Homework 2

Oliver Kis

15576333

19 January 2024

17/20

The University of British Columbia

Electrical and Computer Engineering

Math 220 - Mathematical Proofs

Instructor: Bennett Michael

Solutions

For Homework 2, parts of problems 1, 2 and all of problem 4 and 5 are worth marks.

Problem 1. (6 Points) Prove the following statements:

1. (2 Points) Let $n \in \mathbb{Z}$. If $3 \mid (n\text{-}4)$, then $3 \mid (n^2\text{-}1)$.

   Proof: Let $n \in \mathbb{Z}$, we work with $3 \mid (n\text{-}4)$ to equate it to $3 \mid (n^2\text{-}1)$.
   $n\text{-}4 = 3k -> n^2 = (3k+4)^2 -> n^2 = 9k^2 + 24k + 16 -> n^2 = 9k^2 + 24k + 15 + 1$

   After working through, we get $n^2 - 1 = 3(3k^2 + 8k + 5)$. From our original statement $3 \mid (n\text{-}4)$, we now know that $3 \mid (n^2\text{-}1)$ stands true, as we can divide $n^2 - 1$ by 3.

2. (2 Points) For a, $b \in \mathbb{Z}$: if a and b have the same parity then $a + b - 4$ is even.

   Proof: Let $a, b \in \mathbb{Z}$, this proof requires two cases, one where a and b are **even** and another where they are **odd**

   1) Our first case we let a and b be **even**. If $a = 2k$ and $b = 2l$, $k, l \in \mathbb{Z}$ are both even, there only exists even solutions of $a+b = 2k+2l$. Hence, subtracting -4, an even value, $a + b - 4 \equiv 2k + 2l - 4 \equiv 2(k + l) - 4$ would remain even.

   2) Our second case we let a and b be **odd**. If $a = 2k + 1$ and $b = 2l + 1, k, l \in \mathbb{Z}$, are both odd, there only exists even solutions when added as $a+b-4 \equiv 2k+1+2l+1-4 \equiv 2k + 2l - 2$, which is always even.

   Both cases are true, therefore when a and b have the same parity, $a + b - 4$ is proven even.

3. (2 Points) For $x \in \mathbb{R}$: if $x > 2$ then $\dfrac{8}{x^2 + 2x} < 1$.

   Proof: Let $x \in \mathbb{R}$. By direct proof, looking at the denominator, $x^2 + 2x = x(x + 2)$. From knowing $x > 2$, $x + 2 > 4$, then we get $x(x + 2) > 4x$. Further, $\dfrac{8}{x^2 + 2x} < \dfrac{8}{4x}$ which then becomes $\dfrac{8}{x^2 + 2x} < \dfrac{2}{x} < 1$. Therefore, when $x > 2$ then $1 > \dfrac{2}{x}$, with x always growing bigger, our statement is always less than 1.

Problem 2. (6 Points)

1. (3 Points) Let $n \in \mathbb{Z}$. Prove that if $5n$ is even then n is even.

   Proof: Let $n \in \mathbb{Z}$. Solving by contrapositive, we knowing n is odd means $n = 2k + 1$ for $k \in \mathbb{Z}$.

   $$5n = 5 * 2k + 1 = 10k + 1$$

   Since $10k + 1$ is always going to be odd as it is nearly identical to $2k + 1$, therefore our statement that $5n$ is even then n is even holds true.

2. (3 Points) Let $n \in \mathbb{Z}$. Prove that if 5 divides n and 2 divides n, then 10 divides n.

   Proof: Let $n \in \mathbb{Z}$. Knowing that $n = 5k$ and $n = 2l$, where $k, l \in \mathbb{Z}$, we know $5k = 2n$, therefore $\dfrac{5k}{2} = l$. 5 is not divisible by 2, hence k must be, $k = 2m$ for $m \in \mathbb{Z}$. Going back to $n = 5k$, we replace the k, $n = 5 * 2m \rightarrow n = 10m$, holding the statement true.

Problem 4. (4 Points) Let $n \in \mathbb{Z}$. Prove the following claim:

If 4 divides n - 1, then n is odd and $(-1)^{(n-1)/2} = 1$.

Proof: Let $n \in \mathbb{Z}$. Reworking $4|(n-1)$, we get $n = 4k + 1$, with $k \in \mathbb{Z}$, hence is always odd. Proving our first point, that n is odd. Replace the n, $(-1)^{(4k+1-1)/2} \rightarrow (-1)^{2k}$. We know $2k$ is even, therefore we also conclude that $(-1)^{2k}$, is always positive and 1.

Therefore, both statements are true.

Problem 5. (4 Points) **Definition** : We call an element $x \in \mathbb{R}$ an *integer root* if there exist $k \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that $x^k = m$.

Use this definition to show, for a, $b \in \mathbb{R}$:

if a and b are integer roots, then ab is an integer root.

Proof: Let $a, b \in \mathbb{R}$. $a^{k_1} = m_1$ and $b^{k_2} = m_2$, where $k \in \mathbb{N}$ and $m \in \mathbb{Z}$.

$$ab = a^{k_1} * b^{k_2} = m_1 * m_2$$

$$k = k_1 + k_2$$

$$ab^k = (a^{k_1} * b^{k_2})^k = (m_1 * m_2)^k$$

$$(m_1 * m_2)^k = m_1^k * m_2^k$$

The final statement, with $k \in \mathbb{N}$ indicates $m \in \mathbb{Z}$ is true. Therefore, $m_1^k * m_2^k \equiv m_1 * m_2$, which is just an integer. Hence, the statement $ab = m_1 * m_2$ stands true.