Asymptotic Methods

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1 Introduction

$$f(x) = x + 3 + \frac{1}{x^2}$$

graph showing asymptotically like x+3

$$f(x) - (x+3) \to \frac{1}{x^2}$$
$$g(x) = x^2 + x + 3 + \frac{1}{x^3}$$

The purpose of this course is to analyse how functions, integrals, and solutions to differential equations behave in some asymptotic limit.

1.1 Important Definitions

Definition 1.1. Big $\mathcal{O}$ Notation

For $f : (a, \infty) \to \mathbb{C}/\mathbb{R}$, $g : (a, \infty) \to \mathbb{R}$, with $g(x) > 0$ for $x \geq A > a$, we say

$$f(x) = \mathcal{O}\left(g(x)\right) \text{ as } x \to \infty$$

if $\exists M > 0, B > 0$ st

$$|f(x)| \leq Mg(x), \; x \geq B > A$$

We say

$$f(x) = \mathcal{O}\left(g(x)\right) x \to x_0$$

if $\exists M, \delta > 0$ s.t.

$$|f(x)| \leq Mg(x), \; 0 < |x - x_0| < \delta$$

or alternatively

$$\limsup_{x \to x_0} \frac{|f(x)|}{g(x)} < \infty$$

Observation. If $f(x) = \mathcal{O}\left(g(x)\right), x \to x_0$, then $cf(x) = \mathcal{O}\left(g(x)\right), c \in \mathbb{R}$
Example.
$f(x) = \frac{1}{x^2}\sin(\frac{1}{x}), x \to 0$ is $\mathcal{O}(\frac{\infty}{\S^{\in}})$, as $\limsup_{x \to 0} |\sin(\frac{1}{x})| = 1$

Definition 1.2. Little-o Notation

$f(x) = o\left(g(x)\right)$ as $x \to x_0$ if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

Example.
$f(x) = \frac{1}{x^2}\sin(\frac{1}{x})$, $g(x) = \frac{1}{x^3}$

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

So as $x \to 0$, $f(x) = o\left(g(x)\right)$

Notation. $f(x) = o\left(g(x)\right)$ is sometimes written as $f(x) \ll g(x)$.

2 Asymptotic Series

Definition 2.1. Asymptotic Sequence

$\phi_n : D \subset \mathbb{C} \to \mathbb{C}$, $n = 0, 1, \ldots$ is called an asymptotic sequence as $z \to z_0 \in D$, if

$$\forall n > m, \phi_n(z) = o\left(\phi_m(z)\right) \text{ as } z \to z_0$$

Example.

1. $\phi_n(x) = x^{n-3}$ defines an asymptotic sequence as $x \to 0$.
2. $\phi_n(x) = (x-5)^{-n}$ defines an asymptotic sequence as $x \to \infty$
3. $\phi_n(x) = \frac{1}{x^2}\cos(nx)$ is not asymptotic as $x \to 0$, as $\limsup_{x \to 0} \frac{\cos(nx)}{\cos(mx)} = 1$.

Using simple asymptotic sequences, our goal is to describe the behaviour of much more complicated functions.

Definition 2.2. Asymptotic Expansion

Let $\phi_n : D \subset \mathbb{C} \to \mathbb{C}$ be an asymptotic sequence about z_0.

We say the sum $\sum_{n=0}^{\infty} a_n \phi_n(z)$ is an asymptotic expansion of f(z) as $z \to z_0$ if for all $N \in \mathbb{N}$, as $z \to z_0$,

$$f(z) - \sum_{n=0}^{N} a_n \phi_n(z) = o\left(\phi_N(z)\right)$$

i.e.

$$\lim_{z \to z_0} \frac{f(z) - \sum_{n=0}^{N} a_n \phi_n(z)}{\phi_N(z)} = 0$$

If this holds, we write

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$$

Remark. We do not require that $\sum_{n=0}^{\infty} a_n \phi_n(z)$ converges.

Proposition 2.3. If $f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$, then

$$a_{N+1} = \lim_{z \to z_0} \frac{f(z) - \sum_{n=0}^{N} a_n \phi_n}{\phi_{N+1}}$$

Remark. If $a_n = 0 \forall n$, we have that $\lim_{z \to z_0} \frac{f(z)}{\phi_n(x)} = 0$, so our asymptotic sequence is subdominated and provides no information.

Example.
(Taylor)

$f \in C^{\infty} : [a, b] \to \mathbb{R}$, $f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$

We have that $f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \int_{x_0}^{x} \frac{(x-t)^N}{N!} f^{(N+1)}(t)dt =: R_N(x)$.

$|R_N(x)| \leq \max_{a \leq t \leq b} |f^{(N+1)}(t)| \frac{1}{N!} \int_{x_0}^{x} |x - t|^N = \frac{|x-x_0|^{N+1}}{(N+1)!} \max |f^{(N+1)}(t)|$

So $\left| \frac{R_N(x)}{(x-x_0)^N} \right| \leq \frac{|x-x_0|}{(N+1)!} \max |f^{(N+1)}(t)| \to 0$

Proposition 2.4.

1. If $f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$, $g(z) \sim \sum_{n=0}^{\infty} b_n \phi_n(z)$, then

$$\alpha f + \beta g \sim \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n) \phi_n(z)$$

2. If $f(z) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n$, $g(z) \sim \sum_{n=0}^{\infty} b_n (z - z_0)^n$ then

$$f \cdot g(z) \sim \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

with $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Proposition 2.5. The asymptotic expansion is unique. That is to say, if $f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$, $f(z) \sim \sum_{n=0}^{\infty} b_n \phi_n(z)$, then $a_n = b_n$ everywhere.

Proof.

$$a_0 - b_0 = \frac{(a_0 \phi_0 - f) + (f - b_0 \phi_0)}{\phi_0} \to 0$$

The rest follows inductively. □

Example.

$$f(x) = \begin{cases} e^{-x^2}, x \neq 0 \\ 0, x = 0 \end{cases}$$

Then $f(x)$ subdominates all polynomials, so $f(x) + \sin(x)$ has the same asymptotic expansion as $\sin(x)$ about 0. Hence we can see that while the asymptotic expansion of a function is unique, an asymptotic expansion does not uniquely define a function.

Proposition 2.6. If $f(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n$, then

$$\int_{x_0}^{x} f(\xi) d\xi \sim \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-x_0)^{n+1}$$

Proof. Let $\varepsilon > 0$. $\exists \delta(\varepsilon) > 0$ s.t. if $0 < |\xi - x_0| < \delta$, then

$$\left| f(\xi) - \sum_{n=0}^{N} a_n (\xi - x_0)^n \right| \le \varepsilon |\xi - x_0|^N$$

Then integrating,

$$\begin{aligned}
\left| \int_{x_0}^{x} f(\xi) - \sum_{n=0}^{N} a_n (\xi - x_0)^n d\xi \right| &\le \int_{x_0}^{x} \left| f(\xi) - \sum_{n=0}^{N} a_n (\xi - x_0)^n \right| d\xi \\
&\le \varepsilon \int_{x_0}^{x} |\xi - x_0|^N d\xi \\
&\le \varepsilon \frac{|x - x_0|^{N+1}}{N+1}
\end{aligned}$$

And hence

$$\frac{\left| \int_{x_0}^{x} f(\xi) d\xi - \sum_{n=0}^{N} \frac{a_n (x-x_0)^{n+1}}{n+1} \right|}{|x - x_0|^{N+1}} \le \varepsilon,$$

i.e.

$$\int_{x_0}^{x} f(\xi) d\xi - \sum_{n=0}^{N} a_n (x - x_0)^{n+1} = o\left((x-x_0)^{N+1} \right)$$

□

Exercise. For $x_0 \in (a,b)$, and $f(x)$ continuous on $[a,b]$, let $\{\phi_n\}|_{n=0}^{\infty}$ be an asymptotic sequence of functions on $[a,b]$ s.t. $\phi_n(x) \neq 0$ for $x \neq x_0$. Then,

1. $\int_{x_0}^{x} \phi_n(\xi) d\xi$ is an asymptotic sequence.

2. $\int_{x_0}^{x} f(\xi) d\xi \sim \sum_{n=0}^{\infty} a_n \int_{x_0}^{x} \phi_n(\xi) d\xi$

Exercise. $f(x) = \tan x, x = 0$.

$f(x) = x + \frac{x^3}{3} + \ldots = \sum_{n=1}^{\infty} \frac{2^n (2^{n-1}}{(2n)!} |B_{2n}| x^{2n-1}$

$\phi_n(x) = (\sin x)^n \ x^n$

Then we have asymptotic expansion

$$\tan x \sim \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} (\sin x)^n$$

2.1 Asymptotic Integrals

Gamma Function

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty t^{z-1}e^{-t}dt \qquad Re(z) > 0 \\
&= \frac{t^z}{z}e^{-t}\Big|_0^\infty \int_0^\infty \frac{t^z}{z}e^{-t}dt \\
\Rightarrow \Gamma(z) &= \frac{1}{z}\Gamma(z+1)
\end{aligned}$$

Noting that $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$, we have that $n! = \Gamma(n+1)$.

$\Gamma(n+1) = n! \sim (2\pi n)^{\frac{1}{2}}(\frac{n}{e})^n$

$\Gamma(n) \sim (\frac{2\pi}{n})^{\frac{1}{2}}(\frac{n}{e})^n[n + \frac{1}{12n} + \frac{1}{28n^2} + \ldots]$

$\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}}e^{-t}dt = \int_0^\infty 2e^{-s^2}ds = \sqrt{\pi}$

Stilljes Integral

$S(x) = \int_0^\infty \frac{\rho(t)}{1+xt}dt, \quad x \to 0^+$. Here, $\rho(t)$ is very smooth, and decays very fast at ∞. We have that $S(0) = \lim_{x\to 0} S(x) = \int_0^\infty \rho(t)dt$.

Now,

$$\begin{aligned}
\frac{1}{1+xt} &= \sum_{n=0}^{N}(-xt)^n + \frac{(-1)^{N+1}x^{N+1}t^{N+1}}{1+xt} \\
\frac{1}{1+xt} - \sum_{n=0}^{N}(-xt)^n &= \frac{(-1)^{N+1}x^{N+1}t^{N+1}}{1+xt} \\
\Rightarrow \int_0^\infty \frac{\rho(t)}{1+xt}dt - \sum_{n=0}^{N}\int_0^\infty (-xt)^n\rho(t)dt &= (-x)^{N+1}\int_0^\infty \frac{\rho(t)t^{N+1}}{1+xt}
\end{aligned}$$

Let $C_n = \int_0^\infty t^n\rho(t)dt$ be the moments of $\rho(t)$, and note that $\frac{1}{1+xt} \leq 1$ for $x, t > 0$. Then

$$\left|\int_0^\infty \frac{\rho(t)}{1+xt}dt - \sum_{n=0}^{N}(-1)^nC_nx^n\right| \leq x^{N+1}C^{N+1} = o(x^N) \text{ as } x \to 0$$

Hence

$$\int_0^\infty \frac{\rho(t)}{1+xt}dt \sim \sum(-1)^nC_nx^n$$

2.2 Approximation of Integrals

Theorem 2.7. Let f, g be continuously differentiable, with integrable derivatives on (a, b). Then

$$\int_a^b f'(t)g(t)dt = [f(t)g(t)]_a^b - \int_a^b f(t)g'(t)dt$$

In general,

$$\int_a^b f^{(n)}(t)g(t)dt = \sum_{k=1}^{n}(-1)^{k-1}f^{(n-k)}(t)g^{(k)}(t)\Big|_a^b + (-1)^n\int_a^b f(t)g^{(n)}(t)dt$$

Example.
$\text{erf}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt$. We wish to consider its behaviour for $x \gg 1$.

Let $F(x) = \sqrt{2\pi}(1 - \text{erf}(x))$. Then

$$\begin{aligned}
F(x) &= \int_x^\infty e^{-\frac{t^2}{2}}\frac{t}{t}dt \\
&= -\frac{1}{t}e^{-\frac{t^2}{2}}\Big|_x^\infty - \int_x^\infty \frac{1}{t^2}e^{-\frac{t^2}{2}}dt \\
&= \frac{e^{-\frac{x^2}{2}}}{x} - \int_x^\infty \frac{1}{t^2}e^{-\frac{t^2}{2}}dt \\
&:= \frac{e^{-\frac{x^2}{2}}}{x} - R(x) \\
|R(x)| &= \int_x^\infty \frac{e^{-\frac{t^2}{2}}}{t^2}\frac{t}{t}dt \le \frac{1}{x^3}e^{-\frac{x^2}{2}} \text{ as } \frac{1}{t^3} \le \frac{1}{x^3}
\end{aligned}$$

Hence

$$\left|\frac{F(x) - \frac{e^{-\frac{x^2}{2}}}{x}}{\frac{e^{-\frac{x^2}{2}}}{x}}\right| \le \frac{1}{x^2} \text{ as } x \to \infty$$

Repeating inductively, integrating by parts, we get that

$$\int_x^\infty e^{-\frac{t^2}{2}}dt \sim \frac{e^{-\frac{x^2}{2}}}{x}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{x^{2n}}$$

Example.
Let $f \in C^\infty([a, b])$, with $f(b) \neq 0$. Find the expansion of

$$I(x) = \int_a^b f(t)e^{xt}dt \text{ as } x \to \infty.$$

Intuition: Near $t = b$, is the main contribution. So we expect

$$\int_a^b f(t)e^{xt}dt \sim \int_{b-\varepsilon}^b f(t)e^{xt}dt \sim f(b)\int_{b-\varepsilon}^b e^{xt}dt = f(b)\frac{e^{xb} - e^{x(b-\varepsilon)}}{x} \sim \frac{f(b)e^{xt}}{x}$$

Integrating by parts, it is easy to see that

$$
\begin{aligned}
I(x) &= \sum_{k=1}^{n} f^{(n-1)} \frac{e^{xt}}{x^k}(-1)^{k-1}\Big|_a^b + (-1)^n \int_a^b f^{(n-1)}(t)\frac{e^{xt}}{x^n}dt \\
&= \sum_{k=1}^{n} f^{(n-1)}(b) \frac{e^{xb}}{x^k}(-1)^{k-1} + (-1)^n \int_a^b f^{(n-1)}(t)\frac{e^{xt}}{x^n}dt - \sum_{k=1}^{n} f^{(n-1)}(a)\frac{e^{xa}}{x^k}(-1)^{k-1} \\
&= e^{xb}\left(\sum_{k=1}^{n} f^{(n-1)}(b)\frac{1}{x^k}(-1)^{k-1}\Big|_a^b + (-1)^n \int_a^b f^{(n-1)}(t)\frac{e^{x(t-b)}}{x^n}dt - \sum_{k=1}^{n} f^{(n-1)}(a)\frac{e^{-x(b-a)}}{x^k}(-1)^{k-1}\right)
\end{aligned}
$$

We can see that $e^{-x(b-a)} = o(x^{-m}, m = 1, 2, \ldots$

$$
\begin{aligned}
\left|(-1)^n \int_a^b f^{(n-1)}(t)\frac{e^{x(t-b)}}{x^n}\right| &\le \max_{a\le t\le b} |f^{(n-1)}(t)| \int_a^b \frac{e^{x(t-b)}}{x^n}dt \\
&= \frac{1}{x^{n+1}}(e^{-x(b-a)} - 1)\underbrace{\max_{a\le t\le b}|f^{(n-1)}(t)|}_{:=C_n} \\
&= o\left(\frac{1}{x^n}\right)
\end{aligned}
$$

We can then see that

$$I(x) \sim e^{xb}\sum_{k=1}^{n}\frac{(-1)^{k-1}f^{(k-1)}(b)}{x^k},$$

so in fact

$$I(x) \sim \frac{f(b)}{x}e^{xb}$$

Theorem 2.8. Watson's Lemma

Let $0 < T \le \infty$. Suppose f(t) has an asymptotic expansion about $t = 0$ ($t > 0$)

$$f(t) \sim t^{\alpha}\sum_{n=0}^{\infty} a_n t^{\beta n} \text{ as } t \to 0^+,$$

with $\alpha > -1$, $\beta > 0$.

Assume further that $f(t)$ satisfies one of the following:

1. $|f(t)| \le Ke^{bt}, \quad t \ge 0$
2. $\int_0^T |f(t)|dt < \infty$

Then

$$F(x) = \int_0^T e^{-xt}f(t)dt \sim \sum_{n=0}^{\infty}\frac{a_n\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \text{ as } x \to \infty$$

Proof.

$$F(x) = \int_o^T e^{-xt} f(t)dt = \int_0^\varepsilon e^{-xt} f(t)dt + \int_\varepsilon^T e^{-xt} f(t)dt$$

Recall that

$$\int_0^\infty e^{-xt} t^\lambda dt \overset{u=xt}{=} \int_0^\infty e^{-u} \frac{u^\lambda}{x^\lambda} \frac{du}{x} = \frac{\Gamma(\lambda+1)}{x^{\lambda+1}}$$

We need to show that

$$R_N(x) := F(x) - \sum_{n=0}^N \frac{a_n \Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} = o\left(\frac{1}{x^{\alpha+\beta N+1}}\right) \text{ as } x \to \infty$$

By our above comment,

$$R(x) = F(x) - \sum_{n=0}^N a_n \int_0^\infty e^{-xt} t^{\alpha+\beta n} dt$$

We can rewrite this as

$$\begin{aligned}
R_N(x) &= \int_0^\varepsilon f(t)e^{-xt}dt + \int_\varepsilon^T f(t)e^{-xt}dt - \sum_{n-0}^N a_n \left(\int_0^\varepsilon e^{-xt} t^{\alpha+\beta n+1} dt + \int_\varepsilon^T e^{-xt} t^{\alpha+\beta n}\right) \\
&= \left(\int_0^\varepsilon e^{-xt} \left(f(t) - \sum_{n=0}^N a_n t^{\alpha+\beta n}\right) dt\right) + \int_\varepsilon^T e^{-xt} f(t)dt + \sum_{n=0}^N a_n \int_\varepsilon^T e^{-xt} t^{\alpha+\beta n} dt \\
&:= R_{N_1} + R_{N_2} + R_{N_3}
\end{aligned}$$

Now,

$$\begin{aligned}
|R_{N_3}| &\leq \sum_{n=0}^N |a_n| \int_\varepsilon^T e^{-xt} t^{\alpha+betan} dt \\
&= e^{-\varepsilon x} \sum_{n=0}^N \int_\varepsilon^T e^{-x(t-\varepsilon)} t^{\alpha+\beta n} \\
&\leq \frac{e^{-\varepsilon x}}{x} \sum_{n=0}^N |a_n| \int_0^\infty e^{-u} \left(1 + \frac{u}{\varepsilon x}\right)^{\alpha+\beta n} du\, \varepsilon^{\alpha+\beta n} \text{ using } u = xt \\
&\leq \frac{e^{-\varepsilon x}}{x} \sum_{n=0}^N |a_n| \int_0^\infty e^{-u} \left(1 + \frac{u}{\varepsilon x}\right)^{\alpha+\beta n} du\, \varepsilon^{\alpha} \text{ as } \beta \text{ positive} \\
&\leq \frac{e^{-\varepsilon x}\varepsilon^\alpha}{x} \sum_{n=0}^N |a_n| \int_0^\infty C(\alpha,\beta,N) \left(1 + \left(\frac{u}{\varepsilon x}\right)^{\alpha+\beta n+1}\right) du \\
&= o(x^{-m})\ \forall m \geq 1
\end{aligned}$$

Next,

$$\begin{aligned}
|R_{N_1}| &= \left| \int_0^{\varepsilon} e^{-xt} \left(f(t) - \sum_{n=0}^{N} a_n t^{\alpha+\beta N} \right) dt \right| \\
&\leq \int_0^{\varepsilon} e^{-xt} \eta t^{\alpha+\beta N} dt \\
&\leq \eta \int_0^{\infty} e^{-xt} t^{\alpha+\beta n} dt \text{ for some suitably chosen } \eta(\varepsilon) \\
&= \eta \frac{\Gamma(\alpha+\beta n+1}{x^{\alpha+\beta n+1}} \\
&= o\left(x^{-(\alpha+\beta n+1)} \right)
\end{aligned}$$

In the case that $|f(t)|$ is integrable,

$$\begin{aligned}
|R_{N_2}| &\leq e^{-xt}|f(t)|dt \\
&\leq e^{-\varepsilon x} \int_0^T |f(t)|dt \qquad\qquad = o(x^{-m}) \ \forall m \geq_1
\end{aligned}$$

Finally, in the case that $|f(t)| \leq Ke^{bt}$ asymptotically,

$$\begin{aligned}
|R_{N_2}| &\leq \int_{\varepsilon}^{T} e^{-xt}|f(t)|dt \\
&\leq K \int_{\varepsilon}^{T} e^{-xt} e^{bt} dt \text{ by assumption} \\
&\leq K \frac{e^{-(x-b)\varepsilon}}{x-b} \\
&= o(x^{-m}) \ \forall m \gg 1
\end{aligned}$$

□

If we instead consider complex values,

$$F(z) = \int_0^T e^{-zt} f(t)dt$$

Now,

$$|e^{-zt}| = e^{-xt},$$

and $|x| \geq |z| - |y| \geq |z|(1-\sin\theta) \geq |z|(1-\sin\delta)$, where we enforce that z is at least an angle δ into the RHP, $0 < \delta < \frac{\pi}{2}$.

Like this, we can reformulate Watson's lemma for complex values.
Example.
$F(x) = \int_0^{\infty} e^{-xt} \sin t dt$.

$f(t) = \sin t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}$, $|f(t)| \leq 1$, so we can apply Watson's lemma.

Then

$$I(x) \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{\Gamma(2k+1+1)}{x^{2k+2}}$$

or more neatly, as $k \in \mathbb{N}$,

$$I(x) \sim \frac{1}{x^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{x^{2k}}$$

Example.
One of the solutions of the hypergeometric equation

$$xy_{xx} + (b-x)y_x - ay = 0$$

With $\Re b > \Re a > 0$, is given by

$$M(a,b,x) = \frac{e^x}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{-xt}(1-t)^{1-a} t^{b-a-1} dt$$

So, let $f(t) = (1-t)^{1-a} t^{b-a-1}, \quad 0 \leq t \leq 1$.

$$f(t) = t^{b-a-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(a)}{\Gamma(a-n)} t^n$$

and as such we can apply Watson's Lemma to $I(x)$, getting

$$M(a,b,t) = \sum_{n=0}^{\infty} \left(\frac{(-1)^n \Gamma(a)}{\Gamma(a-n)n!} \frac{\Gamma(b-a+n)}{z^{b+a-n}} \right) \frac{e^x}{\Gamma(a)\Gamma(b-a)} - a)$$

So to leading order,

$$M(a,b,x) = \frac{e^x}{x^{b-a}\Gamma(a)}.$$

3 Asymptotics of Integrals

3.1 Laplace's Method

Consider

$$F(x) = \int_a^b f(t) e^{x\phi(t)} dt,$$

as $x \to \infty$.

Firstly, consider the case that ϕ has a **unique** global maximum at $c \in [a,b]$, such that $\phi'(c) = 0$, and $\phi''(c) < 0$. Assume also that $f(c) \neq 0$

Then we have

$$
\begin{aligned}
F(x) &\sim \int_{c-\varepsilon}^{c+\varepsilon} f(t) e^{x\phi(t)} dt \\
&\sim \int_{c-\varepsilon}^{c+\varepsilon} f(t) e^{x(\phi(c)+\frac{1}{2}\phi''(c)(t-c)^2)} dt \\
&\sim f(c) e^{x\phi(c)} \int_{c-\varepsilon}^{c+\varepsilon} e^{\frac{1}{2}\phi''(c)(t-c)^2} dt \\
&\sim \frac{f(c)e^{x\phi(c)}}{|x\phi''(c)|^{\frac{1}{2}}} \underbrace{\int_{-|x\phi''(c)|^{\frac{1}{2}}}^{|x\phi''(c)|^{\frac{1}{2}}} e^{-\frac{s^2}{2}} ds}_{s=\sqrt{-x\phi''(c)}(t-c)} \\
&\sim \frac{\sqrt{2\pi}}{|x\phi''(c)|^{\frac{1}{2}}} e^{x\phi(c)} f(c)
\end{aligned}
$$

Example.

$$
I(x) = \int_{-\infty}^{\infty} e^{-xt^2} e^{at} dt
$$

So we have $f(t) = e^{at}$, $\phi(t) = -t^2$, and

$$
\begin{aligned}
I(x) &\sim \frac{e^{a\cdot 0} e^0 \sqrt{2\pi}}{(2x)^{\frac{1}{2}}} \\
&\sim \sqrt{\frac{\pi}{x}} \text{ as } x \to \infty
\end{aligned}
$$

It is also quite simple to calculate this explicitly, where we find that this is as expected correct to leading order.

Next, we consider the case of Laplace's method when the global maximum occurs at the endpoint $t = a$, with $\phi'(a)$, $f(a) \neq 0$ (so $\phi'(a) < 0$).

$$
F(x) \sim \int_a^{a+\varepsilon} f(t) e^{x\phi(t)} dx
$$

We see $\phi(t) \sim \phi(a) + \phi'(a)(t-a)$, so that

$$\begin{aligned}
F(x) &\sim \int_a^{a+\varepsilon} f(t) e^{x\left(\phi(a)+\phi'(a)(t-a)\right)} dt \\
&\sim f(a) e^{x\phi(a)} \int_a^{a+\varepsilon} e^{\phi'(a)(t-a)} dt \\
&\sim f(a) e^{x\phi(a)} \int_0^{\varepsilon x|\phi'(a)|} e^{-u} \frac{du}{x|\phi'(a)|} \\
&\sim \frac{f(a)e^{x\phi(a)}}{x|\phi'(a)|} \left(\underbrace{\int_0^\infty e^{-u} du}_{=1} - \underbrace{\int_{\varepsilon x|\phi'(a)|}^\infty e^{-u} du}_{\text{exponentially decays in } x} \right) \\
&\sim \frac{f(a)e^{x\phi(a)}}{x|\phi'(a)|}
\end{aligned}$$

Remark. If our global maximum is instead at $t = b$, with $\phi'(b) > 0$, we see that

$$F(x) \sim \frac{f(b)e^{x\phi(b)}}{x\phi'(b)}$$

Our third case is that we have a unique global maximum of $\phi(t)$ at $c \in (a, b)$, with $\phi'(c) = \phi''(c) = 0$. Assume $f(c) \neq 0$.

$$F(x) = \int_a^b f(t) e^{x\phi(t)} dt \sim \int_{c-\varepsilon}^{c+\varepsilon} f(t) e^{\phi(t)} dt$$

As c is a maximum, we must have that

$$\phi(t) \sim \phi(c) + \frac{\phi^{(p)}(c)}{p!}(t-c)^p,$$

where p is the first (even) power such that $\phi^{(p)}(c) \neq 0$. For a maxima, $\phi^{(p)}(c) < 0$

Then

$$\begin{aligned}
F(x) &\sim \int_{c-\varepsilon}^{c+\varepsilon} f(c) e^{x\left(\phi(c)+\frac{\phi^{(p)}(c)}{p!}(t-c)^p\right)} dt \\
&\sim f(c) e^{x\phi(c)} \int_{c-\varepsilon}^{c+\varepsilon} e^{x\phi^{(p)}(c)} \frac{(t-c)^p}{p!} dt
\end{aligned}$$

Let $s = \left(\frac{x|\phi^{(p)}(c)|}{p!}\right)$. So

$$
\begin{aligned}
F(x) &\sim f(c)e^{x\phi(c)} \int_{-\varepsilon\left(\frac{x|\phi^{(p)}(c)|}{p!}\right)^{\frac{1}{p}}}^{\varepsilon\left(\frac{x|\phi^{(p)}(c)|}{p!}\right)^{\frac{1}{p}}} e^{-s^p} \frac{1}{\left(\frac{x|\phi^{(p)}(c)|}{p!}\right)^{\frac{1}{p}}} ds \\
&\sim \frac{f(c)e^{x\phi(c)}}{\left(\frac{x|\phi^{(p)}(c)|}{p!}\right)^{\frac{1}{p}}} \int_{-\infty}^{\infty} e^{-s^p} ds \\
&\sim \frac{2f(c)e^{x\phi(c)}(p!)^{\frac{1}{p}\Gamma(1+p)}}{(x|\phi^{(p)}(c)|)^{\frac{1}{p}}}
\end{aligned}
$$

Where the last integral (for even p) is left as an exercise to the reader.

Laplace Method - Higher Order Expansion

Motivation: What if $f(c) = 0$?

Again, we assume that $\phi(t)$ has a unique global maximum at $t = c$, $c \in (a, b)$, with $\phi'(c) = 0$, $\phi''(c) < 0$.

So $f(c) \sim f(c) + f'(c)(t-c) + \frac{1}{2}f''(c)(t-c)^2 + \dots$

And $\phi(c) \sim \phi(c) + \frac{1}{2}\phi''(c)(t-c)^2 + \frac{1}{6}\phi'''(c)(t-c)^3 + \frac{1}{24}\phi''''(c)(t-c)^4 + \dots$

Then as before,

$$
\begin{aligned}
F(x) &\sim \int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{x\phi(t)}dt \\
&\sim \int_{c-\varepsilon}^{c+\varepsilon} \left(f(c) + f'(c)(t-c) + \frac{1}{2}f''(c)(t-c)^2\right) e^{x\left(\phi(c)+\frac{1}{2}\phi''(c)(t-c)^2+\frac{1}{6}\phi'''(c)(t-c)^3+\frac{1}{24}\phi''''(c)(t-c)^4\right)}dt \\
&\sim \int_{c-\varepsilon}^{c+\varepsilon} \left(f(c) + f'(c)(t-c) + \frac{1}{2}f''(c)(t-c)^2\right) e^{x\phi(c)}e^{\frac{1}{2}\phi''(c)(t-c)^2}e^{x\left(\frac{1}{6}\phi'''(c)(t-c)^3+\frac{1}{24}\phi''''(c)(t-c)^4\right)}dt \\
&\sim \frac{e^{x\phi(c)}}{(|x\phi''(c)|)^{\frac{1}{2}}} \int_{-\varepsilon|x\phi''(c)|^{\frac{1}{2}}}^{\varepsilon|x\phi''(c)|^{\frac{1}{2}}} E_1(s)e^{-\frac{s^2}{2}} \exp\left(\frac{\phi'''(c)s^3}{6x^{\frac{1}{2}}|\phi''(c)|^{\frac{3}{2}}} + \frac{\phi''''(c)}{24}\frac{s^4}{x|\phi''(c)|^2}\right) ds
\end{aligned}
$$

Where

$$E_1(s) = f(c) + \frac{f'(c)}{(x|\phi''(c)|)^{\frac{1}{2}}}s + \frac{f''(c)s^2}{2x|\phi''(c)|}$$

Now, writing our second exponential as e^z and expanding to second order,

$$F(x) \sim \frac{e^{x\phi(c)}}{(|x\phi''(c)|)^{\frac{1}{2}}} \int_{-\varepsilon|x\phi''(c)|^{\frac{1}{2}}}^{\varepsilon|x\phi''(c)|^{\frac{1}{2}}} E_1(s)E_2(s)ds$$

Where

$$E_2(s) = 1 + \left[\frac{\phi''(c)s^3}{6x^{\frac{1}{2}}|\phi''(c)|^{\frac{3}{2}}} + \frac{\phi''''(c)s^4}{24z|\phi''(c)|^2}\right] + \frac{(\phi'''(c))^2 s^6}{2\cdot 32|\phi''(c)|^3}$$

Note that if c is at an endpoint, our odd terms contribute, but here, we need only consider even terms by symmetry.

So,

$$\begin{aligned}
F(x) &\sim \frac{e^{x\phi(c)}}{(|x\phi''(c)|)^{\frac{1}{2}}} \int_{-\varepsilon|x\phi''(c)|^{\frac{1}{2}}}^{\varepsilon|x\phi''(c)|^{\frac{1}{2}}} \left(f(c) + \frac{1}{x}\left[\frac{f''(c)^2}{2|\phi''(c)|} + \frac{f'(c)\phi'''(c)s^4}{6|\phi''(c)|^2} + \frac{f(c)\phi''''(c)s^4}{24|\phi''(c)|^2} + \frac{f(c)(\phi'''(c))^2 s^6}{721|\phi''(c)|^3}\right]\right) ds \\
&\sim \frac{e^{x\phi(c)}}{(x|\phi''(c)|)^{\frac{1}{2}}}\sqrt{2\pi}\left(f(c) + \frac{1}{x}\left[\frac{f''(c)^2}{2|\phi''(c)|} + \frac{f'(c)\phi'''(c)s^4}{2|\phi''(c)|^2} + \frac{f(c)\phi''''(c)s^4}{8|\phi''(c)|^2} + \frac{f(c)(\phi'''(c))^2 s^6}{24|\phi''(c)|^3}\right]\right)
\end{aligned}$$

Where we used the identity (exercise) that for $p = 2n$ even,

$$\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} s^p ds = \sqrt{2\pi} s^n \Gamma(n + \frac{1}{2})$$

4 Oscillatory Integrals

We wish to consider integrals of the form

$$I(\omega) = \int_a^b f(t)e^{i\omega\phi(t)}dt,$$

as $|\omega| \to \infty$, for $f, \phi \in \mathbb{R}$.

We often call $\phi(t)$ the phase function.

Lemma 4.1. Riemann-Lebesgue Lemma

Let $-\infty \leq a \leq b \leq \infty$, with $|f(t)|$ integrable.

Then

$$I(\omega) = \int_a^b f(t)e^{i\omega t}dt \to 0 \text{ as } \omega \to \infty$$

Proof. Consider first the case where $f \in C^1[a,b]$

$$\begin{aligned}
I(\omega) &= \int_a^b f(t)e^{i\omega t}dt \\
&= \left.\frac{f(t)e^{i\omega t}}{i\omega}\right|_a^b - \frac{1}{i\omega}\int_a^b f'(t)e^{i\omega t}dt \\
|I(\omega)| &\leq \frac{|f(b)| + |f(a)|}{|\omega|} + \frac{1}{|\omega|}(b-a)\max_{a\leq t\leq b}|f'(t)| \\
&\to 0 \text{ as } |\omega| \to \infty
\end{aligned}$$

In our general case, we go back to definitions of Riemann Integrability. $\forall \varepsilon > 0$, there is a partition such that our lower and upper sums satisfying

$$m(t) \le f(t) \le M(t)$$

with

$$\left|\int_a^b f(t) - m(t)dt\right| \le \varepsilon$$
$$\left|\int_a^b M(t) - f(t)dt\right| \le \varepsilon$$
$$\left|\int_a^b M(t) - m(t)dt\right| \le \varepsilon$$

Hence

$$\begin{aligned}
|\int_a^b f(t)e^{i\omega t}dt| &\le \left|\int_a^b (f(t) - m(t))\, e^{i\omega t}dt\right| + \left|m(t)e^{i\omega t}dt\right| \\
&\le \int_a^b f(t) - m(t)dt + \left|\int_a^b m(t)e^{i\omega t}dt\right| \\
&= \varepsilon + \left|\int_a^b m(t)e^{i\omega t}dt\right|
\end{aligned}$$

Taking $\limsup_{\omega\to\infty}$ of both sides, we get the result.

Note that in our first case, if f' is Riemann integrable, we get that $I(\omega) = \mathcal{O}(\frac{1}{\omega})$. □

4.1 Method of Stationary Phase

Proposition 4.2. Consider $f \in C^\infty[a, b]$ Then

$$I(\omega) \sum_{n=0}^{\infty} \frac{(-1)^n}{(i\omega)^{n+1}} \left(e^{i\omega b} f^{(n)}(b) - e^{i\omega a} f^{(n)}(a)\right)$$

Exercise. For $f(t) = \frac{1}{1+t}$, $I(\omega) = \int_0^1 \frac{e^{i\omega t}}{1+t}dt$.

Then

$$f^{(n)}(t) = \frac{(-1)^n n!}{(1+t)^{n+1}}$$

and

$$I(\omega) \sim \sum_{n=0}^{\infty} \frac{n!}{(i\omega)^{n+1}} \left[\frac{e^{i\omega}}{2^{n+1}} - 1\right]$$

Proposition 4.3. Suppose $\phi'(t) \neq 0$ on $[a, b]$, and $f'(t)$ integrable.

Then

$$I(\omega) \sim \frac{1}{i\omega}\left[\frac{f(b)}{\phi'(b)}e^{i\omega\phi(b)} - \frac{f(a)}{\phi'(a)}e^{i\omega\phi(a)}\right]$$

Proof. Let $u = \phi(t)$. Then we can define $t = \phi^{-1}(u)$ by monotonicity.

So

$$I(\omega) = \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(u))\frac{e^{i\omega u}}{\phi'(\phi^{-1}(u))}du$$

□

We now consider the case where $\phi(t)$ has a unique local max/min at $t = c, 1-c$, where $\phi'(c) = 0, \phi''(c) > 0$ In the areas $a < t < c - \varepsilon$ and $c + \varepsilon < t < b$, our contribution is $\mathcal{O}(\frac{1}{\omega})$ as before. So, we consider

$$\begin{aligned}\int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{i\omega\phi(t)}dt &= \int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{i\omega\left(\phi(c)+\frac{1}{2}\phi''(c)(t-c)^2\right)}dt \\ &\sim f(c)\int_{c-\varepsilon}^{c+\omega} e^{i\frac{\omega}{2}\phi''(c)(t-c)^2}dt\end{aligned}$$

We first consider $\phi''(c), \omega > 0$. Let $s = \sqrt{\frac{\omega\phi''(c)}{2}}(t-c)$. Then

$$\begin{aligned}\int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{i\omega\phi(t)}dt &\sim \frac{f(c)e^{i\omega\phi(c)}}{\sqrt{\frac{\omega\phi''(c)}{2}}}\int_{-\varepsilon\sqrt{\frac{\omega\phi''(c)}{2}}}^{\varepsilon\sqrt{\frac{\omega\phi''(c)}{2}}} e^{is^2}ds \\ &\sim \frac{f(c)e^{i\omega\phi(c)}}{\sqrt{\frac{\omega\phi''(c)}{2}}}\int_{-\infty}^{\infty} e^{is^2}ds \\ &= \frac{2f(c)e^{i\omega\phi(c)}}{\sqrt{\frac{\omega\phi''(c)}{2}}}\int_{0}^{\infty} e^{is^2}ds\end{aligned}$$

We have found ourselves a Fresnel integral, $I = \int_0^\infty e^{ixs^2}ds = \frac{1}{2}\sqrt{\frac{\pi}{x}}e^{i\frac{\pi}{4}}$. In general we might like to (for higher order expansions) consider $I_n = \int_0^\infty e^{ixs^{2n}}$. This can be done by considering wedges of angle $\frac{\pi}{2n}$.

So

$$\int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{i\omega\phi(t)}dt \sim \frac{f(c)e^{i\omega\phi(c)}\sqrt{2\pi}}{\sqrt{\omega\phi'(c)}}e^{i\frac{\pi}{4}}$$

and in general, we get

$$I(\omega) \sim f(c)e^{i\omega\phi(c)}\sqrt{\frac{2\pi}{|\omega\phi''(c)|}}e^{i\frac{\pi}{4}\operatorname{sgn}(\omega\phi''(c))} \text{ as } |\omega| \to \infty$$

As opposed to the Laplace's method, every local maxima or minima contributes. If our maxima occurs at the endpoint of the integral, we get a prefactor of $\frac{1}{2}$.

Remark. If c is a max/minx, with $\phi^{(2n)} \neq 0$ our first non-zero derivative, then we get that $I(\omega) \sim \mathcal{O}\left(\frac{1}{|\omega|^{\frac{1}{2n}}}\right)$

Example.
Bessel Function of order 0.

$$J_0(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin t}dt, \quad r \geq 0, \ r \to \infty$$

For $\phi(t) = \sin t$, we have a maximum at $t = \frac{\pi}{2}$, and a minima at $t = \frac{3\pi}{2}$.

At $\frac{\pi}{2}$, our contribution is

$$\frac{1}{2\pi}\sqrt{\frac{2\pi}{r| - \sin(\frac{\pi}{2})}}e^{ir\sin\frac{\pi}{2}(-1)} = \frac{1}{\sqrt{2\pi r}}e^{ir}e^{-i\frac{\pi}{4}},$$

and at $\frac{3\pi}{2}$ our contribution is similarly $\frac{1}{\sqrt{2\pi r}}e^{-ir}e^{i\frac{\pi}{4}}$.

Then $J_0(r) \sim \sqrt{\frac{2}{\pi r}}\cos\left(r - \frac{\pi}{4}\right)$.

4.2 Method of Steepest Descent

We now wish to consider

$$I(x) := \int_C g(z,x)dz$$

about some contour $C \subset \mathbb{C}$. We might like to try deforming our contour within a region of analyticity to some more convenient integral which is equal by Cauchy's Theorem.

We now consider the class of functions given by

$$I(x) = \int_C f(z)e^{x\phi(z)}dz$$

where f, ϕ are analytic in some region $D \supset C$.

Writing $\phi(z) = u(p,q) + iv(p,q)$, we have that $u_p = v_q$, $u_q = -v_p$, and so $\nabla u \cdot \nabla v = 0$. Also, $\nabla u \neq 0 \Leftrightarrow \nabla v \neq 0 \Leftrightarrow \phi'(z) \neq 0$. We say then that ϕ is conformal at z.

Example.
Take $\phi(z) = z^2 = p^2 - q^2 + 2ipq$. Then $|e^{x\phi(z)}| = |e^{xu}|$ for real x.

Ideal Case

Ideally, we can deform $C \to \tilde{C}$ in a manner such that, for $e^{x\phi(z)} = e^{xu(p,q)}e^{iv(p,q)}$, $\tilde{C}$ is a level curve of v, i.e. $v(p,q) = v(p_0,q_0) = \text{const}$.

In this case,

$$I(x) = e^{iv(p_0,q_0)} \int_{\tilde{C}} f(z) e^{iu(p,q)} dz$$

We can then parameterize $\tilde{C}$ as $z(t), \alpha \leq t \leq \beta$, and we can use Laplace's Method on

$$I(x) = e^{ixv(p_0,q_0)} \int_{\tilde{C}} f(z(t)) e^{xu(p(t),q(t))} (\dot{p} + i\dot{q}) dt$$

Added Value

In addition we have by Cauchy-Riemann that $\phi'(z_0) \neq 0 \Leftrightarrow \nabla u(p_0, q_0) \neq 0 \Leftrightarrow \nabla v(p_0, q_0) \neq 0$. Then if $phi'(z_0)$, we have a unique orthogonal intersection of curves with $u =$ const and $v =$ const. The curve $v =$ const is in fact the steepest descent curve of u, that is along the direction of ∇u.

Fixing Things

Suppose $v(p_0, q_0) = v(p_1, q_1)$, where C connects z_0 and z_1. In this case, we cannot deform to a curve $\tilde{C}$ with $v(p, q) =$ const.

Idea. We hope that by continuing two paths of $v =$ const from z_0 and z_1 to infinity, we can make it such that the connecting contour goes to 0, and we magically get the right answer (see example).

Consider instead what happens at a point with $\phi'(z_0)$. Then u (and v) is stationary at (p_0, q_0). Now, u, v are harmonic functions, so $u_{pp} = -u_{qq}$, and all interior stationary points must be saddle points. Then we can find a direction of steepest descent.
Example.
Saddle point

Let $\phi(z) = iz^2$. Then $\phi'(z) = 2iz = 0 \Leftrightarrow z = 0$.

Now, $u = -2pq$, and $v = p^2 - q^2$. $v(0, 0) = 0$. Consider the curve $v(p, q) = 0$. These are given by $p = q$ and $p = -q$.

On these two curves, $u = -2p^2$ or $u = 2p^2$. The former has a maximum at the origin, the latter has a minimum. Then to use steepest descent, we go along the curve $p = q$, where we can use Laplace's method.
Example.
(Bender and Orszage)

Let

$$I(x) = \int_0^1 e^{ixt^2} dt, \quad x \to \infty$$

(Fresnel integral at finite interval)

Then we expect, as t^2 is minimal at the origin, that $I(x) \sim \mathcal{O}\left(\frac{1}{\sqrt{x}}\right)$, with our leading contribution being from $t = 0$.

So, $\phi(z) = iz^2$, $f(z) = 1$, $C = [0, 1]$. We have $u = -2pq$, $v = p^2 - q^2$. Note that $v(0, 0) = 0 \neq v(1, 0) = 1$.

By before, at $z = 0$, our curve of steepest descent is $p = q$, where we go in to the UHP. Now, $\phi'(1) = 2i$, so there is a single line of steepest descent here. To have $v(p,q) = 1$, we have $p^2 - q^2 = 1$, so $p = \pm\sqrt{q^2+1}$. On this curve, we have $u = \mp 2q\sqrt{q^2+1}$. We pick the negative curve for steepest descent.

We consider truncating at $q = T$, connecting the two curves with a segment of $q = T$.

So we let $\Gamma_2(T) = \{z + s + iT : T \leq s \leq \sqrt{T^2+1}\}$.

$$\begin{aligned}
\left| \int_{\Gamma_2(T)} e^{ixz^2} dz \right| &= \left| \int_T^{\sqrt{T^2+1}} e^{ix\left(s^2 - T^2 + 2isT\right)} ds \right| \\
&\leq \left| \int_T^{\sqrt{T^2+1}} e^{-2sT} ds \right| \\
&= \frac{1}{2xT} \left(e^{-2xT^2} - e^{-2xT\sqrt{T^2+1}} \right) \\
&\to 0 \text{ as } T \to \infty
\end{aligned}$$

Let $\Gamma_1(T)$ be our curve of steepest descent, $p = q$.

$$\begin{aligned}
\int_{\Gamma_1} e^{ixz^2} dz &= \int_0^\infty e^{-2q^2x}(1+i)dq \\
&= \frac{1}{2}\sqrt{\frac{\pi}{x}} e^{i\frac{\pi}{4}}
\end{aligned}$$

Finally, we let Γ_3 be our right curve of steepest descent, $p = \sqrt{q^2+1}$.

$$\int_{-\Gamma_3} e^{ixz^2} dz = \int_0^\infty \left(\frac{q}{\sqrt{1+q^2}} + 1 \right) e^{ix} e^{-2q\sqrt{1+q^2}} dq$$

Let $s = 2q\sqrt{1+q^2}$. Then $iz^2 = i - s$, $z^2 = 1 + is$.

So

$$\begin{aligned}
\int_{-\Gamma_3} e^{ixz^2} dz &= \int_0^\infty \frac{e^{(i-s)x} i}{2(1+is)^{\frac{1}{2}}} ds \\
&= \frac{i}{2} e^{ix} \int_0^\infty \frac{e^{-sx}}{(1+is)^{\frac{1}{2}}}
\end{aligned}$$

Now,

$$\frac{1}{(1+is)^{\frac{1}{2}}} = \sum_{n=0}^\infty (-i)^n \frac{s^n}{n!} \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)},$$

And so by Watson's Lemma,

$$\int_{-\Gamma_3} e^{ixz^2} dz \sim \frac{i}{2} e^{ix} \sum_{n=0}^{\infty} (-i)^n \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right) n!} \frac{\Gamma\left(n+1\right)}{x^{n+1}}$$
$$= \frac{e^{i\frac{\pi}{2}}}{2} \sum_{n=0}^{\infty} (-i)^n \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{x^{n+1}}$$

And

$$I(x) \sim \int_{\Gamma_1} - \int_{-\Gamma_3}$$

We go back now to our general form

$$\int_C f(z) e^{x(u+iv)} dz$$

We seek to deform C to reach a saddle point, as there we have multiple curves of steepest descent, giving then the major contribution to the integral from the saddle point.

Suppose C in fact starts from a saddle point and is of steepest descent for u. Then $\phi'(z_0) = 0$. Assume that $\phi''(z_0) \neq 0$.

$$\int_C f(z) e^{x\phi(z)} dz = e^{ixv_0} \int_C f(z) e^{xu(z)} dz$$

As C is a curve of steepest descent, we can happily consider just a small region about z_0 for our majority contribution.

So

$$\int_C f(z) e^{x\phi(z)} dz \sim f(z_0) e^{x\phi(z_0)} \int_{C_\varepsilon} e^{\frac{1}{2}\phi''(z_0)(z-z_0)^2} dz$$

Let $z(t) = z_0 + r(t)e^{i\theta(t)}$. Then $\phi''(z_0)(z-z_0)^2 = \phi''(z_0) r^2 e^{2i\theta(t)}$, and $\phi''(z_0) = |\phi''(z_0)| e^{i\alpha}$. $\Im(\phi(z) - \phi(z_0)) = 0$ on C_ε, as v is constant. So, writing $\phi''(z_0)(z-z_0)^2 = |\phi''(z_0)| r^2 e^{i(2\theta+\alpha)}$, we have that $\sin(2\theta+\alpha) = 0$. Also, $\Re(\phi(z) - \phi(z_0)) < 0$ for $z \neq z_0$ on C_ε. So $\cos(2\theta+\alpha) < 0$, and so $2\theta + \alpha = (2k+1)\pi$, $k = 0, 1$.

Then

$$I(x) \sim f(z_0) e^{x\phi(z_0)} \int_{C_\varepsilon} e^{\frac{x}{2}|\phi''(z_0)| r^2 e^{i(2\theta+\alpha)}} d(re^{i\theta})$$
$$\sim f(z_0) e^{x\phi(z_0)} e^{i\theta} \int_0^{\varepsilon} e^{\frac{x}{2}|\phi''(z_0)| r^2 e^{i(2\theta+\alpha)}} dr$$

Let $s = -i \left(\frac{|\phi''(z_0)| x}{2}\right)^{\frac{1}{2}} r e^{i(\theta + \frac{\alpha}{2})}$

Then

$$
\begin{aligned}
I(x) &\sim f(z_0)e^{i\theta}e^{x\phi(z_0)}i\left(\frac{2}{x|\phi''(z_0)|}\right)^{\frac{1}{2}}e^{-i\frac{\alpha}{2}}e^{-i\theta}\int_0^\infty e^{-s^2}ds \\
&\sim \frac{i}{2}\left(\frac{2\pi}{x\phi''(z_0)}\right)^{\frac{1}{2}}e^{x\phi(z_0)}f(z_0)
\end{aligned}
$$