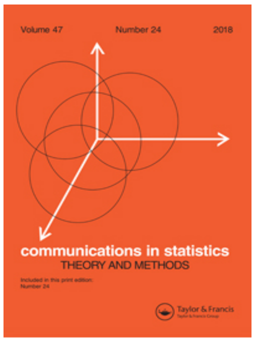

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On some generalized farlie-gumbel-morgenstern distributions-II regression, correlation and further generalizations

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ON SOME GENERALIZED FARLIE-GUMBEL-MORGENSTERN DISTRIBUTIONS - II REGRESSION, CORRELATION AND FURTHER GENERALIZATIONS

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ABSTRACT

Regression and correlation properties of the generalized Farlie-Gumbel-Morgenstern distributions introduced in Johnson and Kotz (1975) are studied. Further generalizations of these distributions are considered.

1. INTRODUCTION

The Farlie-Gumbel-Morgenstern (FGM) system of bivariate distributions has joint cumulative distribution functions (cdf's) of the form

$$F_{12}(x_1,x_2) = F_1(x_1)F_2(x_2)[1 + \alpha\{1 - F_1(x_1)\}\{1 - F_2(x_2)\}], \quad (1.1)$$

where $F_{12}(x_1,x_2)$, $F_j(x_j)$ are used as abbreviations for the cdf's $F_{X_1,X_2}(x_1,x_2) = \Pr[\cap_{j=1}^{2}(X_j \le x_j)]$ and $F_{X_j}(x_j) = \Pr[X_j \le x_j]$ respectively.

We note that if X_1, X_2 have a joint FGM distribution and $Y_1 = h_1(X_1)$, $Y_2 = h_2(X_2)$ are monotonic increasing functions of X_1, X_2 respectively then Y_1 and Y_2 also have a joint FGM distribution. This is easily seen by noting that

$$\Pr[\bigcap_{j=1}^{2}(Y_j \leq h_j(x_j))] = F_{12}(x_1, x_2)$$

and

$$\Pr[Y_j \leq h_j(x_j)] = F_j(x_j) \ .$$

If X_1 and X_2 are each continuous, we can find transformations $Y_1 = h_1(X_1)$, $Y_2 = h_2(X_2)$ so that Y_1 and Y_2 each have a standard uniform distribution. The resulting special FGM distribution (with uniform marginals) was discussed in 1936 by H. Eyraud, which is, we believe, the earliest reference to FGM distributions. (See also Kimeldorf and Sampson (1975).)

In terms of the survival functions

$$S_{12}(x_1, x_2) = \Pr[\bigcap_{j=1}^{2}(X_j > x_j)]; \quad S_j(x_j) = \Pr[X_j > x_j] \ ,$$

(1.1) is equivalent to

$$S_{12}(x_1, x_2) = S_1(x_1)S_2(x_2)[1 + \alpha F_1(x_1)F_2(x_2)] \ . \tag{1.2}$$

In Johnson and Kotz (1975) we generalized (1.1) to the form

$$F_{12\ldots m}(x_1, x_2, \ldots, x_m)$$

$$= \{\prod_{j=1}^{m} F_j(x_j)\}\left\{1 + \sum_{g=2}^{m} \sum_{\{1 \leq j_1 < \ldots < j_g \leq m\}} \cdots \sum \alpha_{j_1 \ldots j_g} \prod_{h=1}^{g} S_{j_h}(x_{j_h})\right\} \ . \tag{2}$$

As in the bivariate case, if $X_1, X_2 \ldots X_k$ have a joint FGM distribution, so do any monotonic increasing functions $Y_j = h_j(X_j)$ $(j=1,2,\ldots,k)$.

For the case when this joint distribution is absolutely continuous, limits for the α's were obtained in Johnson and Kotz (1975). (For the case $m=2$, as in (1.1) we must have $|\alpha| \leq 1$.) Limits on the values of the α's in the general case have been obtained by Cambanis (1976).

In this paper we will assume that the joint distribution is absolutely continuous.

2. REGRESSION

The joint density corresponding to (1.1) is

$$f_{12}(x_1,x_2) = f_1(x_1)f_2(x_2)[1+\alpha\{1-2F_1(x_1)\}\{1-2F_2(x_2)\}], \quad (3)$$

where $f_j(x_j) = dF_j(x_j)/dx_j$ is the density function of X_j $(j=1,2)$. The conditional density of X_1, given X_2, is

$$f_{1|2}(x_1|x_2) = f_1(x_1)[1+\alpha\{1-2F_2(x_2)\}\{1-2F_1(x_1)\}] \quad (4)$$

$$= f_1(x_1)[1+\alpha\{2F_2(x_2)-1\}\{2F_1(x_1)-1\}] \;. \quad (4)'$$

The corresponding cdf is

$$\Pr[X_1 \le x_1 | X_2=x_2] = F_{1|2}(x_1|x_2)$$

$$= F_1(x_1) - \alpha\psi_2(x_2)F_1(x_1)\{1-F_1(x_1)\}, \quad (5)$$

where

$$\psi_2(x_2) = 2F_2(x_2) - 1 \;.$$

If m_2 is a median value of X_2, so that $F_2(m_2) = 1/2$ then $\psi_2(m_2)$ is zero and the conditional distribution of X_1 given $X_2 = m_2$ is the same as its unconditional distribution.

The r-th moment about zero of the conditional distribution (5) is

$$E[X_1^r|X_2=x_2] = E[X_1^r] + \alpha\psi_2(x_2)\int_{-\infty}^{\infty} x^r\{2F_1(x)-1\}f_1(x)dx \;. \quad (6)$$

If the r-th moment of X_1 exists, so does the integral on the right hand side of (6). It can be expressed in terms of the r-th moment of ${}_{2:2}X_1$, the greater of two independent random variables each having the cdf $F_1(x)$.

We have

$$E[{}_{2:2}X_1{}^r] = 2\int_{-\infty}^{\infty} x^r F_1(x)f_1(x)dx \;, \quad (7)$$

and so, from (6)

$$E[X_1^r|X_2=x_2] = E[X_1^r] + \alpha\{E[{}_{2:2}X_1{}^r] - E[X_1^r]\}\psi_2(x_2) \ . \qquad (6)'$$

Taking r = 1 we obtain the regression function

$$E[X_1|X_2=x_2] = E[X_1] + \alpha\{E[{}_{2:2}X_1] - E[X_1]\}\psi_2(x_2) \ . \qquad (8)$$

Since $E[{}_{2:2}X_1] - E[X_1] > 0$ we see that the regression function is an increasing or decreasing function of x_2 according as α is positive or negative. We further note that the regression function is always between the inclusive limits $E[X_1] \pm \alpha\{E[{}_{2:2}X_1] - E[X_1]\}$ and it approaches these limits asymptotically as $x_2 \to \pm\infty$.

The regression is linear if and only if (iff) $\psi_2(x_2)$ (and so $F_2(x_2)$) is a linear function of x_2. This is possible only over finite intervals, and occurs iff X_2 is uniformly distributed over these intervals.

Taking r = 2 in (6)' gives

$$E[X_1^2|X_2=x_2] = E[X_1^2] + \alpha\{E[{}_{2:2}X_1{}^2] - E[X_1^2]\}\psi_2(x_2) \qquad (9)$$

whence

$$\begin{aligned} \operatorname{var}(X_1|X_2=x_2) = \operatorname{var}(X_1) &+ \alpha\{\tau_1(2;2) - 2E(X_1)\tau_1(1;2)\psi_2(x_2) \\ &- \alpha^2\{\tau_1(1;2)\}^2\{\psi_2(x_2)\}^2 \end{aligned} \qquad (10)$$

introducing the notation

$$\tau_j(r;n) = E[{}_{n:n}X_j{}^r] - E[X_j^r], \qquad (11)$$

where

$${}_{n:n}X_j = \text{greatest of } n \text{ independent random variables each distributed as } X_j \ . \qquad (12)$$

If X_1 has a symmetrical distribution then by taking the origin at $E[X_1]$ we get $\tau_1(2;2) = 0$. Consequently

(i) $E[X_1^2|X_2=x_2]$ does not vary with x_2

and (ii) $\operatorname{var}(X_1|X_2=x_2) = \operatorname{var}(X_1) - \{\alpha\tau_1(1;2)\}^2\psi_2(x_2) < \operatorname{var}(X_1)$.

Note that

(a) the value of $\tau_j(1;n)$ for $(X_j+\theta)$ is the same as for X_j (for any θ),

(b) the value of $\tau_j(r;n)$ for θX_j is θ^r times the value for X_j (for any θ).

Table I gives values of $\tau(1;2)$ and $\tau(2;2)$ for a few common distributions. Variation in location and scale parameters can be allowed for by using (a) and (b) above.

TABLE I

Values of $\tau(r;2) = 2\int_{-\infty}^{\infty} x^r F(x)f(x)dx - E[X^r]$

Name	F(x)	τ(1;2)	τ(2;2)
Normal	$(2\pi)^{-1/2}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}dt$	0.5642 $(= \pi^{-1/2})$	0
Uniform	$\begin{cases} 0 & (x\le 0) \\ x & (0<x\le 1) \\ 1 & (x>1) \end{cases}$	0.1667	0.1667
Weibull	$\begin{cases} 0 & (x\le 0) \\ 1-\exp(-x^c) & (x>0) \end{cases}$		
	$c = 0.5$	1.5000	22.5000
(Exponential)	$c = 1$	0.5000	1.5000
	$c = 2$	0.2596	0.5000
Gamma	$\begin{cases} 0 & (x\le 0) \\ \{\Gamma(\alpha)\}^{-1}\int_0^x t^{\alpha-1}e^{-t}dt & (x>0) \end{cases}$		
	$\alpha = 2$	0.7500	3.7500
	$\alpha = 3$	0.9375	6.5625
	$\alpha = 4$	1.09375	9.8438
	$\alpha = 5$	1.2305	13.5352

Turning now to the general case (2) of an m-variate FGM distribution, we note (from (5) of Johnson and Kotz (1975)) that the joint density function of $X_1, X_2, \ldots, X_m$ is

$$f_{12\ldots m}(x_1,x_2,\ldots,x_m) =$$
$$\left\{\prod_{j=1}^{m} f_j(x_j)\right\}[1 + \sum_{g=2}^{m} \sum_{j_1<\cdots<j_g}^{m-g+1} \cdots \sum^{m} \alpha_{j_1\ldots j_g} \prod_{h=1}^{g} \{1-2F_{j_h}(x_{j_h})\}] \quad (13)$$

and the density function of X_1, conditional on $\cap_{j=2}^{m}(X_j=x_j)$ is

$$f_{1|2\ldots m}(x_1|x_2,\ldots,x_m) =$$
$$f_1(x_1)[1+g_1(x_2,\ldots,x_m)\{1-2F_1(x_1)\}], \quad (14)$$

where

$$g_1(x_2\ldots x_m) =$$
$$\left[\sum_{g=1}^{m-1} \sum_{2\le j_1<\cdots<j_g}^{m-g+1} \cdots \sum \alpha_{1j_1\ldots j_g} \prod_{h=1}^{g} \{1-2F_{j_h}(x_{j_h})\}\right](1+D_1)^{-1} \quad (15)$$

and D_1 is obtained from the expression in square brackets by deleting the suffix '1' in each $\alpha_{1j_1\ldots j_g}$, and putting all those resulting α's with only a single suffix equal to zero.

From (12) we see that the conditional distribution of X_1 is always of form

$$f_{1|2\ldots m}(x_1|x_2\ldots x_m) = f_1(x_1)[1+c\{1-2F_1(x_1)\}] . \quad (14)'$$

Iff $|c| \le 1$, (14)' is a proper density function. The cdf corresponding to (14) is

$$F_{1|2\ldots m}(x_1|x_2\ldots x_m) = F_1(x_1) + cF_1(x_1)S_1(x_1) \quad (16)$$

whence

$$|F_{1|2\ldots m}(x_1|x_2\ldots x_m) - F_1(x_1)| \le \frac{1}{4}|c| . \quad (17)$$

The regression function of X_1 on the other (m-1) variables (cf. (8)) is

$$E[X_1|\bigcap_{j=2}^{m}(X_j=x_j)] = E[X_1] - g_1(x_2,\ldots,x_m)\tau_1(1;2) , \quad (18)$$

and the conditional variance (cf. (10)) is

$$\mathrm{var}(X_1|\bigcap_{j=2}^{m}(X_j=x_j)) = \mathrm{var}(X_1) - g_1(x_2,\ldots,x_m)\{\tau_1(2;2)-2\tau_1(1;2)E[X_1]\}$$
$$- \{g_1(x_2,\ldots,x_m)\tau_1(1;2)\}^2 . \quad (19)$$

As in the bivariate case if X_1 has a symmetrical distribution,

$$\mathrm{var}(X_1 | \bigcap_{j=2}^{m} (X_j = x_j)) = \mathrm{var}(X_1) - \{g_1(x_2,\ldots,x_m)\tau_1(1;2)\}^2 \ . \quad (20)$$

We note that for any $\xi_1' < \xi_1''$

$$\Pr[\xi_1' < X_1 \le \xi_1'' | \bigcap_{j=2}^{m} (X_j = x_j)] =$$
$$F_1(\xi_1'') - F_1(\xi_1') + c\{F_1(\xi_1'')S_1(\xi_1'') - F_1(\xi_1')S_1(\xi_1')\}. \quad (21)$$

This lies between the inclusive limits

$$F_1(\xi_1'') - F_1(\xi_1') \pm \{F_1(\xi_1'')S_1(\xi_1'') - F_1(\xi_1')S_1(\xi_1')\} \ . \quad (22)$$

Note that if $F_1(\xi_1'') + F_1(\xi_1') = 1$ so that

$$F_1(\xi_1'')S_1(\xi_1'') = F_1(\xi_1')S_1(\xi_1')$$

then

$$\Pr[\xi_1' \le X_1 \le \xi_1'' | \bigcap_{j=2}^{m} (X_j = x_j)] = \Pr[\xi_1' \le X_1 \le \xi_2'] \ . \quad (23)$$

This means that a symmetrical $100(1-2\alpha)\%$ interval for X_1 (i.e. such that $\Pr[X_1 \le \xi_1'] = \alpha = 1 - \Pr[X_1 \le \xi_1''])$ is also a $100(1-2\alpha)\%$ interval (though not always a symmetrical one) for every conditional distribution of X_1.

3. COVARIANCE AND CORRELATION

From (8)

$$\begin{aligned} \mathrm{cov}(X_1,X_2) &= E[\{X_2 - E[X_2]\}\alpha\tau_1(1;2)\{2F_2(X_2) - 1\}] \\ &= \alpha\tau_1(1;2)\int_{-\infty}^{\infty}(x - E[X_2])\{2F_2(x) - 1\}f_2(x)dx \\ &= \alpha\tau_1(1;2)2\int_{-\infty}^{\infty}(x - E[X_2])F_2(x)f_2(x)dx \\ &= \alpha\tau_1(1;2)\{2\int_{-\infty}^{\infty}xF_2(x)f_2(x)dx - E[X_2]\} \\ &= \alpha\tau_1(1;2)\tau_2(1;2) \ . \end{aligned} \quad (24)$$

The correlation between X_1 and X_2 has the same sign as α, and is, in fact, proportional to α. As an example, using Table I, the correlation between the variables in a bivariate FGM distribution with normal marginal distributions is $\alpha(\sqrt{\pi})^{-2} = \alpha\pi^{-1}$.

The maximum correlation between the variables in a general bivariate FGM distribution is

$$\tau_1(1;2)\tau_2(1;2)\{\text{var}(X_1)\text{var}(X_2)\}^{-1/2}. \qquad (25)$$

We recall from Johnson and Kotz (1975) that with the generalized FGM distribution (2), the joint distribution of any two variables X_i, X_j is bivariate FGM with parameter α_{ij}. The above correlation and regression properties for any pair of variates from a generalized FGM distribution therefore hold with α replaced by α_{ij}.

4. FURTHER GENERALIZATION OF BIVARIATE FGM DISTRIBUTIONS

Farlie (1960) has considered more general systems of bivariate distributions, including cdf's of form

$$F_{12}(x_1,x_2) = F_1(x_1)F_2(x_2)[1 + \alpha S_1^*(x_1)S_2^*(x_2)] , \qquad (26)$$

where $F_j^*(x_j) = 1 - S_j^*(x_j)$ $(j=1,2)$ are cdf's, but not necessarily identical with $F_j(x)$ $(j=1,2)$. A further generalization may be effected by replacing $S_1^*(x_1)S_2^*(x_2)$ by

$$S_{12}^*(x_1,x_2) = 1 - F_1^*(x_1) - F_2^*(x_2) + F_{12}^*(x_1,x_2) , \qquad (26)'$$

where $F_{12}^*(x_1,x_2)$ is a joint cdf, but not necessarily equal to $F_1^*(x_1)F_2^*(x_2)$.

As a particular case of this generalization we suppose that $F_{12}^*(x_1,x_2)$ is itself an FGM distribution with

$$F_{12}^*(x_1,x_2) = F_1^*(x_1)F_2^*(x_2)[1 + \alpha^* S_1^*(x_1)S_2^*(x_2)] , \qquad (27.1)$$

or, equivalently,

$$S_{12}^*(x_1,x_2) = S_1^*(x_1)S_2^*(x_2)[1 + \alpha^* F_1^*(x_1)F_2^*(x_2)] . \qquad (27.2)$$

Then (26)' becomes

$$F_{12}(x_1,x_2)$$
$$= F_1(x_1)F_2(x_2)[1+\alpha S_1^*(x_1)S_2^*(x_2)\{1+\alpha^* F_1^*(x_1)F_2^*(x_2)\}] , \qquad (28.1)$$

or, equivalently,

$$S_{12}(x_1,x_2) = S_1(x_1)S_2(x_2)[1+\alpha F_1^*(x_1)F_2^*(x_2)\{1+\alpha^* S_1^*(x_1)S_2^*(x_2)\}]. \quad (28.2)$$

Of course, α and α^* must each satisfy the conditions for the distribution to be proper. If all the distributions are absolutely continuous, this means that $|\alpha| \le 1$ and $|\alpha^*| \le 1$, at least,

As a yet more special case (though still a generalization of (1)) we can take $F_j^*(x) = F_j(x)$ for $j=1,2$ and obtain

$$\begin{aligned} &F_{12}(x_1,x_2) \\ &= F_1(x_1)F_2(x_2)\{1+\alpha_1 S_1(x_1)S_2(x_2) + \alpha_2 S_1(x_1)F_1(x_1)S_2(x_2)F_2(x_2)\}. \quad (29)\end{aligned}$$

If all distributions are absolutely continuous, $|\alpha_2| \le |\alpha_1| \le 1$.

Now suppose we construct a distribution like (28), taking $F_{12}^*(x_1,x_2)$ to be of form (29). We obtain (cf. Farlie (1960, p.308))

$$\begin{aligned} F_{12}(x_1,x_2) = F_1(x_1)F_2(x_2)[&1+\alpha_1 S_1(x_1)S_2(x_2) \\ &+ \alpha_2 S_1(x_1)F_1(x_1)S_2(x_2)F_2(x_2) \\ &+ \alpha_3\{S_1(x_1)\}^2F_1(x_1)\{S_2(x_2)\}^2F_2(x_2)]. \quad (30)\end{aligned}$$

Successive iterations of this kind lead to joint cdf's of form

$$F_{12}(x_1,x_2) = F_1(x_1)F_2(x_2)[1 + \sum_{j=1}^{k}\alpha_j \prod_{h=1}^{2}\{S_h(x_h)\}^{a_j}\{F_h(x_h)\}^{b_j}] , \quad (31)$$

with $$a_j = \begin{cases} \tfrac{1}{2}(j+1) & (j \text{ odd}) \\ \tfrac{1}{2}j & (j \text{ even}) \end{cases} ; \quad b_j = \begin{cases} \tfrac{1}{2}(j-1) & (j \text{ odd}) \\ \tfrac{1}{2}j & (j \text{ even}) \end{cases} .$$

If all distributions are absolutely continuous we must have

$$|\alpha_h| \le |\alpha_{h-1}| \le \ldots \le |\alpha_1| \le 1 .$$

The joint density function corresponding to (31) is

$$\begin{aligned} &f_{12}(x_1,x_2) \\ &= f_1(x_1)f_2(x_2)[1 + \sum_{j=1}^{k}\alpha_j\{S_1(x_1)S_2(x_2)\}^{a_j-1}\{F_1(x_1)F_2(x_2)\}^{b_j} \\ &\qquad\qquad \times \prod_{i=1}^{2}\{b_j + 1 - (j+1)F_i(x_i)\}]. \quad (32)\end{aligned}$$

(Note that $a_j+b_j = j$ for all j.)

If m_i is a median value of X_i (i.e. $F_i(m_i) = 1/2$), then

$$b_j + 1 - (j+1)F_i(m_i) = \begin{cases} 0 & \text{if } j \text{ is odd} \\ 1/2 & \text{if } j \text{ is even,} \end{cases}$$

and so

$$f_{12}(m_1,m_2) = f_1(m_1)f_2(m_2)(1 + \sum_{g=1}^{h} \alpha_{2g}4^{-(4g-1)}) \tag{33}$$

for $k = 2h$ or $k = 2h+1$.

The series in (33) converges quite rapidly, since $|\alpha_{2h}| \le |\alpha_{2(h-1)}| \le \ldots \le |\alpha_2| \le 1$. The ratio of the $(g+1)$-th term to the g-th term is less than 4^{-4} in absolute magnitude. In fact

$$\left| \frac{f_{12}(m_1,m_2)}{f_1(m_1)f_2(m_2)} - 1 \right| \le \sum_{g=1}^{h} 4^{-(4g-1)} = \frac{4}{255} . \tag{34}$$

We recall from equation (6) of Johnson and Kotz (1975) that for the original FGM distribution, $f_{12}(m_1,m_2) = f_1(m_1)f_2(m_2)$. Inequality (34) shows that this property is *nearly* preserved for the generalized system (32), irrespective of the number of iterations.

The conditional density of X_1 given $(X_2=x_2)$ is

$$\begin{aligned} f_{1|2}(x_1|x_2) = f_1(x_1)[1 + \sum_{j=1}^{k} \alpha_j\{S_2(x_2)\}^{a_j-1}\{F_2(x_2)\}^{b_j} \\ \times \{b_j+1 - (j+1)F_2(x_2)\}\{S_1(x_1)\}^{a_j-1} \\ \times \{F_1(x_1)\}^{b_j}\{b_j+1 - (j+1)F_1(x_1)\}] . \end{aligned} \tag{35}$$

The conditional r-th moment of X_1, given $(X_2=x_2)$ is

$$\begin{aligned} E[X_1^r|X_2=x_2] = E[X_1^r] + \sum_{j=1}^{k} \alpha_j\{S_2(x_2)\}^{a_j-1}\{F_2(x_2)\}^{b_j} \\ \times \{b_j+1 - (j+1)F_2(x_2)\}h_{1,r}(a_j,b_j) , \end{aligned} \tag{36}$$

where

$$h_{i,r}(a,b) = \int_{-\infty}^{\infty} x^r\{(b+1)S_i(x) - aF_i(x)\}\{F_i(x)\}^b\{S_i(x)\}^{a-1}f_i(x)dx$$

$$= \binom{a+b+1}{a}^{-1}\left[\frac{(a+b+1)!}{b!a!}\int_{-\infty}^{\infty} x^r\{F_i(x)\}^b\{S_i(x)\}^a f_i(x)dx\right.$$

$$\left. - \frac{(a+b+1)!}{(b+1)!(a-1)!}\int_{-\infty}^{\infty} x^r\{F_i(x)\}^{b+1}\{S_i(x)\}^{a-1}f_i(x)dx\right]$$

$$= \binom{a+b+1}{a}^{-r}\left\{E[{}_{b+1:a+b+1}X_i^r] - E[{}_{b+2:a+b+1}X_i^r]\right\} \tag{37}$$

and ${}_{s:n}X_i$ denotes a random variable distributed as the s-th smallest of n independent random variables each having the cdf $F_i(x)$. The conditional distribution of X_2 given X_1 can be shown to be

$$F_{1|2}(x_1|x_2) = F_1(x_1) + \sum_{j=1}^{k}\alpha_j\{S_2(x_2)\}^{a_j-1}\{F_2(x_2)\}^{b_j}$$
$$\times \{b_j+1-(j+1)F_2(x_2)\}\{S_1(x_1)\}^{a_j}\{F_1(x_1)\}^{b_j+1} . \tag{38}$$

Recalling that if j is odd, $a_j = b_j+1 = (1/2)(j+1)$ while if j is even $a_j = b_j = (1/2)j$, we see that if $\xi_1' < \xi_1''$ and $F_1(\xi_1') = S_1(\xi_1'')$ then

$$\Pr[\xi_1'<X_1\leq\xi_1''|X_2=x_2] = \Pr[\xi_1'<X_1<\xi_2''] + \sum_{j=1}^{h}\alpha_{2j}\{S_2(x_2)\}^{j-1}\{F_2(x_2)\}^j$$
$$\times \{S_1(\xi_1'')F_1(\xi_1'')\}^j\{F_1(\xi_1'') - S_1(\xi_1'')\} , \tag{39}$$

where h is the integer part of (1/2)k.

It is natural to consider generalizations of (31) to m-variate distributions in which $\Pi_{h=1}^2\{S_h(x_h)\}^{a_j}\{F_h(x_h)\}^{b_j}$ is replaced by

$$\prod_{h=1}^{m}\{S_h(x_h)\}^{a_j}\{F_h(x_h)\}^{b_j} . \tag{40}$$

The analogous generalization of (2), however, is more complicated. It is

$$F_{12\ldots m}(x_1,x_2,\ldots,x_m) = \left\{\prod_{j=1}^{m}F_j(x_j)\right\}\left[1 + \sum_{j=1}^{k}\sum_{h=2}^{m}\sum_{g_1}\cdots\sum_{g_h}\alpha_{jg_1\ldots g_h}\right.$$
$$\left.\times\left\{\prod_{i=1}^{h}S_{g_i}(x_{g_i})\right\}^{a_j}\left\{\prod_{i=1}^{h}F_{g_i}(x_{g_i})\right\}^{b_j}\right], \tag{41}$$

which is analogous to Gumbel's (1958) formula

$$F_{12\ldots m}(x_1,\ldots,x_m) = \{\prod_{j=1}^{m} F_j(x_j)\}[1 + \alpha \prod_{j=1}^{m} S_j(x_j)] .$$

For the joint distribution (41) it is true that any subset of the m variables has a joint cdf of the same form as (41).

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