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On two dependent individual risk models☆

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Abstract

In this paper, we propose two constructions which allow dependence between the risks of an insurance portfolio in the individual risk model. In the first construction, each risk's experience is influenced by an individual and a collective risk factor, as well as a class factor if the portfolio is divided into different classes. The second construction uses copulas. The impact on the cumulative distribution function of the aggregate claim amount and on the stop-loss premium is presented via numerical examples. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The individual risk model can be applied, e.g. in group life insurance, in private health insurance, in car insurance and in other lines of non-life insurance. Usually, the risks of the insurance portfolio are assumed independent. However, there are practical situations for which this assumption is not appropriate. For example, consider a group life insurance or a group health insurance contract issued to a company for a section of its employees working in a mine, on a steel plant, in a paper mill, etc. In these cases, a single event (e.g. explosion, breakdown) influences the risks of the entire portfolio. The risks are therefore statistically dependent (see other examples in Bäuerle and Müller (1998)).

An insurance portfolio is generally divided in different classes. The insureds are classified according to the risk they represent for the insurer. The experience of these classes has traditionally been considered independent. However, there are contexts for which this assumption is not verified. Think for example of housing insurance in an area subject to snowstorms or glazed frost which have a common impact on all risks. In this area, certain residents living close to a river are exposed in spring to flooding catastrophes. As for another group of residents of this area living in contiguous houses, they are all more exposed to fire due to their closeness. These two groups of residents form two classes which are each exposed differently to flooding and fire.

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Let us consider an insurance portfolio divided in m different classes where each class j contains n_j risks ($j = 1, \dots, m$). Let X_{jk} be the claim amount for the policy k in the class j with $k = 1, \dots, n_j$ and S be the aggregate claim amount for the whole portfolio such that

$$S = \sum_{j=1}^{m} \sum_{k=1}^{n_j} X_{jk}.$$

The random variable X_{jk} is defined by

$$X_{jk} = \begin{cases} B_{jk}, & I_{jk} = 1, \\ 0, & I_{jk} = 0, \end{cases} \tag{1}$$

where I_{jk} ($j = 1, \dots, m$ and $k = 1, \dots, n_j$) is a Bernoulli random variable with

$$P(I_{jk} = 1) = q_{jk} \quad \text{and} \quad P(I_{jk} = 0) = p_{jk} = 1 - q_{jk}, \tag{2}$$

and B_{jk} is the claim amount random variable if policy k ($k = 1, \dots, n_j$) in the jth class ($j = 1, \dots, m$) has a claim. Its cumulative distribution function is denoted by $F_{B_{jk}}$. If a positive claim B_{jk} occurs, then I_{jk} equals 1 and otherwise, I_{jk} takes the value of 0. The random vectors $\underline{I} = (I_{11}, \dots, I_{1n_1}, \dots, I_{m1}, \dots, I_{mn_m})$ and $\underline{B} = (B_{11}, \dots, B_{1n_1}, \dots, B_{m1}, \dots, B_{mn_m})$ are supposed to be independent. In group life insurance, B_{jk} could be a random variable degenerated at the death benefit b_{jk}. In health, liability or property insurance, B_{jk} is a non-negative random variable. The cumulative distribution function $F_{X_{jk}}$ of X_{jk} ($j = 1, \dots, m$ and $k = 1, \dots, n_j$) can be written as the mixture

$$F_{X_{jk}} = p_{jk}\Delta_0 + q_{jk}F_{B_{jk}}, \tag{3}$$

where Δ_d is the Dirac function with

$$\Delta_d(x) = \begin{cases} 1, & \text{if } x \geq d, \\ 0, & \text{otherwise.} \end{cases}$$

This construction can be found, e.g. in Bowers et al. (1997), Klugman et al. (1998) and Rolski et al. (1999). Except for the latter, the authors define X_{jk} as $X_{jk} = I_{jk}B_{jk}$ ($j = 1, \dots, m$ and $k = 1, \dots, n_j$).

In actuarial science, we are interested in the computation of the cumulative distribution function F_S of S. Traditionally, this distribution has been studied for a portfolio with one or more classes of independent risks. A certain number of papers have treated the problem of the computation of F_S in various contexts (under the assumption of independent risks), see e.g. Kornya (1983), De Pril (1986, 1988, 1989), Dhaene and De Pril (1994), Waldmann (1994), Dhaene and Vandebroek (1995) and Hipp (1996). Since the beginning of the 1990s, various papers have treated dependency between risks. Among them, Dhaene and Goovaerts (1996, 1997), Müller (1997), Dhaene et al. (2001), Bäuerle and Müller (1998) and Wang and Dhaene (1998) study different orderings between two portfolios of dependent risks in the individual risk model. Based on the concept of comonotonicity, Wang and Dhaene (1998) and Dhaene et al. (2001) find the riskiest stop-loss premiums. Goovaerts and Dhaene (1996) propose a compound Poisson approximation for a portfolio of dependent risks. Denuit et al. (1999) construct stochastic bounds on sums of dependent risks. Wang (1998) (see also the discussion by Meyers (1999)) suggests a set of tools for modeling and combining correlated risk portfolios.

The models proposed by Wang (1998) are mostly applicable within the collective risk model. Our objective is to propose dependence structures which can be applied in the context of the individual risk model. As in Wang (1998), we suggest credible dependent structures and illustrate their numerical applicability, e.g. the numerical computation of F_S is feasible as in the classical independent context. Our work can be considered as a complement to the works of Dhaene and Goovaerts (1996, 1997), Müller (1997), Dhaene et al. (2001), Bäuerle and Müller (1998) and Wang and Dhaene (1998). To the exception of Goovaerts and Dhaene (1996), they mostly examine the qualitative aspects of the introduction of dependence within the individual risk model.

In this paper, we propose two constructions which allow the introduction of dependence between the occurrence random variables I_{jk} ($j = 1, \dots, m$ and $k = 1, \dots, n_j$) leading to dependent risks X_{jk} ($j = 1, \dots, m$ and $k = 1, \dots, n_j$). These constructions greatly facilitate the computation of the cumulative distribution function F_S for a portfolio with dependent risks and also ease the interpretation of the impact of the dependence on F_S. Furthermore, the two proposed constructions do not affect the definition of the marginals of the occurrence random variables.

The paper is organized as follows. In Section 2, we present the first construction used to introduce a dependence relation between the risks (within a given class) and between the classes of a portfolio. In Section 3, we propose the second construction which is based on copulas. Finally, we compare these two models in Section 4. Numerical examples illustrating the impact of the constructions on the cumulative distribution function F_S and on the stop-loss premium are given in each section.

2. A simple model for a portfolio with dependent risks

In this section, we present the first construction allowing dependence among risks of a portfolio. The dependence is introduced via the occurrence random variables I_{jk} ($j = 1, \dots, m$ and $k = 1, \dots, n_j$). We assume that the occurrence of a claim for the kth policy in the jth class is function of three independent risk factors represented by the random variables J_{jk}, J_j, and J_0. They correspond respectively to the individual, the class, and the global risk factors. This simple model is a special case of model 3.1 in Bäuerle and Müller (1998) as well as a special case of the mixture models in Wang (1998).

Consider the random variables I_{jk} ($j = 1, \dots, m$ and $k = 1, \dots, n_j$) defined as

$$I_{jk} = \min(J_{jk} + J_j + J_0, 1), \tag{4}$$

where J_{jk}, J_j, and J_0 are independent Bernoulli random variables with

$$P(J_{jk} = 1) = \tilde{q}_{jk} \quad \text{and} \quad P(J_{jk} = 0) = \tilde{p}_{jk} = 1 - \tilde{q}_{jk}, \qquad P(J_j = 1) = \tilde{q}_j \quad \text{and}$$
$$P(J_j = 0) = \tilde{p}_j = 1 - \tilde{q}_j, \qquad P(J_0 = 1) = \tilde{q}_0 \quad \text{and} \quad P(J_0 = 0) = \tilde{p}_0 = 1 - \tilde{q}_0.$$

Given (4), the random vector $\underline{I}$ has clearly dependent components. The random variables I_{jk} ($j = 1, \dots, m$ and $k = 1, \dots, n_j$) remain however Bernoulli distributed as in (2). The parameter q_{jk} can now be written as a function of $\tilde{q}_{jk}$, $\tilde{q}_j$ and $\tilde{q}_0$. This is confirmed with the probability generating function (pgf) of I_{jk}

$$P_{I_{jk}}(t) = p_{jk} + q_{jk}t,$$

where $p_{jk} = \tilde{p}_0 \tilde{p}_j \tilde{p}_{jk}$ and $q_{jk} = 1 - (1 - \tilde{q}_0)(1 - \tilde{q}_j)(1 - \tilde{q}_{jk})$. If $\tilde{q}_j = \tilde{q}_0 = 0$ ($j = 1, \dots, m$), we get the special case of the individual risk model with independent risks with $q_{jk} = \tilde{q}_{jk}$. If $\tilde{q}_j = 0$ then we have the case of a portfolio with only one class of dependent risks. The random vector $\underline{X} = (X_{11}, \dots, X_{1n_1}, \dots, X_{m1}, \dots, X_{mn_m})$ has now dependent components due to definition (4) of the I_{jk}'s with however unchanged B_{jk}'s. The moment generating function (mgf) of each X_{jk} ($j = 1, \dots, m$ and $k = 1, \dots, n_j$) is

$$M_{X_{jk}}(t) = P_{I_{jk}}(M_{B_{jk}}(t)) = p_{jk} + q_{jk} M_{B_{jk}}(t).$$

In order to obtain the mgf of the aggregate claim amount S, we must find the multivariate mgf of the random vector $\underline{X}$ which in turn is function of the pgf of the random vector $\underline{I}$. The latter is given by

$$P_{\underline{I}}(\underline{t}) = \tilde{p}_0 \left[\prod_{j=1}^{m} \left(\tilde{q}_j \prod_{k=1}^{n_j} t_{jk} + \tilde{p}_j \prod_{k=1}^{n_j} P_{J_{jk}}(t_{jk}) \right) \right] + \tilde{q}_0 \prod_{j=1}^{m} \prod_{k=1}^{n_j} t_{jk}, \tag{5}$$

where $\underline{t} = (t_{11}, \ldots, t_{1n_1}, \ldots, t_{m1}, \ldots, t_{mn_m})$. Given (5), the multivariate mgf of $\underline{X}$ is

$$M_{\underline{X}}(\underline{t}) = P_{\underline{I}}(M_{B_{11}}(t_{11}), \ldots, M_{B_{1n_1}}(t_{1n_1}), \ldots, M_{B_{m1}}(t_{m1}), \ldots, M_{B_{mnm}}(t_{mn_m})). \tag{6}$$

Finally, in order to find the mgf of S, we use (6) in conjunction with the following lemma.

Lemma 1. *Let $M_{Y_1,\ldots,Y_n}(t_1, \ldots, t_n)$ be the multivariate mgf of the vector $(Y_1, \ldots, Y_n)$ given by*

$$M_{Y_1,\ldots,Y_n}(t_1, \ldots, t_n) = E[\mathrm{e}^{t_1 Y_1} \cdots \mathrm{e}^{t_n Y_n}].$$

Then, the mgf of $Z = Y_1 + \cdots + Y_n$ is

$$M_Z(t) = M_{Y_1,\ldots,Y_n}(t, \ldots, t). \tag{7}$$

Proof. $M_Z(t) = E[\mathrm{e}^{tZ}] = E[\mathrm{e}^{t(Y_1+\cdots+Y_n)}] = E[\mathrm{e}^{tY_1} \cdots \mathrm{e}^{tY_n}] = M_{Y_1,\ldots,Y_n}(t, \ldots, t)$. □

From Lemma 1 and (6), the mgf of S is

$$M_S(t) = \tilde{p}_0 \left[\prod_{j=1}^{m} \left(\tilde{q}_j \prod_{k=1}^{n_j} M_{B_{jk}}(t) + \tilde{p}_j \prod_{k=1}^{n_j} P_{J_{jk}}(M_{B_{jk}}(t)) \right) \right] + \tilde{q}_0 \prod_{j=1}^{m} \prod_{k=1}^{n_j} M_{B_{jk}}(t). \tag{8}$$

From (8), one sees that F_S is the convex combination of two cumulative distribution functions, say F_U and F_V,

$$F_S(x) = \tilde{p}_0 F_U(x) + \tilde{q}_0 F_V(x), \quad x \geq 0, \tag{9}$$

where U and V are random variables with mgf $M_U(t) = \prod_{j=1}^{m} (\tilde{q}_j \prod_{k=1}^{n_j} M_{B_{jk}}(t) + \tilde{p}_j \prod_{k=1}^{n_j} P_{J_{jk}}(M_{B_{jk}}(t)))$ and $M_V(t) = \prod_{j=1}^{m} \prod_{k=1}^{n_j} M_{B_{jk}}(t)$. Their respective cumulative distribution functions are

$$F_U = F_{C_1} * \cdots * F_{C_m},$$

and

$$F_V = F_{B_{11}} * \cdots * F_{B_{1n_1}} * \cdots * F_{B_{m1}} * \cdots * F_{B_{mn_m}},$$

where $F_{C_j} = \tilde{q}_j (F_{B_{j1}} * \cdots * F_{B_{jn_j}}) + \tilde{p}_j (F_{D_{j1}} * \cdots * F_{D_{jn_j}})$ $(j = 1, \ldots, m)$, and $F_{D_{jk}} = \tilde{p}_{jk} \Delta_0 + \tilde{q}_{jk} F_{B_{jk}}$ $(k = 1, \ldots, n_j)$. The symbol "$*$" holds for the convolution product between two cumulative distribution functions.

In most cases, there is no explicit form for F_S. One must therefore resort to a numerical approximation. Numerical recursive methods used for the computation of F_S have been studied in the independence case among others, e.g. in Kornya (1983), De Pril (1986, 1988, 1989), Dhaene and De Pril (1994), Waldmann (1994) and Dhaene and Vandebroek (1995). The FFT method can also be used if B_{jk} ($j = 1, \ldots, m$ and $k = 1, \ldots, n_j$) are non-degenerated random variables. This method is explained, e.g. in Klugman et al. (1998) or Rolski et al. (1999). An approximation of F_S by a compound Poisson distribution is also given in the latter references or in Panjer and Willmot (1992).

Given (9), two steps can be made to numerically evaluate F_S. In the first step, the computation of F_U and F_V is made with the appropriate methods given the context. Then, F_S is calculated with (9). If the random variables B_{jk} ($j = 1, \ldots, m$ and $k = 1, \ldots, n_j$) are non-degenerated, the FFT method can be used to approximate F_U and F_V. In the case where the B_{jk}'s are degenerated at b_{jk} one should be aware that the mass $\tilde{q}_0$ is placed at $\sum_{j=1}^{m} \sum_{k=1}^{nj} b_{jk}$. The methods given in De Pril (1986, 1989) and Dhaene and Vandebroek (1995) can be used for the computation of F_U in this case.

Now let us look at the expectation and the variance of S in this dependent context. The expectation of S is not influenced by the dependence between the X_{jk}'s. The variance of S requires an expression for $\mathrm{Cov}(X_{jk}, X_{j'k'})$ for ($j = j'$ and $k \neq k'$) and ($j \neq j'$ and $k \neq k'$).

The covariance between the occurrence random variables I_{jk} and $I_{jk'}$ ($k \neq k'$) related to two risks of the same class j ($j = 1, \dots, m$) is

$$\mathrm{Cov}(I_{jk}, I_{jk'}) = \tilde{q}_0 + \tilde{p}_0\tilde{q}_j + \tilde{p}_0\tilde{p}_j\tilde{q}_{jk}\tilde{q}_{jk'} - q_{jk}q_{jk'},$$

which leads to

$$\mathrm{Cov}(X_{jk}, X_{jk'}) = E[B_{jk}]E[B_{jk'}]\mathrm{Cov}(I_{jk}, I_{jk'}) \quad (j = 1, \dots, m \text{ and } k \neq k'). \tag{10}$$

The covariance between the occurrence random variables I_{jk} and $I_{j'k'}$ ($k = 1, \dots, n_j$ and $k' = 1, \dots, n_{j'}$) of two policies from two different classes j and j' ($j \neq j'$) is given by

$$\mathrm{Cov}(I_{jk}, I_{j'k'}) = \tilde{q}_0 + \tilde{p}_0(\tilde{q}_j + \tilde{p}_j\tilde{q}_{jk})(\tilde{q}_{j'} + \tilde{p}_{j'}\tilde{q}_{j'k'}) - q_{jk}q_{j'k'},$$

from which we obtain

$$\mathrm{Cov}(X_{jk}, X_{j'k'}) = E[B_{jk}]E[B_{j'k'}]\mathrm{Cov}(I_{jk}, I_{j'k'}). \tag{11}$$

We use (10) and (11), and $\mathrm{Var}[X_{jk}] = E^2[B_{jk}]\mathrm{Var}[I_{jk}] + \mathrm{Var}[B_{jk}]E[I_{jk}]$ to obtain the variance of S. Clearly if $\tilde{q}_0 = \tilde{q}_j = 0$ ($j = 1, \dots, m$), then $\mathrm{Cov}(I_{jk}, I_{jk'}) = 0$ and $\mathrm{Cov}(I_{jk}, I_{j'k'}) = 0$ for ($k \neq k'$). The Var[S] in this case corresponds therefore to the variance of S in the individual risk model with independent risks.

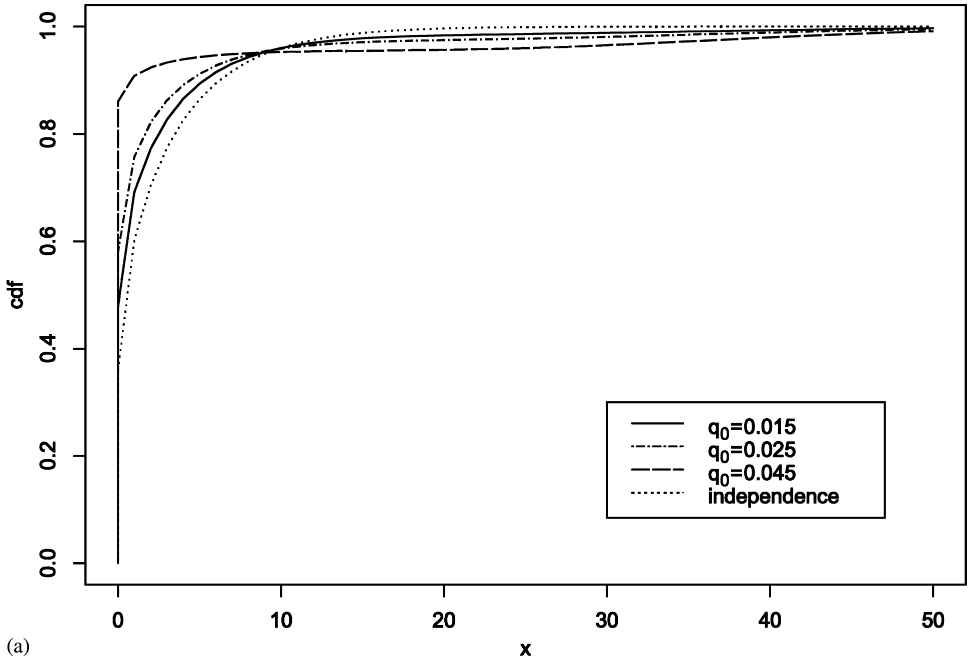
Example 1. We assume portfolios with only one class of 20 identically distributed risks. The occurrence variables $I_1, \dots, I_{20}$ (we omit the subscript j since we have only one class) have a Bernoulli distribution with $q_i = 0.05$ ($i = 1, \dots, 20$). The claim amount random variables $B_1, \dots, B_{20}$ have a gamma distribution with mean 2 and variance 4.

We give in Fig. 1a and b, the exact values of the cumulative distribution functions F_S and the stop-loss premiums $\pi_S(u)$ for four portfolios. They differ in the value attributed to the parameter $\tilde{q}_0$ (of the Bernoulli random variable J_0) which determines the degree of dependence between the risks of the portfolio. We have chosen values of 0, 0.015, 0.025 and 0.045 for $\tilde{q}_0$, where $\tilde{q}_0 = 0$ corresponds to the independent individual risk model. In Table 1, we give $E[S]$ and Var[S] for each portfolio. It is clear from Fig. 1a and b that the dependency significantly influences both F_S and $\pi_S(u)$. In fact, we observe that, for any fixed retention level u, the value of the stop-loss premium increases with $\tilde{q}_0$, the degree of dependence between the risks of the portfolio.

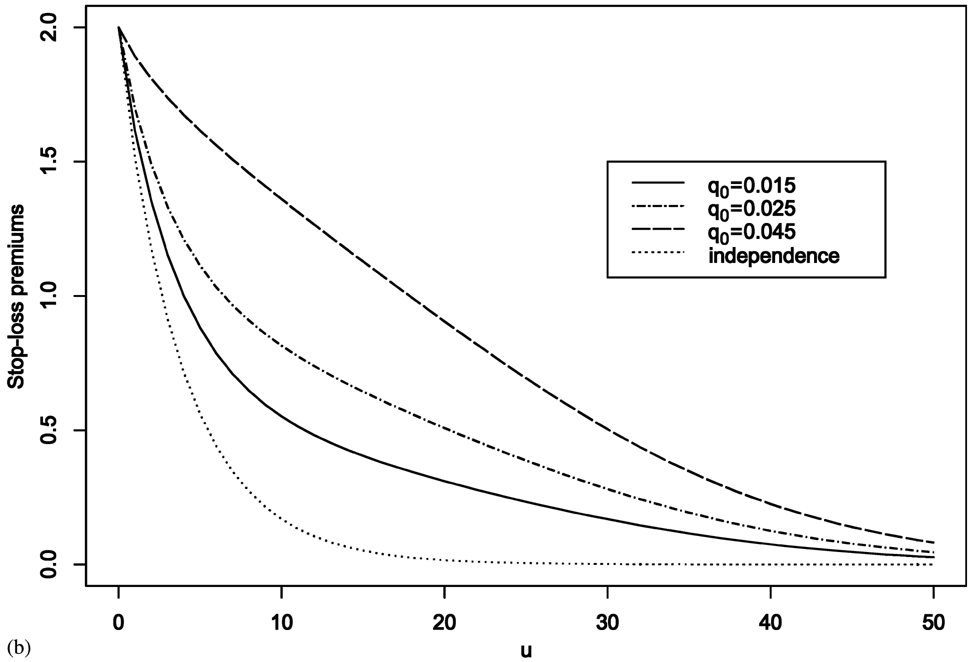
Example 2. We suppose portfolios divided in four classes of five risks. In each class j ($j = 1, \dots, 4$), the occurrence variables $I_{j1}, \dots, I_{j5}$ have a Bernoulli distribution with $q_{jk} = 0.005 + 0.015j$ ($j = 1, \dots, 4$ and $k = 1, \dots, 5$). The claim amount random variables $B_{j1}, \dots, B_{j5}$ have a gamma distribution with mean 2 and variance 4 ($j = 1, \dots, 4$). The level of dependency between the risks of the portfolios is determined by the values attributed to the Bernoulli parameters $\tilde{q}_0$, and $\tilde{q}_j$ related to the risk factors J_0, and J_j, respectively. We have chosen a different value of $\tilde{q}_0$ for each portfolio (0, 0.005, 0.01) and set $\tilde{q}_j = \tilde{q}_{jk} = 0.006\ \forall j$ for the three portfolios. The fourth portfolio is the one with independent risks. The variance of the aggregate claim amount for each portfolio is given in Table 2. The exact values of the cumulative distribution functions F_S and the stop-loss premiums $\pi_S(u)$ for the different portfolios are given in Fig. 2a and b, respectively. Again, we observe the important effect of the dependency on both F_S and $\pi_S(u)$.

Table 1

$\tilde{q}_0$	$E[S]$	Var[S]
Independence	2	11.80
0.015	2	32.69
0.025	2	46.97
0.045	2	76.44

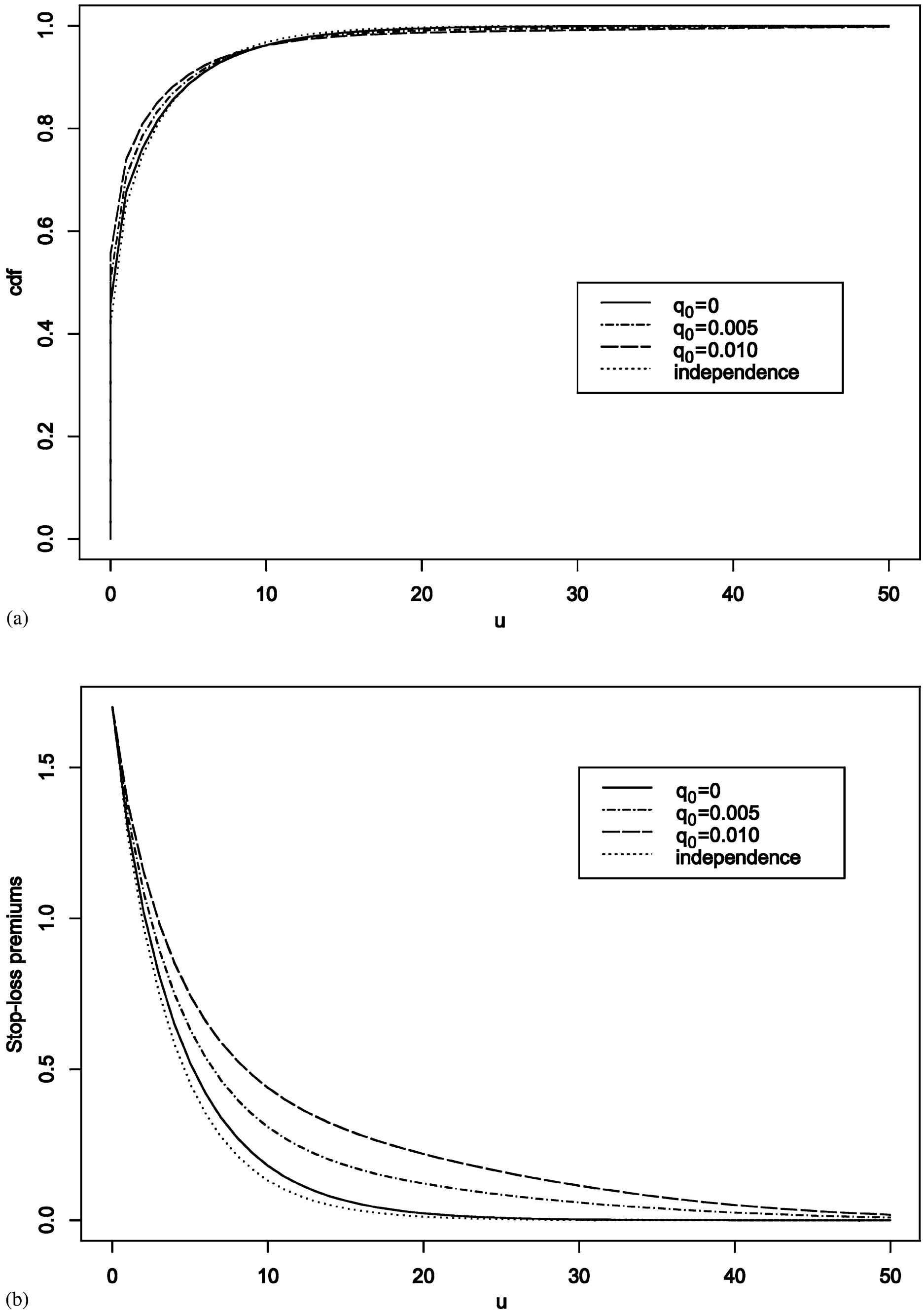


(a)



(b)

Fig. 1. (a) Cumulative distribution functions F_S of S and (b) stop-loss premiums $\pi_S(u)$ for portfolios with dependent risks affected by a global risk factor J_0 and an individual risk factor J_{ii}.



(a)

(b)

Fig. 2. (a) Cumulative distribution functions F_S of S and (b) stop-loss premiums $\pi_S(u)$ for portfolios with dependent risks divided in four classes and affected by a global risk factor J_0, a class risk factor J_j and an individual risk factor J_{jk}.

Table 2

$\tilde{q}_0$	$E[S]$	$\mathrm{Var}[S]$
Independence	1.7	10.03
0	1.7	11.80
0.005	1.7	16.05
0.01	1.7	20.34

3. Dependence model based on copulas

In the present section, we propose a second construction which allows dependence between the risks of an insurance portfolio. We only discuss the case of a portfolio with one class. We consider a portfolio of n risks X_k ($k = 1, \dots, n$) defined as in (1). We assume, as in Section 3, that the occurrence random variables I_k ($k = 1, \dots, n$) are dependent. The relation of dependence between the components of the random vector $\underline{I} = (I_1, \dots, I_n)$ is introduced directly on their joint cumulative distribution function $F_{\underline{I}}$ with marginals F_{I_k} ($k = 1, \dots, n$). We use a copula to separate the dependence structure and the definition of the marginals.

Suppose that $Y_1, \dots, Y_n$ are random variables with continuous marginal distributions $F_{Y_1}, \dots, F_{Y_n}$. Then, for any continuous multivariate distribution $F_{Y_1,\dots,Y_n}$ the following representation holds for a unique copula C:

$$F_{Y_1,\dots,Y_n}(y_1, \dots, y_n) = C(F_{Y_1}(y_1), \dots, F_{Y_n}(y_n)). \tag{12}$$

A copula C is the distribution function of a random vector with uniform-[0, 1] marginals. The advantage of writing $F_{Y_1,\dots,Y_n}$ as in (12) is that it allows to distinguish the definition of the dependence which is made through the copula $C(u_1, \dots, u_n)$ and the definition of the marginals $F_{Y_1}, \dots, F_{Y_n}$.

Good introductions to copulas are given in Joe (1997) and Nelsen (1999). Frees and Valdez (1998) (with the discussion of Genest et al. (1998)) and Wang (1998) give an overview of copulas and a few applications to actuarial science. Numerous copulas can be found in the literature (see e.g. Joe (1997), Nelsen (1999) and references therein). The simplest one is the independent copula

$$C^{\mathrm{Ind}}(u_1, \dots, u_n) = u_1 \times \cdots \times u_n.$$

Two other examples are the Cook–Johnson copula

$$C_\alpha^{\mathrm{CJ}}(u_1, \dots, u_n) = (u_1^{-\alpha} + \cdots + u_n^{-\alpha} - (n-1))^{-1/\alpha}, \quad \alpha > 0, \tag{13}$$

and the Gumbel copula

$$C_\alpha^{\mathrm{G}}(u_1, \dots, u_n) = \exp(-[(-\ln u_1)^\alpha + \cdots + (-\ln u_n)^\alpha]^{1/\alpha}), \quad \alpha \geq 1, \tag{14}$$

where $0 \leq u_k \leq 1$ ($k = 1, \dots, n$).

In (13) and (14), the parameter α indicates the level of dependence between the random variables. Wang (1998) introduces, with a copula, dependency between risks directly on the joint distribution of $\underline{X} = (X_1, \dots, X_n)$. In that case, the cumulative distribution function of S has to be approximated via simulation. Our approach, as we shall see, is different and makes possible the computation of the cumulative distribution function of S without having recourse to simulation. We find the expression of the mgf of S and then we obtain a numerical approximation of the cumulative distribution function of S with the FFT method.

We introduce the dependence between the occurrence random vector $\underline{I}$ in order to have dependent risks X_k ($k = 1, \dots, n$). The joint cumulative distribution function of $\underline{I}$ is now defined with a copula C

$$F_{\underline{I}}(i_1, \dots, i_n) = C(F_{I_1}(i_1), \dots, F_{I_n}(i_n)), \quad i_k = 0 \text{ or } 1,\ k = 1, \dots, n, \tag{15}$$

where F_{I_k} is the marginal cumulative distribution function of I_k with $F_{I_k}(i_k) = 0$, for $i_k < 0$. In this case, however, the copula C is not uniquely determined outside of $\{0, 1\}^n$ (see Nelsen (1999)) since the marginals F_{I_k} are discrete.

For the evaluation of F_S, we need to find $M_S(t)$ which is obtained from $M_{\underline{X}}(t_1, \dots, t_n)$, the multivariate mgf of $\underline{X}$, given by

$$M_{\underline{X}}(t_1, \dots, t_n) = P_{\underline{I}}(M_{B_1}(t_1), \dots, M_{B_n}(t_n)).$$

For the calculation of the multivariate pgf of $\underline{I}$, the explicit expression of $f_{\underline{I}}(i_1, \dots, i_n)$ is required. Since the copulas only apply on the cumulative distribution function, the expression for $f_{\underline{I}}(i_1, \dots, i_n)$ must be derived from the multivariate distribution function $F_{\underline{I}}(i_1, \dots, i_n)$.

In the bivariate case (i.e. $n = 2$), we obtain

$$f_{I_1,I_2}(i_1, i_2) = F_{I_1,I_2}(i_1, i_2) - F_{I_1,I_2}(i_1 - 1, i_2) - F_{I_1,I_2}(i_1, i_2 - 1) + F_{I_1,I_2}(i_1 - 1, i_2 - 1),$$

for $i_k = 0$ or 1 ($k = 1, 2$) and $F_{I_1,I_2}(i_1, i_2) = 0$ if i_1 or $i_2 < 0$.

For any n, we have a general formula for $f_{\underline{I}}(i_1, \dots, i_n)$. This requires some notations. Let $P_r^n = \{s \subset \{1, \dots, n\} | \#s = r\}$ be the set of r-sized subsets of $\{1, \dots, n\}$ with $r = 0, \dots, n$ and let $I : P_r^n \to \{0, 1\}^n$ be the indicator function defined as

$$I(s) = (j_1, \dots, j_n) \quad \text{with} \quad j_k = \begin{cases} 0, & k \notin s, \\ 1, & k \in s. \end{cases}$$

Then, for $\underline{i} = (i_1, \dots, i_n) \in \aleph^n$, where $\aleph$ is the set of non-negative integers, we have

$$f_{\underline{I}}(\underline{i}) = \sum_{r=0}^{n} (-1)^r \sum_{s \in P_r^n} F_{\underline{I}}(\underline{i} - I(s)). \tag{16}$$

Notice that $f_{\underline{I}}(0, \dots, 0) = F_{\underline{I}}(0, \dots, 0)$. In the particular case where the random variables $I_1, \dots, I_n$ are identically distributed, then (16) becomes

$$f_{\underline{I}}(c_{j,n}) = \sum_{k=0}^{j} (-1)^k \binom{j}{k} F_{\underline{I}}(c_{j-k,n}),$$

where $c_{j,n}$ is a vector with any combination of j *one*'s and $n - j$ *zero*'s (e.g. $c_{2,5}$ could be $(1, 1, 0, 0, 0)$ or $(0, 0, 1, 0, 1)$).

Therefore, F_S can be obtained by inverting

$$M_S(t) = \sum_{i_1,\dots,i_n \in \{0,1\}} f_{\underline{I}}(i_1, \dots, i_n)(M_{B_1}(t))^{i_1} \cdots (M_{B_n}(t))^{i_n}.$$

Example 3. We assume the same marginals for the occurrence variables $I_1, \dots, I_{20}$ and the same distribution for the claim amount random variables $B_1, \dots, B_{20}$ as the ones in Example 1. The multivariate distribution of $(I_1, \dots, I_{20})$ is defined with the Cook–Johnson copula (13). The values of α, $E[S]$, $\mathrm{Var}[S]$ are given in Table 3.

Table 3

α	$E[S]$	$\mathrm{Var}[S]$
Independence	2	11.80
1	2	15.24
10	2	36.10
30	2	56.37

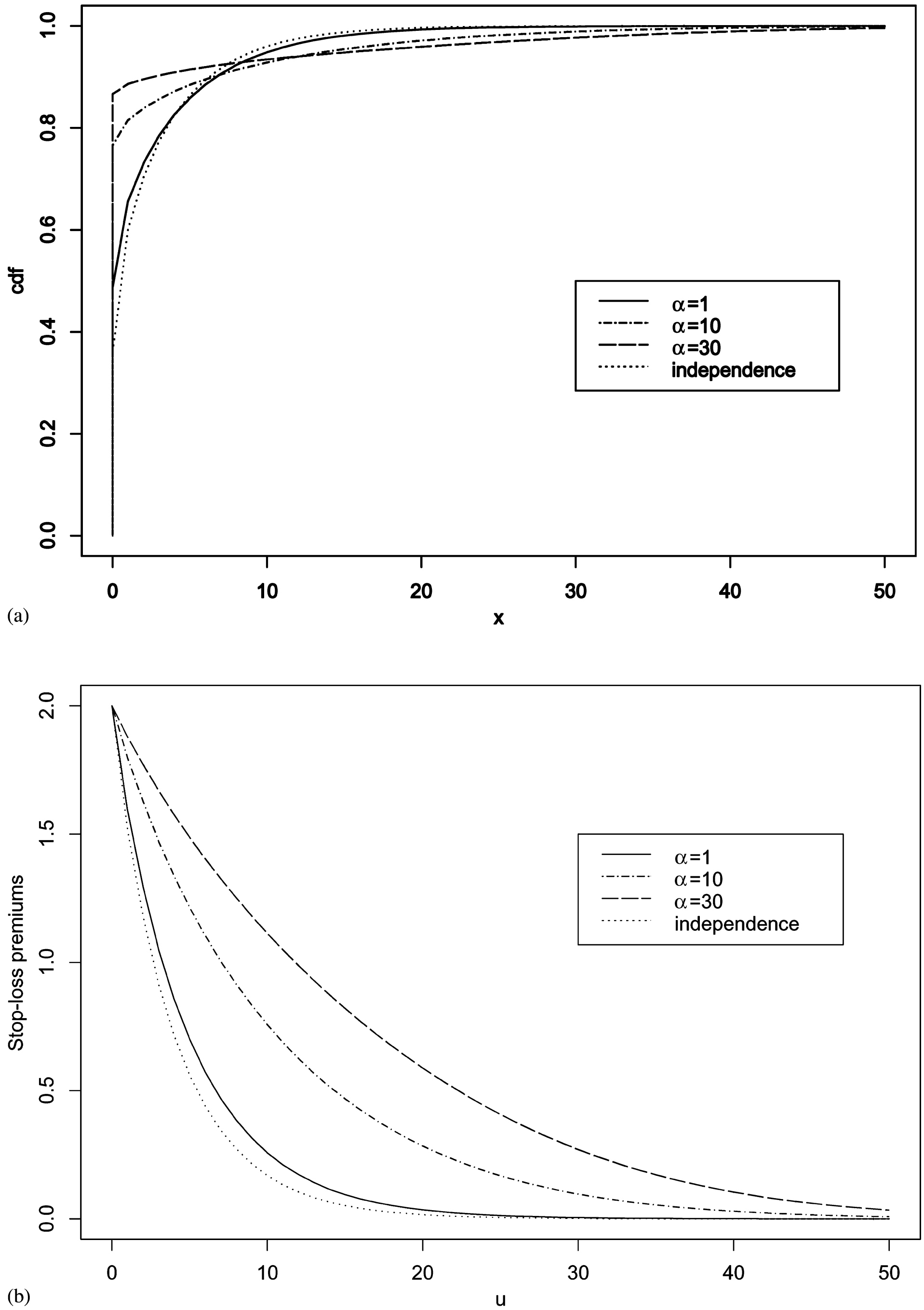


Fig. 3. (a) Cumulative distribution functions F_S of S and (b) stop-loss premiums $\pi_S(u)$ for portfolios with dependent risks obtained via the Cook–Johnson copula.

We give in Fig. 3a and b, respectively, the exact values of the cumulative distribution functions F_S and the stop-loss premiums $\pi_S(u)$ for four portfolios which differ by the degree of dependence given by the parameter α. Fig. 3a and b clearly depict the incidence of the increase of dependency (with α) between the occurrence random variables.

The main advantage of using the model with copulas is the possibility to specify the degree of dependency between the risks of the portfolio. The impact of the introduction, via a copula, of a relation of dependence between the occurrence random variables $I_1, \ldots, I_n$ on the stop-loss premium is illustrated in Fig. 3b. We have made the same observations on F_S and the stop-loss premiums $\pi_S(u)$ in a similar context as the one of Example 3 but with a Gumbel copula.

As in the model presented in Section 2, the expectation of S is not influenced by the introduction of the dependence relation. The variance of S is

$$\mathrm{Var}[S] = \sum_{i=1}^{n} \mathrm{Var}[X_i] + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{Cov}(I_i, I_j) E[B_i] E[B_j],$$

where $\mathrm{Cov}(I_i, I_j)$ $(i \neq j)$ is given by

$$\mathrm{Cov}(I_i, I_j) = f_{I_i, I_j}(1, 1) - q_i q_j. \tag{17}$$

If the occurrence random variables $(I_1, \ldots, I_n)$ are mutually independent, $f_{I_i, I_j}(1, 1) = q_i q_j$ and $\mathrm{Var}[S] = \sum_{i=1}^{n} \mathrm{Var}[X_i]$.

4. Comparison of two models

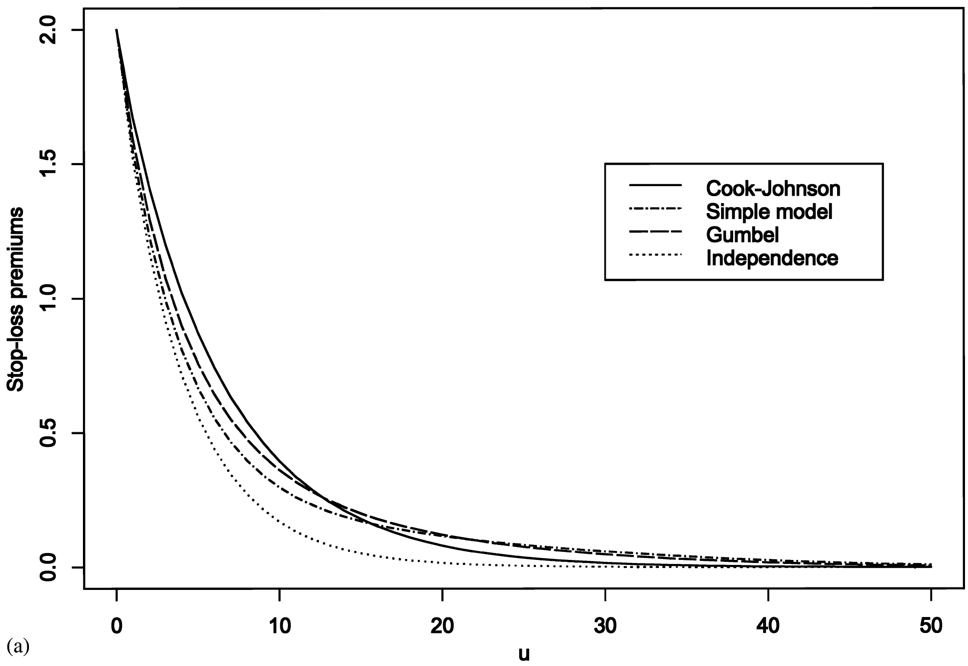
We have proposed and separately examined, in Sections 2 and 3, two constructions which permit the introduction of dependency between risks in the individual risk model. In this section, we numerically compare the behavior of the stop-loss premiums within the two models. We examine how the stop-loss premiums of these two models differ if the I_j's and the B_j's are identically distributed and the covariance between I_j and $I_{j'}$ $(j \neq j')$ is identical for any pair of j and j' in both models. We therefore choose a parameter $\tilde{q}_0$ of the simple model and a parameter α of the copula model such that $\mathrm{Cov}(I_j, I_{j'})$ is identical for both models. The multivariate distribution of $(I_1, \ldots, I_n)$ is determined by the choice of the model but the mean and the variance of the aggregate claim coincide for each one.

Example 4. We assume the same marginals for the occurrence variables $I_1, \ldots, I_{20}$ and the same distribution for the claim amount random variables $B_1, \ldots, B_{20}$ as the ones in Examples 1 and 3. We compare the stop-loss premiums obtained with three different models, the simple model, the Cook–Johnson copula, and the Gumbel copula. The dependence parameters $(\tilde{q}_0, \alpha_{\mathrm{CJ}}, \alpha_{\mathrm{G}})$ of the models are fixed such that the variance of the aggregate claim amount is identical. We consider three values for the variance of S leading to three figures of stop-loss premiums. The parameters for the models are summarized in Table 4.

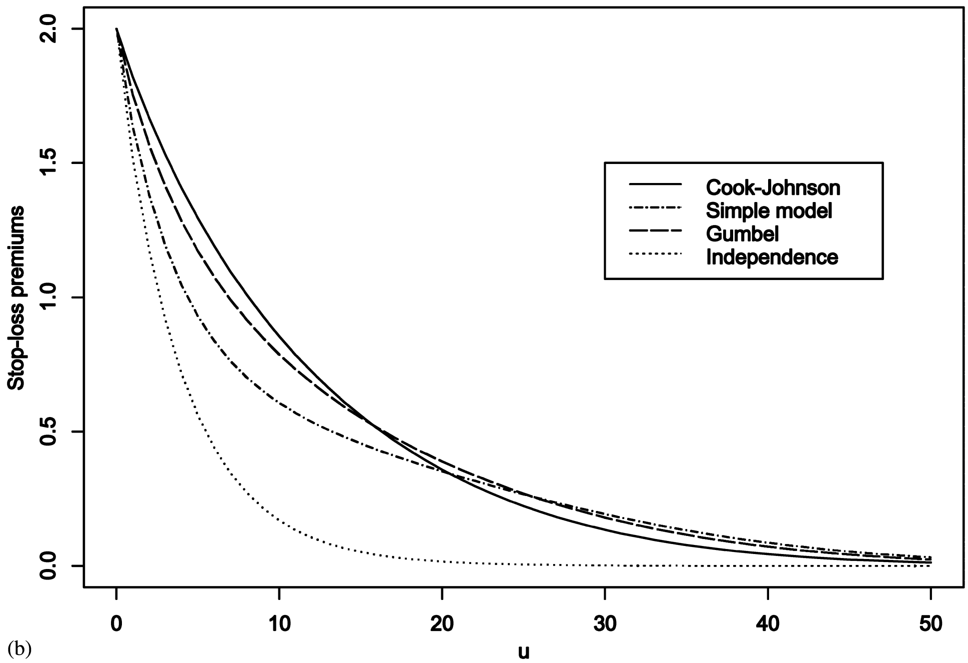
For the three cases examined in this example (and for other values of Var[S] which are not presented here), we have observed the same behavior of the stop-loss premiums for each model. None of the three models lead to a

Table 4

	$\tilde{q}_0$	α_{CJ}	α_{G}	$E[S]$	Var[S]
Independence	–	–	–	2	11.80
Case #1	0.017	13.38	1.1	2	20.42
Case #2	0.005	2.71	1.5	2	41.14
Case #3	0.028	37.72	2.5	2	60.53

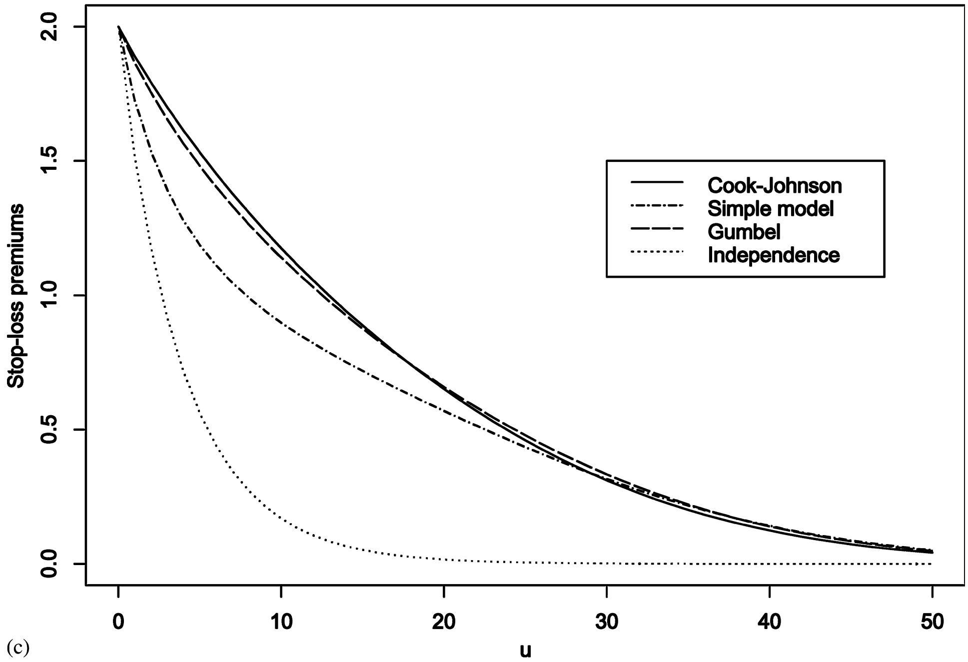


(a)



(b)

Fig. 4. Stop-loss premiums $\pi_S(u)$ for portfolios with dependent risks such that $E[S] = 2$ and (a) $\mathrm{Var}[S] = 20.42$, (b) $\mathrm{Var}[S] = 41.14$ and (c) $\mathrm{Var}[S] = 60.53$.



(c)

Fig. 4. (*Continued*).

dominating stop-loss premium for any retention level u. However, above a certain retention level u^*, we observe that the simple model leads to the maximum stop-loss premium. The value of u^* differs in each of the three cases. One also sees that the stop-loss premium obtained with the Gumbel copula dominates the one obtained with the Cook–Johnson copula for any retention level greater than a certain value u^{**} (which varies from one case to the other). The roles are reversed for small retention levels. The values at which the curves cross one another is different in the three cases (Fig. 4a–c).

5. Conclusion

Our objective was to propose two different dependence structures which can be applied in the context of the individual risk model and to illustrate their numerical applicability. We suggest two constructions which allow the introduction of a dependence relation between the occurrence random variables I_{jk} ($j = 1, \ldots, m$ and $k = 1, \ldots, n_j$) leading to dependent risks X_{jk} ($j = 1, \ldots, m$ and $k = 1, \ldots, n_j$). We show that within these constructions, the computation of the cumulative distribution function F_S for a portfolio of dependent risks is as easily tractable as in the independent case.

In future works, we wish to apply other dependence structures in the context of the individual risk model. Individual risk models based on common mixtures and their links with Archimedian copulas will be examined.

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