Q13.

first, we have a weighted directed graph, and also a destination node t.

Initialization: $d[t]=0$, $d[v]=\infty$, for $v \in V \setminus \{t\}$,

$\pi[v]=\phi$ for $v \in V$.

$S=\phi$

while $S \neq V$,

we choose $u \in V \setminus S$ with minimal value $d[u]$, add it to S,

for each vertex v with $(v,u) \in E$,

if $d[v] > w(v,u) + d[u]$,

set $d[v] = w(v,u) + d[u]$, $\pi[v]=u$.

return $\{d[v], \pi[v]\}$

At the first beginning of each while loop, we have $d(v) = d^*(v)$ for all $v \in S$.

It will show that for all $u \in V$, we have $d(u) = d^*(u)$, when u is added to S.

For upper-bound property, it will never change ~~afer~~ afterwards.

Initialization: $S=\phi$, so the invariant is true.

Maintenance:

For the purpose of contradiction, let u be the first node added to the set S, such that $d(u) \neq d^*(u)$.

we must have $u \neq t$, since t is the first node added to S, and $d(t) = d^*(t) = 0$. we have that $S \neq \phi$ before u is added. There must be a path from u to t, otherwise $d(u) = d^*(u) = \infty$, so, there's a shortest path p from u to t.

prior to adding u to S, p connects a node in $V-S$ to a node in S. Let x denote the last node in p, such that $x \in V-S$ and let y denote x's successor, $y \in S$. we can decompass p into $u \to x \to y \to t$.

We claim that $d(x) = d^*(x)$, when u is added to S. we also see $d(y) = d^*(y)$, since $y \in S$ and u is the first node for which property does not hold.

since $x \to y \to t$ is the shortest path from x to t, when y was relaxed, we had $d(x) = w(x, y) + d^*(y) = d^*(x)$.

We now get the contradiction, since x appears after

u on the shortest path P, and since all weights are non-negative, we must have $d^*(x) \leq d^*(u)$.

so, $d(x) = d^*(x)$

$$\leq d^*(u)$$

$$\leq d(u)$$

Because x and u were in $V-S$, we have $d(u) \leq d(x)$, so, $d(x) = d^*(x) = d^*(u) = d(u)$, which contradicts our definition of u.

Termination:

For a graph with non-negative weights, Dijkstra's algorithm terminates with $d(v) = d^*(v)$ for all $v \in V$.