1 Factor Strength

The concept of factor strength in this project was first introduced by Bailey, Kapetanios, and Pesaran (2016). This initial paper limited the estimation of factor strength in a small scopes, only factors who is strong can be estimated. Bailey, Kapetanios, and Pesaran (2020) extended the method to all observed factors. Pesaran and Smith (2019) provides an empirical application of the factor on estimating asset's risk premia. In general, the factor strength represents the pervasiveness of factor, which is captured by the factor's loading.

1.1 Definition

Consider the following multi-factor model for n different cross-section units and T observations with k factors.

$$x_{it} = a_t + \sum_{j=1}^{k} \beta_{ij} f_{jt} + \varepsilon_{it} \tag{1}$$

In the left-hand side, we have x_{it} denotes the cross-section unit at time t, where $i = 1,2,3,\cdots,n$ and $t = 1,2,3,\cdots T$. In the other hand, a_i is the constant term, which does not variate through the time. f_{jt} of $j = 1,2,3\cdots k$ is factors included into the model, and β_{ij} is the corresponding factor loading. ε_{it} is the stochastic error term.

The factor strength is dependent on how many non-zero loadings a factor can generate. For factor f_{jt} with n different factor loading β_{ij}, we assume that:

$$|\beta_j| > 0 \quad i = 1,2,\ldots,[n^{\alpha_j}]$$

$$|\beta_j| = 0 \quad i = [n^{\alpha_j}]+1,[n^{\alpha_j}]+2,\ldots,n$$

The α_j represents strength of facto f_{jt}. If factor has strength α_j, we will assume that the factor's loadings of first $[n^{\alpha_j}]$ terms are all different from zero, and here $[\cdot]$ is defined as integral operator, it will only take the integral part of inside value.Then for the rest $n - [n^{\alpha_j}]$ term are all equal to zero. Assume for a factor which has strength $\alpha = 1$, the factor's loadings will be non-zero for all cross-section units. We will refer such factor as strong factor. And if we have factor strength $\alpha = 0$,

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it means that the factor cannot generate any loading different from zero, and we will describe such factor as useless. In general term, the more non-zero loading a factor can generate, the stronger the factor's strength is.

1.2 Estimation

To estimate the strength α_j, Bailey et al. (2020) provides a estimation.

Here we consider a simplified model (1), a factor model with only one factor named f with different value f_t at different time. β_{it} is the factor loading of unit i at time t. v_{it} is the stochastic error term.

$$x_{it} = a_i + \beta_{it} f_t + v_{it} \tag{2}$$

Assume we have n different assets and T observations for each assets: $i = 1, 2, 3, \cdots, n$ and $t = 1, 2, 3, \cdots T$. Running the OLS regression for each $i = 1, 2, 3 \cdots, n$, we obtain:

$$x_{it} = \hat{a}_{iT} + \hat{\beta}_{iT} f_t + \hat{v}_{it}$$

For every factor loading $\hat{\beta}_{iT}$, we can exam their significance by constructing a t-test. The t-test statistic will be $t_{iT} = \frac{\hat{\beta}_{iT} - 0}{\hat{\sigma}_{iT}}$. Then the test statistic for the corresponding $\hat{\beta}_i$ will be:

$$t_{iT} = \frac{(\mathbf{f}'\mathbf{M}_\tau \mathbf{f})^{1/2} \hat{\beta}_{iT}}{\hat{\sigma}_{iT}} = \frac{(\mathbf{f}'\mathbf{M}_\tau \mathbf{f})^{-1/2} (\mathbf{f}'\mathbf{M}_\tau \mathbf{x}_i)}{\hat{\sigma}_{iT}}$$

Here, the $\mathbf{M}_\tau = \mathbf{I}_T - T^{-1}\tau\tau'$, and the τ is a $T \times 1$ vector with every elements equals to 1. $\mathbf{f}$ and $\mathbf{x_i}$ are two vectors with: $\mathbf{f} = (f_1, f_2 \cdots, f_T)'$ $\mathbf{x_i} = (x_{i1}, x_{i2}, \cdots, x_{iT})$. The denominator $\hat{\sigma}_{iT} = \frac{\sum_{i=1}^{T} \hat{v}_{it}^2}{T}$.

Using this test statistic, we then defined an indicator function as: $\ell_{i,nT} := \mathbf{1}[|t_{it}| > c(n)]$. If the t-statistic t_{iT} is greater than certain critical value $c_p(n)$, $\hat{\ell}_{i,nT} = 1$. In other word, we will count one if the factor loading $\hat{\beta}_{ij}$ is significant. With the indicator function, we then defined π_{nT} as the proportion of significant factor loading amount to the total factor loadings amount:

$$\hat{\pi}_{nT} = \frac{\sum_{i=1}^{n} \hat{\ell}_{i,nT}}{n} \tag{3}$$

For the critical value $c_p(n)$, rather than use the traditional critical value from student-t distribu-

tion $\Phi^{-1}(1-\frac{P}{2})$, we use:

$$c_p(n) = \Phi^{-1}(1 - \frac{p}{2n^\delta}) \tag{4}$$

Here, $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of a standard normal distribution, P is the size of the test, and δ is a non-negative value represent the critical value exponent. The traditional method to calculate critical value $\Phi^{-1}(1-\frac{p}{2})$ does not take multiple-test problem into the consideration. Here, by adding the n^δ term into the denominator, the new critical value has adjusted the multiple-test problem. Therefore, we will use this critical value $c_p(n)$ to compare with the test statistic to justify does the factor loading is significantly different from zero.

After obtain the $\hat{\pi}_{nT}$, we can use the following formula provided by Bailey et al. (2020) to estimate our strength indicator α_j:

$$\hat{\alpha} = \begin{cases} 1 + \frac{\ln(\hat{\pi}_{nT})}{\ln n} & \text{if } \hat{\pi}_{nT} > 0, \\ 0, & \text{if } \hat{\pi}_{nT} = 0. \end{cases}$$

Whenever we have $\hat{\pi}_{nT}$, the estimated $\hat{\alpha}$ will be equal to zero. From the estimation, we can find out that $\hat{\alpha} \in [0, 1]$

2 Monte Carlo Design

In order to study the limited sample property of factor strength α, we designed several Monte Carlo simulations, to observe the behaviours of the factors strength under different settings. We consider the following data generating process (DGP): a CAPM model with market factor and risk factor

$$x_{it} = q_1(r_{mt} - r_f) + q_2(\sum_{j=1}^{k} \beta_{ij} f_{jt}) + \varepsilon_{it}$$

In the simulation, we consider a dataset has $i = 1, 2, \ldots, n$ different assets, with $t = 1, 2, \ldots, T$ different observations. k different risk factors represents by the subscript j and one market factors are also included.

$q_1(\cdot)$ and $q_2(\cdot)$ are two different functions represent the unknown mechanism of market factor

and other risk factors in pricing asset risk. $(r_{mt} - r_f)$ is the excess market return, calculated by average market return r_{mt} minus risk free return r_{ft}. r_{it} is the stock return, f_{jt} denotes factors other than market factors and β_{ij} is the corresponding factor loading. ε_{it} is random error with structure can be defined in different designs. For each factor, we assume they follow a multinomial distribution with mean zero and a $k \times k$ variance-covariance matrix Σ.

$$\mathbf{f_t} = \begin{pmatrix} f_{i,t} \\ f_{2,t} \\ \vdots \\ f_{k,t} \end{pmatrix} \sim MVN(\mathbf{0}, \Sigma) \quad \Sigma := \begin{pmatrix} \sigma_{f_1}^2, & \rho_{12}\sigma_{f1}\sigma_{f2} & \cdots & \rho_{1k}\sigma_{f1}\sigma_{fk} \\ \rho_{12}\sigma_{f2}\sigma_{f1}, & \sigma_{f2}^2 & \cdots & \rho_{2k}\sigma_{f2}\sigma_{fk} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1k}\sigma_{fk}\sigma_{f1}, & \rho_{k2}\sigma_{fk}\sigma_{f2} & \cdots & \sigma_{fk}^2 \end{pmatrix}$$

The diagonal of matrix Σ indicates the variance of each factor, and the rest represent the correlation among all k factors. In this model, we can control several parts to investigates different scenarios of the simulation:

2.1 Baseline Design

Follow the general model above, we assume both $q_1(a)$ and $q_2(a)$ are linear function:

$$q_1(r_{mt} - r_f) = a_{it} + \beta_{im}(r_{mt} - r_{ft})$$

$$q_2(\sum_{j=1}^{k} \beta_{ij}f_{jt}) = \sum_{j=1}^{k} \beta_{ij}f_{jt}$$

Therefore, if we include the market factor with other risk factors, the model with single factor can be write as:

$$x_{it} = a_{it} + \sum_{j=1}^{k+1} \beta_{ij}f_{ij} + \varepsilon_{it} \tag{5}$$

To generate factor loadings and asset's return, we follow the next procedures: First, we generate the constant term a_{it} which has a uniform distribution from -0.5 to 0.5, $a_{it} \sim U[-0.5, 0.5]$. Then, in this baseline design, we assume the error term $\varepsilon_{it} \sim N(0,1)$, this means that the error term has mean zero and variance equals to one . Next, we will set up the true factor strength α. Because here we allocate the market factor which as factor strength equals to one at the top of every factor, the factor

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68 strength vector will have k+1 elements:

$$\alpha = \begin{pmatrix} 1, & \alpha_2, & \alpha_3, & \cdots, & \alpha_{k+1} \end{pmatrix}$$

69 The other strengths will be adjusted to various values base on the simulation's design. After hav-

70 ing the factor strength, we can calculate for each factors, how many loadings will be significantly

71 different from zero. Since we assume for any random factors j with strength α_j, it will have $[n_j^\alpha]$,

72 so we

73   For each factors, we assume they follow the multinomial standard distribution with mean 0 and

74 variance Σ, but in this baseline design, each factors are independent with each others, and each

75 factors has same variance as one, so the Σ will be:

$$\Sigma = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

76 After that, we will generate the factor loadings from a uniform distribution. In order to make sure

77 every factor loading is sufficiently larger than 0, we set the expected value of those loadings $\mu_\beta =$

78 0.71, $\beta_{ij} \sim IIDU(\mu_\beta - 0.2, \mu_\beta + 0.2)$. Then we randomly assign $n - [n^\alpha]$ factor loadings as zero,

79 to reflect the fact that only $[n^\alpha]$ factor loadings are significantly different from zero. After generate

80 constant term, factor, factor loading, and the error term, we can calculate the simulated asset's return

81 by using the equation (5). With the return and factors, we can re-calculate the factors loading and

82 use the estimation method discussed in section 1.2.

83   For this baseline design, we consider the different combinations of T and n with $T = \{120, 240, 360\}$,

84 $n = \{100, 300, 500\}$. The market factor will have strength $\alpha_m = 1$ all the time, and the strength of

85 the other factor will be $\alpha_x = \{0.5, 0.7, 0.9, 1\}$. For every setting, we will replicate 500 times inde-

86 pendently, all the constant a_{it} and loading β_i will be re-generated for each replication. To exam the

87 goodness of estimation, we calculate the bias between our true underneath factor strength α and the

88 estimated strength $\hat{\alpha}$ as $bias = |\alpha - \hat{\alpha}|$. We also use the bias to calculate the Mean Square Error

(MSE). To calculate the MSE, we will collect the bias for each replication, and then use the formula:

$$MSE = \frac{1}{n} \sum_{i=1}^{500} (bias_i)^2$$

References

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