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TVaR-based capital allocation for multivariate compound distributions with positive continuous claim amounts

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In this paper, we consider a portfolio of n dependent risks $X_1, \ldots, X_n$ and we study the stochastic behavior of the aggregate claim amount $S = X_1 + \cdots + X_n$. Our objective is to determine the amount of economic capital needed for the whole portfolio and to compute the amount of capital to be allocated to each risk $X_1, \ldots, X_n$. To do so, we use a top–down approach. For $(X_1, \ldots, X_n)$, we consider risk models based on multivariate compound distributions defined with a multivariate counting distribution. We use the TVaR to evaluate the total capital requirement of the portfolio based on the distribution of S, and we use the TVaR-based capital allocation method to quantify the contribution of each risk. To simplify the presentation, the claim amounts are assumed to be continuously distributed. For multivariate compound distributions with continuous claim amounts, we provide general formulas for the cumulative distribution function of S, for the TVaR of S and the contribution to each risk. We obtain closed-form expressions for those quantities for multivariate compound distributions with gamma and mixed Erlang claim amounts. Finally, we treat in detail the multivariate compound Poisson distribution case. Numerical examples are provided in order to examine the impact of the dependence relation on the TVaR of S, the contribution to each risk of the portfolio, and the benefit of the aggregation of several risks.

1. Introduction

In this paper, we consider a portfolio of n possibly dependent risks $X_1, \ldots, X_n$ and we study the stochastic behavior of the aggregate claim amount $S = X_1 + \cdots + X_n$. Our objective is to determine the amount of economic capital for the whole portfolio and to compute the amount of capital to be allocated to each risk $X_1, \ldots, X_n$ assuming that $(X_1, \ldots, X_n)$ is modeled by a multivariate compound distribution. To perform that task, we use a top–down approach. Under this approach, we assume a multivariate model for $(X_1, \ldots, X_n)$, we choose a risk measure to evaluate the total capital requirement for the portfolio based on the distribution of S, and we use a capital allocation method to quantify the contribution of each risk.

The usual risk measures are the Value-at-Risk (VaR) and the Tail-Value-at-Risk (TVaR). Artzner et al. (1999) proposed the TVaR, also called Expected Shortfall (ES) in quantitative risk management, as a coherent alternative to the non-coherent risk measure VaR. Applied to continuous random variables (r.v.s.), the TVaR can be defined as the conditional tail expectation (CTE). The TVaR and the CTE differ however in the context of discrete r.v.s. where the CTE is no longer coherent. The differences between these definitions and their properties have been highlighted in Acerbi et al. (2001) and Acerbi and Tasche (2002). See e.g. Acerbi and Tasche (2002) and McNeil et al. (2005) for details on the properties of the risk measures VaR and TVaR.

In this paper, we use the rule based on the TVaR to determine the amount of capital allocated to each risk. When each risk is represented by a continuous r.v., the TVaR-based allocation coincides with the CTE-based allocation rule. See e.g. Tasche (1999) and McNeil et al. (2005) for details on capital allocation rules, including the TVaR-based allocation rule.

In recent years, several authors have used a top–down approach to find closed-form expressions for the amount of capital needed for the whole portfolio based on the TVaR and the contribution to each risk using the TVaR-based allocation rule for some specific multivariate distributions for $(X_1, \ldots, X_n)$: multivariate normal distribution (Panjer, 2002), multivariate elliptical distribution (Landsman and Valdez, 2003 and Dhaene et al., 2008), multivariate gamma distribution (Furman and Landsman, 2005), multivariate Tweedie distribution (Furman and Landsman, 2008), and multivariate Pareto distribution (Chiragiev and Landsman, 2007). In those papers, the dependence between the different lines of business of the insurance company is due to the construction of a multivariate distribution for $(X_1, \ldots, X_n)$. Bargès et al. (2009) propose to introduce such a dependence relation

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with a copula. They obtain closed-form expressions for the TVaR and then the TVaR-based contribution of one risk when the Farlie–Gumbel–Morgenstern copula describes the dependence between the risk marginals. They also present approximation methods to evaluate the TVaR and the TVaR-based allocation in which they use different discretization methods of continuous random variables that are applicable with any copula and any marginals.

In this paper, we focus on the computation of the TVaR and the TVaR-based allocation for multivariate compound distributions. As stated in almost all actuarial textbooks (e.g. Rolski et al., 1999; Klugman et al., 2008), compound distributions are the corner stones of several risk models in risk theory. We consider risk models based on multivariate compound distributions defined with a multivariate counting distribution. To simplify the presentation, the claim amounts are assumed to be continuously distributed. As in e.g. Furman and Landsman (2005, 2008), Chiragiev and Landsman (2007), and Bargès et al. (2009), the capital allocation is based on a top–down approach: we first determine the global amount of capital for the whole portfolio and, using a TVaR-based allocation rule, we determine the amount of capital to be allocated to each risk. For multivariate compound distributions with continuous claim amounts, we provide general formulas for the cumulative distribution function (c.d.f.) of S, the TVaR of S and the contribution to each risk. We obtain closed-form expressions for these quantities for multivariate compound distributions with gamma and mixed Erlang claim amounts. Finally, we treat in detail the multivariate compound Poisson distribution case.

The paper is constructed as follows. In Section 2, we recall the basic definition for the Tail-Value-at-Risk and provide closed-form expressions of the TVaR for special cases (e.g. gamma distribution and mixed Erlang distribution). Then, in Section 3, we recall the TVaR-based allocation rule and we provide results for special cases (e.g. gamma distribution and mixed Erlang distribution) for which closed-form expressions for the contributions are obtained. In Section 4, assuming multivariate compound distributions (with continuous claim amounts) for $(X_1, \ldots, X_n)$, we examine the derivation of general expressions for the c.d.f. of S, the TVaR of S and the contribution of each risk using the TVaR-based allocation rule. Closed-form expressions are also obtained for multivariate compound distributions with gamma claim amounts or mixed Erlang claim amounts. We also consider in detail the multivariate compound Poisson distribution. Numerical examples are provided in order to examine the impact of the dependence relation on the TVaR of S, the contribution to each risk of the portfolio, and the benefit of the aggregation of several risks.

2. Some results related to the TVaR

2.1. Basic definitions

We consider a r.v. X with c.d.f. F_X. The Value-at-Risk at level κ, $0 \le \kappa < 1$, of X is defined by

$$\mathrm{VaR}_\kappa(X) = \inf(x \in \mathbb{R},\ F_X(x) \ge \kappa).$$

We denote the truncated expectation of X by $E\left[X \times 1_{\{X>b\}}\right]$ where 1_A is the indicator function such that $1_A(X) = 1$, if $X \in A$, and $1_A(X) = 0$, if $X \notin A$. The Tail-Value-at-Risk at level κ, $0 \le \kappa < 1$, is defined by

$$\begin{aligned}\mathrm{TVaR}_\kappa(X) &= \frac{1}{1-\kappa}\int_\kappa^1 \mathrm{VaR}_u(X)du \\ &= \frac{E\left[X \times 1_{\{X>\mathrm{VaR}_\kappa(X)\}}\right] + \mathrm{VaR}_\kappa(X)\left(F_X\left(\mathrm{VaR}_\kappa(X)\right) - \kappa\right)}{1-\kappa}, \end{aligned} \tag{1}$$

where $E\left[X \times 1_{\{X>b\}}\right]$ can be expressed as $E[X] - E\left[X \times 1_{\{X\le b\}}\right]$. See e.g. Acerbi (2002), Acerbi and Tasche (2002) and McNeil et al. (2005) for details on the risk measures VaR and TVaR.

When the r.v. X is continuous, it implies that $F_X\left(\mathrm{VaR}_\kappa(X)\right) - \kappa = 0$ and (1) becomes

$$\mathrm{TVaR}_\kappa(X) = \frac{E\left[X \times 1_{\{X>\mathrm{VaR}_\kappa(X)\}}\right]}{1-\kappa} = E\left[X | X > \mathrm{VaR}_\kappa(X)\right], \tag{2}$$

where $E\left[X|X > \mathrm{VaR}_\kappa(X)\right] = CTE_\kappa(X)$ which means that the Tail-Value-at-Risk of a continuous r.v. is equal to its conditional tail expectation (which is not the case generally). In the present work, we prefer to use the term TVaR which is always coherent rather than the term CTE.

The expression for $\mathrm{TVaR}_\kappa(X)$ when $X \sim \mathrm{Gamma}\,(\alpha, \beta)$ is given in Furman and Landsman (2005). To set the notation that will be used in the remainder of the paper, we briefly state the result in the following proposition.

Proposition 1 (*Gamma Distribution*). *Assume that* $X \sim \mathrm{Gamma}$ (α, β) *with probability density function (p.d.f.) and c.d.f.*

$$f_X(x) = h(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

$$F_X(x) = H(x; \alpha, \beta) = \int_0^x h(y; \alpha, \beta)\, dy, \tag{3}$$

with $x > 0$. *The expression for* $E\left[X \times 1_{\{X>b\}}\right]$ *is given by*

$$E\left[X \times 1_{\{X>b\}}\right] = \frac{\alpha}{\beta}\overline{H}(b; \alpha+1, \beta) = E[X]\,\overline{H}(b; \alpha+1, \beta), \tag{4}$$

where $\overline{H}(b; \alpha+1, \beta) = 1 - H(b; \alpha+1, \beta)$. *If* $b = \mathrm{VaR}_\kappa(X)$, *then* (2) *becomes*

$$\mathrm{TVaR}_\kappa(X) = \frac{E[X]\,\overline{H}\left(\mathrm{VaR}_\kappa(X); \alpha+1, \beta\right)}{1-\kappa}. \tag{5}$$

2.2. Compound distribution with continuous claims

We derive in this subsection the expression for the TVaR of a r.v. X which follows a compound distribution. This distribution is frequently used in non-life insurance (e.g. Rolski et al., 1999; Klugman et al., 2008) and can also be applied in a quantitative risk management context (see e.g. McNeil et al., 2005).

Let the r.v. X have a compound distribution with

$$X = \begin{cases} \sum_{k=1}^{M} B_k, & M > 1 \\ 0, & M = 0, \end{cases} \tag{6}$$

where $B_1, B_2, \ldots$ form a sequence of i.i.d. r.v.s. ($B_k \sim B$) and independent of M. We denote the probability mass function (p.m.f) and the probability generating function (p.g.f) of the r.v. M by $\Pr(M = k) = q_k$ for $k \in \mathbb{N}$ and $P_M(s) = E\left[s^M\right] = \sum_{k=0}^\infty q_k s^k$, respectively. We assume the r.v. B to be positive and continuous. In such a context, the c.d.f. of X is given by

$$F_X(x) = q_0 + \sum_{k=1}^\infty q_k \Pr\left(B_1 + \cdots + B_k \le x\right). \tag{7}$$

In non-life insurance, the r.v. M corresponds to the number of claims and the r.v. B_k represents the amount of the kth claim. The moment generating function (m.g.f.) of X is given by

$$M_X(r) = E\left[e^{rX}\right] = P_M\left(M_B(r)\right).$$

To determine the expression for $\mathrm{TVaR}_\kappa(X)$ in this case, we must first find

$$E\left[X \times 1_{\{X>b\}}\right] = \sum_{k=1}^\infty q_k E\left[(B_1 + \cdots + B_k) \times 1_{\{B_1+\cdots+B_k>b\}}\right].$$

Then, we obtain

$$\mathrm{TVaR}_{\kappa}(X) = \frac{1}{1-\kappa}\sum_{k=1}^{\infty} q_k E \times \left[(B_1+\cdots+B_k)\times 1_{\{B_1+\cdots+B_k>\mathrm{VaR}_{\kappa}(X)\}}\right] + \frac{1}{1-\kappa}\mathrm{VaR}_{\kappa}(X)\left(F_X\left(\mathrm{VaR}_{\kappa}(X)\right)-\kappa\right). \quad (8)$$

Note from (7) that the distribution of X is mixed with a probability mass at 0 and with a continuous part for $x > 0$. If $\kappa < q_0$, then $\mathrm{VaR}_{\kappa}(X) = 0$. Moreover, when $\kappa > q_0$, it implies that $\mathrm{VaR}_{\kappa}(X) > 0$ so that $F_X(\mathrm{VaR}_{\kappa}(X)) = \kappa$. The second term in (8) is thus equal to 0 in both cases and the expression for $\mathrm{TVaR}_{\kappa}(X)$ becomes

$$\mathrm{TVaR}_{\kappa}(X) = \frac{1}{1-\kappa}\sum_{k=1}^{\infty} q_k E \times \left[(B_1+\cdots+B_k)\times 1_{\{B_1+\cdots+B_k>\mathrm{VaR}_{\kappa}(X)\}}\right]. \quad (9)$$

The expressions in (7) and (9) are interesting when $B_1+\cdots+B_k$ belongs to a family of distributions closed under convolution as it is illustrated in the following proposition for the gamma distribution.

Proposition 2 (*Compound Distribution with Gamma Claim Amounts*). *Suppose that* $B \sim \mathrm{Gamma}(\alpha, \beta)$. *Using (3), the expression for* F_X *in this particular case is*

$$F_X(x) = q_0 + \sum_{k=1}^{\infty} q_k H(x; \alpha k, \beta). \quad (10)$$

The $\mathrm{VaR}_{\kappa}(X)$ *can be easily determined with any optimization tool (e.g.* `optimize` *in R and* `solver` *in EXCEL). Using (4), we find*

$$E\left[X \times 1_{\{X>b\}}\right] = \sum_{k=1}^{\infty} q_k \frac{k\alpha}{\beta}\overline{H}(b; \alpha k+1, \beta),$$

which, combined to (9), leads to

$$\mathrm{TVaR}_{\kappa}(X) = \frac{1}{1-\kappa}\sum_{k=1}^{\infty} q_k \frac{k\alpha}{\beta}\overline{H}\left(\mathrm{VaR}_{\kappa}(X); \alpha k+1, \beta\right). \quad (11)$$

2.3. Mixed Erlang distribution

We now consider mixtures of Erlang distributions with common scale parameter. Tijms (1994) has shown that this class of distributions can approximate any continuous positive distribution. Willmot and Woo (2007), Lee and Lin (2010), and Willmot and Lin (2011) show the usefulness of this distribution. With several examples, these authors illustrate the versatility of this distribution to model claim amounts and the feasibility to obtain closed-form expressions for various quantities of interest in risk theory. Furthermore, Willmot and Woo (2007) and Willmot and Lin (2011) provide several non trivial examples of distributions which belong to the class of mixed Erlang distributions. They provide a detailed procedure to express e.g. mixtures of exponentials, countable scale and shape mixtures of Erlang distributions, and generalized Erlang distributions in terms of mixed Erlang distributions.

Let Y be a mixed Erlang r.v. with common scale parameter. The p.d.f and c.d.f. of Y are respectively given by

$$f_Y(x) = \sum_{k=1}^{\infty} \zeta_k h(x; k, \beta),$$

$$F_Y(y) = \sum_{k=1}^{\infty} \zeta_k H(x; k, \beta),$$

where ζ_i is the non-negative weight of the ith Erlang distribution in the mixture and β is the common scale parameter. Another and quite useful representation can be given to mixed Erlang distributions. It can be interpreted as a compound distribution such that the mixed Erlang r.v. Y can be written as

$$Y = \begin{cases} \sum_{k=1}^{K} C_k, & K > 0 \\ 0, & K = 0, \end{cases} \quad (12)$$

where $C_k \sim \mathrm{Exp}(\beta)$ and K is a discrete r.v. with p.m.f. $f_K(k) = \Pr(K = k) = \zeta_k$, $k \in \mathbb{N}$, and p.g.f. $P_K(s) = \sum_{k=0}^{\infty} \zeta_k s^k$. We use the notation $Y \sim \mathrm{MixErl}\left(\underline{\zeta}, \beta\right)$ with $\underline{\zeta} = (\zeta_0, \zeta_1, \ldots)$. Note that here we allow a probability mass at 0 and, given (10), we have

$$F_Y(x) = \zeta_0 + \sum_{k=1}^{\infty} \zeta_k H(x; k, \beta),$$

where $\zeta_0 = \Pr(K = 0) \geq 0$. Also, the m.g.f. of Y is $M_Y(r) = P_K(M_C(r))$ with $M_C(r) = \frac{\beta}{\beta - r}$. This interpretation of the mixed Erlang distribution allows us to use (11) to find the expression for $\mathrm{TVaR}_{\kappa}(Y)$:

$$\mathrm{TVaR}_{\kappa}(Y) = \frac{1}{1-\kappa}\sum_{k=1}^{\infty} \zeta_k \frac{k}{\beta}\overline{H}\left(\mathrm{VaR}_{\kappa}(Y); k+1, \beta\right).$$

In the following proposition, we recall without proof a useful result of Willmot and Woo (2007) to set the notation that will be used in what follows.

Proposition 3 (*Compound Distribution with Mixed Erlang Claims*). *We consider a risk X which follows a compound distribution as in (6), where the claim amount* $B_j \sim \mathrm{MixErl}\left(\underline{\zeta}, \beta\right)$ *with* $\underline{\zeta} = (\zeta_0, \zeta_1, \ldots)$. *Then,* $X \sim \mathrm{MixErl}\left(\underline{\xi}, \beta\right)$ *with* $\underline{\xi} = (\xi_0, \xi_1, \ldots)$ *i.e. the r.v. X can be expressed as*

$$X = \begin{cases} \sum_{j=1}^{M^*} C_k, & M^* > 1 \\ 0, & M^* = 0, \end{cases} \quad (13)$$

where the discrete r.v. M^* *is defined as*

$$M^* = \begin{cases} \sum_{j=1}^{M} K_j, & M > 1 \\ 0, & M = 0 \end{cases} \quad (14)$$

and $K_1, K_2, \ldots$ *form a sequence of i.i.d. r.v.s distributed as the r.v. K. The p.m.f. of* M^* *is denoted by* $\Pr(M^* = k) = \xi_k$, $k \in \mathbb{N}$. *The r.v.s.* $C_1, C_2, \ldots$ *are independent and they are exponentially distributed with mean* $\frac{1}{\beta}$. *They are also independent of* M^*. *It means that* $M_X(r) = P_M(M_B(r)) = P_M(P_K(M_C(r))) = P_{M^*}(M_C(r))$ *where* $P_{M^*}(s) = P_M(P_K(s)) = \sum_{k=0}^{\infty} \xi_k s^k$ *is the p.g.f. of* M^*. *The values of* ξ_k $(k \in \mathbb{N})$ *can be computed with a recursive aggregation method (see e.g. Klugman et al., 2008 and Sundt and Vernic, 2009).*

3. Top-down approach and TVaR based allocation rule

We consider a portfolio of n risks. The total amount of claims (or losses) for risk i is denoted by the non-negative r.v. X_i for $i = 1, 2, \ldots, n$ over a given period (e.g. a day, a week, a month or a year). The aggregate claim amount for the whole portfolio over a given period is defined by the r.v. S where $S = X_1 + \cdots + X_n$.

Following the top–down approach, the first step is to determine the amount of capital needed for the whole portfolio. Since it is well known that the VaR is not a coherent risk measure, we choose to determine the amount of capital for the whole portfolio with the TVaR. Then, we use a TVaR-based allocation rule to find the contribution of the ith risk to the aggregate risk of the portfolio representing the part of the capital that is allocated to risk i ($i = 1, 2, \ldots, n$). For $\kappa \in (0, 1)$, this contribution is expressed as follows

$$\mathrm{TVaR}_\kappa(X_i; S) = \frac{E\left[X_i 1_{\{S>\mathrm{VaR}_\kappa(S)\}}\right] + \beta_S E\left[X_i 1_{\{S=\mathrm{VaR}_\kappa(S)\}}\right]}{1-\kappa}, \tag{15}$$

with

$$\beta_S = \begin{cases} \dfrac{\Pr(S \le \mathrm{VaR}_\kappa(S)) - \kappa}{\Pr(S = \mathrm{VaR}_\kappa(S))}, & \text{if } \Pr(S = \mathrm{VaR}_\kappa(S)) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

It can be easily shown that the sum of the n TVaR contributions is equal to the TVaR of the aggregate claim amount i.e.

$$\mathrm{TVaR}_\kappa(S) = \sum_{i=1}^{n} \mathrm{TVaR}_\kappa(X_i; S)$$

(see e.g. Tasche, 1999 and McNeil et al., 2005 for details on the TVaR-based allocation rule).

First, we consider the case where $X_1, \ldots, X_n$ are assumed to be continuous r.v.s. implying that the r.v. S is also continuous. Consequently, the expression of $\mathrm{TVaR}_\kappa(S)$ is provided in (2) and the contribution of the risk X_i given in (15) becomes

$$\mathrm{TVaR}_\kappa(X_i; S) = \frac{E\left[X_i \times 1_{\{S>\mathrm{VaR}_\kappa(S)\}}\right]}{1-\kappa}. \tag{16}$$

Let us now examine the computation of $E\left[X_i \times 1_{\{S>s_0\}}\right]$ and let

$$S_{-i} = \sum_{l=1, l\neq i}^{n} X_l.$$

For $i = 1$, we have $S_{-1} = X_2 + \cdots + X_n$ and hence,

$$E\left[X_1 \times 1_{\{S>s_0\}}\right] = \int_{s_0}^{\infty} E\left[X_1 \times 1_{\{S=s\}}\right] ds$$

where

$$E\left[X_1 \times 1_{\{S=s\}}\right] = \int_0^s x f_{X_1}(x) f_{S_{-1}}(s-x)\, dx.$$

In the two following propositions, we obtain the expressions for the contribution under the TVaR-based allocation rule for two special cases.

Proposition 4 (*Gamma Distribution*)**.** *Let $X_1, \ldots, X_n$ be independent r.v.s. with $X_i \sim \mathrm{Gamma}(\alpha_i, \beta)$. Then, $S = X_1 + \cdots + X_n \sim \mathrm{Gamma}(\alpha_{\mathrm{TOT}}, \beta)$ with $\alpha_{\mathrm{TOT}} = \alpha_1 + \cdots + \alpha_n$ and the contribution of the ith risk is*

$$\mathrm{TVaR}_\kappa(X_i; S) = \frac{\alpha_i}{\beta} \frac{\overline{H}(\mathrm{VaR}_\kappa(S); \alpha_{\mathrm{TOT}}+1, \beta)}{1-\kappa}. \tag{17}$$

Proof. To derive the contribution of X_1, we first need to find $E\left[X_1 \times 1_{\{S=s\}}\right]$ where

$$\begin{aligned} E\left[X_1 \times 1_{\{S=s\}}\right] &= \int_0^s x f_{X_1}(x) f_{S_{-1}}(s-x)\, dx \\ &= \int_0^s x h(x; \alpha_1, \beta)\, h(s-x; \alpha_{\mathrm{TOT}} - \alpha_1, \beta)\, dx \\ &= \frac{\alpha_1}{\beta} h(s; \alpha_{\mathrm{TOT}}+1, \beta). \end{aligned}$$

Then, we obtain

$$\begin{aligned} E\left[X_1 \times 1_{\{S>b\}}\right] &= \int_b^{\infty} E\left[X_1 \times 1_{\{S=s\}}\right] ds \\ &= \frac{\alpha_1}{\beta} \overline{H}(b; \alpha_{\mathrm{TOT}}+1, \beta). \end{aligned} \tag{18}$$

Replacing (18) in (16), we obtain the desired result in (17). One can also verify that, as expected, the sum of the contributions equals the TVaR for the whole portfolio. This result is also given in Furman and Landsman (2005). □

Proposition 5 (*Mixed Erlang Distribution*)**.** *We consider a portfolio of n independent risks $X_1, \ldots, X_n$ where $X_i \sim \mathrm{MixErl}\left(\underline{\zeta}^{(i)}, \beta_i\right)$ with $\underline{\zeta}^{(i)} = \left(\zeta_0^{(i)}, \zeta_1^{(i)}, \ldots\right)$, $\zeta_k^{(i)} = \Pr(K_i = k)$, and $\beta_i = \beta$ for $i = 1, 2, \ldots, n$. The aggregate claim amount is defined by $S = \sum_{l=1}^{n} X_l$. Then, $S \sim \mathrm{MixErl}\left(\underline{\nu}, \beta\right)$ with $\underline{\nu} = (\nu_0, \nu_1, \ldots)$ i.e. the r.v. S also follows a mixed Erlang distribution with $\nu_k = \Pr\left(\sum_{l=1}^{n} K_l = k\right)$ for $k \in \mathbb{N}$. It implies that*

$$F_S(x) = \nu_0 + \sum_{k=1}^{\infty} \nu_k H(x; k, \beta),$$

and

$$\mathrm{TVaR}_\kappa(S) = \frac{1}{1-\kappa} \sum_{k=1}^{\infty} \nu_k \frac{k}{\beta} \overline{H}(\mathrm{VaR}_\kappa(S); k+1, \beta).$$

The allocation for the risk X_i is

$$\mathrm{TVaR}_\kappa(X_i; S) = \frac{1}{1-\kappa} \sum_{k=1}^{\infty} \sum_{j=1}^{k} \zeta_j^{(i)} \nu_{k-j}^{(-i)} \frac{j}{\beta} \overline{H}(\mathrm{VaR}_\kappa(S); k+1, \beta),$$

where $\nu_k^{(-i)} = \Pr\left(\sum_{l=1, l\neq i}^{n} K_l = k\right)$ for $k \in \mathbb{N}$.

Proof. To prove that $S = \sum_{i=1}^{n} X_i \sim \mathrm{MixErl}\left(\underline{\nu}, \beta\right)$, note that

$$\begin{aligned} M_S(r) &= E\left[e^{rS}\right] = \prod_{l=1}^{n} E\left[e^{rX_l}\right] \\ &= \prod_{i=1}^{n} P_{K_l}(M_C(r)) = P_{\sum_{l=1}^{n} K_l}(M_C(r)), \end{aligned}$$

where $P_{\sum_{l=1}^{n} K_l}(s) = \sum_{k=0}^{\infty} \nu_k s^k$ is the p.g.f. of $\sum_{l=1}^{n} K_l$, $\nu_k = \Pr\left(\sum_{l=1}^{n} K_l = k\right)$, and $M_C(r) = \frac{\beta}{\beta - r}$. The values of the components of $\underline{\nu} = (\nu_0, \nu_1, \ldots)$ are obtained with recursive aggregation methods (see e.g. Klugman et al., 2008). Similarly, for $S_{-i} = \sum_{l=1, l\neq i}^{n} X_l \sim \mathrm{MixErl}\left(\underline{\nu}^{(-i)}, \beta\right)$, we have

$$\begin{aligned} M_{S_{-i}}(r) &= E\left[e^{rS_{-i}}\right] = \prod_{l=1, l\neq i}^{n} E\left[e^{rX_l}\right] \\ &= \prod_{l=1, l\neq i}^{n} P_{K_l}(M_C(r)) = P_{\sum_{l=1, l\neq i}^{n} K_l}(M_C(r)), \end{aligned}$$

where $P_{\sum_{l=1, l\neq i}^{n} K_l}(s) = \sum_{k=0}^{\infty} \nu_k^{(-i)} s^k$ is the p.g.f. of $\sum_{l=1, l\neq i}^{n} K_l$ and $\nu_k^{(-i)} = \Pr\left(\sum_{l=1, l\neq i}^{n} K_l = k\right)$ $(k \in \mathbb{N})$ is the kth component of $\underline{\nu}^{(-i)} = \left(\nu_0^{(-i)}, \nu_1^{(-i)}, \ldots\right)$. □

4. Multivariate compound distribution with continuous claim amounts

In the present section, we derive expressions for the c.d.f. of S, the TVaR of S and the contribution of each risk under the TVaR-based allocation rule for a portfolio of n dependent risks with a multivariate compound distribution. We first present general results. Then, we derive analytical results for two choices of claim amount distributions: gamma distribution and mixed Erlang distribution. We also develop analytical results for the multivariate compound Poisson distribution, in general and in the case of mixed Erlang distributions for the claim amounts.

4.1. General results

Let $(X_1, \ldots, X_n)$ be a vector of r.v.s following a multivariate compound distribution where

$$X_i = \begin{cases} \sum_{j_i=0}^{M_i} B_{i,j_i}, & M_i > 0 \\ 0, & M_i = 0, \end{cases} \tag{19}$$

where the joint p.m.f. of $(M_1, \ldots, M_n)$ is given by

$$f_{M_1,\ldots,M_n}(m_1, \ldots, m_n) = \Pr(M_1 = m_1, \ldots, M_n = m_n) = q_{m_1,\ldots,m_n},$$

for $m_1, \ldots, m_n \in N$. Also, for each i, $(B_{i,1}, B_{i,2}, \ldots)$ form a sequence of i.i.d. r.v.s. Also, $(B_{1,1}, B_{1,2}, \ldots), (B_{2,1}, B_{2,2}, \ldots), \ldots, (B_{n,1}, B_{n,2}, \ldots)$ are independent between themselves and independent of $(M_1, \ldots, M_n)$. The r.v.s $B_1, B_2, \ldots, B_n$ are assumed to be continuous and strictly positive.

In this section, we provide the general expression for F_S, $\mathrm{TVaR}_\kappa(S)$ and the contribution under the TVaR-based allocation rule.

Conditioning on the different values of $(M_1, \ldots, M_n)$, the general expression for the c.d.f. of S is given by

$$F_S(x) = q_{0,\ldots,0} + \sum_{\substack{m_1=0 \\ \backslash\{m_1=0,m_2=0,\ldots,m_n=0\}}}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} q_{m_1,\ldots,m_n} \times \Pr\left(\sum_{i_1=1}^{m_1} B_{1,i_1} + \sum_{i_2=1}^{m_2} B_{2,i_2} + \cdots + \sum_{i_n=1}^{m_n} B_{n,i_n} \le x\right), \tag{20}$$

where $\sum_{j=1}^{0} A_j = 0$ by convention.

The general expression for the truncated expectation of S is

$$E\left[S \times 1_{\{S>b\}}\right] = \sum_{\substack{m_1=0 \\ \backslash\{m_1=0,m_2=0,\ldots,m_n=0\}}}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} q_{m_1,\ldots,m_n} \times E\left[\left(\sum_{i_1=1}^{m_1} B_{1,i_1} + \sum_{i_2=1}^{m_2} B_{2,i_2} + \cdots + \sum_{i_n=1}^{m_n} B_{n,i_n}\right) \times 1_{\left\{\left(\sum_{i_1=1}^{m_1} B_{1,i_1} + \sum_{i_2=1}^{m_2} B_{2,i_2} + \cdots + \sum_{i_n=1}^{m_n} B_{n,i_n}\right) > b\right\}}\right]. \tag{21}$$

Using (21), it follows that the general expression for $\mathrm{TVaR}_\kappa(S)$ is

$$\mathrm{TVaR}_\kappa(S) = \frac{1}{1-\kappa} \sum_{\substack{m_1=0 \\ \backslash\{m_1=0,m_2=0,\ldots,m_n=0\}}}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} q_{m_1,\ldots,m_n} \times E\left[\left(\sum_{i_1=1}^{m_1} B_{1,i_1} + \sum_{i_2=1}^{m_2} B_{2,i_2} + \cdots + \sum_{i_n=1}^{m_n} B_{n,i_n}\right) \times 1_{\left\{\left(\sum_{i_1=1}^{m_1} B_{1,i_1} + \sum_{i_2=1}^{m_2} B_{2,i_2} + \cdots + \sum_{i_n=1}^{m_n} B_{n,i_n}\right) > \mathrm{VaR}_\kappa(S)\right\}}\right] \tag{22}$$

and, according to the TVaR allocation rule, the contribution from the risk X_i is

$$\mathrm{TVaR}_\kappa(X_i; S) = \frac{1}{1-\kappa} \sum_{\substack{m_1=0 \\ \backslash\{m_1=0,m_2=0,\ldots,m_n=0\}}}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} q_{m_1,\ldots,m_n} E\left[\left(\sum_{i_i=1}^{m_i} B_{i,i_i}\right) \times 1_{\left\{\left(\sum_{i_1=1}^{m_1} B_{1,i_1} + \sum_{i_2=1}^{m_2} B_{2,i_2} + \cdots + \sum_{i_n=1}^{m_n} B_{n,i_n}\right) > \mathrm{VaR}_\kappa(S)\right\}}\right]. \tag{23}$$

Expressions (20), (22) and (23) are interesting when the claim amounts r.v.s. belong to a family of distributions closed under convolution, such as the gamma distribution and the mixed Erlang distribution.

Various dependence structures for $(M_1, \ldots, M_n)$ can be considered. See e.g. Kocherlakota and Kocherlakota (1992) and Johnson et al. (1997) for bivariate extensions of common discrete distributions such as the Poisson, binomial, and negative binomial distributions. The special case of the multivariate Poisson distribution is examined in detail in Section 4.4. Bivariate discrete distributions for (M_1, M_2) can also be constructed using copulas as illustrated in Example 7.

4.2. Multivariate compound distribution with gamma claim amounts

In the following proposition, we provide analytic expressions for (20), (22) and (23) when the claim amounts from each risk follow gamma distributions with common scale parameter.

Proposition 6. *Assume in (19) that the claim amount $B_i \sim$ Gamma (α_i, β) for $i = 1, 2, \ldots, n$. Expressions (20), (22) and (23)*

$$F_S(x) = q_{0,\ldots,0} + \sum_{\substack{m_1=0 \\ \backslash\{m_1=0,m_2=0,\ldots,m_n=0\}}}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} q_{m_1,\ldots,m_n} \times H\left(x; \sum_{i=1}^{n} m_i\alpha_i; \beta\right), \tag{24}$$

$$\mathrm{TVaR}_\kappa(S) = \frac{1}{1-\kappa} \sum_{\substack{m_1=0 \\ \backslash\{m_1=0,m_2=0,\ldots,m_n=0\}}}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} q_{m_1,\ldots,m_n} \times \frac{\sum_{i=1}^{n} m_i\alpha_i}{\beta} \overline{H}\left(\mathrm{VaR}_\kappa(S); \sum_{i=1}^{n} m_i\alpha_i + 1; \beta\right) \tag{25}$$

and

$$\mathrm{TVaR}_\kappa(X_i; S) = \frac{1}{1-\kappa} \sum_{m_1=1}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} q_{m_1,\ldots,m_n} \times \frac{m_i\alpha_i}{\beta} \overline{H}\left(\mathrm{VaR}_\kappa(S); \sum_{i=1}^{n} m_i\alpha_i + 1; \beta\right). \tag{26}$$

The closed-form expressions obtained in Proposition 6 are numerically illustrated in the next example.

Example 7. We consider a bivariate compound distribution for (X_1, X_2) where the bivariate distribution of (M_1, M_2) is based on a Frank copula

$$C_\alpha(u_1, u_2) = \frac{-1}{\alpha} \ln\left(1 + \frac{\left(e^{-au_1} - 1\right)\left(e^{-au_2} - 1\right)}{\left(e^{-a} - 1\right)}\right),$$

(the dependence parameter being α) and with $M_1 \sim$ Pois $(\lambda = 4)$ and $M_2 \sim$ NBinom$\left(r = 4, q = \frac{1}{2}\right)$. The bivariate cumulative distribution function F_{M_1,M_2} of (M_1, M_2) with marginals F_{M_1} and F_{M_2} is defined as

Table 1
Values of $\mathrm{Cov}(M_1, M_2)$, $\mathrm{corr}(M_1, M_2)$, $\mathrm{Cov}(X_1, X_2)$ and $\mathrm{Var}(S)$.

α	$\mathrm{Cov}(M_1, M_2)$	$\mathrm{corr}(M_1, M_2)$	$\mathrm{Cov}(X_1, X_2)$	$\mathrm{Var}(S)$
−20	−4.8330	−0.8544	−60.4125	329.1742
0	0	0	0	450
20	5.0626	0.8949	63.2825	576.5658

$$F_{M_1,M_2}(m_1, m_2) = C\left(F_{M_1}(m_1), F_{M_2}(m_2)\right) \quad (27)$$

for $(m_1, m_2) \in N^+ \times N^+$. The joint p.m.f. of (M_1, M_2) is given by

$$q_{m_1,m_2} = F_{M_1,M_2}(m_1, m_2) - F_{M_1,M_2}(m_1 - 1, m_2) - F_{M_1,M_2}(m_1, m_2 - 1) + F_{M_1,M_2}(m_1 - 1, m_2 - 1), \quad (28)$$

for $(m_1, m_2) \in N^+ \times N^+$ and where $F_{M_1,M_2}(m_1, m_2) = 0$ if $m_1 = 0$ or $m_2 = 0$. As mentioned in Nelsen (2006), $F_{M_1,M_2}(m_1, m_2) = C\left(F_{M_1}(m_1), F_{M_2}(m_2)\right)$ is defined on the support of (M_1, M_2). A given discrete bivariate distribution does not lead to a unique copula. Copulas can be used to construct discrete bivariate distributions as suggested e.g. in Joe (1997) and as illustrated e.g. in Denuit et al. (2005), Cossette et al. (2002), Genest et al. (2003), Trivedi and Zimmer (2005) and Marceau (2009). This type of structure allows the coupling of various marginals. Genest and Nešlehovà (2007) provide an excellent review on copulas linking discrete distributions. In the conclusion of their paper, they mention that dependence modeling with copulas as in (27) is a valid (and even attractive) approach for constructing bivariate distributions. Many stochastic dependence properties of a copula are inherited by the bivariate model obtained in (27). For example, stochastic ordering relations are preserved. Care should be taken when one considers the parameter estimation of a copula in a discrete setting. Since the estimation of these parameters is not discussed in the present paper, we refer the interested readers to Genest and Nešlehovà (2007) for further details.

We assume that the claim amount r.v.s. $B_1 \sim \mathrm{Gamma}(\alpha = 0.5, \beta = 0.1)$ and $B_2 \sim \mathrm{Gamma}(\alpha = 0.25, \beta = 0.1)$. It implies that $E[X_1] = 20$, $E[X_2] = 10$ and $E[S] = 30$. Also, $\mathrm{Var}(X_1) = 300$ and $\mathrm{Var}(X_2) = 150$. We consider the dependence parameter $\alpha = -20, 0, 20$. In Table 1, we provide the values of $\mathrm{Cov}(M_1, M_2)$, $\mathrm{corr}(M_1, M_2)$, $\mathrm{Cov}(X_1, X_2)$ and $\mathrm{Var}(S)$ for $\alpha = -20, 0, 20$.

The values of $\mathrm{VaR}_\kappa(X_i)$ and $\mathrm{TVaR}_\kappa(X_i)$ $(i = 1, 2)$ are provided in Table 2 for $\kappa = 0.25, 0.5, 0.95, 0.99$ and 0.995.

In Table 3, the values of $\mathrm{VaR}_\kappa(S)$, $\mathrm{TVaR}_\kappa(S)$, and the contributions $\mathrm{TVaR}_\kappa(X_i; S)$ $(i = 1, 2)$ are provided for $\alpha = -20$, 0 and 20.

As expected, the values of $\mathrm{TVaR}_\kappa(S)$ increase as the value of the dependence parameter α increases. This behavior confirms a result obtained in Denuit et al. (2002). Also, the results obtained provide an illustration of the incoherence of the Value-at-Risk since $\mathrm{VaR}_\kappa(X_1) + \mathrm{VaR}_\kappa(X_2) \leq \mathrm{VaR}_\kappa(S)$ for $\kappa = 0.25, 0.5$ and for $\alpha = -20, 0, 20$. For each $i = 1, 2$, for all values of κ and for all values α, the contributions $\mathrm{TVaR}_\kappa(X_i; S)$ obtained under the TVaR-based rule are lower than the values of $\mathrm{TVaR}_\kappa(X_i)$ $(i = 1, 2)$. We observe that, for $\kappa = 0.995$, the benefits resulting from the aggregation of risks are $175.0749 - 109.0645 = 66.0104$, $175.0749 - 121.5988 = 53.4761$, and $175.0749 - 133.5039 = 41.5710$. □

Table 3
Values of $\mathrm{VaR}_\kappa(S)$, $\mathrm{TVaR}_\kappa(S)$, and $\mathrm{TVaR}_\kappa(X_i; S)$ $(i = 1, 2)$.

α	κ	$\mathrm{VaR}_\kappa(S)$	$\mathrm{TVaR}_\kappa(S)$	$\mathrm{TVaR}_\kappa(X_1; S)$	$\mathrm{TVaR}_\kappa(X_2; S)$
−20	0.25	16.6480	36.3532	24.4554	11.8978
−20	0.50	26.4895	43.7817	29.8411	13.9406
−20	0.95	64.6230	78.4609	56.4538	22.0071
−20	0.99	86.9787	100.0683	73.9654	26.1029
−20	0.995	96.1888	109.0645	81.4277	27.6368
0	0.25	14.1330	37.3878	24.9954	12.4341
0	0.50	25.7459	46.1640	30.8236	15.3404
0	0.95	70.6887	86.6448	57.5097	29.1350
0	0.99	96.4965	111.3889	73.5011	37.8878
0	0.995	107.0218	121.5988	80.0301	41.5687
20	0.25	11.6701	38.2026	25.3437	12.8590
20	0.50	24.7506	48.2819	31.6894	16.5925
20	0.95	76.5806	94.5817	59.6693	34.9124
20	0.99	105.7425	122.2356	75.9766	46.2590
20	0.995	117.4702	133.5039	82.5761	50.9278

4.3. Multivariate compound distribution with mixed Erlang claim amounts

Closed-form expressions for (20), (22) and (23) are given in the following proposition when the claim amount distributions are mixed Erlang.

Proposition 8. *In (19), we assume that $B_i \sim \mathrm{MixErl}\left(\underline{\zeta}^{(i)}, \beta\right)$ with $\underline{\zeta}^{(i)} = \left(\zeta_0^{(i)}, \zeta_1^{(i)}, \ldots\right)$, $\zeta_k^{(i)} = \Pr(K_i = k)$ $(k \in \mathbb{N})$, and with $M_{B_i}(t) = P_{K_i}(M_C(r))$ where $P_{K_i}(s) = \sum_{k=0}^{\infty} \zeta_k^{(i)} s^k$. We identify the expression for F_S from the expression of the m.g.f. of S given by*

$$M_S(t) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} q_{m_1,\ldots,m_n} \times M_{B_1}(t)^{m_1} M_{B_2}(t)^{m_2} \cdots M_{B_n}(t)^{m_n}.$$

As in (14), we define the r.v.s $M_1^*, M_2^*, \ldots, M_n^*$ by

$$M_i^* = \begin{cases} \sum_{j_i=1}^{M_i} K_{i,j_i}, & M_i > 0 \\ 0, & M_i = 0, \end{cases}$$

where $K_{i,j_i} \sim K_i$ for $i = 1, 2, \ldots, n$. Then, we can express (X_1, X_2) as

$$X_i = \begin{cases} \sum_{j_i=1}^{M_i^*} C_{i,j_i}, & M_i^* > 0 \\ 0, & M_i^* = 0, \end{cases}$$

where the joint p.m.f. of $\left(M_1^*, M_2^*, \ldots, M_n^*\right)$ is denoted by

$$f_{M_1^*,\ldots,M_n^*}(j_1, \ldots, j_n) = \Pr\left(M_1^* = j_1, \ldots, M_n^* = j_n\right) = \xi_{j_1,\ldots,j_n},$$

for $j_1, j_2, \ldots, j_n \in N$. The r.v.s $C_{i,j_i} \sim \mathrm{Exp}(\beta)$ for all i and j_i. The values of $\xi_{j_1,\ldots,j_n}$ are computed with the following relation:

Table 2
Values of $\mathrm{VaR}_\kappa(X_i)$ and $\mathrm{TVaR}_\kappa(X_i)$ $(i = 1, 2)$.

κ	$\mathrm{VaR}_\kappa(X_1)$	$\mathrm{VaR}_\kappa(X_2)$	$\mathrm{TVaR}_\kappa(X_1)$	$\mathrm{TVaR}_\kappa(X_2)$	$\sum_{i=1}^{2} \mathrm{VaR}_\kappa(X_i)$	$\sum_{i=1}^{2} \mathrm{TVaR}_\kappa(X_i)$
0.25	6.9217	1.0908	25.66058	13.24903	8.0125	38.9096
0.50	15.7603	5.6480	32.8983	16.6516	21.4083	49.5499
0.95	53.9412	34.8943	68.1707	47.2722	88.8355	115.4429
0.99	76.9342	54.8506	90.4176	66.9989	131.7848	157.4165
0.995	86.4245	63.3218	99.6833	75.3916	149.7463	175.0749

$$\xi_{j_1,\ldots,j_n} = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} q_{m_1,\ldots,m_n} \prod_{i=1}^{n} \zeta_{j_i}^{(1)*m_i}, \tag{29}$$

for $j_1, \ldots, j_n \in N$, *where* $\zeta_j^{(i)*m_i} = \Pr\left(K_{1,1} + \cdots + K_{1,m_i} = j\right)$ $(j \in \mathbb{N})$, $\zeta_0^{(i)*0} = 1$, *and* $\zeta_k^{(i)*0} = 0$, *for* $k \neq 0$.

The values of $\zeta_j^{(i)*m_i} = \Pr\left(K_{1,1} + \cdots + K_{1,m_i} = j\right)$ $(j \in \mathbb{N})$ *in* (29) *are obtained with the usual aggregate recursive algorithms (see e.g.* Rolski et al., 1999 *and* Klugman et al., 2008*). Now, we can find the expression for* $F_S(x)$ *which is given by*

$$F_S(x) = \xi_{0,\ldots,0} + \sum_{\substack{j_1=0 \\ \backslash\{j_1=0,j_2=0,\ldots,j_n=0\}}}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \xi_{j_1,\ldots,j_n} H\left(x; \sum_{i=1}^{n} j_i; \beta\right).$$

Then, we obtain

$$E\left[S \times 1_{\{S>b\}}\right] = \sum_{\substack{j_1=0 \\ \backslash\{j_1=0,j_2=0,\ldots,j_n=0\}}}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \xi_{j_1,\ldots,j_n} \times \frac{\sum_{i=1}^{n} j_i}{\beta} \overline{H}\left(b; \sum_{i=1}^{n} j_i + 1; \beta\right).$$

Also, we have

$$E\left[X_i \times 1_{\{S>b\}}\right] = \sum_{\substack{j_1=0 \\ \backslash\{j_1=0,j_2=0,\ldots,j_n=0\}}}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \xi_{j_1,\ldots,j_n} \frac{j_1}{\beta} \overline{H}\left(b; \sum_{i=1}^{n} j_i + 1; \beta\right).$$

Consequently, the expression for $\mathrm{TVaR}_\kappa(S)$ *and the contribution of the risk* X_1 *are respectively given by*

$$\mathrm{TVaR}_\kappa(S) = \frac{1}{1-\kappa} \sum_{\substack{j_1=0 \\ \backslash\{j_1=0,j_2=0,\ldots,j_n=0\}}}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \xi_{j_1,\ldots,j_n} \times \frac{\sum_{i=1}^{n} j_i}{\beta} \overline{H}\left(\mathrm{VaR}_\kappa(S); \sum_{i=1}^{n} j_i + 1; \beta\right),$$

and

$$\mathrm{TVaR}_\kappa(X_i; S) = \frac{1}{1-\kappa} \sum_{j_1=1}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \xi_{j_1,\ldots,j_n} \times \frac{j_1}{\beta} \overline{H}\left(\mathrm{VaR}_\kappa(S); \sum_{i=1}^{n} j_i + 1; \beta\right). \tag{30}$$

4.4. Multivariate compound Poisson distribution

We assume that $(M_1, \ldots, M_n)$ follows a multivariate Poisson distribution which is defined with a common shock as explained in e.g. Johnson et al. (1997) and Lindskog and McNeil (2003). We briefly recall the definition of this distribution. Let $J_0, J_1, \ldots, J_n$ be independent r.v.s. with $J_0 \sim \mathrm{Pois}(\alpha_0)$ and $J_i \sim \mathrm{Pois}(\alpha_i = \lambda_i - \alpha_0)$ with $0 \leq \alpha_0 \leq \min(\lambda_1, \ldots, \lambda_n)$. The components of the random vector $(M_1, \ldots, M_n)$ are defined by

$$M_1 = J_1 + J_0,$$
$$\ldots$$
$$M_n = J_n + J_0.$$

It implies that $M_i \sim \mathrm{Pois}(\lambda_i)$. Also, the joint p.m.f. and the joint p.g.f. of $(M_1, \ldots, M_n)$ are respectively given by

$$q_{m_1,\ldots,m_n} = \sum_{j=0}^{\min(m_1,\ldots,m_n)} e^{-\alpha_0} \frac{\alpha_0^j}{j!} \prod_{i=1}^{n} e^{-(\lambda_i-\alpha_0)} \frac{(\lambda_i - \alpha_0)^{(m_i-j)}}{(m_i - j)!},$$

$$P_{M_1,\ldots,M_n}(s_1, \ldots, s_n) = E\left[s_1^{M_1} \ldots s_n^{M_n}\right] = e^{\alpha_0(s_1 \ldots s_n - 1)} \prod_{i=1}^{n} e^{(\lambda_i-\alpha_0)(s_i-1)}, \tag{31}$$

where α_0 corresponds to the dependence parameter. When $\alpha_0 = 0$, it means that the components of $(M_1, \ldots, M_n)$ are independent.

Since $(M_1, \ldots, M_n)$ follows a multivariate Poisson distribution, it implies that $(X_1, \ldots, X_n)$ has a multivariate compound Poisson distribution.

In the following proposition, the distribution of $S = \sum_{l=1}^{n} X_l$ is identified.

Proposition 9. *Given the definition of* $\left(M_1, \ldots, M_n\right)$ *and* $\left(X_1, \ldots, X_n\right)$, *the r.v.* $S = \sum_{l=1}^{n} X_l$ *follows a compound Poisson distribution with parameters* $\lambda_S = \lambda_1 + \cdots + \lambda_n - (n-1)\alpha_0$ *and*

$$F_D(x) = \frac{\lambda_1 - \alpha_0}{\lambda_S} F_{B_1}(x) + \cdots + \frac{\lambda_n - \alpha_0}{\lambda_S} F_{B_n}(x) + \frac{\alpha_0}{\lambda_S} F_{B_1+\cdots+B_n}(x), \tag{32}$$

i.e. we have

$$S = \begin{cases} \sum_{j=1}^{N} D_j, & N > 1 \\ 0, & N = 0, \end{cases}$$

where $N \sim \mathrm{Pois}(\lambda_S = \lambda_1 + \cdots + \lambda_n - (n-1)\alpha_0)$ *and the common c.d.f. of* D_j *is* F_D.

Proof. Note that

$$\begin{aligned} M_S(r) &= E\left[e^{rS}\right] = E\left[e^{r\sum_{l=1}^{n} X_l}\right] \\ &= M_{X_1,\ldots,X_n}(r, \ldots, r) \\ &= P_{M_1,\ldots,M_n}\left(M_{B_1}(r), \ldots, M_{B_n}(r)\right). \end{aligned} \tag{33}$$

Replacing (31) in (33), we obtain the following expression for the m.g.f. of S

$$\begin{aligned} M_S(r) &= e^{\alpha_0\left(M_{B_1}(r) \ldots M_{B_n}(r) - 1\right)} \prod_{l=1}^{n} e^{(\lambda_l - \alpha_0)\left(M_{B_l}(r)-1\right)} \\ &= e^{\lambda_S(M_D(r)-1)}, \end{aligned}$$

where $\lambda_S = \lambda_1 + \cdots + \lambda_n - (n-1)\alpha_0$ and

$$M_D(r) = \frac{\alpha_0}{\lambda_S} \prod_{l=1}^{n} M_{B_l}(r) + \sum_{l=1}^{n} \frac{(\lambda_l - \alpha_0)}{\lambda_S} M_{B_l}(r). \tag{34}$$

The expression for F_D follows from (34). □

From Proposition 9, the expressions for $F_S(x)$ and $\mathrm{TVaR}_\kappa(S)$ are

$$F_S(x) = \Pr(N = 0) + \sum_{k=1}^{\infty} \Pr(N = k) \Pr(D_1 + \cdots + D_k \leq x), \tag{35}$$

and

$$\mathrm{TVaR}_\kappa(S) = \frac{\sum_{k=1}^{\infty} \Pr(N = k) E\left[(D_1 + \cdots + D_k) \times 1_{\{D_1+\cdots+D_k>\mathrm{VaR}_\kappa(S)\}}\right]}{1-\kappa}, \tag{36}$$

which are more convenient to use for computation than (8) and (7).

To determine the contribution $\mathrm{TVaR}_\kappa(X_i; S)$ from each risk, the application of (23) can be very time-consuming when m becomes large. Fortunately, it is possible to take advantage of the above definition of the distribution of $(M_1, \dots, M_n)$ to determine the expression for the contribution $\mathrm{TVaR}_\kappa(X_i; S)$. We need to find the joint distribution of (X_i, S_{-i}) whose components are defined by

$$X_i = \begin{cases} \sum_{k_i=0}^{M_i} B_{i,j_i}, & M_i > 0 \\ 0, & M_i = 0, \end{cases} \tag{37}$$

and $S_{-i} = \sum_{l=1, l\neq i}^{n} X_l$. The m.g.f. $M_{S-i}(r)$ of S_{-i} is given by

$$\begin{aligned} M_{S_i}(r) &= e^{\alpha_0\left(\prod_{l=1,l\neq i}^{n} M_{B_l}(r)-1\right)} \prod_{l=1,l\neq i}^{n} e^{(\lambda_l-\alpha_0)\left(M_{B_l}(r)-1\right)} \\ &= e^{\alpha_0\left(M_{C_{-i}}(r)-1\right)} e^{\lambda_{-i}\left(M_{D_{-i}}(r)-1\right)}, \end{aligned}$$

where $M_{C_{-i}}(r) = \prod_{l=1,l\neq i}^{n} M_{B_l}(r)$, $M_{D_{-i}}(r) = \sum_{l=1,l\neq i}^{n} \frac{\lambda_l-\alpha_0}{\lambda_{-i}} M_{B_l}(r)$ et $\lambda_{-i} = \sum_{l=1,l\neq i}^{n} \lambda_l - \alpha_0$. It implies that the c.d.f. of C_{-i} and D_{-i} are

$$F_{C_{-i}}(x) = F_{\sum_{l=1,l\neq i}^{n} B_l}(x) \tag{38}$$

and

$$F_{D_{-i}}(x) = \sum_{l=1,l\neq i}^{n} \frac{\lambda_l - \alpha_0}{\sum_{l=1,l\neq i}^{n} (\lambda_l - \alpha_0)} F_{B_l}(x). \tag{39}$$

Let $J_{-i} = \sum_{l=1,l\neq i}^{n} J_l \sim \mathrm{Pois}(\lambda_{-i})$, where J_{-i} is interpreted as the sum of the individual shocks associated to all risks except the ith one. It means that the v.a. S_{-i} can be expressed as the sum of two terms, the first one being associated to the claims related to the individual shocks and the second one being associated to the claims associated to the common shock. Consequently, we have

$$S_{-i} = \begin{cases} \sum_{j=1}^{J_0} C_{-i,j}, & J_0 > 0 \\ 0, & J_0 = 0 \end{cases} + \begin{cases} \sum_{j_{-i}=1}^{J_{-i}} D_{-i,j_{-i}}, & J_{-i} > 0 \\ 0, & J_{-i} = 0, \end{cases} \tag{40}$$

where $D_{-i,1} \sim \cdots \sim D_{-i,n_{-i}-j} \sim D_{-i}$ and $C_{-i,1} \sim \cdots \sim C_{-i,j} \sim C_{-i}$.

Then, we define the pair (M_i, N_{-i}) whose components are defined by

$$M_i = J_i + J_0,$$
$$N_{-i} = J_{-i} + J_0,$$

where J_{-i}, J_i and J_0 are independent. The joint p.m.f. of (M_i, N_{-i}) is given by

$$f_{M_i,N_{-i}}(k_i, n_{-i}) = \sum_{j=0}^{\min(k_i, n_{-i})} f_{J_i}(k_i - j) f_{J_{-i}}(n_i - j) f_{J_0}(j), \tag{41}$$

for $k_i, n_{-i} \in \mathbb{N}$.

Finally, we have all the elements to derive the expression for the contribution $\mathrm{TVaR}_\kappa(X_i; S)$.

Proposition 10. *The expression for the contribution* $\mathrm{TVaR}_\kappa(X_i; S)$ $(i = 1, 2, \dots, n)$ *is Eq. (42), which is given in Box I.*

Proof. Using (41), (37) and (40), we obtain (42). □

4.5. Multivariate compound Poisson distribution with mixed Erlang claim amounts

In the following proposition, we apply (35), (36) and (42) to claim amounts that follow a mixed Erlang distribution.

Proposition 11. *Suppose that* $B_i \sim \mathrm{MixErl}\left(\underline{\zeta}^{(i)}, \beta\right)$ *with* $\underline{\zeta}^{(i)} = \left(\zeta_0^{(i)}, \zeta_1^{(i)}, \dots\right)$, $\zeta_k^{(i)} = \Pr(K_i = k)$ $(k \in \mathbb{N})$, *and with* $M_{B_i}(t) = P_{K_i}(M_C(r))$ *where* $P_{K_i}(s) = \sum_{k=0}^{\infty} \zeta_k^{(i)} s^k$. *Hence, (32) becomes*

$$\begin{aligned} F_D(x) &= \frac{\alpha_0}{\lambda_S}\nu_0 + \sum_{l=1}^{n}\left(\frac{\lambda_1-\alpha_0}{\lambda_S}\zeta_0^{(l)}\right) \\ &\quad + \sum_{k=1}^{\infty}\left(\frac{\alpha_0}{\lambda_S}\nu_k + \sum_{l=1}^{n}\frac{\lambda_1-\alpha_0}{\lambda_S}\zeta_k^{(l)}\right) H(x; k, \beta), \end{aligned} \tag{43}$$

where, as in Proposition 5, $\Pr(K_1 + \cdots + K_n = k) = \nu_k$ *for* $k \in \mathbb{N}$. *Therefore,* $D \sim \mathrm{MixErl}\left(\underline{\tau}; \beta\right)$ *with* $\underline{\tau} = (\tau_0, \tau_1, \dots)$ *and* $\tau_k = \left(\frac{\alpha_0}{\lambda_S}\xi_k + \sum_{l=1}^{n}\frac{\lambda_1-\alpha_0}{\lambda_S}\zeta_k^{(l)}\right)$. *It follows that* $S \sim \mathrm{MixErl}\left(\underline{\xi}; \beta\right)$ *where the computation of the components of* $\underline{\xi}$ *obtained with the components of* $\underline{\tau}$ *is explained in Proposition 5.*

Also, (39) and (38) respectively become

$$\begin{aligned} F_{D_{-i}}(x) &= \sum_{l=1,l\neq i}^{n}\left(\frac{\lambda_1-\alpha_0}{\sum_{l=1,l\neq i}^{n}(\lambda_l-\alpha_0)}\zeta_0^{(l)}\right) \\ &\quad + \sum_{k=1}^{\infty}\left(\sum_{l=1,l\neq i}^{n}\frac{\lambda_1-\alpha_0}{\sum_{l=1,l\neq i}^{n}(\lambda_l-\alpha_0)}\zeta_k^{(l)}\right) H(x; k, \beta), \end{aligned} \tag{44}$$

and

$$F_{C_{-i}}(x) = \nu_0^{(-i)} + \sum_{k=1}^{\infty} \nu_k^{(-i)} H(x; k, \beta), \tag{45}$$

where $\nu_k^{(-i)} = \Pr\left(\sum_{l=1,l\neq i}^{n} K_l = k\right)$ *for* $k \in \mathbb{N}$. *Then, we have* $D_{-i} \sim \mathrm{MixErl}\left(\underline{\tau^{(-i)}}; \beta\right)$ *and* $C_{-i} \sim \mathrm{MixErl}\left(\underline{\nu^{(-i)}}; \beta\right)$ *with* $\underline{\tau^{(-i)}} = \left(\tau_0^{(-i)}, \tau_1^{(-i)}, \dots\right)$, $\underline{\nu^{(-i)}} = \left(\nu_0^{(-i)}, \nu_1^{(-i)}, \dots\right)$, *and* $\tau_k^{(-i)} = \sum_{l=1,l\neq i}^{n} \frac{\lambda_1-\alpha_0}{\sum_{l=1,l\neq i}^{n}(\lambda_l-\alpha_0)}\zeta_k^{(l)}$. *Consequently, we can express (42) as given in Box II, where* $\nu^{(-i)*l} * \tau_m^{(-i)*n_{-i}-l} = \sum_{u=0}^{m} \nu_u^{(-i)*l} \tau_{m-u}^{(-i)*n_{-i}-l}$ *for* $m \in \mathbb{N}$. *Note that* $a * b_k = \sum_{r=0}^{k} a_r * b_{k-r}$ *denote the convolution product of a and b. Also,* a^{*n} *denotes the nth convolution of a with itself.* □

We illustrate the results of Proposition 11 in the two following examples.

Example 12. We provide an illustration of the results obtained for the multivariate compound Poisson distribution. We consider a portfolio of n_1+n_2 risks $X_1, \dots, X_{n_1+n_2}$. We have $S = X_1+X_2+\cdots+X_{n_1+n_2}$ where $X_i \sim \mathrm{CompPoi}\left(\lambda_i, F_{B_i}\right)$, $\lambda_i = 0.003$ for $i = 1, \dots, n_1$ and $\lambda_i = 0.004$, for $i = n_1+1, \dots, n_1+n_2$. Also, we assume that $B_i \sim \mathrm{Gamma}\left(\gamma_i, \frac{1}{1000}\right)$, $\gamma_i = 2$ for $i = 1, \dots, n_1$ and $\gamma_i = 1$, for $i = n_1+1, \dots, n_1+n_2$. The values of $E[X_i]$, $\mathrm{Var}(X_i)$, $\mathrm{VaR}_{0.995}(X_i)$ and $\mathrm{TVaR}_{9.995}(X_i)$ are provided in Table 4 for $i = 1, \dots, n_1+n_2$.

We have $\mathrm{Cov}(X_i, X_j) = \alpha_0 E[B_i] E[B_j]$ for $i \neq j$. Then, $E[S] = n_1 6 + n_2 4$, $\mathrm{Var}(S) = n_1 18000 + n_2 8000 + n_1(n_1-1)\alpha_0 6^2 + n_2(n_2-1)\alpha_0 4^2 + n_1 n_2 6 \times 4$. The values of

$$\mathrm{TVaR}_{\kappa}\left(X_i;S\right)=\frac{\sum_{k_i=0}^{\infty}\sum_{n_{-i}=0}^{\infty}\sum_{j=0}^{\min\left(k_i,n_{-i}\right)} f_{J_i}\left(k_i-j\right)f_{J_{-i}}\left(n_{-i}-j\right)f_{J_0}(j)E\left[\left(\sum_{l=1}^{k_i}B_{i,l}\right)\times 1_{\left\{\sum_{l=1}^{k_i}B_{i,l}+\sum_{m=1}^{n_{-i}-j}D_{-i,m}+\sum_{r=1}^{j}C_{-i,r}>\mathrm{VaR}_{\kappa}(S)\right\}}\right]}{1-\kappa}. \quad (42)$$

Box I.

$$\mathrm{TVaR}_{\kappa}\left(X_i;S\right)=\frac{\sum_{k_i=0}^{\infty}\sum_{n_{-i}=0}^{\infty}\sum_{j=0}^{\min\left(k_i,n_{-i}\right)} f_{J_i}\left(k_i-j\right)f_{J_{-i}}\left(n_{-i}-j\right)f_{J_0}(j)\sum_{k=1}^{\infty}\sum_{l=1}^{k}\zeta_l^{(i)*k_i}\nu^{(-i)*l}*\tau_{k-l}^{(-i)*n_{-i}-l}\frac{l}{\beta}\overline{H}\left(\mathrm{VaR}_{\kappa}(S);k+1,\beta\right)}{1-\kappa},$$

Box II.

Table 4
Values of $E[X_i]$, $\mathrm{Var}(X_i)$, $\mathrm{VaR}_{\kappa}(X_i)$, and $\mathrm{TVaR}_{\kappa}(X_i)$ $(i=1,\dots,n_1+n_2)$.

i	$E[X_i]$	$\mathrm{Var}(X_i)$	$\mathrm{VaR}_{0.995}(X_i)$	$\mathrm{TVaR}_{0.995}(X_i)$	$\mathrm{VaR}_{0.9995}(X_i)$	$\mathrm{TVaR}_{0.9995}(X_i)$
$1,\dots,n_1$	6	18 000	0	1200	3238.266	4478.152
$n_1+1,\dots,n_1+n_2$	4	8 000	0	800	2081.600	3083.598

Table 5
Values of $\mathrm{VaR}_{\kappa}(S)$, $\mathrm{TVaR}_{\kappa}(S)$, $\mathrm{TVaR}_{\kappa}(X_1;S)$ and $\mathrm{TVaR}_{\kappa}(X_{n_1+1};S)$.

α_0	n_1	n_2	$\mathrm{VaR}_{0.995}(S)$	$\mathrm{TVaR}_{0.995}(S)$	$\mathrm{TVaR}_{0.995}(X_1;S)$	$\mathrm{TVaR}_{0.995}\left(X_{n_1+1};S\right)$
0	10	10	3 652.7581	4 878.4333	376.6091	111.2342
	100	100	8 139.8303	9 683.8950	73.4793	23.3596
	500	500	17 492.66	19 695.98	27.89728	11.49468
0.001	10	10	3 435.54973	9 730.0882	678.4794	294.5294
	100	100	7386.3940	67 204.6171	453.8504	218.1958
	500	500	14 831.62	314 154.66	419.5803	208.729
0.002	10	10	3 018.4914	14 541.8393	971.4273	482.7567
	100	100	6 392.1719	124 789.3371	883.8936	413.9998
	500	500	11 685.9	608 843.8	811.4286	406.259

Table 6
Values of $E[X_i]$, $\mathrm{Var}(X_i)$, $\mathrm{VaR}_{\kappa}(X_i)$, and $\mathrm{TVaR}_{\kappa}(X_i)$.

i	$E[X_i]$	$\mathrm{Var}(X_i)$	$\mathrm{VaR}_{0.995}(X_i)$	$\mathrm{TVaR}_{0.995}(X_i)$
$1,\dots,5$	1.4	38	42.6234	55.9980
$6,\dots,10$	4.8	172	59.9921	74.1010

$\mathrm{VaR}_{0.995}(S)$, $\mathrm{TVaR}_{9.995}(S)$ and $\mathrm{TVaR}_{0.995}(X_i;S)$ $(i=1,n_1+1)$ are provided in Table 5.

Note that $\sum_{i=1}^{n_1+n_2}\mathrm{VaR}_{0.995}(X_i)=0$ for all n_1 and n_2; $\sum_{i=1}^{n_1+n_2}\mathrm{TVaR}_{0.995}(X_i)=n_1 1200+n_2 800$ for all n_1 and n_2. We observe that $\mathrm{VaR}_{0.995}(X_i)=0$ for all $i=1,2,\dots,n_1+n_2$ which implies that $\sum_{i=1}^{n_1+n_2}\mathrm{VaR}_{0.995}(X_i)=0$ for all n_1 and n_2. It is interesting to mention that $\mathrm{VaR}_{0.995}(S)>0=\sum_{i=1}^{n_1+n_2}\mathrm{VaR}_{0.995}(X_i)$. It clearly underlines the incoherence of the VaR risk measure. Also, as expected, $\mathrm{TVaR}_{0.995}(S)\leq\sum_{i=1}^{n}\mathrm{TVaR}_{0.995}(X_i)=n_1 1200+n_2 800$ for $n_1,n_2\in\mathbb{N}^+$. The benefit of the aggregation of risks increases with n_1 and n_2 but decreases with α_0. □

Example 13. We consider a portfolio of 10 risks $X_1,\dots,X_{10}$. We have $S=X_1+X_2+\cdots+X_{10}$ where $X_i\sim\mathrm{CompPoi}\left(\lambda_i,F_{B_i}\right)$, $\lambda_i=0.1$ for $i=1,\dots,5$ and $\lambda_i=0.2$, for $i=6,\dots,10$. Also, we assume that

$$F_{B_i}(x)=\frac{7}{10}H(x;1,0.1)+\frac{2}{10}H(x;2,0.1)+\frac{1}{10}H(x;3,0.1),\quad i=1,\dots,5$$

$$F_{B_i}(x)=\frac{1}{10}H(x;1,0.1)+\frac{4}{10}H(x;2,0.1)+\frac{5}{10}H(x;3,0.1),\quad i=6,\dots,10.$$

In Table 6, we provide the values of $E[X_i]$, $\mathrm{Var}(X_i)$, $\mathrm{VaR}_{0.995}(X_i)$, and $\mathrm{TVaR}_{9.995}(X_i)$ for $i=1,\dots,10$.

We have $\mathrm{Cov}\left(X_i,X_j\right)=\alpha_0 E[B_i]E\left[B_j\right]$ for $i\neq j$. Then, $E[S]=31$, $\mathrm{Var}(S)=1050+21\,440\alpha_0$. The results provided in Table 7 are obtained using the closed-form expressions of Proposition 11 for $\alpha_0=0, 0.5, 0.09$.

We observe that the value of $\mathrm{TVaR}_{0.995}(S)$ increases as the value of the dependence parameter increases which implies that the benefit of the risk aggregation decreases. □

4.6. Comments

In several occasions, the distributions of the claim amount r.v.s. $B_1,\dots,B_n$ do not belong to a class of distributions which

Table 7
Values of $\mathrm{VaR}_{\kappa}(S)$, $\mathrm{TVaR}_{\kappa}(S)$, $\mathrm{TVaR}_{\kappa}(X_i;S)$ $(i=1,2,\dots,10)$.

α_0	$\mathrm{VaR}_{0.995}(S)$	$\mathrm{TVaR}_{0.995}(S)$	$\mathrm{TVaR}_{0.995}(X_i;S)$ $(i=1,2,\dots,5)$	$\mathrm{TVaR}_{0.995}(X_{10};S)$ $(i=6,7,\dots,10)$
0	152.4876	175.2395	5.9111	29.1368
0.05	292.8010	345.6295	23.7683	45.3576
0.09	324.1812	395.7055	27.7363	51.4048

are closed under convolution. Inspired by Tijms (1994) and as suggested by Lee and Lin (2010), an interesting strategy would be to approximate the r.v.s. $B_1, \ldots, B_n$ by r.v.s. $\widetilde{B}_1, \ldots, \widetilde{B}_n$ which follow mixed Erlang distributions. The corresponding r.v.s. are denoted by $\widetilde{X}_1, \ldots, \widetilde{X}_n$, and $\widetilde{S} = \widetilde{X}_1 + \cdots + \widetilde{X}_n$. We can compute explicitly $F_{\widetilde{S}}$, $\mathrm{TVaR}_\kappa\left(\widetilde{S}\right)$ and the contributions $\mathrm{TVaR}_\kappa\left(\widetilde{X}_i; \widetilde{S}\right)$ $(i = 1, \ldots, n)$ which approximate F_S, $\mathrm{TVaR}_\kappa(S)$ and the contributions $\mathrm{TVaR}_\kappa(X_i; S)$ $(i = 1 \ldots, n)$.

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