_{X_2}(s-x)dx,$$

$$I_3(s) = -2\theta\int_0^s f_{X_1}(x)\overline{F}_{X_2}(s-x)f_{X_2}(s-x)dx,$$

$$I_4(s) = 4\theta\int_0^s \overline{F}_{X_1}(x)f_{X_1}(x)\overline{F}_{X_2}(s-x)f_{X_2}(s-x)dx.$$

Each term $I_i(s),\ i = 1, 2, 3, 4$, can be expressed as a convolution of two mixed Erlang distributions. In the sequel, we determine the closed-form expression for each $I_i(s),\ i = 1, 2, 3, 4$.

For the first term $I_1(s)$, changing the scale parameter of the distribution of X_1 and X_2 using Lemma 2.4 leads to

$$f_{X_1}(s) = \sum_{j=1}^{\infty}\omega(j, \underline{p}_1, \beta_1, 2\beta_2)h(s; j, 2\beta_2),$$

and

$$f_{X_2}(s) = \sum_{j=1}^{\infty}\omega(j, \underline{p}_2, \beta_2, 2\beta_2)h(s; j, 2\beta_2).$$

According to Lemma 2.2, one can write

$$\begin{aligned}I_1(s) &= (1+\theta)\int_0^s f_{X_1}(x)f_{X_2}(s-x)dx\\ &= (1+\theta)\sum_{j=2}^{\infty}\sigma^{(2)}(j, \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_2),\\ &\quad \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2))h(s; j, 2\beta_2).\end{aligned}$$

For the second term $I_2(s)$, one must first use Lemmas 2.3 and 2.4 to write

$$2\overline{F}_{X_1}(x)f_{X_1}(x) = \sum_{k=1}^{\infty}\omega(k, \underline{\pi}(\underline{p}_1), 2\beta_1, 2\beta_2)h(x; k, 2\beta_2).$$

To perform the convolution in $I_2(s)$, we write $f_{X_2}(x)$ as a pdf of a mixed Erlang rv with scale parameter $2\beta_2$ using Lemma 2.4

$$f_{X_2}(x) = \sum_{k=1}^{\infty}\omega(k, \underline{p}_2, \beta_2, 2\beta_2)h(x; k, 2\beta_2).$$

Then, the convolution formula leads to

$$\begin{aligned}I_2(s) &= -\theta\sum_{j=2}^{\infty}\sigma^{(2)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_1), 2\beta_1, 2\beta_2),\\ &\quad \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2))h(s; j, 2\beta_2).\end{aligned}$$

For $I_3(s)$ and $I_4(s)$, we obtain with similar calculations

$$I_3(s) = -\theta\sum_{j=2}^{\infty}\sigma^{(2)}(j, \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_2), \underline{\pi}(\underline{p}_2))h(s; j, 2\beta_2),$$

and

$$I_4(s) = \theta\sum_{j=2}^{\infty}\sigma^{(2)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_1), 2\beta_1, 2\beta_2), \underline{\pi}(\underline{p}_2))h(s; j, 2\beta_2). \quad \square$$

Since S_2 follows a mixed Erlang distribution, we have closed-form expressions for the TVaR and the stop-loss premium associated with S_2.

Corollary 3.1. *Let (X_1, X_2) have a bivariate distribution defined with the FGM copula and $X_i \sim$ MixErl $\left(\underline{p}_i, \beta_i\right)$, $i = 1, 2$ with $\beta_1 \leq \beta_2$. At a given level $\kappa \in [0, 1]$, the closed-form expression for the TVaR risk measure is given by*

$$\mathrm{TVaR}_\kappa(S_2) = \frac{1}{1-\kappa}\sum_{j=1}^{\infty}p_j^{(2)}\frac{j}{2\beta_2}\overline{H}\left(\mathrm{VaR}_\kappa(S_2); j+1, 2\beta_2\right),$$

and for a given retention $d \in \mathbb{R}^+$, the stop-loss premium is

$$\pi_{S_2}(d) = \sum_{j=1}^{\infty}p_j^{(2)}e^{-2\beta_2 d}\frac{(2\beta_2 d)^j}{j!},$$

where probabilities $\underline{p}^{(2)}$ are as given in (17).

Proof. Based on Proposition 3.2, the expressions for the TVaR risk measure and the stop-loss premium follow from (11) and (12). $\square$

3.2. TVaR-based capital allocation

In the following proposition, we provide the expression for the amount allocated to the risks X_1 and X_2 under the TVaR-based allocation rule.

Proposition 3.3. *Let (X_1, X_2) have a bivariate distribution defined with the FGM copula and $X_i \sim$ MixErl $\left(\underline{p}_i, \beta_i\right)$, $i = 1, 2$ with $\beta_1 \leq \beta_2$. Then, the expression for $\mathrm{TVaR}_\kappa(X_i, S_2)$ at level κ, $0 < \kappa < 1$, is given by*

$$\mathrm{TVaR}_\kappa(X_i, S_2) = \frac{1}{1-\kappa}\sum_{k=1}^{\infty}q_{i,k}^{(2)}\overline{H}\left(\mathrm{VaR}_\kappa(S_2); k, 2\beta_2\right),$$

where

$$\begin{aligned}q_{1,k}^{(2)} &= (1+\theta)E[X_1]\,\sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{p}_1), \beta_1, 2\beta_2),\\ &\quad \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2)) - \theta\Pi_1\sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{\pi}(\underline{p}_1)), 2\beta_1, 2\beta_2),\\ &\quad \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2)) - \theta E[X_1]\,\sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{p}_1), \beta_1, 2\beta_2), \underline{\pi}(\underline{p}_2))\\ &\quad + \theta\Pi_1\sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{\pi}(\underline{p}_1)), 2\beta_1, 2\beta_2), \underline{\pi}(\underline{p}_2))\end{aligned} \tag{18}$$

and

$$\begin{aligned}q_{2,k}^{(2)} &= (1+\theta)E[X_2]\,\sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{p}_2), \beta_2, 2\beta_2),\\ &\quad \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_2)) - \theta\Pi_2\sigma^{(2)}(k, \underline{\alpha}(\underline{\pi}(\underline{p}_2)), \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_2))\\ &\quad - \theta E[X_2]\,\sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{p}_2), \beta_2, 2\beta_2), \underline{\omega}(\underline{\pi}(\underline{p}_1), 2\beta_1, 2\beta_2))\\ &\quad + \theta\Pi_2\sigma^{(2)}(k, \underline{\alpha}(\underline{\pi}(\underline{p}_2)), \underline{\omega}(\underline{\pi}(\underline{p}_1), 2\beta_1, 2\beta_2))\end{aligned} \tag{19}$$

for $k = 3, 4 \ldots$ and $q_{i,k}^{(2)} = 0$, for $i = 1, 2$, $k = 1, 2$ and

$$\Pi_i = E\left[X_i\overline{F}_{X_i}(X_i)\right] = \sum_{j=1}^{\infty}\frac{j}{2\beta_i}\pi(j, \underline{p}_i).$$

Proof. We know that

$$\begin{aligned}\mathrm{TVaR}_\kappa(X_1, S_2) &= \frac{E\left[X_1 1_{\{S_n > \mathrm{VaR}_\kappa(S_n)\}}\right]}{1-\kappa}\\ &= \frac{1}{1-\kappa}\int_{\mathrm{VaR}_\kappa(S_2)}^{\infty}E\left[X_1 1_{\{S_2 = s\}}\right]ds.\end{aligned}$$

The expression for $E\left[X_1 1_{\{S_2=s\}}\right]$ is given by

$$\begin{aligned}E\left[X_1 1_{\{S_2=s\}}\right] &= \int_0^s x f_{\underline{X}}(x, s-x)dx\\ &= J_1(s) + J_2(s) + J_3(s) + J_4(s),\end{aligned}$$

where

$$J_1(s) = (1+\theta)\int_0^s x f_{X_1}(x) f_{X_2}(s-x)dx,$$

$$J_2(s) = -2\theta\int_0^s \overline{F}_{X_1}(x) x f_{X_1}(x) f_{X_2}(s-x)dx,$$

$$J_3(s) = -2\theta\int_0^s x f_{X_1}(x)\overline{F}_{X_2}(s-x) f_{X_2}(s-x)dx,$$

$$J_4(s) = 4\theta\int_0^s \overline{F}_{X_1}(x) x f_{X_1}(x)\overline{F}_{X_2}(s-x) f_{X_2}(s-x)dx.$$

In the sequel, each term $J_i(s)$, for $i = 1, 2, 3, 4$, is expressed as a pdf of a mixed Erlang distribution.

Simple manipulations lead to

$$J_1(s) = (1+\theta)E[X_1]\int_0^s \frac{x f_{X_1}(x)}{E[X_1]} f_{X_2}(s-x)dx.$$

Lemma 2.5 allows us to write $\frac{x f_{X_1}(x)}{E[X_1]}$ as the pdf of a mixed Erlang rv, meaning

$$\frac{x f_{X_1}(x)}{E[X_1]} = \sum_{k=2}^{\infty}\alpha(k, \underline{p}_1) h(x; k, \beta_1). \tag{20}$$

Using Lemma 2.4, we change the scale parameter of the pdf of the mixed Erlang distribution in (20) and we get

$$\begin{aligned} J_1(s) &= (1+\theta)E[X_1]\sum_{k=3}^{\infty}\sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{p}_1), \beta_1, 2\beta_2), \\ &\quad \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2)) h(s; k, 2\beta_2). \end{aligned}$$

For the second term, we write

$$J_2(s) = -\theta\Pi_1\int_0^s \frac{2x\overline{F}_{X_1}(x) f_{X_1}(x)}{\Pi_1} f_{X_2}(s-x)dx,$$

where

$$\begin{aligned} \Pi_1 &= E\left[X_1\overline{F}_{X_1}(X_1)\right] \\ &= \sum_{j=1}^{\infty}\frac{j}{2\beta_1}\pi(j, \underline{p}_1). \end{aligned}$$

According to Lemmas 2.3 and 2.5, we can state that

$$\frac{2x\overline{F}_{X_1}(x) f_{X_1}(x)}{\Pi_1} = \sum_{j=2}^{\infty}\alpha(j, \underline{\pi}(\underline{p}_1)) h(x; j, 2\beta_1).$$

Then, one can write $J_2(s)$ as follows:

$$\begin{aligned} J_2(s) &= -\theta\Pi_1\sum_{k=3}^{\infty}\sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{\pi}(\underline{p}_1)), 2\beta_1, 2\beta_2), \\ &\quad \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_2)) h(s; k, 2\beta_2). \end{aligned}$$

Using similar calculations and according to Lemmas 2.3–2.5, we have the following expressions for $J_3(s)$ and $J_4(s)$:

$$\begin{aligned} J_3(s) &= -\theta E[X_1]\sum_{k=3}^{\infty}\sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{p}_1), \beta_1, 2\beta_2), \\ &\quad \underline{\pi}(\underline{p}_2)) h(s; k, 2\beta_2), \end{aligned}$$

and

$$\begin{aligned} J_4(s) &= \theta\Pi_1\sum_{k=3}^{\infty}\sigma^{(2)}(k, \underline{\omega}(\underline{\alpha}(\underline{\pi}(\underline{p}_1)), 2\beta_1, 2\beta_2), \\ &\quad \underline{\pi}(\underline{p}_2)) h(s; k, 2\beta_2). \end{aligned}$$

One can write

$$\int_0^s x f_{X_1,S_2}(x, s)dx = \sum_{j=1}^{\infty} q_{1,j}^{(2)} h(s; j, 2\beta_2),$$

where the expressions for the probabilities $q_{1,j}^{(2)}$ are as given in (18). Then, we find the desired result for $\mathrm{TVaR}_\kappa(X_1, S_2)$. Similarly, one can derive the expression for $\mathrm{TVaR}_\kappa(X_2, S_2)$ with $q_{2,j}^{(2)}$ as given in (19). □

3.3. Covariance-based capital allocation

A closed-form expression for the amount of capital attributed to risk $i = 1, 2$ under the covariance allocation rule is given in the next proposition.

Proposition 3.4. *Let (X_1, X_2) have a bivariate distribution defined with the FGM copula and $X_i \sim \mathrm{MixErl}\left(\underline{p}_i, \beta_i\right)$, $i = 1, 2$ with $\beta_1 \leq \beta_2$. Then, the contribution $C_\kappa(X_i, S_2)$ at level κ, $0 < \kappa < 1$, is given by*

$$C_\kappa(X_i, S_2) = \sum_{k=1}^{\infty} c_{i,k}\frac{k}{\beta_2}, \tag{21}$$

where

$$\begin{aligned} c_{i,k} &= \omega(k, \underline{p}_i, \beta_i, \beta_2) \\ &\quad + 2\rho_{i,k} p_k^{(2)}\left[\frac{\overline{H}(\mathrm{VaR}_\kappa(S_2); k+1, 2\beta_2)}{1-\kappa} - 1\right], \end{aligned}$$

with

$$\begin{aligned} \rho_{i,k} &= \frac{\sum_{l=1}^{\infty} l\omega(l, \underline{p}_i, \beta_i, \beta_2) - \left(\sum_{l=1}^{\infty} l\omega(l, \underline{p}_i, \beta_i, \beta_2)\right)^2}{\sum_{l=1}^{\infty} l p_l^{(2)} - \left(\sum_{l=1}^{\infty} l p_l^{(2)}\right)^2} \\ &\quad + \frac{\theta\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} lm\left(\pi(l, \underline{\omega}(\underline{p}_1, \beta_1, \beta_2)) - \omega(l, \underline{p}_1, \beta_1, \beta_2)\right)\left(\pi(m, \underline{p}_2) - p_{2,m}\right)}{\sum_{l=1}^{\infty} l p_l^{(2)} - \left(\sum_{l=1}^{\infty} l p_l^{(2)}\right)^2}, \end{aligned}$$

for $i = 1, 2$.

Proof. The amount of capital attributed to risk i, for $i = 1, 2$, is given in (10) by

$$C_\kappa(X_i, S_2) = E[X_i] + \frac{\mathrm{Cov}(X_i, S_2)}{\mathrm{Var}(S_2)}\left(\mathrm{TVaR}_\kappa(S_2) - E[S_2]\right). \tag{22}$$

By Proposition 3.2, we have

$$E[S_2] = \sum_{k=1}^{\infty} p_k^{(2)}\frac{k}{2\beta_2}.$$

Using Lemma 2.4, we have

$$E[X_i] = \sum_{k=1}^{\infty}\omega(k, \underline{p}_i, \beta_i, \beta_2)\frac{k}{\beta_2}.$$

Given Corollary 3.1, (22) becomes

$$\begin{aligned} C_\kappa(X_i, S_2) &= \sum_{k=1}^{\infty}\left[\omega(k, \underline{p}_i, \beta_i, \beta_2) + \frac{\mathrm{Cov}(X_i, S_2)}{2\mathrm{Var}(S_2)} p_k^{(2)}\right. \\ &\quad \left.\times\left[\frac{\overline{H}(\mathrm{VaR}_\kappa(S_2); k+1, 2\beta_2)}{1-\kappa} - 1\right]\right]\frac{k}{\beta_2}. \end{aligned}$$

Table 1
Descriptive statistics of X_1 and X_2.

$E[X_1]$	$E[X_2]$	$\mathrm{Var}[X_1]$	$\mathrm{Var}[X_2]$	$\mathrm{Cov}(X_1, X_2)$
14	12.67	164	106.22	17.98

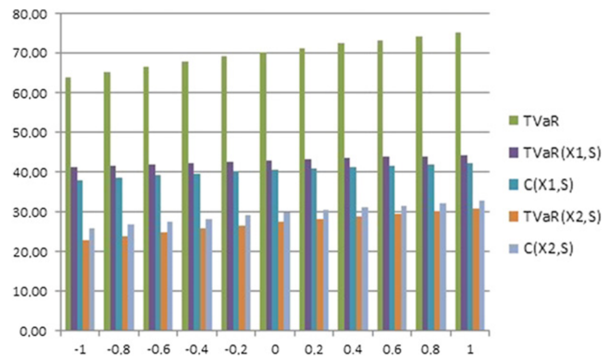


Fig. 1. Capital allocation based on the TVaR and the covariance rules.

We also know that $S_2 \sim \mathrm{MixErl}\left(\underline{p}^{(2)}, 2\beta_2\right)$, and hence

$$\mathrm{Var}(S_2) = \frac{1}{4\beta_2} \sum_{l=1}^{\infty} l p_l^{(2)} - \left(\sum_{l=1}^{\infty} l p_l^{(2)} \right)^2 .$$

Using the expression for $\mathrm{Cov}(X_1, X_2)$ given in Remark 3.1, one may find the expression in (21). $\square$

3.4. Numerical example

Let (X_1, X_2) have a bivariate distribution defined by the FGM copula with $\theta = 0.5$ and mixed Erlang marginals. The pdf of X_1 and X_2 are given by

$$f_{X_1}(x) = 0.6h(x; 1, 0.1) + 0.4h(x; 2, 0.1),$$
$$f_{X_2}(x) = 0.3h(x; 1, 0.15) + 0.5h(x; 2, 0.15) + 0.2h(x; 3, 0.15).$$

The values of the expectations, variances and covariances of X_1 and X_2 are displayed in Table 1.

We calculate the VaR and TVaR for $S_2 = X_1 + X_2$ and also determine the amount allocated under the TVaR- and covariance-based allocation rules for X_1 and X_2.

By Proposition 3.2, we have $S_2 \sim \mathrm{MixErl}\left(\underline{p}^{(2)}, 2\beta_2\right)$ and Table 2 gives the first 40 values of the probabilities $\underline{p}^{(2)} = \left(p_k^{(2)}, k = 1, 2, \ldots\right)$ for a dependence parameter $\theta = 0.5$. It is clear that the value of $p_k^{(2)}$ goes to zero for high values of k.

For different values of κ, we provide in Table 3 the values of $\mathrm{VaR}_\kappa(S_2)$ and $\mathrm{TVaR}_\kappa(S_2)$.

For $\kappa = 0.95$ and for different values of θ, we calculate the amount allocated under the TVaR-based allocation rule and the covariance-based allocation rule. We observe that these allocations show a different behavior according to the dependence relationship measured by the value of the copula parameter θ (see Table 4 and Fig. 1). We also calculate values of the TVaR, TVaR-based capital allocation and covariance-based capital allocation for different values of θ and also at different levels κ. Fig. 2 summarizes the behavior of each measure and illustrates the impact of the copula parameter θ and κ on the ratio $\frac{\mathrm{TVaR}_\kappa(X_1,S_2)}{C_\kappa(X_1,S_2)}$. It is clear that the TVaR increases as θ goes to 1 and one sees that the covariance-based allocation underestimates the capital that should be allocated to risk X_1 especially for high negative dependence.

4. Multivariate distribution defined with the FGM copula and mixed Erlang marginals

In this section, we assume that $(X_1, \ldots, X_n)$ has a multivariate distribution defined by the FGM copula given in (4) and $X_i \sim \mathrm{MixErl}\left(\underline{p}_i, \beta_i\right)$, $i = 1, \ldots, n$. Without loss of generality, we assume that $\beta_i \le \beta_n$, for $i = 1, \ldots, n-1$.

We provide, in the following proposition, a closed-form expression for the covariance between X_i and X_j, for $i \ne j = 1, \ldots, n$.

Proposition 4.1. *Let $(X_1, \ldots, X_n)$ have a multivariate distribution defined with the FGM n-copula and $X_i \sim \mathrm{MixErl}\left(\underline{p}_i, \beta_i\right)$, $i = 1, \ldots, n$ with $\beta_i \le \beta_n$, for $i = 1, 2, \ldots, n-1$. We have the following expression for the covariance $\mathrm{Cov}(X_i, X_j)$, for $i \ne j = 1, \ldots, n$:*

$$\mathrm{Cov}(X_i, X_j) = \frac{\theta_{ij}}{\beta_i \beta_j} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} lm \left(\pi(l, p_i) - p_{i,l}\right) \times \left(\pi(m, p_j) - p_{j,m}\right), \tag{23}$$

where $\theta_{ij} = \theta_{\min(i,j)\max(i,j)}$.

Proof. We know that each k-margin of the FGM n-copula is an FGM k-copula, for $k = 2, \ldots, n$. Particularly, for $i \ne j = 1, \ldots, n$, the dependence structure for (X_i, X_j) is given by a bivariate FGM copula with a dependence parameter $\theta_{ij} = \theta_{\min(i,j)\max(i,j)}$. Using Proposition 3.1, we find the expression given in (23). $\square$

Remark 4.1. Using Lemma 2.4, the covariance formula given in (23) can be written as follows:

$$\mathrm{Cov}(X_i, X_j) = \frac{\theta}{\beta_n^2} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} lm \Big(\pi\left(l, \underline{\omega}(\underline{p}_i, \beta_i, \beta_n)\right) - \omega(l, \underline{p}_i, \beta_i, \beta_n)\Big) \Big(\pi\left(m, \underline{\omega}(\underline{p}_j, \beta_j, \beta_n)\right) - \omega(m, \underline{p}_j, \beta_j, \beta_n)\Big).$$

4.1. Distribution of S_n

In the next proposition, we show that $S_n = \sum_{i=1}^n X_i$ follows a mixed Erlang distribution.

Proposition 4.2. *Let $(X_1, \ldots, X_n)$ have a multivariate distribution defined with the FGM n-copula and $X_i \sim \mathrm{MixErl}\left(\underline{p}_i, \beta_i\right)$, $i = 1, \ldots, n$ with $\beta_i \le \beta_n$, for $i = 1, 2, \ldots, n-1$. Then, $S_n = \sum_{i=1}^n X_i \sim \mathrm{MixErl}\left(\underline{p}^{(n)}, 2\beta_n\right)$, where $\underline{p}^{(n)} = \{p_j^{(n)}, j \in \mathbb{N}^*\}$ with*

$$\begin{aligned} p_j^{(n)} = {} & (1+\zeta)\sigma^{(n)}(j, \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_n), \\ & \underline{\omega}(\underline{p}_2, \beta_2, 2\beta_n), \ldots, \underline{\omega}(\underline{p}_n, \beta_n, 2\beta_n)) \\ & + \sum_{k=2}^{n} \sum_{1 \le j_1 < \cdots < j_k \le n} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{\{i_1, i_2, \ldots, i_{k-l}\} \in C_{J_k}^{k-l}} (-1)^l \\ & \times \sigma^{(n)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_{i_1}), 2\beta_{i_1}, 2\beta_n), \ldots, \underline{\omega}(\underline{\pi}(\underline{p}_{i_{k-l}}), 2\beta_{i_{k-l}}, \\ & 2\beta_n), \underline{\omega}(\underline{p}_{i_{k-l+1}}, \beta_{i_{k-l+1}}, 2\beta_n), \ldots, \underline{\omega}(\underline{p}_{i_n}, \beta_{i_n}, 2\beta_n)), \end{aligned} \tag{24}$$

for $j = n, n+1, \ldots$, and with $p_j^{(n)} = 0$ for $j = 1, \ldots, n-1$.

Also, $\zeta = \sum_{k=2}^n \sum_{1 \le j_1 < \cdots < j_k \le n} (-1)^k \theta_{j_1 j_2 \ldots j_k}$ and $C_{J_k}^{k-l}$ is the set of all combinations of $(k-l)$ elements from $J_k = \{1 \le j_1 < \cdots < j_k \le n\}$.

Table 2
Probabilities $p_k^{(2)}$ for $S_2 = X_1 + X_2$.

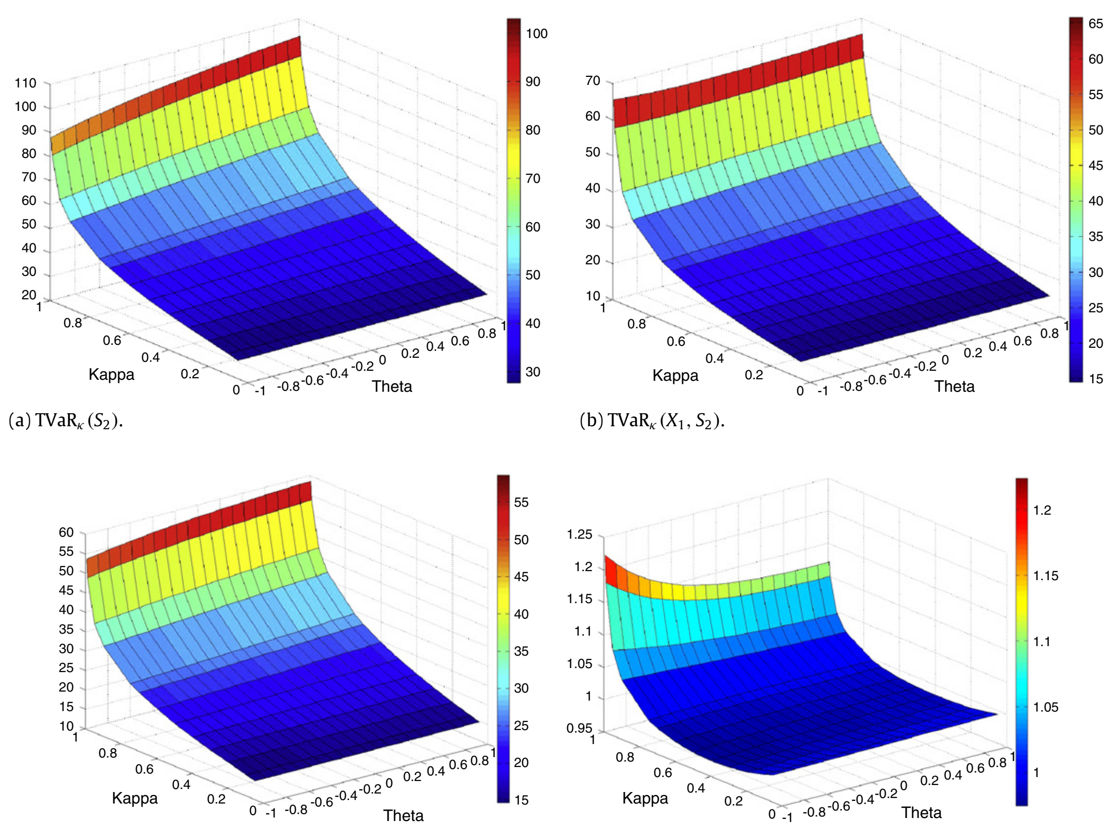
k	1	2	3	4	5	6	7	8	9	10
$p_k^{(2)}$	0	0.045	0.0895	0.1092	0.1060	0.0955	0.0867	0.0794	0.0719	0.0635
k	11	12	13	14	15	16	17	18	19	20
$p_k^{(2)}$	0.0544	0.0452	0.0366	0.0290	0.0225	0.0171	0.0129	0.0096	0.0070	0.0051
k	21	22	23	24	25	26	27	28	29	30
$p_k^{(2)}$	0.0037	0.0027	0.0019	0.0013	0.0009	0.0006	0.0004	0.0003	0.0002	0.0001
k	31	32	33	34	35	36	37	38	39	40
$p_k^{(2)}$	0.0001	8.1E−05	5.6E−05	3.9E−05	2.7E−05	1.8E−05	1.3E−05	8.9E−06	6.2E−06	4.2E−06

Table 3
Value-at-Risk measures and Tail-Value-at-Risk measures for $S_2 = X_1 + X_2$.

κ	0.05	0.10	0.50	0.75	0.90	0.95	0.99	0.995	0.999
$\mathrm{VaR}_\kappa(S_2)$	5.19	7.62	23.09	36.18	50.52	60.21	80.75	89.10	107.83
$\mathrm{TVaR}_\kappa(S_2)$	27.89	29.08	40.08	51.03	63.89	72.91	92.54	100.62	118.61

Table 4
Amounts allocated under the TVaR-based and the covariance-based rules.

θ	−1	−0.8	−0.6	−0.4	−0.2	0	0.2	0.4	0.6	0.8	1
$\mathrm{VaR}_{0.95}(S_2)$	53.08	54.07	55.06	56.04	57.01	57.96	58.88	59.78	60.64	61.48	62.29
$\mathrm{TVaR}_{0.95}(S_2)$	63.81	65.28	66.66	67.96	69.18	70.33	71.42	72.44	73.40	74.31	75.17
$\mathrm{TVaR}_{0.95}(X_1, S_2)$	41.08	41.45	41.79	42.13	42.47	42.80	43.12	43.44	43.75	44.06	44.37
$C_{0.95}(X_1, S_2)$	37.98	38.55	39.08	39.59	40.06	40.50	40.92	41.31	41.67	42.02	42.35
$\mathrm{TVaR}_{0.95}(X_2, S_2)$	22.72	23.83	24.86	25.82	26.71	27.54	28.30	29.00	29.65	30.24	30.80
$C_{0.95}(X_2, S_2)$	25.83	26.73	27.57	28.37	29.12	29.83	30.50	31.13	31.73	32.29	32.82



(a) $\mathrm{TVaR}_\kappa(S_2)$.

(b) $\mathrm{TVaR}_\kappa(X_1, S_2)$.

(c) $C_\kappa(X_1, S_2)$.

(d) $\frac{\mathrm{TVaR}_\kappa(X_1,S_2)}{C_\kappa(X_1,S_2)}$.

Fig. 2. Comparison of the capital allocation based on the TVaR and the covariance rules.

Proof. To simplify the presentation, it is assumed that $n > 2$. Using (1) and (6), the joint pdf of $\underline{X} = (X_1, \dots, X_n)$ is given by

$$f_{\underline{X}}(x_1, \dots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) P(2F_{X_1}(x_1), \dots, 2F_{X_n}(x_n)) \tag{25}$$

and the expression for the pdf of S_n is obtained with

$$f_{S_n}(s) = \int_0^s \int_0^{s-x_1} \dots \int_0^{s-x_1-\dots-x_{n-2}} f_{\underline{X}}(x_1, x_2, \dots, s - x_1 - \dots - x_{n-1}) dx_{n-1} \dots dx_2 dx_1. \tag{26}$$

Using (5), we have

$$\begin{aligned} P(2F_{X_1}(x_1), 2F_{X_2}(x_2), \dots, 2F_{X_n}(x_n)) &= 1 + \sum_{k=2}^{n} \sum_{1 \le j_1 < \dots < j_k \le n} \theta_{j_1 j_2 \dots j_k} (2\overline{F}_{X_{j_1}}(x_{j_1}) - 1) \dots (2\overline{F}_{X_{j_k}}(x_{j_k}) - 1) \\ &= 1 + \zeta + Q(x_1, x_2, \dots, x_n), \end{aligned} \tag{27}$$

where

$$Q(x_1, \dots, x_n) = \sum_{k=2}^{n} \sum_{1 \le j_1 < \dots < j_k \le n} \theta_{j_1 j_2 \dots j_k} \times \sum_{l=0}^{k-1} \sum_{(i_1, i_2, \dots, i_{k-l}) \in C_{J_k}^{k-l}} (-1)^l 2^{k-l} \prod_{m=1}^{k-l} \overline{F}_{X_{i_m}}(x_{i_m}). \tag{28}$$

The joint pdf in (25) hence becomes

$$f_{\underline{X}}(x_1, \dots, x_n) = (1 + \zeta) \prod_{i=1}^{n} f_{X_i}(x_i) + \sum_{k=2}^{n} \sum_{1 \le j_1 < \dots < j_k \le n} \theta_{j_1 j_2 \dots j_k} \times \sum_{l=0}^{k-1} \sum_{\{i_1, i_2, \dots, i_{k-l}\} \in C_{J_k}^{k-l}} (-1)^l 2^{k-l} \prod_{m=1}^{k-l} \overline{F}_{X_{i_m}}(x_{i_m}) \prod_{i=1}^{n} f_{X_i}(x_i). \tag{29}$$

For calculation purposes, we decompose (28) as follows:

$$\begin{aligned} Q(x_1, \dots, x_n) &= \sum_{k=2}^{n-1} \sum_{1 \le j_1 < \dots < j_k \le n-1} \theta_{j_1 j_2 \dots j_k} \times \sum_{l=0}^{k-1} \sum_{(i_1, i_2, \dots, i_{k-l}) \in C_{J_k}^{k-l}} (-1)^l 2^{k-l} \prod_{m=1}^{k-l} \overline{F}_{X_{i_m}}(x_{i_m}) \\ &+ \sum_{k=2}^{n} \sum_{1 \le j_1 < \dots < j_{k-1} \le n-1} \theta_{j_1 j_2 \dots j_n} \times \sum_{l=0}^{k-1} \sum_{(i_1, i_2, \dots, i_{k-l}) \in C_{J_k - \{n\}}^{k-l}} (-1)^l 2^{k-l} \prod_{m=1}^{k-l} \overline{F}_{X_{i_m}}(x_{i_m}) \\ &+ \sum_{k=2}^{n} \sum_{1 \le j_1 < \dots < j_{k-1} \le n-1} \theta_{j_1 j_2 \dots j_n} \times \sum_{l=0}^{k-1} \sum_{(i_1, i_2, \dots, i_{k-l-1}) \in C_{J_k - \{n\}}^{k-l-1}} (-1)^l 2^{k-l} \overline{F}_{X_n}(x_n) \prod_{m=1}^{k-l-1} \overline{F}_{X_{i_m}}(x_{i_m}). \end{aligned} \tag{30}$$

With (27) and (30), the expression for the pdf of S_n in (26) becomes

$$f_{S_n}(s) = (1 + \zeta) \int_0^s \int_0^{s-x_1} \dots \int_0^{s-x_1-\dots-x_{n-2}} \prod_{i=1}^{n-1} f_{X_i}(x_i) \times f_{X_n}(s - x_1 - \dots - x_{n-1}) dx_{n-1} \dots dx_2 dx_1 + I(s), \tag{31}$$

where

$$\begin{aligned} I(s) &= \sum_{k=2}^{n-1} \sum_{1 \le j_1 < \dots < j_k \le n-1} \theta_{j_1 j_2 \dots j_k} \sum_{l=0}^{k-1} \sum_{\{i_1, i_2, \dots, i_{k-l}\} \in C_{J_k}^{k-l}} (-1)^l \\ &\times \int_0^s \int_0^{s-x_1} \dots \int_0^{s-x_1-\dots-x_{n-2}} \prod_{m=k-l+1}^{n-1} f_{X_{i_m}}(x_{i_m}) \\ &\times \prod_{m=1}^{k-l} \left(2\overline{F}_{X_{i_m}}(x_{i_m}) f_{X_{i_m}}(x_{i_m})\right) f_{X_n}(s - x_1 - \dots - x_{n-1}) \\ &\times dx_{n-1} \dots dx_2 dx_1 + \sum_{k=2}^{n} \sum_{1 \le j_1 < \dots < j_{k-1} \le n-1} \theta_{j_1 j_2 \dots j_n} \\ &\times \sum_{l=0}^{k-1} \sum_{(i_1, i_2, \dots, i_{k-l}) \in C_{J_k - \{n\}}^{k-l}} (-1)^l \\ &\times \int_0^s \int_0^{s-x_1} \dots \int_0^{s-x_1-\dots-x_{n-2}} \prod_{m=k-l+1}^{n-1} f_{X_{i_m}}(x_{i_m}) \\ &\times \prod_{m=1}^{k-l} \left(2\overline{F}_{X_{i_m}}(x_{i_m}) f_{X_{i_m}}(x_{i_m})\right) f_{X_n}(s - x_1 - \dots - x_{n-1}) \\ &\times dx_{n-1} \dots dx_2 dx_1 + \sum_{k=2}^{n} \sum_{1 \le j_1 < \dots < j_{k-1} \le n-1} \theta_{j_1 j_2 \dots j_n} \\ &\times \sum_{l=0}^{k-1} \sum_{(i_1, i_2, \dots, i_{k-l-1}) \in C_{J_k - \{n\}}^{k-l-1}} (-1)^l \\ &\times \int_0^s \int_0^{s-x_1} \dots \int_0^{s-x_1-\dots-x_{n-2}} \prod_{m=k-l+1}^{n-1} f_{X_{i_m}}(x_{i_m}) \\ &\times \prod_{m=1}^{k-l-1} \left(2\overline{F}_{X_{j_m}}(x_{j_m}) f_{X_{i_m}}(x_{i_m})\right) \\ &\times (2\overline{F}_{X_n}(s - x_1 - \dots - x_{n-1})) \\ &\times f_{X_n}(s - x_1 - \dots - x_{n-1}) dx_{n-1} \dots dx_2 dx_1. \end{aligned} \tag{32}$$

By replacing (32) in (31), we observe that the expression for the pdf of S_n can be seen as a sum of convolutions of mixed Erlang distributions as in the bivariate case. Hence, with Lemmas 2.1–2.4, one can write

$$f_{S_n}(s) = \sum_{j=n}^{\infty} p_j^{(n)} h(s; j, 2\beta_n),$$

with $p_j^{(n)}$ as given in (24). □

Since S_n follows a mixed Erlang distribution, we have closed-form expressions for the TVaR and the stop-loss premium associated with S_n.

Corollary 4.1. *Let $X_1, \dots, X_n$ be n mixed Erlang distributed rvs with $X_i \sim \text{MixErl}\left(\underline{p}_i, \beta_i\right)$, $i = 1, \dots, n$. Assuming $\beta_i \le \beta_n$, for $i = 1, 2, \dots, n-1$, and a dependence structure for $(X_1, \dots, X_n)$ based on the FGM n-copula, the closed form expression for the TVaR* *risk measure, at a given level $\kappa \in [0,1]$, is given by*

$$\mathrm{TVaR}_\kappa(S_n) = \frac{1}{1-\kappa}\sum_{j=1}^{\infty} p_j^{(n)} \frac{j}{2\beta_n}\overline{H}\left(\mathrm{VaR}_\kappa(S_n); j+1, 2\beta_n\right). \quad (33)$$

For a given retention $d \in \mathbb{R}^+$, the stop-loss premium is given by

$$\pi_{S_n}(d) = \sum_{j=1}^{\infty} p_j^{(n)} e^{-2\beta_n d}\frac{(2\beta_n d)^j}{j!}, \quad (34)$$

where the probabilities $\underline{p}^{(n)}$ are as given in (24).

Proof. Applying the result of Proposition 4.2 with (11) and (5) leads to (33) and (34). □

4.2. TVaR-based capital allocation

The expression for the amount allocated to the risks X_i, $i = 1, \ldots, n$, under the TVaR-based allocation rule is provided in the following proposition.

Proposition 4.3. *Let $(X_1, \ldots, X_n)$ have a multivariate distribution defined with the FGM n-copula and $X_i \sim \mathrm{MixErl}\left(\underline{p}_i, \beta_i\right)$, $i = 1, \ldots, n$ with $\beta_i \le \beta_n$, for $i = 1, 2, \ldots, n-1$. Then, the expression for $\mathrm{TVaR}_\kappa(X_i, S_n)$ at level κ, $0 < \kappa < 1$, is given by*

$$\mathrm{TVaR}_\kappa(X_i, S_n) = \frac{1}{1-\kappa}\sum_{k=1}^{\infty} q_{i,k}^{(n)} \frac{k}{2\beta_n}\overline{H}\left(\mathrm{VaR}_\kappa(S_n); k+1, 2\beta_n\right),$$

with $q_{i,k}^{(n)} = 0$ for $i = 1, \ldots, n$ and

$$\begin{aligned} q_{i,k}^{(n)} = {} & (1+\zeta)E[X_i]\,\sigma^{(n)}(j, \underline{\omega}(\underline{p}_1, \beta_1, 2\beta_n), \ldots, \underline{\omega}(\alpha(\underline{p}_{i_n}), \\ & \beta_{i_n}, 2\beta_n), \ldots, \underline{\omega}(\underline{p}_n, \beta_n, 2\beta_n)) \\ & + \Pi_i \sum_{k=2}^{n} \sum_{1\le j_1<\cdots<j_k\le n} \theta_{j_1 j_2 \ldots j_k} \sum_{l=0}^{k-1} \sum_{\{i_1, i_2, \ldots, i_{k-l}\}\in C_{J_k,i}^{k-l}} (-1)^l \\ & \times \sigma^{(n)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_{i_1}), 2\beta_{i_1}, 2\beta_n), \ldots, \underline{\omega}(\alpha(\underline{\pi}(\underline{p}_i)), \\ & 2\beta_i, 2\beta_n), \ldots, \underline{\omega}(\alpha(\underline{\pi}(\underline{p}_{i_{k-l}})), 2\beta_{i_{k-l}}, 2\beta_n), \\ & \underline{\omega}(\underline{p}_{i_{k-l+1}}, \beta_{i_{k-l+1}}, 2\beta_n), \ldots, \underline{\omega}(\underline{p}_{i_n}, \beta_{i_n}, 2\beta_n)) \\ & + E[X_i] \sum_{k=2}^{n} \sum_{1\le j_1<\cdots<j_k\le n} \theta_{j_1 j_2 \ldots j_k} \\ & \times \sum_{l=0}^{k-1} \sum_{\{i_1, i_2, \ldots, i_{k-l}\}\in C_{J_k,-i}^{k-l}} (-1)^l \sigma^{(n)}(j, \underline{\omega}(\underline{\pi}(\underline{p}_{i_1}), \\ & 2\beta_{i_1}, 2\beta_n), \ldots, \underline{\omega}(\underline{\pi}(\underline{p}_{i_{k-l}}), 2\beta_{i_{k-l}}, 2\beta_n), \\ & \underline{\omega}(\underline{p}_{i_{k-l+1}}, \beta_{i_{k-l+1}}, 2\beta_n), \ldots, \\ & \underline{\omega}(\underline{p}_i, \beta_i, 2\beta_n), \ldots, \underline{\omega}(\underline{p}_{i_n}, \beta_{i_n}, 2\beta_n)), \end{aligned} \quad (35)$$

for $k = n+1, n+2, \ldots$ where

$$\Pi_i = E\left[X_i \overline{F}_{X_i}(X_i)\right] = \sum_{j=1}^{\infty} \frac{j}{2\beta_i}\pi(j, \underline{p}_i),$$

and $\zeta = \sum_{k=2}^{n}\sum_{1\le j_1<\cdots<j_k\le n}(-1)^k\theta_{j_1 j_2 \ldots j_k}$. Also, $C_{J_k,i}^{k-l}$ is the set of all combinations of $(k-l)$ elements from $J_k = \{1 \le j_1 < \cdots < j_k \le n\}$ which includes i. Finally, $C_{J_k,-i}^{k-l}$ is the set of all combinations of $(k-l)$ elements from $J_k = \{1 \le j_1 < \cdots < j_k \le n\}$ which does not include i.

Proof. For $i = 1, 2, \ldots, n$, the capital attributed to the risk i is expressed as

$$\begin{aligned} \mathrm{TVaR}_\kappa(X_i, S_n) &= E[X_i 1_{\{S_n > \mathrm{VaR}_\kappa(S_n)\}}] \\ &= \frac{1}{1-\kappa}\int_{\mathrm{VaR}_\kappa(S_n)}^{+\infty} E[X_i 1_{\{S_n = s\}}]ds, \end{aligned} \quad (36)$$

where

$$\begin{aligned} E[X_i 1_{\{S_n=s\}}] = {} & \int_0^s \int_0^{s-x_1} \cdots \int_0^{s-x_1-\cdots-x_{n-2}} x_i f_{\underline{X}}(x_1, \ldots, \\ & s-x_1-\cdots-x_{n-1})dx_{n-1}\ldots dx_2 dx_1. \end{aligned} \quad (37)$$

From the joint pdf expression in (29), one can conclude that, for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} x_i f_{\underline{X}}(x_1, x_2, \ldots, x_n) = {} & (1+\zeta)\prod_{i=1}^{n} f_{X_i}(x_i) \\ & + \sum_{k=2}^{n}\sum_{1\le j_1<\cdots<j_k\le n}\theta_{j_1 j_2 \ldots j_k}\sum_{l=0}^{k-1}\sum_{\{i_1, i_2, \ldots, i_{k-l}\}\in C_{J_k}^{k-l}} (-1)^l 2^{k-l} x_i \\ & \times \prod_{m=1}^{k-l}\overline{F}_{X_{i_m}}(x_{i_m})\prod_{i=1}^{n} f_{X_i}(x_i). \end{aligned} \quad (38)$$

After some rearrangements, (38) becomes

$$\begin{aligned} x_i f_{\underline{X}}(x_1, x_2, \ldots, x_n) = {} & (1+\zeta)\prod_{i=1}^{n} f_{X_i}(x_i) \\ & + \sum_{k=2}^{n}\sum_{1\le j_1<\cdots<j_k\le n}\theta_{j_1 j_2 \ldots j_k} \\ & \times \sum_{l=0}^{k-1}\sum_{\{i_1, i_2, \ldots, i_{k-l}\}\in C_{J_k,i}^{k-l}} (-1)^l (2x_i f_{X_i}(x_i)\overline{F}_{X_i}(x_i)) \\ & \times \prod_{m=1\neq i}^{k-l} 2f_{X_{i_m}}(x_{i_m})\overline{F}_{X_{i_m}}(x_{i_m}) \prod_{m=k+l+1}^{n} f_{X_{i_m}}(x_{i_m}) \\ & + \sum_{k=2}^{n}\sum_{1\le j_1<\cdots<j_k\le n}\theta_{j_1 j_2 \ldots j_k} \\ & \times \sum_{l=0}^{k-1}\sum_{\{i_1, i_2, \ldots, i_{k-l}\}\in C_{J_k^{\prime}}^{k-l}} (-1)^l (x_i f_{X_i}(x_i)) \\ & \times \prod_{m=1}^{k-l} 2f_{X_{i_m}}(x_{i_m})\overline{F}_{X_{i_m}}(x_{i_m}) \prod_{m=k+l+1\neq i}^{n} f_{X_{i_m}}(x_{i_m}). \end{aligned} \quad (39)$$

Substituting (39) into (37) and using Lemmas 2.3–2.5 lead to the following expression for $E[X_i 1_{\{S_n=s\}}]$:

$$E[X_i 1_{\{S_n=s\}}] = \sum_{k=n+1}^{\infty} q_{i,k}^{(n)} h(s; k, 2\beta_n),$$

where the probabilities $q_{i,k}$ are as given in (35). The desired result follows. □

4.3. Covariance-based capital allocation

In the following proposition, we give the expression for the amount allocated to the risks X_i, $i = 1, \ldots, n$, under the covariance-based allocation rule.

Proposition 4.4. *Let $(X_1, \ldots, X_n)$ have a multivariate distribution defined with the FGM n-copula and $X_i \sim \mathrm{MixErl}\left(\underline{p}_i, \beta_i\right)$, $i =$*

Table 5
Descriptive statistics of X_1, X_2 and X_3.

$E[X_1]$	$E[X_2]$	$E[X_3]$	$\mathrm{Var}[X_1]$	$\mathrm{Var}[X_2]$	$\mathrm{Var}[X_3]$	$\mathrm{Cov}(X_1,X_2)$	$\mathrm{Cov}(X_1,X_3)$	$\mathrm{Cov}(X_2,X_3)$
15	11.33	11	175	84.88	69	10.00	6.11	−2.15

Table 6
Probabilities $p_k^{(3)}$ for $S_3 = X_1 + X_2 + X_3$.

k	1	2	3	4	5	6	7	8	9	10
$p_k^{(3)}$	0	0	0.0021	0.0081	0.0179	0.0299	0.0415	0.0512	0.0583	0.0628
k	11	12	13	14	15	16	17	18	19	20
$p_k^{(3)}$	0.0652	0.0658	0.0648	0.0625	0.0592	0.0551	0.0504	0.0454	0.0404	0.0354
k	21	22	23	24	25	26	27	28	29	30
$p_k^{(3)}$	0.0307	0.0264	0.0224	0.0189	0.0158	0.0131	0.0107	0.0088	0.0071	0.0058
k	31	32	33	34	35	36	37	38	39	40
$p_k^{(3)}$	0.0047	0.0037	0.0030	0.0024	0.0019	0.0015	0.0011	0.0009	0.0007	0.0005
k	41	42	43	44	45	46	47	48	49	50
$p_k^{(3)}$	0.0004	0.0003	0.0002	0.0002	0.0001	0.0001	0.0001	0.00007	0.00006	0.00004

$1, \ldots, n$ with $\beta_i \le \beta_n$, for $i = 1, 2, \ldots, n-1$. Then, the contribution $C_\kappa(X_i, S_n)$ at level κ, $0 < \kappa < 1$, is given by

$$C_\kappa(X_i, S_n) = \sum_{k=1}^{\infty} c_{i,k} \frac{k}{\beta_n}, \tag{40}$$

where

$$c_{i,k} = \omega(k, \underline{p}_i, \beta_i, \beta_n) + 2\rho_{i,k} p_k^{(n)} \times \left[\frac{\overline{H}(\mathrm{VaR}_\kappa(S_n); k+1, 2\beta_n)}{1-\kappa} - 1 \right]$$

with

$$\rho_{i,k} = \frac{\sum_{l=1}^{\infty} l\omega(l, \underline{p}_i, \beta_i, \beta_n) - \left(\sum_{l=1}^{\infty} l\omega(l, \underline{p}_i, \beta_i, \beta_n) \right)^2}{\sum_{l=1}^{\infty} l p_l^{(n)} - \left(\sum_{l=1}^{\infty} l p_l^{(n)} \right)^2} + \frac{\sum_{j=1, j\ne i}^{n} \theta_{i,j} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} lm \left(\pi(l, \underline{\omega}(\underline{p}_j, \beta_i, \beta_n)) - \omega(l, \underline{p}_i, \beta_i, \beta_n) \right) \left(\pi(m, \underline{p}_j) - p_{j,m} \right)}{\sum_{l=1}^{\infty} l p_l^{(n)} - \left(\sum_{l=1}^{\infty} l p_l^{(n)} \right)^2},$$

for $i = 1, \ldots, n$, and $\theta_{i,j} = \theta_{\min(i,j)\max(i,j)}$.

Proof. The amount of capital attributed to risk i, for $i = 1, \ldots, n$, is given in (10) by

$$C_\kappa(X_i, S_n) = E[X_i] + \frac{\mathrm{Cov}(X_i, S_n)}{\mathrm{Var}(S_n)} \left(\mathrm{TVaR}_\kappa(S_n) - E[S_n] \right). \tag{41}$$

Using Lemma 2.4, we have

$$E[X_i] = \sum_{k=1}^{\infty} \omega(k, \underline{p}_i, \beta_i, \beta_n) \frac{k}{\beta_n}.$$

We also know that $S_n \sim \mathrm{MixErl}\left(\underline{p}^{(n)}, 2\beta_n\right)$; hence

$$E[S_n] = \sum_{k=1}^{\infty} p_k^{(n)} \frac{k}{2\beta_n},$$

and

$$\mathrm{Var}(S_n) = \frac{1}{4\beta_n} \sum_{l=1}^{\infty} l p_l^{(n)} - \left(\sum_{l=1}^{\infty} l p_l^{(n)} \right)^2.$$

Given Corollary 4.1, (41) becomes

$$C_\kappa(X_i, S_n) = \sum_{k=1}^{\infty} \left[\omega(k, \underline{p}_i, \beta_i, \beta_n) + \frac{\mathrm{Cov}(X_i, S_n)}{2\mathrm{Var}(S_n)} p_k^{(n)} \times \left[\frac{\overline{H}(\mathrm{VaR}_\kappa(S_n); k+1, 2\beta_n)}{1-\kappa} - 1 \right] \right] \frac{k}{\beta_n}.$$

Using the expression for the covariance given in Remark 4.1, one may find the expression in (40). □

4.4. A numerical application: the trivariate case

We illustrate here our results for the trivariate case via a numerical example. We suppose that the pdfs of X_i, for $i = 1, 2, 3$ are given by

$$f_{X_1}(x) = 0.5h(x; 1, 0.1) + 0.5h(x; 2, 0.1),$$
$$f_{X_2}(x) = 0.3h(x; 1, 0.15) + 0.7h(x; 2, 0.15),$$
$$f_{X_3}(x) = 0.2h(x; 1, 0.2) + 0.4h(x; 2, 0.2) + 0.4h(x; 3, 0.2).$$

We assume that the dependence structure is given by the following FGM 3-copula:

$$C(u_1, u_2, u_3) = u_1 u_2 u_3 P(u_1, u_2, u_3),$$

where

$$\begin{aligned} P(u_1, u_2, u_3) = {} & 1 + \theta_{12}(1-u_1)(1-u_2) \\ & + \theta_{13}(1-u_1)(1-u_3) + \theta_{23}(1-u_2)(1-u_3) \\ & + \theta_{123}(1-u_1)(1-u_2)(1-u_3). \end{aligned}$$

For $n = 3$, (29) becomes

$$\begin{aligned} f_{\underline{X}}(x_1, x_2, x_3) = {} & \prod_{i=1}^{3} f_{X_i}(x_i)[1 + \overline{\theta} + 2(\theta_{123} - \theta_{12} - \theta_{13})\overline{F}_{X_1}(x_1) \\ & + 2(\theta_{123} - \theta_{12} - \theta_{23})\overline{F}_{X_2}(x_2) + 2(\theta_{123} - \theta_{13} - \theta_{23})\overline{F}_{X_3}(x_3) \\ & - 4(\theta_{123} - \theta_{12})\overline{F}_{X_1}(x_1)\overline{F}_{X_2}(x_2) - 4(\theta_{123} - \theta_{13})\overline{F}_{X_1}(x_1)\overline{F}_{X_3}(x_3) \\ & - 4(\theta_{123} - \theta_{23})\overline{F}_{X_2}(x_2)\overline{F}_{X_3}(x_3) + 8\theta_{123}\overline{F}_{X_1}(x_1)\overline{F}_{X_2}(x_2)\overline{F}_{X_3}(x_3)]. \end{aligned}$$

We set $\theta_{12} = 0.3, \theta_{13} = 0.2, \theta_{23} = -0.1, \theta_{123} = 0.15$, which respects conditions (7). Table 5 displays the values of the expectations, variances and covariances of X_1, X_2 and X_3. Table 6 gives the first 50 values of the probabilities $p_k^{(3)}$, $k = 1, 2, \ldots$ and Table 7 displays the results obtained with the closed-form formulas for the VaR and TVaR of S_3. In Table 8, we compare once again the capital allocation using the TVaR-based allocation rule and the covariance-based allocation rule at different levels κ. As in the bivariate case, important differences between the results are obtained with the two different allocation rules. Note that the routine implemented in Matlab takes only a few seconds to produce these results.

Table 7
Value-at-Risk measures and Tail-Value-at-Risk measures for $S_3 = X_1 + X_2 + X_3$.

κ	0.1	0.5	0.6	0.7	0.75	0.85	0.9	0.95	0.99	0.995	0.999
$\mathrm{VaR}_\kappa(S_3)$	15.83	34.48	39.30	44.89	48.16	56.61	62.79	72.63	93.44	101.87	120.75
$\mathrm{TVaR}_\kappa(S_3)$	40.28	52.06	55.86	60.48	63.28	70.73	76.33	85.45	105.15	113.10	129.70

Table 8
Capital allocation based on the TVaR and the covariance rules for the trivariate case.

κ	0.1	0.5	0.6	0.7	0.75	0.85	0.9	0.95	0.99	0.995	0.999
$\mathrm{TVaR}_\kappa(X_1, S_3)$	16.21	21.64	23.58	26.09	27.69	32.25	35.95	42.40	57.82	64.58	80.48
$C_\kappa(X_1, S_3)$	16.53	22.85	24.89	27.37	28.87	32.87	35.87	40.78	51.44	55.83	65.73
$\mathrm{TVaR}_\kappa(X_2, S_3)$	12.13	14.97	15.82	16.82	17.42	18.97	20.09	21.81	25.04	26.17	28.28
$C_\kappa(X_2, S_3)$	12.07	15.14	16.13	17.33	18.06	20.00	21.46	23.84	29.01	31.14	35.95
$\mathrm{TVaR}_\kappa(X_3, S_3)$	11.93	15.45	16.46	17.57	18.17	19.51	20.30	21.28	22.51	22.81	23.28
$C_\kappa(X_2, S_3)$	11.58	13.99	14.77	15.72	16.29	17.82	18.97	20.84	24.91	26.58	30.36
$\mathrm{TVaR}_\kappa(S_3)$	40.28	52.06	55.87	60.49	63.29	70.74	76.35	85.50	105.38	113.57	132.05

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