Further Complex Methods

1 Complex Variables

Definition 1.1. A *neighbourhood* of a point $z \in \mathbb{C}$ is an open set containing z.

Definition 1.2. The extended complex plane $\mathbb{C}_\infty$ or $\overline{\mathbb{C}}$ is defined as $\mathbb{C} \cup \{\infty\}$. All directions lead to ∞, as in the Riemann sphere.

Definition 1.3. A function $f(z)$ is *differentiable* at z if $f'(z) = \lim\limits_{a \to 0} \frac{f(z+a)-f(z)}{a}$ exists (i.e. is the same for all paths $a \to 0$).

Definition 1.4. We say that that $f(z)$ is *analytic/holomorphic/regular* at a point z if it is differentiable in a neighbourhood of z. This definition naturally extends to being analytic in a domain $D \subset \mathbb{C}$.

Proposition 1.5. Cauchy-Riemann Conditions

For $f(z) = u(z) + iv(z)$, with $u, v \in \mathbb{R}$, f is differentiable at z, iff

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Where it exists, this is equivalent to the Wirtinger derivative $\frac{\partial f}{\partial \bar{z}} = 0$, where $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$

Theorem 1.6. Cauchy s Theorem

If $f(z)$ is analytic within and on a closed contour C then $\oint_C f(z) = 0$. Note that the interior is simply connected.

Theorem 1.7. Cauchy s Integral Formula

For $z_0 \in \operatorname{int} C$,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

Consequently,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n-1}} dz,$$

implying that an analytic function is infinitely differentiable. Here, all path integrals are taken anti-clockwise.

Definition 1.8. A function $f(z)$ is *entire* if it is analytic on $\mathbb{C}$ (not $\mathbb{C}_\infty$).

Theorem 1.9. Liouville's Theorem

If f is entire and bounded on $\mathbb{C}_\infty$, then it is constant.

Proof. Consider a circular disk of radius R, i.e. $D = \{z : |z - z_0| < R\}$, and pick M s.t. $|f(z)| < M$.

Then

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \oint_C \frac{|f(z)|}{|z - z_0|^{n-1}} dz \leq \frac{n!M}{2\pi R^{n+1}} \oint_C |dz| \leq \frac{n!M}{R^n}$$

As this holds for all R, z_0, we must have that f' vanishes identically, and so $f(z) = f(0)$. □

1.1 Series expansions

An analytic function has a convergent Taylor expansion about any point within its domain of analyticity:

$$f(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

We can also consider Laurent series for functions $f(z)$ with an isolated singularity about some point z_0, but analytic in a neighbourhood of z_0.

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n, \quad C_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz$$

We can classify the singularity as follows:

$$f(z) = \sum_{n=0}^{\infty} C_n (z - z_0)^n + \sum_{n=1}^{N} C_{-n} (z - z_0)^{-n}$$

Then z_0 is:

1. A regular point (or 0) if $C_{-n} = 0 \, \forall n \geq 1$.

2. A simple pole if $N = 1$

3. A pole of order N if $N > 1$ (here we can write $f = \frac{g}{(z-z_0)^N}$, g analytic)

4. And essential singularity if $N \to \infty$.

The coefficient C_{-1} in our Laurent series is called the *residue* of f at z_0.

For a pole of order N, $C_{-1} = \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} [(z-z_0)^N f(z)] \Big|_{z=z_0}$

Theorem 1.10. Residue Theorem

If f is analytic in a simply-connected domain, except at a finite number of isolated singularities $z_1, \ldots, z_n$, then

$$\oint f(z) dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}(f(z); z_k)$$

Lemma 1.11. The Identation Lemma

Consider a simple pole at z_0.

Then

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z) dz = i(\beta - \alpha) \operatorname{res}(f; z_0),$$

where on C_ε, $z = z_0 + \varepsilon e^{i\theta}$, $\alpha \leq \theta \leq \beta$.

Proof. Consider the Laurent expansion of f about z_0.

$$f(z) = \frac{\operatorname{res}(f; z_0)}{z - z_0} + g(z),$$

where g is analytic in the region $|z - z_0| < r$, $r > 0$.

By continuity of g at z_0, we can choose r small enough such that g is bounded by some $M \in \mathbb{R}$. On $0 < \varepsilon < r$, we have

$$\begin{aligned}
\int_{C_\varepsilon} f(z) dz &= \operatorname{res}(f; z_0) \int_{C_\varepsilon} \frac{dz}{z - z_0} + \int_{C_\varepsilon} g(z) dz \\
&= i \operatorname{res}(f; z_0) \int_{\alpha}^{\beta} i d\theta + \int_{C_\varepsilon} g(z) dz \\
&= i(\beta - \alpha) \operatorname{res}(f; z_0) \text{ in the limit } \varepsilon \to 0, \text{ as } g \text{ is bounded.}
\end{aligned}$$

□

1.2 Functions defined by integrals

Consider $F(z) = \int_C f(z,t)dt$, where C is some contour in $\mathbb{C}$ (not necessarily closed). We wish to find out when such an F is defined and analytic.

Conditions on analyticity

We need to check that:

1. The integrand is continuous in t and z.
2. The integral converges uniformly in each subset of its domain.
3. The integrand is analytic in z for each value of t.

This second condition will not be treated rigorously.
Example.

$$F(z) = \int_{-\infty}^{\infty} e^{-zt^2} dt \left(= \sqrt{\frac{\pi}{z}} \right)$$

The integral converges for $\Re(z) > 0$, and diverges for $\Re(z) < 0$. If $z \in i\mathbb{R}$, then the integrand e^{-iyt} oscillates increasingly rapidly, and $F(z)$ is not absolutely convergent, but conditionally convergent, i.e.

$$\lim_{l \to \infty} \int_{-l}^{l} |e^{-iyt^2}| dt \to \infty,$$

but

$$\lim_{l \to \infty} \int_{-l}^{l} e^{-iyt^2} dt \text{ is finite.}$$

Conditions 13 hold. It can be shown that 2 also holds.
Example.

$$F(z) = \int_0^{\infty} \frac{u^{z-1}}{u+1} du$$

1. Existence: Potential problems when $u = 0, \infty$. The integrand is otherwise well behaved (except for -1, which is outside the range of integration). There are no problematic values of z, as $u^{z-1} = e^{(z-1)\log u}$.

   At $u = 0$, we have $\int_0 u^{z-1} du = \frac{u^z}{z}\Big|_0$

   $$|u^z| = |e^{z \log u}| = e^{x \log u}$$

   So to have this converge (to 0), we require $\Re(z) > 0$.

   At $u = \infty$, $u + 1 \approx u$, and $\int^{\infty} u^{z-2} = \frac{u^{z-1}}{z-1}$

   $$|u^{z-1}| = e^{(x-1)\log u},$$

   so we require $\Re(z) < 1$.

   If $\Re(z) = 0, 1$, we also do not have convergence. Thus $F(z)$ is defined for $0 < \Re(z) < 1$.

2. Analyticity: Conditions 13 are clearly satisfied in $0 < \Re(z) < 1$. 2 probably is.

So $F(z)$ is analytic for $0 < \Re(z) < 1$.

We can evaluate it using a circular keyhole contour. On C_R, $t = Re^{i\theta}$, on C_+, $t = u, \varepsilon < u < R$, on C_-, $t = ue^{2\pi i}, R > u > \varepsilon$, and on C_ε, $t = \varepsilon e^{i\theta}$.

So, $\int_{C_-} \frac{t^{z-1}}{t+1} du = -(e^{2\pi i})^{z-1} F(z)$.

As $0 < \Re(z) < 1$, $\lim_{R\to\infty} R^{1-z} =$, and so our C_R integral goes to 0. Similarly, so too does our C_ε integral.

Therefore,

$$\begin{aligned}(1 - e^{2\pi i(z-1)})F(z) &= 2\pi i \times e^{-i\pi(z-1)} \\ \Rightarrow F(z) &= \frac{\pi}{\sin \pi z}\end{aligned}$$

We will see later that $F(z) = \Gamma(z)\Gamma(1-z)$.

1.3 Analytic Continuation

We have that $F(z) = \int_{-\infty}^{\infty} e^{-zt^2} dt$ is analytic for $\Re(z) > 0$. We would like to know if it is possible to extend its domain of analyticity, and whether such an extension is unique.

Theorem 1.12. Identity Theorem

Let g_1, g_2 be analytic functions in a connected, non-empty, open set $D \subset \mathbb{C}$ with $g_1 = g_2$ in a non-empty open subset $\tilde{D} \subset D$. Then $g_1 = g_2$ on D.

Proof. (sketch)

Expand $g_1 - g_2$ as a Taylor expansion about $z_0 \in \tilde{D}$. Then the series holds in all of D, and is identically 0 in $\tilde{D}$. Therefore $g_1 - g_2 = 0$ on D. □

We can extend this proof by replacing our set $\tilde{D}$ by a contour $\gamma \subset D$.

Definition 1.13. Analytic Continuation

Let D_1, D_2 be open sets with $D_1 \cap D_2 \neq \emptyset$. Let f_1 and f_2 be analytic on D_1 and D_2 respectively, with $f_1 = f_2$ on $D_1 \cap D_2$. Then we say that f_2 is the *analytic continuation* of f_1 from D_1 to D_2.

Proposition 1.14. Our analytic continuation f_2 is unique.

Proof. Suppose there exists $\tilde{f}_2 \neq f_2$ which provides such an analytic continuation with $\tilde{f}_2 = f_1$ on $D_1 \cap D_2$. Define

$$g_1 = \begin{cases} f_1 \text{ on } D_1 \\ f_2 \text{ on } D_2 \end{cases}$$

$$g_2 = \begin{cases} f_1 \text{ on } D_1 \\ \tilde{f}_2 \text{ on } D_2 \end{cases}$$

Then by the identity theorem, $g_1 = g_2$, and so $f_2 = \tilde{f}_2$ and our analytic continuation is unique. □

Proposition 1.15. Monodromy Theorem

If we have open sets D_1, D_3 with $D_1 \cap D_3 = \emptyset$, with a function f_1 defined on D_1. A unique analytic continuation of this to D_3 is possible iff we can analytically continue f_1 through all domains D_2 connecting D_1 and D_3 with $D_1 \cap D_2, D_2 \cap D_3 \neq \emptyset$.

Proof. Left as an exercise to a different reader. □

Methods of Analytic Expansion

1. Taylor expansion

   If we pick z_0 near the boundary of our domain D, we can extend f_1 to a disk $|z - z_0| < r$ for some radius of convergence r.
   Example.
   Note that

$$\begin{aligned} f(z) &= \frac{1}{1-z} \\ &= \frac{1}{1-z_0} \frac{1}{1 - \frac{z-z_0}{1-z_0}} \\ &= \frac{1}{1-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{1-z_0} \right)^n \end{aligned}$$

,

   which converges for $|z - z_0| < |1 - z_0|$.

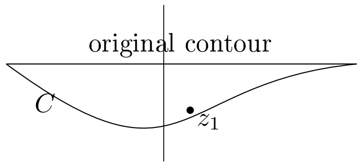
   Now, let $f_1 = \sum_{n=0}^{\infty} z^n$. It is analytic for $|z| < 1$. Let $f_2 = \sum_{n=0}^{\infty} \frac{(z-\frac{i}{2})^n}{(1-\frac{i}{2})^{n+1}}$, analytic on $|z - \frac{i}{2}| < \frac{\sqrt{5}}{2}$. We have that $f_1 = f_2$ on the intersection of the disks, and hence by the identity theorem, f_2 is the analytic continuation of f_1. This can be continued as a chain of disks covering $\mathbb{C} \setminus \{1\}$, to obtain the function $\frac{1}{1-z}$, which has a simple pole at $z = 1$.

   This is known as meromorphic continuation (analytic continuation excluding singularities). However, such extensions are not always possible.
   Example.
   Let $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ is convergent in $|z| < 1$ by ratio test, but its singularities are dense, and analytic continuation is not possible. We call $|z| = 1$ a *natural barrier*.

2. Contour deformation
   Example.
   Let $F(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z} dt$ for $\Im z > 0$.

   We want to continue $F(z)$ to the lower half-plane, but obviously it is not analytic for $\Im z = 0$. So, we might think to re-define F for $\Im z \neq 0$. We shall see shortly why this does not work.

   Pick z_1 with $\Im z_1 < 0$. We wish to continue F into a neighbourhood of z_1 by deforming our path of integration.



   Define
   $$F_1(z) = \int_C \frac{e^{it}}{t-z}$$

   Then F_1 is analytic for all z above z_1. For $\Im z > 0$, we can see by deforming our new path to the real axis, that $F_1 = F$. Therefore, F_1 is the analytic continuation of F into $\Im z < 0$.

   Now, instead consider $G(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z} dt$ for $\Im z \neq 0$. So if $\Im z > 0$, then $G(z) = F(z)$ by definition.

   If $\Im z < 0$, then consider closing the contour with our path C above. We find
   $$F_1(z) - G(z) = 2\pi i e^{iz}$$

   So for $\Im z > 0$, we have that $F = F_1 = G$, and for $\Im z < 0$ we have $F_1 = G - 2\pi i e^{iz}$.

   Hence G jumps by $2\pi i e^{iz}$ as it crosses the real axis.

1.4 Cauchy Principal Value

Idea. Can we say that

$$\int_{-1}^{2} \frac{dx}{x} = \log 2 - \log|-1| = \log 2?$$

Definition 1.16. If $f(x)$ is badly-behaved at $x = c$ and $a < c < b$, we can define the *Cauchy Principal Value* integral by

$$\mathcal{P}\int_a^b f(x)dx := \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^{b} f(x)dx$$

when the limit exists of course.

Example.
Let $I = \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x}dx$, where f is analytic in the upper half-plane and real axis, and $f(x) \to 0$ at infinity.

Closing in the UHP, our C_R contribution vanishes in the limit $R \to \infty$, and our C_ε term contributes $-i\pi f(0)$, where by analyticity of f, our residue at the origin is $f(0)$. Hence

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x} = i\pi f(0)$$

Example.
Let $I = \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}$. Our integrand has a removable singularity at $x = 0$. We can show using standard methods that $I = \pi$.

Alternatively, $I = \Re\mathcal{P}\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}$. Closing this in an arch, we get by indentation lemma that

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x} - i\pi(-i) = 0,$$

So $I = \pi$, and $\mathcal{P}\int_{-\infty}^{\infty} \frac{\sin x}{x^2} = 0$.

Hilbert Transforms

Definition 1.17. The Hilbert transform of $f(x)$ is defined by

$$\mathcal{H}(f)(y) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)dx}{x-y}$$

Remark. Observe that $\mathcal{H}$ is a linear functional.

We shall assume that f has a Fourier decomposition, so we only need to consider the Hilbert transform of $e^{i\omega x}$, and then use linearity of the transform. We will show that

$$\mathcal{H}(e^{i\omega x})(y) = \begin{cases} ie^{i\omega y}, \ \omega > 0 \\ -ie^{i\omega y}, \ \omega < 0 \end{cases} = i\,\mathrm{sgn}(\omega)e^{i\omega y}$$

Integrating in an arch shaped contour about y, we get

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{x-y}dx+\underbrace{\int_{C_R}}_{\to 0}+\underbrace{\int_{C_\varepsilon}}_{=-i\pi e^{i\omega y}}=0$$

And flipping our arch for $\omega<0$, we get a negative sign from the indentation lemma, so the result indeed holds.

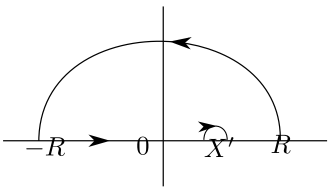
Remark. From this it follows that $\mathcal{H}^2(e^{i\omega x})=-e^{i\omega x}$, so $\mathcal{H}$ is "anti-self-inverse" here.

More (but not completely of course) generally, if $g(y)=\mathcal{H}(f)(y)$, then

$$f(x)=\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}-\frac{g(y)}{y-x}$$

Kramers-Kronig Relations

Let $f=u+iv$ be analytic in $\Im z>0$, with $f\to 0$ as $|z|\to\infty$. Let $x'=z'\in\mathbb{R}$, and consider C as follows:



Then by indentation lemma,

$$\int_C\frac{f(z)}{z-z'}dz=\mathcal{P}\int_{-\infty}^{oo}\frac{f(x)dx}{x-x'}-i(x)=0 \tag{1}$$

For $z\in\mathbb{R}$, we can write $f(z)=f(x,y)=f(x,0)$, with $f(x,0)=u(x,0)+iv(x,0)$.

Hence, taking real and imaginary parts of (1), we obtain

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{u(x)dx}{x-x'} = -\pi v(x')$$
$$\mathcal{P}\int_{-\infty}^{\infty}\frac{v(x)dx}{x-x'} = \pi u(x')$$

Or

$$\mathcal{H}u(x') = -v(x')$$
$$\mathcal{H}v(x') = u(x')$$

These are known as the Kramers-Kronig relations, relating the real and imaginary parts of functions analytic in the upper half plane.
Example.
The Laplace equation in the upper half plane

Let $u(x,y)$ be a harmonic function in $\Im z > 0$. Recall that for $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, we have

$$4444\frac{\partial^2 u}{\partial z \partial \overline{z}}u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Suppose $u \to 0$ for large $|x|, y$.

Consider $F(z) = \frac{\partial u}{\partial z}$. $\frac{\partial F}{\partial \overline{z}}$ implies analyticity for $\Im z > 0$, and so

$$u_x(x,0) = -\mathcal{H}u_y(x,0)$$
$$u_y(x,0) = \mathcal{H}u_x(x,0)$$

1.5 Multivalued Functions

Definition 1.18. A multivalued function $f(z)$ admits more than one value for given z.

Definition 1.19. A point $z = a$ is a branch point of the multivalued function $f(z)$ if f is discontinuous upon traversing in a small circle about $z = a$ i.e. $f(a + re^{2\pi i}) \neq f(a + r)$.

Example.
$f(z) = (1-z^2)^{\frac{1}{2}} = (1-z)^{\frac{1}{2}(1+z)^{\frac{1}{2}}}$ has branch points at $z = \pm 1$.

For $z = 1$, consider $z = 1 + \varepsilon e^{i\theta}, 0 < \varepsilon \ll 1$.

$$f(z) = (-\varepsilon e^{i\theta})^{\frac{1}{2}}(2 + \varepsilon e^{i\theta})^{\frac{1}{2}} \approx \pm i\sqrt{2\varepsilon}e^{i\frac{\theta}{2}}$$

And it is easily seen that this is a branch point.

Note that ∞ is not a branch point (consider $t = \frac{1}{z}$).

We seek to express a multivalued function in terms of a singlevalued function. This is achieved by restricting the region in $\mathbb{C}$ to cut in such a way that the resulting function is singlevalued and continuous.

Definition 1.20. A continuous singlevalued function obtained in this way is called a *branch* of the multivalued function.

Integrating using a branch cut

We seek to evaluate

$$I = \int_{-1}^{1} (1 - x^2)^{\frac{1}{2}} dx$$

by contour integration.

We choose $f(z)$ to be the branch of $(1 - z^2)^{\frac{1}{2}}$ with $f(0^+) = 1$, given in local polars by

$$f(z) = |1 - z^2|^{\frac{1}{2}} e^{\frac{i}{2}\left(\phi_1 + \phi_2 - \frac{\pi}{2}\right)}$$

You then do some boring stuff, and get the answer to be 2π or something.

The arcsin function defined as an integral

Let

$$z = \int_0^{2\pi} \frac{dt}{(1 - t^2)^{\frac{1}{2}}},$$

where $\sqrt{1 - t^2}$ is defined by a branch cut between -1 and 1 as before, such that it takes value 1 at 0^+, and where $0 \leq \arg z < \pi$

See siklos' notes.

2 Special Functions

2.1 The Gamma Function

We are motivated by finding a smooth curve that interpolates the points $f(n) = n!, n \in \mathbb{N}$. We find such a magical function to be given by $f(x) = \Gamma(x + 1)$! We now seek to generalize this in integral form to $\mathbb{C}$.

Let $I(z) = \int_0^\infty t^{z-1} e^{-t} dt$ (Euler's Integral), which converges and is analytic for $\Re z > 0$.

Now,

$$\begin{aligned} I(z + 1) = \int_0^\infty t^z e^{-t} dt &= \left[-t^z e^{-t}\right]_0^\infty + \int_0^\infty z t^{z-1} e^{-t} dt \\ &= (z) \end{aligned}$$

Also, $I(1) = 1$. Hence

$$I(n + 1) = n! I(1) = n!, \; n \in \mathbb{N}$$

So our idea is to define

$$\Gamma(z) = \begin{cases} I(z), & \Re z > 0 \\ \text{Analytic continuation elsewhere} \end{cases}$$

Now, we can see that

$$I(z) = \frac{I(z+1)}{z}$$

is analytic for $\Re(z+1) > 0$, and $z \neq 0$. As such, we can iteratively extend this to

$$I(z) = \frac{I(z+n+1)}{z(z-1)\dots(z+n)},$$

which is analytic for $\Re z > -(n+1)$, $z \neq 0, -1, \dots, -n$.

Hence we can meromorphically continue $\Gamma(z)$ to $\mathbb{C} \setminus \{-n : n \in \mathbb{N}\}$, with simple poles at the negative integers. It is easily seen that $\text{res}(\Gamma(z); -n) = (-1)^n \Gamma(1) \frac{1}{n!} = \frac{(-1)^n}{n!}$

Some alternative definitions and formulae

Proposition 2.1. Euler Product Formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\dots(z+n)}, \qquad \forall z \in \mathbb{C} \setminus (-\mathbb{N})$$

Proof. Firstly, we consider $\Re z > 0$. Recall that $e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n$.

So,

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt \\
&= \lim_{n\to\infty} n^z \left[\frac{(1-\tau)^n \tau^z}{z}\right]_0^1 - \frac{n^\tau}{z}(-n) \int_0^1 (1-\tau)^{n-1} \tau^z d\tau \quad (\tau = \frac{t}{n}) \\
&= \lim_{n\to\infty} 0 + (-1)^n n^z n! \int_0^1 \frac{\tau^{z+n-1}}{z(z+1)\dots(z+n-1)} \\
&= \lim_{n\to\infty} \frac{n! n^z}{z(z-1)\dots(z+n)}
\end{aligned}$$

For $\Re z \leq 0$, it is clear to see that our analytic continuation by $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ continues the product formula, and is indeed analytic. □

Proposition 2.2. Gauss Product Formula

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^z}{1 + \frac{z}{n}}$$

Proof. By the Euler product formula, we can write

$$
\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \frac{1}{z} \frac{n^z}{\frac{z+1}{1}\frac{z+2}{2}\dots\frac{z+n}{n}} \\
&= \frac{1}{z} \lim_{n\to\infty} \frac{\left.\frac{n+1}{n}\right)^z \left(\frac{n}{n-1}\right)^z \dots \left.\frac{2}{1}\right)^z \left(\frac{n}{n+1}\right)^z}{(1+z)(1+\frac{z}{2})\dots(1+\frac{z}{n})}
\end{aligned}
$$

As $\left(\frac{n}{n+1}\right)^z \to 1$ as $n \to \infty$, we obtain the required expression. □

Proposition 2.3. The Weierstrass Canonical Product

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}},$$

where $\gamma = \lim\limits_{n\to\infty} 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n \approx 0.577$ is the Euler-Mascheroni constant.

Proof. Using Euler's product formula,

$$
\begin{aligned}
\frac{1}{\Gamma(z)} &= z \lim_{n\to\infty} \frac{(1+z)(2+z)\dots(n+z)}{n!n^z} \\
&= z \lim_{n\to\infty} e^{-z\log n} (1+z)\left(1+\frac{z}{2}\right)\dots\left(1+\frac{z}{n}\right) \\
&= z \lim_{n\to\infty} e^{-z\left(\log n - (1+\frac{1}{2}+\ldots+\frac{1}{n})\right)} e^{-z\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)} (1+z)\dots\left(1+\frac{z}{n}\right) \\
&= ze^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}}
\end{aligned}
$$

□

Proposition 2.4. Reflection Formula

$$\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z), \quad z \notin \mathbb{Z}$$

Proof. We first consider the case $\Re z \in (0,1)$ so that we can write $\Gamma(z)$ and $\Gamma(1-z)$ can be written in integral form. Using substitutions $t = r\sin^2\theta$, $s = r\cos^2\theta$, we have

$$
\begin{aligned}
\Gamma(z)\Gamma(1-z) &= \int_0^\infty e^{-t}t^{z-1}dt \int_0^\infty e^{-s}s^{-z}ds \\
&= 2\int_0^{\frac{\pi}{2}} (\tan\theta)^{2z-1} d\theta \\
&= \int_0^\infty \frac{u^{z-1}}{u+1} du \\
&= \frac{\pi}{\sin(\pi z)}
\end{aligned}
$$

Where we used the substitution $\tan\theta = u^{\frac{1}{2}}$, and calculated the last integral earlier.

Now, $\Gamma(z)$, $\Gamma(1-z)$ and $\pi(\csc \pi z)$ are analytic for all z except integer points, and they are equal for $\Re z \in (0,1)$, and so the result holds by analytic continuation. $\square$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

2.2 Hankel Representation of $\Gamma(z)$

Proposition 2.5. Hankel Representation

For $z \notin \mathbb{Z} \setminus \mathbb{N}$,

$$\Gamma(z) = \frac{1}{2i\sin(\pi z)} \int_{-\infty}^{0^+} e^t t^{z-1} dt,$$

where $-\pi \leq \arg t \leq \pi$, and the path is called the *Hankel contour*. Note that the function is analytic in both z and t.

Well-Definedness of Hankel Integral

Note that for $\Re z > 0$, the Hankel representation is equal to the Gaussian integral $I(z)$ from earlier. To see this, we collapse the Hankel contour onto the branch cut, and define for $\Re z > 0$,

$$J(x) = \int_{-\infty}^{0^+} e^t t^{z-1} dt = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_\varepsilon},$$

defining contours

$$\gamma_1 : t = xe^{i\pi}, \infty > x > \varepsilon, \quad \gamma_2 : t = xe^{i\pi}, \varepsilon < x < \infty, \quad \gamma_\varepsilon : t = \varepsilon e^{i\theta}, -\pi < \theta < \pi$$

Note that we have

$$\begin{aligned} \int_{\gamma_1} &\to (e^{-\pi})^z \int_{-\infty}^{0} e^{-x} x^{z-1} dx \\ \int_{\gamma_2} &\to (e^{i\pi})^z \int_{0}^{\infty} e^{-x} x^{z-1} dx \\ \int_{\gamma_\varepsilon} &\to 0 \text{ as } \Re z > 0 \text{ and } \varepsilon \to 0 \end{aligned}$$

so we have

$$J(z) = 2i\sin(\pi I(z)$$

Hence the claim is proved by analytic continuation.

Note that for $z \in \mathbb{N}$, the zeroes of $\sin(\pi z)$ are cancelled by the integral, and $t = 0$ is not a branch point, so there are no singularities in the Hankel contour. This suggests that $J(z) = 0$.

Residues of $\Gamma(z)$ in Hankel Representation

In this case with $z \in \mathbb{N}$, we can choose a Hankel contour to be a unit circle enclosing the origin anticlockwise. Now,

$$J(-m) = \int_{|t|=1} e^t t^{-(m+1)} dt = 2\pi i \operatorname{res}\left(e^t t^{-(m+1)}; 0\right)$$

Using Taylor expansion,

$$e^t t^{-(m+1)} = \sum_{n=0}^{\infty} \frac{t^{n-m-1}}{n!},$$

and the residue is then the coefficient of t^{-1}, $m!$. So $J(-m) = \frac{2\pi i}{m!}$.

Thus the residue of $\Gamma(z)$ at $z = -m$ is $\lim\limits_{z \to -m} \frac{z+m}{2i \sin \pi z} J(z) = \frac{2\pi i}{m!} \lim\limits_{z \to -m} \frac{z+m}{2i \sin \pi z} = \frac{(-1)^m}{m!}$ by l'Hôpital as expected.

We now seek to answer whether the Gamma function is the unique analytic interpolation problem of the factorial.

Theorem 2.6. Wielondt's Theorem

If $F(z)$ satisfies:

1. $F(z)$ is analytic for $\Re z > 0$
2. $F(z+1) = zF(z)$
3. $F(z)$ is bounded in $1 \leq \Re z \leq 2$
4. $F(1) = 1$

then $F(z) = \Gamma(z)$.

Lemma 2.7. Define the difference function

$$f(z) := F(z) - \Gamma(z)$$

Then $f(z)$ is entire.

Proof. Properties 1 and 2 imply that $F(z)$ can be meromorphically continued into $\mathbb{C} \setminus (-\mathbb{N})$,

$$F(z) = \frac{F(z+n)}{z(z+1)\dots(z+n-1)}$$

By property 4, $\operatorname{res}(F(z); -n) = \frac{F(1)(-1)^n}{n!}$, which is the same as the gamma function. Hence $f(z)$ has only removable poles, and is in fact entire. ☐

Lemma 2.8. $f(z)$ is bounded in the strip $0 \leq \Re z \leq 1$.

Proof. We first show that $f(z)$ is bounded on $1 \leq \Re z \leq 2$. It suffices to check that $\Gamma(z)$ is.

$$\begin{aligned} |\Gamma(z)| &= \left| \int_0^\infty e^{-t} t^{z-1} dt \right| \\ &\leq \int_0^\infty \left| e^{-t} t^{x+iy-1} \right| dt \\ &= \int_0^\infty e^{-t} t^{x-1} dt \\ &\leq \int_0^\infty e^{-t} t^{2-1} dt \\ &= 1 \end{aligned}$$

We examine our last inequality more closely:

Define $I(x) = \int_0^\infty e^{-t} t^{x-1} dt$ $I(1) = I(2) = 1$. $\frac{d^2 I}{dx^2} > 0$, so $I(x)$ is convex in $[1, 2]$, and the inequality indeed holds.

Now, for $0 \leq \Re z \leq 1$, we can write $f(z) = \frac{f(z+1)}{z}$. As f is bounded on $1 \leq \Re z \leq 2$, we conclude that it is also bounded on $0 \leq \Re z \leq 1$, noting that the pole is removable at the origin. □

We now prove the original theorem.

Proof. (2.6)

Let $S(z) = f(z) f(1-z)$. $S(z)$ is entire by lemma 2.7, and is bounded in $0 \leq \Re z \leq 1$ by lemma 2.8. Indeed, both $f(z)$ and $f(1-z)$ have the same range in this domain by symmetry.

Now, $S(z+1) = f(z+1) f(-z) = z f(z) (-z)^{-1} f(1-z) = -S(z)$. Thus $S(z)$ is bounded in $1 \leq \Re z \leq 2$.

Also, $S(z+2) = S(z)$, so $S(z)$ is periodic with period 2, and so is bounded in $\mathbb{C}$. Hence by Liouville's theorem, we must have that $S(z) = S(1) = f(1) f(0) = (F(1) - \Gamma(1)) f(0) = 0$. Then $f(z) f(1-z) = 0$ for all z. Hence $f(z) \equiv 0$, and $F(z) \equiv \Gamma(z)$.

□

2.3 The Beta Function

Definition 2.9.

$$B(p, q) := \int_0^1 t^{p-1} (1-t)^{q-1} dt, \quad \Re p, \Re q > 0,$$

and is analytically continued in p and q.

Setting $t = \sin^2 \theta$, it is easily shown (on the example sheet) that

$$B(p, q) = 2 \int_0^{\frac{\pi}{2}} \sin^{2p-1} \cos^{2q-1} \theta d\theta$$

Proposition 2.10.
1. $B(p, q) = B(q, p)$
2. $B(1, q) = \frac{1}{q}$
3. $B(p, z + 1) = \frac{z}{p+z} B(p, z)$

Proof. (1) and (2) are trivial. For (3):

$$\begin{aligned} B(p, z + 1) &= \int_0^1 t^{p-1}(1 - t)^{z-1}(1 - t)dt \\ &= B(p, z) - B(p + 1, z) \\ &= B(p, q) - \frac{p}{z} B(p, z + 1) \text{ upon integrating by parts.} \end{aligned}$$

□

This last identity gives us an analytic continuation of the Beta function into $\Re z > -1$, just as we constructed for the Gamma function.

As our continuation is from $B(p, z) = \frac{p+z}{z} B(p, z+1)$, it is easy to see that much like the Gamma function, for fixed p there are simple poles at $z \in (-\mathbb{Z})$.

Proposition 2.11.
4. $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

Notice that for $(n, m) \in \mathbb{N}^2$, $B(n, m) = \frac{(n-1)!(m-1)!}{(n+m-1)!}$

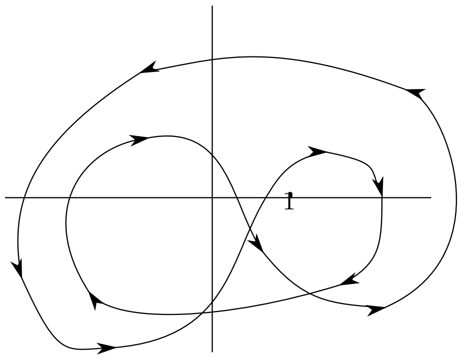
Proof.

$$\begin{aligned} \Gamma(p)\Gamma(q) &= \int_0^\infty e^{-s} s^{p-1} ds \int_0^\infty e^{-t} t^{q-1} \\ &= \Gamma(p + q)(p, q), \quad \text{using } s = r \cos^\theta, t = r \sin^2 \theta \end{aligned}$$

□

Proposition 2.12. Pochhammer Representation (non-examinable)

Let $J(p,q) := \int_P f(t)dt$, where P is Pochhammer's contour



See handout.

2.4 The Zeta function

Definition 2.13.

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \Re z > 1$$

and is analytically continued wherever possible.

Euler showed the well known result that $\zeta(2) = \frac{\pi^2}{6}$.

Proposition 2.14. Integral Representation of $\zeta(z)$

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt, \quad \Re z > 1$$

Proof. Let $t = ns$, for some fixed $n \in \mathbb{N}$, with $s \in \mathbb{R}$.

Then

$$\Gamma(z) = \int_0^\infty n^z s^{z-1} e^{-ns} ds, \quad \Re z > 0$$

Hence

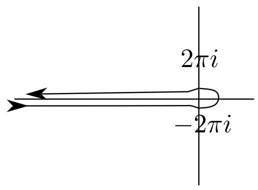
$$\begin{aligned}\zeta(z)\Gamma(z) &= \sum_{n=1}^{\infty}\int_0^{\infty} s^{z-1}e^{-ns} \\ &= \int_0^{\infty} t^{z-1}\sum_{n=1}^{\infty} e^{-nt}dt \\ &= \int_0^{\infty} \frac{t^{z-1}}{e^{-t}-1}dt\end{aligned}$$

□

Proposition 2.15. Hankel Representation

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{0^+} \frac{t^{z-1}}{e^{-t}-1}dt$$

Note that the integrand has simple poles at $2\pi i n$, for $n \in \mathbb{Z}$. We take a branch cut on the negative real axis.



Proof. We show that

$$\frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{0^+} \frac{t^{z-1}}{e^{-t}-1}dt = \frac{1}{\Gamma(z)}\int_0^{\infty} \frac{t^{z-1}}{e^{t}-1}dt,$$

and show that the LHS gives the analytic continuation of the RHS into $\Re z < 1$.

On our bottom line, $t = xe^{-i\pi}$, on the circle, $t = \varepsilon e^{i\theta}$, and on top $t = xe^{i\pi}$. Treating this just as with the Gamma function,

$$
\begin{aligned}
\int_{-\infty}^{0^+} &= \int_{\gamma_1} + \underbrace{\int_{\gamma_\varepsilon}}_{\to 0} + \int_{\gamma_2} \\
&= \left(e^{i\pi z} - e^{-i\pi z}\right) \int_0^\infty \frac{x^{z-1}}{e^x - 1} dx \\
&= 2i \sin \pi z \Gamma(z) \zeta(z) \text{ by (2.14)}
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\Gamma(1-z)}{2\pi i} \int_{-\infty}^{0^+} \frac{t^{z-1}}{e^{-t}-1} dt &= \frac{\Gamma(1-z)}{2\pi i} 2i \sin \pi z \Gamma(z) \zeta(z) \\
&= \pi \frac{\csc \pi z}{\pi} \sin \pi z \zeta(z) \text{ by reflection formula} \\
&= \zeta(z)
\end{aligned}
$$

□

The integral on the LHS is entire in z and smooth in t, and hence provides an analytic continuation of $\zeta(z)$ into $\Re z < 1$.

Proposition 2.16. The ζ-function extends to a meromorphic continuation into $\mathbb{C}$, with the only singular point being a simple pole at $z = 1$ with residue 1.

Proof. Notice that $\Gamma(1-z)$ has simple poles at $z = 1, 2, 3, \ldots$ But $\zeta(z)$ is analytic for $\Re z > 1$ from its series definition. Hence, $z = 1$ is the only singularity of ζ.

The residue

$$
\begin{aligned}
\operatorname{res}(\zeta(z); 1) &= \lim_{z \to 1} \frac{(z-1)\Gamma(1-z)}{2\pi i} \int_{-\infty}^{0^+} \frac{t^{z-1}}{e^{-t}-1} dt \\
&= \lim_{z \to 1} \frac{(z-1)\Gamma(1-z)}{2\pi i} {}_{|z|=\frac{1}{2}} \frac{dt}{e^{-t}-1} dt
\end{aligned}
$$

Note that for $z = -n$, $n \in \mathbb{N}_0$, then $\Gamma(z) = \frac{(-1)^n}{n!} \frac{1}{z+n} +$ analytic function.

So,

$$
\lim_{z \to 1} (z-1)\Gamma(1-z) = \lim_{z \to 1} (z-1) \left(\frac{(-1)^0}{0!} \frac{1}{1-z} + \text{ analytic function} \right) = -1
$$

Also,

$$
{}_{|z|=\frac{1}{2}} \frac{dt}{e^{-t}-1} = 2\pi i \cdot (-1)
$$

Hence $\operatorname{res}(\zeta(z); 1) = 1$. □

What about the zeroes of $\zeta(z)$?

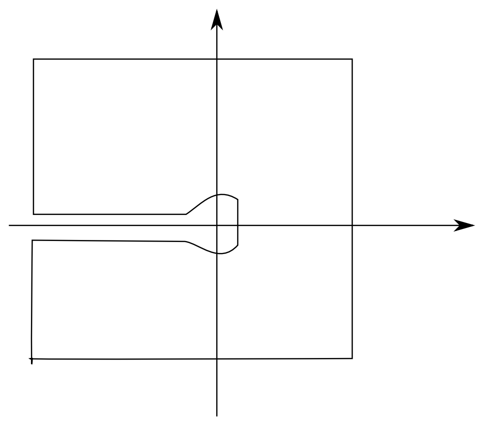
Proposition 2.17. Functional Equation for $\zeta(z)$.

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z)$$

for all z.

Proof. We derive this for $\Re z < 0$, and then use analytic continuation.

We modify the Hankel contour as follows, closing in a rectangle with vertices at $z = \pm R \pm (2n+1)\pi i$:



The integral has a branch cut on the negative real axis, with branch point at 0, and poles at $z = 2\pi i n$, $n \in \mathbb{Z} \setminus \{0\}$. The residues at these points are given by $\frac{1}{(2\pi i n)^{1-z}}$.

$\square$