Homework 4

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Electrical and Computer Engineering

Math 220 - Mathematical Proofs

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Solutions

For Homework 4, Problem 1, 2, parts of 4, and parts of 5 are worth marks.

Problem 1. (2 *Points*) Prove, for $x, y, \in \mathbb{Z}$, that

$$(xy \text{ even and } x + y \text{ even}) \implies (x \text{ even and } y \text{ even})$$

Proof: Let $x, y \in \mathbb{Z}$. We proved this by Contrapositive and several cases. The contrapositive turns out to be (x odd or y odd) $\implies$ (xy odd or $x + y$ odd)

1) Case one

$$x \text{ is odd and } y \text{ is odd} \therefore x = 2k + 1 \text{ and } y = 2l + 1 \text{ where } k, l \in \mathbb{Z}.$$
$$xy = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$$

2) Case two

$$x \text{ is odd and } y \text{ is even} \therefore x = 2k + 1 \text{ and } y = 2l \text{ where } k, l \in \mathbb{Z}.$$
$$x + y = 2k + 1 + 2l = 2(k + l) + 1$$

3) Case three

Identical to case two, but x and y are "flipped" x is even and y is odd.

Hence, all three cases prove the contrapositive, which proves the original statement, as required. //

Problem 2. (2 *Points*) For $a \in \mathbb{R}$, we define the set $S_a = x \in \mathbb{R} : (x \geq 0 \wedge x < a - 2)$. Show that

$$S_a = \emptyset \text{ if and only is } a \in]-\infty, 2]$$

Proof: Let $a, x \in \mathbb{R}$. To prove this, we use the biconditional technique.

$\implies$ If our set is null, we know $x \geq 0$ isn't satisfied and/or $x < a - 2$ isn't satisfied. Using the bounds, when $a = 2$, we know $x < 0$, which doesn't hold with the statement $x \geq 0$

$\impliedby$ With $a \in]-\infty, 2]$, we know $-\infty < a - 2 \leq 0$. Using the maximum, this won't satisfy both $x \geq 0$ and $x < a - 2$

Problem 4. (12 *Points*) For each of the following statements:

- Negate the statement.
- Decide if the original statement is true or false and justify your answer.

1. (2 *Points*) $\forall\, a \in \mathbb{Z}, ((6 \mid a \text{ and } 8 \mid a \implies 48 \mid a)$.
   - $\exists\, a \in \mathbb{Z}$, $((6|a$ and $8|a \implies 48 \nmid a)$.
   - False $\longrightarrow$ Finding gcd(6, 8) is 24, next multiple is 48. These work with the implications, however the next multiple of 72 does not.
2. (2 *Points*) $\forall\, x \in \mathbb{R}, \forall y \in \mathbb{R}, (xy \geq 0 \implies x + y \geq 0)$.
   - $\exists\, x \in \mathbb{R}, \exists\, y \in \mathbb{R}$ s.t. $(xy \geq 0 \implies x + y < 0)$.
   - False $\longrightarrow$ $\forall\,(-x)$ and $\forall\,(-y)$ satisfy $xy \geq 0$, but does not satisfy $(x + y \geq 0)$.
3. (2 *Points*) $\forall\, a, b \in \mathbb{Z}, \forall n \in \mathbb{N}, (6a \equiv 6b \bmod 6n \implies a \equiv b \bmod n)$.
   - $\exists\, a, b \in \mathbb{Z}, \exists\, n \in \mathbb{N}, (6a \equiv 6b \bmod 6n \implies a \not\equiv b \bmod n)$.
   - False $\longrightarrow$ $6a \equiv 6b \bmod 6n$ is equivalent to $6a = 6nk + 6b$, where $k \in \mathbb{Z}$, divide both sides by 6, $a = nk + b$ which is equivalent to $a = b \bmod n$. But, with $\forall\, a, b$, when a is divided by nk, there is a time when b is just zero, not all values.
4. (2 *Points*) $\forall\, a, b \in \mathbb{Z}, (4a \equiv 4b \bmod 24 \implies a \equiv b \bmod 24)$.
   - $\exists\, a, b \in \mathbb{Z}, (4a \equiv 4b \bmod 24 \implies a \not\equiv b \bmod 24)$.
   - False $\longrightarrow$ As shown before this, when $4a$ is divided to have a remainder of zero, there should be no other value of b than zero.
5. (2 *Points*) $\exists\, x \in \mathbb{Z}$ such that $((x > 84) and (x \equiv 75 \bmod 84))$.
   - $\forall\, x \in \mathbb{Z}$ such that $((x \leq 84$ or $x \not\equiv 75 \bmod 84))$.
   - True $\longrightarrow$ When x is above 84, specifically when x is 159, the statement holds.
6. (2 *Points*) $\exists\, x, y \in \mathbb{R}$ such that $(x^2 \geq y^2$ and $x \leq y)$.
   - $\forall\, x, y \in \mathbb{R}$ such that $(x^2 < y^2$ or $x > y)$.
   - True $\longrightarrow$ Take [-1, 1] and the original statement works

Problem 5. (4 *Points*) Let $(u_0, u_1, u_2, u_3, ...)$ be a sequence of real numbers. We write this as $(u_n)_{n\in\mathbb{N}}$. We say that it is:

- bounded above when: $\exists\, A \in \mathbb{R}\, s.t.(\forall\, n \in \mathbb{N}, u_n \leq A)$.
- bounded below wheb: $\exists\, B \in \mathbb{R}\, s.t.(\forall\, n \in \mathbb{N}, u_n \geq B)$.

We say that it converges towards $+\infty$ when

$$\forall\, A > 0, \exists\, m \in \mathbb{N}\, s.t. \forall\, n \in \mathbb{N}, (n \geq m \Longrightarrow u_n > A).$$

and that it converges towards $-\infty$ when

$$\forall\, B < 0, \exists\, m \in \mathbb{N}\, s.t. \forall\, n \in \mathbb{N}, (n \geq m \Longrightarrow u_n < B).$$

1. (1 *Points*) Write in quantifiers the statement:

   $(u_n)_{n\in\mathbb{N}}$ is not bounded below.

2. (1 *Points*) Give an example of sequence of real numbers $(u_n)_{n\in\mathbb{N}}$ which is bounded above but not bounded below.

3. (2 *Points*) Write in quantifiers the statement:

   $(u_n)_{n\in\mathbb{N}}$ does not converge towards $+\infty$.