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Dependent risk models with Archimedean copulas: A computational strategy based on common mixtures and applications

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ABSTRACT

In this paper, we investigate dependent risk models in which the dependence structure is defined by an Archimedean copula. Using such a structure with specific marginals, we derive explicit expressions for the pdf of the aggregated risk and other related quantities. The common mixture representation of Archimedean copulas is at the basis of a computational strategy proposed to find exact or approximated values of the distribution of the sum of risks in a general setup. Such results are then used to investigate risk models in regard to aggregation, capital allocation and ruin problems. An extension to nested Archimedean copulas is also discussed.

1. Introduction

The present paper deals with risk models incorporating dependent components whose dependence structure is induced via an Archimedean copula. Copula theory provides a flexible approach for modeling the dependency relationship between risks. A copula is a multivariate distribution function for which the marginals are standard uniformly distributed. See e.g. Joe (1997) or Nelsen (2007) for further details. One important class of copulas is the class of Archimedean copulas.

A d-dimensional copula C is said to be an *Archimedean copula* if

$$C(u_1,\dots,u_d) = \psi\left(\psi^{-1}(u_1)+\cdots+\psi^{-1}(u_d)\right), \quad \text{for } (u_1,\dots,u_d)\in[0,1]^d. \tag{1}$$

The continuous and strictly decreasing function ψ is called the generator of the copula, where $\psi : [0,\infty) \to [0,1]$, $\psi(0) = 1$ and $\lim_{t\to\infty}\psi(t) = 0$. In the same manner, $\psi^{-1} : [0,1] \to [0,\infty)$, for which $\psi^{-1}(0) = \inf\{t : \psi(t) = 0\}$, where ψ^{-1} is the inverse of the generator ψ. McNeil and Nešlehová (2009) show that (1) is a d-dimensional copula if and only if ψ is a d-monotone function. In this paper, we consider a specific class of Archimedean copulas with completely monotone generators. By using Bernstein's theorem (see e.g. Feller, 1971), it has been shown that such generators correspond to the Laplace–Stieltjes Transform (LST) of a strictly positive rv Θ with cumulative distribution function (cdf) F_Θ, where the LST of the rv Θ is given by

$$\mathcal{L}_\Theta(t) = \int_0^\infty e^{-t\theta}\,dF_\Theta(\theta) = E\left[e^{-t\Theta}\right]. \tag{2}$$

Then, the expression in (1) becomes

$$C(u_1,\dots,u_d) = \mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}(u_1)+\cdots+\mathcal{L}_\Theta^{-1}(u_d)\right). \tag{3}$$

The strictly positive rv Θ, which can be either discrete or continuous, corresponds to a latent mixing rv, and there is a one-to-one relation between its distribution and the expression of an Archimedean copula C. This special class of Archimedean copulas defined in (3) is intimately linked to common mixtures. As explained in Denuit et al. (2006) and Embrechts et al. (2005), the representation of an Archimedean copula C as a common mixture allows us to identify the conditional univariate cumulative distribution functions (cdfs) of $(U_1|\Theta=\theta)$, …, $(U_d|\Theta=\theta)$, where $\underline{U} = (U_1,\dots,U_d)$ and $U_i \sim Unif\,(0,1)$, $i = 1,2,\dots,d$. Using the definition in (2) in combination with (3), we have the following representation of an Archimedean copula C as a common mixture:

$$\begin{aligned} C(u_1,\dots,u_d) &= \psi\left(\psi^{-1}(u_1)+\cdots+\psi^{-1}(u_d)\right) \\ &= \mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}(u_1)+\cdots+\mathcal{L}_\Theta^{-1}(u_d)\right) \\ &= \int_0^\infty \prod_{i=1}^d e^{-\theta\mathcal{L}_\Theta^{-1}(u_i)}\,dF_\Theta(\theta) \end{aligned}$$

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which becomes

$$C(u_1,\ldots,u_d) = F_{\underline{U}}(u_1,\ldots,u_d) = \int_0^\infty \prod_{i=1}^d F_{U_i|\Theta=\theta}(u_i)\,\mathrm{d}F_\Theta(\theta). \quad (4)$$

Then, (4) implies that the conditional cdf of $(U_i|\Theta = \theta)$ is $F_{U_i|\Theta=\theta}(u_i) = \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(u_i)}$, for $i = 1,\ldots,d$, $u_i \in [0,1]$, $\theta > 0$, which leads to the useful representation of the copula C as a common mixture. Examples of the most popular Archimedean copulas (Θ discrete and continuous) are provided in Appendix section (see e.g. Nelsen, 2007 for an extensive list of Archimedean copulas).

The above representation provides a natural sampling algorithm (see e.g. Marshall and Olkin, 1988; Hofert, 2008 and references therein) useful for actuarial science and quantitative risk management purposes. Mixture representations are frequently involved in risk models. For example, in a credit-risk context, Vasicek (1987) was the first to develop the idea of the conditional independence of all defaults upon some market factor. Later, this idea was used by Li (2000), Schönbucher and Schubert (2001), Frey and McNeil (2001) and put into a copula setup. Among the papers that exploit the conditional independence technique with Archimedean and nested Archimedean copulas in a credit risk context one finds for example Schönbucher (2002) and Hofert and Scherer (2011). Copulas in a credit risk setting are also treated with much detail in Cherubini et al. (2004). See references therein. Several researchers have also used the conditional representation of Archimedean and nested Archimedean copulas to derive their sampling algorithms see e.g. Marshall and Olkin (1988), McNeil (2008) and Hofert (2012). Archimedean risk models are also involved in the context of collective risk models with dependence. Thanks to the latter representation, Albrecher et al. (2011) were able to establish explicit formulas for the ruin probability with dependence among claim sizes and among claim inter-occurrence times modeled by Archimedean copulas. Also, a recent work of Jordanova et al. (2017), considered dependent inter-arrival times and exploited Archimedean copulas as treated in Albrecher et al. (2011).

Our objective here is to explore in more depth the mixture representation of an Archimedean copula and its advantages. More precisely, we propose a new strategy relying on (4), to tackle risk assessment problems such as risk aggregation for finite and random sums, capital allocation, ruin problems and so on. We show that this representation allows to avoid entirely or partially Monte Carlo (MC) simulations. Furthermore, this methodology is accurate, exact in specific cases, very flexible, and most importantly, naturally applicable in high dimensions.

The outline of the paper is as follows. We propose in Section 2, a computational methodology based on the common mixture representation of Archimedean copulas in different settings. This strategy allows to derive the distribution of the aggregated risks which can be later used in different applications. Analytic expressions related to the aggregated risks are also derived for some special cases using specific marginal distributions. Section 3 deals with capital allocation issues involving the strategy proposed in Section 2. Random sums are then considered in Section 4. Sections 5 and 6 are devoted to the investigation of ruin problems. Finally, Section 7 discusses the application of the mixture-based strategy in the case of a hierarchical dependence structure based on Archimedean copulas.

2. Computational methodology based on the common mixture representation of Archimedean copulas

2.1. Common mixture representation

Let $\underline{X} = (X_1,\ldots,X_d)$ be a vector of rvs with multivariate distribution defined in terms of a d-dimensional Archimedean copula C given in (3). The multivariate cdf $F_{\underline{X}}$ of $\underline{X}$ can be defined with the copula C and marginal univariate cdfs $F_{X_1},\ldots,F_{X_d}$ as

$$F_{\underline{X}}(x_1,\ldots,x_d) = C\left(F_{X_1}(x_1),\ldots,F_{X_d}(x_d)\right). \quad (5)$$

The common mixture representation of $F_{\underline{X}}$ is given by

$$F_{\underline{X}}(x_1,\ldots,x_d) = \int_0^\infty \prod_{i=1}^d F_{X_i|\Theta=\theta}(x_i)\,\mathrm{d}F_\Theta(\theta) = \int_0^\infty \prod_{i=1}^d \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}\left(F_{X_i}(x_i)\right)}\mathrm{d}F_\Theta(\theta), \quad (6)$$

where

$$F_{X_i|\Theta=\theta}(x_i) = \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}\left(F_{X_i}(x_i)\right)} \ (i = 1,2,\ldots,d). \quad (7)$$

Similarly, we can define the multivariate distribution of $\underline{X}$ through its multivariate survival function with the copula C and marginal univariate survival functions $\overline{F}_{X_1},\ldots,\overline{F}_{X_d}$, i.e.,

$$\overline{F}_{\underline{X}}(x_1,\ldots,x_d) = C\left(\overline{F}_{X_1}(x_1),\ldots,\overline{F}_{X_d}(x_d)\right). \quad (8)$$

Then, the common mixture representation of $\overline{F}_{\underline{X}}$ is given by

$$\overline{F}_{\underline{X}}(x_1,\ldots,x_d) = \int_0^\infty \prod_{i=1}^d \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(x_i)\right)}\mathrm{d}F_\Theta(\theta) = \int_0^\infty \prod_{i=1}^d \overline{F}_{X_i|\Theta=\theta}(x_i)\,\mathrm{d}F_\Theta(\theta), \quad (9)$$

where

$$\overline{F}_{X_i|\Theta=\theta}(x_i) = \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(x_i)\right)} (i = 1,2,\ldots,d). \quad (10)$$

Our main objective in what follows is to find $E[\phi(S)]$, for any integrable univariate function ϕ of the rv $S = X_1 + \cdots + X_d$, or to find $E[\varphi(X_1,\ldots,X_d)]$, for any integrable d-variate function φ. The required steps to derive these quantities are as follows:

1. Find the expressions of the conditional cdfs (7) or the conditional survival functions (10) of $X_i|\Theta = \theta$.
2. Derive the conditional expectations $E[\phi(S)|\Theta = \theta]$ or $E[\varphi(X_1,\ldots,X_d)|\Theta = \theta]$.
3. Find $E[\phi(S)]$ and $E[\varphi(X_1,\ldots,X_d)]$ with

$$E[\phi(S)] = E_\Theta[E[\phi(S)|\Theta]] = \int_0^\infty E[\phi(S)|\Theta = \theta]\mathrm{d}F_\Theta(\theta),$$

and

$$E[\varphi(X_1,\ldots,X_d)] = E_\Theta[E[\varphi(X_1,\ldots,X_d)|\Theta]] = \int_0^\infty E[\varphi(X_1,\ldots,X_d)|\Theta = \theta]\mathrm{d}F_\Theta(\theta),$$

where $(S|\Theta = \theta) = \sum_{i=1}^d (X_i|\Theta = \theta)$.

We present below three examples with specific marginals but any Archimedean copula with generator $\mathcal{L}_\Theta$ in which the required steps just detailed to find the pdf of S are easily performed and allow explicit expressions.

Example 1. Let $\underline{X} = (X_1,\ldots,X_d)$ be a vector of exchangeable Bernoulli rvs where $X_i \sim Bern(q)$ $(i = 1,\ldots,d)$ and

$$F_{\underline{X}}(k_1,\ldots,k_d) = C\left(F_{X_1}(k_1),\ldots,F_{X_d}(k_d)\right),$$

for $k_1,\ldots,k_d \in \{0,1\}$. From (7), we find that

$$(X_i|\Theta = \theta) \sim Bern\left(1 - \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(1-q)}\right),$$

for $i = 1, 2, \ldots, d$. Therefore, $(S|\Theta = \theta)$ follows a binomial distribution with

$$f_{S|\Theta=\theta}(k) = \binom{d}{k}\left(1-\mathrm{e}^{-\theta\mathcal{L}_{\Theta}^{-1}(1-q)}\right)^{k}\mathrm{e}^{-\theta\mathcal{L}_{\Theta}^{-1}(1-q)(d-k)}$$
$$= \binom{d}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\mathrm{e}^{-\theta\mathcal{L}_{\Theta}^{-1}(1-q)(j+d-k)},$$

and we conclude

$$f_S(k) = \binom{d}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\mathcal{L}_{\Theta}\left(\mathcal{L}_{\Theta}^{-1}(1-q)(j+d-k)\right),$$

for $k = 0, 1, 2, \ldots, d$.

Example 2. Let $\underline{X} = (X_1, \ldots, X_d)$ be a vector of exchangeable Bernoulli rvs where $X_i \sim Bern(q)$ ($i = 1, \ldots, d$) and

$$\overline{F}_{\underline{X}}(k_1, \ldots, k_d) = C\left(\overline{F}_{X_1}(k_1), \ldots, \overline{F}_{X_d}(k_d)\right)$$

for $k_1, \ldots, k_d \in \{0, 1\}$. It follows from (10) that

$$(X_i|\Theta = \theta) \sim Bern\left(\mathrm{e}^{-\theta\mathcal{L}_{\Theta}^{-1}(q)}\right)$$

for $i = 1, 2, \ldots, d$. Since

$$f_{S|\Theta=\theta}(k) = \binom{d}{k}\mathrm{e}^{-\theta\mathcal{L}_{\Theta}^{-1}(q)k}\left(1-\mathrm{e}^{-\theta\mathcal{L}_{\Theta}^{-1}(q)}\right)^{n-k}$$
$$= \binom{d}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{j}\mathrm{e}^{-\theta\mathcal{L}_{\Theta}^{-1}(q)(k+j)},$$

we obtain

$$f_S(k) = \binom{d}{k}\sum_{j=0}^{d-k}\binom{d-k}{j}(-1)^{j}\mathcal{L}_{\Theta}\left(\mathcal{L}_{\Theta}^{-1}(q)(k+j)\right),$$

for $k = 0, 1, 2, \ldots, d$. This expression can be found e.g., in Cossette et al. (2002) or on page 321 in Marceau (2013).

Example 3. Let Θ be a strictly positive rv with LST $\mathcal{L}_{\Theta}$. Given $\Theta = \theta$, let $(X_1|\Theta = \theta), \ldots, (X_n|\Theta = \theta)$ conditionally independent rvs where $(X_i|\Theta = \theta)$ follows a geometric distribution with

$$f_{X_i|\Theta=\theta}(k) = \Pr(X_i = k|\Theta = \theta) = \mathrm{e}^{-\theta r k}\left(1-e^{-r\theta}\right)$$

and

$$\overline{F}_{X_i|\Theta=\theta}(k_i) = \Pr(X_i > k_i|\Theta = \theta) = \mathrm{e}^{-\theta r(k_i+1)}\ (i = 1, 2, \ldots, n)$$

for $k = 0, 1, 2, \ldots$ and $r > 0$. Then, $\underline{X} = (X_1, \ldots, X_n)$ follows a multivariate mixed geometric distribution where

$$\overline{F}_{\underline{X}}(k_1, \ldots, k_n) = \mathcal{L}_{\Theta}(r(k_1+1) + \cdots + r(k_n+1))$$

with

$$\overline{F}_{X_i}(k_i) = \mathcal{L}_{\Theta}(r(k_i+1))\ (i = 1, 2, \ldots, n).$$

It implies that

$$\overline{F}_{\underline{X}}(k_1, \ldots, k_n) = \mathcal{L}_{\Theta}\left(\mathcal{L}_{\Theta}^{-1}\left(\overline{F}_{X_1}(k_1)\right) + \cdots + \mathcal{L}_{\Theta}^{-1}\left(\overline{F}_{X_n}(k_n)\right)\right)$$
$$= C\left(\overline{F}_{X_1}(k_1), \ldots, \overline{F}_{X_n}(k_n)\right)$$

where C is an Archimedean copula defined with LST $\mathcal{L}_{\Theta}$.

Now, we are in position to derive the closed-form expression of the pmf of $S_n = \sum_{i=1}^{n} X_n$. Observe that $(S_n|\Theta = \theta)$ follows a negative binomial distribution with

$$f_{X_i|\Theta=\theta}(k) = \Pr(X_i = k|\Theta = \theta)$$
$$= \binom{k+n-1}{k}\mathrm{e}^{-\theta r k}\left(1-e^{-r\theta}\right)^{n}$$
$$= \binom{k+n-1}{k}\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}\mathrm{e}^{-\theta r(j+k)},$$

for $k = 0, 1, 2, \ldots$. Then, it follows that

$$f_S(k) = \binom{k+n-1}{k}\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}\mathcal{L}_{\Theta}(r(j+k)),$$

for $k = 0, 1, 2, \ldots$.

In the following subsection, we consider the family of multivariate mixed exponential distributions which is equivalent to choosing specific combinations of Archimedean copulas and marginals. For this class of multivariate distributions, we derive analytic expressions for the pdf of the sum of risks and other related quantities of interest.

2.2. Closed-form expressions for multivariate mixed exponential distributions

Let $\underline{X}$ follow a multivariate mixed exponential distribution which belongs to the class of multivariate distributions constructed by common frailty as explained in e.g. Marshall and Olkin (1988). Briefly, given $\Theta = \theta$, the conditional distribution of the rv X_i is exponential with mean $\frac{\lambda_i}{\theta}$, i.e.,

$$\overline{F}_{X_i|\Theta=\theta}(x_i) = \mathrm{e}^{\frac{-\theta x_i}{\lambda_i}}, \tag{11}$$

for $i = 1, \ldots, d$. It implies that the marginal survival function of X_i is given by

$$\overline{F}_{X_i}(x_i) = \mathcal{L}_{\Theta}\left(\frac{x_i}{\lambda_i}\right), \tag{12}$$

for $i = 1, \ldots, d$. Also, the multivariate survival function of $\underline{X}$ is given by

$$\overline{F}_{\underline{X}}(x_1, \ldots, x_d) = \mathcal{L}_{\Theta}\left(\frac{x_1}{\lambda_1} + \cdots + \frac{x_d}{\lambda_d}\right),$$

which implies that $\overline{F}_{\underline{X}}$ satisfies (8) with Archimedean copula C given in (3). Clearly, in this setting, the required steps 1, 2, and 3 of the methodology described in Section 2.1 are easily performed. The expression in (10) is clearly given in (11). We first consider the case where $0 < \lambda_1 < \cdots < \lambda_d$. Here, $(S|\Theta = \theta)$ follows a generalized Erlang distribution with pdf

$$f_{S|\Theta=\theta}(x) = \sum_{i=1}^{n}\left(\prod_{j=1, j\neq i}^{n}\frac{\lambda_i}{\lambda_i-\lambda_j}\right)\frac{\theta}{\lambda_i}\mathrm{e}^{-\frac{\theta}{\lambda_i}x}. \tag{13}$$

Then, using (13), the unconditional pdf of S is given by

$$f_S(x) = \int_0^{\infty} f_{S|\Theta=\theta}(x)\mathrm{d}F_{\Theta}(\theta)$$
$$= \sum_{i=1}^{d}\frac{1}{\lambda_i}\left(\prod_{j=1, j\neq i}^{d}\frac{\lambda_i}{\lambda_i-\lambda_j}\right)\left((-1)\frac{\mathrm{d}\mathcal{L}_{\Theta}(t)}{\mathrm{d}t}\bigg|_{t=\frac{x}{\lambda_i}}\right).$$

The bivariate case ($d = 2$) is detailed on page 295 of Marceau (2013). Sarabia et al. (2017) consider the subclass of multivariate mixed exponential distributions in which $\lambda_1 = \cdots = \lambda_d = 1$. In such a case, $(S|\Theta = \theta)$ follows an Erlang distribution with

$$f_{S|\Theta=\theta}(x) = \frac{\theta^d x^{d-1}}{\Gamma(d)}\mathrm{e}^{-\theta x}.$$

Sarabia et al. (2017) find the following closed-form expressions for the pdf and the survival function of the rv S:

$$f_S(x) = \frac{x^{d-1}}{\Gamma(d)}\left\{(-1)^d\frac{\mathrm{d}^d}{\mathrm{d}x^d}\mathcal{L}_{\Theta}(x)\right\} \tag{14}$$

and

$$\Pr(S > x) = \sum_{k=0}^{d-1} \frac{x^k}{k!} \left\{ (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathcal{L}_\Theta(x) \right\}, \ x \in \mathbb{R}^+.$$

From (14), one can deduce the following expression for the TVaR of the rv S:

$$\begin{aligned} TVaR_\kappa(S) &= \frac{E\left[S \times 1_{\{S > VaR_\kappa(S)\}}\right]}{1-\kappa} \\ &= \sum_{j=0}^{d} \frac{d \times (VaR_\kappa(S))^j}{j! \times (1-\kappa)} \left\{ (-1)^{j-1} \frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}} \mathcal{L}_\Theta(x) \Big|_{x=VaR_\kappa(S)} \right\}, \end{aligned}$$

where $VaR_\kappa(S)$ is the solution of $F_S(x) = \kappa$, with $\kappa \in [0, 1)$. The stop-loss function of S can also be expressed in terms of $\mathcal{L}_\Theta$ as follows

$$\begin{aligned} \Pi_S(y) &= E\left[\max(S - y; 0)\right] \\ &= d \sum_{j=0}^{d} \frac{y^j}{j!} \left\{ (-1)^{j-1} \frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}} \mathcal{L}_\Theta(x) \Big|_{x=y} \right\} \\ &\quad - y \sum_{j=0}^{d-1} \frac{y^j}{j!} \left\{ (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}x^j} \mathcal{L}_\Theta(x) \Big|_{x=y} \right\}, \ y \in \mathbb{R}^+. \end{aligned}$$

Sarabia et al. (2017) give explicit formulas for the pdf of S for some multivariate mixed exponential distributions for which the dependence structure is an Archimedean copula with continuous common factor Θ such as the Clayton and Gumbel copulas. In the same context, Dacorogna et al. (2016) use similar explicit formulas to compute analytically risk measures and the associated diversification benefit.

Since there are several much used copulas generated from a discrete mixing rv Θ such as the Ali–Mikhail–Haq (AMH) copula, the Frank copula and the Joe copula, we provide the two following examples to complete those provided in Sarabia et al. (2017). Also, since the derivatives of the generators are known for different Archimedean copulas with discrete or continuous mixing rv (see, e.g., Hofert et al., 2012), the results provided in the following examples follow directly from the definitions just presented above.

Example 4. Let $\underline{X} = (X_1, \ldots, X_d)$ follow a multivariate mixed exponential–geometric distribution, i.e., $\Theta \sim Geo(q)$ with probability mass function (pmf) $f_\Theta(k) = q(1-q)^{k-1}$, for $k \in \mathbb{N}$. Clearly, the dependence structure underlying this multivariate distribution is the AMH copula with dependence parameter $\alpha = 1 - q$. Then, the following properties hold:

1. $\overline{F}_{X_i}(x) = \mathcal{L}_\Theta(x) = \frac{q}{\mathrm{e}^x - (1-q)}, \ x > 0, \ i = 1, \ldots, d.$
2. $\bar{F}_{\underline{X}}(\underline{x}) = \frac{q}{\exp\left(\sum_{i=1}^d x_i\right) - (1-q)}, \ x > 0.$
3. $f_S(x) = \frac{x^{d-1} q}{(1-q)\,\Gamma(d)} Li_{-d}\left((1-q)\mathrm{e}^{-x}\right), \ x > 0.$
4. $\Pr(S > x) = \sum_{k=0}^{d-1} \frac{x^k q}{(1-q) \times k!} Li_{-k}\left((1-q)\mathrm{e}^{-x}\right), \ x > 0.$
5. $TVaR_\kappa(S) = \sum_{j=0}^{d} \frac{dq \times (VaR_\kappa(S))^j}{j! \times (1-\kappa)(1-q)} Li_{1-j}\left((1-q)\mathrm{e}^{-VaR_\kappa(S)}\right), \ \kappa \in [0, 1).$
6. $\Pi_S(y) = d\sum_{j=0}^{d} \frac{y^j q}{j!\,(1-q)} Li_{1-j}\left((1-q)\mathrm{e}^{-y}\right) - y\sum_{j=0}^{d-1} \frac{q\, y^j}{j!\,(1-q)} Li_{-j}\left((1-q)\mathrm{e}^{-y}\right), \ y \in \mathbb{R}^+.$

We denote by "Li" the general polylogarithm function defined as $Li_a(z) = \sum_{d=1}^{\infty} \frac{z^d}{d^a}$ (see e.g. Lewin, 1981 for more details).

Example 5. Let $\underline{X} = (X_1, \ldots, X_d)$ be an d-dimensional vector with multivariate mixed exponential–logarithmic distribution, i.e., $\Theta \sim Log(1 - \mathrm{e}^{-\alpha})$ with pmf $f_\Theta(k) = \frac{(1-e^{-\alpha})^k}{k\alpha}$, for $k \in \mathbb{N}$. The corresponding dependence structure is the Frank copula with dependence parameter α. Then, the following properties hold:

1. $\overline{F}_{X_i}(x) = \mathcal{L}_\Theta(x) = -\frac{\ln(1-(1-\mathrm{e}^{-\alpha})\mathrm{e}^{-x})}{\alpha}, \ x > 0, \ i = 1, \ldots, d.$
2. $\bar{F}_{\underline{X}}(\underline{x}) = -\frac{\ln\left(1-(1-\mathrm{e}^{-\alpha})\mathrm{e}^{-\sum_{i=1}^d x_i}\right)}{\alpha}, \ x > 0.$
3. $f_S(x) = \frac{x^{d-1}}{\Gamma(d)\alpha} Li_{1-d}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^x}\right), \ x > 0.$
4. $\Pr(S > x) = \sum_{k=0}^{d-1} \frac{x^k}{\alpha \times k!} Li_{1-k}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^x}\right), \ x \in \mathbb{R}^+.$
5. $TVaR_\kappa(S) = \sum_{j=0}^{d} \frac{d \times (VaR_\kappa(S))^j}{j! \times (1-\kappa)\alpha} Li_{2-j}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^{VaR_\kappa(S)}}\right), \ \kappa \in [0, 1).$
6. $\Pi_S(y) = d\sum_{j=0}^{d} \frac{y^j}{j!\,\alpha} Li_{2-j}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^y}\right) - y\sum_{j=0}^{d-1} \frac{y^j}{j!\,\alpha} Li_{1-j}\left(\frac{1-\mathrm{e}^{-\alpha}}{\mathrm{e}^y}\right), \ y \in \mathbb{R}^+.$

In this subsection, we have discussed multivariate mixed exponential distributions which, as highlighted in the examples, require specific combinations of the mixing rv Θ and the marginal distributions of X_i ($i = 1, \ldots, d$). The dependence structure defined by the Archimedean copula is governed by Θ and hence only one choice of marginal distribution can be linked through such a dependency framework. This is very limitative since it does not allow to choose the dependence construction without any regard to the specification of the marginals. This is somewhat counterintuitive with copulas being used as flexible tools to build dependence risk models.

In the next three subsections, we provide insight on how to benefit from the mixture representation of an Archimedean copula for any multivariate distribution defined through an Archimedean copula with a discrete mixing rv Θ and discrete marginals. Furthermore, we provide a strategy that can be used when one or both of these rvs are continuous.

2.3. Discrete mixing rv Θ and discrete marginals

Our strategy is to use the conditional independence assumption to identify the conditional distribution of X_i through (7) or (10). This step is usually more difficult for continuous rvs X_i than for discrete ones which is the basis of the computational strategy presented in Section 2.3. We have recourse to discretization methods in the continuous case. The conditional distributions of S given $\Theta = \theta$ are also easier to identify for discrete distributions.

The proposed strategy has many advantages. First, it is easy to implement regardless of the portfolio's dimension. Second, it yields the exact values of F_S for discrete risks X_i, $i = 1, \ldots, d$ and it gives an accurate approximation for continuous ones. The proposed strategy can also be used in the context of many actuarial risk models involving Archimedean dependence which will be discussed in Sections 4–6.

Our strategy is to make use of the conditional independence representation as well as the convolution of independent rvs to derive a simple computation approach for the distribution of S and any integrable function of $\underline{X}$. Given these techniques, risk aggregation problems for a portfolio of dependent risks with a multivariate joint distribution defined with an Archimedean copula or a nested Archimedean copula, as well as other related quantities become easier to deal with.

Let $\underline{X} = (X_1, \ldots, X_d)$ be a vector of discrete rvs such that $X_i \in A = \{0, 1h, 2h, \ldots\}$ ($i = 1, \ldots, d$). The univariate pmf of X_i, the univariate cdf of X_i, the multivariate pmf of $\underline{X}$, and the multivariate cdf of $\underline{X}$ are respectively denoted by $f_{X_i}(k_i h) = \Pr(X_i = k_i h)$, $F_{X_i}(k_i h) = \Pr(X_i \le k_i h)$,

$$f_{\underline{X}}(k_1 h, \ldots, k_d h) = \Pr(X_1 = k_1 h, \ldots, X_d = k_d h),$$

and

$$F_{\underline{X}}(k_1 h, \ldots, k_d h) = \Pr(X_1 \le k_1 h, \ldots, X_d \le k_d h),$$

for $h > 0$ and $k_1, \ldots, k_n \in \mathbb{N}_0$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In theory, exact values of f_S and $E[\varphi(X_1, \ldots, X_d)]$ can be found with

$$f_S(kh) = \sum_{k_1=0}^{k} \cdots \sum_{k_{d-1}=0}^{k-(k_1+\cdots+k_{d-2})} \times f_{\underline{X}}\left(k_1 h, \ldots, k_{d-1} h, \left(k - \sum_{j=1}^{d-1} k_j\right) h\right) \quad (15)$$

and

$$E[\varphi(X_1, \ldots, X_d)] = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \varphi(k_1 h, \ldots, k_d h) \times f_{\underline{X}}(k_1 h, \ldots, k_d h), \quad (16)$$

where

$$f_{\underline{X}}(k_1 h, \ldots, k_d h) = \sum_{i_1=0,1} \cdots \sum_{i_d=0,1} (-1)^{i_1+\cdots+i_d} \times F_{\underline{X}}((k_1 - i_1) h, \ldots, (k_d - i_d) h). \quad (17)$$

The computation of (15), (16), and (17) is feasible when the dimension d of $\underline{X}$ is small (e.g. $d = 2$, 3 or 4). However, it rapidly becomes impracticable when d gets larger.

Assume that the dependence structure of $\underline{X}$ is induced via an Archimedean copula C. In such a case, the multivariate cdf of $\underline{X}$ (or its survival function) is defined with the copula C and the univariate cdfs (or the univariate survival functions) of $X_1, \ldots, X_d$, i.e., (5) and (8) become

$$F_{\underline{X}}(k_1 h, \ldots, k_n h) = C\left(F_{X_1}(k_1 h), \ldots, F_{X_d}(k_d h)\right) \quad (18)$$

or

$$\overline{F}_{\underline{X}}(k_1 h, \ldots, k_d h) = C\left(\overline{F}_{X_1}(k_1 h), \ldots, \overline{F}_{X_d}(k_d h)\right), \quad (19)$$

for $k_i \in \mathbb{N}$ and $i = 1, \ldots, d$.

In the following, let the copula C in either (18) or (19) be an Archimedean copula defined by a discrete mixing rv Θ such that $E[\Theta]$ is finite. Then, we can take advantage of the common mixture representation to compute the exact values of f_S and $E[\varphi(X_1, \ldots, X_d)]$.

Assuming discrete marginals and a discrete mixing rv Θ and using (6), the expression for $F_{\underline{X}}$ in (18) becomes

$$F_{\underline{X}}(k_1 h, \ldots, k_d h) = \sum_{\theta=1}^{\infty} \prod_{i=1}^{d} F_{X_i|\Theta=\theta}(k_i h) f_\Theta(\theta), \quad (20)$$

and, from (7), we have

$$F_{X_i|\Theta=\theta}(k_i h) = \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(F_{X_i}(k_i h)\right)}, \quad (21)$$

for $k_i \in \mathbb{N}_0$, $i = 1, 2, \ldots, d$, and $\theta \in \mathbb{N}$. For $i = 1, 2, \ldots, d$ and for each $\theta \in \mathbb{N}$, we can easily find the values of $f_{X_i|\Theta=\theta}(k_i h)$ with

$$f_{X_i|\Theta=\theta}(k_i h) = \begin{cases} \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(F_{X_i}(0)\right)} & , k_i = 0 \\ \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(F_{X_i}(k_i h)\right)} - \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(F_{X_i}((k_i-1)h)\right)} & , k_i \in \mathbb{N}. \end{cases} \quad (22)$$

Similarly, using (9), the expression for $\overline{F}_{\underline{X}}$ in (19) turns into

$$\overline{F}_{\underline{X}}(k_1 h, \ldots, k_d h) = \sum_{\theta=1}^{\infty} \prod_{i=1}^{d} \overline{F}_{X_i|\Theta=\theta}(k_i h) f_\Theta(\theta), \quad (23)$$

and, from (10), we have

$$\overline{F}_{X_i|\Theta=\theta}(k_i h) = \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(k_i h)\right)}, \quad (24)$$

for $k_i \in \mathbb{N}_0$, $i = 1, 2, \ldots, d$, and $\theta \in \mathbb{N}$. It follows that

$$f_{X_i|\Theta=\theta}(k_i h) = \begin{cases} 1 - \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(0)\right)} & , k_i = 0 \\ \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}((k_i-1)h)\right)} - \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(k_i h)\right)} & , k_i \in \mathbb{N} \end{cases}, \quad (25)$$

for $i = 1, 2, \ldots, d$ and for each $\theta \in \mathbb{N}$.

Then, from either (22) or (25), the expression for $f_{\underline{X}}(k_1 h, \ldots, k_d h)$ is now given by

$$f_{\underline{X}}(k_1 h, \ldots, k_d h) = \sum_{\theta=1}^{\infty} \prod_{i=1}^{d} f_{X_i|\Theta=\theta}(k_i h) f_\Theta(\theta). \quad (26)$$

Let $(S|\Theta = \theta) = \sum_{i=1}^{d} (X_i|\Theta = \theta)$ be the sum of conditionally independent rvs and $f_{S|\Theta=\theta}$ be the corresponding pmf. Due to the representation of $f_{\underline{X}}$ in (26), the expression for the pmf of S can be written as follows

$$f_S(kh) = \sum_{\theta=1}^{\infty} f_{S|\Theta=\theta}(kh) f_\Theta(\theta), \quad k \in \mathbb{N}_0. \quad (27)$$

Since $f_{S|\Theta=\theta}$ corresponds to the convolution product of $f_{X_1|\Theta=\theta}, \ldots, f_{X_d|\Theta=\theta}$, traditional aggregation tools (e.g. FFT and DePril algorithm see e.g. Panjer et al., 2008) from actuarial science can be applied to find values of $f_{S|\Theta=\theta}$ for each θ. It is important to note here that our strategy leads to exact values of f_S contrarily to MC simulation methods. This is also true even when n is large.

Given the representation of $f_{\underline{X}}$ in (26), $E[\varphi(X_1, \ldots, X_d)]$ becomes

$$E[\varphi(X_1, \ldots, X_d)] = \sum_{\theta=1}^{\infty} E[\varphi(X_1, \ldots, X_d) | \Theta = \theta] f_\Theta(\theta), \quad (28)$$

where

$$E[\varphi(X_1, \ldots, X_d) | \Theta = \theta] = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \varphi(k_1 h, \ldots, k_d h) \prod_{i=1}^{d} f_{X_i|\Theta=\theta}(k_i h). \quad (29)$$

Applications of (28) and (29) are provided in Section 3.

The procedure to compute the values of $f_S(kh)$ is summarized in the following algorithm.

Algorithm 6. Computation of the values of f_S.

1. Let θ^* be chosen such that $F_\Theta(\theta^*) \leq 1 - \varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon = 10^{-10}$).
2. Fix $\theta = 1$.
3. For $i = 1, \ldots, d$, calculate either $F_{X_i|\Theta=\theta}(k_i h)$ with (21) or $\overline{F}_{X_i|\Theta=\theta}(k_i h)$ with (24), for $k_i \in \mathbb{N}_0$.
4. For $i = 1, \ldots, d$, calculate $f_{X_i|\Theta=\theta}(k_i h)$ with either (22) ou (25).
5. Using e.g. FFT or DePril's Algorithm, compute $f_{S|\Theta=\theta}(kh)$ for $k \in \mathbb{N}_0$.
6. Repeat steps 3, 4, and 5 for $\theta = 2, \ldots, \theta^*$ where θ^* is chosen such that $F_\Theta(\theta^*) \leq 1 - \varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon = 10^{-10}$).
7. Compute $f_S(kh) = \sum_{\theta=1}^{\theta^*} f_{S|\Theta=\theta}(kh) f_\Theta(\theta)$, for $k \in \mathbb{N}_0$.

Some remarks must be made in regard to the proposed methodology of this section and the tail dependence of an Archimedean copula C. Let us recall that if the lower and upper-tail dependence coefficients exist for an Archimedean copula C with mixing rv Θ, then according to Joe and Hu (1996), these coefficients λ_L and λ_U can be written in terms of $\mathcal{L}_\Theta$ as follows

$$\lambda_L = \lim_{t\to\infty} \frac{\mathcal{L}_\Theta(2t)}{\mathcal{L}_\Theta(t)} = 2 \lim_{t\to\infty} \frac{\mathcal{L}'_\Theta(2t)}{\mathcal{L}'_\Theta(t)}$$

Table 1
Values of the expectation, variance, VaR and TVaR of $S = X_1 + \ldots + X_4$ where $F_{X_1,\ldots,X_4}$ is defined with the Frank copula.

	$\alpha = 1$	$\alpha = 3$	$\alpha = 6$
$E[S]$	10	10	10
$Var(S)$	9.99256	15.15425	19.90096
$VaR_{0.9}(S)$	14	15	16
$VaR_{0.999}(S)$	20	21	23
$TVaR_{0.9}(S)$	15.82535	17.11038	18.04888
$TVaR_{0.999}(S)$	20.88055	22.39553	23.41423

Table 2
Values of the pmf of $S = X_1 + \ldots + X_4$ where $F_{X_1,\ldots,X_4}$ is defined with the Frank copula.

α	s	Exact values of $f_S(s)$ (with (15))	Exact values of $f_S(s)$ (with Algorithm 6)
1	0	1.992354e−05	1.992354e−05
	5	0.03925912	0.03925912
	10	0.1199567	0.1199567
3	0	0.0001013881	0.0001013881
	5	0.0619347088	0.0619347088
	10	0.0887858345	0.0887858345
6	0	0.0003887516	0.0003887516
	5	0.0691872490	0.0691872490
	10	0.0767524930	0.0767524930

Table 3
Values of the expectation, variance, VaR and TVaR of $S = X_1 + \ldots + X_{100}$ where $F_{X_1,\ldots,X_{100}}$ is defined with the AMH copula.

	$\alpha = 0$ (independence)	$\alpha = 0.5$	$\alpha = 0.9$
$E[S]$	100	100	100
$Var(S)$	90	1454.027	2793.690
$VaR_{0.9}(S)$	112	156	172
$VaR_{0.999}(S)$	130	225	242
$TVaR_{0.9}(S)$	116.934	176.206	192.651
$TVaR_{0.999}(S)$	133.277	233.651	250.590

and

$$\lambda_U = 2 - \lim_{t\to 0} \frac{1-\mathcal{L}_\Theta(2t)}{1-\mathcal{L}_\Theta(t)} = 2 - 2\lim_{t\to 0} \frac{\mathcal{L}'_\Theta(2t)}{\mathcal{L}'_\Theta(t)}.$$

For the special case of discrete mixing rvs Θ, the underlying Archimedean copulas cannot have lower tail dependence, i.e. $\lambda_L = 0$. This implies that our proposed methodology does not allow lower tail dependence. Also, if $E[\Theta]$ is finite, then $\lambda_U = 0$ (see Hofert, 2010 page 62 for proof). This means that it is possible to find a finite θ^* for which the methodology described in Algorithm 6 leads to exact values of the pmf of the rv S. In such a case, the methodology works well for notably the AMH and Frank copulas. Note that the proposed methodology cannot be applied when $E[\Theta]$ is infinite (e.g. for Joe's copula) since Algorithm 6 suggests to truncate the distribution of Θ at θ^*. Such a truncation leads to an Archimedean copula with no upper tail dependence which violates the initial assumption.

In the following two examples, we illustrate the accuracy of our computational methodology. More precisely, we present a first example which considers a small portfolio. This allows us to compare the values of f_S obtained with Algorithm 6 and (15). As expected, both results coincide. The second example is somewhat similar but illustrates the applicability of Algorithm 6 to large portfolios.

Example 7. Let $F_{\underline{X}}$ be defined as in (18) with the Frank copula as given in Appendix and $X_i \sim Bin(10, q_i)$, where $q_i = 0.1i$, for $i = 1, 2, 3, 4$. In this case, the mixing rv Θ follows a logarithmic distribution. While values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ are given in Table 1, Table 2 provides the exact values of f_S obtained with (15) and Algorithm 6.

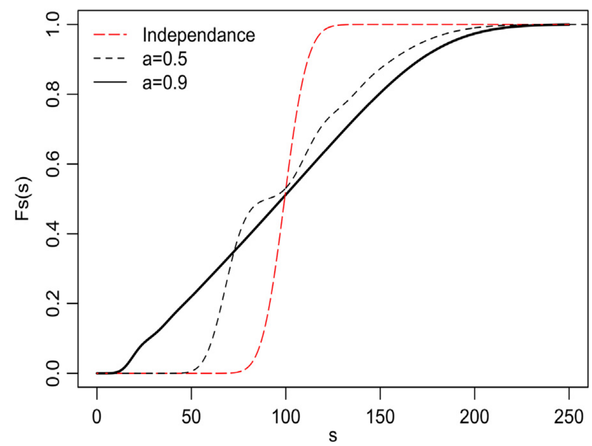

Fig. 1. The cdf of $S = X_1 + \cdots + X_{100}$ where $F_{X_1,\ldots,X_{100}}$ is defined by the AMH copula with $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 0.9$.

Example 8. Let $X_i \sim Bin(10, q_i)$, with $q_i = 0.1$, for $i = 1, \ldots, 100$. The joint cdf $F_{\underline{X}}$ is defined as in (18) with the AMH copula. In Fig. 1, we depict the exact values of F_S obtained with Algorithm 6 ($\alpha = 0.5$ and 0.9). For comparison purposes, the exact values of $F_{S^\perp}$, where $S^\perp$ is the sum of the independent rvs $X_1^\perp, \ldots, X_{100}^\perp$ and $X_i^\perp \sim X_i$ for $i = 1, 2, \ldots, 100$, are also provided. It is well known that the AMH copula introduces a low to moderate positive dependence relation between the rvs $X_1, \ldots, X_{100}$. However, the impact is clearly significant when the number of risks of the portfolio becomes large as illustrated in Fig. 1. In Table 3, we provide the values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$ and $TVaR_\kappa(S)$. Note that the computation time increases as the dependence parameter becomes larger. Indeed, $\theta^* = F_\Theta^{-1}(1-\varepsilon; \alpha)$ increases with α. For example, if $\varepsilon = 10^{-10}$, $\theta^* = 34$ and $\theta^* = 219$ for $\alpha = 0.5$ and $\alpha = 0.9$ respectively.

Example 8 allows us to better understand the impact of the dependence between individual risks on the overall exposure evaluation. As shown in Fig. 1, using a portfolio of 100 risks highlights this significant impact. Indeed, one cannot neglect the dependence, even moderate, for a portfolio of large dimension. Note that the proposed algorithm allowed us to make such a conclusion, which is not possible using the convolution method in (15), since it only applies to small portfolios.

2.4. Discrete mixing rv Θ and continuous marginals

Let us assume that $\underline{X} = (X_1, \ldots, X_d)$ is a vector of continuous positive rvs where the multivariate cdf is defined by an Archimedean copula C as in (5) and the mixing rv Θ is discrete. In this case, the computation of $F_{S|\Theta=\theta}(x)$ or eventually $E[\varphi(X_1, \ldots, X_d)|\Theta = \theta])$ becomes more difficult. Since $(X_1|\Theta = \theta), \ldots, (X_d|\Theta = \theta)$ are conditionally independent rvs, one can apply the tools at hand to compute an accurate (as possible) approximation of $F_{S|\Theta=\theta}(x)$ or $E[\varphi(X_1, \ldots, X_d)|\Theta = \theta]$ for the sum or any function of independent rvs. Note that the approach described here can be easily adapted when the multivariate survival function instead of the multivariate cdf is defined with an Archimedean copula C as in (8) and when the mixing rv Θ is discrete.

Inspired by Bargès et al. (2009), we propose an approximation based on discretization methods and the application of Algorithm 6. This approximation leads to numerical bounds as accurate as one may desire. Indeed, we approximate $\underline{X}$ by $\underline{\widetilde{X}} = (\widetilde{X}_1, \ldots, \widetilde{X}_d)$, a vector of discrete rvs with $\widetilde{X}_i \in A = \{0, 1h, 2h, \ldots\}$, for $i = 1, \ldots, d$ and a discretization step $h > 0$. The multivariate cdf of $\underline{\widetilde{X}}$ is defined with the same copula C

Table 4
Approximated values of the expectation, variance, VaR and TVaR of $S = X_1 + \ldots + X_{40}$ where $F_{X_1,\ldots,X_{40}}$ is defined with the AMH copula and continuous marginals. The discretization steps is either $h = 1$ or $h = 0.1$. For the MC method, the number of simulations is 100 000.

	$\widetilde{S}^{(u,1)}$	$\widetilde{S}^{(u,0.1)}$	$\widetilde{S}^{(l,0.1)}$	$\widetilde{S}^{(l,1)}$	$\widetilde{S}^{(m,1)}$	$\widetilde{S}^{(m,0.1)}$	MC	95%-confidence interval
Expectation	380.3333	398.0033	402.0033	420.3333	400	400	399.8067	[398.8303 ; 400.7831]
Variance	24 667.6236	24 749.6556	24 749.6556	24 667.6236	24 796.9950	24 750.9523	24 817.9510	[24 601.8436 ; 25 036.9318]
$VaR_{0.9}$	606	624	628.4	646	627	626.4	627.3094	[624.8721 ; 629.6699]
$VaR_{0.999}$	975	993.2	997.2	1015	995	995.2	989.2484	[981.2374 ; 1000.4592]
$TVaR_{0.9}$	705.6048	723.3208	727.3191	745.6048	725.7719	725.3294	726.2473	[725.7238 ; 726.7709]
$TVaR_{0.999}$	1027.2863	1043.7218	1047.7156	1067.2863	1047.4537	1045.7465	1039.4606	[1039.1150 ; 1039.8062]

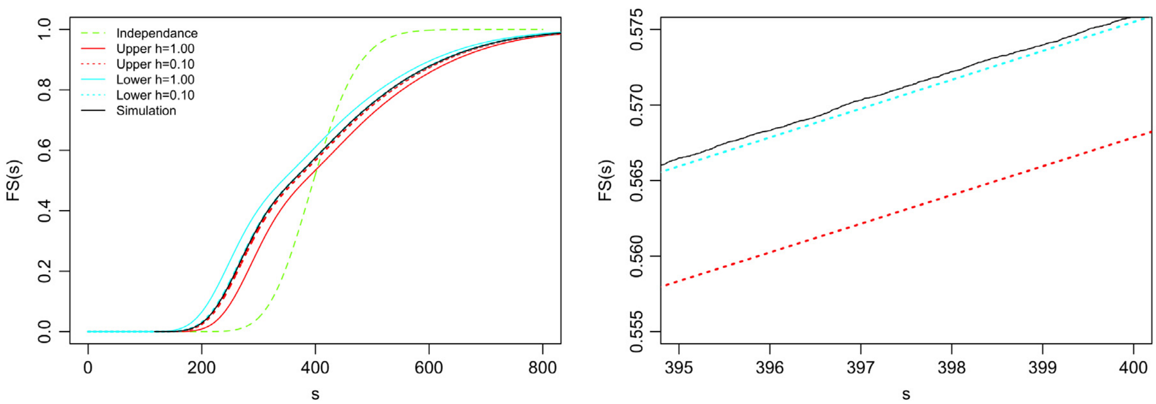

Fig. 2. Approximated values of the cdf of $S = X_1 + \cdots + X_{40}$.

and with marginals $F_{\widetilde{X}_1}$, …, $F_{\widetilde{X}_d}$ obtained with a discretization method, i.e., $F_{\widetilde{\underline{X}}}(k_1h, \ldots, k_dh) = C\left(F_{\widetilde{X}_1}(k_1h), \ldots, F_{\widetilde{X}_d}(k_dh)\right)$, for $k_1, \ldots, k_d \in \mathbb{N}_0$. In this article, we consider the upper, lower and mean preserving discretization methods (see e.g. Müller and Stoyan, 2002 or Bargès et al., 2009 for details).

Example 9. Let $X_i \sim Exp(0.1)$, for $i = 1, 2, \ldots, d$ ($d = 40$). The multivariate cdf of $\underline{X}$, $F_{\underline{X}}$, is defined as in (5) with the AMH copula ($\alpha = 0.5$). For $S = X_1 + \cdots + X_{40}$, we compute the upper and lower bounds to F_S with $h = 1$ and $h = 0.1$. Values of the expectation, variance, VaR and TVaR of the rvs $\widetilde{S}^{(u,1)}$, $\widetilde{S}^{(u,0.1)}$, $\widetilde{S}^{(l,1)}$, $\widetilde{S}^{(l,0.1)}$, $\widetilde{S}^{(m,1)}$, and $\widetilde{S}^{(m,0.1)}$ are given in Table 4. Note that "u", "l", and "m" in the superscripts refer respectively to the upper, lower and mean preserving discretization methods. Also, the approximated values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ ($\kappa = 0.9, 0.999$) obtained using 100 000 MC simulations are provided (with 95%-level confidence intervals given in the last column). Clearly, as the discretization step h goes to 0, the difference between the bounds also tends to 0. Also, notice that the approximated values $\widetilde{VaR}^{MC}_{0.999}(S)$ and $\widetilde{TVaR}^{MC}_{0.999}(S)$ of $VaR_{0.999}(S)$ and $TVaR_{0.999}(S)$ obtained by 100 000 MC simulations are outside the interval defined by the upper and the lower bounds. This is also illustrated in Fig. 2. In the left panel, we provide the values for the cdfs of the rvs $\widetilde{S}^{(u,1)}$, $\widetilde{S}^{(u,0.1)}$, $\widetilde{S}^{(l,1)}$, and $\widetilde{S}^{(l,0.1)}$ and the approximated values of F_S, obtained with 100 000 MC simulations. The close-up on the right panel of Fig. 2 clearly shows that the approximated values of F_S obtained with MC simulations lies out of the upper and lower bounds $F_{\widetilde{S}^{(u,0.1)}}$ and $F_{\widetilde{S}^{(l,0.1)}}$. The proposed approximation has the advantage to allow us to control the precision of the approximation. On the left panel of Fig. 2, the exact values of $F_{S^\perp}$, where $S^\perp$ is the sum of the independent rvs $X_1^\perp$, …, $X_{100}^\perp$ and $X_i^\perp \sim X_i$ for $i = 1, 2, \ldots, 100$, are also depicted.

2.5. Continuous mixing rv Θ

The common mixture representation in (6) and (9) leads to a two-step natural sampling procedure for $\underline{X}$. The first step is to simulate a sampled value of Θ. Then, in the second step, the sampled value of $\underline{X}$ is computed via the conditional distribution of $\underline{X}$ given the sampled value of Θ using either (7) or (10). See, e.g., Marshall and Olkin (1988) or Hofert (2008) for details. We examine two alternatives to this approach: one based on the simulation of the mixing rv Θ and another one on the approximation of Θ by a discrete rv.

We consider a vector of discrete (or discretized) rvs $\underline{X} = (X_1, \ldots, X_d)$ where $X_i \in A = \{0, 1h, 2h, \ldots\}$ ($i = 1, \ldots, d$). Let the rv Z be the estimator of $\varphi(X_1, \ldots, X_d)$ under a MC simulation approach, which is constructed using the two-step sampling procedure of $\underline{X}$. The standard error of the estimator Z is given by $\sqrt{Var(Z)}$. We can use a variance reduction technique, namely the conditional MC simulation method (see, e.g., Lemieux, 2009 or Kroese et al., 2013 for details on this topic) to reduce this standard error. First, one generates sampled values of the rv Θ and then applies Algorithm 6 as follows. We produce m sampled values of Θ, denoted by $\Theta^{(1)}$, …, $\Theta^{(m)}$. Then, (27) becomes $f_S(kh) \simeq \sum_{j=1}^m f_{S|\Theta=\Theta^{(j)}}(kh)\frac{1}{m}$, $k \in \mathbb{N}_0$, where the values of $f_{S|\Theta=\Theta^{(j)}}(kh)$ are computed using e.g. FFT or DePril's Algorithm. Then, $E[Z|\Theta]$ corresponds to the resulting conditional MC approximation of $\varphi(X_1, \ldots, X_d)$. Clearly, the standard error of $E[Z|\Theta]$ is given by $\sqrt{Var(E[Z|\Theta])}$. Since

$$Var(Z) = E[Var(Z|\Theta)] + Var(E[Z|\Theta]), \tag{30}$$

it is clear that

$$\sqrt{Var(E[Z|\Theta])} \leq \sqrt{Var(Z)}. \tag{31}$$

The magnitude of the difference $\sqrt{Var(Z)} - \sqrt{Var(E[Z|\Theta])}$ is analyzed in the following example.

Example 10. Let $\underline{X}^{(C,s,\alpha)} = \left(X_1^{(C,s,\alpha)}, \ldots, X_{50}^{(C,s,\alpha)}\right)$ and $\underline{X}^{(G,s,\alpha)} = \left(X_1^{(G,s,\alpha)}, \ldots, X_{50}^{(G,s,\alpha)}\right)$ be two vectors of rvs with $X_1^{(C,s,\alpha)} \sim X_1^{(G,s,\alpha)} \sim Bin(100, 0.2)$, for $i = 1, \ldots, 50$, where their joint survival functions are defined either with the Clayton copula

Table 5
Values of $Cov(X_i^{(G,s,\alpha)}, X_j^{(G,s,\alpha)})$ and $Cov(X_i^{(G,f,\alpha)}, X_j^{(G,f,\alpha)})$, $i \neq j \in \{1, 2, \ldots, 50\}$.

α	$Cov(X_i^{(G,s,\alpha)}, X_j^{(G,s,\alpha)})$	$Cov(X_i^{(G,f,\alpha)}, X_j^{(G,f,\alpha)})$
1.25	4.893995	5.11945
2	11.01781	11.2806
5	15.01042	15.09308

Table 6
Values of $Cov(X_i^{(C,s,\alpha)}, X_j^{(C,s,\alpha)})$ and $Cov(X_i^{(C,f,\alpha)}, X_j^{(C,f,\alpha)})$, $i \neq j \in \{1, 2, \ldots, 50\}$.

α	$Cov(X_i^{(C,s,\alpha)}, X_j^{(C,s,\alpha)})$	$Cov(X_i^{(C,f,\alpha)}, X_j^{(C,f,\alpha)})$
0.5	5.20697	4.859871
2	11.14602	10.62849
8	14.72897	14.44488

Table 7
Values of $Var(S^{(G,s,\alpha)})$ and $Var(S^{(G,f,\alpha)})$, for $\alpha = 1.5, 2, 5$.

α	$Var(S^{(G,s,\alpha)})$	$Var(S^{(G,f,\alpha)})$
1.25	12 790.29	13 342.65
2	27 793.64	28 437.48
5	37 575.52	37 778.05

Table 8
Values of $Var(S^{(C,s,\alpha)})$ and $Var(S^{(C,f,\alpha)})$, for $\alpha = 0.5, 2, 8$.

α	$Var(S^{(C,s,\alpha)})$	$Var(S^{(C,f,\alpha)})$
0.5	13 557.08	12 706.68
2	28 107.76	26 839.79
8	36 885.98	36 189.97

or the Gumbel copula according to (8). Similarly, let $\underline{X}^{(C,f,\alpha)} = \left(X_1^{(C,f,\alpha)}, \ldots, X_{50}^{(C,f,\alpha)}\right)$ and $\underline{X}^{(G,f,\alpha)} = \left(X_1^{(G,f,\alpha)}, \ldots, X_{50}^{(G,f,\alpha)}\right)$ be two vectors of rvs with $X_1^{(C,f,\alpha)} \sim X_1^{(G,f,\alpha)} \sim Bin\,(100, 0.2)$, for $i = 1, \ldots, 50$, where their joint cdfs are defined either with the Clayton copula or the Gumbel copula according to (5). For each copula, the dependence parameter α of the copula is fixed such that Kendall's tau is equal to 0.2, 0.5, or 0.8. The exact values of the covariances between pairs of rvs for each vectors of rvs are provided in Tables 5 and 6 for the three values of the dependence parameter. We define $S^{(C,s,\alpha)} = \sum_{i=1}^{50} X_i^{(C,s,\alpha)}$, $S^{(G,s,\alpha)} = \sum_{i=1}^{50} X_i^{(G,s,\alpha)}$, $S^{(C,f,\alpha)} = \sum_{i=1}^{50} X_i^{(C,f,\alpha)}$, and $S^{(G,f,\alpha)} = \sum_{i=1}^{50} X_i^{(G,f,\alpha)}$. Clearly, $E[S^{(C,s,\alpha)}] = E[S^{(G,s,\alpha)}] = E[S^{(C,f,\alpha)}] = E[S^{(G,f,\alpha)}] = 1000$, whatever the values of the dependence parameters. In Tables 7 and 8, we also give the exact values of $Var(S^{(C,s,\alpha)})$, $Var(S^{(G,s,\alpha)})$, $Var(S^{(C,f,\alpha)})$, and $Var(S^{(G,f,\alpha)})$ for the three values of their dependence parameter. In Tables 9–14, we provide the approximated values of $\overline{F}_{S^{(C,s,\alpha)}}$, $\overline{F}_{S^{(G,s,\alpha)}}$, $\overline{F}_{S^{(C,f,\alpha)}}$, and $\overline{F}_{S^{(G,f,\alpha)}}$. Those values are computed using both the conditional MC and the MC approaches with $m = 100\,000$ simulations. In parenthesis, we indicate the values of the standard deviation for each approximation. As expected from (31), we observe that the standard error of the approximation based on the conditional MC approach is lower than the corresponding one for the approximation based on the MC approach. For a given multivariate distribution, we observe that the improvement is more significant as the dependence parameter α decreases. The improvement is also more significant for large values of x. However, for a specific value of Kendall's tau, the improvement differs from one multivariate distribution to another. Notably, we observe that the improvement is the least significant for the results associated to $\underline{X}^{(C,s,\alpha)}$ and $\underline{X}^{(G,f,\alpha)}$, meaning that the improvement observed with the conditional MC approach is less significant with multivariate distributions having a non-zero right tail dependence.

Let us now consider the case of the Clayton copula for which Θ is gamma distributed. To obtain f_S, we can approximate the Clayton copula with the shifted negative binomial copula allowing us to use Algorithm 6. This family includes the AMH copula as a special case and the Clayton copula as a limit case (see Cossette et al., 2017 for more details). The idea here is to approximate the rv Θ by a discrete rv $\widetilde{\Theta}$ and apply the proposed methodology of Section 2.3. However, we need to be careful. We cannot blindly apply the three discretization methods used in the previous section because we aim to approximate the copula generated from Θ and not only the distribution of Θ. Thus, finding an appropriate discrete rv $\widetilde{\Theta}$ is not an easy task as we will see in the special case of the Clayton copula.

The multivariate shifted negative binomial copula is defined by

$$C_{\alpha,q_h}^{SNB}(u_1, \ldots, u_d) = \left(q_h \left(\prod_{i=1}^{d} \left(q_h u_i^{-\alpha} + (1 - q_h) \right) - (1 - q_h) \right)^{-1} \right)^{\frac{1}{\alpha}}. \quad (32)$$

The two parameters of the copula are $\alpha > 0$ and $q_h = 1 - e^{-h}$, where $h > 0$ can be seen as a discretization parameter.

The underlying mixing rv, associated to (32) and denoted by $\Theta_{(h)}^{SNB(\alpha)}$, follows a shifted negative binomial distribution, i.e., the rv $\Theta_{(h)}^{SNB(\alpha)}$ is defined as $\Theta_{(h)}^{SNB(\alpha)} = h\left(M_{(h)}^{NB(\alpha)} + \alpha\right)$ where $M_{(h)}^{NB(\alpha)}$ follows a negative binomial distribution, i.e., $M_{(h)}^{NB(\alpha)} \sim NB\left(\frac{1}{\alpha}, q_h\right)$, with $f_{M_{(h)}^{NB(\alpha)}}(k) = \binom{\frac{1}{\alpha}+k-1}{k}(q_h)^{\frac{1}{\alpha}}(1-q_h)^k$, $k \in \mathbb{N}_0$, and $E\left[M_{(h)}^{NB(\alpha)}\right] = \frac{1-q_h}{\alpha q_h}$ with $q_h = 1 - e^{-h}$, $h > 0$. The LST of $\Theta_{(h)}^{SNB\left(\frac{1}{\alpha}\right)}$ is

$$\mathcal{L}_{\Theta_{(h)}^{SNB(\alpha)}}(t) = E\left[e^{-t\Theta_{(h)}^{SNB(\alpha)}}\right] = \left(\frac{e^{-th} - e^{-(t-1)h}}{1 - e^{-(t-1)h}} \right)^{\frac{1}{\alpha}}.$$

Table 9
Approximated values of $\overline{F}_{S^{(C,s,\alpha)}}$ and $\overline{F}_{S^{(C,f,\alpha)}}$, using conditional MC and MC approaches, for $\alpha = 0.5$ ($\tau = 0.2$). The values in parenthesis correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S^{(C,s,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(C,s,\alpha)}}(x_i)$	x_i	CMC approx.$\overline{F}_{S^{(C,f,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(C,f,\alpha)}}(x_i)$
1	1216	0.050527249 (0.198427516)	0.050140000 (0.218234866)	1157	0.050705395 (0.174562788)	0.050730000 (0.219446915)
2	1354	0.010079740 (0.090022495)	0.010160000 (0.100283972)	1201	0.010331348 (0.070988687)	0.010450000 (0.101690220)
3	1332	0.013160376 (0.102769800)	0.013100000 (0.113703647)	1243	0.001055487 (0.018413807)	0.001150000 (0.033892315)
4	1698	0.000108547 (0.009061601)	0.000090000 (0.009486453)	1273	0.000107720 (0.004440821)	0.000060000 (0.007745773)
5	1840	0.000010584 (0.002904670)	0.000020000 (0.004472114)	1296	0.000010881 (0.000807741)	0.000010000 (0.003162278)

Table 10
Approximated values of $\overline{F}_{S^{(C,s,\alpha)}}$ and $\overline{F}_{S^{(C,f,\alpha)}}$, using conditional MC and MC approaches, for $\alpha = 2$ ($\tau = 0.5$). The values in parenthesis correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S^{(C,s,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(C,s,\alpha)}}(x_i)$	x_i	CMC approx.$\overline{F}_{S^{(C,f,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(C,f,\alpha)}}(x_i)$
1	1313	0.049968514 (0.211556800)	0.050030000 (0.218007969)	1232	0.050956074 (0.192522672)	0.050430000 (0.218831657)
2	1468	0.010020219 (0.096953570)	0.010000000 (0.099499241)	1289	0.010297759 (0.081601626)	0.010130000 (0.100137323)
3	1641	0.001008032 (0.030668361)	0.000990000 (0.031448844)	1340	0.001000416 (0.021596196)	0.000870000 (0.029483076)
4	1798	0.000100325 (0.009958315)	0.000100000 (0.009999550)	1374	0.000104887 (0.006010938)	0.000130000 (0.011401070)
5	1958	0.000010000 (0.003162278)	0.000010000 (0.003162278)	1397	0.000012494 (0.001157376)	0.000000000 (-)

Table 11
Approximated values of $\overline{F}_{S^{(C,s,\alpha)}}$ and $\overline{F}_{S^{(C,f,\alpha)}}$, using conditional MC and MC approaches, for $\alpha = 8$ ($\tau = 0.8$). The values in parenthesis correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S^{(C,s,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(C,s,\alpha)}}(x_i)$	x_i	CMC approx.$\overline{F}_{S^{(C,f,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(C,f,\alpha)}}(x_i)$
1	1337	0.050074019 (0.215637266)	0.050050000 (0.218049244)	1303	0.050640550 (0.203357022)	0.050360000 (0.218687788)
2	1489	0.010064309 (0.098773129)	0.010130000 (0.100137323)	1374	0.010921753 (0.090974062)	0.011040000 (0.104490323)
3	1649	0.001000266 (0.030807217)	0.001010000 (0.031764603)	1436	0.001052248 (0.024805081)	0.001090000 (0.032997315)
4	1799	0.000111861 (0.010159259)	0.000120000 (0.010953849)	1472	0.000103602 (0.006558965)	0.000080000 (0.008943959)
5	1960	0.000010000 (0.003162278)	0.000010000 (0.003162278)	1497	0.000010341 (0.001357239)	0.000020000 (0.004472114)

Table 12
Approximated values of $\overline{F}_{S^{(G,s,\alpha)}}$ and $\overline{F}_{S^{(G,f,\alpha)}}$, using conditional MC and MC approaches, for $\alpha = 1.25$ ($\tau = 0.2$). The values in parenthesis correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S^{(G,s,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(G,s,\alpha)}}(x_i)$	x_i	CMC approx.$\overline{F}_{S^{(G,f,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(G,f,\alpha)}}(x_i)$
1	1141	0.050095336 (0.138900179)	0.050220000 (0.218399699)	1227	0.050329739 (0.208588438)	0.050500000 (0.218975408)
2	1179	0.009979292 (0.047470342)	0.009950000 (0.099252688)	1388	0.010100157 (0.095770936)	0.009950000 (0.099252688)
3	1218	0.001031343 (0.009745777)	0.001070000 (0.032693513)	1568	0.001077327 (0.031529811)	0.001040000 (0.032232418)
4	1249	0.000105875 (0.001984137)	0.000110000 (0.010487564)	1745	0.000093758 (0.009441001)	0.000090000 (0.009486453)
5	1276	0.000010431 (0.000391711)	0.000010000 (0.003162278)	1900	0.000010000 (0.003162278)	0.000010000 (0.003162278)

Table 13
Approximated values of $\overline{F}_{S^{(G,s,\alpha)}}$ and $\overline{F}_{S^{(G,f,\alpha)}}$, using conditional MC and MC approaches, for $\alpha = 2$ ($\tau = 0.5$). The values in parenthesis correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S^{(G,s,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(G,s,\alpha)}}(x_i)$	x_i	CMC approx.$\overline{F}_{S^{(G,f,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(G,f,\alpha)}}(x_i)$
1	1252	0.050799403 (0.194516995)	0.051030000 (0.220060045)	1307	0.049717809 (0.211499142)	0.049840000 (0.217615367)
2	1336	0.010132678 (0.082516449)	0.010190000 (0.100430398)	1458	0.010201015 (0.097931511)	0.010140000 (0.100186230)
3	1425	0.001002746 (0.022945213)	0.001040000 (0.032232418)	1630	0.001026755 (0.031203817)	0.001040000 (0.032232418)
4	1495	0.000100023 (0.007070299)	0.000070000 (0.008366349)	1793	0.000100527 (0.009880428)	0.000100000 (0.009999550)
5	1572	0.000010349 (0.001879823)	0.000010000 (0.003162278)	1960	0.000010000 (0.003162278)	0.000010000 (0.003162278)

Table 14
Approximated values of $\overline{F}_{S^{(G,s,\alpha)}}$ and $\overline{F}_{S^{(G,f,\alpha)}}$, using conditional MC and MC approaches, for $\alpha = 5$ ($\tau = 0.8$). The values in parenthesis correspond to the standard errors.

i	x_i	CMC approx.$\overline{F}_{S^{(G,s,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(G,s,\alpha)}}(x_i)$	x_i	CMC approx.$\overline{F}_{S^{(G,f,\alpha)}}(x_i)$	MC approx.$\overline{F}_{S^{(G,f,\alpha)}}(x_i)$
1	1320	0.050086643 (0.210804858)	0.050100000 (0.218152391)	1330	0.050803544 (0.217050585)	0.050620000 (0.219221569)
2	1450	0.010204031 (0.095399818)	0.010190000 (0.100430398)	1475	0.010269781 (0.099412094)	0.010220000 (0.100576601)
3	1600	0.001057080 (0.029683037)	0.001040000 (0.032232418)	1648	0.001008499 (0.031358794)	0.001020000 (0.031921306)
4	1730	0.000102036 (0.009012274)	0.000110000 (0.010487564)	1788	0.000109967 (0.010484446)	0.000110000 (0.010487564)
5	1908	0.000010152 (0.002586474)	0.000020000 (0.004472114)	1920	0.000010000 (0.003162278)	0.000010000 (0.003162278)

The shifted negative binomial copula is an Archimedean copula. Indeed, (32) can be represented as

$$C^{SNB}_{\alpha,q_h}(u_1,\ldots,u_d) = \mathcal{L}_{\Theta^{SNB(\alpha)}_{(h)}}\left(\mathcal{L}^{-1}_{\Theta^{SNB(\alpha)}_{(h)}}(u_1) + \cdots + \mathcal{L}^{-1}_{\Theta^{SNB(\alpha)}_{(h)}}(u_d)\right).$$

When $d = 2$, $q_h = 1-\beta$ and $\gamma = \frac{1}{\alpha}$, (32) becomes

$$C^{SNB}_{\alpha,q_h}(u_1,u_2) = \left((1-\beta)\left(\prod_{i=1}^{2}\left((1-\beta)\,u_i^{-\frac{1}{\gamma}} + \beta\right) - \beta\right)^{-1}\right)^{\gamma} = \frac{u_1 u_2}{\left(1-\beta\left(1-u_1^{\frac{1}{\gamma}}\right)\left(1-u_2^{\frac{1}{\gamma}}\right)\right)^{\gamma}},$$

which corresponds to the so-called bivariate Lomax copula in Balakrishnan and Lai (2009) , bivariate Fang–Fang–Rosen copula in Fang et al. (2000) and Genest and Rivest (2001), and bivariate BB10 copula in Joe (2014). Note that the multivariate copula provided in (2.2) of Fang et al. (2000) does not correspond to the Archimedean copula in (32). The copula in (32) is constructed via the approach proposed by Marshall and Olkin (1988) as for the BB10 copula.

Clearly, when $h \to 0$, we have

$$\lim_{h\to 0} E\left[e^{-t\Theta^{SNB(\alpha)}_{(h)}}\right] = \left(\frac{1}{1+t}\right)^{\frac{1}{\alpha}}, \tag{33}$$

where $\left(\frac{1}{1+t}\right)^{\frac{1}{\alpha}}$ is the LST of the rv $\Theta^{Ga\left(\frac{1}{\alpha},1\right)}$ which follows a gamma distribution, i.e., $\Theta^{\left(\frac{1}{\alpha},1\right)} \sim Gamma\left(\frac{1}{\alpha},1\right)$. Then, given (33), $\Theta^{SNB\left(\frac{1}{\alpha}\right)}_{(h)} \xrightarrow{\mathcal{D}} \Theta^{Ga\left(\frac{1}{\alpha},1\right)}$, as the discretization parameter $h \to 0$, where "$\xrightarrow{\mathcal{D}}$" corresponds to the convergence in distribution.

As a special case, when $\alpha = 1$, the shifted negative binomial copula in (32) becomes the AMH copula. For a fixed $\alpha > 0$, when $h \to 0$ (i.e., $q_h \to 1$), the limit of the shifted negative binomial copula corresponds to the Clayton copula with parameter α, i.e.,

$$\lim_{h\to 0} C^{SNB}_{\alpha,q_h}(u_1,\ldots,u_n) = \left(u_1^{-\alpha} + \cdots + u_n^{-\alpha} - (n-1)\right)^{-\frac{1}{\alpha}} = C^{CLAY}_{\alpha}(u_1,\ldots,u_n).$$

In the following example, we examine the efficiency of the approximation of the Clayton copula with the shifted negative binomial copula in a risk aggregation context.

Example 11. In this example, we compare the performance of the approximation of the Clayton copula by the shifted negative binomial copula. Let $\underline{X}^{(C,\alpha)} = \left(X_1^{(C,\alpha)},\ldots,X_4^{(C,\alpha)}\right)$ and $\underline{X}^{(SNB,\alpha,h)} = \left(X_1^{(SNB,\alpha,h)},\ldots,X_4^{(SNB,\alpha,h)}\right)$ be two vectors of rvs with $X_i^{(C,\alpha)} \sim X_i^{(SNB,\alpha,h)} \sim Bin(10, 0.2)$, for $i = 1,\ldots,4$, where their joint cdfs are defined either with the Clayton copula (with $\alpha = 0.5, 2, 8$) or the shifted negative binomial copula (with $\alpha = 0.5, 2, 8$ and $h = 0.001, 0.0001$) according to (5). We define $S^{(C,\alpha)} = \sum_{i=1}^{4} X_i^{(C,\alpha)}$ and $S^{(SNB,\alpha,h)} = \sum_{i=1}^{4} X_i^{(SNB,\alpha,h)}$. Both the exact values of $\overline{F}_{S^{(C,\alpha)}}$ resulting from (15) and $\overline{F}_{S^{(SNB,\alpha,h)}}$ using the shifted negative binomial copula are given in Tables 15–17. The results of both the conditional and the full MC simulation methods are also provided. The approximation using the shifted negative binomial copula is good and more significant as h decreases and for Kendall taus

Table 15
Approximated values of $\overline{F}_{S(C,0.5)}$, using the negative binomial copula, the conditional MC and MC approaches, for $\alpha = 0.5$ ($\tau = 0.2$). The values in parenthesis correspond to the standard errors.

i	x_i	$\overline{F}_{S(C,0.5)}(x_i)$	$\overline{F}_{S(SNB,0.5,0.001)}(x_i)$	$\overline{F}_{S(SNB,0.5,0.0001)}(x_i)$	CMC approx.$\overline{F}_{S(C,0.5)}(x_i)$	MC approx.$\overline{F}_{S(C,0.5)}(x_i)$
1	13	0.046898732	0.046896056	0.046898466	0.046743618 (0.076386143)	0.045940000 (0.209356048)
2	14	0.022517508	0.022516026	0.022517365	0.022429261 (0.042487275)	0.022260000 (0.147528675)
3	17	0.001310321	0.001310236	0.001310341	0.001301674 (0.003570962)	0.001290000 (0.035893576)
4	19	0.000113916	0.000113946	0.000113956	0.000112915 (0.000370698)	0.000090000 (0.009486453)
5	20	0.000028520	0.000028559	0.000028562	0.000028239 (0.000099466)	0.000020000 (0.004472114)
6	26	0.000000001	0.000000044	0.000000044	0.000000001 (0.000000003)	0.000000000 (-)

Table 16
Approximated values of $\overline{F}_{S(C,2)}$, using the negative binomial copula, the conditional MC and MC approaches, for $\alpha = 2$ ($\tau = 0.5$). The values in parenthesis correspond to the standard errors.

i	x_i	$\overline{F}_{S(C,2)}(x_i)$	$\overline{F}_{S(SNB,2,0.001)}(x_i)$	$\overline{F}_{S(SNB,2,0.0001)}(x_i)$	CMC approx.$\overline{F}_{S(C,2)}(x_i)$	MC approx.$\overline{F}_{S(C,2)}(x_i)$
1	14	0.057715752	0.057710844	0.057715261	0.057828623 (0.134180425)	0.057780000 (0.233328130)
2	16	0.014628680	0.014627145	0.014628527	0.014636998 (0.049205147)	0.014280000 (0.118643257)
3	18	0.002313893	0.002313606	0.002313864	0.002314054 (0.011151686)	0.002190000 (0.046746398)
4	20	0.000220129	0.000220099	0.000220126	0.000220007 (0.001443237)	0.000200000 (0.014140792)
5	22	0.000012529	0.000012527	0.000012529	0.000012508 (0.000103554)	0.000000000 (-)
6	28	1.145583e−10	3.685083e−10	3.685204e−10	1.139485e−10 (0.000000001)	0.000000000 (-)

Table 17
Approximated values of $\overline{F}_{S(C,8)}$, using the negative binomial copula, the conditional MC and MC approaches, for $\alpha = 8$ ($\tau = 0.8$). The values in parenthesis correspond to the standard errors.

i	x_i	$\overline{F}_{S(C,8)}(x_i)$	$\overline{F}_{S(SNB,8,0.001)}(x_i)$	$\overline{F}_{S(SNB,8,0.0001)}(x_i)$	CMC approx.$\overline{F}_{S(C,8)}(x_i)$	MC approx.$\overline{F}_{S(C,8)}(x_i)$
1	16	0.042577614	0.042573043	0.042577157	0.043076480 (0.145774109)	0.042560000 (0.201863949)
2	18	0.011940884	0.011939463	0.011940742	0.012148560 (0.061046175)	0.011880000 (0.108346587)
3	20	0.001991095	0.001990824	0.001991068	0.002048498 (0.015964780)	0.002080000 (0.045559789)
4	22	0.000182527	0.000182499	0.000182525	0.000189279 (0.002184336)	0.000290000 (0.017027002)
5	24	0.000008969	0.000008967	0.000008969	0.000009308 (0.000143841)	0.000000000 (-)
6	29	3.483774e−10	3.672044e−10	3.672627e−10	3.57897e−10 (0.000000008)	0.000000000 (-)

below 0.5. The stronger the dependence relationship, the more the calculation time increases. Once again, the efficiency of the approximation of a copula with non-zero lower tail dependence by another one with $\lambda_L = 0$ is less significant when the related dependence relationship is strong.

To summarize, if the mixing rv Θ is continuous, we can proceed either by conditional MC simulation or by the approximation of the rv Θ with a discrete one. The difference between both approximation methods depends on several elements such as the value of the chosen dependence parameter, the size of the portfolio, the number of simulations, etc. For example, the approximation of the Clayton copula with the shifted negative binomial copula is more accurate compared to the simulation method when the dependence parameter is small. In the latter case, execution times of both methods are comparable. However, in the event of a greater degree of dependence or a greater number of risks, the conditional MC simulation executes faster. Note that the aim here was to only present different approaches to solve this problem, rather than putting them through a comprehensive comparison.

2.6. Portfolio of exchangeable risks

Exchangeability plays an important role in the analysis of homogeneous portfolios in actuarial science and in quantitative risk management, notably in credit risk modeling (e.g. McNeil et al., 2015 and the references therein). Based on De Finetti's Theorem (see De Finetti, 1957) and its extension in Bühlmann (1960), an infinite exchangeable sequence of rvs can be represented as a mixture over a common parameter rv of an infinite sequence of iid rvs (see e.g. Feller, 1971 for more details). Let $\underline{X} = \{X_n, n \in \mathbb{N}\}$ be a sequence of n positive and exchangeable rvs. Define the sequence $\underline{W} = \{W_n, n \in \mathbb{N}\}$, where $W_n = \frac{X_1+\cdots+X_n}{n}$, for $n \in \mathbb{N}$. As discussed in, e.g., Aldous (1985), the common parameter rv drives the asymptotic behavior of $\underline{W}$. In the context of credit risk, Frey and McNeil (2001, 2002) have studied the asymptotic behavior of $\underline{W}$ with the distribution of $(X_1, \ldots, X_n)$ defined in terms of exchangeable Bernoulli mixture models, for $n = 2, 3, \ldots$ (see e.g. Proposition 3.2 in Frey and McNeil, 2002). Their result can be generalized to other dependence models. In the context of Section 2.3, we consider the joint distribution of $(X_1, \ldots, X_n)$ to be defined with an Archimedean copula C with either (18) or (19) for $n = 2, 3, \ldots$. Let Z be a discrete rv with $Z \in \{z_\theta, \theta \in \mathbb{N}\}$ where $z_\theta = E[X|\Theta = \theta] < \infty$ and $f_Z(z_\theta) = f_\Theta(\theta)$, $\theta \in \mathbb{N}$. Clearly, the sequence $\underline{W}$ converges in distribution to the rv Z, i.e.,

$$W_n \overset{\mathcal{D}}{\to} Z. \tag{34}$$

Indeed, we have

$$\mathcal{L}_{W_n}(t) = \sum_{\theta=1}^{\infty} \mathcal{L}_{W_n|\Theta=\theta}(t) f_\Theta(\theta) = \sum_{\theta=1}^{\infty} \left(\mathcal{L}_{X|\Theta=\theta}\left(\frac{t}{n}\right)\right)^n f_\Theta(\theta).$$

Near the origin, $\mathcal{L}_{X|\Theta=\theta}(t) = 1 - z_\theta t + o(t)$ which implies that

$$\begin{aligned}\lim_{n\to\infty} \mathcal{L}_{W_n}(t) &= \sum_{\theta=1}^{\infty} f_\Theta(\theta) \lim_{n\to\infty}\left(1 - z_\theta \frac{t}{n}\right)^n \\ &= \sum_{\theta=1}^{\infty} f_\Theta(\theta)\, \mathrm{e}^{-z_\theta t} = \mathcal{L}_Z(t),\end{aligned}$$

leading to (34).

The result in (34) shows that for a large portfolio of exchangeable risks, the distribution of W_n can be approximated by the distribution of Z. One of the interesting features of the strategy described in Section 2.3 is that the exact values of z_θ ($\theta \in \mathbb{N}$) can be computed as well as its pmf. If the rv X is continuous, discretization methods presented in Section 2.4 are used.

Let us look at a simple example that illustrates the convergence of W_n to Z for a portfolio of 100 exchangeable risks.

Table 18
Values of the expectation, variance, VaR and TVaR of W_n and Z where $F_{X_1,\ldots,X_n}$ is defined with the AMH copula and Poisson marginals.

	$\alpha = 0.5$	$\alpha = 0.9$		$\alpha = 0.5$	$\alpha = 0.9$
$E[W_n]$	5	5	$E[Z]$	5	5
$Var(W_n)$	0.920999	1.886689	$Var(Z)$	0.879797	1.855241
$VaR_{0.9}(W_n)$	6.39	6.74	$VaR_{0.9}(Z)$	6.426813	6.702153
$VaR_{0.9}(W_n)$	7.34	7.71	$VaR_{0.9}(Z)$	7.265279	7.660453
$VaR_{0.9999}(W_n)$	8.31	8.66	$VaR_{0.9999}(Z)$	8.241060	8.579895
$TVaR_{0.9}(W_n)$	6.833850	7.191685	$TVaR_{0.9}(Z)$	6.786601	7.165780
$TVaR_{0.99}(W_n)$	7.599597	7.958826	$TVaR_{0.99}(Z)$	7.535144	7.910267
$TVaR_{0.9999}(W_n)$	8.448255	8.795515	$TVaR_{0.9999}(Z)$	8.347715	8.704634

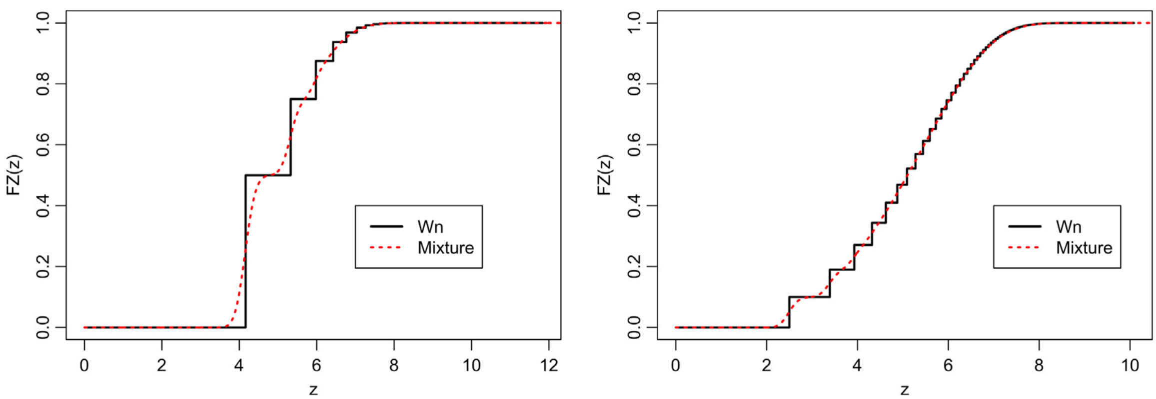

Fig. 3. The cdf of Z and W_n where $F_{X_1,\ldots,X_{100}}$ is defined by Poisson marginals and AMH copula with $\alpha = 0.5$ (left) and $\alpha = 0.9$ (right).

Example 12. Let $(X_1, \ldots, X_{100})$ be a vector of 100 exchangeable rvs with $X_i \sim X \sim Pois(\lambda = 5)$, for $i = 1, \ldots, 100$. The multivariate cdf of $(X_1, \ldots, X_{100})$ is defined with an AMH copula with dependence parameter $\alpha = 0.5$ and $\alpha = 0.9$.

As shown in Table 18 and illustrated in Fig. 3, for large portfolios, the distribution of Z is a good candidate to approximate the behavior of W_n. This result is very important in terms of computation time for very large portfolios since using Z is a faster, more efficient and easier to handle tool in comparison to directly computing W_n.

3. Capital allocation

Capital allocation is fundamental in actuarial science and quantitative risk management. It describes how the capital needed for the whole portfolio can be divided and allocated between risks of the portfolio. It is crucial for an insurance company or a financial institution to evaluate the overall capital charge for a portfolio of risks in order to protect itself from large rare events. The amount of capital needed for the entire portfolio is determined with a chosen risk measure ρ. Among the desired properties for a capital allocation rule (see e.g. Furman and Zitikis, 2008), one has to be fully additive, i.e.,

$$\rho(S) = \sum_{i}^{n} C_i, \tag{35}$$

where $S = X_1 + \cdots + X_n$ and C_i is the contribution of the ith risk to the aggregate risk of the portfolio.

In the present section, we show how to compute the values (exact or approximated) of the contributions C_i with the proposed methodology of Section 2. We consider Euler's capital allocation principle and the weighted risk capital allocation principle using different risk measures.

Let us assume here that we are in the context of Section 2.3, meaning that Θ is a discrete rv defined on $\mathbb{N}$ and that $X_i \in A = \{0, 1h, 2h, \ldots\}$ $(i = 1, \ldots, n)$. Also, assume that the joint cdf of $\underline{X}$ or its joint survival function is defined with the Archimedean copula C as in (18) or (19). If the rvs X_i are continuous, the procedure described in Section 2.4 will be used to find the contributions. For a continuous Θ, we refer to Section 2.5.

To apply Euler's capital allocation rule, we need to assume that the risk measure ρ is positive homogeneous. Let us define

$$L(\underline{\lambda}) = \sum_{i=1}^{n} \lambda_i X_i,$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$. For a given risk measure ρ, the contribution C_i allocated to risk i is given by

$$C_i = \rho(L; X_i) = \lambda_i \frac{\partial}{\partial \lambda_i} \rho\left(L\left(\underline{\lambda}\right)\right)\Big|_{\underline{\lambda}=\underline{1}}, \quad \text{for } i = 1, 2, \ldots, n,$$

where $\underline{1} = (1, \ldots, 1)$. We apply the Euler allocation principle with three different risk measures: covariance, VaR and TVaR (see Table 19 for their expressions). For more details, see e.g. Tasche (1999), Embrechts et al. (2005) or Rosen et al. (2011).

The second capital allocation principle we use is the one proposed by Furman and Zitikis (2008) in which the capital amount required and the contributions are based on weighted risk measures. Furman and Zitikis (2008) propose to compute the capital amount with weighted risk measures and to use weighted allocation methods for determining the contribution of each risk i, $i = 1, 2, \ldots, n$. Let ω be a weight function. Then, the capital amount corresponds to ρ_ω which is defined by $\rho_\omega(S) = \frac{E[S\omega(S)]}{E[\omega(S)]}$, assuming that both expectations exist. Since

$$\frac{E[S\omega(S)]}{E[\omega(S)]} = \frac{E\left[\left(\sum_{i=1}^{n} X_i\right)\omega(S)\right]}{E[\omega(S)]} = \sum_{i=1}^{n} \frac{E[X_i\omega(S)]}{E[\omega(S)]}, \tag{36}$$

the share of the capital allocated to X_i is given by

$$C_i = \rho_\omega(X_i; S) = \frac{E[X_i\omega(S)]}{E[\omega(S)]},$$

for $i = 1, 2, \ldots, n$. Clearly, due to (36), property (35) is satisfied.

Table 19
Contributions under considered allocation rules.

Allocation rule	Contribution C_i
Euler-Covariance	$E[X_i] + \frac{Cov(X_i,S)}{Var(S)}\{\xi(S) - E[S]\}$, with ξ is a chosen risk measure
Euler-VaR	$E[X_i \mid S = VaR_\kappa(S)]$
Euler-TVaR	$\frac{1}{1-\kappa}\left\{E\left[X_i \times 1_{S>VaR_\kappa(S)}\right] + \beta E\left[X_i 1_{S=VaR_\kappa(S)}\right]\right\}$, with $\beta = F_S(VaR_\kappa(S)) - \kappa$
Weighted-Esscher	$\frac{E\left[X_i e^{\eta S}\right]}{E\left[e^{\eta S}\right]}$
Weighted-Kamps	$\frac{E\left[X_i\left(1-e^{-\eta S}\right)\right]}{E\left[1-e^{-\eta S}\right]}$
Weighted-size-biased	$\frac{E\left[X_i S^\eta\right]}{E[S^\eta]}$

Among the several weight functions that can be used to calculate the capital allocation, we consider the following three methods: Esscher with $\omega(s) = e^{\eta s}$, Kamp with $\omega(s) = 1 - e^{-\eta s}$ and size-biased with $\omega(s) = s^\eta$.

Let us address the computation of C_i for the considered Euler capital allocation rules and weighted risk capital allocation rules given in Table 19. One encounters frequently the evaluation of the expectation $E[X_i S]$ (or a variation of it) which poses problem given that X_i and S are dependent. To circumvent this, we rewrite the product of these two rvs as

$$E[X_i S] = E_\Theta\left[E\left[X_i\left(X_i + S_{-i}\right) \mid \Theta\right]\right],$$

where $S_{-i} = X_1 + \cdots + X_{i-1} + X_{i+1} + \cdots + X_n$. The conditional independence of X_i and S_{-i} given Θ simplifies the evaluation of the quantities of interest.

Note that with the VaR risk measure, it is slightly more tedious. In this case, the contribution of risk i is given by

$$C_i = E[X_i \mid S = VaR_\kappa(S)] = E[X_i \mid S = k_0 h] = \frac{E\left[X_i \times 1_{\{S=k_0h\}}\right]}{\Pr(S = k_0 h)},$$

assuming that $\Pr(S = k_0h) > 0$. Then, using our approach, we have

$$\begin{aligned}
E\left[X_i \times 1_{\{S=k_0h\}}\right] &= E_\Theta\left[E\left[X_i \times 1_{\{S=k_0h\}} \mid \Theta\right]\right] \\
&= \sum_{\theta=1}^{\infty}\left(\sum_{j=1}^{k_0} j f_{X_i|\Theta=\theta}(jh) f_{S_{-i}|\Theta=\theta}\left((k_0-j)h\right)\right) \\
&\quad \times f_\Theta(\theta) \\
&= \sum_{\theta=1}^{\infty} E[X_i \mid \Theta = \theta]\left(\sum_{j=1}^{k_0} f^*_{X_i|\Theta=\theta}(jh)\right. \\
&\quad \left. \times\ f_{S_{-i}|\Theta=\theta}\left((k_0-j)h\right)\right) f_\Theta(\theta),
\end{aligned}$$

with $f^*_{X_i|\Theta=\theta}(jh) = \frac{j f_{X_i|\Theta}(j)}{E[X_i|\Theta=\theta]}$, for $j \in \mathbb{N}$. Finally, defining

$$g_{i|\Theta=\theta}(k_0h) = \sum_{j=1}^{k_0} f^*_{X_i|\Theta=\theta}(jh) f_{S_{-i}|\Theta=\theta}\left((k_0-j)h\right)$$

as a result of a convolution product, we obtain

$$E\left[X_i \times 1_{\{S=k_0h\}}\right] = \sum_{\theta=1}^{\infty} E[X_i \mid \Theta = \theta]\, g_{i|\Theta=\theta}(k_0h) f_\Theta(\theta), \tag{37}$$

where the values of $g_{i|\Theta=\theta}(kh)$ can be easily obtained using classic aggregation methods such as FFT.

Example 13. We consider a portfolio of 10 risks, where $X_i - 1 \sim NB\left(r_i = \frac{1+i}{2}, q_i = \frac{1}{1+\frac{8}{1+i}}\right)$, such that $E[X_i] = 5$, for $i = 1, 2, \ldots, 10$. It implies $E[S] = 50$. The multivariate cdf of $\underline{X}$ is defined as in (18) with the AMH copula. For $\alpha = 0$, 0.3 and 0.8, the variance of S is 104.6361, 185.2821 and 343.7994 respectively. In Table 20, we provide the relative contributions of $X_1, \ldots, X_{10}$ (i.e., $\frac{C_i}{\rho(S)}$) for the methods based on Euler's capital allocation rule and the ones under the weighted risk allocation approach assuming a dependence parameter $\alpha = 0.8$. As shown in Table 20, we are able to find exact contributions based on different allocation methods. Consistent results are obtained for $\alpha = 0$ and $\alpha = 0.3$.

4. Random sum of exchangeable risks

Random sums are essential in the description of many fundamental risk models in actuarial science. The so-called frequency-severity model is based on random sums. In this section, the computational methodology exposed in Section 2.3 is used to analyze the distribution of the aggregate claim amount rv S which is defined as the random sum of exchangeable individual claim amounts. Under this risk model, the rv S is defined by $S = \sum_{j=1}^{N} X_j$, where $\underline{X} = \left\{X_j, j \in \mathbb{N}\right\}$ forms a sequence of exchangeable rvs independent of the counting positive discrete rv N. The components of $\underline{X}$ are such that $X_j \in A = \{0, 1h, 2h, \ldots\}$ $(j \in \mathbb{N})$. The notation for the univariate pmf of X_i and the univariate cdf of X_i is given in Section 2. Also, we assume that the joint distribution of $\left(X_1, \ldots, X_j\right)$ is defined via either (18) or (19) with $d = j$.

The risk model considered in this section can be seen as an extension of the risk model described and studied in Section 2 of Albrecher et al. (2011) . Indeed, they consider a risk model defined with a compound Poisson process where the vector of claim amounts follows a multivariate mixed exponential distribution as defined in Section 2.2. Here, the multivariate distribution of the vector of claim amounts can be defined with any Archimedean copula and any marginal distributions for the claim amounts.

To apply the computational methodology, assume that the mixing rv Θ is a strictly positive discrete rv defined on $\mathbb{N}$. Clearly, the aggregate claim amount rv $S \in A$ and the objective is to compute the values of $f_S(kh) = \Pr(S = kh)$, $k \in \mathbb{N}_0$. Given the common mixture representation of either $F_{X_1,\ldots,X_j}$ or $\overline{F}_{X_1,\ldots,X_j}$, we have $f_S(kh) = \sum_{\theta=1}^{\infty} f_{S|\Theta=\theta}(kh) f_\Theta(\theta)$, where $(S|\Theta = \theta) = \sum_{j=1}^{N}\left(X_j|\Theta = \theta\right)$. The usual aggregation algorithms (see e.g. Panjer et al., 2008) such as FFT or Panjer's recursive algorithm can be used to compute the values of $f_{S|\Theta=\theta}(kh)$ for $k \in \mathbb{N}_0$ and for each $\theta = 1, 2, \ldots, \theta^*$ where θ^* is chosen such that $F_\Theta(\theta^*) \leq 1 - \varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon = 10^{-10}$). Again, note that we just need the values of $f_{X|\Theta=\theta}(kh)$, which are given in (22) or (25). Also, the LST of the rv S is given by

$$\begin{aligned}
\mathcal{L}_S(t) &= E\left[e^{-tS}\right] = \sum_{\theta=1}^{\infty} E\left[e^{-tS} \mid \Theta = \theta\right] f_\Theta(\theta) \\
&= \sum_{\theta=1}^{\infty} \mathcal{L}_{S|\Theta=\theta}(t) f_\Theta(\theta),
\end{aligned}$$

Table 20
Contributions of X_i ($i = 1, 2, \ldots, 10$) in % under the 6 allocation methods. The multivariate cdf of $\underline{X}$ is defined with an AMH copula ($\alpha = 0.8$).

i	Covariance $\frac{C_i}{\rho_{0.99}(S)}$	VaR $\frac{VaR_{0.99}(X_i;S)}{VaR_{0.99}(S)}$	TVaR $\frac{TVaR_{0.99}(X_i;S)}{TVaR_{0.99}(S)}$	Esscher ($\eta=0.1$) $\frac{E[X_i\omega(S)]}{E[S\omega(S)]}$	Kamps ($\eta=10^{-6}$) $\frac{E[X_i\omega(S)]}{E[S\omega(S)]}$	Size − biased ($\eta=10$) $\frac{E[X_i\omega(S)]}{E[S\omega(S)]}$
1	11.53	28.52	31.92	14.23	10.55	14.25
2	10.75	11.88	11.65	11.52	10.27	11.58
3	10.32	9.17	8.80	10.45	10.12	10.48
4	10.04	8.15	7.75	9.85	10.01	9.87
5	9.84	7.61	7.21	9.47	9.94	9.47
6	9.69	7.28	6.88	9.21	9.89	9.20
7	9.58	7.05	6.65	9.01	9.85	8.99
8	9.49	6.89	6.49	8.86	9.81	8.84
9	9.41	6.77	6.37	8.75	9.79	8.71
10	9.35	6.68	6.28	8.65	9.77	8.61
$\rho(S)$	76.2266	96	103.0490	89.2213	57.3053	94.1052

with $\mathcal{L}_{S|\Theta=\theta}(t) = P_N\left(\mathcal{L}_{X|\Theta=\theta}(t)\right)$ where

$$\mathcal{L}_{X|\Theta=\theta}(t) = E\left[\mathrm{e}^{-tX}|\Theta=\theta\right] = \sum_{k=0}^{\infty} \mathrm{e}^{-tkh} f_{X|\Theta=\theta}(kh)$$

and $P_N(s) = E\left[s^N\right]$ is the probability generating function of the positive discrete rv N.

Example 14. Let $N \sim Pois(\lambda = 2)$ and $X \sim Gamma(\alpha = 2, \beta = 0.01)$ such that $E[X] = 200$. It implies that $E[S] = 400$. Also, we assume that the joint distribution of $(X_1, \ldots, X_j)$, $j \in \mathbb{N}$, is defined as in (18) with an AMH copula with dependence parameter $\alpha = 0.8$. To obtain the desired results provided in Table 21, we use the upper and lower discretization methods with $h = \frac{1}{20}$. Different MC simulation studies (with 10 million simulations) have been performed. We present results from one of them in the second column of Table 21. From one study to the next, we have observed results that may or may not fall between the upper and lower bounds given in the third and fourth columns of Table 21 which highlights the advantage of the proposed methodology.

5. Renewal risk models with exchangeable inter-claim times

In this section, we consider a general class of continuous-time renewal risk models with exchangeable inter-claim times. This class is an extension of the class discussed in Section 3 of Albrecher et al. (2011). For an insurance portfolio, the surplus process is defined by $\underline{U} = \{U(t), t \geq 0\}$ where the surplus level at time t, $U(t)$, is given by

$$U(t) = u + ct - S(t),$$

where $U(0) = u$ is the initial surplus and c is the premium rate. The aggregate claim amount process, denoted by $\underline{S} = \{S(t), t \geq 0\}$ with $S(t) = \sum_{j=1}^{N(t)} X_j$ is a mixed compound renewal process with exchangeable inter-claim times. The claim number process $\underline{N} = \left\{N(t), t \in \mathbb{R}^+\right\}$ is a mixed renewal process where the inter-claim times $\underline{W} = \left\{W_j, j \in \mathbb{N}\right\}$ form a sequence of exchangeable and strictly positive real-valued rvs. The time between the $(j-1)$th and the jth claim ($j = 2, \ldots$) is defined by the rv W_j with W_1 the time of the first claim. The rvs $\left\{W_j, j \in \mathbb{N}\right\}$, are identically distributed as the canonical rv W, have pdf f_W, cdf F_W, and survival function $\overline{F}_W$.

To simplify the presentation, the joint survival function of $(W_1, W_2, \ldots, W_k)$ is defined with an Archimedean copula as in (8), i.e.,

$$\overline{F}_{W_1,W_2,\ldots,W_k}(x_1, \ldots, x_k) = C\left(\overline{F}_W(x_1), \ldots, \overline{F}_W(x_k)\right), \tag{38}$$

for $k \in \{2, 3, \ldots\}$ and $x_1, \ldots, x_k \geq 0$. The multivariate distribution of $(W_1, \ldots, W_k)$ can also be defined with joint cdf as in (5). The time of arrival of the jth claim is denoted $T_j = W_1 + \cdots + W_j$.

The claim amount rvs $\underline{X} = \left\{X_j, j \in \mathbb{N}\right\}$, where X_j corresponds to the amount of the jth claim, are assumed to be a sequence of strictly positive and iid rvs with pdf f_X and cdf F_X. The sequences $\underline{W}$ and $\underline{X}$ are independent.

The time of ruin is defined by the rv $\tau_u = \inf\{t \geq 0 : U(t) < 0\}$ with $\tau_u = \infty$ if $U(t) \geq 0$ for all $t \geq 0$. The infinite-time ruin probability is $\zeta(u) = \Pr(\tau_u < \infty | U(0) = u)$. Throughout this section, we assume the positive security loading condition $E[cW - X] > 0$ to be verified which ensures that ruin will not occur almost surely. Due to the common mixture representation, (38) is given by

$$\begin{aligned}\overline{F}_{W_1,W_2,\ldots,W_k}(x_1, \ldots, x_k) &= C\left(\overline{F}_W(x_1), \ldots, \overline{F}_W(x_k)\right) \\ &= \int_0^{\infty} \overline{F}_{W|\Theta=\theta}(x_1) \\ &\quad \times \cdots \times \overline{F}_{W|\Theta=\theta}(x_k)\, \mathrm{d}F_{\Theta}(\theta),\end{aligned}$$

where

$$\overline{F}_{W|\Theta=\theta}(x) = \mathrm{e}^{-\theta\psi^{-1}(\overline{F}_W(x))}$$

for $x \geq 0$. As mentioned in Section 3 of Albrecher et al. (2011), $\overline{F}_{W|\Theta=\theta}$ is the canonical survival function of the inter-claim time rvs for an ordinary renewal process. Let ζ_θ be the conditional ruin probability associated to the corresponding renewal process. It implies that the ruin probability ζ can be represented as a mixture, where Θ is the mixing rv, i.e.,

$$\zeta(u) = \int_0^{\infty} \zeta_\theta(u)\, \mathrm{d}F_{\Theta}(\theta). \tag{39}$$

The security loading condition is violated when the mixing rv Θ takes a value larger that $\theta_0 > 0$. We define θ_0 such that $\zeta_\theta(u) = 1$, for $\theta > \theta_0 > 0$. There exists a θ_0 such that $c \times E[W|\Theta=\theta] > E[X]$, for $\theta \in \{1, 2, \ldots, \theta_0\}$, and $c \times E[W|\Theta=\theta] < E[X]$, for $\theta \in \{\theta_0+1, \ldots\}$. Then, (39) becomes

$$\zeta(u) = \int_0^{\theta_0} \zeta_\theta(u)\, \mathrm{d}F_{\Theta}(\theta) + \overline{F}_{\Theta}(\theta_0), \tag{40}$$

for $u \geq 0$. Assuming that ζ_θ could be computed for each $\theta \in \{1, 2, \ldots, \theta_0\}$ of the ordinary renewal process associated to $\overline{F}_{W|\Theta=\theta}$, the value of $\zeta(u)$ can be computed using (40).

In the examples of Section 3 of Albrecher et al. (2011), the authors assume that $(W_1, W_2, \ldots, W_k)$ follows a multivariate mixed exponential distribution as defined in Section 2.2, where the LST of the mixing rv corresponds to the generator of an Archimedean copula. It means that the univariate marginal distribution of the inter-claim time is a univariate mixed exponential distribution. Indeed, the authors consider specific examples of mixed Poisson risk models.

In this section, we show that it is possible to consider any multivariate distribution for $\underline{W}$ defined with any Archimedean

Table 21
Values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ where S is defined as a random sum of dependent rvs.

	$\alpha = 0.8$ (simul)	$IC_{0.05}$	$\alpha = 0.8$ (upper)	$\alpha = 0.8$ (lower)	$\alpha = 0$ (exact)
$E[S]$	400.0359	[399.8034 ; 400.2685]	399.950	400.050	400
$Var(S)$	140775.7887	[140652.4769 ; 140899.2633]	140912.018	140942.018	80000
$VaR_{0.9}(S)$	906.9635	[906.3475 ; 907.5886]	907.000	907.200	873.8748
$VaR_{0.99}(S)$	1644.4705	[1642.6493 ; 1646.4017]	1645.350	1645.600	1470.9808
$VaR_{0.999}(S)$	2317.5620	[2312.0092 ; 2322.2396]	2323.350	2323.650	1992.0052
$VaR_{0.9999}(S)$	2959.6858	[2942.0263 ; 2975.0607]	2967.250	2967.600	2473.3833
$TVaR_{0.9}(S)$	1230.7730	[1230.5806 ; 1230.9654]	1231.206	1231.333	1138.1220
$TVaR_{0.99}(S)$	1939.1160	[1938.9381 ; 1939.2938]	1941.278	1941.494	1699.2458
$TVaR_{0.999}(S)$	2596.5160	[2596.3469 ; 2596.6851]	2603.626	2603.935	2202.1856
$TVaR_{0.9999}(S)$	3223.8834	[3223.7239 ; 3224.0429]	3236.634	3237.032	2672.1090

copula and given marginal distributions. First, let Θ be a strictly positive discrete rv defined on $\mathbb{N}$. As in Albrecher et al. (2011), we limit our analysis to exponentially distributed claim amounts with parameter β. It implies that

$$\zeta_\theta(u) = \frac{\beta - \rho_\theta}{\beta} e^{-\rho_\theta u}, u \geq 0, \tag{41}$$

where ρ_θ is the adjustment coefficient which is the smallest strictly positive solution to the Lundberg relation

$$E\left[e^{r(X-cW)}|\Theta = \theta\right] = E\left[e^{rX}\right] \times E\left[e^{-rcW}|\Theta = \theta\right] = 1 \tag{42}$$

with $E\left[e^{-rcW}|\Theta = \theta\right] = \int_0^\infty e^{-rcx} f_{W|\Theta=\theta}(x)\,dx$ and $f_{W|\Theta=\theta}(x) = -\frac{d\overline{F}_{W|\Theta=\theta}(x)}{dx}$. Given that Θ is a discrete rv and with (41), (40) becomes

$$\zeta(u) = \sum_{\theta=1}^{\theta_0} \Pr(\Theta = \theta) \frac{\beta - \rho_\theta}{\beta} e^{-\rho_\theta u} + \overline{F}_\Theta(\theta_0). \tag{43}$$

The expression in (43) is illustrated in the following example.

Example 15. Let $X \sim Exp(1)$ and $\overline{F}_{W_1,W_2,\ldots,W_k}(x_1,\ldots,x_k)$ be defined as in (38) where C is an AMH copula. It means that Θ follows a geometric distribution with parameter $q = 1-\alpha$ and

$$\mathcal{L}_\Theta(t) = \frac{qe^{-t}}{1-(1-q)e^{-t}} \text{ and } \mathcal{L}_\Theta^{-1}(u) = -\ln\left(\frac{1}{\frac{q}{t}+1-q}\right).$$

Then,

$$\overline{F}_{W|\Theta=\theta}(x) = \left(\frac{1}{qe^x+1-q}\right)^\theta \text{ and}$$
$$f_{W|\Theta=\theta}(x) = \frac{\theta q e^x}{(qe^x+1-q)^{\theta+1}}$$

for $x \geq 0, \theta \in \mathbb{N}$ and $0 < q < 1$. Note that $\overline{F}_{W|\Theta=1}(x) = \frac{1}{qe^x+1-q}$ corresponds to the univariate survival function of the exponential distribution with tilt as defined in Marshall and Olkin (2007).

In this case, we have

$$E[W|\Theta=\theta] = \int_0^\infty \left(\frac{1}{q\,e^x+1-q}\right)^\theta dx = \lim_{t\to\infty}\int_0^t \left(\frac{1}{q\,e^x+1-q}\right)^\theta dx. \tag{44}$$

Let $u = q\,e^x + 1 - q$, then (44) becomes

$$\int_0^\infty \left(\frac{1}{q\,e^x+1-q}\right)^\theta dx = \lim_{t\to\infty}\int_0^t \frac{1}{u^\theta(u-1+q)}du.$$

Using a partial fraction decomposition, we obtain

$$\int_0^\infty \left(\frac{1}{q\,e^x+1-q}\right)^\theta dx = -\frac{\ln(q)}{(1-q)^\theta} - \sum_{k=1}^{\theta-1}\frac{1}{(\theta-k)(1-q)^k}.$$

Also, in order to calculate the adjustment coefficient ρ_θ, we develop the expression of $E\left[e^{-tW}|\Theta=\theta\right]$ as follows

$$E\left[e^{-tW}|\Theta=\theta\right] = \int_0^\infty e^{-tx}\frac{\theta q e^x}{(qe^x+1-q)^{\theta+1}}dx. \tag{45}$$

Let $u = qe^x + 1 - q$, then (45) becomes

$$\begin{aligned}
&\int_0^\infty e^{-tx}\frac{\theta q e^x}{(qe^x+1-q)^{\theta+1}}dx \\
&= \int_1^\infty \frac{\theta q^t (u+q-1)^{-t}}{u^{\theta+1}}du \\
&= \frac{\theta q^t}{(q-1)^t}\int_1^\infty \frac{1}{\left(\frac{u}{q-1}+1\right)^t u^{\theta+1}}du \\
&= \frac{\theta q^t}{(q-1)^t} \times \frac{(q-1)^t\, {}_2F_1([t, t+\theta]; [t+\theta+1]; 1-q)}{t+\theta} \\
&= \frac{\theta q^t\, {}_2F_1([t, t+\theta]; [t+\theta+1]; 1-q)}{t+\theta},
\end{aligned}$$

where ${}_nF_m$ denotes the generalized hypergeometric function defined as follows

$${}_nF_m([a_1,\ldots,a_n];[b_1,\ldots,b_m];z) = \sum_{k=0}^\infty \frac{(a_1)_k\ldots(a_n)_k}{(b_1)_k\ldots(b_m)_k}\frac{z^k}{k!},$$

with $(x)_k = x(x+1)\ldots(x+k-1)$.

Let $\lambda = 1$, $\beta = 1$, and $c = 1.2$. The parameter of the AMH copula is $\alpha = 1-q = 0.9$. We find that $\theta_0 = 5$ and the expression in (43) for $\zeta(u)$ becomes

$$\zeta(u) = \sum_{\theta=1}^{5} 0.9^{\theta-1} \times 0.1 \times (1-\rho_\theta)e^{-\rho_\theta u} + 0.9^5,$$

where the values of ρ_θ for $\theta = 1, 2, \ldots, 5$ are given in the following table:

θ	1	2	3	4	5
ρ_θ	0.7666945	0.5591448	0.3649751	0.1794426	0.0001673811

The values of ρ_θ are computed by numerical optimization using (42) for $\theta = 1, 2, \ldots, 5$. □

We have defined here the multivariate distribution of $\underline{W}$ in terms of the multivariate survival function as in (19). Our strategy, based on common mixtures, allows us however to also do our investigation of ruin models with the multivariate cdf as in (18). Also, it is possible to adapt the strategy discussed in Section 2.5 with a continuous distribution for the mixing rv Θ. The content of this section clearly demonstrates that using our common mixture representation methodology allows to analyze a variety of risk models which opens the door to further research.

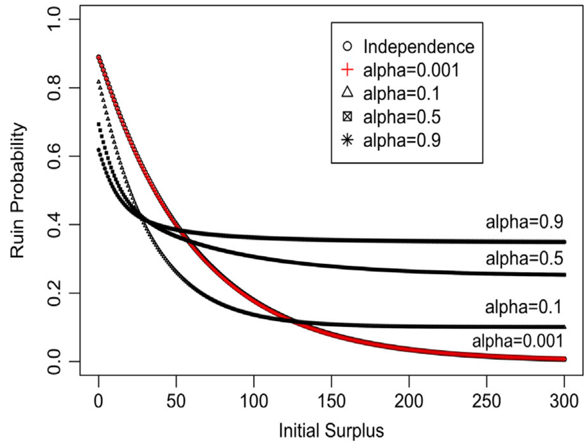

Fig. 4. Ultimate ruin probability for various degrees of dependence.

6. Classical discrete-time risk models with exchangeable losses

In this section, we consider a discrete-time risk model with exchangeable losses. Let $\underline{X} = \{X_k, k \in \mathbb{N}\}$ be a sequence of exchangeable rvs, where X_k is the aggregate loss for a portfolio in period $k \in \mathbb{N}$ with $X_k \sim X$, $k \in \mathbb{N}$. Let $\underline{U} = \{U_k, k \in \mathbb{N}_0\}$ be the surplus process of the portfolio, where U_k corresponds to the surplus level at period $k \in \mathbb{N}$. For $k = 0$, $U_0 = u$ corresponds to the initial amount of capital allocated to the portfolio. Then, at period $k \in \mathbb{N}$, $U_k = U_{k-1} + \pi - X_k = u - \sum_{j=1}^{k} \left(X_j - \pi\right)$. The time of ruin $\tau_u = \inf\{k \in \mathbb{N}, U_k < 0\}$, if $\underline{U}$ goes below 0 at least once, or ∞, if $\underline{U}$ never goes below 0. We define the infinite-time ruin probability by $\zeta(u) = \Pr(\tau_u < \infty)$. To prevent ruin with certainty (i.e., $\zeta(u) = 1$, $u \geq 0$), we assume that the net profit condition is satisfied, i.e., $E[X - \pi] < 0$, where π is the premium income per period with $\pi = (1+\eta)E[X]$. For simplification purposes, we assume $X \in \mathbb{N}_0$, $u \in \mathbb{N}_0$, and $\pi = 1$ with $\pi > E[X]$. Then, with these additional assumptions, the classical discrete-time risk model with exchangeable losses corresponds to an extension of the compound binomial classical risk model (see e.g. Gerber, 1988; Shiu, 1989; Willmot, 1993; De Vylder and Marceau, 1996; Dickson, 1992 for details on the compound binomial risk model).

For $j = 2, 3, \ldots$, the multivariate distribution of $\left(X_1, \ldots, X_j\right)$ is defined with an Archimedean copula C with either (18) or (19) as in Section 2.3. Let $\zeta_\theta(u)$ be the conditional infinite-time ruin probability given $\Theta = \theta$. There exists a θ_0 such that $E[X|\Theta = \theta] < 1$, for $\theta \in \{1, 2, \ldots, \theta_0\}$, and $E[X|\Theta = \theta] > 1$, for $\theta \in \{\theta_0 + 1, \theta_0 + 2, \ldots\}$. Then, when $\theta = \theta_0 + 1, \theta_0 + 2, \ldots$, the solvency condition is not satisfied and the conditional infinite-time ruin probability $\zeta_\theta(u) = 1$, for all $u \in \mathbb{N}_0$. For $\theta = 1, 2, \ldots, \theta_0$, adapting expressions from Cossette et al. (2003), we have

$$\zeta_\theta(u) = \frac{\zeta_\theta(u-1) - \sum_{j=1}^{u} \zeta_\theta(u-j) \times f_{X|\Theta=\theta}(j) - \overline{F}_{X|\Theta=\theta}(u)}{f_{X|\Theta=\theta}(0)}, \quad \text{for } u \in \mathbb{N}_0,$$

with initial value $\zeta_\theta(0) = \frac{E[X|\Theta=\theta] - \Pr(X>0|\Theta=\theta)}{f_{X|\Theta=\theta}(0)}$. The unconditional infinite-time ruin probability $\zeta(u)$ is given by

$$\begin{aligned} \zeta(u) &= \sum_{\theta=1}^{\infty} \zeta_\theta(u) f_\Theta(\theta) \\ &= \sum_{\theta=1}^{\theta_0} \zeta_\theta(u) f_\Theta(\theta) + \overline{F}_\Theta(\theta_0), \quad u \in \mathbb{N}_0. \end{aligned} \tag{46}$$

Example 16. Let X be a non-negative discrete rv with

$$f_X(0) = (1-\delta) + \delta \times f_B(0) \quad \text{and } f_X(k) = \delta \times f_B(k), \; k \in \mathbb{N}, \; \delta \in (0,1),$$

and $B - 1 \sim NB(r, q)$ $(r \in \mathbb{R}^+, q \in (0,1))$ with $E[B] = 1 + r \times \frac{1-q}{q}$. We fix the different parameters as follows: $\delta = 0.1$, $r = 2$, and $q = \frac{1}{5}$, and $E[X] = 0.9 < 1$. Finally, $F_{X_1,\ldots,X_j}$ is defined with an AMH copula as in (18). The ruin probability is calculated for several values of initial capital and different values of dependence parameter α. Results are presented in Fig. 4 from which we can see that for a small dependency parameter, the ruin probability tends to zero. Otherwise, the greater the parameter becomes, the more likely the probability of ruin tends to a value close to 0.35. Once again, we emphasize the significant impact of a low to moderate dependence relation between rvs on the overall portfolio.

In the following example we provide an analytical expression of ζ for a specific loss distribution.

Example 17. Let X be a non-negative discrete rv with

$$f_X(0) = (1-\delta) \text{ and } f_X(2) = \delta, \; k \in \mathbb{N}, \; \delta \in (0, 0.5).$$

Assume that $F_{X_1,\ldots,X_j}$ is defined by (18) with copula C. Using (22), we find

$$f_{X|\Theta=\theta}(0) = 1 - \delta_\theta = \mathrm{e}^{-\theta \mathcal{L}_\Theta^{-1}(f_X(0))}.$$

From Example 3.1 of Willmot (1993), we find

$$\zeta_\theta(u) = \left(\frac{\delta_\theta}{1-\delta_\theta}\right)^{u+1}, \tag{47}$$

for $u \in \mathbb{N}_0$ and $\theta = 1, 2, \ldots, \theta_0$. Replacing (47) in (46), the expression $\zeta(u)$ becomes

$$\zeta(u) = \sum_{\theta=1}^{\theta_0} f_\Theta(\theta) \left(\frac{\delta_\theta}{1-\delta_\theta}\right)^{u+1} + \overline{F}_\Theta(\theta_0), \; u \in \mathbb{N}_0.$$

For illustration purposes, we fix the different parameters as follows: $\delta = 0.4$, implying $E[X] = 0.8 < 1$. Finally, $F_{X_1,\ldots,X_j}$ is defined by (18) where C is a Frank copula with $\alpha = 4$. We obtain that $\theta_0 = 9$ and

$$\zeta(u) = \sum_{\theta=1}^{9} \frac{\left(1-\mathrm{e}^{-4}\right)^\theta}{4\theta} \left(\frac{\delta_\theta}{1-\delta_\theta}\right)^{u+1} + \overline{F}_\Theta(9), \; u \in \mathbb{N}_0,$$

where the values of δ_θ are given in Table 22.

7. Partially nested Archimedean copulas

In the present section, we generalize the proposed strategy of Section 2 and adapt it to hierarchical structures for which at least one of the arguments is an Archimedean copula. More precisely, we consider in detail nested Archimedean copulas. Since there are several ways of nesting copulas and the computational methodology based on the mixing representation is the same for any chosen structure, we will only present a detailed example where we apply the strategy to a partially nested Archimedean copula. See Joe (1997), McNeil (2008) and Hofert (2011) for a general introduction to nested Archimedean copulas.

Let C be a one level partially nested Archimedean copula with d children, i.e., C is of the form

$$\begin{aligned} C\left(\underline{u}\right) &= C(\underline{u}; \psi_0, \psi_1, \ldots, \psi_n) \\ &= C\left(C\left(\underline{u_1}; \psi_1\right), \ldots, C\left(\underline{u_d}; \psi_d\right); \psi_0\right) \\ &= C\left(C\left(u_{1,1}, \ldots, u_{1,n_1}; \psi_1\right), \ldots,\right. \end{aligned} \tag{48}$$

Table 22
Values of δ_θ for different values of θ.

θ	1	2	3	4	5	6	7	8	9
δ_θ	0.92625	0.85793	0.79466	0.73604	0.68176	0.63148	0.58491	0.54177	0.50181

$$\times\ C\left(u_{d,1},\ldots,u_{d,n_d};\psi_d\right);\psi_0\right),$$

with $\underline{u}=\left(\underline{u_1},\ldots,\underline{u_d}\right)$, where $\underline{u_i}=\left(u_{i,1},\ldots,u_{i,n_i}\right)$ for $i=1,\ldots,d$.

An example of the partially nested Archimedean copula with arguments $\underline{u}$ is defined as follows in terms of a multivariate parent copula of dimension d and d child copulas, where the dimension of the i^{th} copula is denoted by n_i:

$$\begin{aligned} C\left(\underline{u}\right) &= C(\underline{u};\psi_0,\psi_1,\ldots,\psi_d) \\ &= C\left(C\left(\underline{u_1};\psi_1\right),\ldots,C\left(\underline{u_d};\psi_d\right);\psi_0\right) \\ &= C\left(C\left(u_{1,1},\ldots,u_{1,n_1};\psi_1\right),\ldots,\right. \\ &\quad\times\ \left.C\left(u_{d,1},\ldots,u_{d,n_d};\psi_d\right);\psi_0\right) \\ &= \psi_0\left(\sum_{i=1}^{d}\psi_0^{-1}\left(\psi_i\left(\sum_{j=1}^{n_i}\psi_i^{-1}\left(u_{i,j}\right)\right)\right)\right). \end{aligned} \tag{49}$$

Since ψ_0 is the LST of a strictly positive rv Θ_0, (49) develops into

$$\begin{aligned} C\left(\underline{u}\right) &= \int_0^\infty \mathrm{e}^{-\theta_0\times\sum_{i=1}^{d}\psi_0^{-1}\left(\psi_i\left(\sum_{j=1}^{n_i}\psi_i^{-1}(u_{i,j})\right)\right)}\mathrm{d}F_{\Theta_0}\left(\theta_0\right) \\ &= \int_0^\infty \prod_{i=1}^{d}\mathrm{e}^{-\theta_0\times\psi_0^{-1}\left(\psi_i\left(\sum_{j=1}^{n_i}\psi_i^{-1}(u_{i,j})\right)\right)}\mathrm{d}F_{\Theta_0}\left(\theta_0\right) \end{aligned}$$

which becomes

$$\begin{aligned} C\left(\underline{u}\right) &= \int_0^\infty \prod_{i=1}^{d}\psi_{0,i}\left(\sum_{j=1}^{n_i}\psi_i^{-1}\left(u_{i,j}\right);\theta_0\right)\mathrm{d}F_{\Theta_0}\left(\theta_0\right) \\ &= \int_0^\infty \prod_{i=1}^{d}\left(\int_0^\infty \prod_{j=1}^{n_i}\mathrm{e}^{-\theta_{0,i}\times\psi_i^{-1}(u_{i,j})}\mathrm{d}F_{\Theta_{0,i}}\left(\theta_{0,i}\right)\right)\mathrm{d}F_{\Theta_0}\left(\theta_0\right), \end{aligned}$$

where $\psi_{0,i}(t)=\mathrm{e}^{-\theta_0\times\psi_0^{-1}\circ\psi_i(t)}$. As mentioned notably in Hofert (2010), $\psi_0^{-1}\circ\psi_i$ must be completely monotone in order to verify the nesting condition.

Let $\underline{k}=\left(k_{1,1},\ldots,k_{1,n_1},\ldots,k_{d,1},\ldots,k_{d,n_d}\right)$. As in (18), let the multivariate distribution of $\underline{X}=\left(\underline{X_1},\ldots,\underline{X_d}\right)$ with $\underline{X_i}=\left(X_{i,1},\ldots,X_{i,n_i}\right)$, for $i=1,\ldots,d$ be defined in terms of its joint cdf as follows

$$\begin{aligned} F_{\underline{X}}\left(\underline{k}h\right) &= C\left(F_{X_{1,1}}\left(k_{1,1}h\right),\ldots,F_{X_{1,n_1}}\left(k_{1,n_1}h\right),\ldots,\right. \\ &\quad\times\ \left.F_{X_{d,1}}\left(k_{d,1}h\right),\ldots,F_{X_{d,n_d}}\left(k_{d,n_d}h\right)\right) \\ &= \int_0^\infty \prod_{i=1}^{d}\left(\int_0^\infty \prod_{j=1}^{n_i}F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}\left(k_{i,j}h\right)\right. \\ &\quad\times\ \left.\mathrm{d}F_{\Theta_{0,i}}\left(\theta_{0,i}\right)\right)\mathrm{d}F_{\Theta_0}\left(\theta_0\right), \end{aligned} \tag{50}$$

where $F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}\left(k_{i,j}\right)=\mathrm{e}^{-\theta_{0,i}\times\psi_i^{-1}\left(F_{X_{i,j}}(k_{i,j}h)\right)}$ for $j=1,2,\ldots,n_i$ and $i=1,\ldots,d$. Note that the multivariate distribution of $\underline{X}$ can also be defined with its joint survival function as in (19).

To apply our methodology in a risk aggregation context, we need to assume that we can identify the distribution of the rv $\Theta_{0,i}$ from $\psi_{0,i}$, i.e., we should be able to write $\psi_{0,i}(t)=\mathcal{L}_{\Theta_{0,i}}(t)$. Also, we assume that $\Theta_0,\Theta_{0,1},\ldots,\Theta_{0,d}$ are strictly positive discrete rvs defined on $\mathbb{N}$ with pmf $f_{\Theta_{0,i}}\left(\theta_{0,i}\right)=\Pr\left(\Theta_{0,i}=\theta_{0,i}\right)$ and cdf $F_{\Theta_{0,i}}\left(\theta_{0,i}\right)=\Pr\left(\Theta_{0,i}\le\theta_{0,i}\right)=\sum_{j=1}^{\theta_{0,i}}f_{\Theta_{0,i}}(j)$ for $\theta_{0,s}\in\mathbb{N}$ and $i=1,2,\ldots,d$. Then, (50) becomes

$$\begin{aligned} &F_{\underline{X}}\left(\underline{k_1}h,\ldots,\underline{k_d}h\right) \\ &\quad= \sum_{\theta_0=1}^{\infty}\prod_{i=1}^{d}\left(\sum_{\theta_{0,i}=1}^{\infty}\prod_{j=1}^{n_i}F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}\left(k_{i,j}h\right)f_{\Theta_{0,i}}\left(\theta_{0,i}\right)\right) \\ &\quad\times f_{\Theta_0}\left(\theta_0\right). \end{aligned} \tag{51}$$

We define $S=\sum_{i=1}^{d}\sum_{j=1}^{n_i}X_{i,j}=\sum_{i=1}^{d}S_i$ where $S_i=\sum_{j=1}^{n_i}X_{i,j}$, for $i=1,\ldots,d$. Then, similarly to (27), we have

$$f_{S_i|\Theta_0=\theta_0}\left(kh\right)=\sum_{\theta_{0,i}=1}^{\infty}f_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}\left(kh\right)f_{\Theta_{0,i}}\left(\theta_{0,i}\right),\quad \text{for } i=1,\ldots,d,$$

and

$$f_S\left(kh\right)=\sum_{\theta_0=1}^{\infty}f_{S|\Theta_0=\theta_0}\left(kh\right)f_{\Theta_0}\left(\theta_0\right),$$

where

$$\left(S|\Theta_0=\theta_0\right)=\sum_{i=1}^{d}\left(S_i|\Theta_0=\theta_0\right),$$

and

$$\left(S|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}\right)=\sum_{j=1}^{n_i}\left(X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}\right).$$

Note that $(S_1|\Theta_0=\theta_0),\ldots,(S_d|\Theta_0=\theta_0)$ are conditionally independent. Within each class $i=1,\ldots,d$, $\left(X_{i,1}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}\right),\ldots,\left(X_{i,n_i}|\Theta_0=\theta_0,\Theta_{0,s}=\theta_{0,i}\right)$ are also conditionally independent. It implies that

$$f_{S|\Theta_0=\theta_0}\left(kh\right)=f_{S_1|\Theta_0=\theta_0}*\ldots*f_{S_d|\Theta_0=\theta_0}\left(kh\right), \tag{52}$$

and

$$\begin{aligned} &f_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}\left(kh\right) \\ &\quad= f_{X_{i,1}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}*\ldots*f_{X_{i,n_i}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}\left(kh\right), \end{aligned} \tag{53}$$

for $k\in\mathbb{N}_0$ and $i=1,\ldots,d$ and where "$*$" denotes the convolution product. Values of (52) and (53) are computed using the same tools (e.g. DePril, FFT, etc.) as the ones mentioned in Section 2.

Algorithm 18. Let $\theta_0\in\left\{1,2,\ldots,\theta_0^*\right\}$.

1. Begin with $\theta_0=1$.
2. For each child copula C_i where $i\in\{1,\ldots,d\}$, let $\theta_{0,i}\in\left\{1,2,\ldots,\theta_{0,i}^*\right\}$ and proceed as follows:
   (a) Begin with $\theta_{0,i}=1$.
   (b) For $j=1,\ldots,n_i$, calculate $F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}\left(k_{i,j}h\right)=\mathrm{e}^{-\theta_{0,i}\psi_i^{-1}\left(F_{X_{i,j}}(k_{i,j}h)\right)}$, for $k_{i,j}\in\mathbb{N}_0$.
   (c) For $j=1,\ldots,n_i$, calculate

$$f_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}\left(k_{i,j}h\right)$$

Table 23
Values of the expectation, variance, VaR and TVaR of $S = X_{1,1} + \ldots + X_{1,40} + X_{2,1} + \ldots + X_{2,40}$ where the joint cdf $F_{X_{1,1},\ldots,X_{1,40},X_{2,1},\ldots,X_{2,40}}$ is as defined in Example 19.

	Exact values	Simulated values	$IC_{0.05}$
$E[S]$	142	141.9887	[141.9304 ; 142.0469]
$Var(S)$	883.6003	883.3098	[880.8666 ; 885.7633]
$VaR_{0.5}(S)$	133	133	[133 ; 134]
$VaR_{0.9}(S)$	186	186	[186 ; 186]
$VaR_{0.99}(S)$	225	225	[225 ; 226]
$VaR_{0.999}(S)$	250	249	[248 ; 250]
$VaR_{0.9999}(S)$	267	267	[266 ; 268]
$TVaR_{0.5}(S)$	165.3440	165.3444	[165.3386 ; 165.3984]
$TVaR_{0.9}(S)$	204.2611	204.4585	[204.4295 ; 204.4874]
$TVaR_{0.99}(S)$	236.2996	236.3021	[236.2140 ; 236.3066]
$TVaR_{0.999}(S)$	257.3535	257.5276	[257.1387 ; 257.5444]
$TVaR_{0.9999}(S)$	273.0259	273.5895	[272.7160 ; 273.805]

$$= \begin{cases} e^{-\theta\psi_i^{-1}\left(F_{X_{i,j}}(k_{i,j}h)\right)} & , k_{i,j} = 0 \\ e^{-\theta\psi_i^{-1}\left(F_{X_{i,j}}(k_{i,j}h)\right)} & \\ \quad - e^{-\theta\psi_i^{-1}\left(F_{X_{i,j}}((k_{i,j}-1)h)\right)} & , k_{i,j} \in \mathbb{N}. \end{cases}$$

(d) Using e.g. FFT or DePril's Algorithm, compute $f_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_i h)$ for $k_i \in \mathbb{N}_0$.
(e) Repeat steps (2b), (2c), and (2d) for $\theta_{0,i} = 2, \ldots, \theta_{0,i}^*$ where $\theta_{0,i}^*$ is chosen such that $F_{\Theta_{0,i}}\left(\theta_{0,i}^*\right) \leq 1 - \varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon = 10^{-10}$).
(f) Compute $f_{S_i|\Theta_0=\theta_0}(k_i h) = \sum_{\theta_{0,i}=1}^{\theta_{0,s}^*} f_{S_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_i h)\, f_{\Theta_{0,i}}\left(\theta_{0,i}\right)$, for $k_i \in \mathbb{N}_0$.

3. Convolute all $f_{S_i|\Theta=\theta}$ for $i = 1, \ldots, d$, to calculate $f_{S|\Theta=\theta}$.
4. Repeat steps (2) and (3) for $\theta_0 = 2, \ldots, \theta_0^*$ where θ_0^* is chosen such that $F_{\Theta_0}\left(\theta_0^*\right) \leq 1-\varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon = 10^{-10}$).
5. Compute $f_S(kh) = \sum_{\theta_0=1}^{\theta_0^*} f_{S|\Theta_0=\theta}(kh) f_{\Theta_0}(\theta_0)$, for $k \in \mathbb{N}_0$.

As mentioned notably in Hofert (2010), the difficult task is to identify the distributions of $\Theta_{0,1}, \ldots, \Theta_{0,d}$. For example, if we assume that $C_{\alpha_0}, C_{\alpha_1}, \ldots, C_{\alpha_d}$ are AMH copulas, with respective parameters $\alpha_0, \alpha_1, \ldots, \alpha_d$ (with $\alpha_0 < \min(\alpha_1, \ldots, \alpha_d)$), then, it implies that $\Theta_0 \sim$ Geo$(1-\alpha_0)$ and $\Theta_{0,i} \sim$ Shifted NB$(\alpha_0, \frac{1-\alpha_i}{1-\alpha_0})$, for $i = 1, \ldots, d$, as shown in Hofert (2010) (see details in Appendix).

In the following example, we consider a two-dimensional partially nested Archimedean copula as defined in (48), where all copulas involved are AMH copulas. The example illustrates the accuracy of the proposed strategy in comparison to the MC simulation method.

Example 19. Consider a portfolio of 80 risks $\underline{X} = (X_{1,1}, \ldots, X_{1,40}, X_{2,1}, \ldots, X_{2,40})$ with multivariate cdf defined as in (51) with $d = 2$ and $n_1 = n_2 = 40$. Assume C_{α_0}, C_{α_1} and C_{α_2} to be AMH copulas with $\alpha_0 = 0.2$, $\alpha_1 = 0.3$, and $\alpha_2 = 0.4$. Let $X_{i,j} \sim Bin(10, q_{i,j})$ where $q_{i,j} = 0.05 \times i + 0.005j$, $i = 1, 2$ and $j = 1, 2, \ldots, 40$. It implies that $E[S] = 142$. Relevant measures of $S = \sum_{i=1}^{2}\sum_{j=1}^{40} X_{i,j}$ can be obtained with Algorithm Algorithm 18 or with MC simulations. Values of the expectation, the variance, the VaR and the TVaR for both methods in addition to their confidence intervals (with a confidence level of 95%) are given in Table 23. We can also calculate the exact values of Pearson's correlation coefficient between different risks. For example $\rho_P(X_{1,1}, X_{1,2}) = 0.07548428$, $\rho_P(X_{2,1}, X_{2,2}) = 0.12173402$ and $\rho_P(X_{1,1}, X_{1,2}) = 0.05336731$. Note that the simulation results (1 million simulations) are very close to the exact values obtained with the proposed approach.

In the following example, we present a specific five-dimensional partially nested Archimedean copula with two nesting levels (see Table 25).

Example 20. Assume a multivariate cdf of $\underline{X} = (X_1, X_2, X_3, X_4, X_5)$ defined with binomial marginals and the following five-dimensional partially nested Archimedean copula:

$$C(u_1, u_2, u_3, u_4, u_5) = C_{\alpha_0}\left(C_{\alpha_1}(u_1, u_2), C_{\alpha_2}\left(u_3, C_{\alpha_3}(u_4, u_5)\right)\right),$$

where C_{α_j}, $j = 0, 1, 2, 3$, correspond to bivariate AMH copulas. Also, $X_i \sim Bin(10, 0.05i)$, $i = 1, 2, \ldots, 5$, and $\alpha_j = 0.1j + 0.2$, $j = 0, 1, 2, 3$. Let $S = \sum_{i=1}^{5} X_i$ which implies $E[S] = 7.5$. In Table 24, we provide the exact and the simulated (10 million MC simulations) values of $f_S(k)$, $k = 0, 1, 2, \ldots, 50$ in addition to their confidence intervals (with a confidence level of 95%). We noticed a fluctuation in the results for several distinct 10 million simulation paths, contrarily to our proposed approach.

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Appendix A. Archimedean copulas

Archimedean copulas defined with a strictly positive discrete mixing rv Θ:

1. Ali–Mikhail–Haq (AMH) family:
   - Copula: $C_\alpha(u_1, \ldots, u_n) = (1-\alpha)\left(\prod_{i=1}^{n}\left((1-\alpha)u_i^{-1} + \alpha\right) - \alpha\right)^{-1}$
   - Parameter: $\alpha \in [0, 1)$
   - Discrete distribution for Θ: Shifted Geometric$(1-\alpha)$
   - Pmf: $f_\Theta(k) = \alpha^{k-1}(1-\alpha)$, $k \in \mathbb{N}$
   - LST: $\mathcal{L}_\Theta(t) = \frac{1-\alpha}{e^t - \alpha}$
   - Inverse of LST: $\mathcal{L}_\Theta^{-1}(u) = \ln\left(\frac{1-\alpha}{u} + \alpha\right)$
   - Particular cases: as $\alpha \to 0$, $C_\alpha(u_1, \ldots, u_n) = C^\perp(u_1, \ldots, u_n)$
2. Frank family:
   - Copula: $C_\alpha(u_1, \ldots, u_n) = \frac{-1}{\alpha}\ln\left(1 - \frac{(1-e^{-u_1\alpha})\times\ldots\times(1-e^{-u_n\alpha})}{(1-e^{-\alpha})^{n-1}}\right)$
   - Parameter: $\alpha \in (0, \infty)$
   - Discrete distribution for Θ: Logarithmic$(1-e^{-\alpha})$
   - Pmf: $f_\Theta(k) = \frac{(1-e^{-\alpha})^k}{k\alpha}$, $k \in \mathbb{N}$
   - LST: $\mathcal{L}_\Theta(t) = -\frac{1}{\alpha}\ln\left(1 - \left(1 - e^{-\alpha}\right)e^{-t}\right)$
   - Inverse of LST: $\mathcal{L}_\Theta^{-1}(u) = -\ln\left(\frac{1-e^{-\alpha u}}{1-e^{-\alpha}}\right)$
   - Particular cases: $C_{\alpha\to 0}(u_1, \ldots, u_n) = C^\perp(u_1, \ldots, u_n)$ and $C_{\alpha\to\infty}(u_1, \ldots, u_n) = C^+(u_1, \ldots, u_n)$
3. Shifted Negative Binomial family:
   - Copula: $C_{r,q}(u_1, \ldots, u_n)$ $= \left(q\left(\prod_{i=1}^{n}\left(qu_i^{\frac{-1}{r}} + 1 - q\right) - (1-q)\right)^{-1}\right)^r$
   - Parameter: $r \in \mathbb{R}^+$ and $q \in (0, 1)$
   - Discrete distribution for Θ: Shifted Negative Binomial(r,q)
   - $\Theta = M + r$ with $M \sim NB(r, q)$
   - Pmf: $f_\Theta(k) = \binom{k-1}{k-r}q^r(1-q)^{k-r}$, $k = r, r+1, \ldots$
   - LST: $\mathcal{L}_\Theta(t) = \left(\frac{qe^{-t}}{1-(1-q)e-t}\right)^r$

Table 24
Values of the pmf of $S = \sum_{i=1}^{5} X_i$ where the multivariate cdf of $\underline{X} = (X_1, X_2, X_3, X_4, X_5)$ is as defined in Example 20.

k	$f_S(k)$ (exact values)	$f_S(k)$ (simulated values)	$IC_{0.05}$
0	0.000808	0.000805	[0.000786 ; 0.000821]
1	0.005795	0.005785	[0.005714 ; 0.005808]
2	0.020111	0.020082	[0.019991 ; 0.020165]
3	0.045814	0.045699	[0.045598 ; 0.045856]
4	0.078337	0.078477	[0.078137 ; 0.078470]
5	0.108726	0.108730	[0.108657 ; 0.109043]
10	0.086310	0.086248	[0.086074 ; 0.086421]
15	0.006728	0.006732	[0.006667 ; 0.006768]

Table 25
Values of expectation, variance, VaR and TVaR of $S = \sum_{i=1}^{5} X_i$ as defined in Example 20.

	Exact values	Simulated values	$IC_{0.05}$
$E[S]$	7.5	7.49949	[7.49771 ; 7.50128]
$Var(S)$	8.31314	8.31013	[8.30286 ; 8.31742]
$VaR_{0.5}(S)$	7	7	[7 ; 7]
$VaR_{0.9}(S)$	11	11	[11 ; 11]
$VaR_{0.99}(S)$	15	15	[15 ; 15]
$VaR_{0.999}(S)$	17	17	[17 ; 17]
$VaR_{0.9999}(S)$	19	19	[19 ; 19]
$TVaR_{0.5}(S)$	9.81112	9.81008	[9.80771 ; 9.81239]
$TVaR_{0.9}(S)$	12.85623	12.85600	[12.85160 ; 12.86020]
$TVaR_{0.99}(S)$	15.75489	15.75197	[15.744530 ; 15.759540]
$TVaR_{0.999}(S)$	17.89402	17.89070	[17.86590 ; 17.91640]
$TVaR_{0.9999}(S)$	19.72388	19.72100	[19.6540 ; 19.7920]

- Inverse of LST: $\mathcal{L}_{\Theta}^{-1}(u) = \ln\left(qu^{-\frac{1}{r}} + (1-q)\right)$
- Particular cases: as $r \to 0$ or $r \to \infty$, $C_{r,q}(u_1, \ldots, u_n) = C^{\perp}(u_1, \ldots, u_n)$. Also, $C_{1,q}(u_1, \ldots, u_n) = C_{1-q}^{AMH}(u_1, \ldots, u_n)$ and when $r \to 0$, $C_{0,0}(u_1, \ldots, u_n) = C^{+}(u_1, \ldots, u_n)$. The most important case is when $q \to 0$, $C_{r,0}(u_1, \ldots, u_n) = C_{1/r}^{Clay}(u_1, \ldots, u_n)$.

Archimedean copulas defined with a strictly positive continuous mixing rv Θ:

1. Clayton family:
   - Copula: $C_{\alpha}(u_1, \ldots, u_n) = \left(u_1^{-\alpha} + \cdots + u_n^{-\alpha} - (n-1)\right)^{-\frac{1}{\alpha}}$
   - Parameter: $\alpha \in (0, \infty)$
   - Continuous distribution for Θ: Gamma($\frac{1}{\alpha}$,1)
   - LST: $\mathcal{L}_{\Theta}(t) = \left(\frac{1}{1+t}\right)^{\frac{1}{\alpha}}$
   - Inverse of LST: $\mathcal{L}_{\Theta}^{-1}(u) = u^{-\alpha} - 1$
   - Particular cases: $C_{\alpha\to 0}(u_1, \ldots, u_n) = C^{\perp}(u_1, \ldots, u_n)$ and $C_{\alpha\to\infty}(u_1, \ldots, u_n) = C^{+}(u_1, \ldots, u_n)$
2. Gumbel family:
   - Copula: $C_{\alpha}(u_1, \ldots, u_n) = \exp\left(-((-\ln(u_1))^{\alpha} + \cdots + (-\ln(u_n))^{\alpha})^{\frac{1}{\alpha}}\right)$
   - Parameter: $\alpha \in [1, \infty)$
   - Continuous distribution for Θ: Positive Stable $\left(\frac{1}{\alpha}, 1, \cos^{\alpha}\left(\frac{\pi}{2\alpha}\right), 1_{\{\alpha=1\}}\right)$
   - LST: $\mathcal{L}_{\Theta}(t) = e^{-t^{\frac{1}{\alpha}}}$
   - Inverse of LST: $\mathcal{L}_{\Theta}^{-1}(u) = (-\ln(u))^{\alpha}$
   - Particular cases: $C_{\alpha\to 1}(u_1, \ldots, u_n) = C^{\perp}(u_1, \ldots, u_n)$ and $C_{\alpha\to\infty}(u_1, \ldots, u_n) = C^{+}(u_1, \ldots, u_n)$.

Appendix B. Nested Archimedean copula

1. **The Nested AMH family**
   - $\psi_{0s}(t) = \mathcal{L}_{\Theta_{0s}}(t) = \left(\frac{1-q_s}{(1-q_0)(e^t - q_s) + q0(1-q_s)}\right)^{\Theta_0}$
   - Distribution of Θ_{0s}: $(\Theta_{0s}|\Theta_0 = \theta) \sim SNB\left(\theta, \frac{1-q_s}{1-q_0}\right)$ with $q_0 \le q_s$.
   - pmf of Θ_{0s}: $f_{\Theta_{0s}|\Theta_0=\theta}(k) = \binom{k-1}{k-\theta}(q^*)^r(1-q^*)^{k-r}$ with $q^* = \frac{1-q_s}{1-q_0}$ and $k \in \{\theta, \theta+1, \ldots\}$.
2. **The Nested Frank family**
   - $\psi_{0s}(t) = \left(\frac{(1-e^{-\alpha_s})e^{-s}}{1-e^{-\alpha_0}}\right)^{\Theta_0}$
   - Distribution of Θ_{0s}: $(\Theta_{0s}|\Theta_0 = \theta) \sim \sum_{i=1}^{\theta} V_i$, with $P(V_i = k) = p_k$ with $\alpha_0 \le \alpha_s$
   - with $p_k = \frac{(1-e^{-\alpha_s})^k}{(1-e^{-\alpha_s})^{\theta_0}} \sum_{j=0}^{\infty} \binom{\theta_0}{j} \binom{j\frac{\alpha_0}{\alpha_s}}{k} (-1)^{j+k}$ for $k \in \{1, 2, \ldots\}$.

References

Albrecher, H., Constantinescu, C., Loisel, S., 2011. Explicit ruin formulas for models with dependence among risks. Insurance Math. Econom. 48 (2), 265–270.

Aldous, D.J., 1985. Exchangeability and related topics. In: École d'Été de Probabilités de Saint-Flour XIII-1983. Springer, pp. 1–198.

Balakrishnan, N., Lai, C.-D., 2009. Continuous Bivariate Distributions. Springer Science & Business Media.

Bargès, M., Cossette, H., Marceau, E., 2009. Tvar-based capital allocation with copulas. Insurance Math. Econom. 45 (3), 348–361.

Bühlmann, H., 1960. Austauschbare stochastische Variabeln und ihre Grenzwertsätze. (Ph.D. thesis).

Cherubini, U., Luciano, E., Vecchiato, W., 2004. Copula Methods in Finance. John Wiley & Sons.

Cossette, H., Gaillardetz, P., Marceau, E., 2002. Common mixture in the individual risk model. Mitt. Schweiz. Aktuarver. 2, 131–157.

Cossette, H., Landriault, D., Marceau, É., 2003. Ruin probabilities in the compound Markov binomial model. Scand. Actuar. J. 2003 (4), 301–323.

Cossette, H., Marceau, E., Mtalai, I., Veilleux, D., 2017. A note on the shifted negative binomial copula. submitted for publication.

Dacorogna, M.M., Elbahtouri, L., Kratz, M., 2016. Explicit diversification beneift for dependent risks.

De Finetti, B., 1957. Su unimpostazione alternativa della teoria collettiva del rischio. In: Transactions of the XVth International Congress of Actuaries, Vol. 2, pp. 433–443.

De Vylder, F., Marceau, E., 1996. Classical numerical ruin probabilities. Scand. Actuar. J. 1996 (2), 109–123.

Denuit, M., Dhaene, J., Goovaerts, M., Kaas, R., 2006. Actuarial Theory for Dependent Risks: Measures, Orders and Models. John Wiley & Sons.
Dickson, D.C., 1992. On the distribution of the surplus prior to ruin. Insurance Math. Econom. 11 (3), 191–207.
Embrechts, P., Frey, R., McNeil, A., 2005. Quantitative risk management. Princeton Series in Finance, Princeton, p. 10.
Fang, K.-T., Fang, H.-B., Rosen, D. v., 2000. A family of bivariate distributions with non-elliptical contours. Comm. Statist. Theory Methods 29 (8), 1885–1898.
Feller, W., 1971. Introduction to probability theory and its appl. II. New York.
Frey, R., McNeil, A.J., 2001. Modelling Dependent Defaults. ETH, Eidgenössische Technische Hochschule Zürich, Department of Mathematics.
Frey, R., McNeil, A.J., 2002. Var and expected shortfall in portfolios of dependent credit risks: conceptual and practical insights. J. Banking Finance 26 (7), 1317–1334.
Furman, E., Zitikis, R., 2008. Weighted risk capital allocations. Insurance Math. Econom. 43 (2), 263–269.
Genest, C., Rivest, L.-P., 2001. On the multivariate probability integral transformation. Statist. Probab. Lett. 53 (4), 391–399.
Gerber, H.U., 1988. Mathematical fun with the compound binomial process. Astin Bull. 18 (02), 161–168.
Hofert, M., 2008. Sampling archimedean copulas. Comput. Statist. Data Anal. 52 (12), 5163–5174.
Hofert, M., 2010. Sampling Nested Archimedean Copulas with Applications to CDO Pricing. Universität Ulm, (Ph.D. thesis).
Hofert, M., 2011. Efficiently sampling nested Archimedean copulas. Comput. Statist. Data Anal. 55 (1), 57–70.
Hofert, M., 2012. A stochastic representation and sampling algorithm for nested Archimedean copulas. J. Stat. Comput. Simul. 82 (9), 1239–1255.
Hofert, M., Mächler, M., Mcneil, A.J., 2012. Likelihood inference for Archimedean copulas in high dimensions under known margins. J. Multivariate Anal. 110, 133–150.
Hofert, M., Scherer, M., 2011. CDO pricing with nested Archimedean copulas. Quant. Finance 11 (5), 775–787.
Joe, H., 1997. Multivariate Models and Multivariate Dependence Concepts. CRC Press.
Joe, H., 2014. Dependence Modeling with Copulas. CRC Press.
Joe, H., Hu, T., 1996. Multivariate distributions from mixtures of max-infinitely divisible distributions. J. Multivariate Anal. 57 (2), 240–265.
Jordanova, P., Nefedova, Y., Stehlík, M., 2017. Risk process approximation with mixing. Appl. Math. Model. 41, 285–298.
Kroese, D.P., Taimre, T., Botev, Z.I., 2013. Handbook of Monte Carlo Methods, Vol. 706. John Wiley & Sons.
Lemieux, C., 2009. Monte Carlo and Quasi-Monte Carlo Sampling. In: Springer Series in Statistics, Springer, New York.
Lewin, L., 1981. Polylogarithms and associated functions.
Li, D.X., 2000. On default correlation: A copula function approach. J. Fixed Income 9 (4), 43–54.
Marceau, E., 2013. Modelisation et Evaluation Quantitative des Risques en Actuariat: Modèles sur une Période, first ed.. Springer.
Marshall, A.W., Olkin, I., 1988. Families of multivariate distributions. J. Amer. Statist. Assoc. 83 (403), 834–841.
Marshall, A.W., Olkin, I., 2007. Life Distributions: Structure of Nonparametric, Semiparametric, and Parametric Families. Springer Science & Business Media.
McNeil, A.J., 2008. Sampling nested Archimedean copulas. J. Stat. Comput. Simul. 78 (6), 567–581.
McNeil, A.J., Frey, R., Embrechts, P., 2015. Quantitative Risk Management: Concepts, Techniques and Tools. Princeton university press.
McNeil, A.J., Nešlehová, J., 2009. Multivariate archimedean copulas, d-monotone functions and -norm symmetric distributions. Ann. Statist. 3059–3097.
Müller, A., Stoyan, D., 2002. Comparison Methods for Stochastic Models and Risks, Vol. 389. Wiley.
Nelsen, R.B., 2007. An Introduction to Copulas. Springer Science & Business Media.
Panjer, H.H., Klugman, S., Willmot, G., 2008. Loss Models: From Data to Decisions. New York: John Wiley & Sons, Inc.
Rosen, D., Saunders, D., Bielecki, T.R., Brigo, D., Patras, F., 2011. Credit risk contributions. Credit Risk Frontiers: Subprime Crisis, Pricing and Hedging, CVA, MBS, Ratings, and Liquidity, pp. 675–720.
Sarabia, J.M., Gómez-Déniz, E., Prieto, F., Jordá, V., 2017. Aggregation of dependent risks in mixtures of exponential distributions and extensions. arXiv preprint arXiv:1705.00289.
Schönbucher, P., 2002. Taken to the limit: simple and not-so-simple loan loss distributions.
Schönbucher, P.J., Schubert, D., 2001. Copula-dependent default risk in intensity models. In: Working Paper, Department of Statistics, Bonn University, Citeseer.
Shiu, E.S., 1989. The probability of eventual ruin in the compound binomial model. Astin Bull. 19 (2), 179–190.
Tasche, D., 1999. Risk contributions and performance measurement. Report of the Lehrstuhl FÜR Mathematische Statistik, TU München.
Vasicek, O.A., 1987. Probability of Loss on Loan Portfolio. Wiley Online Library.
Willmot, G.E., 1993. Ruin probabilities in the compound binomial model. Insurance Math. Econom. 12 (2), 133–142.