Further Complex Methods

1 Complex Variables

Definition 1.1. A *neighbourhood* of a point $z \in \mathbb{C}$ is an open set containing z.

Definition 1.2. The extended complex plane $\mathbb{C}_\infty$ or $\overline{\mathbb{C}}$ is defined as $\mathbb{C} \cup \{\infty\}$. All directions lead to ∞, as in the Riemann sphere.

Definition 1.3. A function $f(z)$ is *differentiable* at z if $f'(z) = \lim\limits_{a \to 0} \frac{f(z+a)-f(z)}{a}$ exists (i.e. is the same for all paths $a \to 0$).

Definition 1.4. We say that that $f(z)$ is *analytic/holomorphic/regular* at a point z if it is differentiable in a neighbourhood of z. This definition naturally extends to being analytic in a domain $D \subset \mathbb{C}$.

Proposition 1.5. Cauchy-Riemann Conditions

For $f(z) = u(z) + iv(z)$, with $u, v \in \mathbb{R}$, f is differentiable at z, iff

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Where it exists, this is equivalent to the Wirtinger derivative $\frac{\partial f}{\partial \bar{z}} = 0$, where $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$

Theorem 1.6. Cauchy s Theorem

If $f(z)$ is analytic within and on a closed contour C then $\oint_C f(z) = 0$. Note that the interior is simply connected.

Theorem 1.7. Cauchy s Integral Formula

For $z_0 \in \operatorname{int} C$,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

Consequently,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n-1}} dz,$$

implying that an analytic function is infinitely differentiable. Here, all path integrals are taken anti-clockwise.

Definition 1.8. A function $f(z)$ is *entire* if it is analytic on $\mathbb{C}$ (not $\mathbb{C}_\infty$).

Theorem 1.9. Liouville's Theorem

If f is entire and bounded on $\mathbb{C}_\infty$, then it is constant.

Proof. Consider a circular disk of radius R, i.e. $D = \{z : |z - z_0| < R\}$, and pick M s.t. $|f(z)| < M$.

Then

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \oint_C \frac{|f(z)|}{|z - z_0|^{n-1}} dz \leq \frac{n!M}{2\pi R^{n+1}} \oint_C |dz| \leq \frac{n!M}{R^n}$$

As this holds for all R, z_0, we must have that f' vanishes identically, and so $f(z) = f(0)$. □

1.1 Series expansions

An analytic function has a convergent Taylor expansion about any point within its domain of analyticity:

$$f(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

We can also consider Laurent series for functions $f(z)$ with an isolated singularity about some point z_0, but analytic in a neighbourhood of z_0.

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n, \quad C_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz$$

We can classify the singularity as follows:

$$f(z) = \sum_{n=0}^{\infty} C_n (z - z_0)^n + \sum_{n=1}^{N} C_{-n} (z - z_0)^{-n}$$

Then z_0 is:

1. A regular point (or 0) if $C_{-n} = 0 \, \forall n \geq 1$.

2. A simple pole if $N = 1$

3. A pole of order N if $N > 1$ (here we can write $f = \frac{g}{(z-z_0)^N}$, g analytic)

4. And essential singularity if $N \to \infty$.

The coefficient C_{-1} in our Laurent series is called the *residue* of f at z_0.

For a pole of order N, $C_{-1} = \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} [(z - z_0)^N f(z)] \Big|_{z=z_0}$

Theorem 1.10. Residue Theorem

If f is analytic in a simply-connected domain, except at a finite number of isolated singularities $z_1, \ldots, z_n$, then

$$\oint f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}(f(z); z_k)$$

Lemma 1.11. The Identation Lemma

Consider a simple pole at z_0.

Then

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)dz = i(\beta - \alpha) \operatorname{res}(f; z_0),$$

where on C_ε, $z = z_0 + \varepsilon e^{i\theta}$, $\alpha \leq \theta \leq \beta$.

Proof. Consider the Laurent expansion of f about z_0.

$$f(z) = \frac{\operatorname{res}(f; z_0)}{z - z_0} + g(z),$$

where g is analytic in the region $|z - z_0| < r$, $r > 0$.

By continuity of g at z_0, we can choose r small enough such that g is bounded by some $M \in \mathbb{R}$. On $0 < \varepsilon < r$, we have

$$\begin{aligned}
\int_{C_\varepsilon} f(z)dz &= \operatorname{res}(f; z_0) \int_{C_\varepsilon} \frac{dz}{z - z_0} + \int_{C_\varepsilon} g(z)dz \\
&= i \operatorname{res}(f; z_0) \int_{\alpha}^{\beta} id\theta + \int_{C_\varepsilon} g(z)dz \\
&= i(\beta - \alpha) \operatorname{res}(f; z_0) \text{ in the limit } \varepsilon \to 0, \text{ as } g \text{ is bounded.}
\end{aligned}$$

$\square$

1.2 Functions defined by integrals

Consider $F(z) = \int_C f(z,t)dt$, where C is some contour in $\mathbb{C}$ (not necessarily closed). We wish to find out when such an F is defined and analytic.

Conditions on analyticity

We need to check that:

1. The integrand is continuous in t and z.
2. The integral converges uniformly in each subset of its domain.
3. The integrand is analytic in z for each value of t.

This second condition will not be treated rigorously.
Example.

$$F(z) = \int_{-\infty}^{\infty} e^{-zt^2} dt \left(= \sqrt{\frac{\pi}{z}}\right)$$

The integral converges for $\Re(z) > 0$, and diverges for $\Re(z) < 0$. If $z \in i\mathbb{R}$, then the integrand e^{-iyt} oscillates increasingly rapidly, and $F(z)$ is not absolutely convergent, but conditionally convergent, i.e.

$$\lim_{l\to\infty} \int_{-l}^{l} |e^{-iyt^2}| dt \to \infty,$$

but

$$\lim_{l\to\infty} \int_{-l}^{l} e^{-iyt^2} dt \text{ is finite.}$$

Conditions 13 hold. It can be shown that 2 also holds.
Example.

$$F(z) = \int_0^{\infty} \frac{u^{z-1}}{u+1} du$$

1. Existence: Potential problems when $u = 0, \infty$. The integrand is otherwise well behaved (except for -1, which is outside the range of integration). There are no problematic values of z, as $u^{z-1} = e^{(z-1)\log u}$.

   At $u = 0$, we have $\int_0 u^{z-1} du = \frac{u^z}{z}\Big|_0$

   $$|u^z| = |e^{z\log u}| = e^{x\log u}$$

   So to have this converge (to 0), we require $\Re(z) > 0$.

   At $u = \infty$, $u+1 \approx u$, and $\int^{\infty} u^{z-2} = \frac{u^{z-1}}{z-1}$

   $$|u^{z-1}| = e^{(x-1)\log u},$$

   so we require $\Re(z) < 1$.

   If $\Re(z) = 0, 1$, we also do not have convergence. Thus $F(z)$ is defined for $0 < \Re(z) < 1$.

2. Analyticity: Conditions 13 are clearly satisfied in $0 < \Re(z) < 1$. 2 probably is.

So $F(z)$ is analytic for $0 < \Re(z) < 1$.

We can evaluate it using a circular keyhole contour. On C_R, $t = Re^{i\theta}$, on C_+, $t = u, \varepsilon < u < R$, on C_-, $t = ue^{2\pi i}, R > u > \varepsilon$, and on C_ε, $t = \varepsilon e^{i\theta}$.

So, $\int_{C_-} \frac{t^{z-1}}{t+1} du = -(e^{2\pi i})^{z-1} F(z)$.

As $0 < \Re(z) < 1$, $\lim\limits_{R\to\infty} R^{1-z} =$, and so our C_R integral goes to 0. Similarly, so too does our C_ε integral.

Therefore,

$$
\begin{aligned}
(1 - e^{2\pi i(z-1)})F(z) &= 2\pi i \times e^{-i\pi(z-1)} \\
\Rightarrow F(z) &= \frac{\pi}{\sin \pi z}
\end{aligned}
$$

We will see later that $F(z) = \Gamma(z)\Gamma(1-z)$.

1.3 Analytic Continuation

We have that $F(z) = \int_{-\infty}^{\infty} e^{-zt^2} dt$ is analytic for $\Re(z) > 0$. We would like to know if it is possible to extend its domain of analyticity, and whether such an extension is unique.

Theorem 1.12. Identity Theorem

Let g_1, g_2 be analytic functions in a connected, non-empty, open set $D \subset \mathbb{C}$ with $g_1 = g_2$ in a non-empty open subset $\tilde{D} \subset D$. Then $g_1 = g_2$ on D.

Proof. (sketch)

Expand $g_1 - g_2$ as a Taylor expansion about $z_0 \in \tilde{D}$. Then the series holds in all of D, and is identically 0 in $\tilde{D}$. Therefore $g_1 - g_2 = 0$ on D. □

We can extend this proof by replacing our set $\tilde{D}$ by a contour $\gamma \subset D$.

Definition 1.13. Analytic Continuation

Let D_1, D_2 be open sets with $D_1 \cap D_2 \neq \emptyset$. Let f_1 and f_2 be analytic on D_1 and D_2 respectively, with $f_1 = f_2$ on $D_1 \cap D_2$. Then we say that f_2 is the *analytic continuation* of f_1 from D_1 to D_2.

Proposition 1.14. Our analytic continuation f_2 is unique.

Proof. Suppose there exists $\tilde{f}_2 \neq f_2$ which provides such an analytic continuation with $\tilde{f}_2 = f_1$ on $D_1 \cap D_2$. Define

$$g_1 = \begin{cases} f_1 \text{ on } D_1 \\ f_2 \text{ on } D_2 \end{cases}$$

$$g_2 = \begin{cases} f_1 \text{ on } D_1 \\ \tilde{f}_2 \text{ on } D_2 \end{cases}$$

Then by the identity theorem, $g_1 = g_2$, and so $f_2 = \tilde{f}_2$ and our analytic continuation is unique. □

Proposition 1.15. Monodromy Theorem

If we have open sets D_1, D_3 with $D_1 \cap D_3 = \emptyset$, with a function f_1 defined on D_1. A unique analytic continuation of this to D_3 is possible iff we can analytically continue f_1 through all domains D_2 connecting D_1 and D_3 with $D_1 \cap D_2, D_2 \cap D_3 \neq \emptyset$.

Proof. Left as an exercise to a different reader. □

Methods of Analytic Expansion

1. Taylor expansion

   If we pick z_0 near the boundary of our domain D, we can extend f_1 to a disk $|z - z_0| < r$ for some radius of convergence r.
   Example.
   Note that

   $$\begin{aligned} f(z) &= \frac{1}{1-z} \\ &= \frac{1}{1-z_0} \frac{1}{1 - \frac{z-z_0}{1-z_0}} \\ &= \frac{1}{1-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{1-z_0} \right)^n \end{aligned}$$

   ,

   which converges for $|z - z_0| < |1 - z_0|$.

   Now, let $f_1 = \sum_{n=0}^{\infty} z^n$. It is analytic for $|z| < 1$. Let $f_2 = \sum_{n=0}^{\infty} \frac{(z-\frac{i}{2})^n}{(1-\frac{i}{2})^{n+1}}$, analytic on $|z - \frac{i}{2}| < \frac{\sqrt{5}}{2}$. We have that $f_1 = f_2$ on the intersection of the disks, and hence by the identity theorem, f_2 is the analytic continuation of f_1. This can be continued as a chain of disks covering $\mathbb{C} \setminus \{1\}$, to obtain the function $\frac{1}{1-z}$, which has a simple pole at $z = 1$.

   This is known as meromorphic continuation (analytic continuation excluding singularities). However, such extensions are not always possible.
   Example.
   Let $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ is convergent in $|z| < 1$ by ratio test, but its singularities are dense, and analytic continuation is not possible. We call $|z| = 1$ a *natural barrier*.

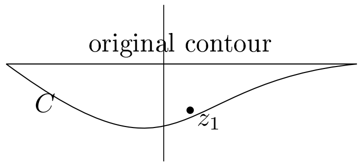
2. Contour deformation

Example.

Let $F(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z} dt$ for $\Im z > 0$.

We want to continue $F(z)$ to the lower half-plane, but obviously it is not analytic for $\Im z = 0$. So, we might think to re-define F for $\Im z \neq 0$. We shall see shortly why this does not work.

Pick z_1 with $\Im z_1 < 0$. We wish to continue F into a neighbourhood of z_1 by deforming our path of integration.



Define

$$F_1(z) = \int_C \frac{e^{it}}{t-z}$$

Then F_1 is analytic for all z above z_1. For $\Im z > 0$, we can see by deforming our new path to the real axis, that $F_1 = F$. Therefore, F_1 is the analytic continuation of F into $\Im z < 0$.

Now, instead consider $G(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z} dt$ for $\Im z \neq 0$. So if $\Im z > 0$, then $G(z) = F(z)$ by definition.

If $\Im z < 0$, then consider closing the contour with our path C above. We find

$$F_1(z) - G(z) = 2\pi i e^{iz}$$

So for $\Im z > 0$, we have that $F = F_1 = G$, and for $\Im z < 0$ we have $F_1 = G - 2\pi i e^{iz}$.

Hence G jumps by $2\pi i e^{iz}$ as it crosses the real axis.

1.4 Cauchy Principal Value

Idea. Can we say that

$$\int_{-1}^{2} \frac{dx}{x} = \log 2 - \log|-1| = \log 2?$$

Definition 1.16. If $f(x)$ is badly-behaved at $x = c$ and $a < c < b$, we can define the *Cauchy Principal Value* integral by

$$\mathcal{P}\int_a^b f(x)dx := \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^{b} f(x)dx$$

when the limit exists of course.

Example.
Let $I = \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x}dx$, where f is analytic in the upper half-plane and real axis, and $f(x) \to 0$ at infinity.

Closing in the UHP, our C_R contribution vanishes in the limit $R \to \infty$, and our C_ε term contributes $-i\pi f(0)$, where by analyticity of f, our residue at the origin is $f(0)$. Hence

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x} = i\pi f(0)$$

Example.
Let $I = \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}$. Our integrand has a removable singularity at $x = 0$. We can show using standard methods that $I = \pi$.

Alternatively, $I = \Re\mathcal{P}\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}$. Closing this in an arch, we get by indentation lemma that

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x} - i\pi(-i) = 0,$$

So $I = \pi$, and $\mathcal{P}\int_{-\infty}^{\infty} \frac{\sin x}{x^2} = 0$.

Hilbert Transforms

Definition 1.17. The Hilbert transform of $f(x)$ is defined by

$$\mathcal{H}(f)(y) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)dx}{x-y}$$

Remark. Observe that $\mathcal{H}$ is a linear functional.

We shall assume that f has a Fourier decomposition, so we only need to consider the Hilbert transform of $e^{i\omega x}$, and then use linearity of the transform. We will show that

$$\mathcal{H}(e^{i\omega x})(y) = \begin{cases} ie^{i\omega y}, \ \omega > 0 \\ -ie^{i\omega y}, \ \omega < 0 \end{cases} = i\,\mathrm{sgn}(\omega)e^{i\omega y}$$

Integrating in an arch shaped contour about y, we get

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{x-y}dx+\underbrace{\int_{C_R}}_{\to 0}+\underbrace{\int_{C_\varepsilon}}_{=-i\pi e^{i\omega y}}=0$$

And flipping our arch for $\omega<0$, we get a negative sign from the indentation lemma, so the result indeed holds.

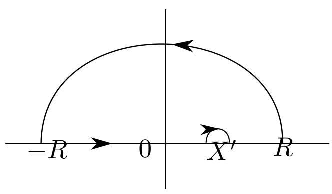
Remark. From this it follows that $\mathcal{H}^2(e^{i\omega x})=-e^{i\omega x}$, so $\mathcal{H}$ is "anti-self-inverse" here.

More (but not completely of course) generally, if $g(y)=\mathcal{H}(f)(y)$, then

$$f(x)=\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}-\frac{g(y)}{y-x}$$

Kramers-Kronig Relations

Let $f=u+iv$ be analytic in $\Im z>0$, with $f\to 0$ as $|z|\to\infty$. Let $x'=z'\in\mathbb{R}$, and consider C as follows:



Then by indentation lemma,

$$\int_C\frac{f(z)}{z-z'}dz=\mathcal{P}\int_{-\infty}^{oo}\frac{f(x)dx}{x-x'}-i(x)=0 \tag{1}$$

For $z\in\mathbb{R}$, we can write $f(z)=f(x,y)=f(x,0)$, with $f(x,0)=u(x,0)+iv(x,0)$.

Hence, taking real and imaginary parts of (1), we obtain

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{u(x)dx}{x-x'} = -\pi v(x')$$
$$\mathcal{P}\int_{-\infty}^{\infty}\frac{v(x)dx}{x-x'} = \pi u(x')$$

Or

$$\mathcal{H}u(x') = -v(x')$$
$$\mathcal{H}v(x') = u(x')$$

These are known as the Kramers-Kronig relations, relating the real and imaginary parts of functions analytic in the upper half plane.
Example.
The Laplace equation in the upper half plane

Let $u(x,y)$ be a harmonic function in $\Im z > 0$. Recall that for $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, we have

$$4444\frac{\partial^2 u}{\partial z \partial \overline{z}}u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Suppose $u \to 0$ for large $|x|, y$.

Consider $F(z) = \frac{\partial u}{\partial z}$. $\frac{\partial F}{\partial \overline{z}}$ implies analyticity for $\Im z > 0$, and so

$$u_x(x,0) = -\mathcal{H}u_y(x,0)$$
$$u_y(x,0) = \mathcal{H}u_x(x,0)$$

1.5 Multivalued Functions

Definition 1.18. A multivalued function $f(z)$ admits more than one value for given z.

Definition 1.19. A point $z = a$ is a branch point of the multivalued function $f(z)$ if f is discontinuous upon traversing in a small circle about $z = a$ i.e. $f(a + re^{2\pi i}) \neq f(a + r)$.

Example.
$f(z) = (1-z^2)^{\frac{1}{2}} = (1-z)^{\frac{1}{2}(1+z)^{\frac{1}{2}}}$ has branch points at $z = \pm 1$.

For $z = 1$, consider $z = 1 + \varepsilon e^{i\theta}, 0 < \varepsilon \ll 1$.

$$f(z) = (-\varepsilon e^{i\theta})^{\frac{1}{2}}(2 + \varepsilon e^{i\theta})^{\frac{1}{2}} \approx \pm i\sqrt{2\varepsilon}e^{i\frac{\theta}{2}}$$

And it is easily seen that this is a branch point.

Note that ∞ is not a branch point (consider $t = \frac{1}{z}$).

We seek to express a multivalued function in terms of a singlevalued function. This is achieved by restricting the region in $\mathbb{C}$ to cut in such a way that the resulting function is singlevalued and continuous.

Definition 1.20. A continuous singlevalued function obtained in this way is called a *branch* of the multivalued function.

Integrating using a branch cut

We seek to evaluate

$$I = \int_{-1}^{1} (1 - x^2)^{\frac{1}{2}} dx$$

by contour integration.

We choose $f(z)$ to be the branch of $(1 - z^2)^{\frac{1}{2}}$ with $f(0^+) = 1$, given in local polars by

$$f(z) = |1 - z^2|^{\frac{1}{2}} e^{\frac{i}{2}\left(\phi_1 + \phi_2 - \frac{\pi}{2}\right)}$$

You then do some boring stuff, and get the answer to be 2π or something.

The arcsin function defined as an integral

Let

$$z = \int_0^{2\pi} \frac{dt}{(1 - t^2)^{\frac{1}{2}}},$$

where $\sqrt{1 - t^2}$ is defined by a branch cut between -1 and 1 as before, such that it takes value 1 at 0^+, and where $0 \leq \arg z < \pi$

See siklos' notes.

2 Special Functions

2.1 The Gamma Function

We are motivated by finding a smooth curve that interpolates the points $f(n) = n!, n \in \mathbb{N}$. We find such a magical function to be given by $f(x) = \Gamma(x + 1)$! We now seek to generalize this in integral form to $\mathbb{C}$.

Let $I(z) = \int_0^\infty t^{z-1} e^{-t} dt$ (Euler's Integral), which converges and is analytic for $\Re z > 0$.

Now,

$$\begin{aligned} I(z + 1) = \int_0^\infty t^z e^{-t} dt &= \left[-t^z e^{-t}\right]_0^\infty + \int_0^\infty z t^{z-1} e^{-t} dt \\ &= (z) \end{aligned}$$

Also, $I(1) = 1$. Hence

$$I(n + 1) = n! I(1) = n!, \; n \in \mathbb{N}$$

So our idea is to define

$$\Gamma(z) = \begin{cases} I(z), & \Re z > 0 \\ \text{Analytic continuation elsewhere} \end{cases}$$

Now, we can see that

$$I(z) = \frac{I(z+1)}{z}$$

is analytic for $\Re(z+1) > 0$, and $z \neq 0$. As such, we can iteratively extend this to

$$I(z) = \frac{I(z+n+1)}{z(z-1)\dots(z+n)},$$

which is analytic for $\Re z > -(n+1)$, $z \neq 0, -1, \dots, -n$.

Hence we can meromorphically continue $\Gamma(z)$ to $\mathbb{C} \setminus \{-n : n \in \mathbb{N}\}$, with simple poles at the negative integers. It is easily seen that $\operatorname{res}(\Gamma(z); -n) = (-1)^n \Gamma(1) \frac{1}{n!} = \frac{(-1)^n}{n!}$

Some alternative definitions and formulae

Proposition 2.1. Euler Product Formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\dots(z+n)}, \qquad \forall z \in \mathbb{C} \setminus (-\mathbb{N})$$

Proof. Firstly, we consider $\Re z > 0$. Recall that $e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n$.

So,

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt \\
&= \lim_{n\to\infty} n^z \left[\frac{(1-\tau)^n \tau^z}{z}\right]_0^1 - \frac{n^\tau}{z}(-n) \int_0^1 (1-\tau)^{n-1} \tau^z d\tau \quad (\tau = \frac{t}{n}) \\
&= \lim_{n\to\infty} 0 + (-1)^n n^z n! \int_0^1 \frac{\tau^{z+n-1}}{z(z+1)\dots(z+n-1)} \\
&= \lim_{n\to\infty} \frac{n! n^z}{z(z-1)\dots(z+n)}
\end{aligned}$$

For $\Re z \leq 0$, it is clear to see that our analytic continuation by $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ continues the product formula, and is indeed analytic. □

Proposition 2.2. Gauss Product Formula

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^z}{1 + \frac{z}{n}}$$

Proof. By the Euler product formula, we can write

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \frac{1}{z} \frac{n^z}{\frac{z+1}{1}\frac{z+2}{2}\dots\frac{z+n}{n}} \\
&= \frac{1}{z}\lim_{n\to\infty} \frac{\left(\frac{n+1}{n}\right)^z \left(\frac{n}{n-1}\right)^z \dots \left(\frac{2}{1}\right)^z \left(\frac{n}{n+1}\right)^z}{(1+z)(1+\frac{z}{2})\dots(1+\frac{z}{n})}
\end{aligned}$$

As $\left(\frac{n}{n+1}\right)^z \to 1$ as $n\to\infty$, we obtain the required expression. $\square$

Proposition 2.3. The Weierstrass Canonical Product

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}},$$

where $\gamma = \lim\limits_{n\to\infty} 1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n \approx 0.577$ is the Euler-Mascheroni constant.

Proof. Using Euler's product formula,

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= z\lim_{n\to\infty}\frac{(1+z)(2+z)\dots(n+z)}{n!n^z} \\
&= z\lim_{n\to\infty} e^{-z\log n}(1+z)\left(1+\frac{z}{2}\right)\dots\left(1+\frac{z}{n}\right) \\
&= z\lim_{n\to\infty} e^{-z\left(\log n-(1+\frac{1}{2}+\ldots+\frac{1}{n})\right)}e^{-z\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)}(1+z)\dots\left(1+\frac{z}{n}\right) \\
&= ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}
\end{aligned}$$

$\square$

Proposition 2.4. Reflection Formula

$$\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z), \quad z\notin\mathbb{Z}$$

Proof. We first consider the case $\Re z\in(0,1)$ so that we can write $\Gamma(z)$ and $\Gamma(1-z)$ can be written in integral form. Using substitutions $t = r\sin^2\theta$, $s = r\cos^2\theta$, we have

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) &= \int_0^\infty e^{-t}t^{z-1}dt\int_0^\infty e^{-s}s^{-z}ds \\
&= 2\int_0^{\frac{\pi}{2}}(\tan\theta)^{2z-1}d\theta \\
&= \int_0^\infty \frac{u^{z-1}}{u+1}du \\
&= \frac{\pi}{\sin(\pi z)}
\end{aligned}$$

Where we used the substitution $\tan\theta = u^{\frac{1}{2}}$, and calculated the last integral earlier.

Now, $\Gamma(z)$, $\Gamma(1-z)$ and $\pi(\csc \pi z)$ are analytic for all z except integer points, and they are equal for $\Re z \in (0,1)$, and so the result holds by analytic continuation. $\square$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

2.2 Hankel Representation of $\Gamma(z)$

Proposition 2.5. Hankel Representation

For $z \notin \mathbb{Z} \setminus \mathbb{N}$,

$$\Gamma(z) = \frac{1}{2i\sin(\pi z)} \int_{-\infty}^{0^+} e^t t^{z-1} dt,$$

where $-\pi \leq \arg t \leq \pi$, and the path is called the *Hankel contour*. Note that the function is analytic in both z and t.

Well-Definedness of Hankel Integral

Note that for $\Re z > 0$, the Hankel representation is equal to the Gaussian integral $I(z)$ from earlier. To see this, we collapse the Hankel contour onto the branch cut, and define for $\Re z > 0$,

$$J(x) = \int_{-\infty}^{0^+} e^t t^{z-1} dt = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_\varepsilon},$$

defining contours

$$\gamma_1 : t = xe^{i\pi}, \infty > x > \varepsilon, \quad \gamma_2 : t = xe^{i\pi}, \varepsilon < x < \infty, \quad \gamma_\varepsilon : t = \varepsilon e^{i\theta}, -\pi < \theta < \pi$$

Note that we have

$$\begin{aligned}
\int_{\gamma_1} &\to (e^{-\pi})^z \int_{-\infty}^{0} e^{-x} x^{z-1} dx \\
\int_{\gamma_2} &\to (e^{i\pi})^z \int_{0}^{\infty} e^{-x} x^{z-1} dx \\
\int_{\gamma_\varepsilon} &\to 0 \text{ as } \Re z > 0 \text{ and } \varepsilon \to 0
\end{aligned}$$

so we have

$$J(z) = 2i\sin(\pi I(z)$$

Hence the claim is proved by analytic continuation.

Note that for $z \in \mathbb{N}$, the zeroes of $\sin(\pi z)$ are cancelled by the integral, and $t = 0$ is not a branch point, so there are no singularities in the Hankel contour. This suggests that $J(z) = 0$.

Residues of $\Gamma(z)$ in Hankel Representation

In this case with $z \in \mathbb{N}$, we can choose a Hankel contour to be a unit circle enclosing the origin anticlockwise. Now,

$$J(-m) = \int_{|t|=1} e^t t^{-(m+1)} dt = 2\pi i \operatorname{res}\left(e^t t^{-(m+1)}; 0\right)$$

Using Taylor expansion,

$$e^t t^{-(m+1)} = \sum_{n=0}^{\infty} \frac{t^{n-m-1}}{n!},$$

and the residue is then the coefficient of t^{-1}, $m!$. So $J(-m) = \frac{2\pi i}{m!}$.

Thus the residue of $\Gamma(z)$ at $z = -m$ is $\lim_{z \to -m} \frac{z+m}{2i \sin \pi z} J(z) = \frac{2\pi i}{m!} \lim_{z \to -m} \frac{z+m}{2i \sin \pi z} = \frac{(-1)^m}{m!}$ by l'Hôpital as expected.

We now seek to answer whether the Gamma function is the unique analytic interpolation problem of the factorial.

Theorem 2.6. Wielondt's Theorem

If $F(z)$ satisfies:

1. $F(z)$ is analytic for $\Re z > 0$
2. $F(z+1) = zF(z)$
3. $F(z)$ is bounded in $1 \leq \Re z \leq 2$
4. $F(1) = 1$

then $F(z) = \Gamma(z)$.

Lemma 2.7. Define the difference function

$$f(z) := F(z) - \Gamma(z)$$

Then $f(z)$ is entire.

Proof. Properties 1 and 2 imply that $F(z)$ can be meromorphically continued into $\mathbb{C} \setminus (-\mathbb{N})$,

$$F(z) = \frac{F(z+n)}{z(z+1)\dots(z+n-1}$$

By property 4, $\operatorname{res}(F(z); -n) = \frac{F(1)(-1)^n}{n!}$, which is the same as the gamma function. Hence $f(z)$ has only removable poles, and is in fact entire. □

Lemma 2.8. $f(z)$ is bounded in the strip $0 \leq \Re z \leq 1$.

Proof. We first show that $f(z)$ is bounded on $1 \leq \Re z \leq 2$. It suffices to check that $\Gamma(z)$ is.

$$\begin{aligned}|\Gamma(z)| &= \left|\int_0^\infty e^{-t}t^{z-1}dt\right| \\ &\leq \int_0^\infty \left|e^{-t}t^{x+iy-1}\right| dt \\ &= \int_0^\infty e^{-t}t^{x-1}dt \\ &\leq \int_0^\infty e^{-t}t^{2-1}dt \\ &= 1\end{aligned}$$

We examine our last inequality more closely:

Define $I(x) = \int_0^\infty e^{-t}t^{x-1}dt$ $I(1) = I(2) = 1$. $\frac{d^2I}{dx^2} > 0$, so $I(x)$ is convex in $[1,2]$, and the inequality indeed holds.

Now, for $0 \leq \Re z \leq 1$, we can write $f(z) = \frac{f(z+1)}{z}$. As f is bounded on $1 \leq \Re z \leq 2$, we conclude that it is also bounded on $0 \leq \Re z \leq 1$, noting that the pole is removable at the origin. □

We now prove the original theorem.

Proof. (2.6)

Let $S(z) = f(z)f(1-z)$. $S(z)$ is entire by lemma 2.7, and is bounded in $0 \leq \Re z \leq 1$ by lemma 2.8. Indeed, both $f(z)$ and $f(1-z)$ have the same range in this domain by symmetry.

Now, $S(z+1) = f(z+1)f(-z) = zf(z)(-z)^{-1}f(1-z) = -S(z)$. Thus $S(z)$ is bounded in $1 \leq \Re z \leq 2$.

Also, $S(z+2) = S(z)$, so $S(z)$ is periodic with period 2, and so is bounded in $\mathbb{C}$. Hence by Liouville's theorem, we must have that $S(z) = S(1) = f(1)f(0) = (F(1) - \Gamma(1))\, f(0) = 0$. Then $f(z)f(1-z) = 0$ for all z. Hence $f(z) \equiv 0$, and $F(z) \equiv \Gamma(z)$.

□

2.3 The Beta Function

Definition 2.9.

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad \Re p, \Re q > 0,$$

and is analytically continued in p and q.

Setting $t = \sin^2\theta$, it is easily shown (on the example sheet) that

$$B(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\cos^{2q-1}\theta d\theta$$

Proposition 2.10. 1. $B(p,q) = B(q,p)$

2. $B(1,q) = \frac{1}{q}$

3. $B(p,z+1) = \frac{z}{p+z}B(p,z)$

Proof. (1) and (2) are trivial. For (3):

$$\begin{aligned}
B(p,z+1) &= \int_0^1 t^{p-1}(1-t)^{z-1}(1-t)dt \\
&= B(p,z) - B(p+1,z) \\
&= B(p,q) - \frac{p}{z}B(p,z+1) \text{ upon integrating by parts.}
\end{aligned}$$

□

This last identity gives us an analytic continuation of the Beta function into $\Re z > -1$, just as we constructed for the Gamma function.

As our continuation is from $B(p,z) = \frac{p+z}{z}B(p,z+1)$, it is easy to see that much like the Gamma function, for fixed p there are simple poles at $z \in (-\mathbb{Z})$.

Proposition 2.11. 4. $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

Notice that for $(n,m) \in \mathbb{N}^2$, $B(n,m) = \frac{(n-1)!(m-1)!}{(n+m-1)!}$

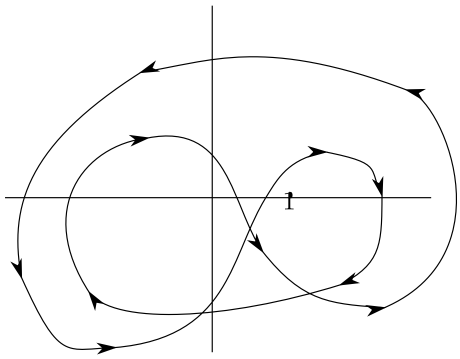
Proof.

$$\begin{aligned}
\Gamma(p)\Gamma(q) &= \int_0^\infty e^{-s}s^{p-1}ds \int_0^\infty e^{-t}t^{q-1} \\
&= \Gamma(p+q)(p,q), \quad \text{using } s = r\cos^\theta, t = r\sin^2\theta
\end{aligned}$$

□

Proposition 2.12. Pochhammer Representation (non-examinable)

Let $J(p,q) := \int_P f(t)dt$, where P is Pochhammer's contour



See handout.

2.4 The Zeta function

Definition 2.13.

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \Re z > 1$$

and is analytically continued wherever possible.

Euler showed the well known result that $\zeta(2) = \frac{\pi^2}{6}$.

Proposition 2.14. Integral Representation of $\zeta(z)$

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt, \quad \Re z > 1$$

Proof. Let $t = ns$, for some fixed $n \in \mathbb{N}$, with $s \in \mathbb{R}$.

Then

$$\Gamma(z) = \int_0^{\infty} n^z s^{z-1} e^{-ns} ds, \quad \Re z > 0$$

Hence

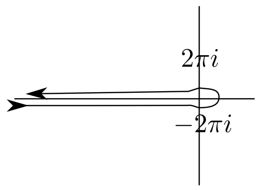
$$\begin{aligned}
\zeta(z)\Gamma(z) &= \sum_{n=1}^{\infty} \int_0^{\infty} s^{z-1} e^{-ns} \\
&= \int_0^{\infty} t^{z-1} \sum_{n=1}^{\infty} e^{-nt} dt \\
&= \int_0^{\infty} \frac{t^{z-1} e^{-t}}{1-e^{-t}} dt \\
&= \int_0^{\infty} \frac{t^{z-1}}{e^t - 1}
\end{aligned}$$

□

Proposition 2.15. Hankel Representation

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i} \int_{-\infty}^{0^+} \frac{t^{z-1}}{e^{-t}-1} dt$$

Note that the integrand has simple poles at $2\pi i n$, for $n \in \mathbb{Z}$. We take a branch cut on the negative real axis.



Proof. We show that

$$\frac{\Gamma(1-z)}{2\pi i} \int_{-\infty}^{0^+} \frac{t^{z-1}}{e^{-t}-1} dt = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt,$$

and show that the LHS gives the analytic continuation of the RHS into $\Re z < 1$.

On our bottom line, $t = xe^{-i\pi}$, on the circle, $t = \varepsilon e^{i\theta}$, and on top $t = xe^{i\pi}$. Treating this just as with the Gamma function,

$$
\begin{aligned}
\int_{-\infty}^{0^+} &= \int_{\gamma_1} + \underbrace{\int_{\gamma_\varepsilon}}_{\to 0} + \int_{\gamma_2} \\
&= \left(e^{i\pi z} - e^{-i\pi z}\right) \int_0^\infty \frac{x^{z-1}}{e^x - 1} dx \\
&= 2i \sin \pi z \Gamma(z) \zeta(z) \text{ by (2.14)}
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\Gamma(1-z)}{2\pi i} \int_{-\infty}^{0^+} \frac{t^{z-1}}{e^{-t}-1} dt &= \frac{\Gamma(1-z)}{2\pi i} 2i \sin \pi z \Gamma(z) \zeta(z) \\
&= \pi \frac{\csc \pi z}{\pi} \sin \pi z \zeta(z) \text{ by reflection formula} \\
&= \zeta(z)
\end{aligned}
$$

$\square$

The integral on the LHS is entire in z and smooth in t, and hence provides an analytic continuation of $\zeta(z)$ into $\Re z < 1$.

Proposition 2.16. The ζ-function extends to a meromorphic continuation into $\mathbb{C}$, with the only singular point being a simple pole at $z = 1$ with residue 1.

Proof. Notice that $\Gamma(1-z)$ has simple poles at $z = 1, 2, 3, \ldots$ But $\zeta(z)$ is analytic for $\Re z > 1$ from its series definition. Hence, $z = 1$ is the only singularity of ζ.

The residue

$$
\begin{aligned}
\operatorname{res}(\zeta(z); 1) &= \lim_{z \to 1} \frac{(z-1)\Gamma(1-z)}{2\pi i} \int_{-\infty}^{0^+} \frac{t^{z-1}}{e^{-t}-1} dt \\
&= \lim_{z \to 1} \frac{(z-1)\Gamma(1-z)}{2\pi i} \oint_{|z|=\frac{1}{2}} \frac{dt}{e^{-t}-1} dt
\end{aligned}
$$

Note that for $z = -n$, $n \in \mathbb{N}_0$, then $\Gamma(z) = \frac{(-1)^n}{n!} \frac{1}{z+n} +$ analytic function.

So,

$$
\lim_{z \to 1} (z-1)\Gamma(1-z) = \lim_{z \to 1} (z-1) \left(\frac{(-1)^0}{0!} \frac{1}{1-z} + \text{ analytic function} \right) = -1
$$

Also,

$$
{}_{|z|=\frac{1}{2}} \frac{dt}{e^{-t}-1} = 2\pi i \cdot (-1)
$$

Hence $\operatorname{res}(\zeta(z); 1) = 1$. $\square$

What about the zeroes of $\zeta(z)$?

Proposition 2.17. Functional Equation for $\zeta(z)$.

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z)$$

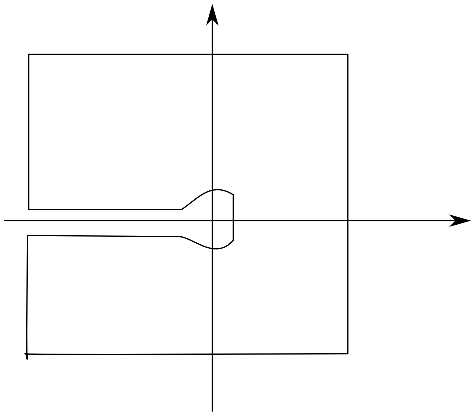
for all z.

Proof. We derive this for $\Re z < 0$, and then use analytic continuation.

We modify the Hankel contour as follows, closing in a rectangle with vertices at

$$z = \pm R \pm (2n+1)\pi i$$

:



Let $J(z) = \int_C \frac{t^{z-1}}{e^{-t}-1} dt$

The integral has a branch cut on the negative real axis, with branch point at 0, and poles at $z = 2\pi i n$, $n \in \mathbb{Z} \setminus \{0\}$. The residues at these points are given by $\lim\limits_{z \to 2\pi i n} \frac{(t-2\pi i n)}{e^{-t}-1} t^{z-1} = \frac{-1}{(2\pi i n)^{1-z}}$ by l'Hôpital's rule.

So, by our Hankel representation,

$$\begin{aligned}
\frac{2\pi i}{\Gamma(1-z)}\zeta(z) &= -2\pi i \sum_{n=1}^{N} (-2\pi i n)^{z-1} - (2\pi i n)^{z-1} \\
&= 2\pi i (2\pi)^{z-1} \frac{(i)^z - (-i)^z}{i} \sum_{n=1}^{N} \frac{1}{n^{1-z}} \\
&\to 2\pi i (2\pi)^{z-1} 2 \sin\left(\frac{\pi z}{2}\right) \zeta(1-z)
\end{aligned}$$

It can be shown that the contributions from the rectangular sides vanish in the limit $R, N \to \infty$. □

It follows easily from the functional equation that $\zeta(-2n) = 0$ for $n \in \mathbb{N}$. We can see that $\zeta(2n) \neq 0$ because the zeroes of the sin are cancelled by the poles of the Gamma function. $\zeta(1+2n) = \Gamma(-2n)\zeta(-2n) \neq 0, andsimilarly \zeta(0) \neq 0.$

The Riemann Hypothesis

The Riemann Hypothesis states that all non-trivial zeroes of the zeta function lie on the critical line $z = \frac{1}{2} + it$.

2.5 Elliptic Functions

Definition 2.18. A *doubly periodic* function $f(z)$ is one that has two periods, ω_1 and ω_2 such that $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$, with $f(z+\omega_1) = f(z+\omega_2) = f(z)$, and more generally $f(z+m\omega_1+n\omega_2) = f(z)$ for all $m, n \in \mathbb{Z}$.

Definition 2.19. A double-periodic function which is meromorphic is called *elliptic*.

Assume wlog that $\omega_1 \in \mathbb{R}$. The parallelogram with sides ω_1 and ω_2 is called a cell, and can be extended to form a lattice of periodicity in $\mathbb{C}$.

Then all zeroes and poles of an Elliptic function correspond to zeroes and poles in any one cell.

Proposition 2.20.
1. The number of zeroes and poles in one cell is finite.
2. An elliptic function with no poles in a cell is constant.

(No proof given)

The Weierstrass $\mathcal{P}$-function

Let $\omega_{m,n} = m\omega_1 + n\omega_2$, and define

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{m,n\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(z-\omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2}$$

It is meromorphic, with infinitely many poles at lattice points.

To show ellipticity, consider $\mathcal{P}(z+\omega_{k,l})$ for given k, l, and shift the summation to be centered there.

Proposition 2.21. The $\mathcal{P}$ function satisfies

$$(\mathcal{P}')^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3,$$

where $g_2 = 60 \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{\omega_{m,n}^4}$, and $g_3 = 140 \sum_{m,m\in\mathbb{Z}^3\setminus(0,0} \frac{1}{\omega_{m,n}^6}$

Proof. See example sheet 3. Hint: Consider $Q(z) = \mathcal{P}(z) - \frac{1}{z^2}$, and argue that Q is analytic and even in the region of $z = 0$. Then, Taylor expand it. $\square$

Rearranging our ODE and integrating, we get

$$z = \int_{w(z)}^{\infty} \frac{ds}{4s^3 - g_2 s - g_3} = \mathcal{P}^{-1}(w) - \alpha,$$

where $w = \mathcal{P}(z + \alpha)$. Thus, $\mathcal{P}$ can be given as the inverse of an elliptic integral (we call this integral and elliptic integral of the first kind). Note - compare this to the relation

$$\arcsin z = \int_0^z \frac{ds}{\sqrt{1 - s^2}}$$

Example.
(General Elliptic ODE)

Consider $w' = (w - a)(w - b)(w - c)(w - d)$. This is solved by an elliptic integral

$$z + \alpha = \int_w^{\infty} \frac{ds}{[(s - a)(s - b)(s - c)(s - d)]^{\frac{1}{2}}}$$

This can be reduced to a more useful form by moving one root to ∞, and shifting the remaining roots as follows. Let $t = \frac{1}{u - d}$. Then

$$\frac{du}{[(u - a)(u - b)(u - c)(u - d)]^{\frac{1}{2}}} \mapsto \frac{-dt}{t^2 \left[(\frac{1}{t} - \tilde{a})(\frac{1}{t} - \tilde{b})(\frac{1}{t} - \tilde{c})\frac{1}{t} \right]^{\frac{1}{2}}} = \frac{Adt}{[(t - t_1)(t - t_2)(t - t_3)]^{\frac{1}{2}}},$$

where A is some constant. Thus, the quartic integrand has been reduced to a cubic with one root moves to ∞ (assuming the original d is not ∞)$. Finally, set t = \mathrm{s} + \gamma$, with γ chosen such that $t_1 + t_2 + t_3$ is moved to 0, and then rescale s such that the leading term has coefficient 4. We then arrive at our previous equation.

Example.
(Euler's Top)

Consider

$$\begin{aligned} w_1' &= w_2 w_3 \\ w_2' &= w_3 w_1 \\ w_3' &= w_1 w_2 \end{aligned}$$

which gives

$$w_2' w_2 - w_3' w_3 = w_1' w_1 - w_3' w_3 = 0,$$

and so there exists constants B, C such that $w_3^2 = w_2^2 + B = w_1^2 + C$

Then we can write

$$(w_3')^2 = (w_3^2 - B)(w_3^2 - C)$$

which is a special case of the elliptic ODE.

3 Solving Differential Equations by Transform Methods

3.1 Solution of ODEs by integral representation

Our idea is to look at general solutions given by integrals in $\mathbb{C}$. We will motivate this with an example.
Example.
(Airy's Equation)

$$\omega''(x) + x\omega(x) = 0$$

In this course, the convention is that we take $+$, often you will see the Airy equation written with a $-$. Let us first attempt to solve via the Fourier transform.

$$\hat{\omega}(k) = \int_{-\infty}^{\infty} e^{-ikx}\omega(x)dx,$$

so

$$-k^2\hat{\omega}(k) + i\hat{\omega}'(k) = 0$$

Solving and inverting gives us

$$\omega(x) = \frac{A}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{ik^3}{3} + ikx\right) dk$$

There exists a second, linearly independent solution which is not bounded for large x and doesn't admit a Fourier transform. Instead, we consider the equation in $\mathbb{C}$, with $\omega(z)$.

Let

$$w(z) = \int_\gamma e^{zt} f(t)dt,$$

where we will determine $f(t)$ uniquely, and then establish suitable contours γ.

Assume $\frac{\partial f}{\partial z} = 0$, then

$$w''(z) = \int_\gamma t^2 e^{zt} f(t)dt$$

Our equation now becomes

$$\int_\gamma (t^2 + z) f(t) e^{zt} dt = 0$$

After integrating $\int_\gamma z f(t) e^{zt} dt$ by parts, we obtain

$$\left[e^{zt} f(t)\right]_\gamma + \int_\gamma e^{zt}\left(t^2 f(t) - \frac{df}{dt}dt\right)$$

Note that if $f(t)$ is entire, and γ is closed, then $\left[e^{zt} f(t)\right]_\gamma = 0$.

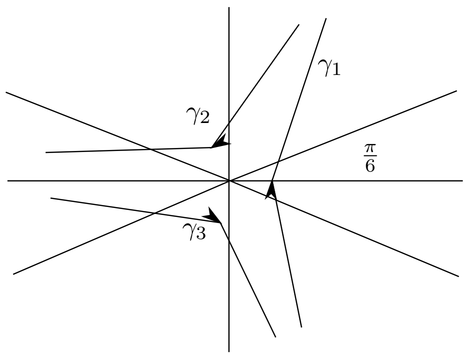
We now pick $f(t)$ such that $t^2 f(t) - \frac{df}{dt} = 0$, and hence

$$f(t) = Ae^{\frac{t^3}{3}}$$

Having found f, we can choose γ such that $\left[e^{zt}e^{\frac{t^3}{3}}\right]_\gamma = 0$. We choose γ which starts/ends at ∞ (a closed contour will give a trivial solution $w \equiv 0$). Set $t = |t|e^{i\theta}$. For large $|t|$, we can write $e^{zt+\frac{t^3}{3}} \sim e^{\frac{t^3}{3}} = e^{\frac{1}{3}|t|^3\cos 3\theta}e^{\frac{i}{3}|t|^3\sin 3\theta}$.

So, we require $\cos 3\theta < 0$ for this to vanish.

As such, a conventional choice of contour is as such:



By Cauchy,

$$\oint_{\gamma_1+\gamma_2+\gamma_3} = 0$$

So only 2 of the contours give linearly independent solutions (no proof of linear independence).

So, we choose any pair of contours.

We deform γ_1 to the Imaginary axis, getting

$$\begin{aligned} w_1(z) &= Ci\int_{-\infty}^{\infty} e^{izy-i\frac{y^3}{3}}\,dy \\ &= C\left[i\underbrace{\int_{-\infty}^{\infty}\cos\left(zy-\frac{y^3}{3}\right)dy}_{\text{real for real z}} - \underbrace{\int_{-\infty}^{\infty}\sin\left(zy-\frac{y^3}{3}\right)dy}_{\text{real for real z}}\right] \end{aligned}$$

which is the solution obtained by Fourier transform.

We from here define Airy's function of the 1st kind:

$$Ai(z) = \frac{1}{\pi}\int_0^{\infty}\cos\left(zy-\frac{y^3}{3}\right)dy$$

Now, instead we deform γ_2 such that it follows the axes as such:

$$\gamma_2 = \{t = iy, y \in (\infty, 0)\} \bigcup \{t = -x, x \in [0, \infty)\}$$

Then

$$w_2(z) = C\left[i\int_\infty^0 e^{izy - i\frac{y^3}{3}}\,dy - \int_0^\infty e^{-zx-\frac{x^3}{3}}\,dx\right]$$

is our second linearly independent solution. We can see that the second integral is unbounded as $z \to -\infty$.

We write

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \left[\sin\left(\frac{x^3}{3} - zx\right) - \exp\left(-zx - \frac{x^3}{3}\right)\right]$$

to be Airy's function of the second kind (real part of the above with $C = -\frac{1}{\pi}$).

3.2 Solving ODE by integral representation: General Method

For equations of the form

$$aw'' + bw' + cw = 0$$

Where $w = w(z)$, and $a, bandc$ are low-order polynomials in z (so that we can integrate by parts),set

$$w(z) = \int_\gamma K(z,t)f(t)dt$$

Substituting back into the ODE, we integrate by parts to eliminate the terms with z^n, choosing $f(t)$ appropriately to do so. The factor $K(z,t)$ is called a *kernel*. The three following kernels are commonly used:

$$e^{zt}\text{(Laplace kernel)}$$
$$(z-t)^\gamma\text{(Euler kernel)}$$
$$t^z\text{(Mellin kernel)}$$

An example of the Euler kernel is given by:

$$w(z) = \int_\gamma t^{a-c}(a-t)^{c-b-1}(t-z)^{-a}dt,$$

which satisfies the hypergeometric equation

$$z(1-z)w'' + [c - (a+b-1)z]w' - abw = 0$$

with constants a, b, c, provided that

$$\left[t^{a-c+1}(1-t)^{c-b}(t-z)^{1-a}\right]_\gamma = 0$$

The integrand might have branch points, possible at $t = 0, 1, z$, depending on the values of a, b, c and whether the exponents are integers. If there are any branch cuts, then of course γ must not cross them (see q6 on ES3).

3.3 Solving PDEs by Integral Transform

Definition 3.1. (Laplace Transform)

For functions which have support on semi-intervals, e.g $f(x) = 0$ for $x < 0$, we define the Laplace transform

$$\hat{f}(p) := \int_0^\infty e^{-px} f(x) dx,$$

with inverse

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} \hat{f}(p) dp$$

where we are integrating over a Bromwich contour such that

$$c > \sup\{x \in \mathbb{R} : \exists y \in \mathbb{R},\ f(x+iy) \text{ is singular}\}$$

Closing the Bromwich contour to the right yields $f(x) = 0$ for $x < 0$. Closing to the left gives us $f(x)$ for $x \geq 0$.

Example.
(Waves on a finite string)

We consider small transverse oscillations, with displacement $y(x)$, satisfying the wave equation

$$y_{tt} - c^2 y_{xx} = 0,$$

with $y, y_t = 0$ for $t < 0$, $y(0,t) = 0$ $y(l,t) = y_0$ (sudden displacement).

(An alternative model for this might be a capacitor plate given some charge).

Taking Laplace transforms (as it is an IVP),

$$p^2 \hat{y}(x,p) = c^2 \hat{y}_{xx}(x,p)$$

And so

$$\hat{y}(x,p) = A(p) \sinh\left(\frac{px}{c}\right) + B(p) \cosh\left(\frac{px}{c}\right)$$

Our boundary conditions give

$$\begin{aligned} \hat{y}(0,p) &= 0 \\ \Rightarrow B(p) &\equiv 0 \end{aligned}$$

And also

$$y(l,t) = y_0 \Rightarrow \hat{y}(l,p) = \int_0^\infty y_0 e^{-pt} dt = \frac{y_0}{p}$$

So

$$A(p) \sinh\left(\frac{lp}{c}\right) = \frac{y_0}{p}$$

Hence

$$\hat{y}(x,p) = \frac{y_0}{p} \frac{\sinh\left(\frac{xp}{c}\right)}{\sinh\left(\frac{lp}{c}\right)}$$

Then

$$y(x,t) = \frac{1}{2\pi i}\int_{\tilde{c}-i\infty}^{\tilde{c}+i\infty} e^{pt}\frac{y_0}{p}\frac{\sinh\left(\frac{xp}{c}\right)}{\sinh\left(\frac{lp}{c}\right)}dp$$

Our integrand has simple poles at $p = \frac{m\pi ci}{l}$. As $|p| \to \infty$ with $\Re p > 0$, our integrand is approximately $\frac{e^{\frac{xp}{c}}e^{pt}}{pe^{\frac{lp}{c}}} = \frac{1}{p}e^{(x-l+ct)\frac{p}{c}}$.

So if $x - l + ct < 0$, we close in the RHP, and so $y(x,t) = 0$ by Cauchy's theorem.

If $-x + l + ct > 0$, we close in the LHP, our radial contributions vanish, and so get that

$$y(x,t) = y_0 \sum_{m=-\infty}^{\infty} \text{res}\left(\frac{e^{pt}}{p}\sinh\left(\frac{xp}{c}\right)\frac{1}{\sinh\left(\frac{lp}{c}\right)}; \frac{m\pi ci}{l}\right)$$

For $m = 0$, by l'Hôpital's rule our residue is

$$\lim_{p\to 0}\frac{\frac{x}{c}\cosh(\frac{xp}{c})e^{pt} + p\sinh(\frac{xp}{c})e^{pt}}{\frac{l}{c}\cosh(\frac{lp}{c})} = \frac{x}{l}$$

For $m \neq 0$, our residue is similarly

$$\frac{\sinh(\frac{im\pi}{c})\exp(i\frac{m\pi c}{l}t)}{im\pi\cosh(im\pi)} = \frac{i\sin(\frac{m\pi x}{l})\exp(i\frac{m\pi c}{l}t)}{im\pi\cos(m\pi)}$$

Hence for $t > \frac{x-l}{c}$,

$$y(x,t) = \frac{x}{l}y_0 + y_0\sum_{m=1}^{\infty}(-1)^m\frac{\sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{m\pi ct}{l}\right)}{m\pi}$$

Note that for $-l + x < ct < l - x$, both our solutions are valid. Thankfully, the Fourier series solution is 0 here (no proof). So, as we expect, our solution 'switches on' at $t = \frac{l-x}{c}$, so the disturbance propagates from the right at speed c.

Remark. For an alternative inversion method, see handout. By this method, we get

$$y(x,t) = y_0\sum_{n=0}^{\infty}\Theta(ct + x - (2n+1)l) - \Theta(ct - x - (2n+1)l),$$

which is equivalent. We can see that this represents a linear superposition of waves caused by reflections at the two ends of the string.

4 Second-Order ODEs in the Complex Plane

We start with what is essentially a revision of IA.

Consider

$$w'' + p(z)w' + q(z)w = 0, \tag{1}$$

where p, q and w are meromorphic on $\mathbb{C}$.

4.1 Classification of Singular Points

Definition 4.1. The point $z = z_0$ is an *ordinary point* of (1) if p and q are both analytic at z_0. Otherwise z_0 is a *singular point.*

If z_0 is a singular point, but $(z - z_0)p(z)$ and $(z - z_0)^2 q(z)$ are analytic at z_0, then z_0 is a *regular singular point.* Otherwise, z_0 is an irregular singular point.

For linear ODEs, the singularities of the solutions are independent of the initial conditions - they are fully determined by p and q. This does not hold for non-linear ODEs. For example,

$$w' + w^2 = 0 \quad \Rightarrow \frac{dw}{w^2} = -dz \quad \Rightarrow w(z) = (z - z_0)^{-1},$$

where the singularity at z_0 is movable, depending on the constant of integration.

We can extend our definitions to $z = \infty$ by setting $z = \frac{1}{t}$ as follows:

By the chain rule,

$$\frac{d^2 w}{dt^2} + \left(\frac{2}{t} - \frac{p\left(\frac{1}{t}\right)}{t^2}\right)\frac{dw}{dt} + \frac{q\left(\frac{1}{t}\right)}{t^4} w = 0$$

Hence $t = 0$ (so $z = \infty$) is an ordinary point if $P(t) := \frac{2}{t} - \frac{p\left(\frac{1}{t}\right)}{t^2}$ is analytic at $t = 0$.

So,

$$p(z) = \frac{2}{z} + \frac{P\left(\frac{1}{z}\right)}{z^2},$$

and similarly

$$q(z) = \frac{Q\left(\frac{1}{z}\right)}{z^4}$$

Where $Q(z)$ and $P(z)$ are analytic at 0.

Similarly, $t = 0 (z = \infty)$ is a regular singular point if

$$2 - \frac{p\left(\frac{1}{t}\right)}{t}, \quad \frac{q\left(\frac{1}{t}\right)}{t^2}$$

are analytic at $t = 0$.

Or equivalently, we can find f, g analytic at 0 such that

$$p(z) = \frac{2}{z} + \frac{1}{z} f\left(\frac{1}{z}\right), \qquad q(z) = \frac{1}{z^2} g\left(\frac{1}{z}\right)$$

Note that p has at most a simple pole at ∞, and q has at most a double pole.
Example.
Legendre's Equation

$$w'' - \frac{2z}{1 - z^2} + \frac{n(n+1)}{1 - z^2} w = 0,$$

which immediately has regular singular points at $z = \pm 1$

We now check the behaviour at ∞.

$$q(z) = \frac{n(n+1)}{1-z^2} = \frac{1}{z^4}\frac{n(n+1)z^4}{1-z^2} := \frac{1}{z^4}g(\frac{1}{z})$$

g is not analytic at $z = \infty$, so this is not an ordinary point.

$$p(z) = \frac{-2z}{1-z^2} = \frac{2}{z} + \frac{f\left(\frac{1}{z}\right)}{z},$$

for $f\left(\frac{1}{z}\right) = -\frac{2}{1-z^2}$, which is analytic at ∞.

Similarly,

$$q(z) = \frac{1}{z^2}g\left(\frac{1}{z}\right), \qquad g\left(\frac{1}{z}\right) := \frac{n(n+1)z^2}{1-z^2},$$

which is analytic at ∞.

Hence $z = \infty$ is a regular singular point of Legendre's Equation.

Remark. In many cases, the differential equations can be approximated at infinity. For example, here $\frac{1}{1-z^2} \sim -\frac{1}{z^2}$, which allows as to see clearly that $z = \infty$ is a regular singular point.

4.2 Indicial Equations

Consider a regular singular point at $z = 0$, and write

$$w(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^n \tag{2}$$

Let $p(z) = \sum_{m=-1}^{\infty} p_m z^m$, $q(z) = \sum_{m=-2}^{\infty} q_m z^m$.

Then our coefficients p_m, q_m determine σ and the coefficients a_n (up to constants). That is,

$$\sum_{n=0}^{\infty}\left[a_n(n+\sigma)(n+\sigma-1)z^{n+\sigma-2} + \sum_{m=-1}^{\infty} p_m a_n (n+\sigma) z^{n+m+\sigma-1} + \sum_{m=-2}^{\infty} q_m a_n z^{n+m+\sigma}\right] = 0$$

Then considering our lowest order coefficients of z, we get an indicial equation for σ:

$$a_0\left[\sigma(\sigma-1) + p_{-1}\sigma + q_{-2}\right] = 0$$

And so assuming $a_0 \neq 0$,

$$\sigma = \frac{1 - p_{-1} \pm \sqrt{(p_{-1}-1)^2 - 4q_{-2}}}{2}$$

And so $w_1(z) = z^{\sigma_1}(a_0 + a_1 z + \ldots)$, and $w_2(z) = z^{\sigma_2}(a_0 + a_1 z \ldots)$. As such, in this context σ is sometimes referred to as the exponent at $z = 0$.

Now, consider a regular singular point at $z = \infty$.

$$w = t^\sigma \sum_{n=0}^\infty a_n t^n = z^{-\sigma} \sum_{n=0}^\infty a_n z^{-n}$$

$$p(z) = \frac{2}{z} + \frac{1}{z} f\left(\frac{1}{z}\right) = \frac{2}{z} + \frac{1}{z} \sum_{n=0}^\infty f_n z^{-n} =: \sum_{m=1}^\infty p_m z^{-m}$$

$$q(z) = \frac{1}{z^2} g\left(\frac{1}{z}\right) =: \sum_{m=2}^\infty q_m z^{-m}$$

So

$$\sum_{n=0}^\infty \left[a_n(-n-\sigma)(-n-\sigma-1)z^{-n-\sigma-2} + \sum_{m=1}^\infty p_m a_n (-n-\sigma) z^{-m-m-\sigma-1} + \sum_{m=2}^\infty q_m a_n z^{-n-m-\sigma} \right] = 0$$

Hence our indicial equation is

$$-\sigma(-\sigma-1) - p_1\sigma + q_2 = 0$$

4.3 Solutions Near a Regular Singular Point

(see handouts also)

Consider a regular singular point at $z = 0$. For $|z| \ll 1$, we can our ODE by

$$w'' + \frac{p_{-1}}{z} w' + \frac{q_{-2}}{z^2} w \approx 0$$

The solutions are $w_1 = z^{\sigma_1}$, $w_2 = z^{\sigma_2}$ if σ_1, σ_2 are distinct, and $w_1 = z^{\sigma_1}$ if they are equal.

Theorem 4.2. Let $z = 0$ be a regular singular point of

$$w'' + pw' + qw = 0$$

Then there are two linearly independent solutions w_1, w_2 such that if

1. $\sigma_1 - \sigma_2 \notin \mathbb{Z}$:
$$w_1(z) = z^{\sigma_1} u_1(z), \quad w_2(z) = z^{\sigma_2} u_2(z),$$
where u_1, u_2 are analytic on some neighbourhood of 0, and $u_i(0) \neq 0$.

2. $\sigma_1 = \sigma_2$:
$$w_1(z) = z^{\sigma_1} u_1(z), \quad w_2(z) = z^{\sigma_2} u_2(z) + w_1(z) \ln z,$$
u_i as above.

3. $\sigma_1 - \sigma_2 \in \mathbb{Z} \setminus \{0\}$ and wlog $\sigma_1 > \sigma_2$:
$$w_1(z) = z^{\sigma_1} u_1(z), \quad w_2(z) = z^{\sigma_2} u_2(z) + c w_1(z) \ln z,$$
where u_i as above, and c is a constant that might be zero.

Proof. We here prove case 1, and leave the other cases to a handout.

Substituting $w(z) = \sum_{n=0}^{\infty} a_n z^{n+\sigma}$ into our ODE, equating coefficients, and assuming $a_0 \neq 0$, we get

$$\begin{aligned} a_0 F(\sigma) &= 0 \\ a_n F(n+\sigma) &= -\sum_{k=0}^{\infty} a_k \left((k+\sigma)p_{n-k-1} + q_{n-k-2}\right) \quad (n > 0) \end{aligned}$$

where $F(x) := x(x-1) + p_{-1}x + q_{-2} = (x-\sigma_1)(x-\sigma_2)$

Since $a_0 \neq 0$, $F(\sigma) = 0$ is the indicial equation, and so

$$\sigma_1 + \sigma_2 = 1 - p_{-1}, \quad \sigma_1\sigma_2 = q_{-2}$$

If $\sigma_1 - \sigma_2 \notin \mathbb{Z}$, then we get a recurrence relation for a_k with two linearly independent solutions. $\square$

Example.
Bessel's Equation

$$w'' + \frac{1}{z}w' + \left(1 - \frac{2}{z^2}\right) w = 0$$

Our non-zero coefficients are $p_-1 = 1, q_0 = 1, q_{-2} = -\gamma^2$. So (ignoring $\gamma = 0$) we obtain (near $z = 0$):

$$\begin{aligned} a_0 F(\sigma) &= a_0(\sigma^2 - \gamma^2) = 0 \\ a_1 F(\sigma+1) &= 0 \\ &\vdots \\ a_n F(\sigma+n) &= -a_{n-2} \end{aligned}$$

so $\sigma_{1,2} = \pm\gamma$

4.4 Fuchsion Equations

We consider second order ODEs as before with at most three regular singular points.

4.4.1 No singular points

Proposition 4.3. There are no ODEs without singular points in $\mathbb{C}_\infty$.

Proof. If there are no singular points, then $p(z)$ is entire on C_∞ and is thus constant by Liouville's theorem. But $z = \infty$ is an ordinary point, so $p(z) = \frac{2}{z} + f\left(\frac{1}{z}\right)$, so it is not a constant. $\square$

4.4.2 One regular singular point

Wlog we can assuming the singular point is at $z = 0$ (as we can move it via Möbius transforms), and all other points are ordinary.

At $z = 0$,

$$zp(z) = P(z), \quad z^2 q(z) = Q(z),$$

where P and Q are entire on $\mathbb{C}_\infty$.

At $z = \infty$,

$$p(z) = \frac{2}{z} + \frac{1}{z^2} A\left(\frac{1}{z}\right), \quad q(z) = \frac{1}{z^4} B\left(\frac{1}{z}\right),$$

where A and B are entire on C_∞.

So $P(z) = 2 + \frac{1}{z} A\left(\frac{1}{z}\right)$ is entire, and so $A = 0$ by Liouville's Theorem. $Q(z) = \frac{1}{z^2} B\left(\frac{1}{z}\right)$ is also entire, and so also $B = 0$.

Hence our ODE takes the form

$$w'' + \frac{2}{z} w' = 0,$$

which has solution

$$w(z) = \alpha + \frac{\beta}{z}$$

for some constants α, β.

Lemma 4.4. A Möbius transformation $M : z \to t$ will transform an ODE of the form

$$w'' + p(z)w' + q(z)w = 0$$

into another ODE of the same form. The singular points move according to $z_0 \mapsto M(z_0)$, with exponents unchanged/

Proof is not given - consider the generators of scaling, translation and inversion.

Thus, if the singular point is at $z = z_0 \neq \infty$, then we let $t = z - z_0$, and trivially our solution is

$$w = \alpha + \frac{\beta}{z - z_0}$$

If $z_0 = \infty$, we let $t = \frac{1}{z}$, and so by chain rule,

$$\frac{d^2 w}{dt^2} = 0 \quad \Rightarrow w = \alpha + \frac{\beta}{t} = \alpha + \beta z$$

4.4.3 Two regular singular points

Wlog we can assume that $z = 0, 1$ are the singular points, with all other points ordinary.

So

$$p(z) = \frac{1 + A}{z} + \frac{1 + B}{z - 1} + \underbrace{P(z)}_{\text{entire}}$$

$z=\infty$ is an ordinary point, so $p(z)=\frac{2}{z}+\mathcal{O}\left(\frac{1}{z^2}\right)$ as $z\to\infty$.

Hence

$$\frac{1+A}{z}+\frac{1+B}{z-1}\to\frac{2}{z},$$

and so $A+B=0$.

As $p(z)\to 0$ at infinity, $P(z)=0$ by Liouville's theorem.

Hence

$$p(z)=\frac{1+A}{z}+\frac{1-A}{z-1}$$

Now, $q(z)=\frac{Cz+D}{z^2}+\frac{Ez+F}{(z-1)^2}+Q_2(z)$, with Q_2 entire. So

$$q(z)=:\frac{Q_1(z)}{z^2(z-1)^2}+Q_2(z),$$

for some polynomial Q_1 of degree at most 3.

As $z=\infty$ is an ordinary point, $z^4q(z)$ is bounded at infinity, and so $Q_2=0$.

Further,

$$\frac{z^4Q_1(z)}{z^2(z-1)^2}\to Q_1(z) \text{ as } z\to\infty,$$

so $Q_1(z)=Q$ constant.

Hence

$$w''+\left(\frac{1+A}{z}+\frac{1-A}{z-1}\right)w'+\frac{Q}{z^2(1-z)^2}w=0,$$

for constant A, where $z=0,1$ are our regular singular points.

Consider exponents in the vicinity of $z=0$. Then we can approximate

$$w''+\frac{1+A}{z}w'+\frac{Q}{z^2}w\sim 0$$

and so get indicial equation

$$\sigma^2+A\sigma+Q=0$$

Similarly, at $z=1$ we get

$$w''+\frac{1-A}{z-1}w+\frac{Q}{(z-1)^2}w\sim 0 \quad\Rightarrow\sigma^2-A\sigma+Q=0$$

We can use Möbius transforms to move the singular points. In particular, we can use $t=\frac{1}{2}$ to map $0\mapsto\infty, 1\mapsto 1$.

So

$$t^4w_{tt}+\left[2t^3-t^2\left(t(1+A)+\frac{t(a-A)}{1-t}\right)\right]w_t+t^4\frac{Q}{(t-1)^2}w=0$$

And we thus obtain

$$w''+\frac{1-A}{t-1}w'+\frac{Q}{(t-1)^2}w=0$$

So our indicial equation at $t = 1$ is given by

$$\sigma^2 - A\sigma + Q = 0,$$

the same as before at $z = 1$, and at $t = \infty$ we get

$$\sigma^2 + A\sigma + Q = 0,$$

the same as before at $z = 0$.

Note that solutions near $z = 0$ take the form $z^{\sigma_1} u_1(z) + z^{\sigma_2} u_2(z)$, where U_i is analytic. The solutions at $t = \infty$ take the form $w(t) = t^{-\sigma_1} u_1\left(\frac{1}{t}\right) + t^{-\sigma_2} u_2\left(\frac{1}{t}\right)$

4.4.4 Three Regular Singular Points - The Papperitz Equation (Riemann's P-Equation)

(also see handout)

We show that there are 8 parameters in the P-equation: 3 from positions of the singular points, 6 for exponents (2 per singular point), but -1 from a constraint on the exponents as they must add up to 1.

Suppose the three regular singular points are at $z = a, b, c$ with all other points regular. Then wlog we can write $p(z)$ in the form

$$p(z) = \frac{1-\alpha-\alpha'}{z-\alpha} + \frac{1-\beta-\beta'}{z-\beta} + \frac{1-\gamma-\gamma'}{z-c} + P(z),$$

where P is entire.

As $z = \infty$ is an ordinary point, then $p(z)$ is bounded at ∞, and so by Liouville's theorem $P(z)$ is constant (wlog 0).

Further,

$$zp(z) \to 2 \text{ as } z \to \infty \quad \Rightarrow (1-\alpha-\alpha') + (1-\beta-\beta') + (1-\gamma-\gamma') = 2,$$

so

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$$

Next, wlog

$$q(z) = \frac{k_\alpha z - k'_\alpha}{(z-\alpha)^2} + \frac{k_\beta z - k'_\beta}{(z-\beta)^2} + \frac{k_\gamma z + k'_\gamma}{(z-\gamma)^2} + Q_2(z)$$

$$=: \frac{Q_1(z)}{(z-a)^2(z-b)^2(z-c)^2} + Q_2(z)$$

where Q_2 is entire, and Q_1 is a polynomial of degree at most 5.

As $z = \infty$ is an ordinary point, $z^4 q(z)$ is bounded as $z \to \infty$, and so by Liouville's theorem, Q_2 is constant (wlog 0), and Q_1 is at most quadratic.

Hence we can write

$$q(z) = \frac{1}{(z-a)(z-b)(z-c)} \left(\frac{q_a}{z-a} + \frac{q_b}{z-b} + \frac{q_c}{z-c} \right)$$

We can now specify $\alpha, \alpha', \beta, \beta', , \gamma'$ by writing

$$\begin{aligned} q_a &= \alpha\alpha'(a-b)(a-c) \\ q_b &= \beta\beta'(b-a)(b-c) \\ q_c &= \gamma\gamma'(c-a)(c-b) \end{aligned}$$

and so we can fix our constants under $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma'$ =1, giving us our 8 degrees of freedom.

This gives the Papperitz equation:

$$\begin{aligned} w'' + &\left(\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right) w' \\ &- \frac{(b-c)(c-a)(a-b)}{(z-a)(z-b)(z-c)} \left[\frac{\alpha\alpha'}{(z-a)(b-c)} + \frac{\beta\beta'}{(z-b)(c-a)} + \frac{\gamma\gamma'}{(z-c)(a-b)}\right] w = 0 \end{aligned}$$

Near $z = a$, the Papperitz equation can be approximated as

$$w'' + \frac{1-\alpha-\alpha'}{z-a} w' + \frac{\alpha\alpha'}{(z-a)^2} w \sim 0,$$

so our indicial equation is

$$\sigma^2 + (\alpha + \alpha')\sigma + \alpha\alpha' = 0,$$

and so α andα' are the exponents are $z = a$, and similarly at b and c.

We write

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

for the $2d$ vector space of solutions of the P-equation. This is known as the Papperitz (P) - symbol.

It is a common abuse of notation to write $w(z) = P\left\{\ldots\right\}$ to mean that w solves the P equation.

4.5 The Hypergeometric Equation

A Möbius transform acts on a P-symbol as follows:

$$(a, b, c, z) \to (a', b', c', z')$$

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} \mapsto P\begin{Bmatrix} a' & b' & c' & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

We also need to transform P-symbol by exponent shifting.

Proposition 4.5.

$$\left(\frac{z-a}{z-b}\right)^{\sigma}\left(\frac{z-b}{z-c}\right)^{\delta} P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} a & b & c & \\ \alpha+\sigma & \beta-\sigma+\delta & \gamma-\delta & z \\ \alpha'+\sigma & \beta-\sigma+\delta & \gamma-\delta & \end{Bmatrix}$$

And our solution

$$w(z) \mapsto \left(\frac{z-a}{z-b}\right)^{\sigma}\left(\frac{z-b}{z-c}\right)^{\delta} w(z)$$

Proof. We prove the case $\delta = 0$, but the proof is easily extended.

Let us prove the equivalent

$$\left(\frac{z-a}{z-b}\right)^{\sigma} P\begin{Bmatrix} a & b & c & \\ \alpha-\sigma & \beta+\sigma & \gamma & z \\ \alpha'-\sigma & \beta+\sigma & \gamma & \end{Bmatrix} = P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

We assume that $w(z)$ is a solution of the RHS, and want to show that $w_1(z) = \left(\frac{z-a}{z-b}\right)^{-\sigma} w(z)$ is a solution to the LHS.

Near $z = a$,

$$w(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+\alpha},$$

so

$$\begin{aligned} w_1(z) &= (z-a)^{\alpha-\sigma}(z-\beta)^{\sigma}\sum_{n=0}^{\infty} a_n(z-a)^n \\ &= (z-a)^{\alpha-\sigma}\sum_{n=0}^{\infty} c_n(z-a)^n \end{aligned}$$

as $(z-b)^{\sigma}$ is analytic at $z = a$.

So α is shifted to $\alpha - \sigma$, and the result follows. ☐

Note: If multiply by$(-b)^{\sigma}$ and take when $b \mapsto \infty$,

$$(z-a)^{\sigma} P\begin{Bmatrix} a & \infty & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} a & \infty & c & \\ \alpha+\sigma & \beta-\sigma & \gamma & z \\ \alpha'+\sigma & \beta'-\sigma & \gamma' & \end{Bmatrix}$$

To obtain the Hypergeometric equation, we perform the following operations.

1. Map $(a, b, c) \mapsto (0, 1, \infty)$ by Möbius transform

2. Shift exponents so $\alpha = \beta = 0$

3. Relabel $\gamma = A$, $\gamma' = B$, $\alpha' = 1 - C$, then $\beta' = C - A - B$

So the P-symbol becomes

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & A & z \\ 1-C & C-A-B & B & \end{matrix}\right\},$$

which corresponds to the hypergeometric equation

$$w'' + \left(\frac{C}{z} + \frac{1+A+B-C}{z-1}\right) w' + \frac{AB}{z(z-1)} w = 0 \tag{†}$$

with regular singular points $0, 1, \infty$. Note that the hypergeometric equation is generally written with lower case letters, not to be confused with our original singular points.

Now, one of the exponents is 0, so there is a solution that is analytic at $z = 0$ with $w(0) = 1$. This is known as the hypergeometric function $F(a, b; c, z) = F(b, a; c, z)$

Proposition 4.6.

$$F(a, b; c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(x)_n = x(x+1)\dots(x+n-1)$, $(x)_0 = 1$ is the *Pochhammer symbol.* The series is convergent in $|z| < 1$. The singular point at $z = 1$ prohibits the series from converging further.

Proof. Since the solution $w(z)$ is analytic, it can be written as a Taylor series

$$w(z) = \sum_{n=0}^{\infty} a_n z^n$$

substituting this into (†) and multiplying by $z - 1$, we get

$$\begin{aligned} n(n-1)a_n z^{n-2} &+ (n+1)na_{n+1}z^{n-1} + cna_n z^{n-2} + c(n+1)a_{n+1}z^{n-1} \\ &+ (1+a+b-c)(na_n z^{n-1} + (n+1)a_{n+1}z^n) + ab(a_n z^{n-1} + a_{n+1}z^n) = 0 \end{aligned}$$

The coefficients of z^{n-1} give

$$n(n-1)a_n + (n_1)na_{n+1} + cna_n - c(n+1)a_{n+1} + (1+a+b-c)na_n + aba_n = 0$$

and thus

$$a_n(n^2 + an + bn + ab) = a_{n+1}(n+1)(c+n)$$

Hence,

$$a_{n+1} = \frac{(a+n)(b+n)}{(c+n)} \frac{1}{n+1} a_n,$$

which gives the claimed result $\square$

Remark. Many special functions are special cases of the Hypergeometric function:

$$\begin{aligned} (1-z)^n &= F(-n, 1; 1, z) \\ \log(1-z) &= zF(1, 1; 2, z) \\ e^z &= \lim_{b \to 0} F(1, b; 1, \frac{z}{b}) \end{aligned}$$

Proposition 4.7. The hypergeometric function has integral representation

$$F(a,b;c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-1}dt$$

for $\Re c > \Re b > 0$ and $|z| < 1$ so that the integral converges at branch points $0, 1, \frac{1}{z}$.

Proof. See example sheet 4. $\square$

4.5.1 Second Solution to the Hypergeometric Equation

Near $z = 0$, there is a solution with exponent $1-c$. It can be expressed in terms of the Hypergeometric function. It has the form $w(z) = z^{1-c}g(z)$, where g is analytic at $z = 0$ and $g(0) = 1$.

We have

$$w(z) = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix},$$

and so by shifting

$$g(z) = P\begin{Bmatrix} 0 & 1 & \infty & \\ c-1 & 0 & a+c+1 & z \\ 0 & c-a-b & b-c+1 & \end{Bmatrix} \equiv P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a-c+1 & z \\ c-1 & c-a-b & b-c+1 & \end{Bmatrix}$$

as the order of our exponents does not matter.

We let $c' = 2-c$, $a' = a+c-a$, $b' = b-c+1$. Then

$$g(z) = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a' & z \\ 1-c' & c'-a'-b' & b' & \end{Bmatrix}$$

Hence

$$w(z) = z^{1-c}F(a-c+1, b-c+1, ; 2-c, 2)$$

is the second solution at $z = 0$.

Note. This works for $c \notin \mathbb{Z}$.

4.5.2 Solutions near $z = \infty$

The two principal branches at $z = \infty$ of a hypergeometric P-function can be written in terms of hypergeometric functions as follows. Note first that the branches are of the form

$$P_a(z) = z^{-a}g_a(z) \qquad \text{and} \qquad P_b(z) = z^{-b}g_b(z)$$

where $g_a(t^{-1})$ and $g_b(t^{-1})$ are analytic at $t = 0$.

Now, $P_a(z)$ is a branch of the hypergeometric P-function

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & & z \\ 1-c & b & c-a-b & \end{matrix}\right\}$$

So by exponent shifting, $g_a(z)$ is a branch of

$$\begin{aligned}
&P\left\{\begin{matrix} 0 & \infty & 1 & \\ a & 0 & 0 & z \\ 1-c+a & b-a & c-a-b & \end{matrix}\right\} \\
&= P\left\{\begin{matrix} \infty & 0 & 1 & \\ a & 0 & & z^{-1} \\ 1-c+a & b-a & c-a-b & \end{matrix}\right\} \text{ by Möbius transform} \\
&= P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z^{-1} \\ b-a & 1-c+a & c-a-b & \end{matrix}\right\} \text{ reordering columns} \\
&\equiv P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a' & 0 & z \\ 1-c' & b' & c'-a'-b' & \end{matrix}\right\}
\end{aligned}$$

where $c' = 1 + a - b$, $b' = 1 - c + a$, and $a' = a$

Now, $g_a(z)$ is analytic at $z^{-1} = 0$ and $g(\infty) = 1$, so $g_a(z)$ must e the principle branch of the above P-function corresponding to the exponent 0 at the point $z^{-1} = 0$, which by definition is a hypergeometric function.

Thus

$$w(z) = z^{-a} F(a, 1-a+c; 1+a-b, z^{-1})$$

The other branch is obtained from this by interchanging a and b. Note that since there are only two linearly independent branches at each point, we can express the analytic continuation of $F(a, b; c, z)$ to large z in the form

$$F(a, b; c, z) = Az^{-a} F(a, 1-a+c; 1+a-b, z^{-1}) + Bz^{-b} F(b, 1-b+c; 1+b-a, z^{-1}),$$

where A and B are constants which can be found using, for example, integral representations of the hypergeometric function.

4.6 Monodromy - the need for a logarithmic term

We briefly revisit our general linear second order ODE

$$w'' + p(z)w' + q(z)w = 0$$

For a regular singular point at $z = 0$, there exist two linearly independent solutions such that if $\sigma_1 = \sigma_2$, then $w_1(z) = z^{\sigma} u_1(z)$, and $w_2(z) = w_1(z) \log z$.

We now justify why the log term arises.

Consider $\mathcal{D} = \{z : 0 < |z| < R\}$ to be the largest punctured disk with no other singularities. Let $z_0 \in \mathcal{D}$. Then z_0 is an ordinary point, so there exists an open disk $\mathcal{D}_0 \subset \mathcal{D}$ which has analytic solutions, wlog $w_1(z) w_2(z)$ which form a Vieta space of dimension 2.

Let $C = \{|z| = |z_0|\}$. We perform analytic continuation using a sequence of disks along C back to $\mathcal{D}_0$. Let $\hat{w}_1(z) = w_1\left(e^{2\pi i} z\right)$, $\hat{w}_2(z) = w_2\left(e^{2\pi i} z\right)$ to be the result of analytic continuation back into $\mathcal{D}_0$.

Then we can find a non-singular matrix M such that

$$\begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \end{pmatrix} = M \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

This is known as the Monodromy matrix.

M has two possible Jordan normal forms:

$$\begin{pmatrix} e^{2\pi i \sigma} & 0 \\ 0 & e^{2\pi i \sigma} \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} e^{2\pi i \sigma} & 0 \\ 1 & e^{2\pi i \sigma} \end{pmatrix}$$

In both cases, $w_1\left(e^{2\pi i} z\right) = \left(e^{2\pi} z\right)^{-\sigma} w_1(z)$. Letting