Intro CMM Hw2 Euler Method

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1 Carbon

The carbon-14 isotope undergoes β^- decay with a half life time of $t_{1/2} = 5700$ years. The amount of carbon-14 atoms remaining after some time will be dependent on the decay constant λ, or it's inverse the time constant for decay τ. These are constants representing the rate of change and they are used as follows:

$$\frac{\Delta N}{\Delta t} = -\lambda N$$

$$\lim_{t \to 0} \frac{\Delta N}{\Delta t} = \frac{dN}{dt} = -\lambda N$$

$$\frac{dN}{N} = -\lambda N dt$$

$$\int_{N_o}^{N} \frac{dN}{N} = -\lambda \int_0^t dt$$

$$\frac{dN}{N} = -\lambda dt$$

$$\ln(N)|_{N_0}^{N} = -\lambda t$$

$$\ln \frac{N}{N_o} = -\lambda t$$

$$\frac{N}{N_o} = e^{-\lambda t}$$

$$N = N_o e^{-\lambda t}$$

$$N = N_o e^{\frac{-t}{\tau}}$$

Where N is the amount of carbon-14 at a given time (this can also be called $N(t)$, N_o is the initial amount of carbon-14, and t is the time. To determine the half life of carbon-14, $t_{1/2}$, an equation from the derivation process can be used such that the atom amount is half of that which was initially present:

$$\ln \frac{N}{N_o} = -\lambda t$$

$$
\begin{aligned}
\ln 0.5 &= -\lambda t_{1/2} \\
\frac{\ln 0.5}{-\lambda} &= t_{1/2} \\
\frac{0.69}{\lambda} &= t_{1/2} \\
\lambda &= 0.00012 \\
\frac{1}{\tau} &= 0.00012 \;\text{ years}^{-1} \\
\tau &= 8300\,\text{years}
\end{aligned}
$$

This is the analytical solution in inspecting the change in carbon-14 atom amount. There exists a numerical calculation which is in the Euler form, which uses discretization to simplify the calculation. This can be done with the use of a for loop or with a recursive formula.

$$N(t + \Delta t) = N(t) - \frac{1}{\tau} N(t) \Delta t$$

Both the analytical and the computational solution are used to model the difference between a continuous and discrete function. One parameter, N_o which is the original number of carbon-14 atoms, is necessary to graph the solutions:

$$N_o = 10^{-12} \text{ kg} \cdot \frac{1000 \text{ g}}{1 \text{ kg}} \cdot \frac{1 \text{ mole}}{14 \text{ g}} \cdot \frac{6.022 * 10^{23} \text{ particles}}{1 \text{ mole}} = 4.3 * 10^{13} \text{ particles}$$

The analytical solution, and the discrete solution with $\Delta t = 10$ and $\Delta t = 1000$ can now be plotted on the Number of Particles Remaining vs. Time graph as seen in Figure 1. The curves overlap, and as such the Euler method may be an appropriate substitute for the analytical solution for small step sizes.

If the analytical solution was plotted against the computational analysis with a step size of 1000, then the solutions would no longer overlap as seen in Figure 2.

Past the second-half life, the analytical solution and the computational solution with Δt clearly differ with values of $N_a = 1.0{*}10^{13}$ atoms and $N_c = 9.3{*}10^{12}$ atoms such that their percent deviation is as follows:

$$\frac{\mid 9.3 * 10^{12} - 1.0 * 10^{13} \mid}{1.0 * 10^{13}} * 100\% = 7.0\%$$

The Euler's method only focuses on the first term, any other terms in the series are ignored, which is why this error exists. The first term, usually contains the greatest magnitude, and the other terms allow for small adjustments to the data value. The deviation from the analytical method is greater than expected as this error will only continue to increase as more time passes, and it is a difference of $7.0 * 10^{11}$ particles. This deviation from the "true" value is greater than 5% which is usually thought of as the cutoff for having significant data. Therefore, the Euler method with step size of $\Delta t = 1000$ is not an acceptable replacement for the analytical solution.

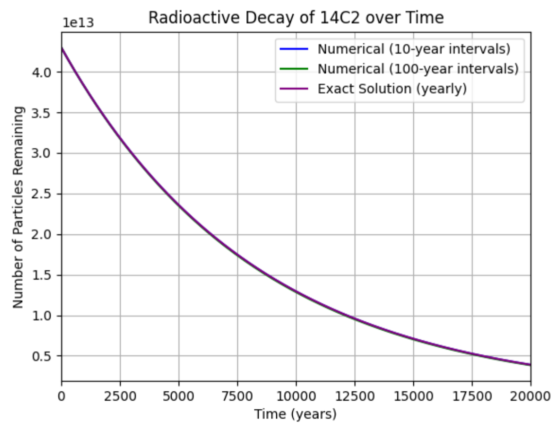

Figure 1: Default plot of the analytical solution and Euler solution for $\Delta t = 10$ and $\Delta t = 100$

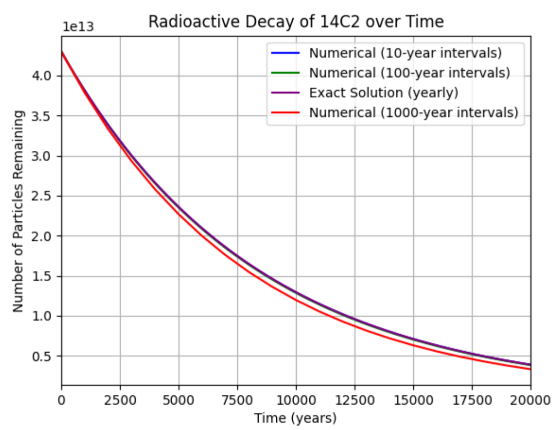

Figure 2: Default plot of the analytical solution and Euler solution for $\Delta t = 10$, $\Delta t = 100$, and $\Delta t = 1000$

2 Golf

Golf balls are optimized to be able to travel long distances based on their shape. The distance they travel is dependent upon the forces acting on them such as gravity, drag and Mangus, and on the angle at which they are hit, if assuming all other parameters are consistent among swings(initial velocity, air density, air density, and the frontal area of the golf ball). Thus, golf ball trajectories will be studied based on the forces the golf ball experiences. There will be four different environments:

1. Ideal trajectory experiencing only gravity.
2. Golf ball experiences drag with coefficient 0.5 and gravity.
3. Golf ball experiences circumstantial drag with a coefficient of 0.5 below $14\frac{m}{s}$ and 7.0/velocity above $14\frac{m}{s}$ and gravity.
4. Golf ball experiences circumstantial drag with a coefficient of 0.5 below $14\frac{m}{s}$ and 7.0/velocity above $14\frac{m}{s}$, gravity, and spin or Mangus force of $\frac{So\omega}{m} * v_y$ in the x direction and $\frac{So\omega}{m} * v_x$ in the y direction.

It is assumed that the golf ball is of mass 46 grams or $m = 0.046$ kg, experiences a gravity of $g = 9.8\frac{m}{s}$ and has an initial velocity of $70\frac{m}{s}$. For the drag force, $F_{drag} = -C\rho Av^2$ where C is the coefficient, $\rho = 1.29\frac{kg}{m^3}$ is the density of air at sea level, $A = 0.0014m^2$ is the frontal area of the golf ball, and v is the velocity of the golf ball.

As seen on Figure 3 the position was updated based on the environment the golf ball experiences. This was done with a small step size, and the position was the first element to be updated with each iteration, followed by the velocity which was influenced by gravity, drag, and then spin. From Figure 3 with $\theta = 45°$ and being the ideal trajectory, it is seen that this would usually give the best-case scenario in terms of going farthest. It is interesting to note, however, that the ideal trajectory does not travel farthest for all four angles. For $\theta = 9°$ it is actually the dimpled ball that experiences gravity, drag, and spin that travels the farthest. Thus, the angle at which the golf ball is projected is critical in determining whether the ideal trajectory is conservative of the outcome.

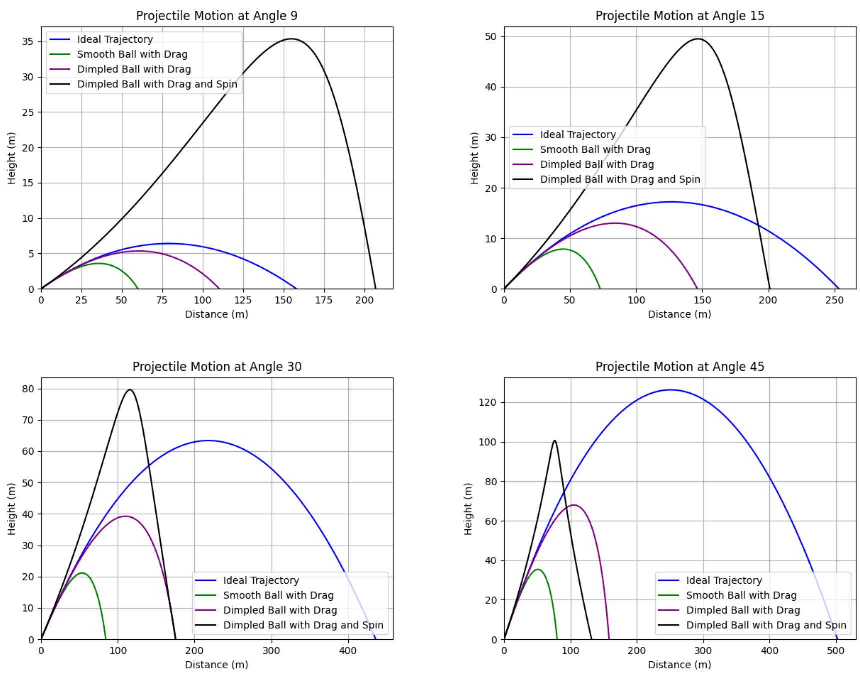

Figure 3: Golf Ball Trajectories for $\theta = 9^\circ$ (top-left), $\theta = 15^\circ$ (top-right), $\theta = 30^\circ$ (bottom-left), $\theta = 45^\circ$ (bottom-right)