

A PDE PERSPECTIVE ON CLIMATE MODELING

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ABSTRACT. This report will provide an overview of climate modeling from a mathematical perspective, particularly with respect to the use of partial differential equations. A visit to the Swedish Meteorological and Hydrological Institute's Rossby Center for climate research in Norrköping, Sweden, is at the foundation of our investigations. An introduction and a brief history section will be followed by a description of the Navier-Stokes equations, which are at the heart of climate-related mathematics, as well as a survey of many of the popular approximations and modeling techniques in use by climate researchers today. Subsequently, a boundary value problem based on the one dimensional compressible Euler equations will be discussed from an analytical as well as a numerical point of view, especially with concern to the well-posedness of the same.

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1. INTRODUCTION

Climate change is one of the greatest global challenges facing humankind in the 21st century. Doubtlessly the study of climate change and its consequences is of immense importance for our future. Global climate models are the best means we have of anticipating likely changes.

In general terms, a climate model can be defined as a mathematical representation of the atmosphere, oceans, and geology of Earth, based on physical, biological, and chemical principles. The equations derived from these principles are solved numerically over grids that span the entire globe or portions of it, using discrete steps in space and time. The time step could be between several minutes and several years, depending on the process being studied, on available computer capacity, and on the choice of numerical method.

The central difficulties that face climate researchers are *stability* and *scalability*; that is, to find stable solutions with respect to small changes in the initial conditions and to increase the resolution of the current models. Even for the models of today with the highest resolution (so called meso-scale models), the numerical grid is too coarse to represent small scale processes such as turbulence in the atmospheric and oceanic boundary layers, interaction with small scale topography features, thunderstorms, cloud microphysics processes, etc. Scientists rely on finely tuned approximations that must walk a thin line between physical accuracy and computational feasibility.

The equations used to build climate models are, of course, partial differential equations (PDEs). The equations are non-linear and are coupled together into systems and their solutions are truly nontrivial. In fact, the existence, uniqueness, and stability of the most general case has yet to be proven. If one can side-step these difficulties by means of approximation and simplification, one may still find that solutions behave chaotically, i.e. small variations in initial conditions give rise to divergent predictions.

The aim of this report is to give an overview of how PDEs are used in the field of climate prediction and to deeper investigate a specific boundary value problem, based on the Euler equations, used to model atmospheric flow. We first give a background and history to the subject, then the Navier-Stokes equations will be presented and put into an appropriate functional setting. An overview of the approximation-methods used by meteorologists to create weather and climate models will be provided. The criticism is then raised, shared at least by some researchers at the Rossby Center, that these approximation-methods are no longer sufficient to accommodate the rising demand for accurate and detailed climate models. We will then investigate the well-posedness of the least approximative model, the Euler equations. Bounded energy will be shown for a general initial boundary value problem; this in the context of being a simplified example of the type of analysis that will be used to create the next generation of climate models. We will show the stability of a numerical scheme used to solve the equations, based on the method of lines, using a summation-by-parts operator for the spacial derivatives, and using a simultaneous-approximation-term to set boundary conditions in a weak manner. Finally the method will be implemented using Matlab, with data retrieved from radiosondes released over Sweden during 2004.

1.1. History

The first attempts at numerically predicting the weather were seen in the 1920s. Prior to that, around the turn of the century, the so called *primitive equations* had been developed by the Norwegian Vilhelm Bjerknes. These are a set of nonlinear differential equations based on the principles of conservation of momentum and mass continuity, combined with a thermal energy equation, and they appear in many models of atmospheric flow of today. Bjerknes perceived their application to weather prediction as early as 1904.

Bjerknes' background was in fluid dynamics and electromagnetism - he is said to have contributed substantially to Hertz work on electromagnetic resonance during his year as an assistant to the latter in Bonn. In 1917 he founded the Geophysical Institute of the University of Bergen where he published his book *On the Dynamics of the Circular Vortex with Applications to the Atmosphere and to Atmospheric Vortex and Wave Motion* in 1921. Carl-Gustaf Rossby, the well renowned Swedish meteorologist and namesake of the Rossby Center, was a student under Bjerknes in Bergen between 1919-1921.

Influenced by the work of Bjerknes, the English scientist Lewis Fry Richardson began, around this time, to work numerically on the meteorological equations found thitherto. His method of finite differences had the fundamental idea of using discrete time-steps in evaluating the equations. These particular time-steps were later subject to research for a trio of mathematicians, Courant, Friedrichs and Lewy, who worked on a stability algorithm for numerical computations of Richardsons kind. Their 1928 paper presented what was to be named the *CFL-criterion* for the length of discrete time-steps for the solutions of a certain types of PDEs to be stable.

Fry Richardson's work resulted in a single trial forecast which unfortunately produced unrealistic values, presumably due to errors in the input data on wind. So the initial response to his work wasn't entirely positive, but in retrospect it is clear that his prescient ideas laid the groundwork for modern forecasting [13].

In the decades after Fry Richardson's efforts, some key developments took place that facilitated the advancement of research in the field. Developments in numerical analysis permitted the implementation of more stable algorithms; the theory of meteorology provided deeper understanding of atmospheric dynamics; the radiosonde was invented, which made more data available in three dimensions, and lastly of course, the improvement of the digital computer. Moreover, World War II created a demand for vast amounts of weather data, which became available for studies afterwards. Furthermore, the number of meteorologists increased by 20-fold in the U.S during the war years [2], and the fifties became the decade when operational numerical weather prediction really began.

The well-known mathematician John von Neumann was involved in the design and construction of an electronic computer at Princeton University between 1946-1952. It was called the Institute for Advanced Studies (IAS) machine and their work had a profound impact on the computer industry. The project consisted of four groups, of which one was dedicated entirely to meteorology. Von Neumann, being familiar with the work of Courant, Friedrichs and Lewy, as well as the publications of Fry Richardson, realized that an automatic computer would be ideal for weather computations. And so it was he who helped initiate this groundbreaking meteorological research group, funded by the U.S Navy. It was called The Joint

Numerical Weather Prediction Unit (JN-WPU) and led by the meteorologist Jule Charney.

This group tried to derive a forecast for a 24-hr total energy change by numerically solving a vorticity equation [1]. This forecast was then compared to observed values and found to be quite similar for the first 23 days, which given the simplicity of the model was considered a great achievement. However, the model was not intended to constitute an accurate forecasting tool, but rather a proof of concept, showing that given a sufficiently skilled parameterization and enough processing power, it should be possible to create accurate mathematical models of the atmosphere.

Among the first electronic general-purpose computers, was the ENIAC (Electronic Numerical Integrator and Computer), situated in Aberdeen, Maryland. This computer, too, was employed by the Princeton meteorology team, and with it they produced one of the first numerical forecasts. The computations at this point were just able to keep up with the weather itself, taking around 24 hours to produce a 24 hour prognosis. The model used the quasi-geostrophic approximation (defined in section 3) and was based on baroclinic equations, a modeling system for which Jule Charney has been credited. The success of the Princeton team encouraged researchers around the world to set up their own centers for numerical weather prediction.

Around the same time in Europe, the Swedish International Meteorological Institute (SMHI), led by Carl-Gustav Rossby, was at the forefront for similar research. This was where the very first real-time barotropic forecast was produced in 1954, on the Binary Electronic Sequence Calculator (BESK). Carl-Gustav Rossby had been present during the startup meetings for the JN-WPU initiated by Von Neumann [2], and he brought these ideas to Stockholm.

In the end of the 1950's in Germany, Karl Heinz Hinkelmann attempted and succeeded to produce a forecast based on the primitive equations, as Richardson had previously attempted. These are a more complete set of equations than those that the common barotropic models were based on, and were thus desirable to use operationally. The primitive equations became more and more common during the 1960's. Additionally, effects of solar radiation, moisture and latent heat could be incorporated to this system of equations, which gave a more complete picture of the weather and climate situation. In 1971 ice melting was taken into consideration for the first time, soon followed by surface temperature data (in 1972) [3].

Because of the complex nature of the equations behind weather forecasts and climate models, statistical methods have been employed since the 1980's. Model Output Statistics (MOS) is the general term for these kinds of methods. They are a means of correcting the deviations of different models' results, as well as the results from the same models differing as a consequence of sensitivity to initial conditions. The present day modeling is a combination of statistical methods, such as MOS and so called ensemble forecasting, the primitive equations and parametrization.

2. THE NAVIER-STOKES EQUATIONS

At the heart of modeling dynamical flow in the atmosphere are the Navier-Stokes equations, which represent the flow of "fluids" such as water, oil, air, and gases. Mathematically, a fluid is represented by three functions: $\rho(\mathbf{x}, t)$, $p(\mathbf{x}, t)$, and $\mathbf{u}(\mathbf{x}, t)$. Density, ρ , and pressure, p , are scalar functions of the position vector

$\mathbf{x} = (x_1, x_2, x_3) \in \Omega$ where Ω is a region of three dimensional space, filled by the fluid. The velocity of a particle of a fluid at point \mathbf{x} at time t is given by the vector-valued function $\mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$.

Given that the fluid in question is Newtonian, the three functions ρ, \mathbf{u}, p are governed by the momentum conservation equation:

$$(2.1) \quad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^3 u_i \frac{\partial \mathbf{u}}{\partial x_i} \right) - \mu \Delta \mathbf{u} - (3\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \nabla p = \mathbf{f},$$

which is a formulation of Newtons second law of motion applied to fluid mechanics. Here μ and λ are physical parameters and $\mathbf{f}(\mathbf{x}, t)$ is a density of force per unit volume. The functions ρ, \mathbf{u}, p also obey the continuity equation:

$$(2.2) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}),$$

which says that the mass of the fluid is conserved. If the fluid is homogeneous and incompressible as well, then ρ is independent of position and time and the equations (2.1) and (2.2) reduce to

$$(2.3) \quad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^3 u_i \frac{\partial \mathbf{u}}{\partial x_i} \right) - \mu \Delta \mathbf{u} + \nabla p = \mathbf{f},$$

$$(2.4) \quad \nabla \cdot \mathbf{u} = 0.$$

Note however that in atmospheric modeling one would assume the fluid (the air) to be compressible. Below we will be concerned with the incompressible case, for simplicity. The convention is to let $\rho = 1$, set $\mu = \nu$, and write (2.3) as

$$(2.5) \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}.$$

The equation (2.5) together with (2.4) are the nondimensional incompressible Navier-Stokes equations. They were first propounded by Claude-Louis Navier in 1822 and, after additional efforts by Poisson (1831) and notably Stokes (1845), were provided a satisfactory justification based upon continuum mechanics concepts [8].

As mentioned above, the Navier-Stokes equations (2.5) and (2.4) describe the motion of usual fluids and, as a consequence, appear in studies of many important physical phenomena. To name a few, they are used in aeronautical sciences, thermo-hydraulics, in plasma physics, in the petroleum industry, and, of course meteorology.

The assumption which leads to these equations is in fact quite simple: that there be a linear local relation between strain rates and stresses. However, the equations are nonlinear because of the convective term $(\mathbf{u} \cdot \nabla) \mathbf{u}$, which is a result of mathematical reasoning and cannot be bypassed through changing the physical model. Regardless of the straightforward physical assumptions behind (2.5) and (2.4) they present a formidable mathematical challenge. Clay Institute [4] has the Navier-Stokes equations on their list of 7 Millenium problems, and awards the person who can show wellposedness for the three dimensional case with a million U.S. Dollars. The analytical problems associated with these equations can generally be categorized into two areas:

- Existence, uniqueness, and regularity.
- Long time behaviour.

The present section will give a brief sketch of these problem areas as outlined in [7], providing a background for a more detailed study of the Euler equations in Section 4.2. First we will present the main boundary value problem associated with the Navier-Stokes equations in its appropriate functional setting.

2.1. A Boundary Value Problem for the Navier-Stokes Equations

Once again, let Ω denote a region of space filled by a fluid described by the Navier-Stokes equations (2.5) and (2.4). Let Ω be an open set of \mathbb{R}^n , $n = \{2, 3\}$, with boundary Γ . Further assume that Ω is located locally on one side of Γ and that Γ is Lipschitz or sometimes of class C^k for a specified k .

Now in order to be able to use these equations, we need to determine a well-posed boundary value problem to go along with them. It is believed [7] that (2.5) and (2.4) must be augmented by the following initial and boundary conditions:

- *Initial condition:*

$$(2.6) \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad \mathbf{u}_0 \text{ given.}$$

- *Boundary condition:*

$$(2.7) \quad \mathbf{u}(\mathbf{x}, t) = \Phi(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma, t > 0, \quad \Omega \text{ bounded, } \Phi \text{ given.}$$

Therefore, for a bounded domain Ω (which is common for applications), the boundary value problem for the incompressible case can be summarized thus:

$$(2.8) \quad \begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f}, & \mathbf{x} \in \Omega, t > 0, \mathbf{f} \text{ given,} \\ \nabla \cdot \mathbf{u} &= 0, & \mathbf{x} \in \Omega, t > 0, \\ \mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}_0(\mathbf{x}), & \mathbf{x} \in \Omega, \mathbf{u}_0 \text{ given,} \\ \mathbf{u}(\mathbf{x}, t) &= \Phi(\mathbf{x}, t), & \mathbf{x} \in \Gamma, t > 0, \Phi \text{ given,} \end{aligned}$$

where for given functions \mathbf{f} , \mathbf{u}_0 , Φ , and constant ν , we would like to solve for the unknown functions \mathbf{u} and p . If Ω is unbounded (and in particular for $\Omega = \mathbb{R}^n$), we add to (2.7) a condition at infinity:

$$\mathbf{u}(\mathbf{x}, t) \rightarrow \psi(\mathbf{x}, t) \quad \text{as } |\mathbf{x}| \rightarrow +\infty, \quad (\psi \text{ given}).$$

The well-posedness of (2.8) is still an open problem, and a technical description of this problem is not within the scope of this paper. We will, however, spend some time touching on the important ideas of smoothness and bounded energy.

2.1.1. Smoothness

In order to be physically viable, solutions to (2.8) must possess the quality of smoothness, which can best be quantified by means of function spaces. One can view a PDE as an operator A acting on appropriate (linear) spaces. This can be symbolically written as

$$A : X \rightarrow Y,$$

where A includes the boundary conditions, if such are present. An advantage of this formulation is that once the PDE is stated in this form, we can often be helped

by the principles of functional analysis to investigate the solvability of the equation. It turns out that so called Sobolev spaces often are suitable for \mathbf{X} , \mathbf{Y} . [18]

Definition 2.1. (Lebesgue Space) Let $L^p(I, w)$ denote the set of all functions f such that for the Lebesgue measure $d\mathbf{x} = dx_1, dx_2, \dots, dx_n$ and domain I :

$$\int_I |f(\mathbf{x})|^p w(d\mathbf{x}) d\mathbf{x} < \infty.$$

For $f \in L^p(I, w)$ define the norm:

$$\|f\|_{p,w} = \left(\int_I |f(\mathbf{x})|^p w(\mathbf{x}) d\mathbf{x} \right)^{1/p}.$$

Definition 2.2. (Sobolev Space) Let $H^m(I)$ denote the space of functions which are $L^2(I)$ (the Lebesgue space of order $p = 2$ and with weight function $w = 1$) along with all their derivatives of order $\leq m$. For $f, g \in H^m(I)$ define the norm and scalar product:

$$(f, g)_m = \sum_{[\alpha] \leq m} \int_I |D^\alpha f \cdot D^\alpha g| d\mathbf{x}, \quad (\text{scalar product}),$$

$$\|f\|_m = \left((f, f)_m \right)^{1/2}, \quad (\text{norm}),$$

where $\alpha_i \in \{1, 2, \dots, n\}$, $[\alpha] = \sum_i^n \alpha_i$, and

$$(2.9) \quad D^\alpha = \prod_i^n D_n^{\alpha_i} = \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \cdot \dots \cdot \partial x_n^{\alpha_n}}.$$

Recall now the setting of the boundary value problem (2.8) and let $H_0^1(\Omega)$ denote the subspace of $H^1(\Omega)$ consisting of functions which vanish on Γ . The norms in Definitions 2.1 and 2.2 are metrics by which we can quantify the smoothness and regularity of functions in Ω . Ultimately, one would like to find solutions to (2.8) which are almost everywhere equal to a continuous function from $[0, T]$ (a bounded time interval) into a functional space V defined as the set of all functions belonging to $H_0^1(\Omega)$ which have null divergence:

$$V = \{\mathbf{v} \in H_0^1(\Omega), \nabla \cdot \mathbf{v} = 0\}.$$

As of yet, however, it is not known exactly how to accomplish this.

2.1.2. Bounded energy

In order to be sure that we obtain physically reasonable solutions for (2.8), we want to make sure that $\mathbf{u}(\mathbf{x}, t)$ does not grow too large as $|\mathbf{x}| \rightarrow \infty$. It is generally thought [4] that this can be obtained through the following restrictions on the boundary data

$$(2.10) \quad |\partial_{\mathbf{x}}^\alpha \mathbf{u}_0(\mathbf{x})| \leq C_{\alpha k} (1 + |\mathbf{x}|)^{-k} \quad \text{on } \mathbb{R}^n, \quad \text{for any } \alpha, k,$$

and on the forcing functions

$$(2.11) \quad |\partial_{\mathbf{x}}^\alpha \partial_t^m \mathbf{f}(\mathbf{x}, t)| \leq C_{\alpha m k} (1 + |\mathbf{x}| + t)^{-k} \quad \text{on } \mathbb{R}^n \times [0, \infty), \quad \text{for any } \alpha, m, k,$$

where $n = 2$ or 3 . The inequalities (2.10) and (2.11) state simply that there exists an upper bound on the derivatives of both the external force term \mathbf{f} and the initial condition \mathbf{u}_0 to an arbitrarily high order. These are both given functions for the

solving of (2.8) and are thus representations of the data used to solve the boundary value problem. Thus (2.10) and (2.11) are criteria that these data must fulfill in order not to jeopardize the well-posedness of (2.8).

Furthermore, solutions to (2.8) must satisfy

$$(2.12) \quad \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t)|^2 dx < C, \quad \text{for all } t \geq 0,$$

in order to be considered physically accurate. That is, solutions must exhibit bounded energy in the sense of the L^2 -norm.

2.2. Existence, Uniqueness, and Regularity

It has been known since the 1930's [7] that the initial value problem for the time-dependent Navier-Stokes equations has a unique smooth solution on a bounded time interval, given that the initial conditions are sufficiently smooth. However, extending this solution for longer time would possibly present a less regular function, and this is a crucial question which is as yet unanswered: will the solution remain smooth for all time? If an affirmative answer could be proved, the question of uniqueness and existence would be considered resolved. If not, it would be important to have information on the nature of the singularities, and on whether the weak solutions would be unique [7]. (A weak solution, or a *generalized solution* to an ODE or a PDE is a solution whose derivatives may not all satisfy the problem but which solves it in some precise sense made clear for a given context.)

2.3. Long Time Behavior

And what happens when $t \rightarrow \infty$, for the time-dependent Navier-Stokes equations? As explained in [7], if the given forces and boundary values of the velocity are small then there exists a unique stable stationary solution and the time-dependent solution converges to it as $t \rightarrow \infty$. But when the forces grow larger, we understand from physical evidence and knowledge of offsets and bifurcations due to chaos, that as $t \rightarrow \infty$, the solution often tends to a time periodic one or to a more complicated *attracting set*, in which case the long time behavior of the solution (representing a "permanent" state) could appear chaotic. This is known to occur even in simple dynamical systems of finite dimension, as is discussed in the following section.

2.4. Instability of Solutions

Nonlinear deterministic equations, such as the Navier-Stokes equations, have been found to exhibit irregular and unexpected behaviour. In 1963, the mathematician E. N. Lorenz produced a simple, nonlinear mathematical model of a weather system that would generate nonperiodic weather patterns, which was a cutting-edge idea at the time. He worked on an early personal computer, the Royal-McBee LGP-30, to perform his computations. Despite its computing power being comparable to a modern pocket calculator, it was capable of solving a set of weather equations involving around a dozen variables, time step by time step, printing out the results every ten seconds. When tweaking the inserted values, he came upon the nonperiodic patterns he sought.

Apparently, a dynamical system with no explicit randomness or uncertainty to begin with, would after a few time steps produce unpredictable motion with only the slightest changes in the initial values. Seen how even the Lorenz equations (as they have become known over time) present chaotic traits, one can just imagine to

what (short, presumably) extent the Navier-Stokes equations on a grid with a million points would be predictable. As previously mentioned, this is the reason why atmospheric models of today use a number of simplifying assumptions, linearizations and statistical methods in order to obtain more well-behaved systems. The remainder of this section will provide a sketch of ideas concerning the unpredictable behavior of solutions to deterministic equations reacting to slightly varying initial conditions.

2.4.1. Phase space

If the states of a system are described by the values of a finite set of time-dependent variables, governed by a set of equations involving functions with continuous first partial derivatives, then such a system can be analyzed by means of its phase space, where each state of the system is represented by a point. Thus a system which is evolving in time is represented by a particle moving in the phase space with a unique trajectory. Analysis of the trajectory, as time goes to infinity, can then be used to observe the evolution of the system. It can be shown using these methods that nonperiodic trajectories are unstable from the point of view of being able to predict future states [5].

Let the state of a system be described by M variables:

$$\mathbf{X} = (X_1, X_2, \dots, X_m),$$

and the system be governed by the set of equations:

$$(2.13) \quad \frac{dX_i}{dt} = F_i(\mathbf{X}),$$

where time t is the single independent variable and the functions F_i possess continuous first partial derivatives. Now consider the *phase-space* of this system, given by an M -dimensional Euclidian space \mathbb{E}^m , with the coordinates X_1, X_2, \dots, X_M . It can be shown that since the partial derivatives $\partial F_i / \partial X_i$ are continuous, if t_0 is any time, and if $X_{1,0}, X_{2,0}, \dots, X_{M,0}$ is any point in \mathbb{E}^m , the equations (2.13) possess a *unique* solution of the form:

$$f_i(X_{1,0}, X_{2,0}, \dots, X_{M,0}, t),$$

valid in some time interval containing t_0 , and satisfying the condition

$$f_i(X_{1,0}, X_{2,0}, \dots, X_{M,0}, t_0) = X_{i,0}.$$

Continuity of the functions f_i imply that there is a unique *trajectory* through each point in \mathbb{E}^m , which describes the state of the system at time t . However, multiple trajectories may approach the same point or curve asymptotically as $t \rightarrow \pm\infty$ and any region of \mathbb{E}^m may continuously deform into another region.

It is beneficial to note that the functions F_i which are candidates for the right hand side of (2.13) must accurately portray the system to be studied. Specifically, it must be considered whether or not the system under scrutiny is conservative, in the sense of energy or some other quantity being constant over time. It is namely shown that trajectories representing *forced dissipative flow* may differ greatly from

those representing *conservative flow*. This is one of the keys to understanding the instability of solutions to equations describing dynamic systems.

The analysis of trajectories within \mathbb{E}^m begins with their classification according to three behavioral characteristics:

- absence or presence of transient properties,
- stability or instability with respect to small modifications,
- absence or presence of periodic behavior.

Now if $P(t)$ denotes a trajectory in \mathbb{E}^m , it can be shown that if the points $P_1(t_0), P_2(t_0) \dots$ approach $P_0(t_0)$ as a limit, then the points $P_1(t_0 + \tau), P_2(t_0 + \tau) \dots$ approach $P_0(t_0 + \tau)$ as a limit; this is due to the continuous deformation of regions within \mathbb{E}^m , as mentioned above. If these trajectories are to represent solutions to the equations governing natural phenomena, it is also necessary to impose the condition of boundedness: that there exist a bounded region R , such that every trajectory remains within R . The qualities of boundedness (of $P(t)$) and continuous deformity (of \mathbb{E}^m) allow for the definition of the *limiting trajectories* of $P(t)$.

Definition 2.3. (Limiting trajectory) The set of trajectories which are formed by the limit points of $P(t)$ are called the limiting trajectories of P .

Definition 2.4. (Central trajectory) A trajectory in phase space which is contained among its own limiting trajectories is called a central trajectory.

It can be shown that central trajectories remain statistically similar to their previous values, given that enough time has passed. A *noncentral* trajectory, however, will remain a certain distance away from any point through which it has previously passed and thus its instantaneous distance from its closest limit point is a transient quality, which becomes arbitrarily small as $t \rightarrow \infty$.

Definition 2.5. (Stable trajectory) A trajectory $P(t)$ is called *stable at point* $P(t_1)$ if any other trajectory passing sufficiently close to $P(t_1)$ at time t_1 remains close to $P(t)$ as $t \rightarrow \infty$.

Because \mathbb{E}^m is continuously deformed as t varies, a trajectory which is stable at one point is stable at every point, and is called a *stable* trajectory. Likewise, a trajectory unstable at one point is unstable at every point, and is called an *unstable* trajectory. Because of the quality of uniqueness, any trajectory passing through a point through which it has already passed must continue to do so as time goes on, and so must be *periodic*.

Definition 2.6. (Quasi-periodic trajectory) A trajectory is defined as quasi-periodic if for some arbitrarily large time interval $[t, t + \tau]$, $P(t + \tau)$ remains arbitrarily close to $P(t)$. A trajectory which is neither periodic or quasi-periodic, is defined as *nonperiodic*.

Now if $P(t)$ is nonperiodic, then $P(t_1 + \tau)$ may be arbitrarily close to $P(t_1)$ at time t_1 ; however, this implies that as $t \rightarrow \infty$, $P(t + \tau)$ will *not* remain close to $P(t)$. Nonperiodic trajectories are representations of deterministic nonperiodic flow, such as solutions to the boundary value problem (2.8). From the details of the definitions 2.3 - 2.6, the following conclusions may be drawn:

- periodic trajectories are central,
- quasi-periodic central trajectories include periodic trajectories whose periods are *incommensurate* from one another,
- those trajectories which approach periodic trajectories asymptotically are quasi-periodic noncentral,
- nonperiodic trajectories may be central or noncentral.

Now if $P_0(t)$ is a stable limiting trajectory of $P(t)$, two distinct points $P(t_1)$ and $P(t_1 + \tau)$ (with τ of arbitrary size), may be found arbitrarily close to any point $P_0(t_0)$. Since $P_0(t)$ is stable, $P(t_1)$ and $P(t_1 + \tau)$ must remain arbitrarily close to the trajectory $P_0(t + t_0 - t_1)$, and hence to each other, as $t \rightarrow \infty$. Thus, any trajectory with a stable limiting trajectory is quasi-periodic. It follows therefore that a stable central trajectory is necessarily quasi-periodic and, equivalently, *a nonperiodic central trajectory is always unstable*.

The implication of this result is devastating to anyone seeking to predict the future state of an observable nonperiodic system, such as the hydrodynamical phenomena which constitute the weather: slightly differing initial conditions can produce vastly different solutions.

3. CONTEMPORARY CLIMATE MODELING

As we have seen in section 2, the Navier-Stokes equations are not currently fully understood. This, of course, poses a problem for mathematicians who wish to develop models for climate and weather phenomena. Other challenges face them as well, not the least of which that they must run their models forward in time much faster than real time. In order to make the runtime realistic for researchers, models are built upon a large number of simplifying assumptions.

These assumptions tend to follow the pattern illustrated in Figure 1. At the root we have the Navier-Stokes equations. If one makes the concession that the atmosphere (or whatever fluid one is modeling) can be considered nonviscous, one arrives at the Euler equations, which are the subject of section 4.2. In this section, we will give an overview of some of the widely used assumptions and approximations that reduce the Euler equations to a more manageable form.

3.1. Approximations used in Climate Modeling

The expressions that meteorologists work with to create forecasts generally have the form

$$(3.1) \quad \mathbf{u}_t + \sum_{i=1}^n \nabla \cdot F_i(\mathbf{u}) = P(\mathbf{u}),$$

where \mathbf{u} is a state vector consisting either of density, velocity and pressure or just velocity and pressure, $n = 2$ or 3 depending on the model. The left hand side is a version (expressed in various different coordinates) of the full compressible version of the Navier-Stokes equations (that is, without the condition (2.4)), and $P(\mathbf{u})$ is a calibration term used to push the method towards a result which is considered realistic. That is, $P(\mathbf{u})$ is a function designed to capture phenomena which are too microscopic to be modeled directly by a macroscopic grid which would cover the entire globe or a large portion of it. In general terms, the operational models

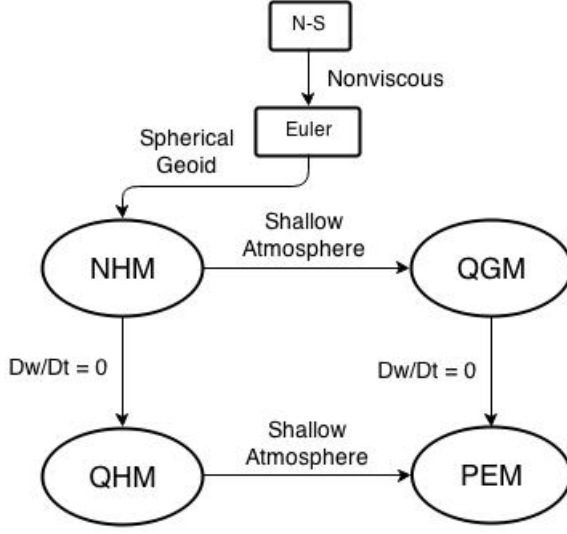


FIGURE 1. Common approximations in climate modeling.

that create the most accurate forecasts today are those which use the most skilled parameterizations $P(\mathbf{u})$.

3.1.1. The spherical geopotential approximation

The first main approximation used in current operational climate models is known as the spherical geopotential approximation. The assumption here is that the Earth can be modeled by a perfect sphere. The common practice is to use spherical coordinates, which do not provide the flexibility of adjusting at will for the potato-like shape of the actual Earth. The model currently in use at the UK Met Office, the HadGEM1, is a Non-Hydrostatic Model (NHM) which uses the spherical geopotential approximation [11].

Expressed in spherical polar coordinates (r, λ, ϕ) , the equations of motion (3.1) become:

$$\begin{aligned}
 (3.2) \quad & \frac{Du}{Dt} - f_r v + f_\phi w - \frac{uv \tan \phi}{r} + \frac{uw}{r} + \frac{\alpha}{r \cos \phi} \frac{\partial p}{\partial \lambda} = P^u, \\
 & \frac{Dv}{Dt} + f_r u - f_\lambda w + \frac{u^2 \tan \phi}{r} + \frac{vw}{r} + \frac{\alpha}{r} \frac{\partial p}{\partial \phi} = P^v, \\
 & \frac{Dw}{Dt} - f_\phi u + f_\lambda v - \frac{(u^2 + v^2)}{r} + g + \alpha \frac{\partial p}{\partial r} = P^w,
 \end{aligned}$$

where $(u, v, w) = (\dot{\lambda} r \cos \phi, \dot{\phi} r, \dot{r})$ are the longitudinal, latitudinal, and vertical wind components, $(f_\lambda, f_\phi, f_r) = (0, 2\omega \cos \phi, 2\omega \sin \phi)$ are the Coriolis terms arising from the rotation of the Earth (the angular speed of rotation given by ω), α is specific volume, and the P terms are the various physical parametrizations. See [11] for more detailed definitions. The material derivative is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{r} \frac{\partial}{\partial \phi} + w \frac{\partial}{\partial r}.$$

3.1.2. The hydrostatic approximation

The central idea behind the hydrostatic approximation is to let the net upward force acting on a parcel of air due to decreasing atmospheric pressure as altitude increases be equal to the downward force due to gravity. This can be written as

$$(3.3) \quad g = \alpha \frac{\partial p}{\partial z}.$$

In Quasi-Hydrostatic Models (QHMs), the vertical component of the material derivative Dw/Dt is neglected, which eliminates vertically propagating acoustic modes [10] and puts the vertical momentum into a form that differs somewhat from (3.3), hence the name *quasi*-hydrostatic. Using this assumption, (3.1) becomes

$$(3.4) \quad \begin{aligned} \frac{Du}{Dt} - f_r v + f_\phi w - \frac{uv \tan \phi}{r} + \frac{uw}{r} + \frac{\alpha}{r \cos \phi} \frac{\partial p}{\partial \lambda} &= P^u, \\ \frac{Dv}{Dt} + f_r u + \frac{u^2 \tan \phi}{r} + \frac{vw}{r} + \frac{\alpha}{r} \frac{\partial p}{\partial \phi} &= P^v, \\ -f_\phi u - \frac{(u^2 + v^2)}{r} + g + \alpha \frac{\partial p}{\partial r} &= 0. \end{aligned}$$

3.1.3. The shallow atmosphere approximation

The shallow atmosphere approximation makes two simplifications to the NHM equations (3.2). The idea is to replace the radial distance r from the center of the Earth with a mean radius a , and to omit the four metric terms that do not explicitly depend on latitude, as well as the two Coriolis terms associated with the horizontal component of planetary vorticity ω . The wind components are redefined as $(u, v, w) = (\lambda a \cos \phi, \phi a, \dot{z})$ and the material derivative becomes:

$$\frac{D_a}{Dt} = \frac{\partial}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \phi} + w \frac{\partial}{\partial z}.$$

Thus for (3.1) we get:

$$(3.5) \quad \begin{aligned} \frac{D_a u}{Dt} - f_r v - \frac{uv \tan \phi}{a} + \frac{uw}{a} + \frac{\alpha}{a \cos \phi} \frac{\partial p}{\partial \lambda} &= P^u, \\ \frac{D_a v}{Dt} + f_r u + \frac{u^2 \tan \phi}{a} + \frac{vw}{a} + \frac{\alpha}{r} \frac{\partial p}{\partial \phi} &= P^v, \\ \frac{D_a w}{Dt} + g + \alpha \frac{\partial p}{\partial z} &= P^w. \end{aligned}$$

Models which make use of the shallow atmosphere approximation but which do not use the hydrostatic approximation are known as quasi-geostrophic models (QGMs). The new system in use at SMHI, Harmonie, is an example of an operational QGM.

The word *quasigeostrophic* describes a system or flow that advances slowly in time as compared to the rotation period of the earth. The *Rossby number* of such a system is considered small (with zero or negligible Rossby number, the system is said to be in geostrophic balance.) This number is defined as

$$Ro = \frac{U}{fL},$$

where U is the velocity scale, f is the Coriolis parameter and L is the horizontal length scale. In summary, a small Rossby number and thus a quasi-geostrophical system is not significantly influenced by the earth's rotation.

3.1.4. Primitive equations models

Those models which use the spherical geoid, shallow atmosphere, and hydrostatic approximations are known as primitive equations models (PEMs) or hydrostatic PEMs. Combining all the approximations together, (3.1) becomes:

$$(3.6) \quad \begin{aligned} \frac{D_a u}{Dt} - f_r v - \frac{uv \tan \phi}{a} + \frac{uw}{a} + \frac{\alpha}{a \cos \phi} \frac{\partial p}{\partial \lambda} &= P^u, \\ \frac{D_a v}{Dt} + f_r u + \frac{u^2 \tan \phi}{a} + \frac{vw}{a} + \frac{\alpha}{r} \frac{\partial p}{\partial \phi} &= P^v, \\ g + \alpha \frac{\partial p}{\partial z} &= 0. \end{aligned}$$

For example, The European Centre for Medium-Range Weather Forecasts (ECMWF) uses an operational PEM.

4. NEXT GENERATION CLIMATE MODELING

4.1. Increased Accuracy and Detail

Climate modeling is undoubtedly moving towards higher resolution and more reliable prediction. The models under development at climate research facilities across the globe will be more accurate and more detailed than those in use today. As grid resolution increases, processing power and high performance computing will take over the importance of the parameterizations and approximations discussed in section 3. Researchers at SMHI are currently developing a model which does not rely on the spherical geopotential, shallow atmosphere, or hydrostatic approximations. The model will use curvilinear coordinates instead of the traditional spherical polar coordinates.

In this section, we will present the theoretical foundation for a boundary value problem associated with the Euler equations. Well-posedness will be shown by means of an energy estimate in L_2 -space. A numerical method will be derived and its convergence to the continuous problem will be shown by means of bounded energy in a norm uniquely defined for the numerical operator. The method will then be implemented on a one dimensional domain.

4.2. A Boundary Value Problem for the Euler Equations

If we can let the viscosity term in the Navier-Stokes equations be approximately zero, the system (2.1 - 2.2) simplifies to the Euler Equations. These are combined with a thermodynamic equation of state for total energy in order to complete the boundary value problem. The central issue when considering the well-posedness of the boundary value problem is to show that the solution depends continuously on the boundary and initial data. Here we will accomplish this by means of an energy estimate of the type (2.12). For the continuous case, as mentioned above, the L_2 -norm will be used. For the numerical method, a norm associated with the summation-by-parts operator will be used.

As before let ρ , u , and p denote density, velocity, and pressure and let their boundary values be given by the functions α , β , and ω . Let Ω be a one dimensional domain with boundary Γ . We will work with the following boundary value problem:

$$\begin{aligned}
 (4.1) \quad & \rho_t + (\rho u)_x = 0, & x \in \Omega, \quad t > 0, \\
 & (\rho u)_t + (\rho u^2 + p)_x = 0, & x \in \Omega, \quad t > 0, \\
 & e_t + ((e + p)u)_x = 0, & x \in \Omega, \quad t > 0, \\
 & \rho(x, t) = \rho_b(x, t), & x \in \Gamma, \quad t > 0, \\
 & u(x, t) = u_b(x, t), & x \in \Gamma, \quad t > 0, \\
 & p(x, t) = p_b(x, t), & x \in \Gamma, \quad t > 0, \\
 & \rho(x, 0) = \rho_I(x), & x \in \Omega, \\
 & u(x, 0) = u_I(x), & x \in \Omega, \\
 & p(x, 0) = p_I(x), & x \in \Omega,
 \end{aligned}$$

where e is the total energy. The functions $\rho_b, u_b, p_b, \rho_I, u_I, p_I$, are known functions representing boundary and initial data. The first three equations can be written in matrix form as:

$$(4.2) \quad \mathbf{u}_t + f(\mathbf{u})_x = 0,$$

where

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, \quad f(\mathbf{u}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (e + p)u \end{bmatrix}.$$

We wish first to decouple the system (4.2) in terms of primitive variables $\mathbf{q} = [\rho \quad u \quad p]^T$. To do this we use the following definition of the total energy:

$$e = \frac{p}{\gamma - 1} + \frac{\rho}{2} u^2,$$

where

$$\gamma = \frac{C_p}{C_v}.$$

Here we have assumed that the atmosphere can be represented by a calorically ideal gas. C_p and C_v are the heat capacities for constant pressure and constant volume respectively. Using these definitions we can rewrite the equations as follows:

$$\begin{aligned}
 \rho_t + (\rho u)_x = 0 &\Rightarrow \rho_t + u\rho_x + \rho u_x = 0, \\
 (\rho u)_t + (\rho u^2 + p)_x = 0 &\Rightarrow (-u\rho_x - \rho u_x)u + \rho u_t + \rho_x u^2 + \rho 2uu_x + p_x = 0, \\
 &\Rightarrow u_t + uu_x + \frac{1}{\rho} p_x = 0.
 \end{aligned}$$

$$\begin{aligned}
e_t + ((e+p)u)_x &= 0 \Rightarrow \left(\frac{p}{\gamma-1} + \frac{\rho}{2}u^2 \right)_t + \left(\left(\frac{p}{\gamma-1} + \frac{\rho}{2}u^2 + p \right)u \right)_x = 0, \\
&\Rightarrow p_t + \frac{\gamma-1}{2} \left[\left(-u\rho_x - \rho u_x \right)u^2 + \rho 2u \left(-uu_x - \frac{1}{\rho}p_x \right) \right] \\
&+ [p_x u + p u_x] + \frac{\gamma-1}{2} \left[\rho_x u^3 + \rho 3u^2 u_x \right] + (\gamma-1)(p_x u + p u_x) = 0, \\
&\Rightarrow p_t + p_x u + \gamma p u_x = 0.
\end{aligned}$$

Thus the system (4.2) can be written as:

$$(4.3) \quad \mathbf{q}_t + \mathbf{A}(\mathbf{q})\mathbf{q} = 0,$$

where

$$\mathbf{q}(x, t) = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}, \quad \mathbf{A}(\mathbf{q}) = \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{bmatrix}, \quad \gamma < 1.$$

4.2.1. Linearization

The system (4.3) contains the nonlinear term $1/\rho$ which, as has been discussed in section 2, can cause problems for the stability and regularity of solutions to (4.1). We will therefore proceed, as in [14], by means of a linearization principle.

Definition 4.1. (Linearization principle) Let $P = P(x, t, u, \partial/\partial x)$ be a quasi-linear differential operator of order m for dimension \mathbb{R}^n of the form

$$P(x, t, u, \partial/\partial x) = \sum_{|\alpha| \leq m} A_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

where $|\alpha|$ is as in (2.9) and the coefficients A fulfill

$$A = A(x, t, u) \in C^{n,n}.$$

Assuming that the initial value problem

$$(4.4) \quad \begin{aligned} u_t &= P(x, t, u, \partial/\partial x)u, \quad x \in \mathbb{R}^n, 0 \leq t \leq T, \\ u(x, 0) &= u_I(x), \quad x \in \mathbb{R}^n, \end{aligned}$$

has a unique solution u_0 for some fixed u_I , we say that the nonlinear problem (4.4) is well-posed at u_0 if there is an $\epsilon > 0$ such that for all smooth functions δu_I with

$$\|\delta u_I\|_2 \leq \epsilon,$$

the perturbed problem

$$\begin{aligned} \tilde{u}_t &= P(x, t, \tilde{u}, \partial/\partial x)\tilde{u}, \quad x \in \mathbb{R}^n, 0 \leq t \leq T, \\ \tilde{u}(x, 0) &= u_I(x) + \delta u_I(x), \quad x \in \mathbb{R}^n, \end{aligned}$$

is also uniquely solvable. For sufficiently small perturbations, solutions to (4.4) can then be written

$$u = u_0 + \tilde{u}.$$

The idea is to regard the atmosphere (or that piece of it which is contained in our domain Ω) as existing in an equilibrium state, described by constant ρ , u , and p , from which it makes only small deviations. Then, as per definition 4.1, the linearization principle holds: the nonlinear problem is well-posed at an equilibrium

point u_0 if a linear problem obtained by linearizing at all functions near u_0 is well-posed.

Let now $\tilde{\mathbf{q}}$ be a small perturbation from an equilibrium point \mathbf{q}_0 and let $\mathbf{q} = \mathbf{q}_0 + \tilde{\mathbf{q}}$. Inserting this into (4.3) gives:

$$\begin{aligned} \frac{d}{dt} \left[\mathbf{q}_0 + \tilde{\mathbf{q}} \right] + \mathbf{A}(\mathbf{q}_0 + \tilde{\mathbf{q}}) \frac{d}{dx} \left[\mathbf{q}_0 + \tilde{\mathbf{q}} \right] &= 0, \\ \Rightarrow \tilde{\mathbf{q}}_t + \mathbf{A}(\mathbf{q}_0) \tilde{\mathbf{q}}_x &= -\mathbf{q}_{0,t} - \mathbf{A}(\mathbf{q}_0) \mathbf{q}_x - \mathbf{A}(\tilde{\mathbf{q}}) \tilde{\mathbf{q}}_x. \end{aligned}$$

Since \mathbf{q}_0 solves (4.3), $\mathbf{q}_{0,t} + \mathbf{A}(\mathbf{q}_0) \mathbf{q}_x = 0$. And if we let $\tilde{\mathbf{q}}$ be a sufficiently *small* perturbation (as in [14]), we can neglect the quadratic terms in $\tilde{\mathbf{q}}$ -variables and set

$$\mathbf{A}(\tilde{\mathbf{q}}) \tilde{\mathbf{q}}_x \approx 0.$$

Thus we get the following, *linearized* system for $\tilde{\mathbf{q}}$:

$$(4.5) \quad \tilde{\mathbf{q}}_t + \mathbf{A}(\mathbf{q}_0) \tilde{\mathbf{q}}_x = 0,$$

where $\mathbf{A}(\mathbf{q}_0)$ is now the constant matrix

$$\mathbf{A}(\mathbf{q}_0) = \begin{bmatrix} u_0 & \rho_0 & 0 \\ 0 & u_0 & 1/\rho_0 \\ 0 & \gamma p_0 & u_0 \end{bmatrix}.$$

Here, $\mathbf{q}_0 = [\rho_0 \ u_0 \ p_0]^T$ are known constants (which can be measured) representing equilibrium density, wind, and pressure, with $\rho_0 > 0$.

We are now approximating the atmosphere with a coupled linear hyperbolic system of partial differential equations and if we can show that the associated linear boundary value problem is well-posed, then as in Definition 4.1 we can claim well-posedness for the non-linear problem in a domain surrounding equilibrium.

4.2.2. Energy estimate

In order to prove stability for (4.3), we need a norm in which solutions to (4.5) can be shown to be nonincreasing with time. This can be done by applying a matrix transformation to the variable $\tilde{\mathbf{q}}$ such that the resulting system is *symmetric*. Symmetry is needed for the analysis-method to properly ensure stability. We wish to impose the following criteria:

$$(4.6) \quad \frac{d}{dt} \|(\mathbf{S}^{-1}) \tilde{\mathbf{q}}\|_2^2 \leq C,$$

where $\|\cdot\|_2$ is the L_2 -norm and the matrix \mathbf{S} has the property that $\mathbf{S}^{-1} \mathbf{A} \mathbf{S}$ is symmetric. Note that C is a constant given by known values at the boundary.

Theorem 1. *Let \mathbf{A}_s be a positive definite Hermitian matrix and let Φ be a given, sufficiently smooth function (the index s denotes the important quality that this matrix is symmetric). Then solutions of the hyperbolic boundary value problem*

$$\begin{aligned} \mathbf{W}_t + \mathbf{A}_s \mathbf{W}_x &= 0, & x \in \Omega, \ t > 0, \\ \mathbf{W}(x, t) &= \Phi(x, t), & x \in \Gamma, \ t > 0, \ \Phi \text{ given,} \end{aligned}$$

satisfy

$$\frac{d}{dt} \|\mathbf{W}\|_2^2 \leq C.$$

Proof. \mathbf{A}_s is Hermitian and $(\mathbf{A}_s) = (\mathbf{A}_s)^T$, therefore

$$\begin{aligned}
\frac{d}{dt} \|\mathbf{W}\|_2^2 &= \frac{d}{dt} \int_{\Omega} \mathbf{W}^T \mathbf{W} dx \\
&= \int_{\Omega} \left(\mathbf{W}_t^T \mathbf{W} + \mathbf{W}^T \mathbf{W}_t \right) dx \\
&= \int_{\Omega} (-\mathbf{A}_s \mathbf{W}_x)^T \mathbf{W} dx + \int_{\Omega} \mathbf{W}^T (-\mathbf{A}_s \mathbf{W}_x) dx \\
&= \left[\mathbf{W}^T (-\mathbf{A}_s)^T \mathbf{W} \right]_{\Gamma} - \int_{\Omega} \mathbf{W}^T (-\mathbf{A}_s)^T \mathbf{W}_x dx + \int_{\Omega} \mathbf{W}^T (-\mathbf{A}_s) \mathbf{W}_x dx \\
&= \left[\mathbf{W}^T (-\mathbf{A}_s) \mathbf{W} \right]_{\Gamma} \\
&= -\Phi^T(x \in \Gamma, t) \mathbf{A}_s \Phi(x \in \Gamma, t) + \Phi^T(x \in \Gamma, t) \mathbf{A}_s \Phi(x \in \Gamma, t) \\
&\leq C.
\end{aligned}$$

□

Note that if we can say that there is no flow through the boundary Γ or if, for example, total inflow is equal to outflow over Γ then we can make the stronger assertion

$$\frac{d}{dt} \|\mathbf{W}\|_2^2 \leq 0.$$

The task therefore is to find a matrix \mathbf{S} such that the transformation \mathbf{S}^{-1} applied to (4.5) gives the system $\mathbf{W}_t + \mathbf{A}(\mathbf{q}_0)_s \mathbf{W}_x = 0$ where $\mathbf{A}(\mathbf{q}_0)_s$ is symmetric. We will do this explicitly for the boundary value problem (4.1). First, however, we will show that the system (4.5) is hyperbolic, as required by Theorem 1.

Definition 4.2. (Hyperbolic PDE) A linear system of the form

$$q_t + \mathbf{A} q_x = 0$$

is called *hyperbolic* if the square matrix \mathbf{A} is diagonalizable with real eigenvalues [15].

To show hyperbolicity we must solve the equation:

$$\det[\mathbf{A}(\mathbf{q}_0) - \lambda \mathbf{I}] = \det \begin{bmatrix} u_0 - \lambda & \rho_0 & 0 \\ 0 & u_0 - \lambda & 1/\rho_0 \\ 0 & \gamma p_0 & u_0 - \lambda \end{bmatrix} = 0,$$

which gives

$$\begin{aligned}
\lambda_1 &= u_0, \\
\lambda_2 &= u_0 + c, \\
\lambda_3 &= u_0 - c,
\end{aligned}$$

where

$$c = \sqrt{\frac{\gamma \cdot p_0}{\rho_0}}$$

is the speed of sound in a polytropic gas. These solve the following equations with associated eigenvectors ϕ_1 , ϕ_2 , and ϕ_3 :

$$\mathbf{A}\phi_1 = \lambda_1\phi_1, \quad \mathbf{A}\phi_2 = \lambda_2\phi_2, \quad \mathbf{A}\phi_3 = \lambda_3\phi_3,$$

giving

$$(4.7) \quad \phi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} \rho_0/c \\ 1 \\ \rho_0 c \end{bmatrix}, \quad \phi_3 = \begin{bmatrix} -\rho_0/c \\ 1 \\ -\rho_0 c \end{bmatrix}.$$

The eigenvectors ϕ_1 , ϕ_2 , and ϕ_3 constitute a complete set in \mathbb{R}^3 , thus (4.5) is hyperbolic.

4.2.3. Symmetrization

We may now turn our attention to constructing the matrix \mathbf{S} . For this we will use the following theorem.

Theorem 2. *Let \mathbf{A} be a square matrix with real eigenvalues and a complete set of eigenvectors and let \mathbf{D} be a diagonal matrix with arbitrary real, positive elements. Let \mathbf{T} be a matrix with real eigenvectors of \mathbf{A} as columns. Then the matrix*

$$\mathbf{H} = (\mathbf{T}^{-1})^T \mathbf{D} \mathbf{T}^{-1}$$

is Hermitian positive definite and can be written

$$\mathbf{H} = \mathbf{L} \mathbf{L}^T,$$

where \mathbf{L} is the lower-triangular matrix which is the Cholesky-factor of \mathbf{H} . Define the matrix \mathbf{S} as

$$\mathbf{S} = (\mathbf{L}^{-1})^T.$$

Then

$$\mathbf{A}_s = \mathbf{S}^{-1} \mathbf{A} \mathbf{S}$$

is Hermitian positive definite.

Proof. For \mathbf{A}_s to be Hermitian, we need to show $(\mathbf{S}^{-1} \mathbf{A} \mathbf{S})^T = \mathbf{S}^{-1} \mathbf{A} \mathbf{S}$. Note that $\mathbf{H} \mathbf{A}$ is symmetric. Using $\mathbf{H} = \mathbf{L} \mathbf{L}^T = (\mathbf{S}^{-1})^T \mathbf{S}^{-1}$ we can write

$$\mathbf{H} \mathbf{A} = (\mathbf{S}^{-1})^T \mathbf{S}^{-1} \mathbf{A} \Rightarrow \mathbf{S}^T \mathbf{H} \mathbf{A} = \mathbf{S}^{-1} \mathbf{A},$$

and therefore

$$\begin{aligned} (\mathbf{S}^{-1} \mathbf{A} \mathbf{S})^T &= \mathbf{S}^T (\mathbf{S}^{-1} \mathbf{A})^T \\ &= \mathbf{S}^T (\mathbf{S}^T \mathbf{H} \mathbf{A})^T \\ &= \mathbf{S}^T (\mathbf{H} \mathbf{A})^T \mathbf{S} \\ &= \mathbf{S}^T \mathbf{H} \mathbf{A} \mathbf{S} \\ &= \mathbf{S}^T (\mathbf{S}^{-1})^T \mathbf{S}^{-1} \mathbf{A} \mathbf{S} \\ &= \mathbf{S}^{-1} \mathbf{A} \mathbf{S}. \end{aligned}$$

□

According to Theorem 2 we need to construct the matrix \mathbf{T} from the eigenvalues of the matrix \mathbf{A} in (4.5). Using (4.7) we get

$$\mathbf{T} = \begin{bmatrix} 1 & \rho_0/c & -\rho_0/c \\ 0 & 1 & 1 \\ 0 & \rho_0 c & -\rho_0 c \end{bmatrix}.$$

To begin with, we give the diagonal matrix \mathbf{D} the symbolic variables d_{11} , d_{22} , and d_{33} , which are assumed positive and real:

$$\mathbf{D} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}.$$

This gives the following calculation for \mathbf{H} :

$$\mathbf{H} = (\mathbf{T}^{-1})^T \mathbf{D} \mathbf{T}^{-1} = \begin{bmatrix} d_{22} & 0 & -\frac{d_{22}}{c^2} \\ 0 & \frac{d_{11} + d_{33}}{4} & \frac{d_{11} - d_{33}}{4\rho_0 c} \\ -\frac{d_{22}}{c^2} & \frac{d_{11} - d_{33}}{4\rho_0 c} & \left(\frac{d_{22}}{c^4} + \frac{d_{33} + d_{11}}{4\rho_0^2 c^2} \right) \end{bmatrix}.$$

Using Cholesky-decomposition we can rewrite \mathbf{H} as the product of a lower-triangular matrix \mathbf{L} and it's transpose:

$$\mathbf{H} = \mathbf{L} \mathbf{L}^T \implies \mathbf{L} = \begin{bmatrix} \sqrt{d_{22}} & 0 & 0 \\ 0 & \sqrt{\frac{d_{11} + d_{33}}{4}} & 0 \\ -\frac{\rho_0 \sqrt{d_{22}}}{\gamma p_0} & \frac{d_{11} - d_{33}}{2\rho_0 c \sqrt{d_{33} + d_{11}}} & \sqrt{\frac{d_{33}}{(d_{22} + 1)\rho_0^2 c^2}} \end{bmatrix}.$$

Now we can define \mathbf{S} as in Theorem 2:

$$\mathbf{S} = (\mathbf{L}^{-1})^T = \begin{bmatrix} \frac{1}{\sqrt{d_{22}}} & 0 & \frac{\rho_0 \sqrt{d_{33} + d_{11}}}{d_{22}c} \\ 0 & \frac{2}{\sqrt{d_{11} + d_{33}}} & \frac{d_{22} - 1}{\sqrt{d_{33}(d_{22} + 1)}} \\ 0 & 0 & \rho_0 c \sqrt{\frac{d_{22} - 1}{d_{33}}} \end{bmatrix}.$$

We now want to choose some appropriate values for the elements of \mathbf{D} that will make the final multiplication palatable. Choosing $d_{11} = d_{33} = 2$ and $d_{22} = \gamma$ gives the following matrix for \mathbf{S} :

$$(4.8) \quad \mathbf{S} = \begin{bmatrix} \frac{1}{\sqrt{\gamma}} & 0 & \frac{2\rho_0}{\gamma c} \\ 0 & 1 & \frac{\gamma - 1}{\sqrt{2(\gamma + 1)}} \\ 0 & 0 & \rho_0 c \sqrt{\frac{\gamma - 1}{2}} \end{bmatrix}.$$

(4.9)

This in turn gives the desired matrix \mathbf{A}_s :

$$\mathbf{A}_s = \mathbf{S}^{-1} \mathbf{A}(\mathbf{q}_0) \mathbf{S} = \begin{bmatrix} u_0 & 0 & \frac{0}{\sqrt{\frac{\gamma p_0}{\rho_0}}} \\ 0 & u_0 & \sqrt{\frac{\gamma p_0}{\rho_0}} \\ 0 & \sqrt{\frac{\gamma p_0}{\rho_0}} & u_0 \end{bmatrix},$$

which is indeed symmetric. Thus we are now free to rewrite the system (4.5) using \mathbf{S} (4.8) as follows:

$$\begin{aligned} \tilde{\mathbf{q}}_t + \mathbf{A}(\mathbf{q}_0) \tilde{\mathbf{q}}_x &= \mathbf{S}^{-1} \tilde{\mathbf{q}}_t + \mathbf{S}^{-1} \mathbf{A} \tilde{\mathbf{q}}_x \\ &= \mathbf{S}^{-1} \tilde{\mathbf{q}}_t + \mathbf{S}^{-1} \mathbf{A} \mathbf{I} \tilde{\mathbf{q}}_x \\ &= \mathbf{S}^{-1} \tilde{\mathbf{q}}_t + \mathbf{S}^{-1} \mathbf{A} \mathbf{S} \mathbf{S}^{-1} \tilde{\mathbf{q}}_x \\ &= \mathbf{W}_t + \mathbf{S}^{-1} \mathbf{A} \mathbf{S} \mathbf{W}_x \\ (4.10) \quad &= \mathbf{W}_t + \mathbf{A}_s \mathbf{W}_x = 0, \end{aligned}$$

and we are guaranteed that the new system (4.10) fulfills the stability criteria (4.6).

4.3. A Numerical Method for solving 1D Euler

The model currently under development at SMHI for solving boundary value problems such as (4.1) follows an approach known as *the method of lines*. The method of lines consists of reducing each PDE to a system of ordinary differential

equations, to which a numerical approximation is applied in successive time steps. This has the advantage that it decouples the spacial and temporal discretization, allowing their order of accuracy to be dealt with separately [15].

We now seek to write the linear symmetric system (4.10) in *semidiscrete* form, using a numerical scheme \mathcal{N} to approximate the spatial derivative. We define \mathcal{N} as a central difference scheme as follows

$$(4.11) \quad \frac{d}{dx}u(x_i, t) = \mathcal{N}(u) = \frac{u(x_{i+1}, t) - u(x_{i-1}, t)}{2\Delta x} + \mathcal{O}(\Delta x^2).$$

If we drop the higher order terms we can say that \mathcal{N} is second-degree accurate and

$$\mathcal{N}(u) \approx \frac{d}{dx}u(x_i, t).$$

We can then write (4.10) in semidiscrete form as

$$(4.12) \quad \mathbf{W}_t + \mathcal{N}(\mathbf{A}_s \mathbf{W}) = 0.$$

Note that since \mathcal{N} is a linear method and \mathbf{A}_s is a scalar matrix, we can include it in the derivative without rewriting \mathbf{A}_s or \mathbf{W} .

$\mathcal{N}(u)$ approximates du/dx by relating the value of u at point x_i to the values on either side: x_{i-1} and x_{i+1} . Therefore when implementing the method, we can use a matrix to accomplish the scheme \mathcal{N} , and the problem reduces to solving a linear system in each time step. This matrix is formalized as a *summation-by-parts (SBP) operator* [16].

4.3.1. Summation-by-Parts.

Definition 4.3. (Summation-By-Parts Operator) The $(n \times n)$ matrix \mathbf{D} is a summation-by-parts operator for the first derivative on the discretized line $\{x_i\}_{i=0}^n$ if it has the form

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{Q},$$

where the $(n \times n)$ matrix \mathbf{P} is a symmetric-positive-definite matrix and \mathbf{Q} is $(n \times n)$ and satisfies

$$\mathbf{Q} + \mathbf{Q}^T = \text{diag}(-1, 0, 0, \dots, 1).$$

This definition of the SBP operator serves another important function as well: it mimics the integration by parts used to prove Theorem 1, and can be used analogously to show energy-stability for the semi discrete case. If we define our SBP operator as

$$\mathbf{D} = \frac{1}{2\Delta x} \begin{bmatrix} -2 & -2 & . & . & . & . & 0 \\ -1 & 0 & 1 & & & & . \\ . & & . & & & & . \\ . & & & . & & & . \\ . & & & & . & & . \\ . & & & & -1 & 0 & 1 \\ 0 & . & . & . & . & -2 & 2 \end{bmatrix},$$

the semi discrete system becomes

$$(4.13) \quad \mathbf{W}_t + \mathbf{D}\mathbf{A}_s\mathbf{W} = 0.$$

4.3.2. The solution vector in long form

The system (4.13) can be further simplified for implementation by being put into long form. This means simply ordering the unknowns consecutively into one long vector, which requires that the matrices in (4.13) be expanded using Kronecker products.

Definition 4.4. (Kronecker product) Let \mathbf{A} and \mathbf{B} be matrices of arbitrary dimension. Define the Kronecker product as:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} A_{11}\mathbf{B} & A_{12}\mathbf{B} & \dots & A_{1n}\mathbf{B} \\ A_{21}\mathbf{B} & A_{22}\mathbf{B} & \dots & A_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}\mathbf{B} & A_{m2}\mathbf{B} & \dots & A_{mn}\mathbf{B} \end{bmatrix}.$$

Note that if \mathbf{A} has dimension $(n \times m)$ and \mathbf{B} has dimension $(i \times j)$, then the Kronecker product $(\mathbf{A} \otimes \mathbf{B})$ has dimension $(ni \times mj)$; note also the following properties: $(\mathbf{A} \otimes \mathbf{B}) + (\mathbf{C} \otimes \mathbf{D}) = (\mathbf{A} + \mathbf{C}) \otimes (\mathbf{B} + \mathbf{D})$ and $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{A}\mathbf{B} \otimes \mathbf{C}\mathbf{D}$ (assuming that the matrix dimensions allow multiplication). Furthermore, $(\mathbf{A} \otimes \mathbf{B})^T = (\mathbf{A}^T \otimes \mathbf{B}^T)$, and if both matrices are invertible then $(\mathbf{A} \otimes \mathbf{B})^{-1} = (\mathbf{A}^{-1} \otimes \mathbf{B}^{-1})$.

Now, let us remember, the system (4.13) is a system of three equations in three unknowns $\mathbf{W} = [\bar{\rho} \ \bar{u} \ \bar{p}]^T$. Let us define now a new vector, \mathbf{v} , which contains the discrete values of \mathbf{W} , point-wise, as elements:

$$\mathbf{v} = [\bar{\rho}_1 \ \bar{u}_1 \ \bar{p}_1 \ \bar{\rho}_2 \ \bar{u}_2 \ \bar{p}_2 \ \dots \ \bar{\rho}_n \ \bar{u}_n \ \bar{p}_n]^T.$$

where $\bar{\rho}_i = \bar{\rho}(x_i, t)$ is the value of $\bar{\rho}$ at grid point i , and so forth. We can then write (4.13) as a single matrix equation:

$$(4.14) \quad \mathbf{v}_t + \mathbf{D} \otimes \mathbf{A}_s \mathbf{v} = 0.$$

4.3.3. The SAT boundary treatment

Now in order to prove the energy estimate, we must augment the SBP operator with a specific boundary treatment that linearly combines the PDE with its boundary condition. To accomplish this we will use the *simultaneous-approximation-term* (SAT) approach. In the SAT method, one adds a source term to the SBP operator, proportional to the difference between the discrete solution and the boundary condition to be fulfilled. That is, the boundary conditions are not imposed *exactly*, which would risk stability if data is not sufficiently smooth. The SAT method imposes a *penalty* term which adjusts its strength based on closeness to the boundary value. Following the notation used in [17], we can build the penalty term using Kronecker products as follows:

$$\text{SAT} = -(\mathbf{P}^{-1}\mathbf{E}_1 \otimes \mathbf{A}^+)(\mathbf{v} - \mathbf{g}) + (\mathbf{P}^{-1}\mathbf{E}_N \otimes \mathbf{A}^-)(\mathbf{v} - \mathbf{g}),$$

where \mathbf{A}^+ is a diagonal matrix with the positive eigenvalues of \mathbf{A}_s in descending order. Likewise \mathbf{A}^- is defined for the negative eigenvalues. The matrices \mathbf{E}_1 and \mathbf{E}_N are such that the penalty term is zero except at the first and final grid points respectively. The vector \mathbf{g} contains the boundary data and has the same structure as \mathbf{v} , that is

$$(4.15) \quad \mathbf{g} = [\rho_b(0, t) \quad u_b(0, t) \quad 0 \dots 0 \quad \rho_b(n, t) \quad u_b(n, t) \quad p_b(n, t)]^T,$$

where the boundaries are at $x = 0$ and $x = n$. Thus we can now write the semi discrete problem as:

$$(4.16) \quad \mathbf{v}_t + \mathbf{D} \otimes \mathbf{A}_s \mathbf{v} = -(\mathbf{P}^{-1}\mathbf{E}_1 \otimes \mathbf{A}^+)(\mathbf{v} - \mathbf{g}) \\ + (\mathbf{P}^{-1}\mathbf{E}_N \otimes \mathbf{A}^-)(\mathbf{v} - \mathbf{g}).$$

Note that due to the structure of \mathbf{E}_1 and \mathbf{E}_N , the system (4.16) reduces to (4.14) at all points except at the boundary. Note also that $-\mathbf{E}_1 + \mathbf{E}_N = \mathbf{Q} + \mathbf{Q}^T$, as defined in Definition 4.3.

4.3.4. Stability for the semi discrete case

With the penalty term in place we are now ready to pursue an energy bound for the discrete case, and for this we will need an appropriate norm. Let

$$\mathbb{P} = \mathbf{P} \otimes \mathbf{I}_N$$

define a norm by $\|\mathbf{v}\|_{\mathbb{P}}^2 = \mathbf{v}^T \mathbb{P} \mathbf{v}$. Here the matrix \mathbf{P} is as in Definition 4.3. The procedure for extracting the energy norm is to multiply (4.16) from the left with $\mathbf{v}^T \mathbb{P}$, multiply the transposed (4.16) from the right with $\mathbb{P} \mathbf{v}$ and then add the two together. We will then be able to show that energy growth is bounded, just as in Theorem 1. The calculation goes as follows:

$$(4.17) \quad \mathbf{v}^T \mathbb{P} \mathbf{v}_t + \mathbf{v}^T \mathbb{P} (\mathbf{D} \otimes \mathbf{A}_s) \mathbf{v} = \mathbf{v}^T \mathbb{P} \mathbf{v}_t + \mathbf{v}^T (\mathbf{P} \otimes \mathbf{I}_N) (\mathbf{P}^{-1} \mathbf{Q} \otimes \mathbf{A}_s) \mathbf{v} \\ = \mathbf{v}^T \mathbb{P} \mathbf{v}_t + \mathbf{v}^T (\mathbf{Q} \otimes \mathbf{A}_s) \mathbf{v} \\ = -\mathbf{v}^T \mathbb{P} (\mathbf{P}^{-1} \mathbf{E}_1 \otimes \mathbf{A}^+) (\mathbf{v} - \mathbf{g}) \\ + \mathbf{v}^T \mathbb{P} (\mathbf{P}^{-1} \mathbf{E}_N \otimes \mathbf{A}^-) (\mathbf{v} - \mathbf{g}) \\ = -\mathbf{v}^T (\mathbf{E}_1 \otimes \mathbf{A}^+) (\mathbf{v} - \mathbf{g}) \\ + \mathbf{v}^T (\mathbf{E}_N \otimes \mathbf{A}^-) (\mathbf{v} - \mathbf{g}),$$

$$\begin{aligned}
(4.18) \quad \mathbf{v}_t^T \mathbb{P} \mathbf{v} + (\mathbf{D} \otimes \mathbf{A}_s \mathbf{v})^T \mathbb{P} \mathbf{v} &= \mathbf{v}_t^T \mathbb{P} \mathbf{v} + \mathbf{v}^T ((\mathbf{P}^{-1} \mathbf{Q})^T \otimes \mathbf{A}_s) (\mathbf{P} \otimes \mathbf{I}_N) \mathbf{v} \\
&= \mathbf{v}_t^T \mathbb{P} \mathbf{v} + \mathbf{v}^T (\mathbf{Q}^T \otimes \mathbf{A}_s) \mathbf{v} \\
&= -((\mathbf{P}^{-1} \mathbf{E}_1 \otimes \mathbf{A}^+) (\mathbf{v} - \mathbf{g}))^T \mathbb{P} \mathbf{v} \\
&\quad + ((\mathbf{P}^{-1} \mathbf{E}_N \otimes \mathbf{A}^-) (\mathbf{v} - \mathbf{g}))^T \mathbb{P} \mathbf{v} \\
&= -(\mathbf{v} - \mathbf{g})^T ((\mathbf{P}^{-1} \mathbf{E}_1)^T \otimes (\mathbf{A}^+)^T) (\mathbf{P} \otimes \mathbf{I}_N) \mathbf{v} \\
&\quad + (\mathbf{v} - \mathbf{g})^T ((\mathbf{P}^{-1} \mathbf{E}_N)^T \otimes (\mathbf{A}^-)^T) (\mathbf{P} \otimes \mathbf{I}_N) \mathbf{v} \\
&= -(\mathbf{v} - \mathbf{g})^T (\mathbf{E}_1^T \otimes \mathbf{A}^+) \mathbf{v} \\
&\quad + (\mathbf{v} - \mathbf{g})^T (\mathbf{E}_N^T \otimes \mathbf{A}^-) \mathbf{v}.
\end{aligned}$$

Adding (4.17) and (4.18) gives

$$\begin{aligned}
\mathbf{v}^T \mathbb{P} \mathbf{v}_t + \mathbf{v}_t^T \mathbb{P} \mathbf{v} + \mathbf{v}^T (\mathbf{Q} \otimes \mathbf{A}_s) \mathbf{v} + \mathbf{v}^T (\mathbf{Q}^T \otimes \mathbf{A}_s) \mathbf{v} &= \frac{d}{dt} \|\mathbf{v}\|_{\mathbf{P}}^2 + \mathbf{v}^T ((-\mathbf{E}_0 + \mathbf{E}_N) \otimes 2\mathbf{A}_s) \mathbf{v} \\
&= -\mathbf{v}^T (\mathbf{E}_1 \otimes \mathbf{A}^+) (\mathbf{v} - \mathbf{g}) \\
&\quad + \mathbf{v}^T (\mathbf{E}_N \otimes \mathbf{A}^-) (\mathbf{v} - \mathbf{g}) \\
&\quad - (\mathbf{v} - \mathbf{g})^T (\mathbf{E}_0^T \otimes \mathbf{A}^+) \\
&\quad + (\mathbf{v} - \mathbf{g})^T (\mathbf{E}_N^T \otimes \mathbf{A}^-) \mathbf{v}.
\end{aligned}$$

Define the following identities

$$\begin{aligned}
\mathbf{M}_N &= \mathbf{E}_N \otimes \mathbf{A}_s, \\
\mathbf{M}_1 &= \mathbf{E}_1 \otimes \mathbf{A}_s, \\
\mathbf{M}_N^+ &= \mathbf{E}_N \otimes \mathbf{A}^+, \\
\mathbf{M}_1^- &= \mathbf{E}_N \otimes \mathbf{A}^-,
\end{aligned}$$

and we can write the energy norm

$$\|\mathbf{v}\|_{\mathbf{P}}^2 = -\mathbf{v}^T (-\mathbf{M}_1 + \mathbf{M}_N) \mathbf{v} - \mathbf{v}^T (\mathbf{M}_1^+ - \mathbf{M}_N^-) (\mathbf{v} - \mathbf{g}) - (\mathbf{v} - \mathbf{g})^T (\mathbf{M}_1^+ - \mathbf{M}_N^-)^T \mathbf{v}.$$

In analogy with the continuous case, this results in bounded growth and implies that the numerical method (4.16) is stable.

4.3.5. The fully discrete form

Having shown stability for the semi discrete case we can now use an explicit method to approximate the time derivative and thereby derive an exact update formula for the values of \mathbf{v} . We use the following numerical scheme (Eular forward):

$$\frac{d}{dt} u(x, t^n) \approx \frac{u(x, t^{n+1}) - u(x, t^n)}{\Delta t}.$$

Inserting this into (4.16) gives the update formula

$$\begin{aligned}
\mathbf{v}^{n+1} = \Delta t & \left(-\mathbf{D} \otimes \mathbf{A}_s \mathbf{v}^n - (\mathbf{P}^{-1} \mathbf{E}_1 \otimes \mathbf{A}^+)(\mathbf{v}^n - \mathbf{g}) \right. \\
(4.19) \quad & \left. + (\mathbf{P}^{-1} \mathbf{E}_N \otimes \mathbf{A}^-)(\mathbf{v}^n - \mathbf{g}) \right) + \mathbf{v}^n.
\end{aligned}$$

Finally the model is ready for implementation.

4.4. Implementation

In order to test the practicality of the numerical method, the linearized one-dimensional system was solved using Matlab. Boundary and initial data were obtained from the Integrated Global Radiosonde Archive, a database administered by the National Climatic Data Center, U.S. Department of Commerce.

Measured values of wind velocity and pressure were taken from radiosondes entering the tropopause once a day from January 2 to December 28, 2004. Density was set at both points as 0.4135 kg/m^3 . A line from the Ekofisk oil field in the North Sea to Sodankylä Village in Finland was used as a domain, with data obtained from balloons released from those positions. Initial values were constructed using the same data. Velocity vectors were projected onto the domain using the regular dot product. The line was discretized into 170 points, approximating 10 km each. A total of 361 time-steps were used, comprising data from most of 2004.

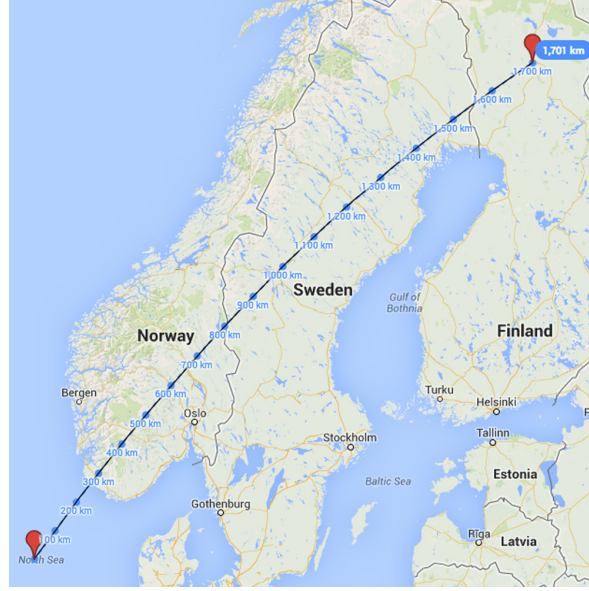


FIGURE 2. A one-dimensional domain between Ekofisk and Sodankylä.

4.4.1. Matlab code.

Following the methodology outlined in section 4.3, the update formula (4.19) can be implemented as follows:

```

for t=1:Nt

    %Simultaneous Approximation Term
    SATl = Pinv*IOls*Xinvs*E(2,2)*Xks*(U_old-U_data);
    SATn = Pinv*IONs*Xinvs*E(3,3)*Xks*(U_old-U_data);

    %Update
    U_new = dt*(-Aks*U_old - SATl + SATn) + U_old;
end

```

Note that the vector `U_old` contains the solutions calculated by the previous time step, and `U_data` is as in (4.15), with the six boundary data elements arranged in the first three and last three slots and zeros in between (updated in each time-step). The following matrices are used to build the SAT:

```

%coefficient matrix (set gamma = 1)
A = zeros(3);
A(1,1) = u0;
A(1,2) = rho0;
A(2,2) = u0;
A(2,3) = 1/rho0;
A(3,2) = p0;
A(3,3) = u0;

%summation by parts operator
e = ones(n,1);
P = dx*diag(e);
P(1,1) = 0.5*P(1,1);
P(n,n) = 0.5*P(n,n);
I = eye(3);
Pinv = kron(inv(P), I);
D = spdiags([-e 0*e e], -1:1, n,n);
D(1,1) = -2;
D(1,2) = 2;
D(n,n) = 2;
D(n, n-1)=-2;
D = D./(2*dx);

%indicator:
IOl=zeros(3*n);
ION=zeros(3*n);
IOl(1,1)=1;
IOl(2,2)=1;
IOl(3,3)=1;
ION(3*n-2,3*n-2)=1;
ION(3*n-1,3*n-1)=1;
ION(3*n,3*n)=1;
IOls = sparse(IOl);
IONs = sparse(ION);

%eigenvalues
[X,E]=eig(A);

```

```

In = eye(n);
Xinv = kron(inv(X), In);
Xinvs = sparse(Xinv);
Xk = kron(X, In);
Xks = sparse(Xk);

%the "long" vectors
U = reshape(W, 3*n, 1);
Ak = kron(D, A);
Aks = sparse(Ak);

```

4.4.2. Simulation results

It is, of course, quite a primitive approximation to use the one dimensional model, and it is not possible to reproduce three dimensional results. Even so, the averages and max-min intervals in Table 1 show the validity of the method. Figures 3 through 5 show the variation of density, wind speed, and pressure across the domain.

TABLE 1. Simulation Results

Case	Max	Min	Avg
Ekofisk ρ (kg/m^3)	0.41	0.41	0.41
Ekofisk u (m/s)	59.2	-48.5	3.6
Ekofisk p (kPa)	42.9	13.7	24.7
Sodankylä ρ (kg/m^3)	0.41	0.41	0.41
Sodankylä u (m/s)	51.7	-37.0	2.41
Sodankylä p (kPa)	45.2	12.2	26.7
Simulated ρ (kg/m^3)	0.42	0.41	0.41
Simulated u (m/s)	42.3	-26.7	4.86
Simulated p (kPa)	37.1	17.2	26.4

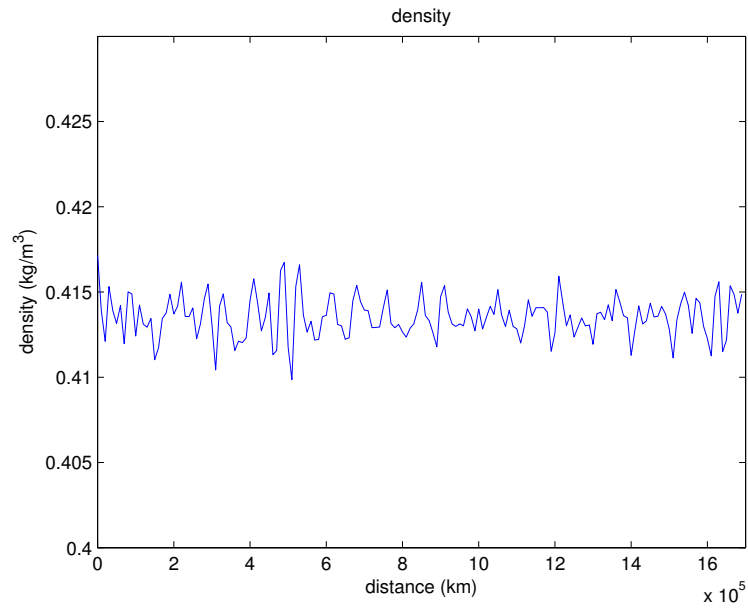


FIGURE 3. Variation of density (kg/m^2), tropopause over Sweden.

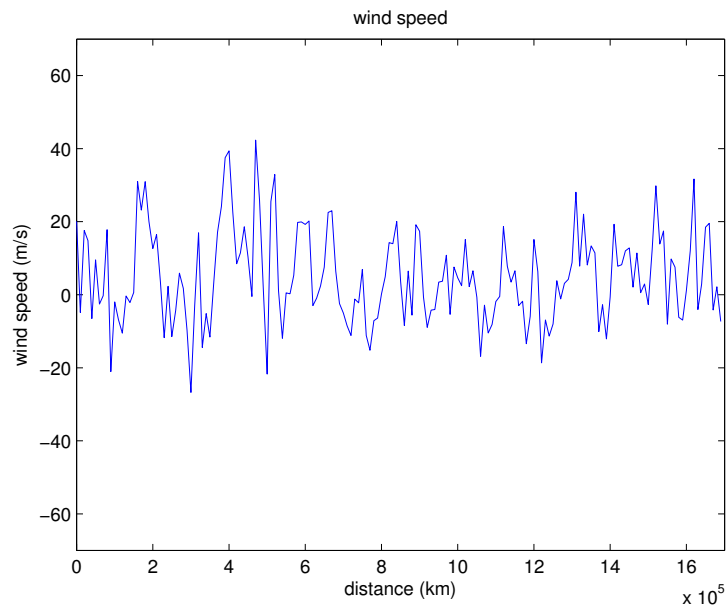


FIGURE 4. Variation of wind speed (m/s), tropopause over Sweden.

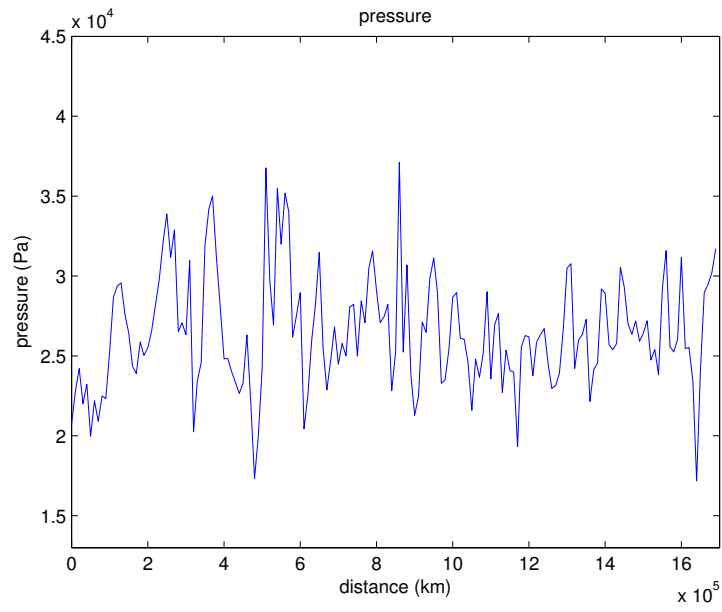


FIGURE 5. Variation of pressure (Pa), tropopause over Sweden.

5. CONCLUSIONS

The boundary value problem and numerical method described in section 4 are easily expandable to the two and three dimensional case. The 3D version of the system (4.2) will have essentially the same structure but with three coefficient matrices and a longer solution vector (five unknowns instead of three):

$$\mathbf{u}_t + f(\mathbf{u})_x + h(\mathbf{u})_y + g(\mathbf{u})_z = 0.$$

Following the procedure outlined in section 4, the energy method can then be applied to show stability in both the continuous and discrete case. A typical use for such a model, for example at SMHI, would be to increase the resolution of an existing model, or to fill in the gaps between data points spread across a wide area. This method can be applied anywhere in the atmosphere, wherever there is boundary data between which one seeks a higher resolution.

The scheme (4.11) has order of accuracy 2 in the interior of the domain and order of accuracy 1 at the boundary, and is therefore called a "2-1" operator. There are a variety of schemes that can be used to approximate the spatial derivatives, and a central difference scheme can be implemented to higher order. The model under development at SMHI uses higher order operators such as the "8-4" operator, which (in the interior) has the form:

$$(5.1) \quad \begin{aligned} N(u_i) = \frac{1}{\Delta x} & \left(\frac{1}{280}u_{i-4} - \frac{4}{105}u_{i-3} + \frac{1}{5}u_{i-2} - \frac{4}{5}u_{i-1} \right. \\ & \left. + \frac{4}{5}u_{i+1} - \frac{1}{5}u_{i+2} + \frac{4}{105}u_{i+3} - \frac{1}{280}u_{i+4} \right) + \mathcal{O}(\Delta x^8). \end{aligned}$$

When used in a three dimensional domain the numerical scheme becomes a "stencil" in which all the values surrounding u_i (in all three directions) are included in the approximation. High resolution three dimensional stencils when written out as in (5.1) may seem like lengthy grotesque sums, but in practice they are merely sparse banded matrices which are stored in the computer memory as vectors. Numerical algorithms need not solve large linear systems, they simply update vector values and feed them to the next iteration. Higher order approximations are also available for time discretization.

In conclusion we would like to reiterate that the methods described in Section 4.2 are not implemented in any operational model used at SMHI or at any climate research center known to our contacts there. The use of rigorous mathematical methods to develop climate models is considered visionary in the field at this time.

6. APPENDIX

Definition 6.1. (Metric space) A metric space is an ordered pair (M, d) where M is a set and d is a metric on M , i.e. a function

$$d : M \times M \rightarrow \mathbb{R},$$

such that for any $x, y, z \in M$, the following holds:

- (1) $d(x, y) \geq 0$ (non-negative),
- (2) $d(x, y) = 0$ iff $x = y$,

- (3) $d(x, y) = d(y, x)$ (symmetry),
- (4) $d(x, y) \leq d(x, y) + d(y, z)$ (triangle inequality).

Definition 6.2. (Complete space) A complete space is a metric space M where every Cauchy sequence of points in M has a limit that is also in M .

Definition 6.3. (Cauchy sequence) A sequence x_1, x_2, x_3, \dots of real numbers is called a Cauchy sequence, if for every positive real number ϵ , there is a positive integer N such that for all natural numbers $m, n > N$

$$|x_m - x_n| < \epsilon.$$

Definition 6.4. (Pre-Hilbert space) A Pre-Hilbert space (or inner product space) is a real vector space with an inner product.

Definition 6.5. (Hilbert space) A Hilbert space is a complete Pre-Hilbert space.

Definition 6.6. (Space of continuous functions) A C^k space is the linear space of all real functions u defined in a domain which together with all their derivatives $D^\alpha u$ of order $|\alpha| < k$ are continuous in the specified domain.

Definition 6.7. (Cholesky decomposition) The Cholesky decomposition of a Hermitian positive-definite matrix \mathbf{A} is a decomposition of the form

$$\mathbf{A} = \mathbf{L}\mathbf{L}^*,$$

where \mathbf{L} is a lower triangular matrix with real and positive diagonal entries and \mathbf{L}^* is the conjugate transpose of \mathbf{L} . Every Hermitian positive-definite matrix (and thus also every real-valued symmetric positive-definite matrix) has a unique Cholesky decomposition.

Definition 6.8. (Hermitian matrix) A Hermitian matrix is a complex-valued square matrix which is equal to its own conjugate transpose. For a Hermitian matrix \mathbf{A} with elements a_{ij} :

$$a_{ij} = \overline{a_{ji}},$$

where $\overline{a_{ji}}$ is the complex conjugate of a_{ji} .

Definition 6.9. (Positive-definite matrix) A positive-definite matrix is a Hermitian matrix for which the product

$$\mathbf{v}^* \mathbf{A} \mathbf{v},$$

is real and positive for all nonzero complex-valued vectors \mathbf{v} with conjugate transpose \mathbf{v}^* . If \mathbf{A} is Hermitian and positive-definite, then all the eigenvalues of \mathbf{A} are real and positive.

REFERENCES

- [1] J. G. Charney, R. Fjörtoft, J. von Neumann, *Numerical Integration of the Barotropic Vorticity Equation* Tellus, Vol. 2, 1950, pp 237-254
- [2] Kristine Harper, Louis W. Uccellini, Eugenia Kalnay, Kenneth Carey, Lauren Morone (May 2007). *2007: 50th Anniversary of Operational Numerical Weather Prediction*. Bulletin of the American Meteorological Society 88 (5): 639650. Bibcode:2007BAMS...88.639H. doi:10.1175/BAMS-88-5-639
- [3] *History of numerical weather prediction*. (2014, January 21). In Wikipedia, The Free Encyclopedia. Retrieved 08:16, May 5, 2014, from url
- [4] Charles L. Fefferman, *Existence and smoothness of the Navier-Stokes equation, official problem formulation*, Clay Institute, 2014
- [5] E. Lorenz, *Deterministic Nonperiodic Flow* Journal of the Atmospheric Sciences, 1963
- [6] R. Kautz, *Chaos - The science of predictable random motion*, Oxford University Press, 2011, New York, p. 145-164, ISBN: 978-0-19-959458-0
- [7] R. Temam, *Navier Stokes Equations and Nonlinear Functional Analysis*, CBMS-NSF regional conference series in applied mathematics, 66. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1995, p.4. ISBN: 0-89871-340-4
- [8] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady state problems*, Springer, 2011, p.2, ISBN: 9783540941729
- [9] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, Science Publishers Inc., 1969, p. 2
- [10] A.A. White, B.J. Hoskins, I. Roulstone, A. Staniforth, *Consistent approximate models of the global atmosphere: shallow, deep, hydrostatic, quasi-hydrostatic, and non-hydrostatic*, Met Office, Exeter, UK, 2005
- [11] T. Davies, M. J. P. Cullen, A. J. Malcolm, M. H. Mawson, A. Staniforth, A. A. White, N. Wood, *A new dynamical core for the Met Offices global and regional modelling of the atmosphere*, Met Office, Exeter, UK, 2004
- [12] J. David Neelin, *Climate Change and Climate Modeling* Cambridge University Press, Cambridge, 2011, p. 76, ISBN: 978-0-521-60243-3
- [13] Peter Lynch, *The Origins of Computer Weather Prediction and Climate Modeling* Journal of Computational Physics, 227, 2007
- [14] H-O Kreiss, J Lorenz, *Initial-Boundary Value Problems and the Navier-Stokes Equations*, Academic Press, 1989, ISBN: 978-0-12-426125-9
- [15] R. LeVeque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge University Press, 2002, ISBN: 13 978-0-521-00924-9
- [16] J.E. Hicken, D.W. Zingg, *Summation-By-Parts Operators and High-Order Quadrature*, Institute for Aerospace Studies, University of Toronto, Toronto, Ontario, M3H 5T6, Canada, Cornell University Library: arXiv:1103.5182v1 [math.NA]
- [17] M. Svärd, J. Nordström *Review of Summation-by-Parts Schemes for Initial-Boundary-Value Problems*, arXiv:4984v1 [math.NA] 20 Nov 2013
- [18] L. C. Evans *Partial Differential Equations*, Volym 19 av Graduate studies in mathematics, ISSN 1065-7339, American Mathematical Society, 1998

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