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Testing the Gaussian copula hypothesis for financial assets dependences

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Abstract

Using one of the key properties of copulas that they remain invariant under an arbitrary monotonic change of variable, we investigate the null hypothesis that the dependence between financial assets can be modelled by the Gaussian copula. We find that most pairs of currencies and pairs of major stocks are compatible with the Gaussian copula hypothesis, while this hypothesis can be rejected for the dependence between pairs of commodities (metals). Notwithstanding the apparent qualification of the Gaussian copula hypothesis for most of the currencies and the stocks, a non-Gaussian copula, such as the Student copula, cannot be rejected if it has sufficiently many 'degrees of freedom'. As a consequence, it may be very dangerous to embrace blindly the Gaussian copula hypothesis, especially when the coefficient of correlation between the pairs of assets is too high, such that the tail dependence neglected by the Gaussian copula can became large, leading to the ignoring of extreme events which may occur in unison.

1. Introduction

determination of the dependence between assets underlies many financial activities, such as risk assessment and portfolio management, as well as option pricing and hedging. Following Markovitz (1959), the covariance and correlation matrices have, for a long time, been considered as the main tools for quantifying the dependence between assets. But the dimension of risk captured by the correlation matrices is only satisfactory for elliptic distributions and for moderate risk amplitudes (Sornette *et al* 2000b). In all other cases, this measure of risk is severely incomplete and can lead to a very strong underestimation of the real incurred risks (Embrechts *et al* 1999).

Although the unidimensional (marginal) distributions of asset returns are reasonably constrained by empirical data and their tails are more or less satisfactorily described by a power law with the tail index ranging between 2 and 4 (de Vries 1994, Lux 1996, Pagan 1996, Guillaume *et al* 1997, Gopikrishnan *et al* 1998, McNeil and Frey 2000), by stretched exponentials (Laherrère and Sornette 1999, Gouriéroux and Jasiak 1999, Sornette *et al* 2000a, 2000b), or by log–Weibull distributions (Malevergne *et al* 2003), no equivalent results have been obtained for *multivariate* distributions of asset returns. Indeed, a brute force determination of multivariate

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distributions is unreliable due to the limited data set (the curse of dimensionality), while the sole knowledge of marginals (one-point statistics) of each asset is not sufficient to obtain information on the multivariate distribution of these assets which involves all the n-points statistics.

Some progress may be expected from the concept of copulas, recently proposed to be useful for financial applications (Embrechts *et al* 2001, Frees and Valdez 1998, Haas 1999, Klugman and Parsa 1999). This concept has the desirable property of decoupling the study of the marginal distribution of each asset from the study of their collective behaviour or dependence. Indeed, the dependence between assets is entirely embedded in the copula, so a copula allows for a simple description of the dependence structure between assets independently of the marginals. For instance, assets can have power law marginals and a Gaussian copula or alternatively Gaussian marginals and a non-Gaussian copula, and any possible combination thereof. Therefore, the determination of the multivariate distribution of assets can be performed in two steps: (i) an independent determination of the marginal distributions using standard techniques for distributions of a single variable; (ii) a study of the nature of the copula characterizing completely the dependence between the assets. This exact separation between the marginal distributions and the dependence is potentially very useful for risk management or option pricing and sensitivity analysis, since it allows for testing several scenarios with different kinds of dependence between assets while the marginals can be set to their well-calibrated empirical estimates. Such an approach has been used by Embrechts *et al* (2001) to provide various bounds for the value-at-risk of a portfolio made of dependent risks, and by Rosenberg (1999) or Cherubini and Luciano (2000) to price and to analyse the pricing sensitivity of binary digital options or options on the minimum of a basket of assets.

A fundamental limitation of the copula approach is that there is in principle an infinite number of possible copulas (Genest and MacKay 1986, Genest 1987, Genest and Rivest 1993, Joe 1993, Nelsen 1998) and, up to now, no general empirical study has determined the classes of copulas that are acceptable for financial problems. In general, the choice of a given copula is guided both by the empirical evidence and the technical constraints, i.e., the number of parameters necessary to describe the copula, the possibility of obtaining efficient estimators of these parameters, and also the possibility offered by the chosen parametrization to allow for tractable analytical calculation. It is indeed sometimes more advantageous to prefer a simple copula to one that fits the data better, provided that we can clearly quantify the effects of this substitution.

In this vein, the first goal of the present paper is to ask whether the Gaussian copula provides a sufficiently good statistical approximation of the unknown true copula. This question is at the heart of many financial problems since the Gaussian copula underlies almost all current financial theories. First, the Gaussian copula obviously appears in the traditional theories relying on the multivariate Gaussian description. In addition, it is also widely used in the most recent financial applications such as in the modelling of the dependent defaults[5] as exemplified by the model of CreditMetrics or KMV, for instance, or such as the pricing of credit derivatives (Li 1999, 2000). There is thus a real need for a test of the Gaussian copula to model financial dependences. Our second goal is to draw the consequences of the parametrization involved in the Gaussian copula in terms of potential over/underestimation of the risks, in particular for large and extreme events.

The paper is organized as follows.

In section 2, we first recall some important general definitions and theorems about copulas that will be useful in the following. We then introduce the concept of tail dependence that will allow us to quantify the probability that two extreme events might occur simultaneously. We define and describe the two copulas that will be at the core of our study: the Gaussian copula and the Student copula and compare their properties particularly in the tails.

In section 3, we present our statistical testing procedure which is applied to pairs of financial time series. First of all, we determine a test statistics which leads us to compare the empirical distribution of the data with a χ^2-distribution using a bootstrap method. We also test the sensitivity of our procedure by applying it to the synthetic multivariate Student time series. This allows us to determine the minimum statistical test value needed to be able to distinguish between a Gaussian and a Student copula, as a function of the number of degrees of freedom and of the correlation strength.

Section 4 presents the empirical results obtained for the following assets which are combined pairwise in the test statistics:

- 6 currencies,
- 6 metals traded on the London Metal Exchange,
- 22 stocks chosen among the largest companies quoted on the New York Stock Exchange.

We show that the Gaussian copula hypothesis is very reasonable for most stocks and currencies, while it is hardly compatible with the description of multivariate behaviour for metals.

Section 5 summarizes our results and concludes.

2. Generalities about copulas

2.1. Definitions and important results about copulas

This section does not pretend to provide a rigorous mathematical exposition of the concept of copula. We only recall a few basic definitions and theorems that will be useful in the following (for more information about the concept of copula, see for instance Nelsen (1998)).

We first give the definition of a copula of n random variables.

Definition 1 (Copula). *A function $C\colon [0,1]^n \longrightarrow [0,1]$ is an n-copula if it enjoys the following properties:*

- $\forall u \in [0,1]$, $C(1,\ldots,1,u,1,\ldots,1) = u$,

[5] Following the recommendations of the Basle Committee on Supervision Banking (2001), the Gaussian copula must be chosen to model the dependence between defaults.

- $\forall u_i \in [0, 1]$, $C(u_1, \ldots, u_n) = 0$ *if at least one of the* u_i *equals zero,*
- C *is grounded and* n*-increasing, i.e., the* C*-volume of every box whose vertices lie in* $[0, 1]^n$ *is positive.*

It is clear from this definition that a copula is nothing but a multivariate distribution with support in $[0,1]^n$ and with uniform marginals. The fact that such copulas can be very useful for representing multivariate distributions with arbitrary marginals is seen from the following result.

Theorem 1 (Sklar's theorem). *Given an* n*-dimensional distribution function* F *with* continuous *marginal (cumulative) distributions* $F_1, \ldots, F_n$*, there exists a* unique n*-copula* C: $[0, 1]^n \longrightarrow [0, 1]$ *such that*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{1}$$

This theorem provides both a parametrization of multivariate distributions and a construction scheme for copulas. Indeed, given a multivariate distribution F with marginals $F_1, \ldots, F_n$, the function

$$C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)) \tag{2}$$

is automatically a n-copula[6]. This copula is the copula of the multivariate distribution F. We will use this method in the following to derive the expressions of standard copulas such as the Gaussian copula or the Student copula.

A very powerful property of copulas is their invariance under arbitrary strictly increasing mapping of the random variables:

Theorem 2 (Invariance theorem). *Consider* n *continuous random variables* $X_1, \ldots, X_n$ *with copula* C*. Then, if* $g_1(X_1), \ldots, g_n(X_n)$ *are strictly increasing on the ranges of* $X_1, \ldots, X_n$*, the random variables* $Y_1 = g_1(X_1), \ldots, Y_n = g_n(X_n)$ *have exactly the same copula* C*.*

It is this result that shows us that the full dependence between the n random variables is completely captured by the copula, independently of the shape of the marginal distributions. This result is the basis of our statistical study presented in section 3.

2.2. Dependence between random variables

The dependence between two time series is usually described by their correlation coefficient. This measure is fully satisfactory only for elliptic distributions (Embrechts *et al* 1999), which are functions of a quadratic form of the random variables, when one is interested in moderately sized events. However, an important issue for risk management concerns the determination of the dependence of the distributions in the tails. Practically, the question is whether it is more probable that large or extreme events occur simultaneously or in contrast more or less independently. This is referred to as the presence or absence of 'tail dependence'.

The tail dependence is also an interesting concept in studying the *contagion* of crises between markets or countries. These questions have recently been addressed by Ang and Cheng (2002), Longin and Solnik (2001) and Starica (1999) among several others. Large negative moves in a country or market are often found to imply large negative moves in others.

Technically, we need to determine the probability that a random variable X is large, knowing that the random variable Y is large.

Definition 2 (Tail dependence 1). *Let* X *and* Y *be random variables with continuous marginals* F_X *and* F_Y*. The (upper) tail dependence coefficient of* X *and* Y *is, if it exists,*

$$\lim_{u \to 1} \Pr\{X > F_X^{-1}(u) | Y > F_Y^{-1}(u)\} = \lambda \in [0, 1]. \tag{3}$$

In words, given that Y *is very large (which occurs with probability* $1 - u$*), the probability that* X *is very large at the same probability level* u *defines asymptotically the tail dependence coefficient* λ*.*

It turns out that this tail dependence is a pure copula property which is independent of the marginals. Let C be the copula of the (*assumed continuous*) variables X and Y, then

Theorem 3. *If the bivariate copula* C *is such that*

$$\lim_{u \to 1} \frac{\bar{C}(u, u)}{1 - u} = \lambda \tag{4}$$

exists (where $\bar{C}(u, u) = 1 - 2u + C(u, u)$*), then* C *has an upper tail dependence coefficient* λ*.*

If $\lambda > 0$, the copula presents tail dependence and large events tend to occur simultaneously, with the probability λ. In contrast, when $\lambda = 0$, the copula has no tail dependence in this sense and large events appear to occur essentially independently. There is however a subtlety in this definition of tail dependence. To make it clear, first consider the case where for large X and Y the distribution function $F(x, y)$ factorizes such that

$$\lim_{x,y \to \infty} \frac{F(x, y)}{F_X(x) F_Y(y)} = 1. \tag{5}$$

This means that, for X and Y sufficiently large, these two variables can be considered as independent. It is then easy to show that

$$\lim_{u \to 1} \Pr\{X > F_X^{-1}(u) | Y > F_Y^{-1}(u)\} = \lim_{u \to 1} 1 - F_X(F_X^{-1}(u)) \tag{6}$$

$$= \lim_{u \to 1} 1 - u = 0, \tag{7}$$

so independent variables really have no tail dependence, as one could expect.

Unfortunately, the converse does not hold: a value $\lambda = 0$ does not automatically imply true independence, namely that $F(x, y)$ satisfies equation (5). Indeed, the tail independence

[6] The quantile function F_i^{-1} of the distribution F_i can be defined by

$$F_i^{-1}(u) = \inf\{x | F_i(x) \geqslant u\}, \qquad \forall u \in (0, 1).$$

When the distribution function F_i is strictly increasing, F_i^{-1} denotes the usual inverse of F_i. In fact, any quantile function can be chosen. But, for non-continuous margins, the copula (2) will depend upon the precise quantile function which will be selected.

criterion $\lambda = 0$ may still be associated with an absence of factorization of the multivariate distribution for large X and Y. In a weaker sense, there may still be a dependence in the tail even when $\lambda = 0$. Such behaviour is for instance exhibited by the Gaussian copula, which has zero tail dependence according to definition 2 but nevertheless does not have a factorizable multivariate distribution, since the non-diagonal term of the quadratic form in the exponential function does not become negligible in general as X and Y go to infinity. To summarize, the *tail independence*, according to definition 2, is not equivalent to the *independence in the tail* as defined in equation (5).

After this brief review of the main concepts underlying copulas, we now present two special families of copulas: the Gaussian copula and the Student copula.

2.3. The Gaussian copula

The Gaussian copula is the copula derived from the multivariate Gaussian distribution. Let Φ denote the standard normal (cumulative) distribution and $\Phi_{\rho,n}$ the n-dimensional Gaussian distribution with correlation matrix ρ. Then, the Gaussian n-copula with correlation matrix ρ is

$$C_\rho(u_1, \dots, u_n) = \Phi_{\rho,n}(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)), \quad (8)$$

whose density

$$c_\rho(u_1, \dots, u_n) = \frac{\partial C_\rho(u_1, \dots, u_n)}{\partial u_1 \dots \partial u_n} \quad (9)$$

reads

$$c_\rho(u_1, \dots, u_n) = \frac{1}{\sqrt{\det \rho}} \exp\left(-\frac{1}{2} y(u)^t (\rho^{-1} - \mathrm{Id}) y(u)\right) \quad (10)$$

with $y_k(u) = \Phi^{-1}(u_k)$. Note that theorem 1 and equation (2) ensure that $C_\rho(u_1, \dots, u_n)$ in equation (8) is a copula.

As we said before, the Gaussian copula does not have a tail dependence:

$$\lim_{u \to 1} \frac{\bar{C}_\rho(u, u)}{1 - u} = 0, \qquad \forall \rho \in (-1, 1). \quad (11)$$

This result is derived for example in Embrechts *et al* (2001). But this does not mean that the Gaussian copula goes to the independent (or product) copula $\Pi(u_1, u_2) = u_1 u_2$ when (u_1, u_2) goes to one. Indeed, consider a distribution $F(x, y)$ with a Gaussian copula:

$$F(x, y) = C_\rho(F_X(x), F_Y(y)). \quad (12)$$

Its density is

$$f(x, y) = c_\rho(F_X(x), F_Y(y)) f_X(x) f_Y(y), \quad (13)$$

where f_X and f_Y are the densities of X and Y. Thus,

$$\lim_{(x,y)\to\infty} \frac{f(x, y)}{f_X(x) f_Y(y)} = \lim_{(x,y)\to\infty} c_\rho(F_X(x), F_Y(y)), \quad (14)$$

which should equal 1 if the variables X and Y are independent in the tail. Reasoning in the quantile space, we set $x = F_X^{-1}(u)$ and $y = F_Y^{-1}(u)$, which yields

$$\lim_{(x,y)\to\infty} \frac{f(x, y)}{f_X(x) f_Y(y)} = \lim_{u \to 1} c_\rho(u, u). \quad (15)$$

Using equation (10), it is now obvious that $c_\rho(u, u)$ goes to one when u goes to one, if and only if $\rho = 0$, which is equivalent to $C_{\rho=0}(u_1, u_2) = \Pi(u_1, u_2)$ for every (u_1, u_2). When $\rho > 0$, $c_\rho(u, u)$ goes to infinity, while for ρ negative, $c_\rho(u, u)$ goes to zero as $u \to 1$. Thus, the dependence structure described by the Gaussian copula is very different from the dependence structure of the independent copula, except for $\rho = 0$.

The Gaussian copula is completely determined by the knowledge of the correlation matrix ρ. The parameters involved in the description of the Gaussian copula are very simple to estimate, as we shall see in the following.

In our tests presented below, we focus on pairs of assets, i.e., on Gaussian copulas involving only two random variables. Obviously, for risk management purposes, baskets or portfolios of $n > 2$ assets must be considered. Our restriction to pairs of assets is not crucial since the testing procedure exposed in section 3 can be applied to any number of assets and it is only for the simplicity of the exposition that we will present the case of pairs of assets. Moreover, testing the Gaussian copula hypothesis for two random variables is not restrictive because it gives useful information for a larger number of dependent variables constituting a large basket or portfolio. Indeed, let us assume that each pair (a, b), (b, c), and (c, a) has a Gaussian copula, and in addition that the copula of the triplet (a, b, c) is elliptical, which is a reasonable assumption. Then, the triplet (a, b, c) also has a Gaussian copula. This result generalizes to an arbitrary number of random variables[7].

2.4. The Student copula

The Student copula is derived from the Student multivariate distribution. Given a multivariate Student distribution $T_{\rho,\nu}$ with ν degrees of freedom and a shape[8] matrix ρ:

$$T_{\rho,\nu}(\boldsymbol{x}) = \frac{1}{\sqrt{\det \rho}} \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})(\pi\nu)^{n/2}} \times \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{\mathrm{d}\boldsymbol{x}}{(1 + \frac{x^t \rho^{-1} x}{\nu})^{\frac{\nu+n}{2}}}, \quad (16)$$

the corresponding Student copula reads

$$C_{\rho,\nu}(u_1, \dots, u_n) = T_{\rho,\nu}(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_n)), \quad (17)$$

[7] An elliptical distribution (and an elliptical copula) is fully determined by the knowledge of its mean, its shape (or covariance) matrix, and the generator of its type. Once the distributions of every pair of random variables (X_i, X_j), $i, j \in \{1, \dots, N\}$, are known, the type of the generator is fixed and the mean and the shape matrix of the joined distribution of $(X_1, X_2, \dots, X_N)$ can be reconstructed.

[8] The shape matrix ρ is proportional to the correlation matrix when ν is larger than two, namely when the second moments of the variables X_i exist.

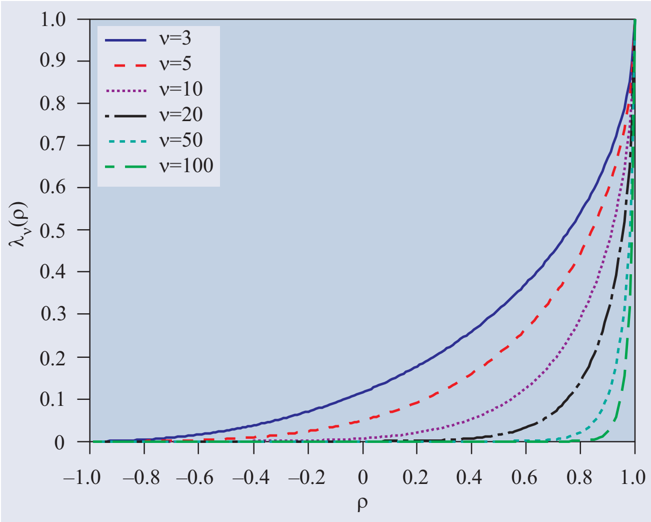

Figure 1. The upper tail dependence coefficient $\lambda_\nu(\rho)$ for the Student copula with ν degrees of freedom as a function of the correlation coefficient ρ, for different values of ν.

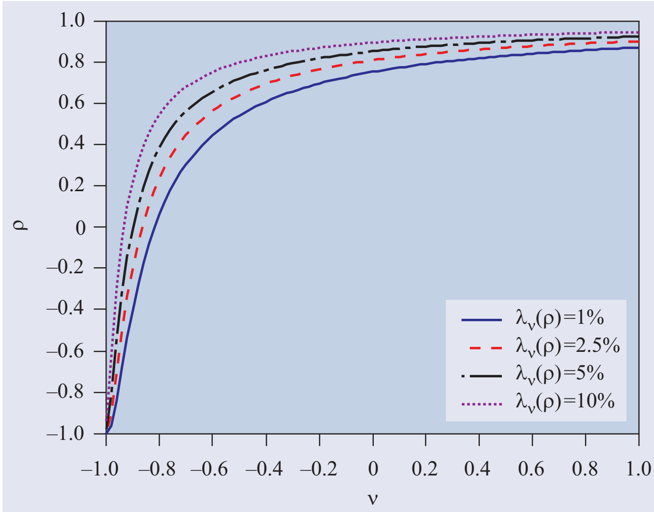

Figure 2. The maximum value of the correlation coefficient ρ as a function of ν, below which the tail dependence $\lambda_\nu(\rho)$ of a Student copula is smaller than a given small value, here taken equal to $\lambda_\nu(\rho) = 1, 2.5, 5$ and 10%. The choice $\lambda_\nu(\rho) = 5\%$ for instance corresponds to 1 event in 20 for which the pair of variables are asymptotically coupled. At the $1 - \lambda_\nu(\rho)$ probability level, values of $\lambda \leqslant \lambda_\nu(\rho)$ are indistinguishable from 0, which means that the Student copula can be approximated by a Gaussian copula.

where t_ν is the univariate Student distribution with ν degrees of freedom. The density of the Student copula is thus

$$c_{\rho,\nu}(u_1, \cdots, u_n) = \frac{1}{\sqrt{\det \rho}} \frac{\Gamma(\frac{\nu+n}{2})[\Gamma(\frac{\nu}{2})]^{n-1}}{[\Gamma(\frac{\nu+1}{2})]^n} \times \frac{\prod_{k=1}^{n}(1+\frac{y_k^2}{\nu})^{\frac{\nu+1}{2}}}{(1+\frac{y^t\rho^{-1}y}{\nu})^{\frac{\nu+n}{2}}}, \quad (18)$$

where $y_k = t_\nu^{-1}(u_k)$.

Since the Student distribution tends to the normal distribution when ν goes to infinity, the Student copula tends to the Gaussian copula as $\nu \to +\infty$. In contrast to the Gaussian copula, the Student copula for ν finite presents a tail dependence given by

$$\lambda_\nu(\rho) = \lim_{u\to 1} \frac{\bar{C}_{\rho,\nu}(u,u)}{1-u} = 2\bar{t}_{\nu+1}\left(\frac{\sqrt{\nu+1}\sqrt{1-\rho}}{\sqrt{1+\rho}}\right), \quad (19)$$

where $\bar{t}_{\nu+1}$ is the complementary cumulative univariate Student distribution with $\nu+1$ degrees of freedom (see Embrechts *et al* (2001) for the proof). Figure 1 shows the upper tail dependence coefficient as a function of the correlation coefficient ρ for different values of the number ν of degrees of freedom. As expected from the fact that the Student copula becomes identical to the Gaussian copula for $\nu \to +\infty$ for all $\rho \neq 1$, $\lambda_\nu(\rho)$ exhibits a regular decay to zero as ν increases. Moreover, for ν sufficiently large, the tail dependence is significantly different from 0 only when the correlation coefficient is sufficiently close to 1. This suggests that, for moderate values of the correlation coefficient, a Student copula with a large number of degrees of freedom may be difficult to distinguish from the Gaussian copula from a statistical point of view. This statement will be made quantitative in the following.

Figure 2 presents the same information in a different way by showing the maximum value of the correlation coefficient ρ as a function of ν, below which the tail dependence $\lambda_\nu(\rho)$ of a Student copula is smaller than a given small value, here taken equal to 1, 2.5, 5 and 10%. The choice $\lambda_\nu(\rho) = 5\%$ for instance corresponds to 1 event in 20 for which the pair of variables are asymptotically coupled. At the 95% probability level, values of $\lambda_\nu(\rho) \leqslant 5\%$ are indistinguishable from 0, which means that the Student copula can be approximated by a Gaussian copula.

The description of a Student copula relies on two parameters: the correlation matrix ρ, as in the Gaussian case, and in addition the number of degrees of freedom ν. The estimation of the parameter ν is rather difficult and this has an important impact on the estimated value of the correlation matrix. As a consequence, the Student copula is more difficult to calibrate and use than the Gaussian copula.

3. Testing the Gaussian copula hypothesis

In view of the central role that the Gaussian paradigm has played and still plays in particular in finance, it is natural to start with the simplest choice of dependence between different random variables, namely the Gaussian copula. It is also a natural first step as the Gaussian copula imposes itself in an approach which consists in (1) performing a non-linear transformation on the random variables into normal random variables (for the marginals) which is always possible and (2) invoking a maximum entropy principle (which amounts to adding the least additional information in the Shannon sense) to construct the multivariable distribution of these Gaussianized random variables (Sornette *et al* 2000a, 2000b, Andersen and Sornette 2001).

In the following, we will denote by H_0 the null hypothesis according to which the dependence between two (or more) random variables X and Y can be described by the Gaussian copula.

3.1. Test statistics

We now derive the test statistics which will allow us to reject or not our null hypothesis H_0 and state the following proposition:

Proposition 1. *Assuming that the N-dimensional random vector $\boldsymbol{x} = (x_1, \ldots, x_N)$, with distribution function F and marginals F_i, satisfies the null hypothesis H_0, then, the variable*

$$z^2 = \sum_{j,i=1}^{N} \Phi^{-1}(F_i(x_i))(\rho^{-1})_{ij}\Phi^{-1}(F_j(x_j)), \tag{20}$$

where the matrix ρ is

$$\rho_{ij} = \mathrm{cov}[\Phi^{-1}(F_i(x_i)), \Phi^{-1}(F_j(x_j))], \tag{21}$$

follows a χ^2-distribution with N degrees of freedom.

To prove the proposition above, first consider an N-dimensional random vector $\boldsymbol{x} = (x_1, \ldots, x_N)$. Let us denote by F its distribution function and by F_i the marginal distribution of each x_i. Let us now assume that the distribution function F satisfies H_0, so that F has a Gaussian copula with correlation matrix ρ while the F_i can be any distribution function. According to theorem 1, the distribution F can be represented as

$$F(x_1, \ldots, x_N) = \Phi_{\rho,N}(\Phi^{-1}(F_1(x_1)), \ldots, \Phi^{-1}(F_N(x_N))). \tag{22}$$

Let us now transform the x_i into normal random variables y_i:

$$y_i = \Phi^{-1}(F_i(x_i)). \tag{23}$$

Since the mapping $\Phi^{-1}(F_i(\cdot))$ is obviously increasing, theorem 2 allows us to conclude that the copula of the variables y_i is identical to the copula of the variables x_i. Therefore, the variables y_i have normal marginal distributions and a Gaussian copula with correlation matrix ρ. Thus, by definition, the multivariate distribution of the y_i is the multivariate Gaussian distribution with correlation matrix ρ:

$$G(\boldsymbol{y}) = \Phi_{\rho,N}(\Phi^{-1}(F_1(x_1)), \ldots, \Phi^{-1}(F_N(x_N))) \tag{24}$$
$$= \Phi_{\rho,N}(y_1, \ldots, y_N), \tag{25}$$

and $\boldsymbol{y}$ is a Gaussian random vector. From equations (24), (25), we obviously have

$$\rho_{ij} = \mathrm{cov}[\Phi^{-1}(F_i(x_i)), \Phi^{-1}(F_j(x_j))]. \tag{26}$$

Consider now the random variable

$$z^2 = \boldsymbol{y}^{\mathrm{t}}\rho^{-1}\boldsymbol{y} = \sum_{i,j=1}^{N} y_i(\rho^{-1})_{ij}y_j, \tag{27}$$

where $\cdot^{\mathrm{t}}$ denotes the transpose operator. It is well known that the variable z^2 follows a χ^2-distribution with N degrees of freedom. Indeed, since $\boldsymbol{y}$ is a Gaussian random vector with covariance matrix[9] ρ, it follows that the components of the vector

$$\tilde{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y}, \tag{28}$$

are *independent* normal random variables. Here, $\boldsymbol{A}$ denotes the square root of the matrix ρ^{-1}, obtained by the Cholevsky decomposition, so $\boldsymbol{A}^{\mathrm{t}}\boldsymbol{A} = \rho^{-1}$. Thus, the sum $\tilde{\boldsymbol{y}}^{\mathrm{t}}\tilde{\boldsymbol{y}} = z^2$ is the sum of the squares of N independent normal random variables, which follows a χ^2-distribution with N degrees of freedom.

3.2. Testing procedure

The testing procedure used in the following is now described. We consider two[10] financial series ($N = 2$) of size T: $\{x_1(1), \ldots, x_1(t), \ldots, x_1(T)\}$ and $\{x_2(1), \ldots, x_2(t), \ldots, x_2(T)\}$. We assume that the vectors $\boldsymbol{x}(t) = (x_1(t), x_2(t))$, $t \in \{1, \ldots, T\}$, are independent and identically distributed with distribution F, which implies that the variables $x_1(t)$ (or $x_2(t)$), $t \in \{1, \ldots, T\}$, are also independent and identically distributed, with distributions F_1 (or F_2)[11].

The cumulative distribution $\hat{F}_i$ of each variable x_i, which is estimated empirically, is given by

$$\hat{F}_i(x_i) = \frac{1}{T}\sum_{k=1}^{T} \mathbf{1}_{\{x_i(k)\leqslant x_i\}}, \tag{29}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function, which equals one if its argument is true and zero otherwise. We use these estimated cumulative distributions to obtain the Gaussian variables $\hat{y}_i$ as

$$\hat{y}_i(k) = \Phi^{-1}(\hat{F}_i(x_i(k))), \qquad k \in \{1, \ldots, T\}. \tag{30}$$

The sample covariance matrix $\hat{\rho}$ is estimated by the expression

$$\hat{\rho} = \frac{1}{T}\sum_{i=1}^{T} \hat{\boldsymbol{y}}(i) \cdot \hat{\boldsymbol{y}}(i)^{\mathrm{t}} \tag{31}$$

which allows us to calculate the variable

$$\hat{z}^2(k) = \sum_{i,j=1}^{2} \hat{y}_i(k)(\hat{\rho}^{-1})_{ij}\hat{y}_j(k), \tag{32}$$

[9] Up to now, the matrix ρ was named the *correlation matrix*. But in fact, since the variables y_i have unit variance, their correlation matrix is also their *covariance matrix*.

[10] As explained in section 2.3, the case $N = 2$ is not restrictive at all, even if it could, *a priori*, appear of limited interest. For portfolio analysis and risk management purposes, a larger basket of assets should be considered. However, the testing procedure exposed here can be applied to any number of assets, and it is only for the sake of simplicity of the exposition that we have restricted our investigation to the bivariate case.

[11] The assumption of independently distributed data is not very realistic. Indeed, it is well known that, while daily returns are uncorrelated, their volatilities exhibit long-range dependence. It would thus seem natural to first filter the data using an ARCH or GARCH process (as in Patton (2001)) and then to apply our testing procedure to the residuals. However, this filtering approach has an important drawback as it does not let the dependence structure, i.e., the copula, through unchanged. Thus, the copula of the residuals is not the same as the copula of the raw returns. Moreover, the copula of the residuals changes with the chosen filter. Residuals are not the same when one filters the data with an ARCH, a GARCH, or a multifractal random walk (Muzy *et al* 2000, 2001). We have chosen to perform a model-free analysis, keeping the initial spirit of the copula description, and thus we have not filtered the data. The price to pay for such a model-free approach is a weakening of the power of the statistical test due to the presence of (temporal) dependence between data.

as defined in (27) for $k \in \{1, \dots, T\}$, which should be distributed according to a χ^2-distribution if the Gaussian copula hypothesis is correct.

The usual way of comparing an empirical with a theoretical distribution is to measure the distance between these two distributions and to perform the Kolmogorov test or the Anderson–Darling (Anderson and Darling 1952) test (for a better accuracy in the tails of the distribution). The Kolmogorov distance is the maximum local distance along the quantile which most often occurs in the bulk of the distribution, while the Anderson–Darling distance puts the emphasis on the tails of the two distributions by a suitable normalization. We propose to complement these two distances by two additional measures which are defined as averages of the Kolmogorov distance and of the Anderson–Darling distance respectively:

$$\text{Kolmogorov: } d_1 = \max_z |F_{z^2}(z^2) - F_{\chi^2}(z^2)|; \tag{33}$$

average Kolmogorov:

$$d_2 = \int |F_{z^2}(z^2) - F_{\chi^2}(z^2)| \, \mathrm{d}F_{\chi^2}(z^2); \tag{34}$$

$$\text{Anderson–Darling: } d_3 = \max_z \frac{|F_{z^2}(z^2) - F_{\chi^2}(z^2)|}{\sqrt{F_{\chi^2}(z^2)[1 - F_{\chi^2}(z^2)]}}; \tag{35}$$

average Anderson–Darling:

$$d_4 = \int \frac{|F_{z^2}(z^2) - F_{\chi^2}(z^2)|}{\sqrt{F_{\chi^2}(z^2)[1 - F_{\chi^2}(z^2)]}} \, \mathrm{d}F_{\chi^2}(z^2). \tag{36}$$

The Kolmogorov distance d_1 and its average d_2 are more sensitive to the deviations occurring in the bulk of the distributions. In contrast, the Anderson–Darling distance d_3 and its average d_4 are more accurate in the tails of the distributions. We present our statistical tests for these four distances in order to be as complete as possible with respect to the different sensitivities of the tests.

The distances d_2 and d_4 are not in common use in statistics, so let us justify our choice. One usually uses distances similar to d_2 and d_4 but which differ by the square instead of the modulus of $F_{z^2}(z^2) - F_{\chi^2}(z^2)$ and lead respectively to the ω-test and the Ω-test, whose statistics are theoretically known. The main advantage of the distances d_2 and d_4 with respect to the more usual distances ω and Ω is that they are simply equal to the average of d_1 and d_3. This averaging is very interesting and provides important information. Indeed, the distances d_1 and d_3 are mainly controlled by the point that maximizes the argument within the $\max(\cdot)$ function. They are thus sensitive to the presence of an outlier. On averaging, d_2 and d_4 become less sensitive to outliers, since the weight of such points is only of order $1/T$ (where T is the size of the sample) while it equals one for d_1 and d_3. Of course, the distances ω and Ω also perform a smoothing since they are averaged quantities too. However, as already said, they are averages of the square of d_1 and d_3, and taking the square can lead to the undesired overweighting of the largest events. Of course, such an overweighting of large events can be interesting when one wants to particularly focus on tail events. In fact, a trade-off between the sensitivity to (desired) tail events and to (undesired) outliers must be found. In our opinion, the distances d_2 and d_4 provide us with more convenient trade-offs compared with the more traditional omega distances. Recall that the omega distances based on the average of a square function are chosen because they allow one to derive explicitly the theoretical asymptotic statistics for the ω- and Ω-tests. In contrast, using the modulus of $F_{z^2}(z^2) - F_{\chi^2}(z^2)$ instead of its square in the expressions for d_2 and d_4, no theoretical test statistics can be derived analytically. One can say that the main advantage of the standard distances ω and Ω with respect to the distances d_2 and d_4 introduced here is the theoretical knowledge of their distributions. However, this advantage disappears in our present case in which the covariance matrix is not known *a priori* and needs to be estimated from the empirical data: indeed, the exact knowledge of all the parameters is necessary in the derivation of the theoretical statistics of the ω- and Ω-tests (as well as the Kolmogorov test). Therefore, we cannot directly use the results of these standard statistical tests. As a remedy, we propose a bootstrap method (Efron and Tibshirani 1986), whose accuracy is proved by Chen and Lo (1997) to be at least as good as that given by asymptotic methods used to derive the theoretical distributions. For the present work, we have determined that the generation of 10 000 synthetic time series was sufficient to obtain a good approximation of the distribution of distances described above. Since a bootstrap method is needed to determine the test statistics in every case, it is convenient to choose functional forms different from the usual ones in the ω- and Ω-tests as they provide an improvement with respect to statistical reliability, as obtained with the distances d_2 and d_4 introduced here.

To summarize, our test procedure is as follows.

(1) Given the original time series $\boldsymbol{x}(t)$, $t \in \{1, \dots, T\}$, we generate the Gaussian variables $\hat{\boldsymbol{y}}(t)$, $t \in \{1, \dots, T\}$.

(2) We then estimate the covariance matrix $\hat{\rho}$ of the Gaussian variables $\hat{\boldsymbol{y}}$, which allows us to compute the variables $\hat{z}^2$ and then measure the distance of its estimated distribution to the χ^2-distribution.

(3) Given this covariance matrix $\hat{\rho}$, we generate numerically a time series of T Gaussian random vectors with the same covariance matrix $\hat{\rho}$.

(4) For the time series of Gaussian vectors synthetically generated with covariance matrix $\hat{\rho}$, we estimate its sample covariance matrix $\tilde{\rho}$.

(5) With each of the T vectors of the synthetic Gaussian time series, we associate the corresponding realization of the random variable z^2, called $\tilde{z}^2(t)$.

(6) We can then construct the empirical distribution for the variable $\tilde{z}^2$ and measure the distance between this distribution and the χ^2-distribution.

(7) Repeating the steps (3)–(6) 10 000 times, we obtain an accurate estimate of the cumulative distribution of distances between the distribution of the synthetic Gaussian variables and the theoretical χ^2-distribution. This cumulative distribution represents the test statistic, which will allow us to reject or not the null hypothesis H_0 at a given significance level.

(8) The significance of the distance obtained at step (2) for the true variables—i.e., the probability of observing, at random and under H_0, a distance larger than the empirically estimated distance—is finally obtained by a simple reading of the complementary cumulative distribution estimated at step (7).

3.3. Sensitivity of the method

Before presenting the statistical tests, it is important to investigate the sensitivity of our testing procedure. More precisely, can we distinguish for instance between a Gaussian copula and a Student copula with a large number of degrees of freedom, for a given value of the correlation coefficient? Formally, denoting by H_ν the hypothesis according to which the true copula of the data is the Student copula with ν degrees of freedom, we want to determine the minimum significance level allowing us to distinguish between H_0 and H_ν.

3.3.1. Importance of the distinction between Gaussian and Student copulas. This question has important practical implications because, as discussed in section 2.4, the Student copula presents a significant tail dependence while the Gaussian copula has no asymptotic tail dependence. Therefore, if our tests are unable to distinguish between a Student and a Gaussian copula, we may be led to choose the latter for the sake of simplicity and parsimony and, as a consequence, we may underestimate severely the dependence between extreme events if the correct description turns out to be the Student copula. This may have catastrophic consequences in risk assessment and portfolio management.

Figure 1 provides a quantification of the dangers incurred by mistaking a Student copula for a Gaussian one. Consider the case of a Student copula with $\nu = 20$ degrees of freedom with a correlation coefficient ρ lower than 0.3–0.4; its tail dependence $\lambda_\nu(\rho)$ turns out to be less than 0.7%, i.e., the probability that one variable becomes extreme knowing that the other one is extreme is less than 0.7%. In this case. the Gaussian copula with zero probability of simultaneous extreme events is not a bad approximation of the Student copula. In contrast, let us take a correlation ρ larger than 0.7–0.8 for which the tail dependence becomes larger than 10%, corresponding to a non-negligible probability of simultaneous extreme events. The effect of tail dependence becomes of course much stronger as the number ν of degrees of freedom decreases.

These examples stress the importance of knowing whether our testing procedure allows us to distinguish between a Student copula with $\nu = 20$ (or less) degrees of freedom and a given correlation coefficient $\rho = 0.5$, for instance, and a Gaussian copula with an appropriate correlation coefficient ρ'.

3.3.2. Statistical test on the distinction between Gaussian and Student copulas. To address this question, we have generated 1000 pairs of time series of size $T = 1250$, each pair of random variables following a Student bivariate distribution with ν degrees of freedom and a coefficient of correlation between the two simultaneous variables of the same pair ρ, while the variables along the time axis are all independent. We then applied the previous testing procedure to each of the pairs of time series.

Specifically, for each pair of Student time series, we construct the marginal distributions and transform the Student variables $x_i(k)$ into their Gaussian counterparts $y_i(k)$ via the transformation (23). For each pair $(y_1(k), y_2(k))$, $k \in \{1, \ldots, T\}$, we estimate its correlation matrix, then construct the time series with T realizations of the random variable $z^2(k)$ defined in (27). The set of T variables z^2 then allows us to construct the distribution of z^2 (with $N = 2$) and to compare it with the χ^2-distribution with two degrees of freedom. We then measure the distances d_1, d_2, d_3 and d_4 defined by (33)–(36) between the distribution of z^2 and the χ^2-distribution.

The significance p_i of these distances d_i is calculated by generating 1000 Gaussian time series with a correlation matrix equal to the correlation matrix estimated from the original Student time series, according to the steps (3)–(8) of the testing procedure described in section 3.2. Given a Student time series with distance d_i, the significance of this distance is

$$p_i = \frac{1}{1000} \sum_{k=1}^{1000} 1_{\{d_i(z^2(y^{(k)}), \chi^2) > d_i\}}, \tag{37}$$

where $y^{(k)} = (y_t^{(k)})_{1 \leqslant t \leqslant T}$ denotes the kth replication of a bivariate Gaussian time series of length T and correlation coefficient equal to the correlation coefficient estimated from the original Student time series.

Repeating this protocol 1000 times for the Student time series with the same ν and ρ, we then construct the cumulative distribution function $D_i(p)$, $i \in \{1, 2, 3, 4\}$, of the significance p obtained for each of the four distances d_1, d_2, d_3 and d_4. It thus allows us to get the minimum significance level p such that we can discriminate a Student copula with ν degrees of freedom and correlation coefficient ρ from a Gaussian copula with the same correlation coefficient, at the confidence level $D_i(p)$, according to the test based upon distance d_i. For instance, the minimum significance level such that we can discriminate a Student copula with ν degrees of freedom from a Gaussian copula with the same correlation coefficient, at the α-confidence level, according to distance d_i, is given by $D_i(p_\alpha) = \alpha$. A small value of p_α corresponds to a clear distinction between Student and Gaussian vectors, at the α-confidence level, as it is improbable that Gaussian vectors exhibit a distance larger than found for the Student vectors.

The cumulative distributions $D_i(p)$ for each of the four distances d_i, $i \in \{1, 2, 3, 4\}$, are shown in figure 3 for $\nu = 4$ degrees of freedom and in figure 4 for $\nu = 20$ degrees of freedom, for five different values of the correlation coefficient $\rho = 0.1$, 0.3, 0.5, 0.7, and 0.9. The very steep increase observed for almost all cases in figure 3 reflects the fact that most of the 1000 Student vectors with $\nu = 4$ degrees of freedom have a small p, i.e., their copula is easily distinguishable from the Gaussian copula. The same cannot be stated for Student vectors with $\nu = 20$ degrees of freedom. Note also that the distances d_1, d_2, and d_4 give essentially the same result while the Anderson–Darling distance d_3 is more sensitive to ρ, especially for small ν.

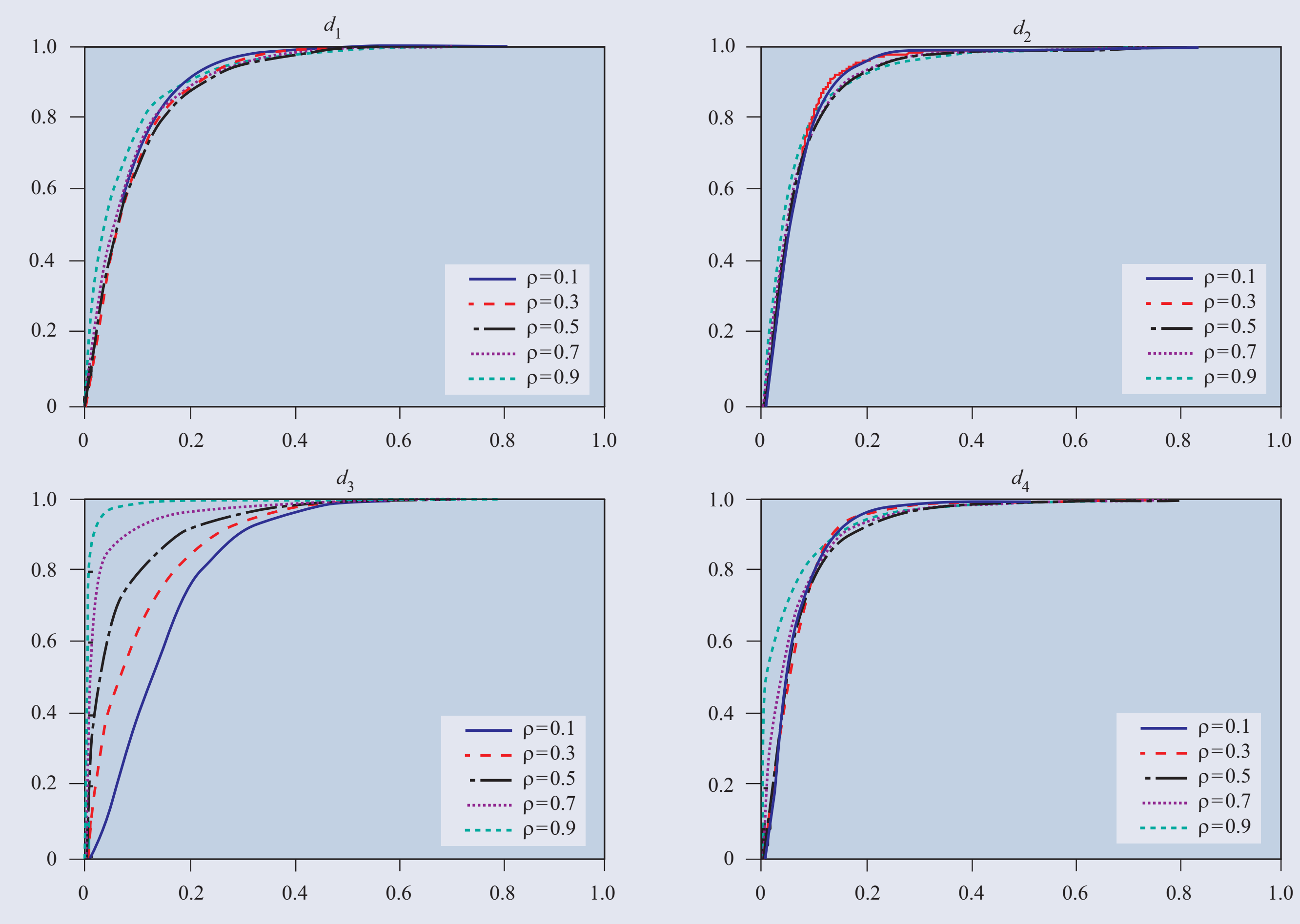

Figure 3. The cumulative distribution function $D(p)$ obtained as the fraction of Student pairs with $\nu = 4$ degrees of freedom that exhibit a value of at least p for the probability that Gaussian vectors can have a similar or larger distance. See the text for a detailed description of how $D(p)$ is defined and constructed. Each panel corresponds to one of the four distances d_i, $i \in \{1, 2, 3, 4\}$, defined in the text by equations (33)–(36). In each panel, we construct the cumulative distribution function $D(p)$ for five different values of the correlation coefficient $\rho = 0.1$, 0.3, 0.5, 0.7 and 0.9 of the Student copula.

Fixing for instance the confidence level at $\alpha = 95\%$, we can read from each of these curves in figures 3 and 4 the minimum $p_{95\%}$-value necessary to distinguish a Student copula with a given ν from a Gaussian copula. This $p_{95\%}$ is the abscissa corresponding to the ordinate $D(p_{95\%}) = 0.95$. These values $p_{95\%}$ are reported in table 1, for different values of the number ν of degrees of freedom ranging from $\nu = 3$ to 50 and correlation coefficients $\rho = 0.1$–0.9. The values of $p_{95\%}(\nu, \rho)$ reported in table 1 are the minimum values that the statistical significance p should take in order to be able to reject the hypothesis that a Student copula with ν degrees and correlation ρ can be mistaken for a Gaussian copula at the 95% confidence level.

The results of table 1 are also depicted in figures 5 and 6 and represent the 'power' of the test. The statistical power is usually defined as the probability of rejection of the null hypothesis when false. Here, we have not exactly represented the conventional statistical power of the test, but, more precisely, the minimum significance level allowing us to discriminate between H_0 (the Gaussian copula) and the alternative hypothesis $H_{(\nu^{-1})}$ (Student copula with ν degrees of freedom).

The abscissa in figures 5 and 6 gives the inverse ν^{-1} of the number ν of degrees of freedom, which provides a natural 'distance' between the Gaussian copula hypothesis $H_0 = H_{(\nu^{-1}=0)}$ and the Student copula hypothesis $H_{(\nu^{-1})}$. The typical shape of these curves is a sigmoid, starting from a value very close to one for $\nu^{-1} \to 0$, decreasing as ν^{-1} increases, and going to 0 as ν^{-1} becomes large enough. This typical shape simply expresses the fact that it is easy to separate a Gaussian copula from a Student copula with a small number of degrees of freedom, while it is difficult and even impossible for too large a number of degrees of freedom.

Figure 5 shows us that the distances d_1, d_2 and d_3 are not sensitive to the value of the correlation coefficient ρ, while the discriminating power of d_3 increases with ρ. In figure 6, we note that d_2 and d_4 have the same discriminating power for all ρ (which makes them somewhat redundant) and that they are the most efficient for differentiating H_ν from H_0 for small ρ. When ρ is about 0.5, d_2, d_3 and d_4 (and maybe d_1) are equivalent with respect to the differential power, while for large ρ, d_3 becomes the most discriminating with high significance.

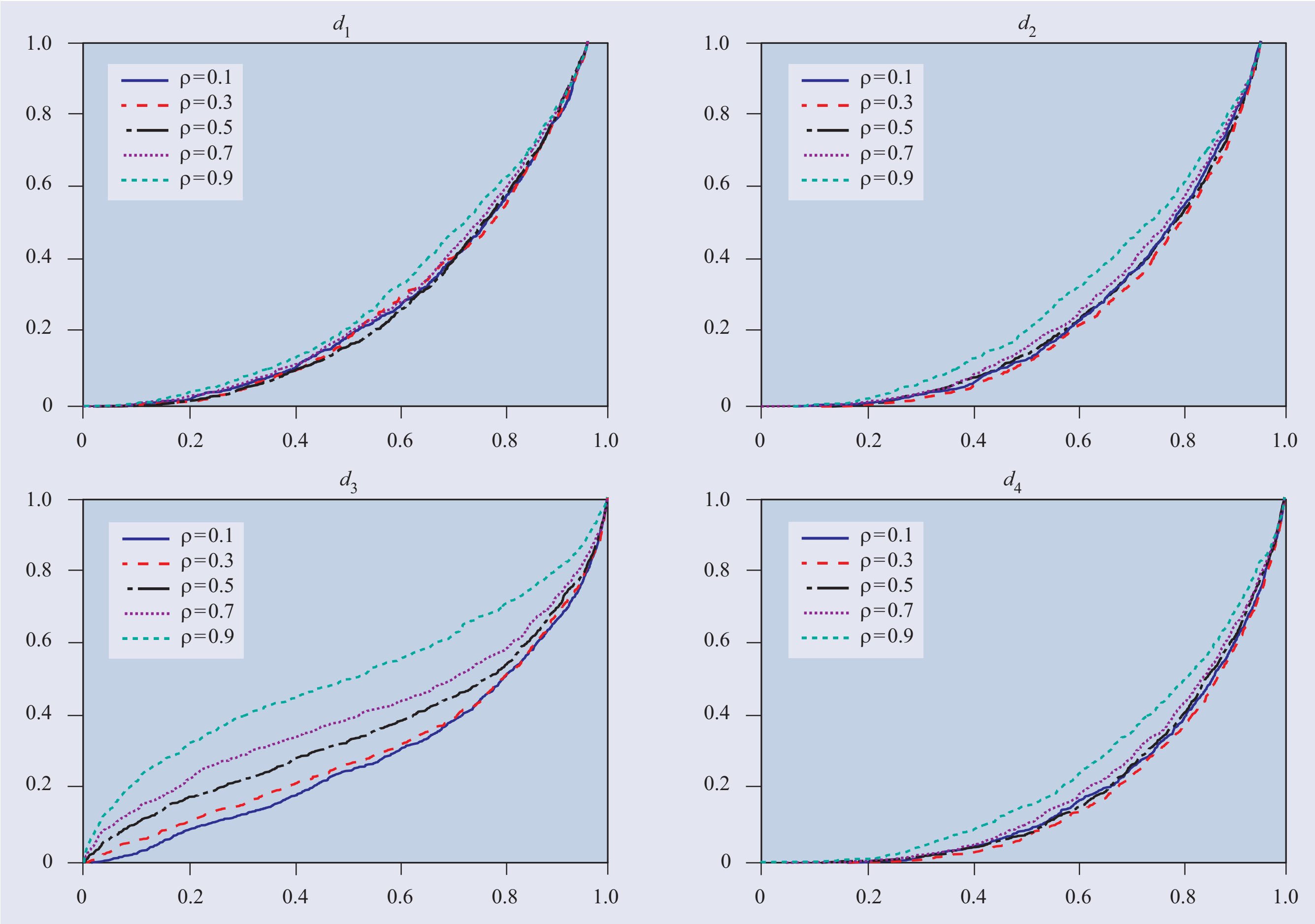

Figure 4. As figure 3, but for Student distributions with $\nu = 20$ degrees of freedom.

This study of the test sensitivity involves a non-parametric approach and the question may arise of why it should be preferred to a direct parametric test involving for instance the calibration of the Student copula. First, a parametric test of copulas would face the 'curse of dimensionality', i.e., the estimation of functions of several variables. With the limited data set available, this does not seem a reasonable approach. Second, we have taken the Student copula as an example of an alternative to the Gaussian copula. However, our tests are independent of this choice and aim mainly at testing the rejection of the Gaussian copula hypothesis. They are thus of a more general nature than would be a parametric test which would be forced to choose one family of copulas with the problem of excluding others. The parametric test would then be exposed to the criticism that the rejection of a given choice might not be of a general nature.

In the following, we will choose the level of 95% as the level of rejection, which leads us to neglect one extreme event out of twenty. This is not unreasonable in view of the other significant sources of errors resulting in particular from the empirical determination of the marginals and from the presence of outliers for instance.

4. Empirical results

We investigate the following assets:

- foreign exchange rates,
- metals traded on the London Metal Exchange,
- stocks traded on the New York Stock Exchange.

4.1. Currencies

The sample that we have considered is made up of the daily returns for the spot foreign exchanges for six currencies[12]: the Swiss franc (CHF), the German mark (DEM), the Japanese yen (JPY), the Malaysian ringgit (MYR), the Thai baht (THA), and the British pound (UKP). All the exchange rates are expressed against the US dollar. The time interval runs over ten years, from 25 January 1989 to 31 December 1998, so each sample contains 2500 data points.

We apply our test procedure to the entire sample and to two sub-samples of 1250 data points such that the first one covers the time interval from 25 January 1989 to 11 January 1994 and the second one from 12 January 1994 to 31 December 1998. The results are presented in tables 2–4 and depicted in figures 7–9.

[12] The data come from the historical database of the Federal Reserve Board.

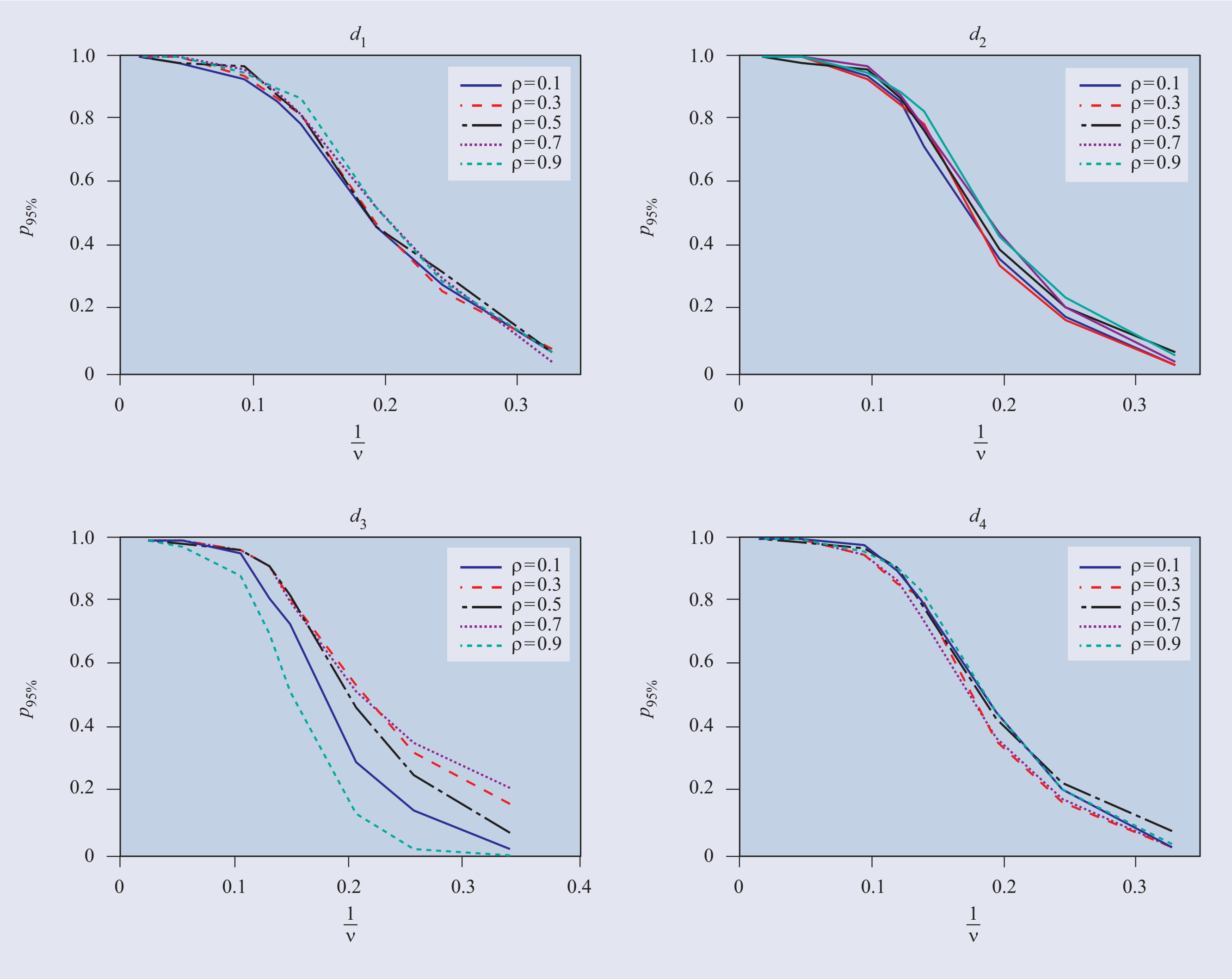

Figure 5. A graph of the minimum significance level $p_{95\%}$ necessary to distinguish the Gaussian copula hypothesis H_0 from the hypothesis of a Student copula with ν degrees of freedom, as a function of $1/\nu$, for a given distance d_i and various correlation coefficients $\rho = 0.1$, 0.3, 0.5, 0.7 and 0.9.

Tables 2–4 give, for the total time interval and for each of the two sub-intervals, the probability $p(d)$ of obtaining from the Gaussian hypothesis a deviation between the distribution of the z^2 and the χ^2-distribution with two degrees of freedom larger than the observed one for each of the 15 pairs of currencies according to the distances d_1–d_4 defined by (33)–(36).

Figures 7–9 organize the information shown in the tables 2–4 by representing, for each distance d_1–d_4, the number of currency pairs that give a test value p within a bin interval of width 0.05. A clustering close to the origin signals a significant rejection of the Gaussian copula hypothesis.

At the 95% significance level, table 2 and figure 7 show that only 40% (according to d_1 and d_3) but 60% (according to d_2 and d_4) of the tested pairs of currencies are compatible with the Gaussian copula hypothesis over the entire time interval. During the first half-period from 25 January 1989 to 11 January 1994 (table 3 and figure 8), 47% (according to d_3) and up to about 75% (according to d_2 and d_4) of the tested currency pairs are compatible with the assumption of Gaussian copula, while during the second sub-period from 12 January 1994 to 31 December 1998 (table 4 and figure 9), between 66% (according to d_1) and about 75% (according to d_2, d_3 and d_4) of the currency pairs remain compatible with the Gaussian copula hypothesis. These results raise several points both from a statistical and from an economic point of view.

We first note that the most significant rejection of the Gaussian copula hypothesis is obtained for the distance d_3, which is indeed the most sensitive to the events in the tail of the distributions. The test statistics given by this distance can indeed be very sensitive to the presence of a single large event in the sample, so much so that the Gaussian copula hypothesis can be rejected only because of the presence of this single event (outlier). The difference between the results given by d_3 and d_4 (the averaged d_3) are very significant in this respect. Consider for instance the case of the German mark and the Swiss franc. During the time interval from 12 January 1994 to 31 December 1998, we check in table 4 that the non-rejection probability $p(d)$ is very significant according to d_1, d_2 and d_4 ($p(d) \geqslant 31\%$) while it is very low according to d_3: $p(d) = 0.05\%$, and should lead to the rejection of the Gaussian

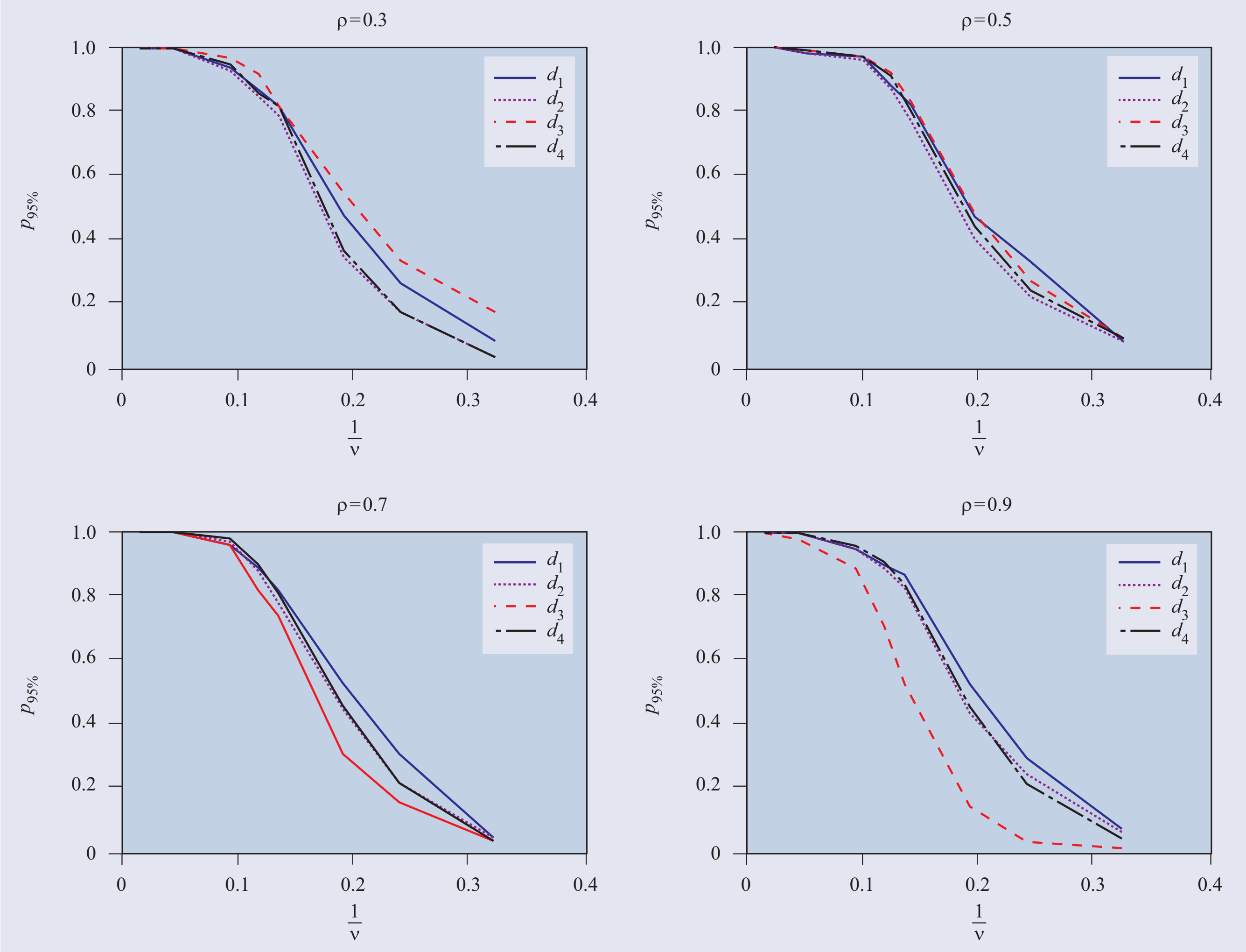

Figure 6. As figure 5, but comparing different distances for the same correlation coefficient ρ.

copula hypothesis. This suggests the presence of an outlier in the sample.

To check this hypothesis, we show in the upper panel of figure 10 the function

$$f_3(t) = \frac{|F_{z^2}(z^2(t) - F_{\chi^2}(\chi^2(t))|}{\sqrt{F_{\chi^2}(\chi^2)[1 - F_{\chi^2}(\chi^2)]}}, \tag{38}$$

used in the definition of the Anderson–Darling distance $d_3 = \max_z f_3(z)$ (see definition (35)), expressed in terms of time t rather than z^2. The functions have been computed over the two time sub-intervals separately.

Apart from three extreme peaks occurring on 20 June 1989, 19 August 1991 and 16 September 1992 during the first time sub-interval and one extreme peak on 10 September 1997 during the second time sub-interval, the statistical fluctuations measured by $f_3(t)$ remain small and of the same order. Excluding the contribution of these outlier events to d_3, the new statistical significance derived according to d_3 becomes similar to those obtained with d_1, d_2, and d_4 on each sub-interval. From the upper panel of figure 10, it is clear that the Anderson–Darling distance d_3 is equal to the height of the largest peak corresponding to the event on 19 August 1991 for the first period and to the event on 10 September 1997 for the second period. These events are depicted by a circled dot in the two lower panels of figure 10, which represent the return of the German mark versus the return of the Swiss franc over the two time periods considered.

The event on 19 August 1991 is associated with the coup against Gorbachev in Moscow: the German mark (the Swiss franc) lost 3.37% (0.74%) in daily annualized value against the US dollar. The 3.37% drop of the German mark is the largest daily move of this currency against the US dollar over the whole first period. On 10 September 1997, the German mark appreciated by 0.60% against the US dollar while the Swiss franc lost 0.79% which represents a moderate move for each currency, but a large joint move. This event is related to the contradictory announcements of the Swiss National Bank about the monetary policy, which put an end to a rally of the Swiss franc along with the German mark against the US dollar.

Thus, neglecting the large moves associated with major historical events or events associated with unexpected incoming information[13], which cannot be taken into account

[13] The outlier nature of the event on 19 August 1991 has been clearly demonstrated by Sornette *et al* (2003).

Table 1. The values $p_{95\%}(\nu, \rho)$ shown in this table give the minimum values that the significance p should take in order to be able to reject the hypothesis that a Student copula with ν degrees and correlation ρ is indistinguishable from a Gaussian copula at the 95% confidence level. $p_{95\%}$ is the abscissa corresponding to the ordinate $D(p_{95\%}) = 0.95$ shown in figures 3 and 4. p is the probability that pairs of Gaussian random variables with the correlation coefficient ρ have a distance (between the distribution of z^2 and the theoretical χ^2-distribution) equal to or larger than the corresponding distance obtained for the Student vector time series. A small p corresponds to a clear distinction between Student and Gaussian vectors, as it is improbable that Gaussian vectors exhibit a distance larger than found for the Student vectors. Different values of the number ν of degrees of freedom ranging from $\nu = 3$ to 50 and of the correlation coefficient $\rho = 0.1$ to 0.9 are shown. Let us take for instance the example with $\nu = 4$ and $\rho = 0.3$. The table indicates that p should be less than about 0.3 (or 0.2) according to the distances d_1 and d_3 (or d_2 and d_4) for being able to distinguish this Student copula from the Gaussian copula at the 95% confidence level. This means that less than 20–30% of Gaussian vectors should have a distance for their z^2 larger than the one found for the Student. See the text for further explanations.

ν	ρ	0.1	0.3	0.5	0.7	0.9
	d_1	0.07	0.08	0.07	0.04	0.07
3	d_2	0.03	0.03	0.07	0.04	0.06
	d_3	0.22	0.17	0.08	0.03	0.01
	d_4	0.03	0.03	0.08	0.03	0.04
	d_1	0.28	0.26	0.32	0.30	0.29
4	d_2	0.18	0.17	0.21	0.21	0.24
	d_3	0.36	0.33	0.26	0.15	0.03
	d_4	0.18	0.17	0.23	0.21	0.21
	d_1	0.46	0.47	0.46	0.52	0.52
5	d_2	0.36	0.34	0.39	0.44	0.43
	d_3	0.52	0.54	0.47	0.30	0.14
	d_4	0.37	0.36	0.43	0.45	0.45
	d_1	0.78	0.81	0.81	0.81	0.86
7	d_2	0.71	0.78	0.76	0.77	0.82
	d_3	0.80	0.81	0.82	0.73	0.52
	d_4	0.75	0.81	0.79	0.80	0.83
	d_1	0.85	0.86	0.87	0.88	0.89
8	d_2	0.85	0.84	0.86	0.87	0.88
	d_3	0.91	0.91	0.91	0.81	0.70
	d_4	0.86	0.85	0.90	0.89	0.90
	d_1	0.92	0.93	0.96	0.95	0.94
10	d_2	0.93	0.92	0.95	0.96	0.94
	d_3	0.96	0.96	0.96	0.95	0.88
	d_4	0.94	0.94	0.96	0.97	0.95
	d_1	0.97	0.99	0.97	0.99	0.99
20	d_2	0.99	0.99	0.97	0.99	0.99
	d_3	0.99	0.99	0.98	0.99	0.97
	d_4	0.99	0.99	0.98	0.99	0.99
	d_1	0.99	0.99	0.99	0.99	0.99
50	d_2	0.99	0.99	0.99	0.99	0.99
	d_3	0.99	0.99	0.99	0.99	0.99
	d_4	0.99	0.99	0.99	0.99	0.99

by a statistical study, we obtain, for d_3, significance levels compatible with those obtained with the other distances. We can thus conclude that, according to the four distances, during the time interval from 12 January 1994 to 31 December 1998, the Gaussian copula hypothesis cannot be rejected for the couple German mark/Swiss franc.

However, the non-rejection of the Gaussian copula hypothesis does not always have minor consequences and may even lead to serious problems in stress scenarios. As shown in section 3.3, the non-rejection of the Gaussian copula hypothesis does not exclude, at the 95% significance level, that the dependence of the currency pairs may be accounted for by a Student copula with adequate values of ν and ρ. Still considering the pair German mark/Swiss franc, we see in table 1 that, according to d_1, d_2 and d_4, a Student copula with about five degrees of freedom allows one to reach the test values given in table 4. But, with the correlation coefficient $\rho = 0.92$ for the German mark/Swiss franc couple, the Gaussian copula assumption could give rise to a neglect of the tail dependence

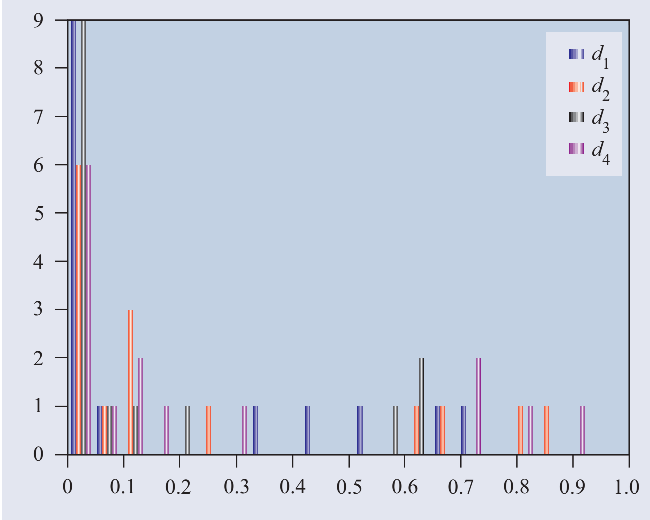

Figure 7. For each distance d_1–d_4 defined in equations (33)–(36), this figure shows the number of currency pairs that give a given p (shown on the abscissa) within a bin interval of width 0.05 for different currencies over a ten-year time interval from 25 January 1989 to 31 December 1998. p is the probability that pairs of Gaussian random variables with the same correlation coefficient ρ have a distance (between the distribution of z^2 and the theoretical χ^2-distribution) equal to or larger than the corresponding distance obtained for each currency pair. A clustering close to the origin signals a significant rejection of the Gaussian copula hypothesis.

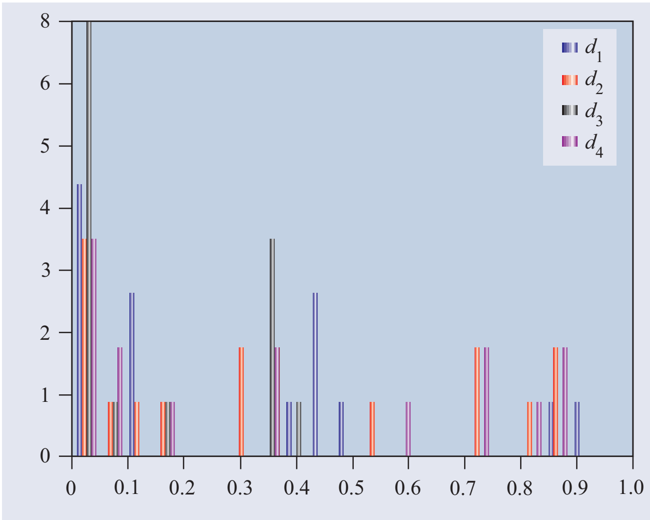

Figure 8. As figure 7, but for currencies over a five-year time interval from 25 January 1989 to 11 January 1994.

Table 2. Each row gives the statistics of our test for each of the 15 pairs of currencies over a ten-year time interval from 25 January 1989 to 31 December 1998. The column $\hat{\rho}$ gives the empirical correlation coefficient for each pair determined as in section 3.1 and defined in (31). The columns d_1, d_2, d_3 and d_4 give the probabilities of obtaining, from the Gaussian hypothesis, a deviation between the distribution of the z^2 and the χ^2-distribution with two degrees of freedom larger than the observed one for the currency pair according to the distances d_1–d_4 defined by (33)–(36).

		$\hat{\rho}$	d_1 (%)	d_2 (%)	d_3 (%)	d_4 (%)
CHF	DEM	0.92	1.01	0.67	0.00	0.72
CHF	JPY	0.53	34.40	27.10	2.32	28.30
CHF	MYR	0.23	72.70	87.10	57.70	92.60
CHF	THA	0.21	3.08	9.47	3.31	9.52
CHF	UKP	0.69	0.28	0.18	0.06	0.13
DEM	JPY	0.54	2.26	13.30	10.00	15.10
DEM	MYR	0.26	42.50	67.70	62.20	73.50
DEM	THA	0.24	6.53	13.50	3.26	13.20
DEM	UKP	0.72	0.17	0.04	0.00	0.04
JPY	MYR	0.31	2.45	6.34	22.60	6.86
JPY	THA	0.34	0.00	0.00	3.24	0.00
JPY	UKP	0.41	2.85	3.72	5.22	3.09
MYR	THA	0.40	0.00	0.00	2.22	0.00
MYR	UKP	0.21	69.40	79.40	62.30	83.10
THA	UKP	0.15	52.20	62.30	3.21	70.50

Table 3. As table 2, but for currencies over a five-year time interval from 25 January 1989 to 11 January 1994.

		$\hat{\rho}$	d_1 (%)	d_2 (%)	d_3 (%)	d_4 (%)
CHF	DEM	0.92	1.73	1.33	0.00	1.31
CHF	JPY	0.55	13.40	14.90	38.30	14.10
CHF	MYR	0.32	84.70	70.00	35.60	74.00
CHF	THA	0.17	44.00	71.00	3.53	71.10
CHF	UKP	0.79	0.31	0.10	0.00	0.05
DEM	JPY	0.56	2.46	9.43	16.30	9.26
DEM	MYR	0.35	93.20	79.50	35.10	79.50
DEM	THA	0.21	43.60	87.70	3.47	87.40
DEM	UKP	0.82	0.00	0.00	0.00	0.00
JPY	MYR	0.34	49.00	54.90	36.60	59.40
JPY	THA	0.27	38.90	30.60	3.37	35.90
JPY	UKP	0.53	0.09	1.66	6.72	1.67
MYR	THA	0.29	10.80	8.71	3.42	9.30
MYR	UKP	0.33	11.20	28.60	35.40	34.50
THA	UKP	0.21	43.40	86.20	3.13	86.70

Table 4. As table 2, but for currencies over a five-year time interval from 12 January 1994 to 31 December 1998.

		$\hat{\rho}$	d_1 (%)	d_2 (%)	d_3 (%)	d_4 (%)
CHF	DEM	0.92	31.50	31.10	0.05	34.10
CHF	JPY	0.52	58.40	64.40	1.98	67.40
CHF	MYR	0.16	71.10	91.50	88.30	92.20
CHF	THA	0.25	1.10	3.87	10.50	3.34
CHF	UKP	0.53	9.75	10.30	23.30	9.29
DEM	JPY	0.53	36.30	54.00	1.77	65.40
DEM	MYR	0.18	35.50	50.00	58.40	56.70
DEM	THA	0.28	1.28	2.18	10.80	1.51
DEM	UKP	0.56	11.50	11.00	30.20	10.60
JPY	MYR	0.29	7.63	21.40	6.67	22.30
JPY	THA	0.38	0.00	0.02	3.09	0.02
JPY	UKP	0.28	46.20	23.00	12.30	20.70
MYR	THA	0.44	0.05	0.12	5.34	0.12
MYR	UKP	0.11	59.40	74.40	69.50	78.20
THA	UKP	0.12	1.26	7.66	11.90	6.51

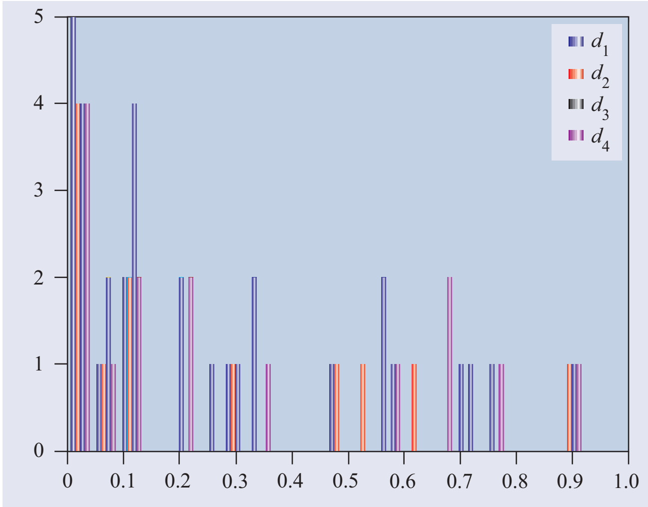

Figure 9. As figure 7, but for currencies over a five-year time interval from 12 January 1994 to December 1998.

coefficient $\lambda_5(0.92) = 63\%$ according to the Student copula prediction. Such a large value of $\lambda_5(0.92)$ means that when an extreme event occurs for the German mark it also occurs for the Swiss franc with a probability equal to 0.63. Therefore, a stress scenario based on a Gaussian copula assumption would fail to account for such coupled extreme events, which may represent as many as two thirds of all the extreme events, if it turned out that the true copula was the Student copula with five degrees of freedom. In fact, with such a value of the correlation coefficient, the tail dependence remains high even if the number of degrees of freedom reaches twenty or more (see figure 1).

The case of the Swiss franc and the Malaysian ringgit offers a striking difference. For instance, in the second half-period, the test statistics $p(d)$ are greater than 70% and even reach 91% while the correlation coefficient is only $\rho = 0.16$, so a Student copula with 7–10 degrees of freedom can be mistaken for the Gaussian copula (see table 1). Even in the most pessimistic situation $\nu = 7$, the choice of the Gaussian copula amounts to neglecting a tail dependence coefficient $\lambda_5(0.16) = 4\%$ predicted by the Student copula. In this case, stress scenarios based on the Gaussian copula would predict uncoupled extreme events, which would be shown wrong only once out of twenty five times.

These two examples show that, more so than the number of degrees of freedom of the Student copula necessary to describe the data, the key parameter is the correlation coefficient.

From an economic point of view, the impact of regulatory mechanisms between currencies or monetary crises can be well identified by the rejection or absence of rejection of our null hypothesis. Indeed, consider the couple German mark/British pound. During the first half-period, their correlation coefficient is very high ($\rho = 0.82$) and the Gaussian copula hypothesis is strongly rejected according to the four distances. In contrast, during the second half-period, the correlation coefficient significantly decreases ($\rho = 0.56$) and none of the four distances allows us to reject our null hypothesis. Such a non-stationarity can be easily explained.

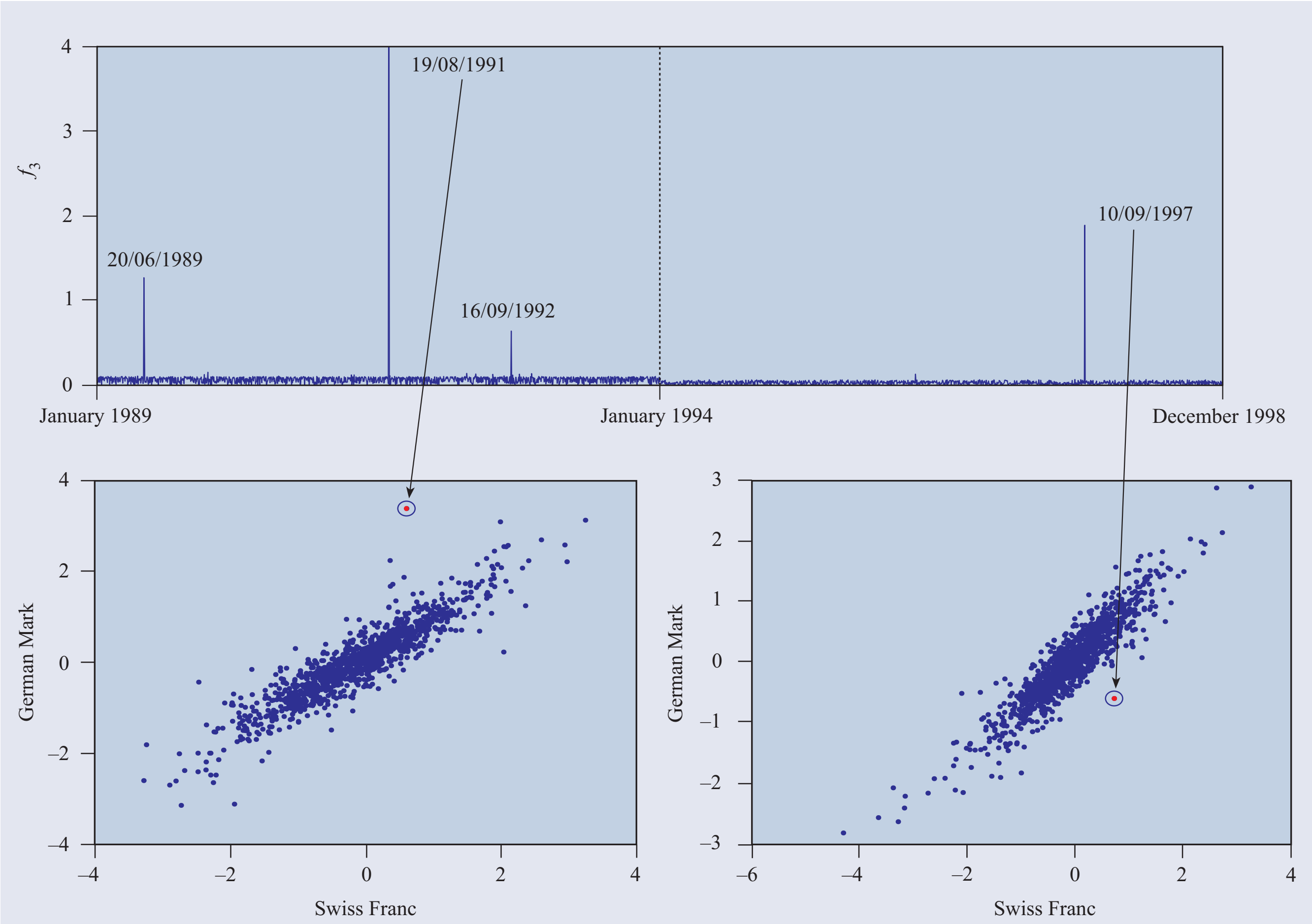

Figure 10. The upper panel represents the graph of the function $f_3(t)$ defined in (38) used in the definition of the distance d_3 for the couple Swiss franc/German mark as a function of time t, over the time intervals from 25 January 1989 to 11 January 1994 and from 12 January 1994 to 31 December 1998. The two lower panels represent the scatter plot of the return of the German mark versus the return of the Swiss franc during the two previous time periods. The circled dot, in each figure, shows the pair of returns responsible for the largest deviation of f_3 during the time interval considered.

Indeed, on 1 January 1990, the British pound entered the European monetary system (EMS), so the rate of exchange between the German mark and the British pound was not allowed to fluctuate beyond a margin of 2.25%. However, due to a strong speculative attack, the British pound was devalued on September 1992 and had to leave the EMS. Thus, between January 1990 and September 1992, the exchange rate of the German mark and the British pound was confined within a narrow spread, incompatible with the Gaussian copula description. After 1992, the British pound exchange rate floated with respect to German mark, and the dependence between the two currencies decreased, as shown by their correlation coefficient. In this regime, we can no longer reject the Gaussian copula hypothesis.

The impact of major crises on the copula can be also clearly identified. Such a case is exhibited by the couple Malaysian ringgit/Thai baht. Indeed, during the period from January 1989 to January 1994, these two currencies have only undergone moderate and weakly correlated ($\rho = 0.29$) fluctuations, so our null hypothesis cannot be rejected at the 95% significance level. In contrast, during the period from January 1994 to October 1998, the Gaussian copula hypothesis is strongly rejected. This rejection is obviously due to the persistent and dependent ($\rho = 0.44$) shocks incurred by the Asian financial and monetary markets during the seven months of the Asian Crisis from July 1997 to January 1998 (Baig and Goldfajn 1998, Kaminsky and Schmukler 1999).

These two cases show that the Gaussian copula hypothesis can be considered reasonable for currencies in the absence of regulatory mechanisms and of strong and persistent crises. They also allow us to understand why the results of the test over the entire sample are so much weaker than the results obtained for the two sub-intervals: the time series are strongly non-stationary.

4.2. Commodities: metals

We consider a set of six metals traded on the London Metal Exchange: aluminium, copper, lead, nickel, tin and zinc. Each sample contains 2270 data points and covers the time interval from 4 January 1989 to 30 December 1997. The results are synthesized in table 5 and in figure 11.

Table 5 gives, for each of the 15 pairs of commodities, the probability $p(d)$ of obtaining from the Gaussian hypothesis

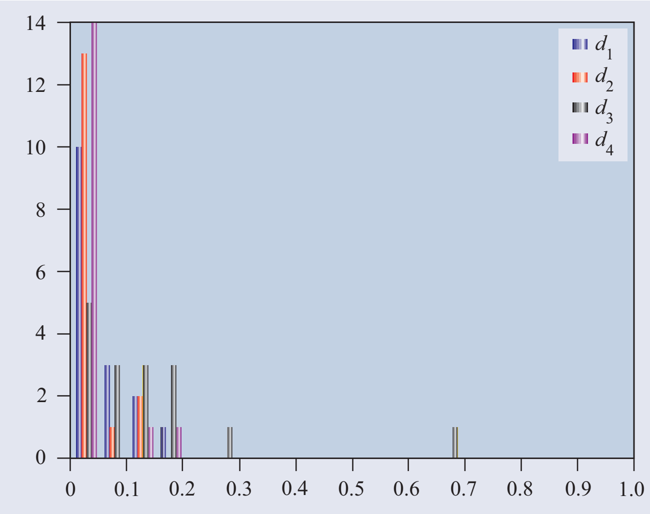

Figure 11. As figure 7, but for metals over a nine-year time interval from 4 January 1989 to 30 December 1997.

Table 5. As table 2, but for metals over a nine-year time interval from 4 January 1989 to 30 December 1997.

		$\hat{\rho}$	d_1 (%)	d_2 (%)	d_3 (%)	d_4 (%)
Aluminium	Copper	0.46	6.46	4.48	1.45	4.00
Aluminium	Lead	0.35	11.40	5.01	17.00	4.59
Aluminium	Nickel	0.36	0.33	0.51	3.41	0.62
Aluminium	Tin	0.34	13.40	13.80	1.25	15.90
Aluminium	Zinc	0.36	0.23	0.22	6.21	0.23
Copper	Lead	0.35	4.71	1.74	17.90	1.34
Copper	Nickel	0.38	4.91	4.60	14.80	3.80
Copper	Tin	0.32	19.40	13.50	65.30	14.70
Copper	Zinc	0.40	3.24	2.05	17.50	1.94
Lead	Nickel	0.32	6.71	3.78	27.40	3.62
Lead	Tin	0.33	7.86	4.04	4.91	3.31
Lead	Zinc	0.42	0.02	0.01	4.59	0.03
Nickel	Tin	0.35	0.91	0.92	8.70	0.76
Nickel	Zinc	0.33	0.08	0.34	8.91	0.35
Tin	Zinc	0.31	0.53	2.02	10.30	1.75

a deviation between the distribution of the z^2 and the χ^2-distribution with two degrees of freedom larger than the observed one for the commodity pair according to the distances d_1–d_4 defined by (33)–(36).

Figure 11 organizes the information shown in table 5 by representing, for each distance, the number of commodity pairs that give a test value p within a bin interval of width 0.05. A clustering close to the origin signals a significant rejection of the Gaussian copula hypothesis.

According to the three distances d_1, d_2 and d_4, at least two thirds and up to 93% of the set of 15 pairs of commodities is inconsistent with the Gaussian copula hypothesis. Surprisingly, according to the distance d_3, at the 95% significance level, two thirds of the set of 15 pairs of commodities remains compatible with the Gaussian copula hypothesis. This is the reverse of the previous situation found for currencies. These test values lead one to globally reject the Gaussian copula hypothesis.

Moreover, the largest value obtained for the distance d_3 is $p = 65\%$ for the pair copper–tin, which is significantly smaller than the 80 or 90% reached for some currencies over a similar time interval. Thus, even in the few cases where the Gaussian copula assumption is not rejected, the test values obtained are not really sufficient for distinguishing between the Gaussian copula and a Student copula with $\nu = 5$–6 degrees of freedom. In such a case, with correlation coefficients ranging between 0.31 and 0.46, the tail dependence neglected by keeping the Gaussian copula is no less than 10% and can reach 15%. One extreme event out of seven or ten might occur simultaneously on both marginals, which would be missed by the Gaussian copula.

To summarize, the Gaussian copula does not seem a reasonable assumption for metals, and it has not appeared necessary to test these data over a smaller time interval.

Table 6. As table 2, but for stocks over a ten-year time interval from 8 February 1991 to 29 December 2000.

		$\hat{\rho}$	d_1 (%)	d_2 (%)	d_3 (%)	d_4 (%)
AMAT	PFE	0.15	7.41	11.20	0.84	11.40
C	SUNW	0.28	25.60	48.70	10.90	53.90
F	GE	0.33	25.20	27.40	11.50	29.00
GM	IBM	0.21	14.90	38.50	16.20	41.80
HWP	SBC	0.12	42.30	16.90	25.20	17.20
INTC	MRK	0.17	24.80	10.90	64.60	10.40
KO	SUNW	0.14	14.10	10.10	21.20	9.35
MDT	T	0.16	12.10	28.10	8.41	29.80
MRK	XOM	0.19	15.40	15.00	11.20	14.50
MSFT	SUNW	0.44	3.40	1.85	0.26	1.74
PFE	WMT	0.27	4.24	4.12	15.40	3.74
T	WCOM	0.27	5.67	8.02	5.44	9.07
TXN	WCOM	0.28	47.90	37.70	15.20	37.50
WMT	XOM	0.20	0.32	0.00	6.02	0.00

4.3. Stocks

We now study the daily returns distributions for 22 stocks among the largest companies quoted on the New York Stock Exchange[14]: Appl. Materials (AMAT), AT&T (T), Citigroup (C), Coca Cola (KO), EMC, Exxon-Mobil (XOM), Ford (F), General Electric (GE), General Motors (GM), Hewlett-Packard (HWP), IBM, Intel (INTC), MCI WorldCom (WCOM), Medtronic (MDT), Merck (MRK), Microsoft (MSFT), Pfizer (PFE), Procter&Gamble (PG), SBC Communication (SBC), Sun Microsystem (SUNW), Texas Instruments (TXN), Wal-Mart (WMT).

Each sample contains 2500 data points, covers the time interval from 8 February 1991 to 29 December 2000, and has been divided into two sub-samples of 1250 data points, such that the first one covers the time interval from 8 February 1991 to 18 January 1996 and the second one that from 19 January 1996 to 20 December 2000. The results for fifteen randomly chosen pairs of assets are presented in tables 6–8 while the results obtained for the entire set are represented in figures 12–14.

At the 95% significance level, figure 12 shows that 75% of the pairs of stocks are compatible with the Gaussian copula hypothesis. Figure 13 shows that over the time interval from February 1991 to January 1996, this percentage becomes larger than 99% for d_1, d_2 and d_4, while it equals 94% according to

[14] The data come from the Center for Research in Security Prices (CRSP) database.

Table 7. As table 2, but for stocks over a five-year time interval from 8 February 1991 to 18 January 1996.

		$\hat{\rho}$	d_1 (%)	d_2 (%)	d_3 (%)	d_4 (%)
AMAT	PFE	0.10	58.30	58.10	11.80	63.80
C	SUNW	0.23	46.60	59.40	43.40	61.60
F	GE	0.31	87.30	78.70	15.40	84.80
GM	IBM	0.21	60.00	65.30	10.30	52.70
HWP	SBC	0.11	87.30	80.60	28.40	85.90
INTC	MRK	0.13	85.90	82.10	5.48	86.50
KO	SUNW	0.20	35.30	59.80	45.10	67.90
MDT	T	0.14	90.90	89.80	16.80	91.50
MRK	XOM	0.12	53.60	62.10	12.00	61.80
MSFT	SUNW	0.40	26.80	13.80	16.00	13.90
PFE	WMT	0.23	29.40	46.60	14.10	52.30
T	WCOM	0.19	79.20	93.60	4.95	94.90
TXN	WCOM	0.23	91.00	98.30	10.00	99.30
WMT	XOM	0.22	71.60	67.10	7.35	68.90

Table 8. As table 2, but for stocks over a five-year time interval from 19 January 1996 to 29 December 2000.

		$\hat{\rho}$	d_1 (%)	d_2 (%)	d_3 (%)	d_4 (%)
AMAT	PFE	0.19	29.60	33.90	3.10	39.50
C	SUNW	0.31	71.20	65.80	94.70	70.80
F	GE	0.34	38.00	23.60	32.20	21.80
GM	IBM	0.21	3.05	17.90	23.70	21.90
HWP	SBC	0.11	34.70	61.30	71.70	64.00
INTC	MRK	0.20	13.10	20.60	55.70	20.50
KO	SUNW	0.10	68.90	34.40	85.90	35.20
MDT	T	0.19	42.80	61.10	50.10	57.90
MRK	XOM	0.23	35.70	66.40	11.30	73.80
MSFT	SUNW	0.46	5.79	7.60	0.08	8.07
PFE	WMT	0.30	23.10	21.20	55.90	19.80
T	WCOM	0.33	12.00	13.70	17.30	14.00
TXN	WCOM	0.31	56.30	40.60	46.40	41.70
WMT	XOM	0.19	16.10	5.38	3.78	4.94

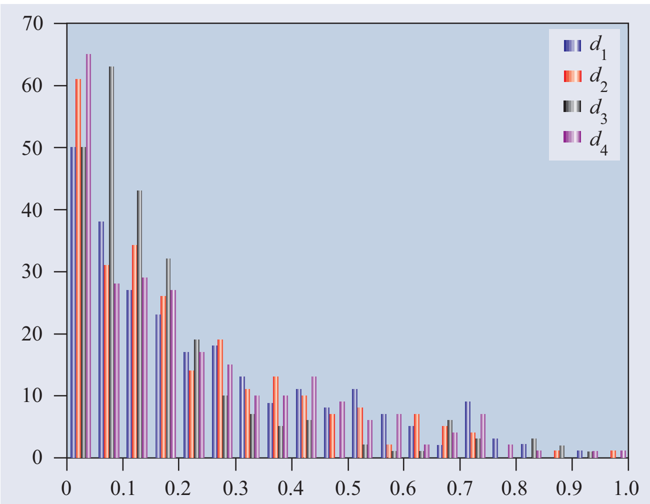

Figure 12. As figure 7, but for stocks over a ten-year time interval from 8 February 1991 to 29 December 2000.

d_3. It is striking that, during this period, according to d_1, d_2 and d_4, more than a quarter of the stocks obtain a test value p larger than 90%, so we can assert that they are completely inconsistent with the Student copula hypothesis for Student copulas with less than ten degrees of freedom. Among this set of stocks, not a single one has a correlation coefficient larger

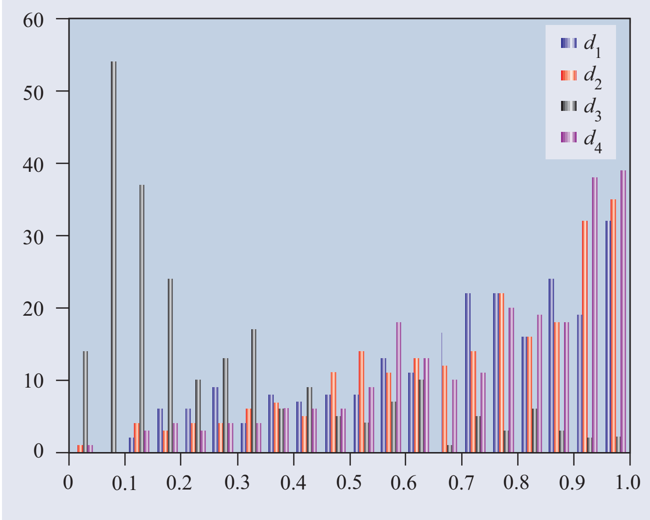

Figure 13. As figure 7, but for stocks over a five-year time interval from 8 February 1991 to 18 January 1996.

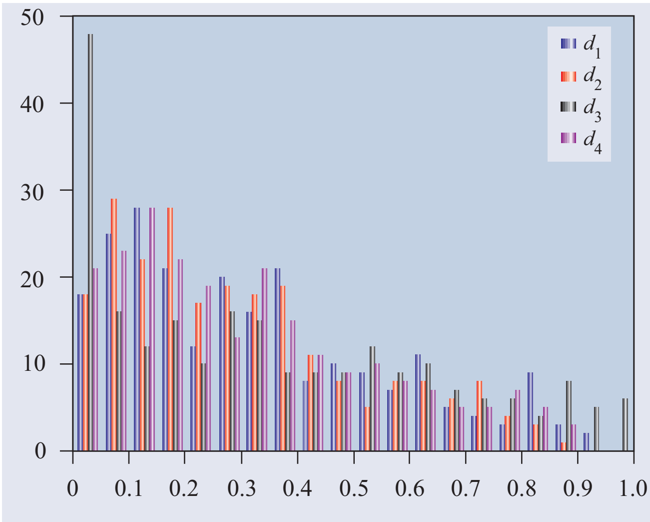

Figure 14. As figure 7, but for stocks over a five-year time interval from 19 January 1996 to 30 December 2000.

than 0.4, so a scenario based on the Gaussian copula hypothesis leads to neglecting a tail dependence of less than 5% as would be predicted by the Student copula with ten degrees of freedom. In addition, about 80% of the pairs of stocks lead to a test value p larger than 50% according to the distances d_1, d_2 and d_4, so as many as 80% of the pairs of stocks are incompatible with a Student copula with a number of degrees of freedom less than or equal to 5. Thus, for correlation coefficients smaller than 0.3, the Gaussian copula hypothesis leads to neglecting a tail dependence less than 10%. For correlation coefficients smaller than 0.1 which corresponds to 13% of the total number of pairs, the Gaussian copula hypothesis leads to neglecting a tail dependence less than 5%.

Figure 14 shows that, over the time interval from January 1996 to December 2000, 92% of the pairs of stocks are compatible with the Gaussian copula hypothesis according to d_1, d_2 and d_4 and more than 79% according to d_3. About a quarter of the pairs of stocks have a test value p larger than 50% according to the four measures, and thus are inconsistent with a Student copula with less than five degrees of freedom.

For completeness, we present in table 9 the results of the tests performed for five stocks belonging to the computer area: Hewlett-Packard, IBM, Intel, Microsoft, and Sun Microsystem. We observe that, during the first half-period, all the pairs of stocks obey the Gaussian copula hypothesis at the 95% significance level. The results are rather different for the second half-period, since about 40% of the pairs of stocks reject the Gaussian copula hypothesis according to d_1, d_2 and d_3. This is probably due to the existence of a few shocks, notably associated with the crash of the 'new economy' in March–April 2000.

On the whole, it appears however that there is no systematic rejection of the Gaussian copula hypothesis for stocks within the same industrial area, notwithstanding the fact that one can expect stronger correlations between such stocks than for currencies for instance.

5. Conclusion and comparison with other studies

We have studied the null hypothesis that the dependence between pairs of financial assets can be modelled by the Gaussian copula.

Our test procedure is based on the following simple idea. Assuming that the copula of two assets X and Y is Gaussian, then the multivariate distribution of (X, Y) can be mapped into a Gaussian multivariate distribution, by a transformation of each marginal into a normal distribution, which leaves the copula of X and Y unchanged. Testing the Gaussian copula hypothesis is therefore equivalent to the more standard problem of testing a two-dimensional multivariate Gaussian distribution. We have used a bootstrap method to determine and calibrate the test statistics. Four different measures of distances between distributions, more or less sensitive to the departure in the bulk or in the tail of distributions, have been proposed to quantify the probability of rejection of our null hypothesis.

Our tests have been performed over three types of asset: currencies, commodities (metals), and stocks. In most cases, for currencies and stocks, the Gaussian copula hypothesis cannot be rejected at the 95% confidence level. For currencies, according to three of the four distances at least,

- 40% of the pairs of currencies, over a ten-year time interval (due to non-stationary data),
- 67% of the pairs of currencies, over the first five-year time interval,
- 73% of the pairs of currencies, over the second five-year time interval

are compatible with the Gaussian copula hypothesis. For stocks, we have shown that

- 75% of the pairs of stocks, over a ten-year time interval,
- 93% of the pairs of stocks, over the first five-year time interval,
- 92% of the pairs of stocks, over the second five-year time interval

are compatible with the Gaussian copula hypothesis. In contrast, the Gaussian copula hypothesis cannot be considered as reasonable for metals: between 66 and 93% of the pairs of metals reject the null hypothesis at the 95% confidence level.

Notwithstanding the apparent qualification of the Gaussian copula hypothesis for most of the currencies and the stocks that we have analysed, we must bear in mind the fact that a non-Gaussian copula cannot be rejected. In particular, we have shown that a Student copula can always be mistaken for a Gaussian copula if its number of degrees of freedom is sufficiently large. Then, depending on the correlation coefficient, the Student copula can predict a non-negligible tail dependence which is completely missed by the Gaussian copula assumption. In other words, the Gaussian copula predicts no tail dependences and therefore does not account for extreme events that may occur simultaneously but nevertheless too rarely to modify the test statistics. To quantify the probability for neglecting such events, we have investigated the situations where one is unable to distinguish between the Gaussian and Student copulas for a given number of degrees of freedom. Our study leads to the conclusion that it may be very dangerous to embrace blindly the Gaussian copula hypothesis when the correlation coefficient between the pair of assets is too high, as the tail dependence neglected by the Gaussian copula can be as large as 0.6. In this respect, the case of the Swiss franc and the German mark is striking. The test values p obtained are very significant (about 33%), so we cannot mistake the Gaussian copula for a Student copula with less than 5–7 degrees of freedom. However, their correlation coefficient is so high ($\rho = 0.9$) that a Student copula with, say, $\nu = 30$ degrees of freedom still has a large tail dependence.

This remark shows that it is highly desirable to test for other non-Gaussian copulas, such as the Student copula. Breymann *et al* (2003) have recently shown that the dependence structure of the couple German mark/Japanese yen is (slightly?) better described by a Student copula with about six degrees of freedom (for daily returns) than by a Gaussian copula, according to the Akaike information criterion. This result is compatible with our results while making them more precise, since our tests cannot reject a Student copula with more than 3–4 degrees of freedom, as seen from table 2 which corresponds to a time interval comparable with that used by Breymann *et al* (2003). However, the stationarity of the data over such a long period is not well ascertained, as demonstrated by the results in tables 3–4, where an important increase of the significance of the non-rejection of the Gaussian copula hypothesis is observed during the second time interval compared with the first one. In both cases, however, the significance levels remain consistent with the non-rejection of a Student copula with about six degrees of freedom.

In the study by Mashal and Zeevi (2002), the dependence between stocks is claimed to be significantly better accounted for by a Student copula with 11–12 degrees of freedom than by a Gaussian copula. Again, these results are compatible with ours. However, in contrast to the case of currencies, it is questionable in this case whether a significant improvement

Table 9. As table 2, but for stocks belonging to the informatic sector, over two time intervals of five years each.

			$\hat{\rho}$	d_1 (%)	d_2 (%)	d_3 (%)	d_4 (%)
Time interval from 8 February 1991 to 18 January 1996	HWP	IBM	0.34	33.60	22.60	33.30	23.50
	HWP	INTC	0.46	30.10	47.30	51.20	52.10
	HWP	MSFT	0.41	76.30	47.20	32.30	45.30
	HWP	SUNW	0.40	29.60	29.80	76.60	35.40
	IBM	INTC	0.30	48.10	35.40	4.18	33.40
	IBM	MSFT	0.24	39.30	66.10	58.80	70.70
	IBM	SUNW	0.29	96.50	97.10	34.60	98.60
	INTC	MSFT	0.47	25.90	14.50	4.50	15.30
	INTC	SUNW	0.40	48.10	38.60	4.47	39.50
	MSFT	SUNW	0.40	26.80	13.80	16.60	13.90
Time interval from 19 January 1996 to 29 December 2000	HWP	IBM	0.46	2.02	3.21	0.96	3.96
	HWP	INTC	0.44	2.88	4.89	0.06	5.80
	HWP	MSFT	0.37	5.23	9.88	33.60	11.80
	HWP	SUNW	0.45	56.60	56.50	10.80	62.30
	IBM	INTC	0.43	5.34	3.31	1.68	2.44
	IBM	MSFT	0.39	1.00	0.95	2.28	0.88
	IBM	SUNW	0.46	23.50	15.60	33.80	14.90
	INTC	MSFT	0.57	31.80	16.10	11.50	17.10
	INTC	SUNW	0.50	6.68	3.55	0.01	4.37
	MSFT	SUNW	0.46	5.79	7.60	0.08	8.07

has been obtained by describing the dependence between stocks in terms of a Student copula. Indeed, as underlined in section 4.3, coefficients of correlation between two stocks are hardly greater than 0.4, so the tail dependence of the Student copula with 11–12 degrees of freedom is about 2.5% or less (see figures 1 and 2). In view of all the different sources of uncertainty during the estimation process in addition to the non-stationarity of the data, we doubt that such a description eventually leads to concrete improvement for any practical purposes.

Finally, we would like to stress that the determination of the coefficient of tail dependence must be studied on its own. Indeed, as we have seen, copula estimations yield poor estimates of this quantity, and are mainly based on the *a priori* assumption of the existence or not of such a tail dependence. Therefore, we think that it is necessary to develop tests that are specific to the detection of a possible tail dependence between two time series. Some results concerning stocks have been obtained by Malevergne and Sornette (2002a, 2002b) and indicate the existence of a tail dependence ranging between about 5 and 15% during the time period considered in the present study (see Malevergne and Sornette 2002a). Such estimates of the coefficient of tail dependence are barely compatible with estimates performed under the Student copula hypothesis and more generally under the elliptical copula assumption. Thus, as a conservative conclusion and in view of the different studies concerning this problem, we think that the Gaussian copula provides the most parsimonious description of the dependence between stock returns, apart from crisis periods. In such periods, the Student copula does not provide a really better practical model since it turns out that it still underestimates the dependence of tail events.

To our knowledge, no direct investigation of the tail dependences between currencies has yet been performed. Thus, we cannot conclude similarly as for stocks and assert that the Student copula still underestimates the tail dependence between currencies. As a consequence, for the risk management of currencies, prudence leads us to recommend using the Student copula rather than the Gaussian copula.

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